MOTIVIC ACTION CONJECTURES

These are notes for a series of two talks at Imperial College London:

- Talk 1 (June 2, 2023): singular cohomology of locally symmetric spaces,
- Talk 2 (June 9, 2023): cohomology of vector bundles on Shimura varieties.

Talk 1: Cohomology of arithmetic groups & periods of automorphic forms.

(see also Venkatesh's Takagi lectures)

§0. Very brief introduction.

Motivating question: How do algebraic cycles interact with the Langlands program?

We will study an example of this phenomenon:

multiple contributions of an aut. form to cohomology $\leftrightarrow$ algebraic cycles

Global Langlands correspondence:

$$\left\{\begin{matrix}\text{automorphic}\\ \text{forms/reps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Galois}\\ \text{representations}\end{matrix}\right\} \xleftarrow[\text{real.}]{\text{étale}} \left\{\text{motives}\right\}$$

$$\pi \longmapsto \rho_\pi$$

$Ad^* M_\pi$ = dual adjoint motive

π = $\underbrace{\text{automorphic}}_{\text{(packet of tempered)}}$ rep. of $G(\mathbb{A})$

$$\rightsquigarrow \underbrace{H^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K, \mathbb{Q}\right)_\pi}_{\text{(Hecke algebra acts as on }\pi\text{)}} \cong \bigoplus_{i=0}^{\delta} \underbrace{H_\pi^{i_{min}+i}}_{\dim = \binom{\delta}{i}} \quad \overset{?}{\circlearrowleft} \quad \underbrace{\Lambda^* \overbrace{H^1_{\mathcal{M}}(Ad^* M_\pi, \mathbb{Q}(1))^\vee}^{\dim = \delta}}_{\text{motivic coh. group} \approx \text{algebraic cycles}}$$

$\dim \Lambda^i (H^1_{\mathcal{M}})^\vee = \binom{\delta}{i}$

Plan:

1. Computing $H^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K, \mathbb{C}\right)$ (Matsushima's formula, Borel–Wallach).
2. The fundamental problem: explain contributions by extra endomorphisms.
3. The extra endomorphisms:
   - over $\mathbb{C}$ (Prasanna–Venkatesh),
   - over $\mathbb{Q}_p$ (Venkatesh).
4. The extra endomorphisms over $\mathbb{Q}$ (conjecturally): motivic action.

§1. Cohomology of arithmetic groups.

G = reductive alg. gp / $\mathbb{Q}$ $\quad$ ($SL_{n,\mathbb{Q}}$, $R_{k/\mathbb{Q}} SL_{n,k}$, Sp_{2n}, ...)

<u>Locally sym. space</u>: $X_G := G(\mathbb{Q})\backslash G(\mathbb{A}) / K_\infty K$ $\qquad$ $K_\infty \subseteq G(\mathbb{R})$, $K \subseteq G(\mathbb{A}_f)$
$\qquad$ max'l cpct $\qquad$ cpct open

real manifold, no complex structure in general

<u>Goal</u>: Compute $H^*(X_G, \mathbb{C})$. Note: $H^*(\Gamma, \mathbb{C})$ for some $\Gamma \leq G(\mathbb{Q})$
when $X_G = \Gamma\backslash S$.

<u>Motivating examples.</u> (Keep on the board!)

(0) $GL_{2,\mathbb{Q}}$: Eichler–Shimura $\qquad$ $\mathcal{H}_2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

$X_G = \Gamma\backslash\mathcal{H}_2$ $\qquad$ $S_k(\Gamma) \xrightarrow{\omega} H^1(\Gamma\backslash\mathcal{H}_2, \mathbb{C})$ $\qquad$ $i_{min} = 1$

$\boxed{\delta = 0}$

modular curve $\qquad$ $f(z) \longmapsto \omega_f := f(z)\,dz$

(not interesting for us)

(1) $GL_{2,k}$, $k = \mathbb{Q}(\sqrt{-d})$: Bianchi modular forms

$\mathcal{H}_3$ = hyperbolic 3-space $= \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$

$\circlearrowleft$

$\Gamma \subseteq GL_2(\mathbb{Q}) \subseteq GL_2(\mathbb{C})$ $\qquad$ with metric $ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$

$X_G = \Gamma\backslash\mathcal{H}_3$ real threefold (no algebraic structure)

<u>Bianchi modular form</u> f of weight 2 : $f : \mathcal{H}_3 \longrightarrow \mathbb{C}^3$ $\quad$ $(f = (f_1, f_2, f_3))$
& transformation property under Γ.

$$S_2(\Gamma) \xrightarrow{\omega^1} H^1(\Gamma\backslash\mathcal{H}_3, \mathbb{C}) \ni \omega_f^1 := f_1 \frac{dz}{t} - f_2 \frac{dt}{t} + f_3 \frac{d\bar{z}}{t}$$

$i_{min} = 1$

$\boxed{\delta = 1}$

$$S_2(\Gamma) \xrightarrow{\omega^2} H^2(\Gamma\backslash\mathcal{H}_3, \mathbb{C}) \ni \omega_f^2 := f_1 \frac{dt \wedge dz}{t^2} - f_2 \frac{dz \wedge d\bar{z}}{t^2} + f_3 \frac{dt \wedge d\bar{z}}{t^2}$$

(2) $GL_{2,K}$, K/F CM, $[F:\mathbb{Q}]=2$

$$\rightsquigarrow \quad \Gamma \subseteq GL_2(\mathcal{O}_K) \subseteq GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \circlearrowright \mathbb{H}_3^2 \quad \rightsquigarrow X_\Gamma = \Gamma \backslash \mathbb{H}_3^2 \text{ real 6-fold, no alg. str.}$$

$$\begin{array}{lll} & \xrightarrow{\omega^2} H^2(\Gamma\backslash \mathbb{H}_3^2, \mathbb{C}) \ni \omega_f^2 & 1 \\ S_{2,2}(\Gamma) & \overset{\omega_1^3}{\underset{\omega_2^3}{\rightrightarrows}} H^3(\Gamma\backslash \mathbb{H}_3^2, \mathbb{C}) \ni \omega_{f,1}^3, \omega_{f,2}^3 & 2 \\ & \xrightarrow{\omega^4} H^4(\Gamma\backslash \mathbb{H}_3^2, \mathbb{C}) \ni \omega_f^4 & 1 \end{array}$$

$i_{min} = 2$

$\boxed{\delta = 2}$

Thm. (1) Matsushima's formula:

$$H^*(\mathfrak{g}, K; C^\infty(S/\Gamma)) = \bigoplus_{\pi \subseteq L^2(S/\Gamma)} m(\pi, \Gamma) \cdot H^*(\mathfrak{g}, K; \pi)$$

(2) Borel–Wallach: If $\pi =$ tempered ($\pi \subseteq L^{2+\varepsilon}$ $\forall \varepsilon > 0$), then $\delta = \mathrm{rk}(G) - \mathrm{rk}(K)$, $i_{min} = \dfrac{\dim(S/\Gamma) - \delta}{2}$

and:
- $H^i(\mathfrak{g}, K; \pi) = 0$ if $i \notin [i_{min}, i_{min}+\delta]$
- $\dim H^i(\mathfrak{g}, K; \pi) = \binom{\delta}{i - i_{min}}$ if $i \in [i_{min}, i_{min}+\delta]$.

(For $\pi =$ non-tempered, contributions look different.)

Def. $\delta = \mathrm{rk}\, G - \mathrm{rk}\, K$ is the defect. Sometimes called l_0.

§2. The fundamental problem.

∃ extra endomorphisms:

$$\begin{array}{ccccccccc}
 & H^{i_{min}}(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}}) & \overset{?}{\rightrightarrows} & H^{i_{min}+1}(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}}) & \overset{?}{\rightrightarrows} & H^{i_{min}+i}(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}}) & \overset{?}{\rightrightarrows} & H^{i_{min}+\delta}(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}}) \\
\dim & d & & d\cdot\delta & & d\cdot\binom{\delta}{i} & & d
\end{array}$$

s.t. $H^*(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}})$ is all explained in terms of $H^{i_{min}}(\Gamma, \cancel{\mathbb{C}}_{\mathbb{Q}})$?

Endomorphisms currently there: Hecke operators, Lefschetz operators.

§3. Extra endomorphisms over $\mathbb{C}$ & over $\mathbb{Q}_p$.

<u>Over $\mathbb{C}$:</u> • In examples, clear:

(1) $$H^1(\mathcal{H}_3/\Gamma, \mathbb{C}) \longrightarrow H^2(\mathcal{H}_3/\Gamma, \mathbb{C})$$
$$\omega_f^1 \longmapsto \omega_f^2$$

These can be descr. geometrically as Hodge * operators; c.f. Venkatesh's Takagi lectures.

(2) $$\begin{tikzcd}
 & H^3(\mathcal{H}_3^2/\Gamma, \mathbb{C}) \arrow[dr] & \\
H^2(\mathcal{H}_3^2/\Gamma, \mathbb{C}) \arrow[ur] \arrow[dr] & & H^4(\mathcal{H}_3^2/\Gamma, \mathbb{C}) \\
 & H^3(\mathcal{H}_3^2/\Gamma, \mathbb{C}) \arrow[ur] &
\end{tikzcd}$$

$$\omega_f^2 \longmapsto \omega^3_{f,1} \qquad \omega^3_{f,1} \longmapsto \omega_f^4 \qquad \omega^3_{f,2} \longmapsto \omega_f^4$$

$$\omega_f^2 \longmapsto \omega^3_{f,2} \qquad \omega^3_{f,1} \longmapsto \omega_f^4 \qquad \omega^3_{f_2} \longmapsto \omega_f^4$$

• In general: go back to Borell-Wallach & define the endomorphism at the level of $(\mathfrak{g}, K)$-cohomology (see Prasanna-Venkatesh for details).

> <u>Result (Prasanna-Venkatesh).</u> $\Lambda^*(\mathbb{C}^\delta) \circlearrowright H^*(\Gamma, \mathbb{C})_{temp}$
>
> δ = defect

($\mathbb{C}^\delta$ will be a canonical v.sp., see later)

<u>Over $\mathbb{Q}_p$.</u> Think about $H^*(\Gamma, \mathbb{Q})$.

<u>If</u> we had a lot of classes in $H^1(\Gamma, \mathbb{Q})$, we could cup with them to define maps $H^*(\Gamma, \mathbb{Q}) \longrightarrow H^{*+1}(\Gamma, \mathbb{Q})$, but alas not many classes in $H^*(\Gamma, \mathbb{Q})$.

Instead, look at $H^1(\Gamma, \mathbb{Z}/q)$ for various q.

Replacing Γ by $\Gamma' \leq \Gamma$, we can always find classes in $H^1(\Gamma', \mathbb{Z}/q)$.

<u>Example:</u> $SL_2(\mathbb{Z}) \geq \Gamma_0(p)$ for $q \mid p-1$

$$\rightsquigarrow \alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a \bmod p \in (\mathbb{Z}/p)^\times \longrightarrow \mathbb{Z}/q$$

gives $\alpha \in H^1(\Gamma_0(p), \mathbb{Z}/q)$.

<u>Hecke operators.</u> Given $\Gamma_1, \Gamma_2 \leq \Gamma$ finite index & $\varphi : \Gamma_1 \xrightarrow{\simeq} \Gamma_2$,

$$H^j(\Gamma) \xrightarrow{\text{res}} H^j(\Gamma_2) \xrightarrow{\varphi^*} H^j(\Gamma_1) \xrightarrow{\text{cores}} H^j(\Gamma)$$

<u>Example:</u> $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_1 = \Gamma_0(p)$

$\Gamma_2 = \Gamma^0(p) = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \bmod p \right\}$

φ = conjugation by $\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ the above process gives T_p on $H^1(\Gamma)$.

<u>Derived Hecke operators.</u> Given $\alpha \in H^i(\Gamma_1, \mathbb{Z}/q)$,

$$\begin{tikzcd} H^j(\Gamma, \mathbb{Z}/p) \arrow[r, "\text{res}"] \arrow[rrrr, bend right, "T_{q,\alpha}"'] & H^j(\Gamma_2) \arrow[r, "\varphi^*"] & H^j(\Gamma_1) \arrow[r, "\cup\alpha"] & H^{j+i}(\Gamma_1) \arrow[r] & H^{j+i}(\Gamma) \end{tikzcd}.$$

derived Hecke operator

Recall: can think of

$$\begin{tikzcd} S/\Gamma_1 \arrow[rr, dashed, "\varphi"] \arrow[dr] & & S/\Gamma_2 \arrow[dl] \\ & S/\Gamma & \end{tikzcd}$$

"Shimura cover".

$\rightsquigarrow$ similarity to the Taylor–Wiles method!

<u>Result (Venkatesh)</u>: Using a limiting process, pass to $H^*(\Gamma, \mathbb{Z}_p)_{temp}$, and tensor with $\mathbb{Q}_p$ $\rightsquigarrow$ algebra of derived operators is isom. to

$$\Lambda^* \mathbb{Q}_p^\delta \circlearrowright H^*(\Gamma, \mathbb{Q}_p)_{temp}.$$

assuming:
- $\exists$ Galois rep. ass. to torsion coh. classes,
- local-to-global compatibility,
- technical conditions at p (OK for p large).

($\mathbb{Q}_p^\delta$ will be a canonical v.sp., see later)

§4. Extra operators over $\mathbb{Q}$.

[Ignore powers of π & $\sqrt{\mathbb{Q}^\times}$ for simplicity.]

So far, have:
- $\Lambda^* \mathbb{C}^\delta \circlearrowright H^*(\Gamma, \mathbb{C})_{temp}$,
- $\Lambda^* \mathbb{Q}_p^\delta \circlearrowright H^*(\Gamma, \mathbb{Q}_p)_{temp}$.

<u>Q</u>: What about $H^*(\Gamma, \mathbb{Q})$?

Let's go back to example (1): $GL_{2,K}$, $K = \mathbb{Q}(\sqrt{d})$.

f = Bianchi m.f. of wt 2 $\longleftrightarrow$ E/K elliptic curve

Recall: the action was $H^1(\mathcal{H}_3/\Gamma, \mathbb{C})_f \longrightarrow H^2(\mathcal{H}_3/\Gamma, \mathbb{C})_f$

$$\omega_f^1 \longmapsto \omega_f^2$$

Define periods $u_f^1, u_f^2 \in \mathbb{C}$ such that $\omega_f^i / u_f^i \in \underbrace{H^i(\mathcal{H}_3/\Gamma, \mathbb{Q})_f}_{\text{1-dimensional}}$.

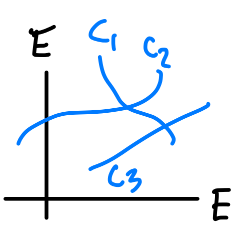
These seem to be closely linked to the geometry of E:

- <u>Conj. (Cremona–Whitley).</u> $u_f^1 \sim \int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma}$

See forthcoming work w/ Kartik for this phrasing.

- <u>Conj. (explicit version of Beilinson + ε)</u>

$C_i \subseteq E \times E$
$f_i \in \mathcal{O}_{C_i}^\times$
$\sum \operatorname{div}(f_i) = 0$
$$\rightsquigarrow \quad u_f^2 \sim \sum_i \int_{C_i, \mathbb{C}} \log|f_i| \cdot \tfrac{1}{2}\left[\omega^\sigma \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^\sigma} \wedge \omega^{\bar\sigma}\right]$$



$$\left(\begin{array}{l} \varepsilon = \text{Urban's thesis}: \quad \omega_f^1 \cdot \omega_f^2 \sim L(f, \mathrm{Ad}, 1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{adjoint } L\text{-function} \end{array} \right)$$

Conclusion: rational action (up to above conjectures)

$$\begin{array}{ccc} H^1(X, \mathbb{Q}) & \longrightarrow & H^2(X, \mathbb{Q}) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ \dfrac{\omega_f^1}{\int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma}} & \longmapsto & \dfrac{\omega_f^2}{\sum_i \int_{C_i(\mathbb{C})} \log|f_i| \cdot \frac{1}{2}\left(\omega^\sigma \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^\sigma} \wedge \omega^{\bar\sigma}\right)} \end{array}$$

$\rightsquigarrow$ the data $\{(C_i, f_i)\}$ defines a "motivic cohomology class":

$$\{(C_i, f_i)\}_i \in H^1_{\mathcal{M}}(E \times E, \mathbb{Q}(1)).$$

General story.

π = aut. rep. of $G(\mathbb{A})$ $\rightsquigarrow$

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, "\rho_\pi"] \arrow[dr, bend right, "\mathrm{Ad}^*\rho_\pi"'] & {}^L G(\overline{\mathbb{Q}}_\ell) \arrow[d, "\mathrm{Ad}^*"] \\ & GL({}^L\mathfrak{g}^*) \end{tikzcd}$$

adjoint rep.

$$\rightsquigarrow L(\pi, \mathrm{Ad}, s) = L(\mathrm{Ad}^* \rho_\pi, s).$$

Assume: $\exists$ motive M whose Galois rep. is $\mathrm{Ad}^* \rho_\pi$.

& consider $H^1_\mu := H^1_\mu(M, \mathbb{Q}(1))$ motivic cohomology group, higher Chow group.

$\mathbb{Q}$-vector space

Example (1). f = BMF wt 2 $\longleftrightarrow$ E/K EC

$\rightsquigarrow M = (\mathrm{Sym}^2 H^1(E))(1)$, realized within $E \times E$

$$\rightsquigarrow H^1_\mu(M, \mathbb{Q}(1)) = H^3_\mu(\mathrm{Sym}^2 H^1(E), \mathbb{Q}(2)) \subseteq H^3_\mu(E \times E, \mathbb{Q}(2))$$

$$\left\{ \{(C_i, f_i)\}_i : \begin{array}{l} C_i \subseteq E \times E \text{ irred. curve} \\ f_i \in \mathbb{C}_i^\times \\ \sum \mathrm{div} f_i = 0 \end{array} \right\} \Big/ \{\ldots\}$$

Beilinson : $\exists$ regulator map $\mathrm{reg}_{\mathbb{C}} : H^1_{\mathcal{M}}(\mathrm{Ad}^*M, \mathbb{Q}(1)) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \underbrace{H^1_{\mathcal{D}}(\mathrm{Ad}^*M_{\mathbb{R}}, \mathbb{R}(1))}_{\text{Deligne cohomology}}$

Example (1). $\left\langle \mathrm{reg}_{\mathbb{C}}(\{(C_i, f_i)\}), \frac{1}{2}(\omega^{\sigma} \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^{\sigma}} \wedge \omega^{\bar\sigma}) \right\rangle_{PD}$

$$= \sum_i \int_{C_{i,\mathbb{C}}} \log|f_i| \cdot \frac{1}{2}(\omega^{\sigma} \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^{\sigma}} \wedge \omega^{\bar\sigma})$$

Conj. (Beilinson). (a) $\mathrm{reg}_{\mathbb{C}}$ is an isomorphism

(b) $L(\pi, \mathrm{Ad}, 1)$ is explicitly related to $\det(\mathrm{reg}_{\mathbb{C}})$.

Example (1). (a) $\dim(H^1_{\mathcal{M}}) = 1$

(b) $L(f, \mathrm{Ad}, 1) \sim \left(\int_{E^{\sigma}(\mathbb{C})} \omega^{\sigma} \wedge \overline{\omega^{\sigma}}\right)\left(\sum_i \int_{C_{i,\mathbb{C}}} \log|f_i| \cdot \frac{1}{2}(\omega^{\sigma} \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^{\sigma}} \wedge \omega^{\bar\sigma})\right)$

Fact. π = temp. aut. rep. of $G(\mathbb{A})$, $\delta = \mathrm{rk}\, G - \mathrm{rk}\, K$ defect

$\Rightarrow$ $\dim(H^1_{\mathcal{D}}) = \delta$, i.e. $H^1_{\mathcal{M}} \cong \mathbb{Q}^{\delta}$ according to (a).

(Assume (a) above).

Conjecture (Prasanna - Venkatesh). π = tempered (trivial weight)

Define action of $\Lambda^*(H^1_{\mathcal{D}})^{\vee} \circlearrowleft H^*(\Gamma, \mathbb{R})_{\pi}$ as before

$\rightsquigarrow$ get action of $\Lambda^*(H^1_{\mathcal{M}})^{\vee} \otimes \mathbb{R}$ via $\mathrm{reg}_{\mathbb{C}}^{\vee} : (H^1_{\mathcal{M}})^{\vee} \otimes \mathbb{R} \longrightarrow (H^1_{\mathcal{D}})^{\vee}$.

Then $\Lambda^*(H^1_{\mathcal{M}})^{\vee} \circlearrowleft H^*(\Gamma, \mathbb{Q})_{\pi}$ (i.e. rational action).

Example (2). $\mathrm{reg}_{\mathbb{C}}^{\vee} : (H^1_{\mathcal{D}})^{\vee} \longrightarrow (H^1_{\mathcal{M}})^{\vee}$ 2×2 matrix

$\rightsquigarrow$ this predicts rationality of some linear comb. of $\omega^3_{f,11}, \omega^3_{f,12}$.

Note : $\Lambda^2(H^1_{\mathcal{M}})^{\vee}$ gives action $H^2 \longrightarrow H^4$

& its rationality is $\Leftrightarrow$ Beilinson (b).

What about $\mathbb{Q}_p$?

$\exists$ $\mathrm{reg}_p : H^1_{\mathcal{M}}(M, \mathbb{Q}(1)) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_p \longrightarrow \underbrace{H^1_f(\mathrm{Ad}^* \rho_\pi, \mathbb{Q}_p(1))}$

Bloch-Kato Selmer gp

which is conjecturally an isomorphism.

Conj. (Venkatesh).

Define action of $\Lambda^*(H^1_f)^\vee \circlearrowright H^*(\Gamma, \mathbb{Q}_p)_\pi$ as before.

$\rightsquigarrow$ get action of $\Lambda^*(H^1_{\mathcal{M}})^\vee \otimes \mathbb{Q}_p$ via $\mathrm{reg}_p^\vee : (H^1_{\mathcal{M}})^\vee \to (H^1_f)^\vee$.

Then $\Lambda^*(H^1_{\mathcal{M}})^\vee \circlearrowright H^*(\Gamma, \mathbb{Q})_\pi$ acts rationally.

More specifically: Venkatesh considers

$\tilde{\mathbb{T}}_{\mathbb{Z}/p^n} \subseteq \mathrm{End}(H^*(\Gamma, \mathbb{Z}/p^n))$ gen. by $T_{q,\alpha}$ $\forall q, \alpha$.

$\tilde{\mathbb{T}}_{\mathbb{Z}_p} \subseteq \mathrm{End}(H^*(\Gamma, \mathbb{Z}_p)_\pi)$ s.t. mod. p^n land in $\tilde{\mathbb{T}}_{\mathbb{Z}/p^n}$.

Thm (Venkatesh). Under assumptions,

$H^*(X, \mathbb{Z}_p)_\pi$ = free over $\tilde{\mathbb{T}}$ & $\tilde{\mathbb{T}} \cong \Lambda^*(H^1_f)^\vee$.

(I don't know how to make such a nice explicit statement out of this... The action over $\mathbb{Q}_p$ is constructed inexplicitly

$\Rightarrow$ can only get an explicit statement for $\mathbb{Z}/p^n\mathbb{Z}$; see next talk for a slightly different example of this.)

Things I did not talk about but should have:

- Galatius-Venkatesh: $\Lambda^*(H^1_f)^\vee \cong (\pi_* \mathcal{R}) \otimes \mathbb{Q}_p$

  $\mathcal{R}$ = derived deformation ring for $\bar{\rho}_\pi$ mod p

  and $\pi_* \mathcal{R} \circlearrowright H_*(X, \mathbb{Z}_p)_\pi$

- Hansen–Thorne, Khare–Ronchetti:

  $\ell = p$, i.e. p-adic version of this story.

Next time: fresh start $\rightsquigarrow$ can we repeat this story for cohomology of vector bundles over Shimura varieties?

Talk 2: Cohomology of vector bundles over Shimura varieties.

Last time: Goal: explain multiple contributions to cohomology in terms of algebraic cycles.

π = tempered aut. rep. for $G(\mathbb{A}) \rightsquigarrow M_\pi$ motive, $M = \mathrm{Ad}^* M_\pi$ dual adjoint motive

Conj (Prasanna–Venkatesh, Venkatesh). $\Lambda^* H^1_{\mathcal{M}}(M, \mathbb{Q}(1))^\vee \circlearrowright H^*(X_G, \mathbb{Q})_\pi$

s.t.
- over $\mathbb{C}$, explicit action of $\Lambda^*(H^1_\mathcal{D})^\vee$,
- over $\mathbb{Z}/p^n$, explicit action by derived Hecke operators,
- over $\mathbb{Q}_p$, action of $\Lambda^*(H^1_f)^\vee$ via patching from $\mathbb{Z}/p^n$.

$\rightsquigarrow$ explicit version over $\mathbb{C}$ for Bianchi modular forms $f_0 \leftrightarrow E/K = \mathbb{Q}(\sqrt{d})$:

$\alpha \in H^1_\mathcal{M}$ corresponds to $\{(C_i, \varphi_i)\}$, $C_i \subseteq E \times E$ curve, φ_i fcn. on C_i, $\sum \operatorname{div} \varphi_i = 0$

$$\begin{array}{ccc} H^1(X, \mathbb{Q}) & \longrightarrow & H^2(X, \mathbb{Q}) \\ \cup\!\!\!\!\!\mid & & \cup\!\!\!\!\!\mid \\ \dfrac{\omega_f^1}{\int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma}} & \longmapsto & \dfrac{\omega_f^2}{\sum_i \int_{C_i(\mathbb{C})} \log|\varphi_i| \cdot \frac{1}{2}\left(\omega^\sigma \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^\sigma} \wedge \omega^{\bar\sigma}\right)} \end{array}$$

Today: Is there a similar story for "coherent cohomology", i.e. cohomology of vector bundles over Shimura varieties?

Plan: Examples!

(1) Modular forms ($GL_{2,\mathbb{Q}}$).

(2) Hilbert modular forms ($GL_{2,F}$).

(3) Siegel modular forms (GSp_4).

(4) Why are things harder? (Generalities on coherent cohomology.)

§1. Modular forms & coherent cohomology.

$f : \mathcal{H} \longrightarrow \mathbb{C}$ cusp form of wt k, level $\Gamma = \Gamma_1(N)$

$\Gamma \backslash \mathcal{H} \times \mathbb{C}$ $\quad \gamma(z,\tau) = \left(\frac{az+b}{cz+d}, (cz+d)^k \cdot \tau\right)$

$\downarrow \qquad (z, f(z))$

$\qquad\quad \uparrow$ section

$\Gamma \backslash \mathcal{H} \qquad z$

$\xrightarrow{\text{upgrade to } \mathbb{Q}}$

ω^k $\quad \omega = $ Hodge bundle, $\omega^{\otimes 2} \cong \Omega^1_X$

$\downarrow$

$X_{\mathbb{Q}}$ modular curve $/\mathbb{Q}$

Upshot: If f has FC in E, then: $f \in H^0(X_{\mathbb{Q}}, \omega^k) \otimes E$.

Other cohomology classes associated with f?

Facts. ① $f(-\bar{z}) \cdot y^{k-2}\, d\bar{z}$ defines a class $\omega_f^\infty \in H^1(X_{\mathbb{C}}, \omega^{2-k})$.

② If $T_p f = a_p f$ $\forall p \nmid N$, then $T_p \omega_f^\infty = a_p \omega_f^\infty$ $\forall p \nmid N$;

we write: $\omega_f^\infty \in H^1(X_{\mathbb{C}}, \omega^{2-k})_f$ "f-isotypic component".

③ These are all the cohomology classes associated to f.

Secretly: $\pi_\infty|_{SL_2(\mathbb{R})} \cong \underbrace{D^+_{k-1}}_{H^0} \oplus \underbrace{D^-_{k-1}}_{H^1}$ & $f \longmapsto \omega_f^\infty$ corresponds to $w_\infty = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$.

Key difficulty: no such map w_∞ was needed in Prasanna–Venkatesh!

Langlands program.

$$\left\{\begin{matrix}\text{automorphic}\\ \text{forms / reps}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Galois}\\ \text{representations}\end{matrix}\right\}$$

f modular form of wt $k \geq 1$ $\rightsquigarrow$ $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell)$

$f \in H^0(X, \omega^k)_f$, $\omega_f^\infty \in H^1(X_{\mathbb{C}}, \omega^{2-k})_f$

DEGENERACIES FOR $k=1$.

f of wt $k=1$

$f \in H^0(X, \omega)_f$, $\omega_f^\infty \in H^1(X_{\mathbb{C}}, \omega)_f$

GOAL. $\longleftrightarrow$

$\rho_f : \mathrm{Gal}(L/\mathbb{Q}) \to GL_2(E)$

factors thru a finite $L/\mathbb{Q}$

Consider Stark unit group associated with $\mathrm{Ad}^0 \rho_f$:

$$U_f = \mathrm{Hom}(\mathrm{Ad}^0 \rho_f, \mathcal{O}_L^\times \otimes E) \cong H^1_{\mathcal{M}}(\mathrm{Ad}^* M, \mathbb{Q}(1))$$

from last time!

Fact. $\dim U_f = 1$. In fact, $c \in U_f^\vee \otimes \mathbb{C}$ explicit basis element.

Def. $U_f^\vee \otimes \mathbb{C} \ni c$ explicit element $\Rightarrow$ $U_f^\vee \otimes \mathbb{C} \circlearrowright H^*(X_{\mathbb{C}}, \omega)_f$ by

$$c * f = \omega_f^\infty$$

I.e. $H^*(X_{\mathbb{C}}, \omega)_f =$ free $\Lambda^* U_f^\vee \otimes \mathbb{C}$-module of rk 1.

Conj. This action descends to $\Lambda^* U_f^\vee \circlearrowright H^*(X_{\mathbb{Q}}, \omega)_f$.

Explicitly: $u_f^\vee \in U_f^\vee$ acts by
$$\begin{array}{ccc} H^0(X, \omega)_f & \longrightarrow & H^1(X, \omega)_f \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ f & \longmapsto & \dfrac{\omega_f^\infty}{\log|u_f|} \end{array}$$

(See H.; implicit in Harris–Venkatesh.)

Prop. This conjecture is equivalent to Stark's conjecture for $\mathrm{Ad}^0 \rho_f$.

In particular, it is true in CM/RM cases.

Pf. Need to check that $\dfrac{\omega_f^\infty}{\log|u_f|}$ is rational in cohomology.

Use Serre duality: $\left\langle f, \dfrac{\omega_f^\infty}{\log|u_f|}\right\rangle_{SD} = \dfrac{\langle f,f\rangle}{\log|u_f|} \sim_{E^\times_{,\pi}} \dfrac{L(f,\mathrm{Ad},1)}{\log|u_f|} \sim_{E^\times_{,\pi}} 1.$ $\square$

Harris–Venkatesh: action modulo p^n.

Action of derived Hecke operators:

q prime, $q\equiv 1 \bmod p^n$, $\alpha : (\mathbb{Z}/q)^\times \twoheadrightarrow \mathbb{Z}/p^n = \Delta$

$$\begin{tikzcd} X_1(q) \arrow[d] \arrow[dr] & \\ X_0(q) & X_1(q)^\Delta \arrow[l] \end{tikzcd} \rightsquigarrow \mathfrak{S}_\alpha \in H^1_{et}(X_0(q), \mathbb{Z}/p^n) \rightsquigarrow \mathfrak{S}_\alpha^{\mathrm{Zar}} \in H^1_{\mathrm{Zar}}(X_0(q)_{\mathbb{Z}/p^n}, \mathcal{O})$$

$$\begin{tikzcd} & X_0(q) \arrow[r, "\cup \mathfrak{S}_\alpha^{\mathrm{Zar}}"'] \arrow[dl, "\pi_1"'] & X_0(q) \arrow[dr, "\pi_2"] & \\ X & & & X \end{tikzcd}$$

Def. $T_{q,\alpha} : H^0(X_{\mathbb{Z}/p^n}, \omega) \longrightarrow H^1(X_{\mathbb{Z}/p^n}, \omega)$ <u>derived Hecke operator</u>

given by $T_{q,\alpha} = \pi_{2,*} \circ (\cup\, \mathfrak{S}_\alpha^{\mathrm{Zar}}) \circ \pi_1^*$

Conj. (Harris–Venkatesh). $\exists$ action $\Lambda^* U_f^\vee \circlearrowright H^*(X,\omega)_f$ s.t. for $u_f^\vee \in U_f^\vee$, $\forall$ p, n & $q \equiv 1 \bmod p^n$

$\begin{pmatrix}\text{reduction of } u_f^\vee \\ \bmod p^n\end{pmatrix}$ acts via $T_{q,\alpha}$.

Evidence:

→ Numerical: Harris–Venkatesh, Marcil

→ Theoretical: in CM & RM cases (as above over $\mathbb{C}$)

- Darmon–Harris–Rotger–Venkatesh
- Lecouturier
- Robin Zhang

§2. Hilbert modular forms. $F = \mathbb{Q}(\sqrt{d})$ for simplicity

$f : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ Hilbert modular form wt (k_1, k_2)

Fact. The contr. of f to coherent cohomology are:

- $f \in H^0(X_{\mathbb{Q}}, \omega^{k_1, k_2})_f$
- $\omega_f^1 := f(-\bar{z}_1, z_2)\, y_1^{k_1-2}\, d\bar{z}_1 \in H^1(X_{\mathbb{C}}, \omega^{2-k_1, k_2})_f$
- $\omega_f^2 := f(z_1, -\bar{z}_2)\, y_2^{k_2-2}\, d\bar{z}_2 \in H^1(X_{\mathbb{C}}, \omega^{k_1, 2-k_2})_f$
- $\omega_f^{1,2} := f(-\bar{z}_1, -\bar{z}_2)\, y_1^{k_1-2}\, y^{k_2-2}\, d\bar{z}_1 \wedge d\bar{z}_2 \in H^2(X_{\mathbb{C}}, \omega^{2-k_1, 2-k_2})_f$

$\rightsquigarrow$ restrict to $k_1 = k_2 = 1$ so that

$\rho_f : \mathrm{Gal}(L/F) \longrightarrow GL_2(E)$ Artin representation.

<u>Secretly:</u> $\pi_\infty = \pi_{\infty,1} \otimes \pi_{\infty,2}$ & $\pi_{\infty,i}|_{SL_2,\mathbb{R}} \cong D_{k-1}^+ \oplus D_{k-1}^-$.

Fact: $U_f := \mathrm{Hom}_{\mathrm{Gal}(L/F)}(\mathrm{Ad}^0 \rho_f, \mathcal{O}_L^\times \otimes E)$ has dimension 2 and:

$$\begin{array}{ccccc} U_f^\vee \otimes \mathbb{C} & \cong & U_{f,1}^\vee & \oplus & U_{f,2}^\vee \\ & & \Psi & & \Psi \\ & & c_1 & & c_2 \end{array} \quad \text{(decomp. into lines)}.$$

Def. Action of $\Lambda^* U_f^\vee \otimes \mathbb{C}$ on $H^*(X_{\mathbb{C}}, \omega)_f$ by:

$$c_1(f) = \omega_f^1, \quad c_2(f) = \omega_f^2 \quad \text{etc.}$$

<u>Secretly:</u>
c_1 acts by $\left(\begin{bmatrix}1 & \\ & -1\end{bmatrix}, \begin{bmatrix}1 & \\ & 1\end{bmatrix}\right)$ at ∞
c_2 acts by $\left(\begin{bmatrix}1 & \\ & 1\end{bmatrix}, \begin{bmatrix}1 & \\ & -1\end{bmatrix}\right)$ at ∞

Conj (H). This action descends to $\Lambda^* U_f^\vee \circlearrowright H^*(X_{\mathbb{Q}}, \omega)_f$.

Explicitly, $\exists$ units $u_{11}, u_{12}, u_{21}, u_{22} \in \mathcal{O}_L^\times$ s.t.

(1) $$\frac{-\log|u_{21}|\,\omega_f^1 + \log|u_{22}|\,\omega_f^2}{\log|u_{11}|\log|u_{22}| - \log|u_{12}|\log|u_{21}|} \in H^1(X, \omega)_f \quad \text{(\& another similar class)}$$

(2) $$\frac{\omega_f^{1,2}}{\log|u_{11}|\log|u_{22}| - \log|u_{12}|\log|u_{21}|} \in H^2(X, \omega)_f.$$

Evidence:

- **Thm (H).** Assuming Stark's conjecture:
  - (2) is true
  - the determinant of the basis in (1) is rational.

- Consider $\underset{\text{modular curve}}{C} \hookrightarrow \underset{\text{Hilbert mod. surface}}{X}$ & restrict cohomology classes to C.

  $\rightsquigarrow$ Numerical evidence when f = base change from $\mathbb{Q}$.

Rmk. Also have action modulo p^n & a Harris-Venkatesh conjecture but it's hard to get evidence for it.

§3. Siegel modular forms.

(joint work in progress w/ Prasanna)

f = holomorphic Siegel modular form, wt (k_1, k_2)

$\Rightarrow$ have $[f] \in H^0(X, \mathcal{E}(k_1,k_2))$

$[w_\infty f] \in H^3(X_{\mathbb{C}}, \mathcal{E}(3-k_2, 3-k_1))$

Note: $k_1 = 3-k_2$, $k_2 = 3-k_1 \Rightarrow k_1 = k_2 = \frac{3}{2}$ $\lightning$

$$w_\infty = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

Secretly: $\pi_\infty \big|_{Sp_4, \mathbb{R}} \cong D^+_{(k_1,k_2)} \oplus D^-_{(k_1,k_2)}$ & w_∞ interchanges them.

But the A-packet associated to f is bigger.

$$\left\{ \begin{matrix} \text{aut. reps} \\ \text{of } GSp_4(\mathbb{A}) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Galois reps} \\ G_{\mathbb{Q}} \longrightarrow GSp_4(\mathbb{Q}_\ell) \end{matrix} \right\}$$

$f \rightsquigarrow \pi^h$ (holomorphic) $\longmapsto \rho_f : G_{\mathbb{Q}} \longrightarrow GSp_4(\mathbb{Q}_\ell)$

$\exists\ \pi^g$ (generic) $\longmapsto \rho_f$

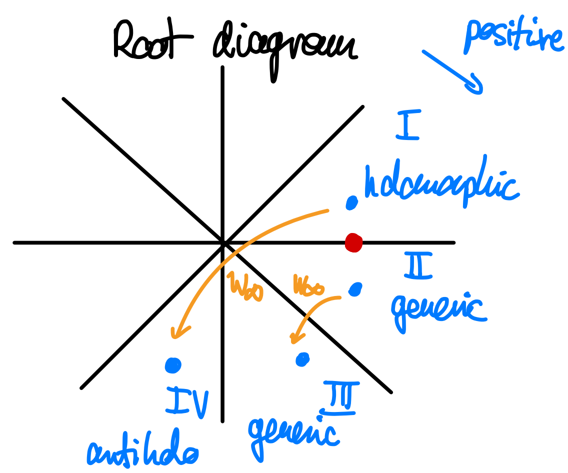
$\Pi = \{\pi^h, \pi^g\}$ = A-packet of ρ_f

$\rightsquigarrow$ $[f] \in H^0(X, \mathcal{E}(k_1,k_2))$ $\longleftarrow \pi^h$

f^W = Whittaker normalized: $[f^W] \in H^1(X_{\mathbb{C}}, \mathcal{E}(k_1, 4-k_2))$ $\longleftarrow \pi^g$

$[w_\infty f^W] \in H^2(X_{\mathbb{C}}, \mathcal{E}(k_2-1, 3-k_1))$ $\longleftarrow \pi^g$

$[w_\infty f] \in H^3(X_{\mathbb{C}}, \mathcal{E}(3-k_2, 3-k_1))$ $\longleftarrow \pi^h$



$k_2 = 2 \Rightarrow 4-k_2 = k_2$ degeneracy

Take $(k_1, k_2) = (2,2)$ for simplicity.

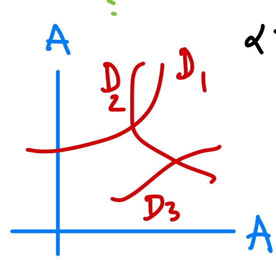
$$\left\{ \begin{matrix} \text{aut. reps} \\ \text{of } GSp_4(\mathbb{A}) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Galois} \\ \text{reps} \end{matrix} \right\} \longleftarrow \{\text{motives}\}$$

f = holomorphic SMF, wt $(2,2)$, paramod. level $\xleftrightarrow{\text{(Brumer-Kramer)}}$ A = abelian surface$/\mathbb{Q}$

$$H^*(X_{\mathbb{C}}, \mathcal{E}(2,2))_\Pi = \underset{\substack{\psi \\ [f]}}{H^0_\Pi} \oplus \underset{\substack{\psi \\ [f^W]}}{H^1_\Pi}$$

$\circlearrowleft$?

$H^1_{\mathcal{M}}(M, \mathbb{Q}(1)) \subseteq H^3_{\mathcal{M}}(A \times A, \mathbb{Q}(2))$



$\alpha = \{(D_i, f_i)\}$ s.t.

- $D_i \subseteq A \times A$ irred. 3-fold,
- f_i = mero. fun. on D_i,
- $\sum \operatorname{div}(f_i) = 0$.

(Ignore powers of π & factors in $(\overline{\mathbb{Q}}^{ab})^\times$.)

Regulator map $\mathrm{reg}_{\mathbb{C}} : H^1_{\mathcal{M}} \longrightarrow H^1_{\mathcal{D}}$

$\rightsquigarrow$ explicit element in $H^1_{\mathcal{D}}$ given by

$$\underset{\text{basis}}{\omega_1, \omega_2 \in H^0(A, \Omega^1_A)} \rightsquigarrow \eta := (\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) - (\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2) \in H^1_{\mathcal{D}}$$

$$\rightsquigarrow \eta^\vee \in (H^1_{\mathcal{D}})^\vee \text{ corresponding element}$$

and define action of $\eta^\vee$ on $H^*(X, \mathcal{E}_{2,2})_\pi$ by

$$H^0(X, \mathcal{E}_{2,2})_\pi \longrightarrow H^1(X, \mathcal{E}_{2,2})_\pi$$
$$[f] \longmapsto [f^W].$$

Compare to explicit [PV] conjecture!

<u>Conj. (H.-Prasanna).</u> The resulting action of $(H^1_{\mathcal{M}})^\vee$ is rational.

Explicitly, for $\alpha = \{(D_i, f_i)\}$ as before:

$$H^0(X, \mathcal{E}_{2,2})_\pi \longrightarrow H^1(X, \mathcal{E}_{2,2})_\pi$$
$$[f] \longmapsto \frac{[f^W]}{\sum_i \int_{D_i(\mathbb{C})} \log|f_i| \cdot \eta^\vee}.$$

<u>Thm (H.-Prasanna).</u> Assume:

$$\left\{\begin{array}{l} \bullet \text{ Beilinson's conj. for } \mathrm{Ad}\,M(f) \\ \bullet \text{ Deligne's conj. for } M(f) \end{array}\right. \implies \text{Conj. is true.}$$

(This relies on a few difficult results $\Rightarrow$ thus quite deep.)

<u>Special case:</u> $A = R_{F/\mathbb{Q}} E$, $[F:\mathbb{Q}] = 2$, E/F elliptic curve

$$\implies \mathrm{Sym}^2 H^1(A) \cong R_{F/\mathbb{Q}} \mathrm{Sym}^2 H^1(E) \oplus \mathrm{Asai}_{F/\mathbb{Q}} H^1(E)$$

dim	10	=	6	+	4
F = real quadratic	$H^1_{\mathcal{M}}$	=	0	$\oplus$	$H^1_{\mathcal{M}}$
F = imag. quadratic	$H^1_{\mathcal{M}}$	=	$H^1_{\mathcal{M}}$	$\oplus$	0

where is motivic coh. non-trivial?

Case 1. $F/\mathbb{Q}$ real quadratic.

f_0 = Hilbert modular form of wt $(2,2)$ [cohomological; no motivic action]

Note that $H^2_{et}(X_{GL_2,F,\bar{\mathbb{Q}}}, \mathbb{Q})_f \cong \mathrm{Asai}_{F/\mathbb{Q}}\, \rho_f \rightsquigarrow$ Asai motive in $X_0 = X_{GL_2,F}$

Ramakrishnan: Constructs classes in $H^3_{\mathcal{M}}(X_0, \mathbb{Q}(2))$

by considering X_0 = Hilbert modular surface

$C_i \subseteq X_F$ modular curve

$f_i \in k(C)^\times$ modular unit

$\rightsquigarrow \{(C_i, f_i)\} \in H^3_{\mathcal{M}}(X_0, \mathbb{Q}(2))$ motivic coh. class.

Thm (Ramakrishnan). $$L(f, \mathrm{Asai}, 2) \sim \sum_i \int_{C_i(\mathbb{C})} \log|f_i| \cdot (\omega^\sigma \otimes \overline{\omega^{\sigma c}} - \overline{\omega^\sigma} \otimes \omega^{\sigma c})$$

(Note: still can't prove $\dim H^3_{\mathcal{M}}(\mathrm{Asai}_{F/\mathbb{Q}} M(f), \mathbb{Q}(2)) = 1$.)

Thm (H.-Prasanna). Assuming $\dim H^1_{\mathcal{M}} = 1$ (as expected), Conj. is true for f assoc. with f_0 via Yoshida lifting.

Analogous to RM/CM cases before: it seems you can prove something if you can construct the motivic class.

Case 2. $F/\mathbb{Q}$ imaginary quadratic

f_0 = Bianchi modular form of wt $(2,2)$ $\longleftrightarrow$ E/F elliptic curve

Last time:
$$\begin{array}{ccc} H^1_{\mathcal{D}}(\mathrm{Ad}\, M(f_0)_{\mathbb{R}}, \mathbb{R}(1))^\vee & \circlearrowleft & H^*(X_{GL_2,F}, \mathbb{R}) \\ \downarrow \mathrm{reg}^\vee & & \uparrow \\ H^1_{\mathcal{M}}(\mathrm{Ad}\, M(f_0), \mathbb{Q}(1))^\vee & \overset{?}{\circlearrowleft} & H^*(X_{GL_2,F}, \mathbb{Q}) \end{array}$$

Now, f = Yoshida lift of f_0 ; $A = R_{F/\mathbb{Q}} E$. Then:
$$\begin{array}{ccc} H^1_{\mathcal{D}}(\mathrm{Ad}\, M(f_0), \mathbb{R}(1)) & \xrightarrow{\cong} & H^1_{\mathcal{D}}(\mathrm{Ad}\, M(f), \mathbb{R}(1)) \\ \cong?\uparrow \mathrm{reg}_{\mathbb{C}} & \circlearrowleft & \cong?\uparrow \mathrm{reg}_{\mathbb{C}} \\ H^1_{\mathcal{M}}(\mathrm{Ad}\, M(f_0), \mathbb{Q}(1)) \otimes \mathbb{R} & \xrightarrow{\cong?} & H^1_{\mathcal{M}}(\mathrm{Ad}\, M(f), \mathbb{Q}(1)) \otimes \mathbb{R} \end{array}$$

<u>Thm (H.-Prasanna).</u> (a) Via $H^1_{\mathcal{D}}(\mathrm{Ad}\,M(f_0), \mathbb{R}(1))^\vee \cong H^1_{\mathcal{D}}(\mathrm{Ad}\,M(f), \mathbb{R}(1))^\vee$

$$\begin{tikzcd} H^1(X_{GL_2,F}, \mathbb{C})_{f_0} \arrow[r, "\Theta_1"] \arrow[d, "\substack{\text{Prasanna}\\\text{Venkatesh action}}"'] \arrow[dr, phantom, "\circlearrowleft"] & H^0(X_{GSp_4,\mathbb{C}}, \mathcal{E}_{2,2})_f \arrow[d, "\substack{\text{H.}\\\text{Prasanna}\\\text{action}}"] \\ H^2(X_{GL_2,F}, \mathbb{C})_{f_0} \arrow[r, "\Theta_2"] & H^1(X_{GSp_4,\mathbb{C}}, \mathcal{E}_{2,2})_f \end{tikzcd}$$

commutes!

Note: this is completely unconditional!

(b) Assuming $H^1_{\mathcal{M}}(\mathrm{Ad}\,M(f_0), \mathbb{Q}(1))^\vee \cong H^1_{\mathcal{M}}(\mathrm{Ad}\,M(f), \mathbb{Q}(1))^\vee$,

i.e. rank part of Beilinson's conjecture,

same diagram for motivic actions.

In particular, H.-Prasanna conj. $\Rightarrow$ Prasanna–Venkatesh conj. in this case.

<u>Other relevant works on coherent cohomology:</u>

- Oh: "volume" version of the conjecture,
- Atanasov: derived Hecke operators on unitary Shimura varieties.

Open questions:

1. General coherent cohomology motivic action.

Difficulty: The contributions to different degrees come from different members of $\Pi = \{\pi_1, \ldots, \pi_n\}$ A-packet

$\Rightarrow$ no good way to normalized automorphic embeddings

$$\begin{matrix} \pi_1 \hookrightarrow \\ \pi_2 \hookrightarrow \\ \vdots \\ \pi_n \hookrightarrow \end{matrix} \; G(\mathbb{A})$$

Siegel case:

$\pi^h \hookrightarrow GSp_4(\mathbb{A})$ rational norm.

$\pi^g \hookrightarrow GSp_4(\mathbb{A})$ Whittaker norm.

2. What is the p-adic analogue of these conjectures?

E.g. f = wt 1 mod. form

Want: $H^0(X, \omega) \otimes \mathbb{Q}_p \longrightarrow H^1(X, \omega) \otimes \mathbb{Q}_p$ "natural map"

$$f \longmapsto \omega_f^p$$

s.t. $\dfrac{\omega_f^p}{\log_p(u_f)} \in H^1(X, \omega)$

(Ongoing work w/ Wang-Erickson; forthcoming work of Oh.)

3. Is there a "motivic action" for torsion classes?

(Note: no "motive" $\Rightarrow$ not a well-posed question...)