

**ON THE ARITHMETIC OF ELLIPTIC CURVES VIA TRIPLE
PRODUCTS OF MODULAR FORMS
(IWASAWA 2019, BORDEAUX)**

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These are notes from a mini course ON THE ARITHMETIC OF ELLIPTIC CURVES VIA TRIPLE PRODUCTS OF MODULAR FORMS taught by Victor Rotger during June 19–20, 2019. It was a part of the *lwasawa 2019* conference, June 19–28, 2019, in Bordeaux. They were L^AT_EX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from June 24, 2019. Check for the latest version of these notes at

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1. LECTURE 1: JUNE 19

1.1. The BSD Conjecture. The goal of these lectures is to describe some recent progress towards the Birch–Swinnerton-Dyer conjecture in cases when the ground field is not \mathbb{Q} and in the rank 2 case. These are cases where Heegner points are not available, so they are not covered by the methods of Gross–Zagier [GZ86] and Kolyvagin [Kol90]. We start with a review of the conjecture and these classical methods.

Let E/\mathbb{Q} be an elliptic curve of conductor N_E . For ℓ prime, the Tate module is

$$V_\ell(E) = \left(\varprojlim_n E[\ell^n] \right) \otimes \mathbb{Q}_\ell.$$

The absolute Galois group $G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $V_\ell(E)$ and yields

$$\varrho_{E,\ell}: G_\mathbb{Q} \rightarrow \text{Aut}(V_\ell(E)) \cong \text{GL}_2(\mathbb{Q}_\ell).$$

The family of Galois representations $\{V_\ell(E)\}_\ell$ is a *compatible system* of Galois representations (in the sense that for any $p \neq \ell$, the characteristic polynomial of Frob_p has \mathbb{Z} -coefficients, independent of ℓ).

Definition 1.1. The L -function of the elliptic curve E is

$$L(E, s) = L(\{V_\ell(E)\}_\ell, s) = \prod_p (\text{Char}_{\text{Frob}_p}(p^{-s}))^{-1}.$$

Here $\ell \neq p$ and $\text{Frob}_p \in \text{End}(V_\ell(E)_{I_p})$ is the arithmetic Frobenius at p acting on the space of co-invariants of $V_\ell(E)$ for the action of the inertia group I_p .

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Conjecture 1.2 (Birch–Swinnerton-Dyer (BSD)). *The order of vanishing of the L -function at $s = 1$ is the rank of $E(\mathbb{Q})$:*

$$\text{ord}_{s=1} L(E, s) = \text{rank } E(\mathbb{Q}).$$

Remark 1.3. Here, $L(E, s)$ has analytic continuation to the complex plane with a functional equation relating $s \mapsto 2 - s$. This follows from the modularity theorem due to Wiles and Taylor–Wiles. Therefore, $s = 1$ is the point of symmetry of the functional equation.

More generally, let H/\mathbb{Q} be a finite Galois extension and let $\rho: \text{Gal}(H/\mathbb{Q}) \rightarrow \text{GL}_n(L)$ be an Artin representation of degree n (here L/\mathbb{Q} is a finite extension).

Definition 1.4. The L -function of E twisted by ρ is

$$L(E, \rho, s) = L(\{V_\ell(E) \otimes \rho\}_\ell, s)$$

Definition 1.5. The ρ -isotypic component of $E(H)$ is

$$E(H)[\rho] = \text{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes L).$$

Conjecture 1.6 (Equivariant BSD). *The L -function $L(E, \rho, s)$ admits analytic continuation and satisfies a functional equation relating $L(E, \rho, s)$ to $L(E, \rho^\vee, 2 - s)$ and*

$$\text{ord}_{s=1} L(E, \rho, s) = \dim_L E(H)[\rho].$$

Remark 1.7. We usually say $\text{ord}_{s=1} L(E, \rho, s)$ is the analytic rank $r_{\text{an}}(E, \rho)$ and $\dim_L E(H)[\rho]$ is the algebraic rank $r(E, \rho)$.

Remark 1.8. These conjectures are all instances of more general conjectures (Beilinson, Bloch–Kato). The L -functions are *motivic L -functions* assigned to subquotients of étale cohomology of some varieties by the same process as above. The right hand sides are dimensions of motivic cohomology groups, which are certain explicit vector spaces, recovering the above in the case of elliptic curves.

These more general conjectures will be *behind the scenes* of the next lectures, even when do not mention them explicitly.

Theorem 1.9 (Kato, 1985–1990). *Let $\rho: \text{Gal}(H/\mathbb{Q}) \rightarrow L^\times$ be a character. If $r_{\text{an}}(E, \rho) = 0$, then $r(E, \rho) = 0$.*

Theorem 1.10 (Gross–Zagier, Kolyvagin, 1987). *Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, and $\psi: \text{Gal}(H/K) \rightarrow L^\times$ where H/K is abelian, H/\mathbb{Q} is Galois and dihedral (so the character ψ is anticyclotomic).*

Let $\rho_\psi = \text{Ind}(\psi): \text{Gal}(H/\mathbb{Q}) \rightarrow \text{GL}(V_\psi) \cong \text{GL}_2(L)$. Here,

$$\rho_\psi|_{\text{Gal}(H/K)} = \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix}$$

where $\psi': \text{Gal}(H/K) \rightarrow L^\times$ and $\psi'(\sigma) = \psi(c\sigma c)$ for complex conjugation c .

Then, if $r_{\text{an}}(E, \rho_\psi) = 0$, then $r(E, \rho_\psi) = 0$. Also, if $r_{\text{an}}(E, \rho_\psi) = 1$, then $r(E, \rho_\psi) = 1$.

Gross–Zagier [GZ86] proved that if $r_{\text{an}}(E, \rho_\psi) = 1$, then there is an infinite order point on $E(H)[\rho]$, so $r(E, \rho_\psi) \geq 1$. Kolyvagin [Kol90] proved that in that case, the rank is actually 1.

Questions.

- (1) Can we extend results of this type to other Artin representations?
- (2) Can we extend them to $r_{\text{an}}(E, \rho) > 1$?

1.2. Modular curves and modular forms. For $N \geq 1$, $X_1(N)$ is the compactification of the moduli space of (A, P) where

- A is an elliptic curve,
- P is a point of exact order N in A .

By the theory of Shimura varieties, this admits a natural model over \mathbb{Q} .

Similarly, $X_0(N)$ is the compactification of the moduli space of (A, C) where

- A is an elliptic curve,
- C is a cyclic subgroup of A of order N .

There is a forgetful map

$$\begin{aligned} X_1(N) &\rightarrow X_0(N) \\ (A, P) &\mapsto (A, \langle P \rangle). \end{aligned}$$

For $k \geq 0$, let $M_k(N)$ be the space of modular forms of weight k on $X_1(N)$. Geometrically,

$$M_k(N) = H^0(X_1(N), \omega^{\otimes k}),$$

where ω is a line bundle on $X_1(N)$ whose fiber at a non-cuspidal point $x = (A, P)$ is Ω_A^1 . The fiber of ω at cusps is defined by means of the Tate curve and logarithmic poles are allowed. See [Kat73] for a detailed account of the theory of geometric modular forms.

If $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}^\times$ is a Dirichlet character, $M_k(N, \chi)$ is the space of modular forms of nebentype χ under the action of diamond operators. Then

$$M_k(N) = \bigoplus_{\chi} M_k(N, \chi).$$

Let $f \in S_k(N, \chi)$ be a normalized primitive Hecke eigenform (a newform of level N). If $f = \sum_{n \geq 1} a_n(f)q^n$ is the q -expansion, let $L(f) = \mathbb{Q}(\{a_n(f)\}_{n \geq 1})$, which is a finite extension of \mathbb{Q} .

By Eichler–Shimura ($k = 2$), Deligne ($k > 2$), Deligne–Serre ($k = 1$), there exists a Galois representation

$$\rho_{f, \ell}: G_{\mathbb{Q}} \rightarrow \text{GL}(V_{f, \ell}) \cong \text{GL}_2(L_{\ell}(f)),$$

where $L_{\ell}(f)$ is the completion of $L(f)$ in $\overline{\mathbb{Q}}_{\ell}$.

Remark 1.11. The original work of Eichler–Shimura (see [Shi71]) realizes it in the Jacobian of $X_1(N)$. Deligne [Del73] uses the cohomology of Kuga–Sato varieties whose fibers are products of elliptic curves. Deligne–Serre [DS74] use congruences to reduce the case $k = 1$ to the other cases.

If $k = 2$ and $L(f) = \mathbb{Q}$, $\{V_\ell(f)\} = \{V_\ell(E)\}$ for some elliptic curve E/\mathbb{Q} . Conversely, given E/\mathbb{Q} there exists $f \in S_2(N)$ such that $V_\ell(f) = V_\ell(E)$ (this is the famous modularity theorem, proved by Wiles and Taylor–Wiles).

If $k = 1$, ρ_f is an Artin representation, meaning that it factors through a finite extension and takes values in $L(f)$:

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\rho_f} & \mathrm{GL}_2(L_\ell(f)) \\ & \searrow & \nearrow \\ & \mathrm{Gal}(H/\mathbb{Q}) & \xrightarrow{\rho_f} \mathrm{GL}_2(L(f)). \end{array}$$

Conversely (under some mild hypotheses), if $\rho: \mathrm{Gal}(H/\mathbb{Q}) \rightarrow \mathrm{GL}_2(L)$ is such that $\det(\rho(c)) = -1$, then $\{V_\rho\}_\ell = \{V_\ell(f)\}_\ell$ for some $f \in M_1(N)$. This follows from the work of many mathematicians, most notably including Khare and Wintenburger.

We can finally recast the Theorem of Gross–Zagier and Kolyvagin 1.10 in terms of modular forms.

Theorem 1.12 (Gross–Zagier, Kolyvagin). *Let $f = f_E \in S_2(N)$ be the modular form attached to E and $g = g_\psi \in M_1(N)$ be the weight one form attached to ρ_ψ . Let $L(f, g, s) = L(\{V_\ell(f) \otimes V_\ell(g)\}_\ell, s)$ be Rankin’s L -function attached to (f, g) . It is known to have analytic continuation and functional equation (for any two modular forms f and g).*

If $\mathrm{ord}_{s=1} L(f, g, s) = r \in \{0, 1\}$, then $\dim_L E(H)[\rho_\psi] = r$.

2. LECTURE 2: JUNE 19

2.1. Triple product L -function. We saw that if E/\mathbb{Q} is an elliptic curve, then there is an attached modular form $f = f_E \in S_2(N_E)$.

Let $g \in S_1(N_g, \chi)$, $h \in S_1(N_h, \chi^{-1})$. The *triple product L -function of eigenforms (f, g, h) of weights $(2, 1, 1)$* is

$$L(f, g, h, s) = L(\{V_\ell(f) \otimes V_\ell(g) \otimes V_\ell(h)\}_\ell, s).$$

This is known as Garrett’s L -function, who proved that it admits analytic continuation and a functional equation relating s to $2 - s$.

Conjecture 2.1 (BSD for triple product L -function). *We have that*

$$\mathrm{ord}_{s=1} L(f, g, h, s) = \dim_L E(H)[\rho_g \otimes \rho_h]$$

where $\rho_g, \rho_h: G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(L)$ are the Artin representations attached to f and g , and H is a field such that both of them factor through $\mathrm{Gal}(H/\mathbb{Q})$.

Special cases.

- (I) Let $K = \mathbb{Q}(\sqrt{\pm D})$ be a real or imaginary quadratic field and $\psi_g, \psi_h: G_K \rightarrow L^\times$ be character of finite orders (and if K is real, take ψ_g, ψ_h to be mixed signature). Then $V_g = \mathrm{Ind}(\psi_g)$, $V_h = \mathrm{Ind}(\psi_h)$ are odd 2-dimensional Artin representations and hence, by modularity, there exist

$$g \in M_1(N_g, \chi_g), h \in M_1(N_h, \chi_h)$$

such that

$$\rho_g \cong \text{Ind}(\psi_g), \quad \rho_h \cong \text{Ind}(\psi_h).$$

Remark 2.2. Here, g and h are cusp forms if the representations V_g, V_h are irreducible; for example, $\psi_g \neq \psi'_g$ and same for h .

Assume that $\chi := \chi_g = \chi_h^{-1}$ and g, h are cusp forms. Then the triple product L -function factors as

$$L(f, g, h, s) = L(f/K, \psi_1, s) \cdot L(f/K, \psi_2, s)$$

where $\psi_1 = \psi_g \psi_h$ and $\psi_2 = \psi_g \psi'_h$, because

$$V_g \otimes V_h = \text{Ind}(\psi_1) \oplus \text{Ind}(\psi_2).$$

(II) Take $g \in \overline{S_1(N, \chi)}$ arbitrary, $h = g^* = g \otimes \chi^{-1}$, characterized by the fact that $a_\ell(h) = \overline{a_\ell(g)}$ for $(\ell, N_g) = 1$. Then

$$L(f, g, h, s) = L(f, s) \cdot L(f, \text{ad}^0(g), s).$$

Question. Can one construct a point $P_{f,g,h} \in E(H)[\rho_g \otimes \rho_h]$ which is non-zero when $L(f, g, h, 1) = 0$?

In order to (attempt to) answer this question, it is helpful to look at the wider setting of triples

$$f_k \in S_k(N_f), \quad f_\ell \in S_\ell(N_g, \chi), \quad h_m \in S_m(N_h, \chi^{-1}).$$

Assume throughout that the local signs of the functional equation of $L(f_k, g_\ell, h_m, s)$ at finite primes are all $\epsilon_v(f_k, g_\ell, h_m) = +1$. This is always the case for instance if $(N_f, N_g, N_h) = 1$.

Then there is a functional equation:

$$L(f_k, g_\ell, h_m, s) = \epsilon(f, g, h) L(f_k, g_\ell, h_m, k + \ell + m - 2 - s)$$

(after completing $L(s)$ at ∞), where

$$\epsilon(f, g, h) = \begin{cases} +1 & \text{if one of the weights dominates the others} \\ & \text{i.e. if } k \geq \ell + m \text{ or } \ell \geq k + m \text{ or } m \geq k + \ell \\ -1 & \text{otherwise, i.e. } (k, \ell, m) \text{ is balanced.} \end{cases}$$

Note that (k, ℓ, m) is balanced exactly when the three numbers k, ℓ, m are the sides of a triangle.

The center of symmetry is $c = \frac{k+\ell+m-2}{2}$.

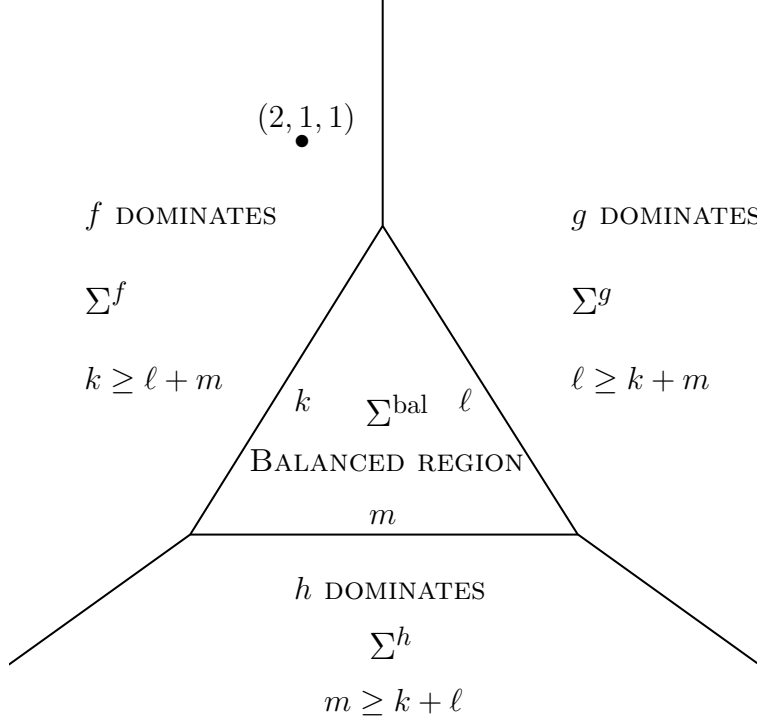
- At $(k, \ell, m) = (2, 1, 1)$, $k = 2$ is dominant, because $2 \geq 1 + 1$. Therefore, $\epsilon(f, g, h) = +1$, so $\text{ord}_{s=1} L(f, g, h, s)$ is even, so we expect the algebraic rank $r(E, \rho_g \otimes \rho_h)$ to be even.
- In (I) and (II), we have $L(f, g, h, s) = L_1(s)L_2(s)$ and $r_{\text{an}}(L_1)$ and $r_{\text{an}}(L_2)$ have the same parity. In particular, we may recover this from Gross–Zagier in a particular case.
- When (k, ℓ, m) is balanced, so that $\epsilon = -1$, the analytic rank $r_{\text{an}}(f, g, h)$ is odd, so BSD for higher dimensional varieties (Bloch–Kato) predicts the existence of at least one $0 \neq P_{f,g,h} \in \text{CH}(\text{Kuga–Sato varieties})$. Together with Darmon in [DR14], we

propose a candidate for $P_{f,g,h}$ and conjecture that the height of $P_{f,g,h}$ is a non-zero factor multiplied by $L'(f, g, h, c)$; cf. Theorem 4.1.

The goal is to define a p -adic analogue of the first non-zero term of $L(f_k, g_\ell, h_m, s)$ in the Taylor expansion at $s = c$.

Let $(k, \ell, m) \in \Sigma^g$, so $\ell \geq k + m$. Then $\epsilon(f, g, h) = +1$. In this case, generically, $L(f, g, h, s)$ will not vanish at $s = 1$ and hence we can hope to interpolate these values p -adically.

We draw the regions in \mathbb{Z}^3 corresponding to the three weight dominating and to when they are balanced as follows:



The idea is to construct a p -adic L -function that interpolate the values of $L(f_k, g_\ell, h_m, c)$ at

$$\{(k, \ell, m) \mid \ell \geq k + m \text{ and } k, \ell, m \geq 2\} \subseteq \Sigma^g$$

and then evaluate it outside of the region of interpolation, as for instance at $(2, 1, 1) \in \Sigma^f$. It is also interesting to investigate what evaluation at other points outside the region of interpolation gives, but we will only mention this in passing (cf. Theorem 4.1 for evaluation in balanced region).

2.2. Algebraicity result. The key to constructing the p -adic L -function is to establish that the values of the L -functions are algebraic (at least up to an explicit period).

Theorem 2.3 (Harris–Kudla [HK91], Ichino [Ich08], Watson [Wat02]). *If $\ell = k + m$,*

$$L(f_k, g_\ell, h_m, c) = (*) \langle f_k \cdot h_m, g_\ell \rangle^2.$$

Equivalently,

$$\frac{L(f_k, g_\ell, h_m, c)^{1/2}}{\langle g_\ell, g_\ell \rangle} = (*) \frac{\langle f_k \cdot h_m, g_\ell \rangle}{\langle g_\ell, g_\ell \rangle}.$$

Here, $(*)$ is an explicit constant.

The right hand side may be interpreted algebraically (using Poincaré duality and algebraic modular forms), so it is suitable for p -adic interpretation. Note also that $f_k \cdot h_m$ is a weight ℓ modular form and hence can be expressed in terms of the orthogonal basis of S_ℓ , which gives:

$$\frac{\langle f_k \cdot h_m, g_\ell \rangle}{\langle g_\ell, g_\ell \rangle} = \begin{array}{l} \text{coefficient of } f_k \cdot h_m \text{ at } g_\ell \\ \text{in the spectral decomposition} \\ \text{of } S_\ell(N, \chi) \end{array}$$

Therefore, this quantity belongs to

$$L(f, g, h) = \mathbb{Q}(\{a_n(f), a_n(g), a_n(h)\}_{n \geq 1}) \subseteq \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}_p}.$$

We need a slightly stronger version of this theorem, which applies to all weight $\ell \geq k + m$. To do this, we recall the Maass–Shimura operator δ on the space of real-analytic modular forms $S_k^{C^\infty}(N)$. It is defined as follows:

$$\delta : S_k^{C^\infty}(N) \rightarrow S_{k+2}^{C^\infty}(N), \quad \delta(f(z)) = \frac{1}{2\pi i} \left(\frac{\partial f}{\partial z} + \frac{k}{2iy} f \right)$$

where $z = x + iy$.

We are now ready to state the full version of Theorem 2.3.

Theorem 2.4 (Harris–Kudla, Ichino, Watson: Part II). *If $\ell = k + m + 2t$ for $t \geq 0$,*

$$\frac{L(f_k, g_\ell, h_m, c)^{1/2}}{\langle g_\ell, g_\ell \rangle} = (*) \frac{\langle \delta^t(f_k) \cdot h_m, g_\ell \rangle}{\langle g_\ell, g_\ell \rangle}.$$

By the same argument as above (due to Shimura), one shows that

$$\frac{\langle \delta^t(f_k) \cdot h_m, g_\ell \rangle}{\langle g_\ell, g_\ell \rangle} \in \overline{\mathbb{Q}}.$$

For any $r \geq 0$, define the *space of nearly holomorphic modular forms of weight k , level N , and order r* as

$$N_k^r(N) = \left\{ \sum_{i=0}^r \frac{1}{y^i} \cdot h_i \mid h_i \text{ holomorphic} \right\} \subseteq S_k^{C^\infty}(N).$$

For any $t \leq r$, if $f \in S_k(N)$ then $\delta^t(f) \in N_{k+2t}^r(N)$.

The goal for the next lecture is to define a p -adic L -function by using p -adic families and give a p -adic interpretation of the above formulas.

3. LECTURE 3: JUNE 20

The references for the next two lectures include [Urb14], [DR14], [DLR15], but we will try to give more specific references.

Recall from the previous lecture that we have the following spaces:

$$\begin{array}{ccccccc}
 H^0(X, \omega^k) & \xrightarrow{=} & M_k(N) = N_k^0 & \hookrightarrow & N_K^1 & \hookrightarrow & \cdots \longrightarrow N_k^r \hookrightarrow H^0(X^{C^\infty}, \omega^k) \\
 & & & & \delta \uparrow & & \delta^r \uparrow \\
 & & & & M_{k-2} & & M_{k-2r}
 \end{array}$$

In [Urb14], Urban proves that there is a sheaf H_k^r on X together with a map of sheaves $H_k^r \rightarrow \omega^k$ over X^{C^∞} such that the induced map

$$H^0(X, H_k^r) \rightarrow N_k^r \subseteq H^0(X^{C^\infty}, \omega^k)$$

is an isomorphism. The map $H_k^r \rightarrow \omega^k$ exists thanks to the Hodge decomposition which for any elliptic curve A/\mathbb{C} says that

$$H_{\text{dR}}^1(A/\mathbb{C}) = \Omega^1(A) \oplus \overline{\Omega}^1(A).$$

We want to give meaning to the quantify $\langle \delta^t(f_k) \cdot h_m, g_\ell \rangle$ appearing in Theorem 2.4 over \mathbb{C}_p .

- Classical modular forms with coefficients in \mathbb{C}_p are

$$M_k(N, \mathbb{C}_p) = H^0(X/\mathbb{C}_p, \omega^k).$$

- Note that $\delta^t(f_k) \in H^0(X/\mathbb{C}_p, H_{k+2t}^t)$ makes sense, but there is no Hodge decomposition over \mathbb{C}_p and no map $H_k^r \rightarrow \omega^k$ over X/\mathbb{C}_p . We give a way to get around this difficulty next.

Let $A/\mathcal{O}_{\mathbb{C}_p}$ be an *ordinary* elliptic curve, i.e. $\overline{A}[p] \cong \mathbb{Z}/p\mathbb{Z}$. Then

$$H_{\text{dR}}^1(A) = \Omega^1(A) \oplus U$$

where U is the free $\mathcal{O}_{\mathbb{C}_p}$ -submodule of rank 1 on which Frob_p acts invertibly by multiplication by $a_p(A) \in \mathcal{O}_{\mathbb{C}_p}^\times$. This is the *unit-root* decomposition on this elliptic curve.

Therefore, define

$$X_{/\mathbb{Q}_p}^{\text{ord}} = \{x = [A, P] \in X(\mathbb{C}_p) \mid A \text{ is ordinary}\}.$$

We define p -adic modular forms as

$$M_k^{(p)}(N) = H^0(X^{\text{ord}}, \omega^k).$$

Note that the space X^{ord} is smaller, so the sheaf ω^k has more sections over it. Therefore, these are not just classical modular forms with coefficients in \mathbb{C}_p , as above.

Remark 3.1. Jan Vonk's lecture course will be devoted to overconvergent modular forms, which are sections of ω^k on a slightly larger variety $X_\varepsilon \supset X^{\text{ord}}$ (where we allow ourself to glue an annulus of width $\varepsilon > 0$ around the ordinary region). While these ideas are certainly present in the background, and are actually necessary to justify some of the claims, we do not make explicit reference to them in this mini course and rather refer to Vonk's lectures for that.

There is a map of sheaves $H_k^r \rightarrow \omega^k$ over X^{ord} and hence we have a map

$$N_k^r \subseteq H^0(X^{\text{ord}}, H_k^r) \rightarrow H^0(X^{\text{ord}}, \omega^k) = M_k^{(p)}(N).$$

We may thus regard $\delta^t(f_k)$ in $M_{k+2t}^{(p)}(N)$. On q -expansions,

$$\delta^t \left(\sum_{n \geq 1} a_n q^n \right) = \sum_{n \geq 1} a_n n^t q^n.$$

Altogether, we have shown that

$$\langle \delta^t(f_k) \cdot h_m, g_\ell \rangle$$

makes sense in \mathbb{C}_p for $k, \ell, m \in \mathbb{Z}_{\geq 2}$. We still need to allow the weights k, ℓ, m and the exponent t to vary p -adically.

From now on, assume that f_k, g_ℓ, h_m are ordinary at p , i.e.

$$a_p(f_k), a_p(g_\ell), a_p(h_m) \in \mathcal{O}_{\mathbb{C}_p}^\times.$$

We can factor

$$T^2 - a_p(f_k)T + p^{k-1} = (T - \alpha_f)(T - \beta_f)$$

with $a_f \in \mathcal{O}_{\mathbb{C}_p}^\times$ (and similarly for g_ℓ, h_m).

Hida constructed p -adic families of modular forms, known as *Hida families*:

$$\underline{f} \in \mathbb{Z}_p[[T]][[q]]$$

(actually, we should replace $\Lambda = \mathbb{Z}_p[[T]]$ with a finite extension of it, but we keep it this way to simplify notation), which can be written as

$$\underline{f} = \sum_{n \geq 1} a_n(\underline{f}) q^n \quad \text{where } a_n(\underline{f}) \in \Lambda,$$

such that:

- for all $w \in \mathbb{Z}_{\geq 2}$, $\underline{f}_w = \sum_{n \geq 1} (a_n(\underline{f}))|_{T=(1+p)^{w-1}} q^n \in S_w(Np)$ (this is the *weight w specialization*),
- $\underline{f}_k(q) = \underbrace{f_k(q) - \beta_f f_k(q^p)}_{f_k^\alpha} \in S_k(Np)$, and $U_p(f_k^\alpha) = \alpha_j f_k^\alpha$.

The specializations at $w \notin \mathbb{Z}$ are overconvergent modular forms; we do not discuss this here further.

We still have to extend the δ^t operator p -adically. If $n \in \mathbb{Z}_p$ is divisible by p , the map $t \mapsto n^t$ does not make sense in \mathbb{Z}_p . However, we can define the p -depletion of f to be

$$f_k^{[p]}(q) = \sum_{\substack{n \geq 1 \\ (p,n)=1}} a_n(f_k) q^n.$$

A priori, it is not clear this is even a modular form. It turns out, however, that this is an algebraic operation: we have Hecke operators U_p and V_p whose effect on q -expansions is:

$$U_p(f) = \sum a_{pn} q^n, \quad V_p(f) = \sum a_n q^{pn} = f(q^p)$$

and

$$f_k^{[p]} = (1 - V_p U_p) f_k.$$

The “family” $\{\delta^t(f_k^{[p]})\}_{t \in \mathbb{Z}_p}$ make sense, in the precise sense that there is $\delta^\bullet(f_k^{[p]}) \in \mathbb{Z}_p[[T]][[q]]$ such that for all $t \in \mathbb{Z}_{\geq 0}$, $\delta^\bullet(f_k^{[p]})|_{T=(1+p)^t-1} = \delta^t(f_k^{[p]})$.

Remark 3.2. This is **not** a Hida family. It is exactly what we wrote above: an element of $\Lambda[[q]]$ with the described specialization.

Example 3.3. Take $k = 2$, $t = -1$ to get:

$$\delta^{-1}(f^{[p]}) = \sum_{(p,n)=1} \frac{a_n}{n} q^n.$$

This is a p -adic modular form of weight 0, called the *Coleman primitive* of f .

Let $S_k^{(p),\text{ord}}(N) \subseteq S_k^{(p)}(N)$ be the subset of f such that $a_p(f) \in \mathcal{O}_{\mathbb{C}_p}^\times$. Hida defined the *ordinary projector*:

$$e_{\text{ord}} = \lim_{n \rightarrow \infty} U_p^{n!} : S_k^{(p)}(N) \rightarrow S_k^{(p),\text{ord}}(N)$$

which satisfies

$$e_{\text{ord}}|_{S_k^{(p),\text{ord}}(N)} = \text{id}.$$

If $\underline{f}, \underline{h}$ are Hida families, we can consider

$$e_{\text{ord}}(\delta^\bullet(\underline{f}^{[p]}) \cdot \underline{h}) \in \mathbb{Z}_p[[T, X, Z]][[q]]$$

whose specialization at $k, m \in \mathbb{Z}_{\geq 2}$ and at any $t \in \mathbb{Z}$ such that $\ell = k + m + 2t \geq 2$ is

$$e_{\text{ord}}(\delta^t(f_k^{[p]}) \cdot h_m) \in S_\ell^{(p),\text{ord}}(Np).$$

By Coleman’s classicality theorem¹, this is an element of $S_\ell(Np)$.

In symbols,

$$e_{\text{ord}}(\delta^\bullet(\underline{f}^{[p]}) \cdot \underline{h}) \in \underline{S}(N, \mathbb{Z}_p[[T, X, Z]])$$

(this is called a Λ -adic family, where $\Lambda = \mathbb{Z}_p[[T, X, Z]]$). Note that we may write $\mathbb{Z}_p[[T, X, Z]] = \mathbb{Z}_p[[X, Y, Z]]$ because T determined by X, Y, Z .

Define

$$\underline{S}(N)[[g]] = \{\phi \in \underline{S}(N) \mid \text{for every } (\ell, N) = 1, (T_\ell - a_\ell(g))^n(\phi) = 0 \text{ for some } n \geq 1\}$$

(where the coefficients are as above), a generalized eigenspace. By multiplicity one,

$$\underline{S}(N)[[g]] = \mathbb{Q}_p((X, Y, Z)) \cdot \underline{g}.$$

There is a natural projector

$$\underline{S}(N) \xrightarrow{\pi_g} \underline{S}(N)[[g]]$$

and define

$$\mathcal{L}_p^g(\underline{f}, \underline{g}, \underline{h}) \cdot \underline{g} = \pi_g \left[e_{\text{ord}}(\delta^\bullet(\underline{f}^{[p]}) \cdot \underline{h}) \right].$$

This defines

$$\mathcal{L}_p^g : \mathbb{Z}_p^3 \rightarrow \mathbb{C}_p.$$

¹A p -adic overconvergent modular form of classical weight $k \geq 2$ is a classical modular form.

By construction, if $(k, \ell, m) \in (\mathbb{Z}_{\geq 2})^3$,

$$\mathcal{L}_p^g(k, \ell, m) = \frac{\text{the coefficient at } g_\ell \text{ of } e_{\text{ord}}(\delta^t f_k^{[p]} \cdot h_m)}{\langle g_\ell, g_\ell \rangle} = \frac{\langle e_{\text{ord}}(\delta^t f_k^{[p]} \cdot h_m), g_\ell \rangle}{\langle g_\ell, g_\ell \rangle}.$$

If moreover, $\ell \geq k + m$, this is equal to

$$(*) \cdot \frac{L(f_k, g_\ell, h_m, c)^{1/2}}{\langle g_\ell, g_\ell \rangle}$$

by Harris–Kudla’s formula 2.4 described in the second lecture.

We have hence achieved the p -adic interpolation of the square root of the special values of this L -function.

4. LECTURE 4: JUNE 20

Recall that $\{(k, \ell, m) \mid \ell \text{ dominates and } k, \ell, m \geq 2\} \subseteq \Sigma^g$ is the region of interpolation of $\mathcal{L}_p^g(k, \ell, m)$.

If $(k, \ell, m) \in \Sigma^{\text{bal}}$, then $L(f_k, g_\ell, h_m, c) = 0$ because $\epsilon = -1$. We expect that

$$L'(f_k, g_\ell, h_m, c) \stackrel{?}{=} \text{height of } P_{f,g,h}$$

in an appropriate sense. This is a Gross–Zagier type formula. When the weights are $(2, 2, 2)$ this has been proven by S. Zhang, W. Zhang and X. Yuan [YZZ12].

The principle is that $\mathcal{L}_p^g(k, \ell, m) \in \mathbb{C}_p$ plays the role of $L'(f_k, g_\ell, h_m, c)$. (Indeed, they are both the first non-zero coefficients in the Taylor expansion).

Here is particular result in this direction, which we call a *p -adic Gross–Zagier formula*.

Theorem 4.1 (Darmon–Rotger [DR14]). *For $(k, \ell, m) \in \Sigma^{\text{bal}}$*

$$\mathcal{L}_p^g(k, \ell, m) = (\text{Euler factor}) \cdot (p\text{-adic Bloch Kato logarithm of } (P_{f,g,h})).$$

We have proved roughly half of this theorem already. Indeed, we have an explicit formula for the value of the p -adic L -function and we just need to prove it agrees with the right hand side in the theorem, which is done in [DR14] by resorting to Besser’s finite-polynomial cohomology.

4.1. Special value at $(2, 1, 1)$. What is the value of \mathcal{L}_p^g at $(2, 1, 1)$? This situation was investigated in [DLR15], [DLR16], and [GG19].

Let E/\mathbb{Q} be an elliptic curve and $f = f_E \in S_2(N)$ and p be a prime coprime to N such that E is ordinary at p . Let $\underline{f} \in \Lambda[[q]]$ be a Hida family passing through f^α . Take $g \in S_1(N, \chi)$, $h \in S_1(N, \chi^{-1})$ such that

$$T^2 - a_p(g)T + \chi(p) = (T - \alpha_g)(T - \beta_g)$$

and $\alpha_g, \beta_g \in \mathcal{O}_{\mathbb{C}_p}^\times$. Let $\underline{g}, \underline{h} \in \Lambda[[q]]$ be Hida families through g^α, h^α .

According to the definition of $\mathcal{L}_p^g(\underline{f}, \underline{g}, \underline{h})$, we must look at the space

$$\underline{S}(N, \mathbb{Q}_p[[X, Y, Z]])[[g]]$$

and specialize at $k = 2$, $\ell = 1$, $m = 1$, $t = -1$, so we get:

$$S_1^{(p),\text{ord}}(Np, \mathbb{C}_p)[[g^\alpha]].$$

This is where

$$e_{\text{ord}}(d^{-1}(f_E^{[p]}) \cdot h^\alpha)$$

belongs to. Before, for weights ≥ 2 , we could invoke here Coleman's classicality result. However, for weight 1, classicality may not hold, but we instead have the following result.

Theorem 4.2 (Bellaïche–Dimitrov [BD16]). *Assume $\alpha_g \neq \beta_g$ and assume $\rho_g \not\cong \text{Ind}(\psi)$ for $\psi: G_K \rightarrow L^\times$, a character of a real quadratic field in which p splits. Then*

$$S_1^{(p),\text{ord}}(Np)[[g^\alpha]] = \mathbb{C}_p g^\alpha$$

(where the left hand side is the space of overconvergent ordinary modular forms).

Remark 4.3. In terms of the eigencurve, this theorem is saying that the eigenspace is smooth and étale over the weight space at these particular weight 1 points. This is how Bellaïche and Dimitrov phrase their theorem.

Assume that the hypothesis of the theorem hold. Then the value of \mathcal{L}_p^g at $(2, 1, 1)$ is given by the formula:

$$\mathcal{L}_p^g(2, 1, 1) = \text{coefficient of } e_{\text{ord}}(d^{-1}(f^{[p]}) \cdot h^\alpha) \text{ at } g^\alpha.$$

Conjecture 4.4 (Darmon–Lauder–Rotger [DLR15]). *Assume*

$$L(E, \rho_g \otimes \rho_h, 1) = L'(E, \rho_g \otimes \rho_h, 1) = 0, \quad L''(E, \rho_g \otimes \rho_h, 1) \neq 0.$$

Then

$$\mathcal{L}_p^g(2, 1, 1) = \underbrace{c}_{\in L} \cdot \det \begin{pmatrix} \log_{E,p}(P_1) & \log_{E,p}(P_2) \\ \log_{E,p}(Q_1) & \log_{E,p}(Q_2) \end{pmatrix} \cdot \frac{1}{\log_p(u_g)}$$

for some (unknown) constant $c \in L$ and P_1, P_2, Q_1, Q_2, u_g described shortly below.

Since $r_{\text{an}}(E, \rho_g \otimes \rho_h) = 2$, by BSD we expect that

$$\dim_L \text{Hom}_{G_{\mathbb{Q}}}(V_g \otimes V_h, E(H) \otimes L) = 2.$$

Pick a basis:

$$\text{Hom}_{G_{\mathbb{Q}}}(V_g \otimes V_h, E(H) \otimes L) = \langle P, Q \rangle_L,$$

so

$$P, Q: V_g \otimes V_h \rightarrow E(H) \otimes L.$$

Pick also a basis $V_g \otimes V_h = \langle e_{\alpha\alpha}, e_{\alpha\beta}, e_{\beta\alpha}, e_{\beta\beta} \rangle$ where $\text{Frob}_p(e_{\alpha\alpha}) = \alpha_g \alpha_h e_{\alpha\alpha}$ and similarly for $e_{\alpha\beta}, e_{\beta\alpha}, e_{\beta\beta}$. Define

$$\begin{aligned} P_1 &= P(e_{\beta\alpha}), & P_2 &= P(e_{\beta\beta}), \\ Q_1 &= P(e_{\beta\alpha}), & Q_2 &= Q(e_{\beta\beta}) \end{aligned}$$

in $E(H) \otimes L$.

Finally, $u_g \in \mathcal{O}_{H_g}^\times[\text{ad}^0(g)] \otimes L$ is characterized by $\text{Frob}_p(u_g) = \frac{\beta_g}{\alpha_g} \otimes u_g$, where H_g is the field cut out by $\text{ad}^0(g)$. This is a *Stark unit*.

4.2. Evidence for the conjecture.

- (1) Numerical evidence in many cases (Lauder’s algorithm),
- (2) The main theoretical evidence is [DLR15, Theorem 3.1] which proves conjecture when $g = \text{Ind}(\psi_g)$, $h = \text{Ind}(\psi_h)$, where $\psi_g, \psi_h: G_K \rightarrow L^\times$ are characters of an imaginary quadratic field K . This can be done by proving a factorization of the p -adic L -function into Katz p -adic L -functions and using results of Bertolini–Darmon–Prasanna.
- (3) If we replace f_E with an Eisenstein series of weight 2, there is a natural generalization of this conjecture, where P_i, Q_i are replaced with Stark units. This is done in [DLR16].
- (4) Gatti and Guitart [GG19] generalize these conjectures to higher weights and provide theoretical and numerical evidence.
- (5) Rivero and Rotger [RR18] prove it when $h = g^*$. The proof uses exceptional zeros and Hida’s *improved* p -adic L -function.

REFERENCES

- [BD16] Joël Bellaïche and Mladen Dimitrov, *On the eigencurve at classical weight 1 points*, Duke Math. J. **165** (2016), no. 2, 245–266, doi:10.1215/00127094-3165755, https://doi.org/10.1215/00127094-3165755. MR 3457673
- [Del73] Pierre Deligne, *Formes modulaires et représentations de $\text{GL}(2)$* , 55–105. Lecture Notes in Math., Vol. 349. MR 0347738
- [DLR15] Henri Darmon, Alan Lauder, and Victor Rotger, *Stark points and p -adic iterated integrals attached to modular forms of weight one*, Forum Math. Pi **3** (2015), e8, 95, doi:10.1017/fmp.2015.7, https://doi.org/10.1017/fmp.2015.7. MR 3456180
- [DLR16] ———, *Gross-Stark units and p -adic iterated integrals attached to modular forms of weight one*, Ann. Math. Qué. **40** (2016), no. 2, 325–354, doi:10.1007/s40316-015-0042-6, https://doi.org/10.1007/s40316-015-0042-6. MR 3529185
- [DR14] Henri Darmon and Victor Rotger, *Diagonal cycles and Euler systems I: A p -adic Gross-Zagier formula*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 779–832, doi:10.24033/asens.2227, https://doi.org/10.24033/asens.2227. MR 3250064
- [DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids 1*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975), http://www.numdam.org/item?id=ASENS_1974_4_7_4_507_0. MR 0379379
- [GG19] Francesca Gatti and Xavier Guitart, *On the elliptic stark conjecture in higher weight*, 2019, Preprint, arXiv:arXiv:1903.02430.
- [GZ86] Benedict H. Gross and Don B. Zagier, *Heegner points and derivatives of L -series*, Invent. Math. **84** (1986), no. 2, 225–320, doi:10.1007/BF01388809, https://doi.org/10.1007/BF01388809. MR 833192
- [HK91] Michael Harris and Stephen S. Kudla, *The central critical value of a triple product L -function*, Ann. of Math. (2) **133** (1991), no. 3, 605–672, doi:10.2307/2944321, https://doi.org/10.2307/2944321. MR 1109355
- [Ich08] Atsushi Ichino, *Trilinear forms and the central values of triple product L -functions*, Duke Math. J. **145** (2008), no. 2, 281–307, doi:10.1215/00127094-2008-052, https://doi.org/10.1215/00127094-2008-052. MR 2449948
- [Kat73] Nicholas M. Katz, *p -adic properties of modular schemes and modular forms*, 69–190. Lecture Notes in Mathematics, Vol. 350. MR 0447119
- [Kol90] V. A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 435–483. MR 1106906
- [RR18] Óscar Rivero and Victor Rotger, *Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form*, 2018, Preprint, arXiv:arXiv:1806.10022.

- [Shi71] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971, Kanô Memorial Lectures, No. 1. MR 0314766
- [Urb14] Eric Urban, *Nearly overconvergent modular forms*, Iwasawa theory 2012, Contrib. Math. Comput. Sci., vol. 7, Springer, Heidelberg, 2014, pp. 401–441. MR 3586822
- [Wat02] Thomas Crawford Watson, *Rankin triple products and quantum chaos*, ProQuest LLC, Ann Arbor, MI, 2002, Thesis (Ph.D.)–Princeton University. MR 2703041
- [YZZ12] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang, *Triple product L -series and Gross–Kudla–Schoen cycles*, 2012, preprint, available at: <http://math.mit.edu/~wz2113/math/online/triple.pdf>.