L-FUNCTIONS

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Contents

Introduction

These notes, meant as an introduction to the theory of L-functions, form a report from the author's Undergraduate Research Opportunities project at Imperial College London under the supervision of Professor David Helm.

The theory of L-functions provides a way to study arithmetic properties of integers (and, more generally, integers of number fields) by first, translating them to analytic properties of certain functions, and then, using the tools and methods of analysis, to study them in this setting.

The first indication that the properties of primes could be studied analytically came from Euler, who noticed the factorization (for a real number $s > 1$):

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This was later formalized by Riemann, who defined the ζ-function by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for a complex number s with $\mathrm{Re}(s) > 1$ and proved that it admits an analytic continuation by means of a functional equation. Riemann established a connection between its zeroes and the distribution of prime numbers, and it was also proven that the Prime Number Theorem is equivalent to the fact that no zeroes of ζ lie on the line $\mathrm{Re}(s) = 1$. Innocuous as it may seem, this function is very difficult to study (the Riemann Hypothesis, asserting that the non-trivial zeroes of ζ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, still remains one of the biggest open problems in mathematics).

The ζ-function was extended further by Dirichlet, who proved that for coprime integers $a, m \in \mathbb{Z}$, there are infinitely many primes in the arithmetic progression $(a + dm)_{d\in\mathbb{Z}}$. This theorem is clearly difficult to approach algebraically and Dirichlet's idea was to restate it in analytic terms. He considered a function that is defined just like ζ, but contains coefficients: for any function $\chi\colon \mathbb{Z} \to \mathbb{C}$, the *Dirichlet L-function* is defined by:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

To ensure that the Euler factorization

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

holds, we assume that the function χ is *multiplicative*, i.e. $\chi(mn) = \chi(m)\chi(n)$ for $(m, n) = 1$. The way Dirichlet proved the theorem is by showing that

$$\prod_\chi L(s, \chi),$$

where χ varies over multiplicative functions with period m, i.e. $\chi(a + m) = \chi(a)$, has a pole at $s = 1$. One can show that this statement implies that the series

$$\sum_{p\equiv a \bmod m} \frac{1}{p}$$

(where the sum is over primes p such that $p \equiv 1 \bmod m$) diverges, so there must be infinitely many primes in the progression $(a + dm)_{d\in\mathbb{Z}}$. In fact, it gives a stronger result about the density of primes in the arithmetic progression (Dirichlet's Density Theorem). For the proofs of these theorems, see [Ser73, Chap. VI].

The aim of this exposition is to generalize Dirichlet L-functions further to encapture not only primes in $\mathbb{Q}$, but also primes in finite algebraic extensions of $\mathbb{Q}$, so-called *number fields*. In Section 1, we provide the necessary background on algebraic number theory. The reader already familiar with [Ser79, Chap. I–III] or [Lan94, Chap. I–III] can skip ahead to Section 2. In Sections 2 and 3, we define the analog of Dirichlet L-functions for an abelian extension—*Hecke L-functions*—and, following Tate's thesis [CF86, Chap. XV], use Fourier analysis in number fields to obtain a functional equation. In Sections 4 and 5, we use class field theory to define L-functions for non-abelian extensions—*Artin L-functions*—that agree with the Hecke L-functions in the abelian case, and use representation theory to obtain a function equation. Our presentation largely follows [Sny02].

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1. Algebraic Number Theory

We have seen that we can use the Dirichlet L-functions to study rational primes (primes of $\mathbb{Z}$). We would like to generalize the ideas to study primes in other rings. We start by providing a background on algebraic number theory. Throughout most of this section, we will follow [Ser79, Chap. I–III] and [Lan94, Chap. I–III]—we will refer to particular sections where applicable.

1.1. Factoring Primes in Extensions.

In this section, we follow [Ser79, Chap. I§5] [Lan94, Chap. I§5,7].

Suppose A is a Dedekind domain (a Noetherian integrally closed domain such that for every prime ideal $\mathfrak{p} \neq 0$ of A, $A_\mathfrak{p}$ is a discrete valuation ring) and K is its field of fractions. Moreover, suppose L is a separable extension of K of degree n, and let B be the integral closure of A in L. Then B is also a Dedekind domain.

We will denote by I_K the set of fractional ideals of K. Recall that for any $\mathfrak{a} \in I_K$ we have a unique factorization into primes

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(\mathfrak{a})},$$

where we denote by $v_\mathfrak{p}$ the valuation at a prime ideal $\mathfrak{p}$ of A (if $\mathfrak{a}$ is an ideal of A, then $v_\mathfrak{p}(\mathfrak{a})$ is the highest power of $\mathfrak{p}$ dividing $\mathfrak{a}$, and we extend this to any $\mathfrak{a} \in I_K$).

Definition 1.1. If $\mathfrak{P}$ is a non-zero prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$, then we say that $\mathfrak{P}$ *divides* $\mathfrak{p}$ or that $\mathfrak{P}$ *lies above* $\mathfrak{p}$, and we write $\mathfrak{P}|\mathfrak{p}$.

Definition 1.2. Suppose $\mathfrak{P}$ divides $\mathfrak{p}$ in the extension L/K. The *ramification index* $e_\mathfrak{P}$ of $\mathfrak{P}$ over $\mathfrak{p}$ in the extension L/K is the exponent of $\mathfrak{P}$ in the factorization of $\mathfrak{p}$ into prime ideals of B. In other words:

$$\mathfrak{p} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_\mathfrak{P}}.$$

The *residue degree* $f_\mathfrak{P}$ of $\mathfrak{P}$ over $\mathfrak{p}$ in the extension L/K is the degree of the extension of residues fields, $B/\mathfrak{P}$ of $A/\mathfrak{p}$; symbolically

$$f_\mathfrak{P} = [B/\mathfrak{P} : A/\mathfrak{p}].$$

We may sometimes write $e_{\mathfrak{P}/\mathfrak{p}}$ and $f_{\mathfrak{P}/\mathfrak{p}}$ when the extension L/K is not implicitly clear.

Proposition 1.3. *Both the ramification index and the residue degree are multiplicative in towers: if $\mathfrak{P}|\mathfrak{q}|\mathfrak{p}$ in the tower of extensions $L/M/K$, then*

$$e_{\mathfrak{P}/\mathfrak{p}} = e_{\mathfrak{P}/\mathfrak{q}} e_{\mathfrak{q}/\mathfrak{p}},$$

$$f_{\mathfrak{P}/\mathfrak{p}} = f_{\mathfrak{P}/\mathfrak{q}} f_{\mathfrak{q}/\mathfrak{p}}.$$

Proof. The proof is clear: one simply writes down the factorizations in the extensions for the first equality and the tower of extensions of residue fields for the second equality. □

Definition 1.4. When there is only one prime ideal $\mathfrak{P}$ dividing $\mathfrak{p}$ and $f_{\mathfrak{P}} = 1$, then L/K is *totally ramified* at $\mathfrak{p}$. In that case

$$\mathfrak{p} = \mathfrak{P}^{e_{\mathfrak{P}}}.$$

When $e_{\mathfrak{P}} = 1$ and $B/\mathfrak{P}$ is separable over $A/\mathfrak{p}$, then L/K is *unramified* at $\mathfrak{P}$. If L/K is unramified at every $\mathfrak{P}$ lying above $\mathfrak{p}$, then L/K is *unramified* at $\mathfrak{p}$. Finally, if $e_{\mathfrak{P}} = f_{\mathfrak{P}} = 1$, then $\mathfrak{p}$ *splits completely* in L and there are exactly $[L : K]$ primes of B above $\mathfrak{p}$.

Proposition 1.5. *Let $\mathfrak{p}$ be a non-zero prime ideal of A. Then*

$$[L : K] = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}} f_{\mathfrak{P}}.$$

Proof. We may assume that A is a discrete valuation ring by localizing at $\mathfrak{p}$. Then B is a free A-module of rank $n = [L : K]$, and $B/\mathfrak{p}$ is an $A/\mathfrak{p}$-vector space of dimension n by [Lan94, Prop. 19, Chapter I]. Moreover,

$$\mathfrak{p} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_{\mathfrak{P}}},$$

and, since $\mathfrak{p} \subseteq \mathfrak{P}^{e_{\mathfrak{P}}}$, we have homomorphisms

$$B \to B/\mathfrak{p} \to B/\mathfrak{P}^{e_{\mathfrak{P}}}$$

which together yield

$$\varphi\colon B \to B/\mathfrak{p} \to \prod_{\mathfrak{P}|\mathfrak{p}} B/\mathfrak{P}^{e_{\mathfrak{P}}}.$$

Each $B/\mathfrak{P}^{e_{\mathfrak{P}}}$ is an $A/\mathfrak{p}$-vector space, and hence so is the direct product. We have that

$$\ker\varphi = \{b \in B \mid b \in \mathfrak{P}^{e_{\mathfrak{P}}} \text{ for each } \mathfrak{P}|\mathfrak{p}\} = \prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}^{e_{\mathfrak{P}}} = \mathfrak{p}.$$

The Chinese reminder theorem shows that φ is surjective. Therefore

$$B/\mathfrak{p} \cong \prod_{\mathfrak{P}|\mathfrak{p}} B/\mathfrak{P}^{e_{\mathfrak{P}}}$$

as an $A/\mathfrak{p}$-vector space. To prove the proposition, it is enough to show that the dimension of $B/\mathfrak{P}^{e_{\mathfrak{P}}}$ over $A/\mathfrak{p}$ is $e_{\mathfrak{P}} f_{\mathfrak{P}}$. Let π be a generator of $\mathfrak{P}$ in B and $j \geq 1$ be an integer. Note that $\mathfrak{P}^j/\mathfrak{P}^{j+1} \subseteq \mathfrak{P}^j/\mathfrak{p}\mathfrak{P}^j$, since $\mathfrak{P}^{j+1} \supseteq \mathfrak{p}\mathfrak{P}^j$, so we can view $\mathfrak{P}^j/\mathfrak{P}^{j+1}$ as an $A/\mathfrak{p}$-vector space. In fact, the map

$$B/\mathfrak{P} \to \mathfrak{P}^j/\mathfrak{P}^{j+1}$$

given by $b \mapsto \pi^j b$ is an $A/\mathfrak{p}$-isomorphism. Therefore, we have a decomposition series

$$1 \subseteq \mathfrak{P}^{e_{\mathfrak{P}}-1}/\mathfrak{P}^{e_{\mathfrak{P}}} \subseteq \mathfrak{P}^{e_{\mathfrak{P}}-2}/\mathfrak{P}^{e_{\mathfrak{P}}} \subseteq \ldots \subseteq \mathfrak{P}/\mathfrak{P}^{e_{\mathfrak{P}}} \subseteq B/\mathfrak{P}^{e_{\mathfrak{P}}}$$

with each quotient isomorphic to $B/\mathfrak{P}$. Hence:

$$[B/\mathfrak{P}^{e_{\mathfrak{P}}} : A/\mathfrak{p}] = \sum_{i=0}^{e_{\mathfrak{P}}-1} [B/\mathfrak{P} : A/\mathfrak{P}] = e_{\mathfrak{P}} f_{\mathfrak{P}},$$

as requested. □

Definition 1.6. Let M be the Galois closure of L, and let $G = \mathrm{Gal}(M/K)$ and $H = \mathrm{Gal}(M/L)$, so that G/H can be identified with the embeddings $L \hookrightarrow M$ which preserve K. We define the *(relative) norm* $N_{L/K} \colon I_L \to I_K$ to be the multiplicative function

$$N_{L/K}\mathfrak{a} = \prod_{\sigma \in G/H} \sigma\mathfrak{a}.$$

Note that the norm sends a principal ideal αB to the principal ideal

$$\left(\prod_{\sigma \in G/H} \sigma\alpha \right) A,$$

so can also define a map $N_{L/K} \colon L \to K$ by

$$N_{L/K}\alpha = \prod_{\sigma \in G/H} \sigma\alpha.$$

1.2. **Primes in Galois Extensions.** We keep the assumptions from the previous section but assume further that L/K is Galois.

Proposition 1.7. *Fix a prime $\mathfrak{p}$ of K. The Galois group $\mathrm{Gal}(L/K)$ acts transitively on the primes $\mathfrak{P}$ above $\mathfrak{p}$ in L/K.*

Proof. Let $\mathfrak{P}|\mathfrak{p}$ and suppose that there exists a prime ideal $\mathfrak{P}'|\mathfrak{p}$ such that $\mathfrak{P}' \neq \sigma\mathfrak{P}$ for all $\sigma \in \mathrm{Gal}(L/K)$. Then by the Chinese remainder theorem, there exists $\alpha \in B$ such that

$$\alpha \equiv 0 \bmod \mathfrak{P}'$$

and for any $\sigma \in \mathrm{Gal}(L/K)$

$$\alpha \equiv 1 \bmod \sigma\mathfrak{P}.$$

The norm

$$N_{L/K}\alpha = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma\alpha \in \mathfrak{P}' \cap A = \mathfrak{p}$$

since it is in A and α is in $\mathfrak{P}'$ above $\mathfrak{p}$. However, $\alpha \notin \sigma\mathfrak{P}$, so $\sigma\alpha \notin \mathfrak{P}$ for all $\sigma \in \mathrm{Gal}(L/K)$, so their product $N_{L/K}\alpha \notin \mathfrak{P}$, which contradicts the fact that $N_{L/K}\alpha \in \mathfrak{p} = \mathfrak{P} \cap A$. ☐

Corollary 1.8. *Let $\mathfrak{p}$ be a non-zero prime ideal of A. For any $\mathfrak{P}$, $\mathfrak{P}'$ above $\mathfrak{p}$, we have*

$$e_{\mathfrak{P}} = e_{\mathfrak{P}'} \text{ and } f_{\mathfrak{P}} = f_{\mathfrak{P}'},$$

so the integers $e_{\mathfrak{P}}$ and $f_{\mathfrak{P}}$ are independent on the choice of $\mathfrak{P}$ above $\mathfrak{p}$. We can hence denote them by $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$, and if $g_{\mathfrak{p}}$ is the number of primes ideals $\mathfrak{P}$ dividing $\mathfrak{p}$, then

$$[L : K] = e_{\mathfrak{p}} f_{\mathfrak{p}} g_{\mathfrak{p}}.$$

Proof. The first part follows from proposition 1.7, the second part from Proposition 1.5. ☐

Definition 1.9. The *decomposition group* of a prime $\mathfrak{P}$ in L/K is the subgroup of $\mathrm{Gal}(L/K)$ fixing $\mathfrak{P}$:

$$D_{\mathfrak{P}} = D_{\mathfrak{P}}(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma\mathfrak{P} = \mathfrak{P}\}$$

We can define a homomorphism

$$\varepsilon\colon D_{\mathfrak{P}} \to \mathrm{Gal}\left((B/\mathfrak{P})/(A/\mathfrak{p})\right)$$

by reducing an element $\sigma \in D_{\mathfrak{P}} \subseteq \mathrm{Gal}(L/K)$ to an automorphism of $B/\mathfrak{P}$ fixing $A/\mathfrak{p}$, since $\mathfrak{p} = \mathfrak{P} \cap A$.

Definition 1.10. The *inertia group* of a prime $\mathfrak{P}$ in L/K is

$$I_{\mathfrak{P}} = \ker \varepsilon \subseteq D_{\mathfrak{P}}.$$

Proposition 1.11. *The residue extension $B/\mathfrak{P}$ over $A/\mathfrak{p}$ is normal** *and the homomorphism*

$$\epsilon\colon D_{\mathfrak{P}} \to \mathrm{Gal}\left((B/\mathfrak{P})/(A/\mathfrak{p})\right)$$

is surjective. In particular,

$$D_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}\left((B/\mathfrak{P})/(A/\mathfrak{p})\right).$$

Proof. To simplify notation, let $\overline{B} = B/\mathfrak{P}$ and $\overline{A} = A/\mathfrak{p}$. To show $\overline{B}/\overline{A}$ is normal, take any irreducible polynomial $\overline{f}$ with at least one root $\overline{b}$ in $\overline{B}$. We will show that it splits completely in $\overline{B}$. By definition, $\overline{f}$ is the minimal polynomial of $\overline{b}$. Take any $b \in B$ that reduces to $\overline{b}$ in $\overline{B}$, and let g be the minimal polynomial of b in L/K. Since b is integral, we know that $g \in B[X]$, so we have the reduction $\overline{g} \in \overline{B}[X]$. Since L/K is separable, the polynomial g splits completely, and hence $\overline{g}$ splits completely in $\overline{B}$. Therefore, $\overline{f}|\overline{g}$ splits completely in $\overline{B}$.

Now, we turn to surjectivity of ϵ. Let $G = \mathrm{Gal}(L/K)$. Choose $\overline{a}$ to be a generator of the largest separable extension $\overline{B}_{\mathrm{sep}}$ of $\overline{A}$ within $\overline{B}$. By Chinese remainder theorem, there is a representative a of $\overline{a}$ which belongs to all the prime ideals $\sigma\mathfrak{P}$ for $\sigma \in G \setminus D_{\mathfrak{P}}$. Consider the minimal polynomial of a

$$f(X) = \prod_{\sigma \in G} (X - \sigma(a))$$

which reduces to

$$\overline{f}(X) = \prod_{\sigma \in G} \left(X - \overline{\sigma(a)}\right).$$

The non-zero roots of $\overline{f}(X)$ are of the form $\overline{\sigma(a)}$ for $\sigma \in D_{\mathfrak{P}}$: for $\sigma \notin D_{\mathfrak{P}}$ we know that $\overline{\sigma(a)} = 0$, since $\sigma(a) \in \mathfrak{P}$. But we know that $f(\overline{a}) = 0$, so the minimal polynomial

$$m(X) = \prod_{\tau \in \mathrm{Gal}(\overline{B}_{\mathrm{sep}}/\overline{A})} (X - \tau(\overline{a}))$$

for $\overline{a}$ divides $\overline{f}(X)$. In particular, any conjugate $\tau(\overline{a})$ is equal to $\overline{\sigma(a)}$ for some $\sigma \in \mathrm{Gal}(L/K)$. This proves surjectivity of ϵ. ∎

Corollary 1.12. *The number of elements of $I_{\mathfrak{P}}$ is the ramification index $e_{\mathfrak{P}/\mathfrak{p}}$*

Proof. By Proposition 1.11:

$$\#D_{\mathfrak{P}}/\#I_{\mathfrak{P}} = f_{\mathfrak{P}/\mathfrak{p}}$$

*Since $B/\mathfrak{P}$ over $A/\mathfrak{p}$ may in general not be separable, the group $\mathrm{Gal}\left((B/\mathfrak{P})/(A/\mathfrak{p})\right)$ is in fact the group $\mathrm{Gal}\left((B/\mathfrak{P})_{\mathrm{sep}}/(A/\mathfrak{p})\right)$, the Galois group of the largest separable subextension of $B/\mathfrak{P}$ over $A/\mathfrak{p}$.

and since $\mathrm{Gal}(L/K)$ acts transitively on the primes of $\mathfrak{P}$ above $\mathfrak{p}$, by Corollary 1.8 we have that

$$\#D_{\mathfrak{P}} = e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}},$$

which shows that $\#I_{\mathfrak{P}} = e_{\mathfrak{P}/\mathfrak{p}}$ □

This hints that the inertia group of a prime $\mathfrak{P}$ measures the ramification at $\mathfrak{P}$. In fact, much more is true.

Proposition 1.13. *Let $K_{\mathfrak{p}}$ and $L_{\mathfrak{P}}$ be the completions of K and L under the valuations associated to the primes $\mathfrak{p}$ and $\mathfrak{P}$ respectively. Then largest unramified subextension of $L_{\mathfrak{P}}$ over $K_{\mathfrak{p}}$ is $L_{\mathfrak{P}}^{I_{\mathfrak{P}}}$.*

Proof. Let K and L be fields complete under valuations v_K and v_L with maximal ideals $\mathfrak{p}$ and $\mathfrak{P}$, respectively. We then write $G = \mathrm{Gal}(L/K)$, $I = I_{\mathfrak{P}}$, and claim that L/L^I is totally ramified and L^I/K is unramified. We will denote by C the valuation ring of L^I, and by $\mathfrak{q}$ the prime of L^I over $\mathfrak{p}$; altogether:

$$\begin{array}{cccc}
B/\mathfrak{P} & B & L & \mathfrak{P} \\
| & | & | & | \\
C/\mathfrak{q} & C & L^I & \mathfrak{q} \\
| & | & | & | \\
A/\mathfrak{p} & A & K & \mathfrak{p}
\end{array}$$

First, take any $\sigma \in I$. Then $\epsilon(\sigma) = 1 \in \mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big)$ by definition of I. Therefore any $\sigma \in I = \mathrm{Gal}(L^I/L)$ induces the identity on $B/\mathfrak{P}$, which shows that $B/\mathfrak{P} = C/\mathfrak{q}$. Hence $f_{\mathfrak{P}/\mathfrak{q}} = 1$ and L/L^I is totally ramified.

We have a surjective homomorphisms $G \to \mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big)$ with kernel I and $\mathrm{Gal}(L^I/K) \to \mathrm{Gal}\big((C/\mathfrak{q})/(A/\mathfrak{p})\big)$ with kernel $I_{\mathfrak{q}/\mathfrak{p}}$, which together yield

$$G/I \cong \mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big) = \mathrm{Gal}\big((C/\mathfrak{q})/(A/\mathfrak{p})\big) \cong (G/I)/I_{\mathfrak{q}/\mathfrak{p}}.$$

Thus $e_{\mathfrak{q}/\mathfrak{p}} = \#I_{\mathfrak{q}/\mathfrak{p}} = 1$ by Corollary 1.12, showing that L^I/K is unramified. □

1.3. **Frobenius Elements and the Artin Map.** Assume furthermore that the residue field $A/\mathfrak{p}$ is finite, so it is isomorphic to $\mathbb{F}_q$ for some q. Fix a prime $\mathfrak{P}$ above $\mathfrak{p}$. Then we know that $B/\mathfrak{P}$ is a degree $f = f_{\mathfrak{P}}$ extension of a finite field $A/\mathfrak{p} \cong \mathbb{F}_q$, so $B/\mathfrak{P} \cong \mathbb{F}_{q^f}$. Furthermore, the Galois group $\mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big)$ is cyclic, generated by the Frobenius automorphism

$$x \mapsto x^q.$$

By Proposition 1.11, we know that

$$D_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big),$$

so $D_{\mathfrak{P}}/I_{\mathfrak{P}}$ is cyclic generated by an element that maps to the Frobenius automorphism $x \mapsto x^q$.

Definition 1.14. The element of $D_{\mathfrak{P}}/I_{\mathfrak{P}}$ that maps to the Frobenius automorphism $x \mapsto x^q$ in $\mathrm{Gal}\big((B/\mathfrak{P})/(A/\mathfrak{p})\big)$ is called the *Frobenius element* or *Frobenius substitution* for $\mathfrak{P}$ and we denote it by $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$ or simply $\mathrm{Frob}_{\mathfrak{P}}$.

Proposition 1.15. *If $\mathfrak{P}'$ and $\mathfrak{P}$ are primes above $\mathfrak{p}$ in L/K and $\sigma \in \mathrm{Gal}(L/K)$ satisfies $\sigma\mathfrak{P}' = \mathfrak{P}$, then*

$$\begin{aligned} D_{\mathfrak{P}'} &= \sigma^{-1} D_{\mathfrak{P}} \sigma, \\ I_{\mathfrak{P}'} &= \sigma^{-1} I_{\mathfrak{P}} \sigma, \\ \mathrm{Frob}_{\mathfrak{P}'} &= \sigma^{-1} \mathrm{Frob}_{\mathfrak{P}} \sigma. \end{aligned}$$

Proof. This is immediate. □

Sometimes, we will only be interested in the decomposition group, the inertia group, and the Frobenius element up to conjugation. In that case, we will use the notation $D_{\mathfrak{p}} = D_{\mathfrak{P}}$, $I_{\mathfrak{p}} = I_{\mathfrak{P}}$, $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{P}}$ for some chosen prime $\mathfrak{P}$ above $\mathfrak{p}$.

Proposition 1.16. *If $\mathfrak{P}|\mathfrak{q}|\mathfrak{p}$ in a tower of extensions $K \subseteq M \subseteq L$, then we can choose representatives $\varphi_{\mathfrak{P}/p} \in \mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$, $\varphi_{\mathfrak{P}/\mathfrak{q}} \in \mathrm{Frob}_{\mathfrak{P}/\mathfrak{q}}$, $\varphi_{\mathfrak{q}/\mathfrak{p}} \in \mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}$, so that*

$$\varphi_{\mathfrak{P}/\mathfrak{q}} = \varphi_{\mathfrak{P}/\mathfrak{p}}^{f_{\mathfrak{q}/\mathfrak{p}}}$$

and if E/K is Galois, then the image of $\varphi_{\mathfrak{P}/\mathfrak{p}}$ in $\mathrm{Gal}(E/K)$ is $\varphi_{\mathfrak{q}/\mathfrak{p}}$.

Proof. This is immediate. □

Suppose a prime $\mathfrak{p}$ is unramified, i.e. $e_{\mathfrak{p}} = 1$ and $I_{\mathfrak{P}} = \{1\}$. Then $\mathrm{Frob}_{\mathfrak{p}}$ is an element of $D_{\mathfrak{P}} \subseteq \mathrm{Gal}(L/K)$, so we can define the following map for L/K abelian.

Definition 1.17. Suppose L/K is abelian, and let $\mathfrak{m}$ be a product of primes of K divisible by all the primes in $\mathfrak{p}$ that ramify. Let $I_K^{\mathfrak{m}}$ be the fractional ideals of K that are coprime to $\mathfrak{m}$. The *Artin map*

$$\mathrm{Frob}\colon I_K^{\mathfrak{m}} \to \mathrm{Gal}(L/K)$$

is defined by $\mathrm{Frob}(\mathfrak{p}) = \mathrm{Frob}_{\mathfrak{p}}$ for any prime $\mathfrak{p}$ coprime to $\mathfrak{m}$, and extended multiplicatively to all of $I_K^{\mathfrak{m}}$.

This map will be the starting point of Section 4 on Class Field Theory and it will enable us to define L-functions on characters of abelian Galois extensions.

1.4. **Discriminant and Different.**

We review two invariants associated to a separable extension L of a field K, the discriminant and the different. We follow [Ser79, Chap. III] but omit the proofs in this section.

We keep the assumptions of the previous sections. Moreover, we assume (for notation purposes) that the ring $A \subseteq K$ is implicitly clear (the important example for our sake will be K and $A = \mathcal{O}_K$, the ring of integers; see Section 1.5). Note that this also fixes the choice of the rings $B \subseteq L$ and $C \subseteq M$, where $K \subseteq L \subseteq M$ is a tower of extensions.

Similarly to defining the norm as the product of Galois conjugates of elements of L, we can consider their sum. We let M be the Galois closure of L, and $G = \mathrm{Gal}(M/K)$, $H = \mathrm{Gal}(M/L)$, so that G/H can be identified with embeddings $L \hookrightarrow M$ which preserve K.

Definition 1.18. The *trace* is a function $\mathrm{Tr}\colon L \to K$ given by

$$\mathrm{Tr}(x) = \sum_{\sigma \in G/H} \sigma(x).$$

The trace is a surjective and the bilinear form $\mathrm{Tr}(xy)$ is non-degenerate.

Definition 1.19. Let $\{e_i\}$ be a basis of B over A as a free A-module. The *discriminant* of L/K is

$$\Delta_{L/K} = \Delta_{B/A} = \det(\mathrm{Tr}(e_i e_j)).$$

There is also an alternative description of the discriminant given in [Bou03, Prop. 12, Chap. V§10]:

$$\Delta_{L/K} = (\det(\sigma e_i))^2$$

where σ runs over cosets G/H.

Definition 1.20. The *codifferent* of L over K is the fractional ideal

$$\mathfrak{d}_{L/K}^{-1} = \{y \in L \mid \mathrm{Tr}(xy) \in A \text{ for all } x \in B\}.$$

Its inverse $\mathfrak{d}_{L/K}$ is called the *different* of L over K.

Proposition 1.21 ([Ser79, Prop. 6, Chap. III])**.** *We have that* $\Delta_{L/K} = N_{L/K}\mathfrak{d}_{L/K}$.

Corollary 1.22. *The discriminant* $\Delta_{L/K}$ *is contained in* A.

Proposition 1.23 (Discriminant and different in towers, [Ser79, Prop. 8, Chap. III])**.** *Suppose* M/L *is a separable extension of finite degree* n. *Then*

$$\mathfrak{d}_{M/K} = \mathfrak{d}_{M/L}\mathfrak{d}_{L/K} \text{ and } \Delta_{M/K} = (\Delta_{M/L})^n \cdot N_{L/K}(\Delta_{M/L}).$$

Finally, we state a proposition that allows us to easily compute the different.

Proposition 1.24 ([Ser79, Cor. 2, Prop. 11, Chap. III])**.** *Suppose* x *is an* A*-generator of* B *and* f *is the minimal ideal of* x. *If* f' *is the derivative of* f, *then:*

$$\mathfrak{d}_{L/K} = (f'(x)).$$

1.5. **Number Fields.** In this paper, we will focus mostly on a particular example of the fields considered before, namely number fields.

Definition 1.25. A *number field* K is a finite algebraic extension of $\mathbb{Q}$.

Definition 1.26. We say that $a \in K$ is an *integer* of K (or *integral* over K) if it is a root of a monic polynomial with coefficients in $\mathbb{Z}$. The set of all integers of K forms the *ring of integers* $\mathcal{O}_K$ of K.

The following proposition ensures that number fields satisfy the assumptions of the preceding parts of the section.

Proposition 1.27. *Let* K *be a number field. Then* $\mathcal{O}_K$ *is a Dedekind domain, with the valuations* $v_K = \mathrm{ord}_{\mathfrak{p}}$ *where* $\mathfrak{p}$ *is a prime ideal, and* K *is its fields of fractions. Moreover, if* L *is a finite separable extension of* K, *then the integral closure of* $\mathcal{O}_K$ *in* L *is* $\mathcal{O}_L$.

Proof. The only part of the proposition that is not immediate is that $\mathcal{O}_K$ is a Dedekind domain. By [Lan94, Chap. I, Th. 2], it is enough to check that $\mathcal{O}_K$ is Noetherain, integrally closed, and such that every non-zero prime ideal is maximal. The first two assertions are clear, so we will only show that every non-zero prime is maximal.

First, note that for $0 \neq a \in \mathcal{O}_K$, the map $\cdot a\colon \mathcal{O}_K \to \mathcal{O}_K$ given by $x \mapsto xa$ is injective. Therefore, the cokernel of the map is finite, and hence (a) is a submodule of $\mathcal{O}_K$ of the same rank, which means that $\mathcal{O}_K/(a)$ is finite. Now, take any non-zero prime ideal of $\mathfrak{p}$ of $\mathcal{O}_K$ with $a \in \mathfrak{p}$. Then $\mathfrak{p}$ reduces to an ideal $\overline{\mathfrak{p}}$ in $A/(a)$, and

$$A/\mathfrak{p} \cong (A/(a))/(\overline{\mathfrak{p}}).$$

Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is a domain, so $(A/(a))/(\overline{p})$ is a finite domain, so it is a field. Hence $A/\mathfrak{p}$ is also a field, and $\mathfrak{p}$ is maximal. □

We will be interested in the different absolute values on a number field K. We list a few notions that will be useful later, but for a broader discussion of absolute values and completions, see [Lan94, Chap. 2]. By Ostrowski's Theorem for number fields (see: [Con]), we know there are two possible absolute values:

- For any prime $\mathfrak{p}$ of $\mathcal{O}_K$, we have the $\mathfrak{p}$-adic absolute value:
$$|a|_{\mathfrak{p}} = \begin{cases} N\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(a)} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$
- For any embedding $v\colon K \hookrightarrow \mathbb{C}$, we have an absolute value induced by the embedding.

This motivates the following definition.

Definition 1.28. Let K be a number field. A *finite prime* of K is a prime of $\mathcal{O}_K$. An *infinite prime* of K is an embedding $v\colon K \hookrightarrow \mathbb{C}$. Moreover, if the image of v lies in $\mathbb{R}$, it is a *real infinite prime*, and if the image of v contains an element of $\mathbb{C} \setminus \mathbb{R}$, it is a *complex infinite prime*.

Now suppose L/K is an extension of number fields. An infinite prime w of L *lies above* an infinite prime v of K if w agrees with v on K.

Instead of finite and infinite primes, these are sometimes called finite and infinite places.

Definition 1.29. The absolute value associated to a real prime $v\colon K \hookrightarrow \mathbb{R}$ is the standard absolute value. The absolute value associated to a complex prime $v\colon K \hookrightarrow \mathbb{C}$ is the **square** of the standard absolute value.

We will sometimes identify the infinite primes with the absolute values defined above, without explicitly stating that they come from a complex embedding.

2. Hecke L-functions

We now have enough background to generalize the ζ-functions and L-functions to any number field K.

Definition 2.1. The *Dedekind ζ-function* for a number field K is defined as the series

$$\zeta_K(s) = \sum_{\mathfrak{a}} N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

for all $\mathrm{Re}(s) > 1$, where the sum varies over all ideals of $\mathcal{O}_K$, and the product varies over all primes ideals of $\mathcal{O}_K$.

As with the Riemann ζ-function, we are interested in describing an analytic continuation of the Dedekind ζ-function by means of a functional equation. Of course, this could be done directly, but we will do it by defining the more general Hecke L-functions (which will be equal to the Dirichlet ζ-functions for the trivial character), and exhibiting a functional equation for them.

We would like to generalize the ζ-function by adding coefficients given by values of a character—just as the Dirichlet L-function is a generalization of the Riemann ζ-function.

Definition 2.2. A *character* of a group G is a group homomorphism $\chi\colon G \to \mathbb{C}^\times$.

Note that in the introduction we introduced Dirichlet L-functions for multiplicative function with period m. This corresponds exactly to a character $\chi\colon (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{Z}$ which is extended to any n not coprime to m by setting $\chi(n) = 0$. We call such a character a *Dirichlet character* modulo m.

Our aim is to try to redefine a character in a similar way for a general number field K.

Definition 2.3. A formal product $\mathfrak{m}$ of finite and real primes, where the finite primes appear with non-negative multiplicity and the real primes appear with multiplicity 0 or 1, is a *modulus*. For such an $\mathfrak{m}$, we define:

$$I_K^{\mathfrak{m}} = \{\mathfrak{a} \in I_K \mid \mathfrak{a} \text{ coprime to } \mathfrak{m}\}$$

and

$$P_K^{\mathfrak{m}} = \{a\mathcal{O}_K \mid a \equiv 1 \bmod \mathfrak{m} \text{ and for any real infinite prime } v|\mathfrak{m} \text{ we have } v(a) > 0\}.$$

Then the *ray class group* of K is the quotient $I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}$.

Proposition 2.4. *The ray class group is finite.*

The reader can refer to [Lan94, Th. 7, Chap. VI] for the proof.

We will define the generalized character on the ray class group. To justify this is indeed a generalization of the above, we analyze the case $K = \mathbb{Q}$ in detail.

Example 2.5 ($K = \mathbb{Q}$)**.** The finite primes of K are just rational primes and there is one infinite prime of K, the subset embedding $\infty\colon \mathbb{Q} \hookrightarrow \mathbb{C}$. Fix $m \in \mathbb{Z}$ (which is a product of finite primes of K) and let us work backwords to find a modulus $\mathfrak{m}$ such that $I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}} \cong \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^\times$. Note that $I_K^{\mathfrak{m}}$ only depends on the finite primes dividing $\mathfrak{m}$; if m' is the finite part of $\mathfrak{m}$, then

$$I_K^{\mathfrak{m}} = \{a\mathbb{Z} \mid (a, m') = 1\}.$$

Therefore, the natural choice for m' is m. We are left with two choices for $\mathfrak{m}$: m and $m\infty$. Suppose $\mathfrak{m} = m\infty$. We can then define a surjective homomorphism $\varphi\colon I_K^{\mathfrak{m}} \to \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^\times$ by setting

$$\varphi(a\mathbb{Z}) = a + m\mathbb{Z}$$

for any $a \in \mathbb{Z}$ with $(a, m) = 1$ and $a > 0$. Note that

$$\ker \varphi = \{a\mathbb{Z} \mid (a,m) = 1, a > 0, \text{ and } a+m\mathbb{Z} = 1+m\mathbb{Z}\} = \{a\mathbb{Z} \mid a \equiv 1 \bmod m \text{ and } a > 0\} = P_K^{m\infty}.$$

Therefore, if we let $\mathfrak{m} = m\infty$, we obtain

$$\frac{I_K^{m\infty}}{P_K^{m\infty}} \cong \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^{\times}.$$

This shows that it is crucial to allow the modulus $\mathfrak{m}$ to have infinite factors. Indeed, if $\mathfrak{m} = m$, then the homomorphism φ would not be well-defined (indeed, $a\mathbb{Z} = (-a)\mathbb{Z}$ but $a + m\mathbb{Z} \neq -a + m\mathbb{Z}$). Instead, we could adjust the above argument to show

$$\frac{I_K^{\mathfrak{m}}}{P_K^{\mathfrak{m}}} \cong \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^{\times} / \{\pm 1\},$$

so the characters could only be defined up to a sign.

Definition 2.6. A *generalized Dirichlet character* modulo a modulus $\mathfrak{m}$ for a number field K is a character χ of the ray class group $I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}$. Such a character χ is *primitive* if $\ker \chi$ is trivial.

Any generalized Dirichlet character extends to a multiplicative, complex-valued function on $I_K^{\mathfrak{m}}$ (and on I_K by setting $\chi(\mathfrak{a}) = 0$ for $\mathfrak{a}$ not coprime to $\mathfrak{m}$).

We wish to generalize this notion further to account for infinite primes of K. So first, let

$$K^{\infty} = \prod_v K_v$$

be the product over all infinite primes v of completions of K with respect to v. So if r_1 is the number of real embeddings (real primes) and r_2 is the number of pairs of complex conjugate embeddings (complex primes), then the dimension of K^{∞} as an $\mathbb{R}$-vector space is $r_1 + 2r_2$. Then we can define an embedding $K^{\times} \hookrightarrow K^{\infty}$.

Definition 2.7. Suppose $I_K/P_K = \{a_1, \ldots, a_n\}$. A function $\chi\colon I_K^{\mathfrak{m}} \to \mathbb{C}^{\times}$ is a *Hecke Grossencharacter* modulo $\mathfrak{m}$ if it can be written of the form

$$\chi(a_i(a)) = \chi_{\text{cl}}(a_i)\chi_f(a)\chi_{\infty}(a)$$

where χ_{cl} is a character of I_K/P_K, χ_f is a character of $(\mathcal{O}_K/\mathfrak{m})^{\times}$, and χ_{∞} is a character of K^{∞}.

We will see in Section 3 that this definition can be restated using the *ideles*.

We can finally define the Hecke L-function.

Definition 2.8. For a Hecke Grossencharacter $\chi\colon I_K \to \mathbb{C}^{\times}$, we define the *Hecke L-function* of χ by

$$L(s, \chi) = \sum_{\mathfrak{a} \in I_K} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s}}$$

for $\text{Re}(s) > 1$.

Definition 2.9. If χ is a Hecke Grossencharacter modulo some modulus $\mathfrak{m}$ for some number field K, then its *conductor* $\mathfrak{f} = \mathfrak{f}(\chi)$ is the smallest modulus such that χ factors through $I_K^{\mathfrak{f}}/P_K^{\mathfrak{f}}$. We denote the finite part of $\mathfrak{f}$ by $\mathfrak{f}_0$.

In order to establish the functional equation for the Hecke L-functions, we have intoduce local factors at the infinite primes.

Definition 2.10. Suppose χ is a Hecke Grossencharacter with conductor $\mathfrak{f}$ and v is an infinite prime of the number field K. Then we define the *local factor* of the L-series at v as follows:

$$L_v(s,\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \text{if } v \text{ is a real prime not dividing } \mathfrak{f}, \\ \Gamma_{\mathbb{R}}(s+1) = \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}) & \text{if } v \text{ is a real prime dividing } \mathfrak{f}, \\ \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) & \text{if } v \text{ is a complex prime.} \end{cases}$$

Theorem 2.11 (Hecke)**.** *If χ is a Hecke Grossencharacter modulo $\mathfrak{m}$ for a number field K, then the completed abelian L-function*

$$\Lambda(\chi,s) = (|\Delta_{K/\mathbb{Q}}|N\mathfrak{f}(\chi)_0)^{s/2}\prod_v L_v(s,\chi)\prod_{\mathfrak{p}\nmid\mathfrak{f}}\frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}$$

(where v ranges over all infinite primes) can be analytically continued to a holomorphic function (unless χ is trivial in which case it is meromorphic with poles at 0 and 1) with the functional equation

$$\Lambda(\chi,s) = \varepsilon(\chi)\Lambda(\overline{\chi},1-s)$$

for some $\varepsilon(\chi)$ with $|\varepsilon(\chi)| = 1$.

Instead of proving the theorem using the classic approach taken by Hecke (for the original proof, see [Hec83, pp. 178–197]), we will follow Tate's thesis [CF86, Chap. XV] and develop the theory of Fourier analysis in number fields.

3. Tate's Thesis: Fourier Analysis in Number Fields and Hecke's Zeta Functions

We follow the reprint of Tate's thesis in [CF86, Chap. XV] and the notes [Buz09]. An alternative approach can be found in [RV99].

3.1. Haar Measure and Abstract Fourier Analysis.

In this section, we review the theory of Fourier analysis on a locally compact Hausdorff topological group G. The reader familiar with this material can skip ahead to Section 3.2. We follow the presentation in [Buz09, Chap. 2], but more details (and proofs) can be found in [RV99, Chap. 1, 3]. Given a function $f\colon G \to \mathbb{C}$, we will define a function $\hat{f}\colon \widehat{G} \to \mathbb{C}$ on the group $\widehat{G}$ of characters of G. By identifying the groups $\widehat{\widehat{G}}$ with G, we will exhibit an Inversion Formula that will send $\hat{f}$ back to f. We start by recalling two examples.

Example 3.1 ($G = \mathbb{R}$)**.** If $G = \mathbb{R}$, then clearly $\widehat{G} = \mathbb{R}$: any character is an exponential function and we can associate to $x \in \mathbb{R}$ the character $\xi \mapsto e^{2\pi i x\xi}$. In this case, we follow the standard definition of a Fourier transform: for $f\colon \mathbb{R} \to \mathbb{C}$, we define $\hat{f}\colon \mathbb{R} \to \mathbb{C}$ by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx.$$

We then have the Inversion Formula

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi.$$

Example 3.2 (G is a finite abelian group)**.** For $f\colon G \to \mathbb{C}$, define $\hat{f}\colon \widehat{G} \to \mathbb{C}$ by

$$\hat{f}(\chi) = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)}.$$

In order to obtain an Inversion Formula, we need an identification of the groups G and $\widehat{\widehat{G}}$: we identify $g \in G$ with the function $\widehat{G} \to \mathbb{C}$ given by

$$\chi \mapsto \chi(g^{-1}).$$

Clearly, this yields an isomorphism of the two groups. Moreover:

$$\begin{aligned}
\hat{\hat{f}}(g) &= \tfrac{1}{|\widehat{G}|} \sum_{\chi \in \widehat{G}} \hat{f}(\chi)\overline{\chi(g^{-1})} \\
&= \tfrac{1}{|G|} \sum_{\chi \in \widehat{G}} \left(\tfrac{1}{|G|} \sum_{h \in G} f(h)\overline{\chi(h)} \right) \chi(g) && \text{by definition} \\
&= \tfrac{1}{|G|} \sum_{h \in G} \left(\tfrac{1}{|G|} \sum_{\chi \in \widehat{G}} \chi(g)\overline{\chi(h)} \right) f(h) \\
&= \tfrac{1}{|G|} f(g) && \text{by orthonormality of characters}
\end{aligned}$$

which yields the required Inversion Formula.

In order to generalize these examples to a wider variety of groups, we will need to generalize the notion of integration (which corresponded to summation in the finite case). We first recall the definition of a locally compact Hausdorff topological group.

Definition 3.3. A *topological group* G is a group endowed with a topology such that the multiplication map $(g, g') \mapsto gg'$ and the inversion map $g \mapsto g^{-1}$ are continuous.

Definition 3.4. A topological space X is *Hausdorff* if for any distinct points $x, y \in X$, there exist disjoint open neighborhoods U of x and V of y.

Definition 3.5. A topological space X is *locally compact* if every $x \in X$ has a compact neighborhood.

For the rest of the section, we let G be a locally compact Hausdorff topological group and

$$\mathcal{K}(G) = \{ f\colon G \to \mathbb{R} \mid f \text{ is continous and has compact support} \}.$$

In order to exhibit that $\mathcal{K}(G)$ is a wide class of functions, we apply Urysohn's Lemma from point-set topology to obtain that $\mathcal{K}(G)$ seperates points: for any distinct $g, h \in G$, there exists a function $f \in \mathcal{K}(G)$ such that $f(g) \neq f(h)$.

Definition 3.6. A *Haar integral (measure)* on G is a non-zero linear map $\mu\colon \mathcal{K}(G) \to \mathbb{R}$ such that

(1) $\mu(f) \geq 0$ for any $f \in \mathcal{K}(G)$ such that $f(x) \geq 0$ for all $x \in G$ and $f(x) > 0$ for some $x \in G$.

(2) $\mu(f) = \mu(f^x)$ for any $x \in G$, where $f^x(g) = f(gx^{-1})$.

(Note that this definition makes sense because $f \in \mathcal{K}(G)$ implies that $f^x \in \mathcal{K}(G)$.)

Theorem 3.7. *If G is a locally compact Hausdorff topological group, then a Haar integral exists on G, and if μ_1, μ_2 are Haar integrals, then for some $c > 0$ we have $c\mu_1 = \mu_2$.*

If μ is a Haar integral on G, we will write

$$\mu(f) = \int_G f(x)d\mu(x).$$

Theorem 3.8 (Fubini's Theorem)**.** *If G, H are locally compact Hausdorff topological groups with Haar integrals μ, ν, respectively, and $f \in \mathcal{K}(G \times H)$, then*

$$\int_G \left(\int_H f(x,y)d\nu(y) \right) d\mu(x), \quad \int_H \left(\int_G f(x,y)d\mu(x) \right) d\nu(y)$$

exist, are equal, and are both Haar measures on $G \times H$.

We will want to extend the range of integral functions to a wider class that $\mathcal{K}(G)$. For that sake, define

$$U = \{f\colon G \to \mathbb{R} \cup \{\infty\} \mid f \text{ is a pointwise limit of a sequence } f_1 \leq f_2 \leq \cdots \text{ of } f_i \in \mathcal{K}(G)\}$$

If $f \in U$, then $\mu(f) = \lim_n \mu(f_n)$ is well-defined and independent of the choice of f_n. Set $-U = \{-f \mid f \in U\}$ and $\mu(-f) = -\mu(f)$ for $-f \in -U$.

Definition 3.9. A function $f\colon G \to \mathbb{R} \cup \{\pm\infty\}$ is *summable* if there exist $g \in -U$ and $h \in U$ with $g \leq f \leq h$ and

$$\sup\{\mu(g) \mid g \leq f \text{ and } g \in -U\} = \inf\{\mu(h) \mid h \geq f \text{ and } h \in U\}.$$

The common value is defined to be $\mu(f) \in \mathbb{R}$.

Definition 3.10. We let

$$\mathcal{L}^1(G) = \{f\colon G \to \mathbb{R} \mid f \text{ is summable}\}$$

with the norm given by $||f|| = \mu(|f|)$. A function $f \in \mathcal{L}^1(G)$ is *null* if $||f|| = 0$ and we define

$$L^1(G) = \frac{\mathcal{L}^1(G)}{\{f \mid ||f|| = 0\}}.$$

Furthermore, we can define $\mathcal{L}^p(G)$ as the set of functions such that $|f|^p$ is summable with the norm $||f||_p = \mu(|f|^p)^{1/p}$, and similarly for $L^p(G)$.

The Haar measure also defines a measure on G: $A \subseteq G$ is *measurable* if χ_A, the characteristic function of A, is summable, and in that case we the define $\mu(A) = \mu(\chi_A)$.

Suppose, moreover, that G is abelian. In order to define a Fourier transform that satisfies an Inversion Formula, we must first introduce a topology on

$$\widehat{G} = \{\chi\colon G \to S^1 \mid \chi \text{ continuous group homomorphism}\}.$$

Suppose $C \subseteq G$ is compact, V is a neighborhood of the identity in S^1. Then define

$$W(C, V) = \{\chi \in \widehat{G} \mid \chi(C) \subseteq V\}.$$

We let $\{W(C, V)\}$ be a base of neighborhoods of the identity in $\widehat{G}$; explicitly, $U \subseteq \widehat{G}$ is open if and only if for all $\psi \in U$ there exist C, V such that

$$W(C, V)\psi \subseteq U.$$

Lemma 3.11. *If G is an abelian topological group, then the above construction makes $\widehat{G}$ into an abelian topological group.*

Proof. The proof is clear: one simply checks that $W(C, V)$ forms a neighborhood basis. □

Proposition 3.12. *Let G be an abelian topological group. Then*

(1) *If G is discrete, then $\widehat{G}$ is compact.*
(2) *If G is compact, then $\widehat{G}$ is discrete.*

Theorem 3.13. *If G is an abelian locally compact Hausdorff topological group, then so is $\widehat{G}$.*

Definition 3.14. Fix an abelian locally compact Hausdorff topological group G and a Haar measure on G. If $f \in L^1(G)$, then define the *Fourier transform* $\hat{f} \in \widehat{G}$ of f by

$$\hat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx.$$

Note that this definition actually makes sense: since $f \in L^1(G)$ and $|f(x)\overline{\chi(x)}| = |f(x)|$, it is easy to check that the integrand is also in $L^1(G)$.

Example 3.15 ($G = \mathbb{R}$)**.** Any character $\chi \colon \mathbb{R} \to S^1$ is given by $\chi(x) = e^{2\pi i x\xi}$ for some $\xi \in \mathbb{R}$. We then have

$$\hat{f}(\chi) = \int_{\mathbb{R}} f(x)\overline{\chi(x)}dx = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx,$$

which is consistent with the regular definition of the Fourier transform (see: Example 3.1) by identifying χ with ξ. Note that the definitons of the Fourier transform in this case vary and all of them are captured within this general theory: multiplying the integral by a constant corresponds to choosing a different Haar measure and using $e^{ix\xi}$ instead of $e^{2\pi i x\xi}$ corresponds to a different identification of $\widehat{\mathbb{R}}$ with $\mathbb{R}$.

Example 3.16 (G is a finite abelian group)**.** A Haar measure on G is given by the average

$$\mu(f) = \frac{1}{|G|}\sum_{g\in G} f(g),$$

indeed

$$\mu(f^x) = \frac{1}{|G|}\sum_{g\in G} f(gx^{-1}) = \frac{1}{|G|}\sum_{g'\in G} f(g') = \mu(f).$$

Therefore

$$\hat{f}(\chi) = \frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi(g)},$$

as expected from Example 3.2.

We might have also introduced the topology on $\widehat{G}$ differently: it is the weakest topology that makes every $\widehat{f}$ continuous, the *transform topology.*

Theorem 3.17 (Plancharet Theorem)**.** *For an abelian locally compact Hausdorff topological group G, one can extend the Fourier transform uniquely to an isometric isomorphism*

$$\hat{\cdot}\colon L^2(G) \to L^2(G),$$

i.e. for any $f \in L^2(G)$ we have that

$$\int_G |f(x)|^2 d\mu(x) = \int_{\widehat{G}} |\hat{f}(\chi)|^2 d\hat{\mu}(\chi).$$

Theorem 3.18 (Pontrjagin duality)**.** *If G is an abelian locally compact Hausdorff topological group G, then the obvious map $G \to \widehat{\widehat{G}}$ is a homeomorphism and group isomorphism.*

Definition 3.19. We define

$$\mathcal{B}^1(G) = \{f \in L^1(G) \mid \hat{f} \in L^1(\widehat{G}) \text{ and } f, \hat{f} \text{ are continuous}\}.$$

with the absolute value given above.

Theorem 3.20 (Fourier Inversion Formula)**.** *Fix Haar measures on G and $\widehat{G}$. Then there exists $c > 0$ such that if $f \in \mathcal{B}^1(G)$, and we identify G with $\widehat{\widehat{G}}$, then*

$$\hat{\hat{f}}(x) = cf(x^{-1})$$

for any $x \in G$. In particular, for any choice of Haar measure on G, there is a unique choice of Haar measure on $\widehat{G}$ for which $c = 1$.

Definition 3.21. Fix a measure on G. The measure on $\widehat{G}$ for which $c = 1$ in the Fourier Inversion Formula 3.20 is the *dual measure.*

3.2. **The Local Theory.** We start by developing the theory in the local setting. Let K be the completion of an algebraic number field at a prime v. Thus K is either $\mathbb{R}$ or $\mathbb{C}$ if v is infinite, or K is $\mathfrak{p}$-adic if $v = \mathfrak{p}$ is finite.

In the latter case, the ring of integers $\mathcal{O}$ of K has a single prime ideal, $\mathfrak{p}$, with a residue class field $\mathcal{O}/\mathfrak{p}$ of $N\mathfrak{p}$ elements.

We introduce the following valuation on K:

$$|\alpha| = \begin{cases} \text{ordinary absolute value} & \text{if } K \text{ is real,} \\ \text{square of ordinary absolute value} & \text{if } K \text{ is complex,} \\ (N\mathfrak{p})^{-\operatorname{ord}_\mathfrak{p}(\alpha)} & \text{if } K \text{ is } \mathfrak{p}\text{-adic.} \end{cases}$$

Note that K is locally compact and in fact a subset $B \subseteq K$ is relatively compact if and only if it is bounded in absolute value.

3.2.1. *Additive Characters and Measure.* In this section, we investigate the additive subgroup K^+ of K.

Lemma 3.22. *If $\chi\colon K^+ \to \mathbb{C}$ is a non-trivial character of K^+, then for any $\eta \in K^+$, the function $\xi \mapsto \chi(\eta\xi)$ is also a character. Moreover, $\eta \mapsto \chi(\eta -)$ is an isomorphism, both topological and algebraic, of K^+ with its character group.*

Proof. We prove this lemma in several steps.

(1) For fixed $\eta \in K^+$, $\chi(\eta-)$ is a character of K^+:
$$\chi(\eta(\xi_1 + \xi_2)) = \chi(\eta\xi_1 + \eta\xi_2) = \chi(\eta\xi_1)\chi(\eta\xi_2).$$
(2) The map $\eta \mapsto \chi(\eta-)$ is an algebraic homomorphism:
$$\chi((\eta_1 + \eta_2)\xi) = \chi(\eta_1\xi + \eta_2\xi) = \chi(\eta_1\xi)\chi(\eta_2\xi)$$
(3) The map is a monomorphism: if $\chi(\eta\xi) = 1$ for all ξ, then $\eta K^+ \neq K^+$, so $\eta = 0$.
(4) The characters of the form $\chi(\eta-)$ are everywhere dense in the character group: if $\chi(\eta\xi) = 1$ for all η, then $K^+\xi \subseteq K^+$, so $\xi = 0$.
(5) The map $\eta \mapsto \chi(\eta-)$ is continuous and open. It is enough to show these properties at the identity elements. We denote by B_r the ball of radius r at 0 in K^+.

To show that the map is continuous, fix any basis neighborhood of the identity, i.e. $W(C, V)$ for a compact set C in K^+ and an open neighborhood V of 1 in $\mathbb{C}$. Since C is bounded, $C \subseteq B_M$ for some M. By continuity of χ, there exists $\varepsilon > 0$ such that
$$\chi(B_\varepsilon) \subseteq V.$$
Then for any $\eta \in B_{\varepsilon/M}$, we have that
$$\eta C \subseteq B_\varepsilon$$
and hence $\chi(\eta C) \subseteq V$, or equivalently $\chi(\eta-) \in W(C, V)$, showing continuity.

To show that the map is open, fix $\varepsilon > 0$. We will show that there exist C, V such that $\chi(\eta-) \in W(C, V)$ for any $\eta \in B_\varepsilon$, or in other words $\chi(\eta C) \subseteq V$. Take any neighborhood V of 1 in $\mathbb{C}$. Since χ is continuous, there exists $\delta > 0$ such that $\chi(B_\delta) \subseteq V$. Therefore, letting $C = \{\xi \in K^+ \mid |\xi| \leq \varepsilon/\delta\}$, we obtain that $\chi(\eta C) \subseteq V$ for any $\eta \in B_\varepsilon$, which is the required property.
(6) Characters of the form $\chi(\eta-)$ comprise a locally compact subgroup of the character group. However, from general topology, we know that locally compact implies complete which implies closed. Together with (4), this shows that the mapping is onto.

This completes the proof. □

To fix the identificaion of K^+ with $\widehat{K^+}$ given by the lemma, we need to choose *special* character.

If v is infinite, let $R = \mathbb{R}$. We then define $\lambda\colon R \to \mathbb{R}/\mathbb{Z}$ by
$$\lambda(x) = -x \bmod \mathbb{Z}.$$
If $v = \mathfrak{p}$ is finite, then let p be a rational prime dividing $\mathfrak{p}$, and $R = \mathbb{Q}_p$, the completion of $\mathbb{Q}$ at p. For $x \in R$, let n be such that $p^n x$ is integral, and choose $m \in \mathbb{Z}$ such that $m \equiv p^n x \bmod p^n$. We then define $\lambda\colon R \to \mathbb{R}/\mathbb{Z}$ by
$$\lambda(x) = m/p^n.$$
Note that λ has the property that $\lambda(x) - x$ is a p-adic integer:
$$\lambda(x) - x = m/p^n - x = (m - xp^n)/p^n$$
and $m \equiv p^n x \bmod p^n$.

Lemma 3.23. *The map λ is non-trivial, continuous, and additive.*

Proof. The infinite case is trivial. For the finite case, it is clear that λ is non-trivial and continuous. To show that it is additive, take $x, x' \in R$ and suppose $p^n x$ and $p^{n'} x'$ are integral and $m \equiv p^n x \bmod p^n$, $m' \equiv p^{n'} x' \bmod p^{n'}$. Without loss of generality, suppose $n' \geq n$. Then $p^{n'}(x + x')$ is integral and

$$mp^{n'-n} + m' \equiv p^{n'}(x + x') \bmod p^{n'}.$$

Hence

$$\lambda(x) + \lambda(x') = \frac{m}{p^n} + \frac{m'}{p^{n'}} = \frac{mp^{n'-n} + m'}{p^{n'}} = \lambda(x + x'),$$

as requested. □

Definition 3.24. We define the map $\Lambda\colon K^+ \to \mathbb{R}/\mathbb{Z}$ by

$$\Lambda(\xi) = \lambda(\mathrm{Tr}_{K/R}\xi),$$

where $\mathrm{Tr}_{K/R}\colon K \to R$ is the trace map.

Finally, we fix an identification of K^+ with its character group.

Theorem 3.25. *The map $\chi\colon K^+ \to S^1$ given by $\chi(\xi) = e^{2\pi i \Lambda(\xi)}$ is a non-trivial character of K^+. Then K^+ is naturally its own character group by identifying $\eta \in K^+$ with $\chi(\eta -)$, i.e. the character given by*

$$\xi \mapsto e^{2\pi i \Lambda(\eta\xi)}.$$

Proof. This follows from Lemmas 3.22 and 3.23, and the fact that the trace map is additive and continuous. □

Lemma 3.26. *Suppose $v = \mathfrak{p}$ is finite. Then the character $e^{2\pi i \Lambda(\eta -)}$ is trivial on $\mathcal{O}_K$ if and only if $\eta \in \mathfrak{d}^{-1}$, where $\mathfrak{d} = \mathfrak{d}_{K/R}$ is the different of K over R.*

Proof. Recall that by definition

$$\mathfrak{d}^{-1} = \{x \in K \mid \mathrm{Tr}_{K/R}(xy) \in \mathcal{O}_K \text{ for all } x \in \mathcal{O}_R\}.$$

Therefore, the character is trivial on $\mathcal{O}_K$ if and only if $\lambda(\mathrm{Tr}_{K/R}(\eta\mathcal{O}_R)) = \Lambda(\eta\mathcal{O}_R) = 0$, which is equivalent to $\mathrm{Tr}(\eta\mathcal{O}_R) \subseteq \mathcal{O}_K$, i.e. $\eta \in \mathfrak{d}^{-1}$. □

Now let μ be a Haar measure on K^+. The measure is invariant under addition and we wish to investigate its behavior under multiplication.

Lemma 3.27. *If we define $\mu_1(M) = \mu(\alpha M)$ for $\alpha \neq 0$, $\alpha \in K$ and M a measurable set in K^+, then μ_1 is a Haar measure, and there exists a number $\varphi(\alpha) > 0$ such that $\mu_1 = \varphi(\alpha)\mu$.*

Proof. Since $\xi \mapsto \alpha\xi$ is an automorphism of K^+, both algebraic and toplogical, and since the Haar measure is uniquely determined by the structure up to a constant, $\mu_1 = \varphi(\alpha)\mu$ for some $\varphi(\alpha)$. □

Lemma 3.28. *We have that $\varphi(\alpha) = |\alpha|$, i.e. $\mu(\alpha M) = |\alpha|\mu(M)$.*

Proof. If $K = \mathbb{R}$, this is clear. If $K = \mathbb{C}$, this is true, since we have chosen $|\alpha|$ to be the square of the regular absolute value.

If K is $\mathfrak{p}$-adic, $\mathcal{O}_K$ is both compact and open, so $0 < \mu(\mathcal{O}_K) < \infty$. Therefore, it suffices to compare $\mu(\mathcal{O}_K)$ and $\mu(\alpha\mathcal{O}_K)$. For α integral, there are $N(\alpha\mathcal{O}_K)$ cosets of α in $\mathcal{O}_K$, and hence

$$\mu(\alpha\mathcal{O}_K) = N(\alpha\mathcal{O}_K)^{-1}\mu(\mathcal{O}_K) = |\alpha|\mu(\mathcal{O}_K).$$

For α non-integral, we can factorize $\alpha\mathcal{O}_K$ into primes and apply the same method. □

For the integral, this yields:

$$\int f(\xi)d\mu(\xi) = |\alpha| \int f(\alpha\xi)d\mu(\xi).$$

We will now fix a particular Haar measure on K^+ that will be used throughout the rest of the section. We can do it in a way that makes the constant in the Fourier inversion equal to 1. Explicitly:

$$d\xi = \begin{cases} \text{the ordinary Lebesgue measure on the real line} & \text{if } K = \mathbb{R}, \\ \text{twice the ordinary Lebesgue measure on the complex plane} & \text{if } K = \mathbb{C}, \\ \text{the measure for which } \mathcal{O}_K \text{ has measure } \Delta^{-1/2} & \text{if } K \text{ is } \mathfrak{p}\text{-adic}. \end{cases}$$

Theorem 3.29 (Inversion Formula for number fields)**.** *If we define the Fourier transform $\hat{f}$ of a function $f \in L_1(K^+)$ by*

$$\hat{f}(\eta) = \int f(\xi)e^{-2\pi i\Lambda(\eta\xi)}d\xi$$

then with the above choice of the measure, the Inversion Formula

$$f(\xi) = \int \hat{f}(\eta)e^{2\pi i\Lambda(\xi\eta)}d\eta = \hat{\hat{f}}(-\xi)$$

holds for any $f \in \mathcal{B}^1(K^+)$.

Proof. By the general Fourier Inversion Formula 3.20, we know that $f(\xi) = c\hat{\hat{f}}(-\xi)$ for some $c > 0$. We only need to check that $c = 1$. This is achieved by considering a particular function:

$$f(\xi) = \begin{cases} e^{-\pi|\xi|^2} & \text{for } K = \mathbb{R}, \\ e^{-\pi|\xi|} & \text{for } K = \mathbb{C}, \\ \chi_{\mathcal{O}_K}, \text{ the characteristic function of } \mathcal{O}_K & \text{for } K\ \mathfrak{p}\text{-adic}. \end{cases}$$

The necessary calculations, showing that $c = 1$, can be found in Section 3.2.4. □

3.2.2. *Multiplicative Characters and Measure.* In this section, we investigate the multiplicative subgroup $K^\times$ of K.

Let $U = \{\alpha \in K^\times \mid |\alpha| = 1\}$, the compact set of *units* in $K^\times$. Note that U is also open if K is $\mathfrak{p}$-adic.

Definition 3.30. A *quasi-character* is any continuous multiplicative map $c\colon K^\times \to \mathbb{C}^\times$, bounded or unbounded. A quasi-character c is *unramified* if c is trivial on U. A quasi-character is a *character* if $|c(\alpha)| = 1$ for any $\alpha \in K^\times$.

Lemma 3.31. *The unramified quasi-characters $c\colon K^\times \to \mathbb{C}^\times$ are of the form $c(\alpha) = |\alpha|^s := e^{s\log|\alpha|}$, where s is a complex number which is*

1. *determined uniquely by c, if v is infinite,*
2. *determined uniquely by c up to an integer multiple of $2\pi i/\log N\mathfrak{p}$, if $v = \mathfrak{p}$ is finite.*

Proof. Clearly, for any s, $|-|^s$ is an unramified quasi-character. Conversely, if c is an unramified quasi-character, then for $\alpha, \alpha' \in K^\times$ with $|\alpha| = |\alpha'|$, we have that $c(\alpha/\alpha') = 1$ since $\alpha/\alpha' \in U$, so $c(\alpha) = c(\alpha')$. Thus c is only dependent on $|\alpha|$.

We now treat the finite and the infinite case seperately. For v infinite, c is a multiplicative function $\mathbb{R}_{\geq 0} \to \mathbb{C}$ and we know that these are indeed the exponential functions, determined uniquely by s.

For $v = \mathfrak{p}$ finite, the image of the valuation are powers of $N\mathfrak{p}$, i.e. $\{(N\mathfrak{p})^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$, so we are looking for additive characters on $\mathbb{Z}$. However, any additive function on $\mathbb{Z}$ is given by multiplication by elements of $\mathbb{Z}$, which corresponds to exponentiation of $N\mathfrak{p}$, as requested. Finally, s is clearly only determined by c up to a multiple of $2\pi i/\log N\mathfrak{p}$. $\square$

For v infinite, we can write any $\alpha \in K^\times$ uniquely as $\alpha = \tilde{\alpha}\varrho$ with $\tilde{\alpha} = \alpha/|\alpha| \in U$ and $\varrho = |\alpha| > 0$. For $v = \mathfrak{p}$ finite, we fix an element π with $\mathrm{ord}_{\mathfrak{p}}\pi = 1$ (a uniformizer), and we can then write any $\alpha \in K^\times$ uniquely as $\alpha = \tilde{\alpha}\varrho$ with $\tilde{\alpha} = \alpha\pi^{-\mathrm{ord}_{\mathfrak{p}}\alpha} \in U$ and $\varrho = \pi^{\mathrm{ord}_{\mathfrak{p}}\alpha}$. In either case, we have a continuous projection $K^\times \twoheadrightarrow U$ given by $\alpha \mapsto \tilde{\alpha}$, i.e. a function onto U which is constant on U.

Theorem 3.32. *The quasi-characters of $K^\times$ are the maps $c\colon K^\times \to \mathbb{C}^\times$ given by $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$, where $\tilde{c}$ is any character of U, uniquely determined by c, and s is determined by c as in Lemma 3.31.*

Proof. A map of the given type is indeed a quasi-character. Conversely, given a quasi-character c, we define $\tilde{c} = c_{|U}$, the restriction of C to U. Then $\tilde{c}$ is a quasi-character of U, and therefore a character of U, since U is compact. But now

$$\alpha \mapsto \frac{c(\alpha)}{\tilde{c}(\tilde{\alpha})}$$

is an unramified character of $K^\times$, and hence Lemma 3.31 completes the proof. $\square$

Therefore, we have reduced the problem of classifying the quasi-characters of $K^\times$ to finding the characters of U.

- If $K = \mathbb{R}$, then $U = \{1, -1\}$, and the characters are given by $\alpha \mapsto \alpha^n$ for $n \in \{0, 1\}$.
- If $K = \mathbb{C}$, then $U = S^1$ and the characters are given by $\alpha \mapsto \alpha^n$ for $n \in \mathbb{Z}$.
- If K is $\mathfrak{p}$-adic, then the subgroups $1 + \mathfrak{p}^n$ for $n > 0$ of U form a fundamental system of neighborhoods of 1 in U. Therefore, we must have $\tilde{c}(1 + \mathfrak{p}^n) = 1$ for sufficiently large n. Selecting n to be minimal with this property ($n = 0$ if $\tilde{c} = 1$), we call the ideal $\mathfrak{f} = \mathfrak{p}^n$ is the *conductor* of $\tilde{c}$. Then $\tilde{c}$ is a character of the finite group $U/(1 + \mathfrak{f})$ and hence is determined by a finite character table.

Definition 3.33. If $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$, then $\sigma = \sigma(c) = \mathrm{Re}(s)$ is the *exponent* of c.

Note that a quasi-character is a character if and only if its exponent is 0.

We will select a multiplicative Haar measure $\mathtt{d}\alpha$ on $K^\times$ by relating it to the chosen additive Haar measure $d\xi$ on K^+.

If $g \in L^1(K^\times)$, then the function given by $\alpha \mapsto g(\alpha)|\alpha|^{-1}$ is also in $L^1(K^\times)$. Thus we may define on $L^1(K^\times)$ a functional

$$\Phi(g) = \int_{K^+\setminus\{0\}} g(\xi)|\xi|^{-1} d\xi.$$

For any $\beta \in K^\times$, we have that

$$\Phi(\beta g) = \int_{K^+\setminus\{0\}} g(\beta\xi)|\xi|^{-1} d\xi = \Phi(g),$$

using the substitution $\xi \mapsto \beta^{-1}\xi$ and the fact that $d(\beta^{-1}\xi) = |\beta|^{-1} d\xi$ by Lemma 3.28. Thus Φ is a non-trivial, positive, and invariant under translation, so it is a Haar measure on $K^\times$, which we will denote by $\mathtt{d}_1\alpha$:

$$\int g(\alpha)\mathtt{d}_1\alpha = \int_{K^+\setminus\{0\}} g(\xi)|\xi|^{-1} d\xi.$$

Lemma 3.34. *A function g is in $L_1(K^\times)$ if and only if the function given by $\xi \mapsto g(\xi)|\xi|^{-1}$ is in $L_1(K^+ \setminus \{0\})$, and for these functions:*

$$\int_{K^\times} g(\alpha)\mathtt{d}_1\alpha = \int_{K^+\setminus\{0\}} g(\xi)|\xi|^{-1} d\xi.$$

Proof. This is clear from the definition of $\mathtt{d}_1$. □

We also define a multiplicative measure that gives the subgroup U measure 1 in the infinite case and measure $\Delta^{-1/2}$ in the finite case:

$$\mathtt{d}\alpha = \begin{cases} \mathtt{d}_1\alpha = \frac{d\alpha}{|\alpha|} & \text{if } v \text{ is infinite,} \\ \frac{N\mathfrak{p}}{N\mathfrak{p}-1}\mathtt{d}_1\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1}\frac{d\alpha}{|\alpha|} & \text{if } v = \mathfrak{p} \text{ is finite.} \end{cases}$$

Lemma 3.35. *If $v = \mathfrak{p}$ is finite, then*

$$\int_U \mathtt{d}\alpha = \Delta^{-1/2}.$$

Proof. Note that we can write U as the disjoint union of $N\mathfrak{p} - 1$ additive cosets of $\mathfrak{p}\mathcal{O}_K$ in U:

$$U = \coprod_{\alpha \neq 1} (\alpha + \mathfrak{p}\mathcal{O}_K),$$

which yields for the additive measure μ:

$$\mu(U) = (N\mathfrak{p} - 1)\mu(\mathfrak{p}\mathcal{O}_K) = \frac{N\mathfrak{p} - 1}{N\mathfrak{p}}\mu(\mathcal{O}_K) = \frac{N\mathfrak{p} - 1}{N\mathfrak{p}}\Delta^{-1/2}$$

by Lemma 3.28 and our selection of the additive measure. Therefore:

$$\int_U \mathtt{d}_1\alpha = \int_U |\xi|^{-1} d\xi = \int_U d\xi = \mu(U) = \frac{N\mathfrak{p} - 1}{N\mathfrak{p}}\Delta^{-1/2},$$

as requested. □

3.2.3. *Local ζ-functions.* We define the class of functions Z to be

$$Z = \{f \in \mathcal{B}^1(K^+) \mid f(\alpha)|\alpha|^\sigma, \hat{f}(\alpha)|\alpha|^\sigma \in L^1(K^\times) \text{ for } \sigma > 0\}.$$

Definition 3.36. The *ζ-function* of K corresponding to $f \in Z$ is a function of quasi-characters of $K^\times$, defined for all quasi-characters c with $\sigma(c) > 0$ by

$$\zeta(f, c) = \int f(\alpha)c(\alpha)\mathtt{d}\alpha.$$

We say that two quasi-characters c_1, c_2 are *equivalent* if c_1/c_2 is unramified. By Lemma 3.31, we know that an equivalence class of quasi-characters consists of characters of the form

$$c(\alpha) = c_0(\alpha)|\alpha|^s,$$

where c_0 is a fixed representative and s is a complex variable. We may view such an equivalence class as a Riemann surface (with variable s). If v is an infinite prime, then the surface is the whole complex plane. If $v = \mathfrak{p}$ is finite, then s is determined up to an integer multiple of $2\pi i/\log N\mathfrak{p}$, so the Riemann surface is $\mathbb{C}$ modulo $(2\pi i/\log N\mathfrak{p})\mathbb{Z}$.

Therefore, we can view the set of all quasi-characters as a collection of Riemann surfaces and hence talk about the *holomorphy* of a function defined on the space of all quasi-characters of exponent greater than 0.

Lemma 3.37. *A ζ-function is holomorphic in the domain of all quasi-characters of exponent greater than 0.*

Proof. We want to show that for each c with $\sigma(c) > 0$:

$$s \mapsto \int f(\alpha)c(\alpha)|\alpha|^s\mathtt{d}\alpha$$

is a holomorphic function of s for s near 0. This is clear: the integral is absolutely convergent for s near 0, and has a derivative for s near 0. □

We will show that ζ-functions have a single-valued meromorphic analytic continuation to the domain of all quasi-characters by means of a functional equation. To this effect, we first prove a crucial lemma.

Lemma 3.38. *For c in the domain $0 < \sigma(c) < 1$ and $\hat{c}(\alpha) = |\alpha|c^{-1}(\alpha)$ we have*

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \zeta(\hat{f}, \hat{c})\zeta(g, c)$$

for any functions $f, g \in Z$.

Proof. We have

$$\zeta(f, c)\zeta(\hat{g}, \hat{c}) = \int f(\alpha)c(\alpha)\mathtt{d}\alpha \cdot \int \hat{g}(\beta)c^{-1}(\beta)|\beta|\mathtt{d}\beta$$

with both integrals converging absolutely for c with $0 < \sigma(c) < 1$. We can hence rewrite it as a double integral over the product $K^\times \times K^\times$:

$$\int\int f(\alpha)\hat{g}(\beta)c(\alpha\beta^{-1})|\beta|\mathtt{d}(\alpha, \beta)$$

and the change variables $(\alpha,\beta)\mapsto(\alpha,\alpha\beta)$, under which $\mathtt{d}(\alpha,\beta)$ is invariant, gives

$$\int\int f(\alpha)\hat{g}(\alpha\beta)c(\beta^{-1})|\alpha\beta|\mathtt{d}(\alpha,\beta).$$

Then Fubini's Theorem 3.8 shows that

$$\zeta(f,c)\zeta(\hat{g},\hat{c})=\int\left(\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\mathtt{d}\alpha\right)c(\beta^{-1})|\beta|\mathtt{d}\beta.$$

Since we can transform the right hand side of the equation, $\zeta(\hat{f},\hat{c})\zeta(g,c)$, in the same manner by replacing f with g, we only have to show that the inner integral

$$\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\mathtt{d}\alpha$$

is symmetric in f and g. Note that

$$\begin{aligned}\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\mathtt{d}_1\alpha &= \int f(\xi)\hat{g}(\xi\beta)d\xi && \text{by definition}\\ &= \int f(\xi)\left(\int g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}d\eta\right)d\xi && \text{by definition}\\ &= \int\int f(\xi)g(\eta)e^{-2\pi i\Lambda(\xi\beta\eta)}d(\xi,\eta) && \text{by Fubini's Theorem 3.8}\end{aligned}$$

and since the last integral is obviously symmetric in f and g, so is

$$\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\mathtt{d}\alpha=(\text{constant})\cdot\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\mathtt{d}_1\alpha,$$

which completes the proof. □

Theorem 3.39. *A ζ-function has an analytic continuation in the domain of all quasi-characters given by a functional equation of the type*

$$\zeta(f,c)=\varrho(c)\zeta(\hat{f},\hat{c}).$$

The factor $\varrho(c)$, which is independent of f, is a meromorphic function of all quasi-characters defined in the domain $0<\sigma(c)<1$ by the functional equation itself, and for all quasi-characters by analytic continuation. (Note that $\sigma(\hat{c})=\sigma(c)-1$.)

Proof. In Section 3.2.4, for each equivalence class C of quasi-characters, we will exhibit an explicit $f_C\in Z$ such that

$$\varrho(c)=\frac{\zeta(f_C,c)}{\zeta(\hat{f}_C,\hat{c})}$$

is well-defined (denominator is not identically 0) for c in the strip $0<\sigma(c)<1$. The function ϱ defined this way will be a meromorphic function described on C with an analytic continuation given by a functional equation.

This will complete the proof of the theorem. Since C is any equivalence class of quasi-characters, $\varrho\colon C\to\mathbb{C}$ is defined for all quasi-characters. Finally, for any $f\in Z$, $c\in C$ with $0<\sigma(c)<1$, we have that

$$\begin{aligned}\zeta(f,c) &= \zeta(f,c)\zeta(\hat{f}_C,\hat{c})/\zeta(\hat{f}_C,\hat{c})\\ &= \zeta(\hat{f},\hat{c})\zeta(f_C,c)/\zeta(\hat{f}_C,\hat{c}) && \text{by Lemma 3.38}\\ &= \zeta(\hat{f},\hat{c})\varrho(c)\end{aligned}$$

as requested. □

Before going on to the promised computations of special ζ-functions, we note some properties of ϱ that follow directly from the functional equation.

Corollary 3.40. *For any quasi-character c with exponent $0 < \sigma(c) < 1$:*

1. $\varrho(\hat{c}) = \frac{c(-1)}{\varrho(c)}$,
2. $\varrho(\overline{c}) = c(-1)\overline{\varrho(c)}$.
3. $|\varrho(c)| = 1$ *if* $\sigma(c) = 1/2$.

Proof. For (1), we have that:

$$\begin{aligned}
\zeta(f,c) &= \varrho(c)\zeta(\hat{f},\hat{c}) && \text{by Theorem 3.39}\\
&= \varrho(c)\varrho(\hat{c})\zeta(\hat{\hat{f}},\hat{\hat{c}}) && \text{by Theorem 3.39}\\
&= \varrho(c)\varrho(\hat{c})\int f(-\alpha)c(\alpha)\mathtt{d}\alpha && \text{by the Inversion Formula 3.29}\\
&= \varrho(c)\varrho(\hat{c})\int f(\alpha)c(-\alpha)\mathtt{d}\alpha && \text{substituting } \alpha \mapsto -\alpha\\
&= c(-1)\varrho(c)\varrho(\hat{c})\zeta(f,c)
\end{aligned}$$

which yields the result.

For (2), we have that

$$\begin{aligned}
\overline{\varrho(c)\zeta(\hat{f},\hat{c})} &= \overline{\zeta(f,c)} && \text{by Theorem 3.39}\\
&= \zeta(\hat{f},\hat{c}) &&\\
&= \varrho(\hat{c})\zeta(\hat{\overline{f}},\hat{\overline{c}}) && \text{by Theorem 3.39}\\
&= \varrho(\hat{c})c(-1)\overline{\zeta(\hat{\overline{f}},\hat{\overline{c}})} && \text{since } \hat{\overline{f}}(\alpha) = \overline{\hat{f}}(-\alpha) \text{ and } \hat{\overline{c}}(\alpha) = \overline{\hat{c}}(\alpha)\\
&= \varrho(\overline{c})c(-1)\overline{\zeta(\hat{f},\hat{c})}
\end{aligned}$$

which yields the result.

Finally, (3) follows from combining (1) and (2). First, note that if $\sigma(c) = 1/2$, then

$$c(\alpha)\overline{c}(\alpha) = |c(\alpha)|^2 = |\alpha| = c(\alpha)\hat{c}(\alpha)$$

and hence $\overline{c}(\alpha) = \hat{c}(\alpha)$. Therefore

$$|\varrho(c)|^2 = \varrho(c)\overline{\varrho(c)} = c(-1)\overline{\varrho(c)} \cdot \varrho(c)/c(-1) = \varrho(\overline{c})/\varrho(\hat{c}) = 1,$$

so $|\varrho(c)| = 1$. □

3.2.4. *Calculations for Special ζ-functions.* We finally exhibit the special ζ-functions for each equivalence class of quasi-characters. We treat the cases $K = \mathbb{R}$, $K = \mathbb{C}$, and K $\mathfrak{p}$-adic seperately.

<u>Case 1: $K = \mathbb{R}$.</u>

ξ real variable	α non-zero real variable
$\Lambda(\xi) = -\xi$	$\lvert\alpha\rvert$ ordinary absolute value
$d\xi$ ordinary Lebesgue integral	$\mathtt{d}\alpha = \frac{d\alpha}{\lvert\alpha\rvert}$

In this case, we have two equivalence classes of characters

class of quasi-characters	special function	Fourier transform
$\lvert - \rvert^s$	$f(\xi) = e^{-\pi\xi^2}$	$\hat{f}(\xi) = f(\xi)$
$\mathrm{sign}(-)\lvert - \rvert^s$	$f_\pm(\xi) = \xi e^{-\pi\xi^2}$	$\hat{f}_\pm(\xi) = i f_\pm(\xi)$

We compute the ζ-functions associated to the two special functions:

$$\zeta(f, |-|) = \int f(\alpha)|\alpha|^s \frac{d\alpha}{|\alpha|} = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}|\alpha|^{s-1} d\alpha = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^{s-1}d\alpha = \pi^{-s/2}\Gamma(s/2),$$

$$\zeta(f_\pm, \pm|-|) = \int f_\pm(\alpha)\mathrm{sign}(\alpha)|\alpha|^s \frac{d\alpha}{|\alpha|} = 2\int_0^\infty e^{-\pi\alpha^2}\alpha^s d\alpha = \pi^{-(s+1)/2}\Gamma((s+1)/2)$$

and to their Fourier transforms:

$$\zeta(\hat{f}, |\hat{-}|^s) = \zeta(f, |-|^{-1}) = \pi^{-(1-s)/2}\Gamma((1-s)/2)$$

$$\zeta(\hat{f}_\pm, \pm|\hat{-}|^s) = \zeta(if_\pm, \pm|-|^{1-s}) = i\pi^{-(2-s)/2}\Gamma((2-s)/2).$$

Therefore, we can express the function ϱ explicitly

$$\varrho(|-|^s) = \frac{\pi^{-s/2}\Gamma(s/2)}{\pi^{-(1-s)/2}\Gamma((1-s)/2)} = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s),$$

$$\varrho(\pm|-|^s) = \frac{\pi^{-(s+1)/2}\Gamma((s+1)/2)}{i\pi^{-(2-s)/2}\Gamma((2-s)/2)} = -i2^{1-s}\pi^{-s}\sin(\pi s/2)\Gamma(s).$$

Note that they are both meromorphic functions with an analytic continuation, as requested.

Case 2: $K = \mathbb{C}$.

$\xi = x + iy$ complex variable	$\alpha = re^{i\theta}$ non-zero complex variable
$\Lambda(\xi) = -2\mathrm{Re}(\xi) = -2x$	$\lvert\alpha\rvert = r^2$
$d\xi = 2\lvert dxdy\rvert$	$\mathtt{d}\alpha = \frac{d\alpha}{\lvert\alpha\rvert} = \frac{2}{r}\lvert drd\theta\rvert$

In this case, we have an equivalence class of quasi-characters for every $n \in \mathbb{Z}$

representative of the nth class	special function	Fourier transform
$c_n(\alpha) = c_n(re^{i\theta}) = e^{in\theta}$	$f_n(\alpha) = f_n(re^{i\theta}) = r^{\lvert n\rvert}e^{-in\theta}e^{-2\pi r^2}$	$\hat{f}_n(\alpha) = i^{\lvert n\rvert}f_{-n}(\alpha)$

where the formula for the Fourier transform is proved by induction—we omit the proof here.

We compute the ζ-functions associated to the nth special function f_n:

$$\begin{aligned}\zeta(f_n, c_n|-|^s) &= \int f_n(\alpha)c_n(\alpha)|\alpha|^s \mathtt{d}\alpha \\ &= \int_0^\infty\int_0^{2\pi} r^{2(s-1)+|n|}e^{-2\pi r^2}2rdrd\theta \\ &= 2\pi\int_0^\infty (r^2)^{s-1+|n|/2}e^{-2\pi r^2}d(r^2) \\ &= (2\pi)^{1-s+|n|/2}\Gamma(s+|n|/2),\end{aligned}$$

and to its Fourier transform $\hat{f}_n$:

$$\zeta(\hat{f}_n, c_n\hat{|-|}) = \zeta(i^{|n|}f_{-n}, c_{-n}|-|^{1-s}) = i^{|n|}(2\pi)^{s+|n|/2}\Gamma(1-s+|n|/2).$$

Therefore, we can express the function ϱ explicitely:

$$\varrho(c_n|-|^s) = (-i)^{|n|}\frac{(2\pi)^{1-s}\Gamma(s+|n|/2)}{(2\pi)^s\Gamma(1-s+|n|/2)}.$$

Case 3: K is $\mathfrak{p}$-adic.

ξ a $\mathfrak{p}$-adic variable	$\alpha = \tilde{\alpha}\pi^n$, non-zero $\mathfrak{p}$-adic variable, π fixed uniformizer
$\Lambda(\xi) = \lambda(\mathrm{Tr}(\xi))$	$\lvert\alpha\rvert = (N\mathfrak{p})^{-n}$
$d\xi$ so that $\mathcal{O}_K$ has measure $\Delta^{-1/2}$	$\mathsf{d}\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1}\frac{d\alpha}{\lvert\alpha\rvert}$ so that U has measure $\Delta^{-1/2}$

In this case, we have an equivalence class of quasi-characters for every $n \in \mathbb{Z}$

representative of the nth class	special function
$c_n(\alpha)$ a character with conductor $\mathfrak{p}^n$ such that $c_n(\pi) = 1$	$f_n(\xi) = \begin{cases} e^{2\pi i\Lambda(\xi)} & \text{for } \xi \in \mathfrak{d}^{-1}\mathfrak{p}^{-n} \\ 0 & \text{otherwise} \end{cases}$

Moreover, the Fourier transform of f_n is:

$$\hat{f}_n(\xi) = \begin{cases} \Delta^{1/2}(N\mathfrak{p})^n & \text{for } \xi \equiv 1 \bmod \mathfrak{p}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed:

$$\hat{f}_n(\xi) = \int f_n(\eta)e^{-2\pi i\Lambda(\xi\eta)}d\eta = \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{-2\pi i\Lambda((\xi-1)n)}d\eta$$

and $e^{-2\pi i\Lambda((\xi-1)n)}$ is a character which is trivial if and only if $\xi \equiv 1 \bmod \mathfrak{p}^n$. Moreover, the measure of the compact subgroup $\mathfrak{d}^{-1}\mathfrak{p}^{-n}$ is $(N\mathfrak{d})^{1/2}(N\mathfrak{p})^n = \Delta^{1/2}(N\mathfrak{p})^n$.

To calculate the ζ-functions, we deal with the unramified and ramified cases reperately. Let A_m be the annulus of elements of order m and $\mathfrak{d} = \mathfrak{p}^d$.

For $n = 0$, the only character of type c_0 is the trivial one, and f_0 is the characteristic function of $\mathfrak{d}^{-1}$. We will show that

$$\zeta(f_0, |-|^s) = \int_{\mathfrak{d}^{-1}} |\alpha|^s \mathsf{d}\alpha = \frac{\Delta^{s-1/2}}{1 - N\mathfrak{p}^{-s}}.$$

Note that $\mathfrak{p}^{m+1} = \mathfrak{p}A_m$ and hence

$$(N\mathfrak{p})^{-(m+1)}\Delta^{-1/2} = \mu(\mathfrak{p}^{m+1}) = \mu(\mathfrak{p}A_M) = (N\mathfrak{p})^{-1}\mu(A_m)$$

which shows that $\mu(A_m) = (N\mathfrak{p})^{-m}\Delta^{-1/2}$. We can decompose

$$\mathfrak{d}^{-1} = \mathfrak{p}^{-d} = \coprod_{m=-d}^{\infty} A_m$$

and hence

$$\begin{aligned}
\zeta(f_0, |-|^s) &= \sum_{m=-d}^{\infty} \int_{A_m} |\alpha|^s \mathtt{d}\alpha \\
&= \sum_{m=-d}^{\infty} \int_{A_m} |\alpha|^{s-1} d\alpha \\
&= \sum_{m=-d}^{\infty} (N\mathfrak{p})^{-m(s-1)} \mu(A_m) \\
&= \sum_{m=-d}^{\infty} (N\mathfrak{p})^{-m(s-1)} (N\mathfrak{p})^{-m} \Delta^{-1/2} \\
&= \sum_{m=-d}^{\infty} (N\mathfrak{p})^{-ms} \Delta^{-1/2} \\
&= \tfrac{N\mathfrak{p}^{ds}}{1-N\mathfrak{p}^{-s}} \Delta^{-1/2} \\
&= \tfrac{\Delta^{s-1/2}}{1-N\mathfrak{p}^{-s}}
\end{aligned}$$

Since $\hat{f}_0 = \Delta^{1/2} \cdot \chi_{\mathcal{O}_K}$, we similarly obtain

$$\zeta(\hat{f}_0, \hat{|-|^s}) = \zeta(\hat{f}_0, |-|^{1-s}) = \Delta^{1/2} \int_{\mathcal{O}_K} |\alpha|^{1-s} \mathtt{d}\alpha = \frac{1}{1 - N\mathfrak{p}^{s-1}}.$$

Now suppose c_n is ramified, i.e. $n > 0$. We have that

$$\zeta(f_n, c_n|-|^s) = \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) |\alpha|^s \mathtt{d}\alpha = \sum_{m=-d-n}^{\infty} N\mathfrak{p}^{-ms} \int_{A_m} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) \mathtt{d}\alpha.$$

We claim that for $m > -d - n$:

$$\int_{A_m} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) \mathtt{d}\alpha = 0.$$

First, if $m \geq -d$, then $A_m \subseteq \mathfrak{d}^{-1}$, so $e^{2\pi i \Lambda(\alpha)} = 1$ on A_m and the integral becomes

$$\int_{A_m} c_n(\alpha) \mathtt{d}\alpha = \int_U c_n(\alpha \pi^v) \mathtt{d}\alpha = \int_U c_n(\alpha) \mathtt{d}\alpha = 0,$$

since c_n is ramified and hence non-trivial on U.

The case $-d > m > -d - n$ takes some more work. We break up A_m into additive cosets $A_m/\mathfrak{d}^{-1}$, disjoint sets of the type

$$\alpha_0 + \mathfrak{d}^{-1} = \alpha_0 + \mathfrak{p}^{-d} = \alpha(1 + \mathfrak{p}^{-d-m}).$$

On each such set $\Lambda = \Lambda(\alpha_0)$ is constant and

$$\int_{\alpha_0 + \mathfrak{d}^{-1}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) \mathtt{d}\alpha = e^{2\pi \Lambda(\alpha_0)} \int_{\alpha_0 + \mathfrak{d}^{-1}} c_n(\alpha) \mathtt{d}\alpha.$$

We only have to show that the last integral is 0. We have

$$\int_{\alpha_0+\mathfrak{d}^{-1}} c_n(\alpha)\mathtt{d}\alpha = \int_{\alpha_0(1+\mathfrak{p}^{-d-m})} c_n(\alpha)\mathtt{d}\alpha = \int_{1+\mathfrak{p}^{-d-m}} c_n(\alpha\alpha_0)\mathtt{d}\alpha = c_n(\alpha_0)\int_{1+\mathfrak{p}^{-d-m}} c_n(\alpha)\mathtt{d}\alpha.$$

The last integral is the integral of a character c_n over a multiplicative subgroup $1+\mathfrak{p}^{-d-m}$. We only have to show that the character is non-trivial. Indeed, $-d > m$ implies $\mathfrak{p}|\mathfrak{p}^{-d-m}$ and hence $1+\mathfrak{p}^{-d-m}$ is a multiplicative subgroup of $K^\times$. Finally, $m > -d-n$ implies that the conductor $\mathfrak{p}^n$ does not divide $\mathfrak{p}^{-d-m}$, so c_n is non-trivial on $1+\mathfrak{p}^{-d-m}$.

Therefore, we have shown that

$$\zeta(f_n, c_n|-|^s) = N\mathfrak{p}^{(d+n)s}\int_{A_{-d-n}} e^{2\pi i\Lambda(\alpha)}c_n(\alpha)\mathtt{d}\alpha.$$

To rewrite this, let $\{\varepsilon\}$ be a set of representatives of the quotient $u/(1+\mathfrak{p}^n)$ so that

$$U = \coprod_{\varepsilon}\varepsilon(1+\mathfrak{p}^n).$$

Then:

$$A_{-d-n} = U\pi^{-d-n} = \coprod_{\varepsilon}\varepsilon\pi^{-d-n}(1+\mathfrak{p}^n) = \coprod_{\varepsilon}(\varepsilon\pi^{-d-n}+\mathfrak{d}^{-1}).$$

On each of these sets, $c_n = c_n(\varepsilon)$ is constant, and $\Lambda = \Lambda(\varepsilon\pi^{-d-n})$ is constant. Therefore:

$$\zeta(f_n, c_n|-|^s) = N\mathfrak{p}^{(d+n)s}\left(\sum_{\varepsilon} c_n(\varepsilon)e^{2\pi i\Lambda(\varepsilon\pi^{-d-n})}\right)\int_{1+\mathfrak{p}^n}\mathtt{d}\alpha.$$

Finally:

$$\zeta(\hat{f}_n, c_n\widehat{|-|^s}) = \zeta(\hat{f}_n, c_n^{-1}|-|^{1-s})$$

and $\hat{f}_n = \Delta^{1/2}N\mathfrak{p}^n\cdot\chi_{1+\mathfrak{p}^n}$. Since $c_n(\alpha)^{-1}|\alpha|^{1-s} = 1$ on $1+\mathfrak{p}^n$, we have that

$$\zeta(\hat{f}_n, c_n\widehat{|-|^s}) = \Delta^{1/2}N\mathfrak{p}^n\int_{1+\mathfrak{p}^n}\mathtt{d}\alpha,$$

which is a constant.

We can express the function ϱ explicitely. First,

$$\varrho(|-|^s) = \Delta^{s-1/2}\frac{1-N\mathfrak{p}^{s-1}}{1-N\mathfrak{p}^{-s}}.$$

Moreover, if c is ramified with conductor $\mathfrak{f}$ such that $c(\pi)=1$, then

$$\varrho(c|-|^s) = N(\mathfrak{d}\mathfrak{f})^{s-1/2}\varrho_0(c),$$

where

$$\varrho_0(c) = N\mathfrak{f}^{-1/2}\sum_{\varepsilon} c(\varepsilon)\exp\left\{2\pi i\Lambda\left(\varepsilon\pi^{-\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}\mathfrak{f})}\right)\right\}$$

is the *root number* and has absolute value 1.

3.3. **Abstract Restricted Direct Product.** In this section, we develop the abstract theory of restricted direct products. This theory will allow us to recover a global field K as a (discrete) subgroup of a product of its completions at finite and infinite primes v. This will yield the required globalization of the previous section.

We follow Tate's Thesis [CF86, Chap. XV], but the reader can refer to [Lan94, Chap. VII] for an alternative treatment of the abstract restricted direct product, the ideles, and the adeles.

Let $\{v\}$ be a set of indices and suppose for each v, we are given a Hausdorff locally compact abelian group G_v, and for almost all[†] v, a fixed subgroup $H_v \leq G_v$, which is open and compact. We form a new abstract group

$$G = \{a = (\dots, a_v, \dots) \mid a_v \in G_v \text{ with } a_v \in H_v \text{ for almost all } v\} \subseteq \prod_v G_v$$

under component-wise multiplication. We will define a topology on G. Let S be any finite set of indices v, including at least the indices for which H_v is not defined. Then define

$$G_S = \{a \in G \mid a_v \in H_v \text{ for } p \notin S\} \leq G$$

and in fact

$$G_S = \prod_{v \in S} G_v \times \prod_{v \notin S} H_v,$$

a product of locally compact groups, almost all of which are compact. Then G_S is a locally compact group in the product topology. We define a fundamental system of neighborhoods of 1 in G to be the set of neighbourhoods of 1 in G_S. Then the resulting topology is independent of S.

Lemma 3.41. *The set of* parallelotopes $N = \prod_v N_v$, *where* N_v *is a neighborhood of 1 in* G_v *and* $N_v = H_v$ *for almost all* v *is a fundamental system of neighborhoods in* G*.*

Proof. By definition of the product topology, a neighborhood of 1 in G_S contains a paralellotope of that type. Conversely, since $N_v = H_v$ for almost all v, the intersection

$$\left(\prod_v N_v\right) \cap G_S = \prod_{v \in S} N_v \times \prod_{v \notin S} (N_v \cap H_v)$$

is a neighborhood of 1 in G_S. □

Note that any G_S is open in G and the subspace topology of is indeed the product topology. Moreover, any compact neighborhood of 1 in G_S is a compact neighborhood of 1 in G, so G is locally compact.

Definition 3.42. We call G the *restricted direct product* of the groups G_v relative to the subgroups H_v.

We have the natural embedding of G_v as a subgroup of G:

$$a_v \mapsto (1, 1, \dots, 1, a_v, 1, \dots).$$

[†]In this chapter, by almost all we always mean all but finitely many.

Moreover, since the components a_v of any $a \in G$ are in H_v for almost all v, G is the union of subgroups of the type G_S. This reduces the investigation of G to just studying groups G_S.

We do this by introducing compact subgroups G^S of G_S:

$$G^S = \{a \in G \mid a_v = 1 \text{ for } v \in S,\ a_v \in H_v \text{ for } v \notin S\} \cong \prod_{v \notin S} H_v.$$

The we can interpret G_S as

$$G_S = \left(\prod_{v \in S} G_v\right) \times G^S,$$

the first product being finite.

Lemma 3.43. *A subset $C \subseteq G$ has a compact closure if and only if it is contained in some $\prod_v B_v$ for $B_v \subseteq G_v$ compact for all v, with $B_v = H_v$ for almost all v.*

Proof. The prescribed sets are clearly compact. Conversely, note that any compact subset of G is contained in some G_S, since $\{G_S\}$ is an open cover for G, and the union of finitely many sets of the form G_S is again of the form G_S. Now, any compact subset of G_S is contained in the product of the projections onto G_v, so it is of the required form. □

3.3.1. *Characters.* We want to study quasi-characters $c\colon G \to \mathbb{C}^\times$ of abstract restricted direct products G. We denote by c_v the restriction of c to G_v, a quasi-character of G_v.

Lemma 3.44. *The quasi-character c_v is trivial on H_v for almost all v, and for any $a \in G$*

$$c(a) = \prod_v c_v(a_v).$$

Note that the product is a priori infinite but since c_v is trivial on H_v for almost all v, almost all the factors are 1, so it is in fact a finite product. As we will see, this phenomenon will occur numerous times in this chapter, so we will usually omit this discussion in the future.

Proof. Let U be a neighborhood of 1 in $\mathbb{C}$, containing no multiplicative subgroup except $\{1\}$ and $N = \prod_v N_v$ be a neighborhood of 1 in G such that $C(N) \subseteq U$. Select S containing all v for which $N_v \neq H_v$. Then $G^S \subseteq N$, so $c(G^S) \subseteq U$ is a multiplicative subgroup, and hence $c(G^S) = \{1\}$. But this shows that $c(H_v) = \{1\}$ for $v \notin S$.

Now, fix $a \in G$ and impose on S that $a \in G_S$, so that we can write

$$a = \left(\prod_{v \in S} a_v\right) a^S$$

with $a^S \in G^S$. We then have

$$c(a) = \prod_{v \in S} c(a_v) \cdot c(a^S) = \prod_{v \in S} c_v(a_v) = \prod_v c_v(a_v),$$

since $c_v(a_v) = 1$ for $v \notin S$. □

In fact, the converse also holds: any suitable family of quasi-characters of G_v will yield a quasi-character of G.

Lemma 3.45. *Let c_v be a given quasi-character of G_v for each v, with c_v trivial for almost all v. Then*

$$c(a) = \prod_v c_v(a_v)$$

is a quasi-character of G.

Once again, note that the product is finite.

Proof. Note that c is clearly multiplicative.

To see it is continuous, take $S \supseteq \{v \mid c_v(H_v) \neq \{1\}\}$ and let $s = \#S$. Fix a neighborhood U of 1 in $\mathbb{C}$ and choose a (smaller) neighborhood V such that $V^s \subseteq U$. By continuity of c_v, we can choose a neighborhood N_v of 1 in G_v such that $c_v(N_v) \subseteq V$ for $v \in S$. Moreover, we let $N_v = H_v$ for $v \notin S$. Then clearly:

$$c\left(\prod_v N_v\right) \subseteq c\left(\prod_{v \in S} N_v\right) \subseteq V^s \subseteq U,$$

showing continuity of c. □

Now, we restrict our consideration to characters. We will show how to express $\widehat{G}$, the character group of G, as a restricted direct product of character groups $\widehat{G_v}$ of G_v.

Note that c given by $c(a) = \prod_v c_v(a_v)$ is a character if and only if each c_v is a character. For v where H_v is defined, let $H_v^* \subseteq \widehat{G_v}$ be the subgroup characters trivial on H_v:

$$H_v^* = \{c_v \in \widehat{G_v} \mid c_v(H_v) = \{1\}\}.$$

Note that if H_v is compact, then $\widehat{H_v} \cong \widehat{G_v}/H_v^*$ is discrete, so H_v^* is open. Moreover, if H_v is open, then G_v/H_v is discrete, so $\widehat{G_v/H_v} \cong H_v^*$ is compact.

Theorem 3.46. *The restricted direct product of $\widehat{G_v}$ relative to H_v^* is naturally isomorphic (algebraically and topologically) to the character group $\widehat{G}$ of G.*

Proof. We may identify $c = (\ldots, c_v, \ldots)$ with the character

$$c(a) = \prod_v c_v(a_v),$$

and hence Lemmas 3.44 and 3.45 provide an algberaic isomorphism between the restriced direct product and $\widehat{G}$. We only have to check that the topology on the character group $\hat{G}$ introduced in Section 3.1 (particularly, Lemma 3.11) agrees with the topology on the restricted direct product. Indeed, the following statements are equivalent

(1) $c = (\ldots, c_v, \ldots)$ is in a neighborhood of 1 as a character of G,
(2) for some $C \subseteq G$ compact (i.e. by Lemma 3.43, $C = \prod_v B_v$ for $B_v \subseteq G_v$ compact for all v with $B_v = H_v$ for almost all v), and $U \subseteq S^1$ neighborhood of 1,

$$c\left(\prod_v B_v\right) = c(C) \subseteq U$$

(3) $c_v(B_v) \subseteq U_v$ for some neighborhood U_v of 1 in S_1 whenever $H_v \neq B_v \subseteq G_v$ is compact, and $c_v(B_v) = \{1\}$ everywhere else,
(4) the character c_v is in a neighborhood of 1 in $\widehat{G_v}$ for a finite number of v, and $c_v \in H_v^*$ for the other v,
(5) the character $c = (\ldots, c_v, \ldots)$ is in a neighborhood of 1 in the restricted direct product of $\widehat{G_v}$ with respect to H_v^*.

Thus the defining systems of neighborhood agree in the two topologies, and hence the topologies agree. □

3.3.2. *Measure.* We will introduce a measure on the restricted direct product. Choose a Haar measure da_v on each G_v such that

$$\int_{H_v} da_v = 1 \qquad \text{for almost all } v.$$

We wish to define a Haar measure da on G for which, in a sense, $da = \prod_v da_v$. To do this, select a finite S as before, and consider

$$G_S = \left(\prod_{v\in S} G_v\right) \times G^S.$$

Then we can define a measure on G_S

$$da_S = \left(\prod_{v\in S} da_v\right) \cdot da^S,$$

where da^S is the measure on the compact subgroup G^S for which

$$\int_{G^S} da^S = \prod_{v\notin S} \int_{H_v} da_v.$$

(As always, this is actually a finite product.) Since G_S is an open subgroup of G, a Haar measure on G is determined by the requirement $da = da_S$ on G_S.

To see that da is independent on the choice of S, let $T \supseteq S$ be a larger set of indices. Then $G_S \subseteq G_T$ and we want to show that da_T and da_S coincide on G_S. We can use the decomposition

$$G^S = \left(\prod_{v\in T\setminus S} H_v\right) \times G^T$$

to conclude that

$$da^S = \prod_{v\in T\setminus S} da_v \cdot da^T.$$

(Indeed, they both give G^S the same measure.) Thus:

$$da_S = \prod_{v\in S} da_v \cdot da^S = \prod_{v\in S} da_v \cdot \prod_{v\in T\setminus S} da_v \cdot da^T = da_T.$$

We have hence determined a unique Haar measure on G, denoted symbolically by

$$da = \prod_v da_v.$$

Lemma 3.47. *If f is a function on G, then*

$$\int f(a)da = \lim_S \int_{G_S} f(a)da_S$$

(where the limit is over all finite S) if one of the following holds:

(1) *f is measurable and non-negative, in which case $+\infty$ is allowed as the value of the integral,*
(2) *$f \in L_1(G)$, in which case the values of the integrals are complex numbers.*

Proof. In either case, $\int f(a)da$ is the limit of $\int_B f(a)da$ for compact sets $B \subseteq G$ and each such compact set is contained in some G_S by Lemma 3.43. □

Lemma 3.48. *For each v, let $f_v \in L_1(G_v)$ be a continuous function such that f_v is trivial on H_v for almost all v. If we define $f \colon G \to \mathbb{C}$ by*

$$f(a) = \prod_v f_v(a_v),$$

then

(1) *f is continuous on G,*
(2) *for any set S containing at least those v for which $f_v(H_v) \neq \{1\}$ or $\int_{H_v} da_v \neq 1$, we have*

$$\int_{G_S} f(a)da = \prod_{v \in S} \left(\int_{G_v} f_v(a_v)da_v \right).$$

Proof. For (1), note that the function f is continuous on any G_S, and hence also on G.

For (2), note that for any $a \in G_S$ we have $f(a) = \prod\limits_{v \in S} f_v(a_v)$. Then

$$\begin{aligned} \int\limits_{G_S} f(a)da &= \int\limits_{G_S} f(a)da_S \\ &= \int\limits_{G_S} \left(\prod_{v\in S} f_v(a_v) \right) \prod_{v \in S} da_v \cdot da^S \\ &= \prod_{v \in S} \left(\int\limits_{G_v} f_v(a_v)da_v \right) \cdot \int\limits_{G^S} da^S \\ &= \prod_{v \in S} \left(\int\limits_{G_v} f_v(a_v)da_v \right) \end{aligned}$$

where the last equality follows from

$$\int_{G^S} da^S = \prod_{v \notin S} \left(\int_{H_v} da_v \right) = 1$$

by our choice of da_v on G_v. □

Theorem 3.49. *If f_v and f satisfy the assumption of the preceeding lemma (Lemma 3.48), and moreover*

$$\prod_v \left(\int |f_v(a_v)| da_v \right) < \infty,$$

then $f \in L_1(G)$ and

$$\int f(a)da = \prod_v \left(\int f_v(a_v) da_v \right).$$

Proof. We combine the two preceeding lemmas (Lemmas 3.47 and 3.48): first for the function $|f|$ to see that $f \in L_1(G)$, then for the function f to evaluate $\int f(a)da$. □

Finally, we apply the results to Fourier transforms. Let dc_v be the measure on $\widehat{G_v}$ dual to the measure da_v on G_v. If χ_{H_v} is the characteristic function of H_v, $\chi_{H_v^*}$ the characteristic function of H_v^*, then

$$\hat{\chi_{H_v}}(c_v) = \int \chi_{H_v}(a_v)\overline{c_v(a_v)} da_v = \int_{H_v} da_v \cdot \chi_{H_v^*}.$$

In particular, using the Fourier Inversion Formula 3.20

$$\int_{H_v^*} dc_v = \int_{H_v} da_v \cdot \int_{H_v^*} dc_v = 1$$

for almost all v, and we may define $dc = \prod_v dc_v$ on $\widehat{G}$.

Lemma 3.50. *If $f_v \in \mathcal{B}_1(G_v)$ for all v, with $f_v = \chi_{H_v}$ for almost all v, then f given by $f(a) = \prod_v f_v(a_v)$ has Fourier transform*

$$\hat{f}(c) = \prod_v \hat{f}_v(c_v)$$

and $f \in \mathcal{B}_1(G)$.

Proof. Apply Theorem 3.49 to $f(a)\overline{c(a)} = \prod_v f_v(a_v)\overline{c_v(a_v)}$ to see that

$$\hat{f}(c) = \int f(a)\overline{c(a)} da = \prod_v \left(\int f_v(a_v)\overline{c_v(a_v)} da_v \right) = \prod_v \hat{f}(c).$$

Since $f_v \in \mathcal{B}_1(G_v)$, $\hat{f}_v \in L_1(\widehat{G_v})$ for all v. For almost all v, $\hat{f}_v = \chi_{H_v^*}$, so $\hat{f} \in L_1(\widehat{G})$, and hence $f \in \mathcal{B}_1(G)$. □

Corollary 3.51. *The measure $dc = \prod_v c_v$ is dual to $da = \prod_v da_v$.*

Proof. Using the lemma, we obtain the Inversion Formula using the component-wise Inversion Formulas. □

3.4. **Adeles and Ideles.** In this chapter, let K be a number field and v a generic prime of K. We let K_v be the completion of K at v, and index the notation of Section 3.2 with v for the local field K_v. So we also write $\mathcal{O}_\mathfrak{p}$ for the ring of integers of $K_\mathfrak{p}$, Λ_v for the function Λ for K_v, and similarly $\mathfrak{d}_v$, Δ_v, $|-|_v$, c_v, ...

Definition 3.52. The additive group A_K of *adeles* of K is the restricted direct product of K_v^+ relative to the subgroups $\mathcal{O}_\mathfrak{p}$, defined for finite primes $v = \mathfrak{p}$. The group A_K with component-wise multiplcation is the *ring of adeles*.

We also write $A_K = A_K^f \times K^\infty$, where

- A_K^f are the *finite adeles*: the restricted direct product over the finite primes,
- K^∞ are the *infinite adeles*: $\prod_v K_v$, a finite product of the completions of K at v.

Crucially, we will be able to recover K as a discrete subgroup of A_K, which we will see later.

Definition 3.53. The units $A_K^\times$ of the ring of adeles A_K are *ideles*.

Lemma 3.54. *The ideles are the restricted direct product of $K_v^\times$ with respect to the subgroups $\mathcal{O}_v^\times$.*

Proof. Clearly, any element of the restricted direct product is an idele. Conversely, if $x \in A_K$ has an inverse, then all the x_v are non-zero and the inverse is $(\ldots, x_v^{-1}, \ldots)$. Since both $(\ldots, x_v, \ldots)$ and $(\ldots, x_v^{-1}, \ldots)$ are in A_K, $x_v \in \mathcal{O}_k$ for almost all v and $x_v^{-1} \in \mathcal{O}_k$ for almost all v. Therefore, $x_v \in \mathcal{O}_v^\times$ for almost all v, and hence $(\ldots, x_v^{-1}, \ldots)$ is in the restricted direct product. □

The topology we give to $A_K^\times$ is the one coming from the restricted direct product construction. In particular, $A_K^\times \subseteq A_K$, but the topology on $A_K^\times$ is stronger than the subset topology.

3.4.1. *Additive theory.* From Theorems 3.25, 3.46, and Lemma 3.26, we see that the character group of A_K is naturally the restricted direct product of K_v^+ relative to the subgroups $\mathfrak{d}_v^{-1}$. Since $\mathfrak{d}_\mathfrak{p}^{-1} = \mathcal{O}_\mathfrak{p}$ for almost all $\mathfrak{p}$, this product is A_K again. An element $\eta = (\ldots, \eta_v, \ldots) \in V$ is identified with the character

$$x = (\ldots, x_v, \ldots) \mapsto \prod_v \exp(2\pi i \Lambda_v(\eta_v x_v)) = \exp\left(2\pi i \sum_v \Lambda_v(\eta_v x_v)\right).$$

Therefore, we define $\Lambda(x) = \sum_v \Lambda_v(x_v)$ to get the following theorem.

Theorem 3.55. *The ring of adeles A_K is naturally its own (additive) character group under the identification*

$$\eta \in A_K \longleftrightarrow x \mapsto e^{2\pi i \Lambda(\eta x)}.$$

Moreover, we introduce the measure $dx = \prod_v dx_v$ on A_K described in the previous section, where dx_v are the local, self-dual measures on K_v. It is self-dual by Corollary 3.51. Therefore, the Fourier Inversion Formula 3.20 becomes:

Theorem 3.56. *If for a function $f \in L_1(A_K)$ we define the Fourier transform*

$$\hat{f}(\eta) = \int f(x) e^{-2\pi i \Lambda(\eta x)} dx,$$

then for $f \in \mathcal{B}_1(A_K)$ the Inversion Formula

$$f(x) = \int \hat{f}(\eta) e^{2\pi i \Lambda(x\eta)} d\eta$$

holds.

Recall that locally $d(\alpha_v \xi_v) = |\alpha_v|_v d\xi_v$ for $\alpha_v \in K_v^\times$ by Lemma 3.27. In order to generalize this to the global case of adeles, note that $x \mapsto ax$ is an automorphism of A_K if and only if $a \in A_K^\times$ is an idele.

Lemma 3.57. *For an idele a we have $d(ax) = |a|dx$, where $|a| = \prod_v |a_v|_v$.*

Proof. If $N = \prod_v N_v$ is a compact neighborhood of 0 in A_K, then by Theorem 3.49 applied to characteristic functions we have that

$$\int_N dx = \prod_v dx_v,$$

$$\int_{aN} dx = \prod_v \int_{a_v N_v} dx_v.$$

We can use $d(a_v x_v) = |a_v|_v dx_v$ (Lemma 3.27) to obtain

$$\int_{aN} dx = \prod_v \int_{a_v N_v} dx_v = \prod_v |a_v|_v \int_{N_v} dx_v = \prod_v |a_v|_v \int_n dx = |a| \int_N dx,$$

as requested. ☐

We now wish to recover our original global field K as a subring of A_K. We can embed K in A_K diagonally:

$$K \ni \xi \mapsto \xi = (\xi, \dots, \xi, \dots) \in A_K.$$

We will work towards proving the following theorem, which characterizes K as a subset of A_K.

Theorem 3.58. *The subspace topology on K coming from the diagonal embedding $K \hookrightarrow A_K$ is the discrete topology, and the quotient A_K/K is compact.*

In order to prove the theorem, we will construct a *fundamental domain* for K in A_K.

Let S_∞ be the set of infinite primes of K.

Lemma 3.59. *We have that $K \cap (A_K)_{S_\infty} = \mathcal{O}_K$.*

Proof. Recall that

$$(A_K)_{S_\infty} \cong \left(\prod_{v \in S_\infty} K_v^+ \right) \times \prod_{\mathfrak{p} \notin S_\infty} \mathcal{O}_\mathfrak{p}.$$

We have to show that $\xi \in K$ is in $\mathcal{O}_K$ if and only if $\xi \in \mathcal{O}_\mathfrak{p}$ for any finite prime $\mathfrak{p}$. Indeed, take any $0 \neq \xi \in K$ and write the fractional ideal (ξ) as $\prod_\mathfrak{p} \mathfrak{p}^{a_\mathfrak{p}}$. Then it is clear that $\xi \in \mathcal{O}_K$ if and only if $a_\mathfrak{p} \geq 0$ for all $\mathfrak{p}$, which in turn is equivalent to $\xi \in O_\mathfrak{p}$ for any $\mathfrak{p}$ finite. □

Recall that we write K^∞ for the *infinite part* of A_K, i.e. the product $\prod_{v \in S_\infty} K_v$ of r_1 real lines and r_2 complex planes, where r_1 is the number of real primes, and r_2 is the number of complex primes. It is naturally a vector space over $\mathbb{R}$ of dimension $n = r_1 + 2r_2$, the degree $[K : \mathbb{Q}]$. For $x \in A_K$, we let x^∞ be the projection of x onto K^∞.

Lemma 3.60. *The image of $\mathcal{O}_K$ in K^∞ is a lattice and (with respect to our choice of measure) the measure of the fundamental domain in this lattice is $\sqrt{|\Delta|}$.*

Remark 3.61. A fundamental domain for a lattice $L \subseteq \mathbb{R}^n$ is a connected set S with non-empty interior such that: for all $x \in \mathbb{R}^n$ there exists a unique $\lambda \in \Lambda$, $s \in S$ such that $\lambda + s = x$. One popular choice of a fundamental domain is to write down a basis $\{e_1, \ldots, e_n\}$ for L and let $S = \{\lambda_i e_i \mid 0 \leq \lambda_i < 1\}$, a *fundamental parallelogram* for L. Our choice of fundamental domain for $\mathcal{O}_K$ will parallel this setting.

Proof of Lemma 3.60. The fact that $\mathcal{O}_K$ is a lattice in K^∞ is clear: we can simply choose a $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ for $\mathcal{O}_K$ over $\mathbb{Z}$, and project it to get a basis $\{e_1^\infty, \ldots, e_n^\infty\}$ for the image of $\mathcal{O}_K$ in K^∞.

If K is totally real (i.e. $r_2 = 0$, all the primes are real), this is immediate: the volume of the fundamental domain of a lattice in $\mathbb{R}^n$ is the absolute value of the determinant of the matrix whose form a basis of the lattice:

$$|\det(\sigma_i e_j)| = \sqrt{|\Delta|}$$

where σ_i ranges over the field maps $K \to \mathbb{C}$.

If K is complex, $\sigma(x + iy)$ and in the usual discriminant calculation that gives the volume of the lattice we will see a contribution from both σ and $\overline{\sigma}$, the complex conjugate. Therefore, choosing coordinates x, y of $\mathbb{R}^2 \cong \mathbb{C}$, we see that

$$\begin{pmatrix} x + iy \\ x - iy \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the absolute value of the determinant of the matrix is 2. Therefore, in the ordinary Lebesgue measure, the volume of the fundamental domain will be $\sqrt{|\Delta|}2^{-r_2}$. However, for any complex prime v, we have chosen the measure on $K_v = \mathbb{C}$ to be **twice** the ordinary Lebesgue measure, and hence the contribution of 2^{-r_2} cancels, yielding the result. □

Fix a $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ for $\mathcal{O}_K$ and consider the projections $\{e_1^\infty, \ldots, e_n^\infty\}$ which give a basis for the lattice $\mathcal{O}_K$ in K^∞. We define

$$D^\infty = \left\{ \sum_{j=1}^n \lambda_j e_j^\infty \mid 0 \leq \lambda_j < 1 \right\}.$$

Definition 3.62. The *additive fundamental domain* $D \subseteq A_K$ is the set

$$D = \{x \in (A_K)_{S_\infty} \mid x^\infty \in D^\infty\} = D^f \times D^\infty$$

with $D^f = \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}$.

Note that D^f is open and hence closed, and thus

$$\overline{D} = D^f \times \overline{D^\infty},$$
$$\mathring{D} = D^f \times \mathring{D}^\infty,$$

so $\overline{D}$ is compact and $\mathring{D}$ is non-empty.

The following proposition justifies the name additive fundamental domain.

Proposition 3.63.

(1) *Any $x \in A_K$ is congruent to a unique element of D modulo the field K; symbolically:*

$$V = \coprod_{\xi \in K} (\xi + D).$$

(2) *The set D has measure 1.*

Proof. For (1), fix $x \in A_K$. We deal with the finite primes $\mathfrak{p}$ first. By definition, $x_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ for almost all finite primes $\mathfrak{p}$. Choose some $0 \neq \xi_1 \in \mathcal{O}$ such that $(\xi_1)_\mathfrak{p} x_p \in \mathcal{O}_\mathfrak{p}$ for all finite places: we simply ensure that ξ_1 is divisible for a sufficiently large power of $\mathfrak{p}$ for the finitely many places $\mathfrak{p}$ where $x_\mathfrak{p} \notin O_\mathfrak{p}$.

Now, for each $\mathfrak{p}|(\xi_1)$, suppose $\mathfrak{p}^{a_\mathfrak{p}}$ divides (ξ_1) exactly and consider the system of congruences

$$\xi_2 \equiv \xi_1 x_\mathfrak{p} \bmod \mathfrak{p}^{a_\mathfrak{p}}.$$

The Chinese Remainder Theorem guarantees the existence of a solution $\xi_2 \in \mathcal{O}_K$. Then setting $\xi = \frac{\xi_2}{\xi_1} \in K$, we obtain that

$$x_\mathfrak{p} - \xi \in \mathcal{O}_\mathfrak{p}$$

for all $\mathfrak{p}|(b)$. But $x_\mathfrak{p}$ and ξ are integral at all finite places $\mathfrak{p} \not| (b)$, so $x - \xi \in D^f$.

Finally, we can adjust the infinite components of ξ to ensure they lie in D^∞ (since D^∞ is a fundamental domain for $\mathcal{O}$ over K^∞.)

For uniqueness, take $x \in A_K$ and $d_1, d_2 \in D$, $\xi_1, \xi_2 \in K$ such that

$$x = d_1 + \xi_1 = d_2 + \xi_2.$$

Let

$$t = d_1 - d_2 = \xi_2 - \xi_1 \in (D - D) \cap K$$

Looking at the finite primes, we see $t \in K$ is in $D^f - D^f = D^f$, so it is integral at all the finite primes: $t \in \sum_{i=1}^n \mathbb{Z} e_i$. Looking at infinite places, we see that $t \in D^\infty - D^\infty$, so

$$t = \sum_{i=1}^n \lambda_i e_i \quad \text{for } \lambda_i \in (-1, 1).$$

Since $t \in \sum_{i=1}^n \mathbb{Z} e_i$, this yields $\lambda_i = 0$ for all i, and thus $t = 0$.

For (2), we compute the measure of D, noting that $D = D^\infty \times V_{S_\infty}$. We have

$$\int_D dx = \int_D dx_{S_\infty} = \int_{D^\infty \times V_{S_\infty}} dx^\infty dx^{S_\infty} = \int_{D^\infty} dx^\infty \cdot \int_{V^{S_\infty}} dx^{S_\infty} = \sqrt{\Delta} \prod_{\mathfrak{p} \notin S_\infty} (\Delta_\mathfrak{p})^{-1/2}$$

by Lemma 3.60. Finally, we note that the absolute discriminant $\Delta = \Delta_{K/\mathbb{Q}}$ is the product of local discriminants $\Delta_{\mathfrak{p}} = \Delta_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ (see: [Ser79, Cor. to Prop. 10, Chap. III]), which gives the required equality. □

This immediately yields Theorem 3.58, which said that K is discrete in A_K and A_K/K is compact.

Proof of Theorem 3.58. We know that D has a non-empty interior $\mathring{D}$ and by Proposition 3.63:

$$A_K = \coprod_{\xi \in K} (\xi + D).$$

To show discreteness of K, we show that there is a neighborhood of 0 in A_K whose intersection with K is $\{0\}$. Fix an adele $d \in \mathring{D}$ and consider $\mathring{D} - d$. This is an open set in A_K, containing 0. Conversely, if $\xi \in K$ is in $\mathring{D} - d$, then $d' - d = \xi$ for some $d' \in \mathring{D}$, whence $d' = d + \xi$, so $\xi = 0$ by uniqueness.

The compactness of the quotient follows from the fact that A_K/K is a continuous image of the compact set $\overline{D}$, the map being surjective by Proposition 3.63. □

Proposition 3.64. *The identification of A_K with $\widehat{A_K}$ given by Theorem 3.56 sends the closed subgroup K isomorphically onto the closed subgroup K^* of characters of $\widehat{A_K}$ which are trivial on K.*

Proof. We need to check that:

(1) If $\xi \in K$, then $\Lambda(\xi) = 0$.
(2) If $\xi \in A_K$, $\Lambda(\alpha\xi) = 0$ for all $\alpha \in K$, then $\xi \in K$.

We can easily reduce (1) to the rational case:

$$\Lambda(\xi) = \sum_v \Lambda_v(\xi) = \sum_v \lambda_v(\mathrm{Tr}(\xi)) = \sum_w \lambda_w \left(\sum_{v|w} \mathrm{Tr}_v(\xi) \right) = \sum_w \lambda_w(\mathrm{Tr}(\xi))$$

(where v runs over all primes of K and w runs over all primes of $\mathbb{Q}$), because the trace is the sum of local traces. Since $\mathrm{Tr}(\xi) \in \mathbb{Q}$, we only have to show that

$$\lambda(x) = \sum_w \lambda_p(x) = 0$$

for $x \in \mathbb{Q}$. Clearly, $\Lambda(n) = 0$ for $n \in \mathbb{Z}$ and by additivity of λ (Lemma 3.23), we only have to show that $\lambda(1/p^e) = 0$ for $p \in \mathbb{Q}$ prime, $e \geq 1$. We have that

$$\lambda_q(1/p^e) = 0 \quad \text{for } q \neq p \text{ rational prime,}$$
$$\lambda_p(1/p^e) = 1/p^e,$$
$$\lambda_\infty(1/p^e) = -1/p^e,$$

which shows the desired equality.

To prove (2), note first that if we identify $\widehat{A_K}$ with A_K via the isomorphism from Theorem 3.56, we obtain $K \subseteq K^*$ by (1). Any character in K^* factors through A_K/K, so K^* is the character group of A_K/K. Since A_K/K is compact by Theorem 3.58, K^* is discrete; indeed:

$W(A_K/K, U)$ consists only of the trivial character for U small enough. Therefore, K^* is a discrete, closed subgroup of A_K. Hence K^*/K is discrete in A_K/K, and since it is also compact, it is finite. But K^* is a vector space over K, and since K is not a finite field, we must have $(K^* : K) = 1$, so $K^* = K$. □

Finally, we want to study functions on A_K/K and their integrals.

Definition 3.65. A function $\varphi\colon A_K \to \mathbb{C}$ is *periodic* if $\varphi(x+\xi) = \varphi(x)$ for any $x \in A_K$, $\xi \in k$.

Any periodic function represents a unique function on the compact space A_K/K. We will abuse the notation and write φ both for the periodic function on A_K and the function on A_K/K. We can define a natural Haar measure on A_K/K by letting

$$\int_{A_K/K} \varphi(x)dx = \int_D \varphi(x)dx,$$

where D is the fundamental domain discussed before.

Recall from the proof of Proposition 3.64 that K is naturally the character group of A_K/K. The Fourier transform of $\varphi \in L(A_K/K)$ can hence be written explicitly as

$$\hat{\varphi}(\xi) = \int_D \varphi(x) e^{-2\pi i \Lambda(\xi x)} dx$$

for $\xi \in K$.

Lemma 3.66. *If φ is continuous, periodic, and $\sum_{\xi\in K} |\hat{\varphi}(\xi)| < \infty$, then*

$$\varphi(\xi) = \sum_{\eta \in K} \hat{\varphi}(\eta) e^{2\pi i \Lambda(\xi\eta)}$$

for any $\xi \in K$.

Proof. This is just the Fourier Inversion Formula 3.20, recalling that K is discrete by Theorem 3.58, so the Haar measure on K is given by the sum over elements of K. □

Lemma 3.67. *If $f \in L_1(A_K)$ is continuous and $\sum_{\eta\in K} f(x+\eta)$ is uniformly convergent for $x \in D$, then for φ given by $\varphi(x) = \sum_{\eta\in K} f(x+\eta)$ we have $\hat{\varphi}(\xi) = \hat{f}(\xi)$ for any $\xi \in K$.*

Proof. We have that

$$\begin{aligned}
\hat{\varphi}(\xi) &= \int_D \varphi(x)e^{-2\pi i\Lambda(\xi x)}dx \\
&= \int_D \left(\sum_{\eta\in K} f(x+\eta)e^{-2\pi i\Lambda(x\xi)}\right)dx \\
&= \sum_{\eta\in K}\int_D f(x+\eta)e^{-2\pi i\Lambda(x\xi)}dx && \text{since } D \text{ is compact and convergence is uniform} \\
&= \sum_{\eta\in K}\int_{\eta+D} f(x)e^{-2\pi i\Lambda(x\xi-\eta\xi)}dx && \text{substituting } x\mapsto x-\eta \\
&= \sum_{\eta\in K}\int_{\eta+D} f(x)e^{-2\pi i\Lambda(x\xi)}dx && \text{by Proposition 3.64} \\
&= \int_K f(x)e^{-2\pi i\Lambda(x\xi)}dx && \text{by Proposition 3.63} \\
&= \hat{f}(\xi)
\end{aligned}$$

for any $\xi \in K$. □

Combining the two previous lemmas, we obtain the *Poisson summation formula*, which the reader may be familiar with from standard Fourier analysis.

Theorem 3.68 (Poisson summation formula)**.** *If* $f\colon A_K \to \mathbb{C}$ *satisfies:*

(1) $f \in L_1(A_K)$ *is continuous,*
(2) $\sum_{\xi\in K} f(x+\xi)$ *is uniformly convergent for* $x \in D$*,*
(3) $\sum_{\xi\in K} |\hat{f}(\xi)|$ *is convergent,*

then

$$\sum_{\xi\in K}\hat{f}(\xi) = \sum_{\xi\in K} f(\xi).$$

If we consider the function $x \mapsto f(ax)$ for an idele $a \in A_K^\times$, we obtain a stronger version of the formula, which looks like a number-theoretic analogue of the Riemann-Roch Theorem.

Theorem 3.69 (Riemann-Roch)**.** *If* $f\colon A_K \to \mathbb{C}$ *satisfies:*

(1) $f \in L_1(A_K)$ *is continuous,*
(2) $\sum_{\xi\in K} f(a(x+\xi))$ *is convergent for all* $a \in A_K^\times$*,* $x \in A_K$*, and the convergence is uniform on* D*,*
(3) $\sum_{\xi\in K} |\hat{f}(a\xi)|$ *is convergent for all ideles* $a \in A_K^\times$*,*

then

$$\frac{1}{|a|}\sum_{\xi\in K}\hat{f}(\xi/a) = \sum_{\xi\in K} f(a\xi).$$

Proof. We only have to show that, for an idele $a \in A_K^\times$, the function given by $g(x) = f(ax)$ satisfies the conditions of the Poisson summation formula 3.68. Indeed, conditions (1) and (2) are clear. Moreover,

$$\begin{aligned}\hat{g}(x) &= \int f(a\eta)e^{-2\pi i\Lambda(x\eta)}d\eta \\ &= \frac{1}{|a|}\int f(\eta)e^{-2\pi i\Lambda(x\eta/a)}d\eta \quad \text{substituting } \eta\mapsto\eta/a \\ &= \frac{1}{|a|}\hat{f}(x/a)\end{aligned}$$

so condition (3) holds. □

3.4.2. *Multiplicative theory.* In this section, we restrict our attention to the multiplicative group $A_K^\times$ of ideles and develop similar results to the previous section. In doing so, we will achieve a *globalization* of the results on the local multiplicative characters and measure in Section 3.2.2.

As before, the quasi-characters of $A_K^\times$ are $c\colon A_K^\times \to \mathbb{C}$ given by

$$c(a) = \prod_v c_v(a_v)$$

where c_v are local quasi-characters which are unramified at almost all v. For a measure $\mathtt{d}a$ on $A_K^\times$, we take

$$\mathtt{d}a = \prod_v \mathtt{d}a_v,$$

where $\mathtt{d}a_v$ is the local multiplicative measure. Finally, we embed $K^\times \hookrightarrow A_K^\times$ diagonally.

Theorem 3.70 (Product formula). *We have that* $|\alpha| = \prod_v |\alpha_v|_v = 1$ *for* $\alpha \in K^\times$.

Proof. If μ is the additive measure on A_K, for any $\alpha \in A_K^\times$, we have that

$$\mu(\alpha D) = |\alpha|\mu(D)$$

by Lemma 3.57.

Since $\alpha K^+ = K^+$, αD is also an additive fundamental domain for K in A_K. Now, note that

$$D = \coprod_{\xi\in K} D\cap(\xi+\alpha D) \text{ and } \alpha D = \coprod_{\xi\in K}(-\xi+D)\cap\alpha D$$

and the elements $d = \xi + \alpha d' \in D$ correspond to the elements $\alpha d' \in -\xi + D$, so

$$\mu(D\cap(\xi+\alpha D)) = \mu((-xi+D)\cap\alpha D),$$

and hence

$$\mu(D) = \mu(\alpha D) = |\alpha|\mu(D).$$

Hence $|\alpha| = 1$, since $\mu(D) \neq 0$. □

Remark 3.71. We already proved that $\mu(D) = 1$ in Proposition 3.63. In hindsight, this was in fact unnecessary—we could have proceeded up until now without knowing the measure of D and the theorem we just prove would guarantee that $\mu(D) = 1$.

We have a continuous, surjective, multiplicative homomorphism $A_K^\times \to \mathbb{R}_{>0}^\times$ given by $\alpha \mapsto |\alpha|$. Therefore, we cannot hope that $A_K^\times/K^\times$ is compact, because is has $\mathbb{R}_{>0}$ as a continuous image. However, we instead consider the kernel $J = \{a \mid |a| = 1\}$, a closed subgroup of $A_K^\times$.

Recall that in the local case K_v, we could write any $\alpha \in K_v^\times$ uniquely as

$$\alpha = \begin{cases} \tilde{\alpha}\varrho \text{ for } |\tilde{\alpha}|_v = 1 \text{ and } \varrho > 0 & \text{if } v \text{ infinite,} \\ \tilde{\alpha}\pi^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)} \text{ for } |\tilde{\alpha}|_{\mathfrak{p}} = 1 \text{ and } \pi \text{ uniformizer} & \text{if } v = \mathfrak{p} \text{ finite.} \end{cases}$$

Similarly, we would like to represent any idele $a \in A_K^\times$ as a multiple of an element of J. In other words, we will select an arbitrary subgroup T of $A_K^\times$ such that

$$I = T \times J.$$

To this effect, choose an infinite prime v_0 of K and let

$$T = \{a \in A_K^\times \mid a_{v_0} > 0 \text{ and } a_v = 1 \text{ for } v \neq v_0\}.$$

Any $a \in T$ is determined uniquely by $|a|$, which gives an isomorphism of T with $\mathbb{R}_{>0}$. We will identify any element $t \in T$ with a number $t > 0$; explicitly, if we write the v_0-component of t first, then $t > 0$ stands for

$$\begin{array}{ll} (t, 1, 1, 1, \ldots) & \text{for } v_0 \text{ real,} \\ (\sqrt{t}, 1, 1, 1, \ldots) & \text{for } v_0 \text{ complex.} \end{array}$$

Now, we can write any idele $a \in A_K^\times$ uniquely as

$$a = |a| \cdot b \text{ with } |a| \in T \text{ and } b = a|a|^{-1} \in J,$$

and it is clear that $A_K^\times = T \times J$.

In order to select a measure $\mathtt{d}b$ on J, we take the measure $\mathtt{d}t = \frac{dt}{t}$ on T and require that $\mathtt{d}a = \mathtt{d}t \cdot \mathtt{d}b$ in the sense of Fubini's Theorem, i.e. for $f \in L_1(A_K^\times)$, we have that:

$$\int_{A_K^\times} f(a)\mathtt{d}a = \int_0^\infty \left(\int_J f(tb)\mathtt{d}b \right) \frac{dt}{t} = \int_J \left(\int_0^\infty f(tb)\frac{dt}{t} \right) \mathtt{d}b.$$

The Product formula 3.70 shows that $K^* \subseteq J$ and, as in the additive case, one can show that J/K^* is compact, using a multiplicative fundamental domain.

Proposition 3.72. *There exists a fundamental domain E for $J/K^\times$, i.e.*

$$J = \coprod_{\alpha \in K^\times} \alpha E,$$

with measure

$$\frac{2^{r_1}(2\pi)^{r_2} hR}{\sqrt{|\Delta|}w},$$

where $h = \#I_K/P_K$, R is the regulator *of K, w the number of roots of unity in K.*

We omit the proof here, but it can be found in [CF86, Theorem 4.3.2, Chap. XV]. It relies on some results from class field theory and some theorems from classical algebraic number theory (such as Dirichlet's Unit Theorem).

Corollary 3.73. *The subgroup $K^\times \subseteq J$ is discrete in J (and thus in $A_K^\times$), and the quotient group $J/K^\times$ is compact.*

This is the multiplicative analog of Theorem 3.58 we were looking for.

We are not actually interested in all quasi-characters of $A_K^\times$ but only those that are trivial on $K^\times$. We will use the word quasi-characters to refer to those. A quasi-character on J is hence a quasi-character on the quotient $J/K^\times$, which is compact by Corollary 3.73. Hence, a quasi-character on J is a character. Furthermore, if a character is trivial on J, then it is in fact a character of T, so it is of the form $|-|^s$ for some complex s.

Once again, for each quasi-character c, there is a unique real number $\sigma = \sigma(c)$ such that $|c(a)| = |a|^\sigma$ for any $a \in A_K^\times$. We call it the *exponent* of c. A quasi-character is a character if and only if its exponent is 0.

3.5. **Main Results.** We are finally able to globalize the results of Section 3.2. We first define a class of functions for which we will define ζ-functions.

Let $\mathcal{Z}$ denote the set of all functions $f\colon A_K \to \mathbb{C}$ that satisfy

($\mathcal{Z}_1$) Both f and $\hat{f}$ are continuous functions in $L_1(A_K)$, i.e. $f \in \mathcal{B}_1(V)$.

($\mathcal{Z}_2$) The series
$$\sum_{\zeta\in K} f(a(x+\xi)) \text{ and } \sum_{\zeta\in K} \hat{f}(a(x+\xi))$$
are both convergent for any idele $a \in A_K^\times$ and adele $x \in A_K$, the convergence being uniform in the pair (a, x) for x ranging over D and a ranging over any fixed compact subset of $A_K^\times$.

($\mathcal{Z}_3$) Both $A_K^\times \ni a \mapsto f(a)|a|^\sigma$ and $A_K^\times \ni a \mapsto \hat{f}(a)|a|^\sigma$ are in $L_1(A_K^\times)$ for $\sigma > 1$.

Note that if f continuous on A_K, then f is continuous on $A_K^\times$, because the topology on $A_K^\times$ is stronger than the subspace topology.

In view of ($\mathcal{Z}_1$) and ($\mathcal{Z}_2$), the Riemann-Roch Theorem 3.69 is valid for functions in $\mathcal{Z}$. The purpose of ($\mathcal{Z}_3$) is defining the ζ-functions.

Definition 3.74. The *ζ-function* of K corresponding to $f \in \mathcal{Z}$ is a function of quasi-characters, defined for all quasi-characters with $\sigma(c) > 1$ by
$$\zeta(f, c) = \int f(a)c(a)\mathtt{d}a.$$

We will call two quasi-characters that coincide on J *equivalent*. An equivalence class of quasi-characters consists of all quasi-characters c of the form $c(a) = c_0(a)|a|^s$ for a fixed representative c_0, $s \in \mathbb{C}$ determined uniquely by c. As in the local theory, we can now view an equivalence class of quasi-characters as a Riemann surface (with the variable s).

It is obvious that for quasi-characters of exponent greater than 1, the ζ-functions are holomorphic. Moreover, once again, we have an analytic continuation to the entire space of quasi-characters.

Theorem 3.75 (Analytic Continuation and Functional Equation of the ζ-Functions)**.** *By analytic continuation, we may extend the definition of any ζ-function, $\zeta(f, c)$, to the domain of all quasi-characters. The extended function is single valued and holomorphic, expect at $c(a) = 1$ and $c(a) = |a|$ where it has simple poles with residues $-\kappa f(0)$ and $+\kappa f(0)$ (where*

κ is the volume of the multiplicative fundamental domain). Moreover, $\zeta(f,c)$ satisfies the functional equation

$$\zeta(f,c) = \zeta(\hat{f},\hat{c}),$$

where $\hat{c}(a) = |a|c^{-1}(a)$, as in the local theory.

To prove the theorem, first note that for c with $\sigma(c) > 1$, we have that

$$\zeta(f,c) = \int_{A_K^\times} f(a)c(a)\mathtt{d}a = \int_0^\infty \left(\int_J f(tb)c(tb)\mathtt{d}b \right) \frac{dt}{t},$$

since $A_K^\times = T \times J$, as was established before. Let us hence define

$$\zeta_t(f,c) = \int_J f(tb)c(tb)\mathtt{d}b.$$

First, note that since $\zeta(f,c)$ converges for $\sigma(c) > 1$, so the integral $\zeta_t(f,c)$ will converge (at least for almost all t). Moreover, for $b \in J$, $|b| = 1$ by definition, so $|c(tb)| = t^\sigma$ is constant on J. Therefore, if $\zeta_t(f,c)$ converges for one quasi-character c, then it converges for all of them.

The first step in the proof of the theorem will be to establish a functional equation for $\zeta_t(f,c)$.

Lemma 3.76. *For all quasi-characters c, we have:*

$$\zeta_t(f,c) + f(0)\int_E c(tb)\mathtt{d}b = \zeta_{1/t}(\hat{f},\hat{c}) + \hat{f}(0)\int_E \hat{c}((1/t)b)\mathtt{d}b.$$

Proof. We first claim that

$$\zeta_t(f,c) + f(0)\int_E c(tb)\mathtt{d}b = \int_E \left(\sum_{\xi \in K} f(\xi tb) \right) c(tb)\mathtt{d}b. \tag{1}$$

Indeed:

$$\begin{aligned}
\zeta_t(f,c) + f(0)\int_E c(tb)\mathtt{d}b &= \sum_{\alpha \in K^\times} \left(\int_{\alpha E} f(tb)c(tb)\mathtt{d}b \right) + f(0)\int_E c(tb)\mathtt{d}b && \text{by Proposition 3.72} \\
&= \sum_{\alpha \in K^\times} \left(\int_E f(\alpha tb)c(tb)\mathtt{d}b \right) + f(0)\int_E c(tb)\mathtt{d}b && \mathtt{d}b \text{ multiplicative} \\
&= \int_E \left(\sum_{\alpha \in K^\times} f(\alpha tb) \right) c(tb)\mathtt{d}b + f(0)\int_E c(tb)\mathtt{d}b && \text{sum unif. convergent} \\
&= \int_E \left(\sum_{\xi \in K} f(\xi tb) \right) c(tb)\mathtt{d}b.
\end{aligned}$$

By following the same steps for $1/t$, $\hat{f}$, $\hat{c}$ instead, we obtain also that

$$\zeta_{1/t}(\hat{f},\hat{c}) + \hat{f}(0)\int_E \hat{c}((1/t)b)\mathtt{d}b = \int_E \left(\sum_{\xi \in K} \hat{f}(\xi(1/t)b) \right) \hat{c}((1/t)b)\mathtt{d}b \tag{2}$$

We only have to transform the right hand side of equation (1) to the right hand side of (2). Fortunately, we are in the right setting to use the Riemann-Roch Theorem 3.69:

$$\begin{aligned} \int_E \left(\sum_{\xi \in K} f(\xi tb) \right) c(tb) \mathtt{d}b &= \int_E \left(\sum_{\xi \in K} \hat{f}(\xi/(tb)) \right) c(tb)/|tb| \mathtt{d}b && \text{Riemann-Roch Theorem 3.69} \\ &= \int_E \left(\sum_{\xi \in K} \hat{f}(\xi(1/t)b) \right) \hat{c}((1/t)b) \mathtt{d}b && \text{substituting } b \mapsto 1/b \end{aligned}$$

This yields the desired result. ☐

Finally, we establish the integral $\int_E c(tb)\mathtt{d}b$ that appears in the above lemma.

Lemma 3.77. *For all quasi-characters c, we have that*

$$\int_E c(tb)\mathtt{d}b = \begin{cases} \kappa t^s & \text{if } c(a) = |a|^s \\ 0 & \text{otherwise (i.e. } c \text{ is non-trivial on } J) \end{cases}$$

Proof. Note that $\int_E c(tb)\mathtt{d}b = c(t)\int_E c(b)\mathtt{d}b$ and the latter integral is the integral over $J/K^\times$ of the character represented by c. Accordingly, if c is non-trivial on J, then the integral is 0.

If c is trivial on J, then $c(a) = |a|^s$ for some s, and recalling that $|b| = 1$ for $b \in J$, we have that the value of integral is κt^s, where κ is the measure of E. ☐

Proof of Theorem 3.75. For c of exponent greater than 1, we write

$$\zeta(f,c) = \int_0^\infty \zeta_t(f,c)\frac{dt}{t} = \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(f,c)\frac{dt}{t} \tag{3}$$

and deal with the two integrals in the sum separately.

Note that:

$$\int_1^\infty \zeta_t(f,c)\frac{dt}{t} = \int_{|a|\geq 1} f(a)c(a)\mathtt{d}a,$$

so it converges quicker for smaller exponents of c. Since it converges for c with $\sigma(c) > 1$ by assumption, it converges for all c.

Therefore, we are left with the integral $\int_0^1 \zeta_t(f,c)\frac{dt}{t}$. The idea is to use Lemma 3.76 to transform it into an integral over the range 1 to ∞, thereby obtaining the functional equation for $\zeta(f,c)$.

We need to distinguish between the case when c is trivial on J and when it is not trivial on J. However, in both of them we will proceed in the same way. Let us hence define

$$\delta = \begin{cases} 1 & \text{if } c \text{ trivial on } J \\ 0 & \text{otherwise} \end{cases}$$

and use it to make the distinction. We then have

$$\int_0^1 \zeta_t(f,c)\frac{dt}{t} = \int_0^1 \zeta_{1/t}(\hat{f},\hat{c})\frac{dt}{t} + \delta\left(\int_0^1 \kappa\hat{f}(0)(1/t)^{1-s}\frac{dt}{t} - \int_0^1 \kappa f(0)t^s\frac{dt}{t}\right)$$

by Lemmas 3.76 and 3.77.

Note that if c is trivial on J, then $c(a) = |a|^s$, and $\mathrm{Re}(s) = \sigma(c) > 1$. This is necessary for the expression in the bracket to make sense. Evaluating the two integrals in the bracket, we get

$$\int_0^1 \kappa\hat{f}(0)(1/t)^{1-s}\frac{dt}{t} = \frac{\kappa\hat{f}(0)}{s-1} \text{ and } \int_0^1 \kappa f(0)t^s\frac{dt}{t} = \frac{\kappa f(0)}{s}.$$

Thus, substituting $t \mapsto \frac{1}{t}$, we get that

$$\int_0^1 \zeta_t(f,c) = \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \delta\left(\frac{\kappa\hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right).$$

In light of the above, equation (3) becomes

$$\zeta(f,c) = \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \delta\left(\frac{\kappa\hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s}\right)$$

and the two integrals are homogeneous for all c. This expression therefore gives the desired analytic continuation of $\zeta(f,c)$ to the domain of all quasi-characters. We can read off the poles from it directly. Moreover, for $c(a) = |\alpha|^s$, $\hat{c}(a) = |a|^{1-s}$, we see that even the form of the equation is unchanged under

$$(f,c) \mapsto (\hat{f},\hat{c}),$$

so we obtained the desired functional equation $\zeta(f,c) = \zeta(\hat{f},\hat{c})$. □

3.6. **Functional equation for Hecke L-functions.** We can finally apply the theory developed in this section to obtain a functional equation for the Hecke L-functions. To do this, we will exhibit for each equivalence class of quasi-characters C, an explicit function in $f \in \mathcal{Z}$ such that $\zeta(f,c)$ is the (completed) Hecke L-function, and Theorem 3.75 will yield the analytic continuation and functional equation, i.e. Theorem 2.11. Note that this section is necessary for another purpose—it will show that our theory is *non-empty*: i.e. there are elements $f \in \mathcal{Z}$ such that $\zeta(f,c)$ is non-trivial.

The approach we will take is to build up function $f \in \mathcal{Z}$ from the local functions we have seen in Section 3.2.4. We parallel the layout of that section, but now, with adeles at our disposition, we do not have to split into cases, as before.

The Equivalence Classes of Quasi-characters. We can represent each class by a character. We will first classify all the characters.

We fix an arbitrary finite set S, containing the set S_∞ of infinite primes, and restrict our attention to characters unramified outside S. A character c of this type is given by

$$c(a) = \prod_v c_v(a_v)$$

for local characters c_v satisfying:

(1) c_v is unramified for $v \notin S$,
(2) $\prod_v c_v(\alpha) = 1$ for $\alpha \in K^\times$ (by the Product formula 3.70).

To construct such characters concretely, we write for $v \in S$:

$$c_v(a_v) = \tilde{c_v}(\tilde{a_v})|a_v|_v^{it_v},$$

where $\tilde{c_v}$ is a character of U_v, and t_v is a real number.

For $v \in S$, we define

$$c^*(a) = \prod_{v \notin S} c_v(a_v)$$

and interpret c^* as coming from an ideal character. Namely, we have a map $\varphi_S \colon A_K^* \to I_K^S$ (where I_K^S are the ideals prime to S) given by

$$\varphi_S(a) = \prod_{v \notin S} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(a)}.$$

Its kernel is $(A_K^*)_S$ and c^* is trivial on $(A_K^*)_S$, so

$$c^*(a) = \chi(\varphi_S(a)),$$

where χ is some character of $(A_K^*)^S$.

Therefore, we have now written our character c as

$$c(a) = \prod_{v \in S} \tilde{c_v}(\tilde{a_v}) \cdot \prod_{v \in S} |a_v|_v^{it_v} \cdot \chi(\varphi_S(a)).$$

We wish to construct such characters by selecting $\tilde{c_v}$, t_v, and χ, in a way that guarantees $c(\alpha) = 1$ for $\alpha \in K^\times$.

First, we look at S-units of K, i.e. $\varepsilon \in K^\times \cap I_S$ such that $\varphi_S(\varepsilon) = \mathcal{O}_K$. Assume $\#S = m+1$ and ε_0 is a primitive root of unity in K. Dirichlet's unit theorem says that the groups of S-units modulo the roots of unity is a free abelian group on m generators, so let $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be a basis for this quotient. Now, c is trivial on S-units if and only if $c(\varepsilon_j) = 1$ for $0 \le j \le m$.

The requirement $c(\varepsilon_0) = 1$ is a simple condition on the $\tilde{c_v}$:

$$\prod_{v \in S} \tilde{c_v}(\varepsilon_0) = 1 \tag{4}$$

Hence, we first select a set $\tilde{c_v}$ of characters for $v \in S$ which satisfies equation (4).

The requirements $c(\varepsilon_j) = 1$ for $1 \le j \le m$, give conditions on the t_v:

$$\prod_{v \in S} |\varepsilon_j|_v^{it_v} = \prod_{v \in S} \tilde{c_v}^{-1}((\tilde{\varepsilon_j})_v)$$

which is satisfied if and only if t_v solve the system of real linear equations

$$\sum_{v \in S} t_v \log|\varepsilon_j|_v = i \log\left(\prod_{v \in S} \tilde{c_v}((\tilde{\varepsilon_j})_v)\right) \quad \text{for } 1 \le j \le m \tag{5}$$

for **some** value of the logarithms on the right hand side. (A solutions always exists because the rank of the matrix $(\log|\varepsilon_j|_v)$ is m.)

We now select a set of values for those logarithms and a set of numbers t'_v satisfying the system of equations (5).

Since $\sum_{v \in S} \log |\varepsilon_j|_v = 0$ for all j, the most general solution is then $t_v = t'_v + t$ for any $t \in \mathbb{R}$.

Having selected $\tilde{c_v}$ and t_v, what are the possible choices for the ideal character χ? Requirement (2), the product formula, means that χ must satisfy

$$\chi(\varphi_S(\alpha)) = \prod_{v \in S} \tilde{c}^{-1}(\tilde{\alpha_v})|\alpha|_v^{-it_v} \tag{6}$$

for all $\alpha \in K^\times$. What are the ideals $\varphi_S(\alpha)$ for $\alpha \in K^\times$? They are obtained from principal ideals by cancelling all the prime from S in the factorization. They form a subgroup H_S of index $h_S \leq h$ in I_K^S, where h is the class number of K, and $h_S = 1$ for large enough S. Thus, selecting the character χ amounts to selecting a character on H_S—we select χ from one of the finite number h_S of extensions on the chosen character on H_S to the whole of I_K^S.

Any character χ is, in fact, a Hecke Grossencharacter, and conversely, any Hecke Grossencharacter is a character χ. We are hence on the right track to proving Hecke's Theorem 2.11.

The Corresponding Functions. Suppose we have selected a character c of the form

$$c(a) = \prod_v c_v(a_v) = \prod_{v \in S} \tilde{c_v}(\tilde{a_v})|a_v|^{it_v} \cdot \chi(\varphi_s(a)),$$

unramified outside $S \supseteq S_\infty$. We will find a function $f \in \mathcal{Z}$ such that $\zeta(f, c)$ is non-trivial on the equivalence class $C = \{c(-)|-|^s \mid s \in \mathbb{C}\}$ (and is in fact the completed Hecke L-function).

To this effect, we use the local ζ-function from Section 3.2.4. Let Z_v be the class of functions Z defined in Section 3.2.4 for the field Z_v. For $v \in S$, let $f_v \in Z_v$ be the function f_v exhibited in Section 3.2.4 for the surface containing c_v. For $\mathfrak{p} \notin S$, let $f_\mathfrak{p}$ be the characteristic function of the set $\mathcal{O}_\mathfrak{p}$. Then define the function f by

$$f(x) = \prod_v f_v(x_v)$$

for any $x \in A_K$. (As always, this is actually a finite product for any $x \in A_K$.)

We need to show that $f \in \mathcal{Z}$, and we will accomplish this in the course of the computations of the Fourier transforms and ζ-functions.

Their Fourier Transforms. By Lemma 3.50, we have that $f \in \mathcal{B}_1(A_K)$ (so f satisfies axiom $(\mathcal{Z}_1)$), and

$$\hat{f}(x) = \prod_v \hat{f}_v(x_v).$$

Importantly, this shows that $\hat{f}$ is a function of the same type as f: for $v \in S$, the local factor is $\hat{f}_v$; the difference appears at those $v \notin S$ where $\mathfrak{d}_v \neq 1$, whence $\hat{f}_v(x_v)$ equals to $\Delta^{-1/2} \cdot \chi_{\mathfrak{d}_v^{-1}}$.

The ζ-functions. As promised, we will first ensure that $f \in \mathcal{Z}$, i.e. axioms $(\mathcal{Z}_2)$ and $(\mathcal{Z}_3)$ hold. We check both of them only for f, not for $\hat{f}$, as we have already established that f and $\hat{f}$ are of the same type, and hence it is easy to adjust the argument for f to $\hat{f}$.

Axiom $(\mathcal{Z}_2)$. This follows easily after unraveling the definition of f. Indeed:

- For $\mathfrak{p} \notin S$: $f_\mathfrak{p} = \chi_{\mathcal{O}_\mathfrak{p}}$ vanishes outside a compact set $\mathcal{O}_\mathfrak{p}$ (for $\hat{f}$, this compact set is $\mathfrak{d}_\mathfrak{p}^{-1}$),
- For $\mathfrak{p} \in S$ finite: $f_\mathfrak{p}(\xi) = e^{2\pi i \Lambda(\xi)} \cdot \chi_{\mathfrak{d}_\mathfrak{p}^{-1}\mathfrak{f}^{-1}}$, where $\mathfrak{f}$ is the conductor of $c_\mathfrak{p}$, vanishes outside a compact set $\mathfrak{d}^{-1}\mathfrak{f}^{-1}$ (for $\hat{f}$, this compact set is $1+\mathfrak{f}$).
- For $v \notin S$ infinite, f_v decays exponentially to 0 as x_v goes to infinity.

Then convergence and uniform convergence is clear (for a longer discussion, refer to [CF86, pp. 345–346]).

Axiom $(\mathcal{Z}_3)$. We will show that $|f(a)||a|^\sigma$ is summable for $\sigma > 1$ on $A_K^\times$, and hence $|\hat{f}(a)||a|^\sigma$ will be summable as well, since it is of the same form. We will use Theorem 3.49 for the product

$$|f(a)||a|^\sigma = \prod_v |f_v(a_v)||a|_v^\sigma,$$

almost all of which are 1 on U_v.

For $\mathfrak{p} \notin S$, we will show that

$$\int_{K_\mathfrak{p}^\times} |f_\mathfrak{p}(a_\mathfrak{p})|_\mathfrak{p} |a_\mathfrak{p}|_\mathfrak{p}^\sigma \mathrm{d}a_\mathfrak{p} = \frac{\Delta_\mathfrak{p}^{-1/2}}{1-(N\mathfrak{p})^{-\sigma}}.$$

Indeed, recall that $f_\mathfrak{p}$ is the characteristic function of $\mathcal{O}_\mathfrak{p}$ by definition. We split $\mathcal{O}_\mathfrak{p} \setminus \{0\}$ according to the values of $|a_\mathfrak{p}|_\mathfrak{p}$, i.e.

$$\mathcal{O}_\mathfrak{p} \setminus \{0\} = \coprod_{m=0}^{\infty} \mathfrak{p}^m \cdot U_\mathfrak{p}$$

to obtain

$$\int_{K_\mathfrak{p}^\times} |f_\mathfrak{p}(a_\mathfrak{p})|_\mathfrak{p} |a_\mathfrak{p}|_\mathfrak{p}^\sigma \mathrm{d}a_\mathfrak{p} = \sum_{m=0}^{\infty} \int_{\mathfrak{p}^m U_\mathfrak{p}} |a_\mathfrak{p}|_\mathfrak{p}^\sigma \mathrm{d}a_\mathfrak{p} = \sum_{m=0}^{\infty} \int_{\mathfrak{p}^m U_\mathfrak{p}} (N\mathfrak{p})^{-\sigma m} \mathrm{d}a_\mathfrak{p} = \frac{\Delta_\mathfrak{p}^{-1/2}}{1-(N\mathfrak{p})^{-\sigma}},$$

since the multiplicative measure of $U_\mathfrak{p}$ is $\Delta_\mathfrak{p}^{-1/2}$ by Lemma 3.35. The summability now follows: the product

$$\prod_{\mathfrak{p} \notin S} \frac{1}{1 - N\mathfrak{p}^{-\sigma}}$$

is known to converge for $\sigma > 1$, and the rest is a finite product.

Therefore, we have finally checked that $f \in \mathcal{Z}$.

Having established summability, we use Theorem 3.49 to express $\zeta(f, c)$ as a product of local ζ-functions

$$\zeta(f, c) = \prod_v \zeta_v(f_v, c_v)$$

for any quasi-character $c = \prod_v c_v$ of exponent $\sigma > 1$. If c now denotes the character we have previously selected, we have

$$c(a) = \prod_v c_v(a_v) = \prod_{v \in S} c_v(a_v) \cdot \chi(\varphi_S(a)).$$

We can compute explitiely the local factors for $v \notin S$:

$$\zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}|-|_{\mathfrak{p}}^{s}) = \int_{\mathcal{O}_{\mathfrak{p}}} c_{\mathfrak{p}}(a_{\mathfrak{p}})|a_{\mathfrak{p}}|_{\mathfrak{p}}^{s} \mathrm{d}a_{\mathfrak{p}} = \sum_{m=0}^{\infty} \chi(\mathfrak{p}^{m}) N\mathfrak{p}^{-ms} \Delta_{\mathfrak{p}}^{-1/2} = \frac{\Delta_{\mathfrak{p}}^{-1/2}}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}},$$

because for $\mathfrak{p} \in S$, $c_{\mathfrak{p}}(a_{\mathfrak{p}}) = \chi(\varphi_S(a_{\mathfrak{p}})) = \chi(\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}} a_{\mathfrak{p}}})$.

Now, recall that the Hecke L-function for the Grossencharacter χ was:

$$L(s,\chi) = \prod_{\mathfrak{p} \notin S} \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}.$$

Moreover, the ζ-function can be written as:

$$\zeta(f, c|-|^{s}) = \prod_{v \in S} \zeta_v(f_v, c_v|-|_v^{s}) \cdot \prod_{\mathfrak{p} \notin S} \Delta_{\mathfrak{p}}^{-1/2} L(s,\chi).$$

Once again, we do a similar computation to obtain:

$$\zeta(\hat{f}, c\widehat{|-|^{s}}) = \prod_{v \in S_\infty} \zeta_v(\hat{f}_v, c_v\widehat{|-|_v^{s}}) \prod_{\mathfrak{p} \notin S} \chi(\mathfrak{d}_{\mathfrak{p}}) \Delta_{\mathfrak{p}}^{-s} \cdot L(1-s, \overline{\chi})$$

Finally, the functional equations:

$$\zeta(\hat{f}, \hat{c}) = \zeta(f,c) \text{ and } \zeta_v(f_v, c_v) = \varrho_v(c_v)\zeta_v(\hat{f}_v, \hat{c_v})$$

yield for $L(s,\chi)$ the functional equation

$$L(1-s, \overline{\chi}) = \prod_{v \in S} \varrho_v(\tilde{c_v}|-|_v^{s+it_v}) \cdot \prod_{v \notin S} \Delta_{\mathfrak{p}}^{s-1/2} \overline{\chi}(\mathfrak{d}_{\mathfrak{p}}) L(s,\chi).$$

It is not hard to see that this yields that completed Hecke L-functions and the functional equation for them, as in the statement of Hecke's Theorem 2.11.

We end this section by remarking that we could have selected a different (easier) function $f \in \mathcal{Z}$, which would yield a functional equation for the Dirichlet L-series or the Dedekind ζ-function directly.

4. Class Field Theory

We initially defined the generalized Dirichlet character on the ray class group. It turns out that it is in fact a character of a Galois group of an abelian Galois extension. This subtle connection is exhibited using class field theory. We only state a few necessary definitions and theorems—for a formal introduction to class field theory, see [Lan94, Part Two].

We first note that a character $\chi \colon I_k^{\mathfrak{m}}/P_K^{\mathfrak{m}} \to \mathbb{C}^\times$ might have a non-trivial kernel H, in which case it is in fact a character of the quotient group $I_k^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$. Hence this is the group that we are in fact interested in.

Definition 4.1. Let H be a subgroup of the ray class group. The quotient $I_k^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$ is called a *class group*. A field extension L of K is a *class field* for this class group if $\mathrm{Gal}(L/K) \cong I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$.

Theorem 4.2. *For any class group, there exists a unique class field.*

For the proof, see [Lan94, Chap. XI].

In general, if L/K is an abelian extension and $\mathfrak{m}$ is a modulus divisible by all primes of K that ramify in L, then we defined the Artin map (Definition 1.17) by

$$\mathrm{Frob}\colon I_L^{\mathfrak{m}} \to \mathrm{Gal}(L/K),$$

$$\mathfrak{p} \mapsto \mathrm{Frob}_{\mathfrak{p}}.$$

Theorem 4.3 (Artin Reciprocity)**.** *The Artin map* Frob *is surjective. Moreover, for some* $\mathfrak{m}$*, divisible by all ramified primes of* K*, we have that* $P_K^{\mathfrak{m}} \subseteq \ker(\mathrm{Frob})$ *. In particular, for some subgroup* $H \subseteq I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}$*, we have*

$$\mathrm{Gal}(L/K) \cong I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}H,$$

i.e. L *is a class field for the class group* $I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$*.*

For the proof, see [Lan94, Theorems 1–3, Chap. X§2].

We can now restate the definitions of a generalized Dirichlet character and a conductor in this terminology.

Proposition 4.4. *Suppose* L/K *is a class field associated to* $\mathfrak{m}$ *and* H*. A generalized Dirichlet character is a character of the Galois group* $\mathrm{Gal}(L/K) \cong I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$ *and the conductor is* $\mathfrak{f}$ *is the minimal modulus* $\mathfrak{m}$ *divisible by all ramified primes such that* $P_K^{\mathfrak{m}}$ *is contained in the kernel of the Artin map.*

Proof. This is simply a restatement of the previous definitions in terms of class field theory, so the proof is a straightforward application of the preceding theorems. □

This gives another statement of Hecke's theorem 2.11 in the special case of a generalized Dirichlet character. In fact, by Artin Reciprocity 4.3, we know that the Hecke L-function can be defined and has a holomorphic analytic continuation for any non-trivial character of the Galois group of an abelian extension.

Suppose we are given an abelian extension of number fields L/K. Recall that we have defined (Definition 2.1) the Dedekind ζ-function of the number field K to be

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}}.$$

We also know that L is a class field of some class group $G = I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}H$, and for any character χ of G we have defined the Hecke L-function $L(s,\chi)$. There is a nice relation between the two functions.

Theorem 4.5 (Weber)**.** *Let* L/K *be an abelian extension of number fields and* $\widehat{G}$ *denote the group of characters of the class group* G*. Then*

$$\zeta_L(s) = \prod_{\chi \in \widehat{G}} L(s,\chi).$$

Proof. We want to show that

$$\prod_{\mathfrak{P}} \frac{1}{1 - N\mathfrak{P}^{-s}} = \prod_{\chi}\prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}$$

by verifying it locally, prime by prime. So fix an unramified prime $\mathfrak{p}$. If there are g primes $\mathfrak{P}$ above $\mathfrak{p}$ and $f = f_{\mathfrak{P}/\mathfrak{p}}$, then $N\mathfrak{P} = N\mathfrak{p}^f$ and by Corollary 1.8 we have $[L : K] = gf$. We will show that

$$\prod_{\mathfrak{P}|\mathfrak{p}} \frac{1}{1 - N\mathfrak{P}^{-s}} = \prod_{\chi} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}}.$$

First, note that

$$\prod_{\mathfrak{P}|\mathfrak{p}} \frac{1}{1 - N\mathfrak{P}^{-s}} = \left(\frac{1}{1 - (N\mathfrak{p}^{-s})^f}\right)^g = \left(\prod_{\zeta_f} \frac{1}{1 - \zeta_f N\mathfrak{p}^{-s}}\right)^g,$$

where the product is over fth roots of unity ζ_f.

Now, recall that the class group G isomorphic to the Galois group $\mathrm{Gal}(L/K)$ via the Artin map Frob. Since the order of $\mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{Gal}(L/K)$ is f, the order of $\mathfrak{p}$ in G is f as well. Therefore for any character χ of G we have that

$$\chi(\mathfrak{p})^f = 1,$$

or, in other words, $\chi(\mathfrak{p}) = \zeta_f$, some fth root of unity. Finally, by decomposing G as a product of cyclic groups, one can show that each of the roots of unity appears as $\chi(\mathfrak{p})$ equally often. Since $\#G = [L : K] = gf$, we obtain that

$$\prod_{\chi} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}} = \left(\prod_{\zeta_f} \frac{1}{1 - \zeta_f N\mathfrak{p}^{-s}}\right)^g,$$

which shows the equality for unramified primes.

Finally, we show that the ramified case reduces to the unramified case. Suppose $\mathfrak{p}$ is a ramified prime. We have to show that

$$\prod_{\mathfrak{P}|\mathfrak{p}} \frac{1}{1 - N\mathfrak{P}^{-s}} = \prod_{\chi} \frac{1}{1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}},$$

where χ runs over all the characters of the group $G' = I_K^{\mathfrak{m}'}/P_K^{\mathfrak{m}'}H$ for $\mathfrak{m}'$, the part of $\mathfrak{m}$ that is relatively prime to $\mathfrak{p}$. However, we know that G' is the class group of some class field M sitting between K and L. Therefore, we have reduced the problem to showing the equality for the extension M/K. Since M/K is unramified, we are done. □

Corollary 4.6. *We have*

$$\frac{\zeta_L(s)}{\zeta_K(s)} = \prod_{\chi \in \widehat{G}\setminus\{\chi_0\}} L(s, \chi)$$

where χ_0 is the trivial character, the function is holomorphic on the entire complex plane.

Proof. The equality follows from $L(s, \chi_0) = \zeta_K(s)$ for the trivial character χ_0 and the holomorphy of the function follows from the holomorphy of each component $L(s, \chi)$ for $\chi \neq \chi_0$ by Theorem 2.11. □

Moreover, Hecke proved that the equality in Theorem 4.5 holds for the completed L-functions, $\Lambda(s,\chi)$, from Theorem 2.11.

Theorem 4.7 (Hecke)**.** *Let L/K be an abelian extension of number fields and $\widehat{G}$ denote the group of characters of G. Then for the completed abelian L-functions $\Lambda(s,\chi)$ we have*

$$\zeta_L(s) = \prod_{\chi\in\widehat{G}} \Lambda(s,\chi).$$

By Weber's Theorem 4.5, only only has to check the equality for infinite primes and for the exponential factor. This is done, for example, in [Sny02, Theorem 2.1.2]

5. Artin L-functions

In this section, we follow [Sny02, Chap. 2]. We will make free use of basic results from representation theory, although we will state some of the necessary definitions and theorems. The background can be found in [FH91, Part I] and [Ser77].

We wish to generalize L-functions to characters of any Galois group of an extension of number fields L/K by exploiting the Artin Reciprocity 4.3. Recall that by surjectivity of the Artin map, any element of the Galois group $\mathrm{Gal}(L/K)$ is a product of Frobenius elements for primes $\mathfrak{p}$ of K. We wish to restate the definition of the local factors in the product defining the L-function.

We want to first define the local factor at an unramified prime $\mathfrak{p}$, i.e. at the element $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(L/K)$. Suppose we are given an irreducible representation $\rho\colon \mathrm{Gal}(L/K) \to \mathrm{GL}(V)$ for a complex vector space V with character $\chi\colon \mathrm{Gal}(L/K) \to \mathbb{C}$. If $\mathrm{Gal}(L/K)$ is abelian as before, then the representation is 1-dimensional, and so

$$\det(1 - N\mathfrak{p}^{-s}\rho(\mathrm{Frob}_\mathfrak{p}))^{-1} = (1 - N\mathfrak{p}^{-s}\rho(\mathrm{Frob}_\mathfrak{p}))^{-1} = (1 - N\mathfrak{p}^{-s}\chi(\mathrm{Frob}_\mathfrak{p}))^{-1}$$

is the local factor at $\mathfrak{p}$ of the abelian L-function. Now, regardless of whether $\mathrm{Gal}(L/K)$ is abelian or not, we may define the local factor at an unramified prime $\mathfrak{p}$ to be

$$L_\mathfrak{p}(s,\chi) = \det(1 - N\mathfrak{p}^{-s}\mathrm{Frob}_\mathfrak{p})^{-1}.$$

Now, suppose the prime $\mathfrak{p}$ ramifies in L/K and $\mathfrak{P}$ lies above $\mathfrak{p}$. Then the Frobenius element $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$ does not determine a unique element of $\mathrm{Gal}(L/K)$; indeed

$$\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}} \in D_\mathfrak{P}/I_\mathfrak{P},$$

where $D_\mathfrak{P}$ is the decomposition group and $I_\mathfrak{P}$ is the inertia group. While $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$ does not determine an action on V, we can consider the subspace $V^{I_\mathfrak{P}}$ of V invariant under the action of $I_\mathfrak{P}$. The action of ρ of G on $V^{I_\mathfrak{P}}$ factors through the action $\rho^{I_\mathfrak{P}}$ of $D_\mathfrak{P}/I_\mathfrak{P}$ on $V^{I_\mathfrak{P}}$, so the element $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}} \in D_\mathfrak{P}/I_\mathfrak{P}$ acts on $V^{I_\mathfrak{P}}$. Altogether, this motivates the following definition.

Definition 5.1. Let L/K be a Galois extension of number fields and $\rho\colon \mathrm{Gal}(L/K) \to \mathrm{GL}(V)$ be a representation of the Galois group of L/K. Then the *local factor* of the *Artin L-function* associated to V at a finite prime $\mathfrak{p}$ of K is

$$L_\mathfrak{p}(s,V;L/K) = \frac{1}{\det(1 - N\mathfrak{p}^{-s}\rho^{I_\mathfrak{p}}(\mathrm{Frob}_\mathfrak{p}))}.$$

Note that this definition actually makes sense. The Frobenius $\mathrm{Frob}_{\mathfrak{p}}$ was defined as $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$ by selecting any prime $\mathfrak{P}$ over $\mathfrak{p}$; selecting a different prime $\mathfrak{P}'$ over $\mathfrak{p}$ conjugates the Frobenius by an element of the Galois group according to Proposition 1.15, and the determinant is invariant under conjugation.

We will sometimes simplify the notation. We will write $L_{\mathfrak{p}}(s, V)$ for $L_{\mathfrak{p}}(s, V; L/K)$ when the extension is implicitly clear. Moreover, if χ is the character of the representation V, then we will sometimes write $L_{\mathfrak{p}}(s, \chi)$ for $L_{\mathfrak{p}}(s, V)$. Finally, if we have fixed a representation $\rho\colon \mathrm{Gal}(L/K) \to \mathrm{GL}(V)$, then we will write $\mathrm{Frob}_{\mathfrak{p}}$ for the element $\rho^{I_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathrm{GL}(V^{I_{\mathfrak{p}}})$. In that case, we will make the vector space $V^{I_{\mathfrak{p}}}$ in the determinant explicit by writing the local factor as

$$\frac{1}{\det(1 - N\mathfrak{p}^{-s}\mathrm{Frob}_{\mathfrak{p}}; V^{I_{\mathfrak{p}}})}.$$

We still have to verify that the local factors of the Artin L-functions agree with the local factors of the abelian L-functions.

Proposition 5.2. *If* $\mathrm{Gal}(L/K)$ *is abelian and* ρ *is an representation with character* χ*, then*

$$L_{\mathfrak{p}}(s, \chi) = \frac{1}{1 - N\mathfrak{p}^{-s}\chi(\mathfrak{p})},$$

so the local factors of the Artin L-functions coincide with the local factors of the Hecke L-functions.

We first prove a lemma.

Lemma 5.3. *If H is a normal subgroup of* $G = \mathrm{Gal}(L/K)$ *and χ is a character of G/H, then*

$$L_{\mathfrak{p}}(s, \chi; L^H/K) = L_{\mathfrak{p}}(s, \chi; L/K).$$

Proof. Fix a prime $\mathfrak{q}$ above $\mathfrak{p}$ in L^H/K and a prime $\mathfrak{P}$ above $\mathfrak{q}$ in L/L^H. We then know that $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}$ restricted to L^H is $\mathrm{Frob}_{\mathfrak{P}/\mathfrak{q}}$ by Proposition 1.16. Moreover, $I_{\mathfrak{P}} = I_{\mathfrak{q}}H$. But this means that

$$\rho^{I_{\mathfrak{P}}}(\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}) = \rho^{I_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}),$$

so

$$L_{\mathfrak{p}}(s, \chi; L^H/K) = \frac{1}{\det(1 - N\mathfrak{p}^{-s}\rho^{I_{\mathfrak{q}}}(\mathrm{Frob}_{\mathfrak{q}/\mathfrak{p}}))} = \frac{1}{\det(1 - N\mathfrak{p}^{-s}\rho^{I_{\mathfrak{P}}}(\mathrm{Frob}_{\mathfrak{P}/\mathfrak{p}}))} = L_{\mathfrak{p}}(s, \chi; L/K)$$

as requested. □

Proof of Proposition 5.2. This follows clearly from the preceeding lemma. Let $N = \ker\chi$, so that χ is an irreducible (and hence 1-dimensional) character of G/H. Then we know that

$$\begin{aligned} L_{\mathfrak{p}}(s, \chi; L/K) &= L_p(s, \chi; L^H/K) && \text{by Lemma 5.3} \\ &= (\det(1 - N\mathfrak{p}^{-s}\mathrm{Frob}_{\mathfrak{p}}; V^{I_{\mathfrak{p}}}))^{-1} && \text{by definition} \\ &= (1 - N\mathfrak{p}^{-s}\chi(\mathrm{Frob}_{\mathfrak{p}}))^{-1} && \text{since } \rho \text{ is 1-dimensional} \\ &= (1 - N\mathfrak{p}^{-s}\chi(\mathfrak{p}))^{-1} && \text{by Artin Reciprocity 4.3} \end{aligned}$$

where we abuse the notation and denote the character of both the ideal group and the isomorphic Galois group by χ (the isomorphism given by Artin Reciprocity 4.3). □

Definition 5.4. Let L/K be a Galois extension of number fields and V a representation of $\mathrm{Gal}(L/K)$. The *Artin L-function* of V is defined for $\mathrm{Re}(s) > 1$ by the Euler product

$$L(s, V) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, V)$$

where $L_{\mathfrak{p}}(s, V)$ is the local factor corresponding to the finite prime $\mathfrak{p}$.

As in the case of the Hecke L-functions, we will be interested in extending the function meromorphically to the entire complex plane. It will turn out, all the necessary information about the group is encoded by 1-dimensional representations (which are in fact representations of abelian subgroups of $\mathrm{Gal}(L/K)$) and can be recovered using additivity and induction. Since the Artin L-functions coincide with Hecke L-functions in the abelian case, we will be able to use these methods to extend Hecke's Theorems 2.11 and 4.7 to Artin L-functions.

5.1. **Additivity.** First of all, we will ensure that the Artin L-function is additive in the sense that

$$L_{\mathfrak{p}}(s, \chi_1 + \chi_2) = L_{\mathfrak{p}}(s, \chi_1) L_{\mathfrak{p}}(s, \chi_2)$$

for characters χ_1, χ_2 of $\mathrm{Gal}(L/K)$.

We will restate the definition of a local factor in a way that will make this explicit.

Proposition 5.5 (Artin)**.** *For a character* χ *or* $\mathrm{Gal}(L/K)$ *we have*

$$L_{\mathfrak{p}}(s, \chi) = \exp\left(-\sum_{j=1}^{\infty} \frac{\chi(\mathrm{Frob}_{\mathfrak{p}}^j)}{j} N\mathfrak{p}^{-sj} \right).$$

Proof. Recall that

$$L_{\mathfrak{p}}(s, \chi) = (\det(1 - N\mathfrak{p}^{-s}\mathrm{Frob}_{\mathfrak{p}}; V^{I_{\mathfrak{p}}}))^{-1}.$$

First, note that $\rho^{I_{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{p}})$ is diagonalizable: its minimal polynomial divides the polynomial $X^{\#(G_{\mathfrak{P}}/I_{\mathfrak{P}})} - 1$, so its roots are distinct. So if $\lambda_1, \ldots, \lambda_n$ are its eigenvalues, then

$$\det(1 - N\mathfrak{p}^{-s}\mathrm{Frob}_{\mathfrak{p}}; V^{I_{\mathfrak{p}}}) = \prod_{i=1}^{n} (1 - N\mathfrak{p}^{-s}\lambda_i).$$

Therefore:

$$\log L_{\mathfrak{p}}(s, \chi) = -\sum_{i=1}^{n} \log(1 - N\mathfrak{p}^{-s}\lambda_i) = -\sum_{i=1}^{n}\sum_{j=1}^{\infty} \frac{\lambda_i^j}{j} N\mathfrak{p}^{-sj} = -\sum_{j=1}^{\infty} \frac{\chi(\mathrm{Frob}_{\mathfrak{p}}^j)}{j} N\mathfrak{p}^{-sj},$$

as requested. □

Corollary 5.6 (Artin)**.** *For characters* χ_1, χ_2 *of* $\mathrm{Gal}(L/K)$ *we have*

$$L_{\mathfrak{p}}(s, \chi_1 + \chi_2) = L_{\mathfrak{p}}(s, \chi_1) L_{\mathfrak{p}}(s, \chi_2)$$

and hence

$$L(s, \chi_1 + \chi_2) = L(s, \chi_1) L(s, \chi_2).$$

Proof. The equation is clear for the reformulation of the local factor in Proposition 5.5. □

Note that we could have taken the equation in Proposition 5.5 as the definition of the local factor of the Artin L-function; however, the approach we have taken is easier to work with later on.

In particular, given any class function ψ, we can write it as a linear combination of irreducible characters $\chi = \sum_{\chi} r_\chi \chi$, and we could define

$$L(s,\psi) = \prod_{\chi} L(s,\chi)^{r_\chi}.$$

5.2. **Induction.** In this subsection, we will describe induction, a method for obtaining a representation of an entire group G from a representation of its subgroup H, and apply it to L-functions.

Note that given a representation V of G, we can clearly restrict it to a representation of a subgroup H. We will denote it by $\mathrm{Res}_H^G V$, or simply $\mathrm{Res}_H V$. Similarly, any representation of H induces a representation of G.

Definition 5.7. Suppose that H is a subgroup of G and W is a representation of H. The *induced representation* from H to G is the vector space

$$\mathrm{Ind}_H^G W = \{f\colon G \to W \mid f(hg) = hf(g) \text{ for all } h \in H, g \in G\}.$$

with the action of G given by

$$(gf)(g') = f(gg').$$

If χ is the character of W, then we will write $\mathrm{Ind}_H^G \chi$ for the character of the induced representation $\mathrm{Ind}_H^G W$.

Remark 5.8. There is also a more explicit description of the induced representation. For a coset $\sigma \in G/H$ with representative $g_\sigma \in G$, we write

$$\sigma W = \{g_\sigma w \mid w \in W\}$$

(this makes sense, since for any $h \in H$, we have $hW = W$.) Then we define

$$\mathrm{Ind}_H^G W = \bigoplus_{\sigma \in G/H} \sigma W.$$

To define an action, we note that for any $g \in G$ we have $g \cdot g_\sigma = g_\tau \cdot h$ for some $\tau \in G/H$ and $h \in H$, and for any $g_\sigma w \in \sigma W$ we let

$$g\,(g_\sigma w) = g_\tau(hw).$$

Intuitively, since any element of G is in a coset of G/H, we let the coset act by permuting the components in the direct product of σW and the particular representative from H act on a component according to the action of H on W.

One way to work out the character of the induced representation is to relate it to other characters using the following reciprocity law.

Theorem 5.9 (Frobenius Reciprocity, [FH91, Cor. 3.20])**.** *Suppose φ is a character of a group G and ψ is a character of a subgroup H of G. Then*

$$(\varphi, \mathrm{Ind}_H^G \psi) = (\mathrm{Res}_H^G \varphi, \psi).$$

The crucial fact is that Artin L-functions are well-behaved under induction.

Theorem 5.10 (Artin)**.** *Let $K \subseteq M \subseteq L$ be a tower of extensions of number fields with $\mathrm{Gal}(L/K) = G$ and $\mathrm{Gal}(L/M) = H$. If χ is a character H, then*

$$L(s, \chi; L/M) = L(s, \mathrm{Ind}_H^G \chi; L/K).$$

The main tool in the proof will be Mackey's theorem, which describes explicitly the composition of restriction with induction.

Theorem 5.11 (Mackey's Theorem)**.** *If H and G' are subgroups of G, W is a representation of H, then for any representatives $\{\tau\}$ of the double cosets $G' \backslash G/H$, we have that*

$$\mathrm{Res}_{G'} \mathrm{Ind}_H^G W \cong \bigoplus_\tau \mathrm{Ind}_{G' \cap \tau H \tau^{-1}}^{G'} W^\tau$$

where W^τ is a representation of $\tau H \tau^{-1}$ given by $\rho_\tau(x) = \rho(\tau^{-1} x \tau)$.

For the proof of the Theorem, see [Ser77, Chap. 7].

To prove Theorem 5.10, we prove it for local factors of the Artin L-function.

Lemma 5.12. *Let $K \subseteq M \subseteq L$ be a tower of extensions of number fields with $\mathrm{Gal}(L/K) = G$ and $\mathrm{Gal}(L/M) = H$. Fix a prime $\mathfrak{p}$ of K. If χ is a character H, then*

$$\prod_{\mathfrak{q} | \mathfrak{p}} L_\mathfrak{q}(s, \chi; L/M) = L_\mathfrak{p}(s, \mathrm{Ind}_H^G \chi; L/K),$$

where the product is over finite primes $\mathfrak{q}$ of M above $\mathfrak{p}$.

Proof. Let us introduce some notation. Suppose χ is a character of a representation W of H and $V = \mathrm{Ind}_H^G W$. Moreover, suppose the primes above $\mathfrak{p}$ in M/K are $\mathfrak{q}_1, \dots, \mathfrak{q}_g$ and choose a prime $\mathfrak{P}_i$ above $\mathfrak{q}_i$ in L/M for each i. Moreover, we let $D_i = D_{\mathfrak{P}_i/\mathfrak{p}}$, the decomposition group, and $I_i = I_{\mathfrak{P}_i/\mathfrak{p}}$, the inertia group.

Since $\mathrm{Frob}_{\mathfrak{P}_1} \in D_1/I_1$, we have that

$$L_\mathfrak{p}(s, V; L/K) = \det(1 - \mathrm{Frob}_{\mathfrak{P}_1} N\mathfrak{p}^{-s}; V^{I_1})^{-1} = \det(1 - \mathrm{Frob}_{\mathfrak{P}_1} N\mathfrak{p}^{-s}; (\mathrm{Res}_{D_1} V)^{I_1})^{-1}.$$

We are now in a position to apply Mackey's Theorem 5.11 to $\mathrm{Res}_{D_1} V = \mathrm{Res}_{D_1} \mathrm{Ind}_H^G W$.

For that sake, we will choose representatives for the double cosets $D_1 \backslash G/H$. Fix any representatives $\{\sigma_i\}$ of $D_1 \backslash G$. They are determined by their value on $\mathfrak{P}_1$; indeed, if $\sigma_i \mathfrak{P}_1 = \sigma_j \mathfrak{P}_1$, then $\sigma_i \sigma_j^{-1} \mathfrak{P}_1 = \mathfrak{P}_1$ and hence $\sigma_i \sigma_j^{-1} \in D_1$. Moreover, G acts transitively on the primes $\mathfrak{P}$ above $\mathfrak{p}$ in L/K, and hence $\{\sigma_i\}$ does as well. To choose representatives of $D_1 \backslash G/H$, we have to choose representatives $\{\tau_j\}$ of $\{\sigma_i H\}$. Since σ_i determines $\sigma_i \mathfrak{P}_1$, a unique prime above $\mathfrak{p}$ in L/K, it also determines a unique prime $\sigma_i \mathfrak{q}_1$ above $\mathfrak{p}$ in M/K. However, H acts transitively on the primes $\mathfrak{P}$ above $\mathfrak{q}$ in L/M, so each representative τ_j is determined uniquely by its value on $\mathfrak{q}_1$. Therefore, we have g representatives $\{\tau_i\}_{i=1}^g$ of the double cosets and we may assume that $\tau_i \mathfrak{q}_i = \mathfrak{q}_1$. Since $\mathfrak{P}_i$ above $\mathfrak{q}_i$ were chosen arbitrarily, we choose them in a way that ensures $\tau_i \mathfrak{P}_i = \mathfrak{P}_1$ as well.

Therefore, Mackey's Theorem 5.11 yields

$$\mathrm{Res}_{D_1}\mathrm{Ind}_H^G W = \bigoplus_{i=1}^{g} \mathrm{Ind}_{D_1\cap\tau_i H\tau_i^{-1}}^{D_1} W^{\tau_i},$$

and hence

$$\begin{aligned} L_{\mathfrak{p}}(s,V;L/K) &= \prod_{i=1}^{g} \det\left(1-\mathrm{Frob}_{\mathfrak{P}_1} N\mathfrak{p}^{-s}; \left(\mathrm{Ind}_{D_1\cap\tau_i H\tau_i^{-1}}^{D_1} W^{\tau_i}\right)^{I_1}\right)^{-1} \\ &= \prod_{i=1}^{g} \det\left(1-\tau_i^{-1}\mathrm{Frob}_{\mathfrak{P}_1}\tau_i N\mathfrak{p}^{-s}; \left(\mathrm{Ind}_{\tau_i^{-1}D_1\tau_i\cap H}^{\tau_i^{-1}D_1\tau_i} W\right)^{\tau_i^{-1}I_1\tau_i}\right)^{-1}, \end{aligned}$$

by conjugating each term by τ_i^{-1}. We now recall that Proposition 1.15 shows that $\tau_i^{-1}D_1\tau_i = D_i$, $\tau_i^{-1}I_1\tau_i = I_i$, and $\tau_i^{-1}\mathrm{Frob}_{\mathfrak{P}_1}\tau_i = \mathrm{Frob}_{\mathfrak{P}_i}$. Hence:

$$L_{\mathfrak{p}}(s,V;L/K) = \prod_{i=1}^{g} \det\left(1-\mathrm{Frob}_{\mathfrak{P}_i} N\mathfrak{p}^{-s}; \left(\mathrm{Ind}_{D_i\cap H}^{D_i} W\right)^{I_i}\right)^{-1}$$

Finally, note that $D_i \cap H = H_i$ and $I_i \cap H = I_i'$ are respectively the decomposition and inertia groups of $\mathfrak{P}_i/\mathfrak{q}_i$ in the extension L/M. Therefore:

$$\prod_{i=1}^{g} L_{\mathfrak{q}_i}(s,W;L/M) = \prod_{i=1}^{g} \det\left(1-\mathrm{Frob}_{\mathfrak{P}_i/\mathfrak{q}_i} N\mathfrak{q}_i^{-s}; W^{I_i'}\right)^{-1}.$$

We have thus reduced the theorem to the case $g = 1$, i.e. the tower of extensions $L^{D_i} \subseteq L^{H_i} \subseteq L$. Explicitly, it is enough to show that

$$\det\left(1-\mathrm{Frob}_{\mathfrak{P}_i} N\mathfrak{p}^{-s}; (\mathrm{Ind}_{H_i}^{D_i} W)^{I_i}\right) = \det\left(1-\mathrm{Frob}_{\mathfrak{P}_i}^{f_i} N\mathfrak{q}_i^{-s}; W^{I_i'}\right) \tag{7}$$

where $f_i = f_{\mathfrak{q}_i/\mathfrak{p}}$, since $\mathrm{Frob}_{\mathfrak{P}_i}^{f_i} \subseteq \mathrm{Frob}_{\mathfrak{P}_i/\mathfrak{q}_i}$ by Proposition 1.16 and hence the two sets determine the same action.

The rest of the proof is a straightforward determinant calculation. The group D_i/I_i is cyclic generated by $\mathrm{Frob}_{\mathfrak{P}_i}$ and the group H_i/I_i' is cyclic generated by $\mathrm{Frob}_{\mathfrak{P}_i}^{f_i}$. Therefore, the quotient can be written as

$$(D_i/I_i)/(H_i/I_i') = \{1, \mathrm{Frob}_{\mathfrak{P}_i}, \ldots, \mathrm{Frob}_{\mathfrak{P}_i}^{f_i-1}\}.$$

Then, following Remark 5.8, we can write

$$\left(\mathrm{Ind}_{H_i}^{D_i} W\right)^{I_i} = \bigoplus_{j=0}^{f_i-1} \mathrm{Frob}_{\mathfrak{P}_i}^{j} W^{I_i'}$$

with the action of $\mathrm{Frob}_{\mathfrak{P}_i} \in D_i/I_i$ given by

$$\mathrm{Frob}_{\mathfrak{P}_i}(w_1, w_2, \ldots, w_{f_i}) = \left(\mathrm{Frob}_{\mathfrak{P}_i}^{j} w_{f_i}, w_1, w_2, \ldots, w_{f_i-1}\right).$$

Choose a basis for W and let A be the matrix representing $\mathrm{Frob}_{\mathfrak{P}_i}^{j} \in H_i/I_i'$ acting on $W^{I_i'}$. Then, by the discussion above, the matrix representing $\mathrm{Frob}_{\mathfrak{P}_i}$ acting on $\left(\mathrm{Ind}_{H_i}^{D_i} W\right)^{I_i}$ is given by

$$\begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Therefore:

$$\det\left(1 - \mathrm{Frob}_{\mathfrak{P}_i} N\mathfrak{p}^{-s}; (\mathrm{Ind}_{H_i}^{D_i} W)^{I_i}\right) = \det \begin{pmatrix} I & -N\mathfrak{p}^{-s} & 0 & \cdots & 0 \\ 0 & I & -N\mathfrak{p}^{-s} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -N\mathfrak{p}^{-s} \\ -AN\mathfrak{p}^{-s} & 0 & 0 & \cdots & I \end{pmatrix}.$$

We calculate the determinant by adding $N\mathfrak{p}^{-s}$ times the second row to the first row, then adding $N\mathfrak{p}^{-2s}$ times the second row to the first row, and continuing this way until we obtain

$$\det\left(1 - \mathrm{Frob}_{\mathfrak{P}_i} N\mathfrak{p}^{-s}; (\mathrm{Ind}_{H_i}^{D_i} W)^{I_i}\right) = \det \begin{pmatrix} I - AN\mathfrak{p}^{-f_i s} & 0 & 0 & \cdots & 0 \\ 0 & I & -N\mathfrak{p}^{-s} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -N\mathfrak{p}^{-s} \\ -AN\mathfrak{p}^{-s} & 0 & 0 & \cdots & I \end{pmatrix}.$$

Then we expand by the first row of the matrix to get

$$\det\left(1 - \mathrm{Frob}_{\mathfrak{P}_i} N\mathfrak{p}^{-s}; (\mathrm{Ind}_{H_i}^{D_i} W)^{I_i}\right) = \det(I - AN\mathfrak{p}^{-sf_i}) = \det(1 - \mathrm{Frob}_{\mathfrak{P}_i}^{f_i} N\mathfrak{q}_i^{-s}; W^{I_i'}),$$

where the last equality follows from $N\mathfrak{q}_i = N\mathfrak{p}^{f_i}$. This shows equation (7) holds and hence completes the proof. □

Proof of Theorem 5.10. We apply Lemma 5.12 to obtain

$$L(s, \mathrm{Ind}_H^G \chi; L/K) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \mathrm{Ind}_H^G \chi; L/K) = \prod_{\mathfrak{p}} \prod_{\mathfrak{q}|\mathfrak{p}} L_{\mathfrak{q}}(s, \chi; L/M) = L(s, \chi; L/M),$$

where $\mathfrak{p}$ are primes of K and $\mathfrak{q}$ are primes of M. □

5.3. **Brauer's Theorem and extending abelian L-functions.** We will now use Brauer's Theorem to show that we can use additivity and induction to express an Artin L-function as a product of abelian L-functions. Moreover, we will show that this factorization is unique.

Theorem 5.13 (Brauer's Theorem)**.** *Each character of G is a linear combination with integer coefficients of characters induced from 1-dimensional characters.*

For the proof of this theorem, see [Ser77, Chap. 10].

Remark 5.14. Artin himself was only able to prove the weaker statement that any character is a linear combination with rational coefficients of characters induced from 1-dimensional characters, know sometimes as the Artin Theorem.

Theorem 5.15. *An Artin L-function is a product of abelian L-functions.*

Proof. Suppose χ is a character of a representation of $\mathrm{Gal}(L/K)$, and let $\chi_1, \ldots, \chi_n$ be the 1-dimensional representations of G. Since χ_i are 1-dimensional representations, they are representations of abelian subgroups H_i of G. Then by Brauer's Theorem 5.13 we can express χ as a linear combination

$$\chi = \sum_{i=1}^{n} r_i \mathrm{Ind}_{H_i}^{G} \chi_i$$

with r_i non-negative integers. Then by Corollary 5.6 and Theorem 5.10, we have

$$L(s, \chi) = L\left(s, \sum_{i=1}^{n} r_i \mathrm{Ind}_{H_i}^{G} \chi_i\right) = \prod_{i=1}^{n} L(s, \mathrm{Ind}_{H_i}^{G} \chi_i)^{r_i} = \prod_{i=1}^{n} L(s, \chi_i)^{r_i}$$

and we have expressed $L(s, \chi)$ as a product of abelian L-functions. □

Note that this already yields a meromorphic continuation and functional equation for the Artin L-function; however, it does not give an explicit formulation. One thing we still have to ensure is that expressing χ as another linear combination does not change the product of abelian L-functions.

For a group G we will denote its group of characters by $\widehat{G}$, the vector space of all class functions on G by $\mathrm{Fun}_{\mathbb{C}}[G]$, and the subgroup generated by $\sigma \in G$ by H_σ.

Suppose we are given a collection $f_\sigma \colon \widehat{H_\sigma} \to \mathbb{C}$ for $\sigma \in G$ of additive functions. We want to find out when they determine a unique additive $\mathbb{C}$-linear function

$$f \colon \mathrm{Fun}_{\mathbb{C}}[G] \to \mathbb{C}$$

such that for any $\sigma \in G$ and any character $\chi \colon H_\sigma \to \mathbb{C}$ we have

$$f(\mathrm{Ind}_{H_\sigma}^{G} \chi) = f_g(\chi).$$

First, recall that we have an inner product on $\mathrm{Fun}_{\mathbb{C}}[H]$ for a finite group H given by

$$(\varphi, \psi) = \frac{1}{|H|} \sum_{\tau \in H} \varphi(\tau) \overline{\psi(\tau)}$$

and the irreducible characters of H form an orthonormal basis for $\mathrm{Fun}_{\mathbb{C}}[H]$ with respect to this inner product.

Lemma 5.16. *For a finite group H, the map $F \colon \mathrm{Fun}_{\mathbb{C}}[H] \to \mathrm{Fun}_{\mathbb{C}}[H]^*$ to the dual space $\mathrm{Fun}_{\mathbb{C}}[H]^*$ of $\mathrm{Fun}_{\mathbb{C}}[H]$ given by*

$$F(\psi)(\varphi) = (\varphi, \psi)$$

is a (semi-linear) isomorphism.

Proof. Clearly, the resulting map $F(\psi)$ is linear:

$$F(\psi)(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 (\varphi_1, \psi) + \lambda_2 (\varphi_2, \psi)$$

and F is semi-linear:

$$F(\lambda_1 \psi_1 + \lambda_2 \psi_2)(\varphi) = (\varphi, \lambda_1 \psi_1) + (\varphi, \lambda_2 \psi_2) = \overline{\lambda_1}(\varphi, \psi_1) + \overline{\lambda_2}(\varphi, \psi_2).$$

To show that F is injective, take any $\psi \in \mathrm{Fun}_{\mathbb{C}}[H]$ and suppose $F(\psi) = 0$. We have that $\psi = \sum_i \lambda_i \chi_i$ for some χ_i are irreducible characters of H, $\lambda_i \in \mathbb{C}$. However, for any i, we have

$$0 = F(\psi)(\chi_i) = (\chi_i, \psi) = \left(\chi_i, \sum_i \lambda_i \chi_i\right) = \lambda_i$$

by orthogonality of characters, and hence $\psi = 0$.

Finally, the dimensions of $\mathrm{Fun}_{\mathbb{C}}[H]$ and $\mathrm{Fun}_{\mathbb{C}}[H]^*$ are equal, so the injective semi-linear map must be an isomorphism. □

Using this lemma, we can restate the above problem in terms of class functions. Indeed, any function $f_\sigma \colon \widehat{H_\sigma} \to \mathbb{C}$ is given by $F(\psi_\sigma) = (-, \psi_\sigma)$ for some $\psi_\sigma \in \mathrm{Fun}_{\mathbb{C}}[H_\sigma]$ and the function f is given by $F(\psi) = (-, \psi)$ for some $\psi \in \mathrm{Fun}_{\mathbb{C}}[G]$. The following proposition provides a criterion for extending the collection of ψ_σ to a class function ψ.

Proposition 5.17. *A collection $\psi_\sigma \in \mathrm{Fun}_{\mathbb{C}}[H_\sigma]$ for all $\sigma \in G$ induces a unique class function $\psi \in \mathrm{Fun}_{\mathbb{C}}[G]$ such that*

$$\mathrm{Res}^G_{H_\sigma} \psi = \psi_\sigma$$

for all $g \in G$ exactly when:

1. $\mathrm{Res}^{H_\sigma}_{H_{\sigma^k}} \psi_\sigma = \psi_{\sigma^k}$ *for all integers k,*
2. $\psi_\sigma(\sigma) = \psi_{\tau\sigma\tau^{-1}}(\tau\sigma\tau^{-1})$ *for any $\sigma, \tau \in G$.*

Proof. First, suppose that a function ψ extends the functions ψ_σ. Then

$$\psi(\sigma) = \mathrm{Res}^G_{H_\sigma} \psi(\sigma) = \psi_\sigma(\sigma).$$

This proves uniqueness of ψ; we will have to prove that ψ defined by the above equation is a well-defined class function that extends the functions ψ_σ. The second condition ensures that ψ is a class function; indeed for any $\sigma \in G$ and $\tau \in G$ we have

$$\psi(\tau\sigma\tau^{-1}) = \psi_{\tau\sigma\tau^{-1}}(\tau\sigma\tau^{-1}) = \psi_\sigma(\sigma) = \psi(\sigma).$$

Moreover, the first condition shows that $\mathrm{Res}^G_{H_\sigma} \psi = \psi_\sigma$, since for any $\sigma^k \in H_\sigma$

$$\psi(\sigma^k) = \psi_{\sigma^k}(\sigma^k) = \mathrm{Res}^{H_\sigma}_{H_{\sigma^k}} \psi_\sigma(\sigma^k) = \psi_\sigma(\sigma^k),$$

so the function ψ extends the functions ψ_σ. □

We now know that $f_\sigma = (-, \psi_\sigma)$ and $f = (-, \psi)$ for the ψ extending ψ_σ. We check that this function f indeed satisfies the desired property: for any $\sigma \in G$ and any character $\chi \in \widehat{H_\sigma}$ we have

$$f(\mathrm{Ind}^G_{H_\sigma} \chi) = (\mathrm{Ind}^G_{H_\sigma} \chi, \psi) = (\chi, \mathrm{Res}^G_{H_\sigma} \psi) = (\chi, \psi_\sigma) = f_\sigma(\chi),$$

where we use the Frobenius Reciprocity 5.9.

We now turn to the case of Artin L-functions.

Theorem 5.18. *The Artin L-function is the unique function that extends the abelian L-functions for characters χ_σ of the cyclic subgroups H_σ by additivity and induction.*

The proof of the theorem is technical and notationally difficult, but it is essentially a straightforward application of the previous results of this section, particularly, Proposition 5.17.

Proof. First, note that the abelian L-functions are multiplicative, not additive, so for $G = \mathrm{Gal}(L/K)$, $\sigma \in G$ and $\chi \in \widehat{H_\sigma}$, we will be considering the additive functions

$$\widehat{L}(s,\chi) = \log L(s,\chi; L/L^{H_\sigma}).$$

We first claim that for

$$\psi_\sigma(\tau) = |H_\sigma| \sum_{\mathfrak{p}} \sum_{j \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{p}}^j} j^{-1} N\mathfrak{p}^{-\bar{s}j},$$

where the first sum ranges over primes of L^{H_σ} and the second sum ranges over j from 1 to ∞ such that $\tau \in \mathrm{Frob}_{\mathfrak{p}}^j$, we have that

$$\widehat{L}(s,\chi) = (\chi, \psi_\sigma).$$

Indeed:

$$\begin{aligned}
(\chi,\psi_\sigma) &= \tfrac{1}{|H_g\sigma|} \sum_{\tau \in H_\sigma} \chi(\tau)\overline{\psi_\sigma(\tau)} \\
&= \sum_{\mathfrak{p}} \sum_{\tau \in H_\sigma} \sum_{j \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{p}}^j} \chi(\tau) j^{-1} N\mathfrak{p}^{-sj} \\
&= \sum_{\mathfrak{p}} \sum_{j=1}^{\infty} \tfrac{\chi(\mathrm{Frob}_{\mathfrak{p}}^j)}{j} N\mathfrak{p}^{-sj} \\
&= \widehat{L}(s,\chi)
\end{aligned}$$

where sums are over primes $\mathfrak{p}$ of L^{H_g} and the last equality follows from Proposition 5.5. Therefore, we only have to check that ψ_g satisfies the two assumptions of Proposition 5.17.

Let us first show (1) holds, i.e. for any $\tau \in H_{\sigma^k}$ we have that

$$\psi_\sigma(\tau) = \psi_{\sigma^k}(\tau).$$

In what follows, we let $\mathfrak{p}$ range over all primes of L^{H_σ} and $\mathfrak{q}$ range over all primes of $L^{H_{\sigma^k}}$. Suppose $\mathfrak{p}$ factors in $L^{H_{\sigma^k}}/L^{H_\sigma}$ as

$$\mathfrak{p} = \mathfrak{q}_1^{e_{\mathfrak{p}}} \dots \mathfrak{q}_n^{e_{\mathfrak{p}}}$$

and recall that $N\mathfrak{q}_i = N\mathfrak{p}^{f_{\mathfrak{p}}}$, $\mathrm{Frob}_{\mathfrak{q}_i} = \mathrm{Frob}_{\mathfrak{p}}^{f_{\mathfrak{p}}}$. We take $\tau \in H_{\sigma^k}$ and transform $\psi_\sigma(\tau)$ to show it is equal to $\psi_{\sigma^k}(\tau)$.

$$\begin{aligned}
\psi_\sigma(\tau) &= |H_\sigma| \sum_{\mathfrak{p}} \sum_{j \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{p}}^j} j^{-1} N\mathfrak{p}^{-\bar{s}j} \\
&= |H_\sigma| \sum_{\mathfrak{p}} \sum_{i=1}^{g_{\mathfrak{p}}} e_{\mathfrak{p}}^{-1} \sum_{j \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{q}_i}^{f_{\mathfrak{p}} j}} j^{-1} N\mathfrak{p}^{-\bar{s}j} \\
&= |H_\sigma| \sum_{\mathfrak{p}} \sigma_{\mathfrak{p}}^{-1} e_{\mathfrak{p}}^{-1} \sum_{m \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{q}_i}^{m}} f_{\mathfrak{p}}^{-1} m^{-1} N\mathfrak{q}_i^{-\bar{s}m} && \text{substituting } m = f_{\mathfrak{p}} j \\
&= |H_{\sigma^k}| \sum_{\mathfrak{q}} \sum_{m \text{ s.t. } \tau \in \mathrm{Frob}_{\mathfrak{q}}^{m}} m^{-1} N\mathfrak{q}^{-\bar{s}m} && \text{by Corollary 1.8} \\
&= \psi_{\sigma^k}(\tau)
\end{aligned}$$

which shows (1).

To finish the proof, we have to show (2), i.e. for any $\sigma, \tau \in G$ we have that

$$\psi_\sigma(\sigma) = \psi_{\tau\sigma\tau^{-1}}(\tau\sigma\tau^{-1})$$

Note that $H_{\tau\sigma\tau^{-1}} = \tau H_\sigma \tau^{-1}$ and hence the primes $\mathfrak{p}$ of L^{H_σ} correspond to the primes $\tau\mathfrak{p}$ of $L^{H_{\tau\sigma\tau^{-1}}}$. Moreover, the Frobenius satisfies $\mathrm{Frob}_{\tau\mathfrak{p}} = \tau^{-1}\mathrm{Frob}_{\mathfrak{p}}\tau$ by Proposition 1.15, so $\tau \in \mathrm{Frob}_{\mathfrak{p}}$ if and only if $\tau^{-1}\sigma\tau \in \mathrm{Frob}_{\mathfrak{p}}$, and we are done. □

5.4. **Factorization of Artin ζ-function.** We have now enough information to generalize Weber's Theorem 4.5 about the factorization of the ζ-function in terms of L-functions.

Theorem 5.19 (Artin)**.** *If L/K is a Galois extension of number fields, then*

$$\zeta_L(s) = \prod_\chi L(s,\chi)^{\chi(1)},$$

the product ranging over all irreducible characters of $\mathrm{Gal}(L/K)$*.*

Proof. Since Artin L-functions extend abelian L-functions (Proposition 5.2), we have that

$$\zeta_L(s) = L(s, \chi_0; L/L)$$

for the trivial character χ_0. By the induction formula (Theorem 5.10), we see that

$$\zeta_L(s) = L(s, \mathrm{Ind}_{\{1\}}^G \chi_0, L/K).$$

Finally, by Frobenius Reciprocity 5.9, for any irreducible character χ, we have that

$$(\chi, \mathrm{Ind}_{\{1\}}^G \chi_0)_G = (\mathrm{Res}_{\{1\}}^G \chi, \chi_0)_{\{1\}} = \chi(1),$$

so

$$\mathrm{Ind}_{\{1\}}^G \chi_0 = \sum_\chi \chi(1)\chi.$$

By additivity (Corollary 5.6), we obtain

$$\zeta_L(s) = L\left(s, \sum_\chi \chi(1)\chi; L/K\right) = \prod_\chi L(s,\chi)^{\chi(1)}$$

which completes the proof. □

First, note that this agrees with the abelian case (Weber's Theorem 4.5), since $\chi(1) = \dim\chi = 1$, if χ is a character of an abelian group.

Moreover, note that this means that the quotient $\frac{\zeta_L(s)}{\zeta_K(s)}$ is a meromorphic function. However, we could previously conclude that $\frac{\zeta_L(s)}{\zeta_K(s)}$ is a holomorphic function, but this is not any more the case: we only know that $L(s,\chi)$ is a meromorphic function. Artin conjectured that $L(s,\chi)$ is a holomorphic function for any χ, and this is still an open problem today.

5.5. **Local Factors at Infinite Primes.** We now turn to describing the local factors of the completed Artin L-function that extend the local factors appearing in the completed abelian L-function (see Definition 2.10 and Theorem 2.11).

Recall from Definition 2.10 that:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$
$$\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s).$$

Suppose v is an infinite prime of K divisible by an infinite prime w of L. By definition D_w, the decomposition group of w over v, is either trivial (if v and w are both real or both complex) or complex conjugation (if v is real and w is complex). Therefore, for a 1-dimensional character χ of $\mathrm{Gal}(L/K)$, we can restate the Definition 2.10 as follows:

$$L_v(s,\chi) = \begin{cases} \Gamma_{\mathbb{C}}(s) & \text{if } v \text{ is complex,} \\ \Gamma_{\mathbb{R}}(s) & \text{if } v \text{ is real and } \mathrm{Res}_{D_w}\chi \text{ is trivial,} \\ \Gamma_{\mathbb{R}}(s+1) & \text{if } v \text{ is real and } \mathrm{Res}_{D_w}\chi \text{ is non-trivial.} \end{cases}$$

We want to extend this definition to the non-abelian case. So let L/K be a Galois extension, V be a representation of $\mathrm{Gal}(L/K)$ with character χ, and $w|v$ be infinite primes. We are only interested in the local factor, so the definition should only depend on $\mathrm{Res}_{D_w}V$ (as in the abelian case). Since D_w only has two elements, its two irreducible characters are the trivial one χ_0 and the sign one χ_-, so we can write

$$\mathrm{Res}_{D_w}\chi = (\mathrm{dim}V^{D_w})\chi_0 + (\mathrm{codim}V^{D_w})\chi_-,$$

To ensure additivity, it is natural to use the following definition.

Definition 5.20. If v is an infinite prime, then we define the *local factor* of the Artin L-series at v to be

$$L_v(s,V) = \begin{cases} \Gamma_{\mathbb{C}}(s)^{\chi(1)} & \text{if } v \text{ is complex,} \\ \Gamma_{\mathbb{R}}(s)^{\mathrm{dim}V^{D_w}}\Gamma_{\mathbb{R}}(s+1)^{\mathrm{codim}V^{D_w}} & \text{if } v \text{ is real.} \end{cases}$$

We still have to check that these local factors behave well under induction.

Theorem 5.21. *Let $K \subseteq M \subseteq L$ be a tower of extensions of number fields with* $\mathrm{Gal}(L/K)$ *and* $\mathrm{Gal}(L/M) = H$. *If χ is a character of H and v is an infinite prime of K, then*

$$\prod_{w|v} L_w(s,\chi;L/M) = L_v(s,\mathrm{Ind}_H^G\chi;L/K),$$

where the product is over infinite primes w of M above v.

Proof. The proof is similar to the proof of the corresponding theorem for finite primes, Theorem 5.10. Let W be a representation of H and $V = \mathrm{Ind}_H^G W$. Fix some infinite prime v and suppose it factors in M as $w_1w_2\dots w_g$. For each i, choose some $u_i \in L$, sitting over w_i. Let D_i denote the decomposition group of u_i/v. As before, we can take coset representatives $\{\tau_i\}$ of $D_1\backslash G/H$ such that $\tau_i u_i = u_1$. Then $D_i = \tau_i^{-1}D_1\tau_i$ and $H_i = D_i \cap H$ is the decomposition group for u_i/w_i.

We apply Mackey's Theorem 5.11 to obtain

$$\mathrm{Res}_{G_1}V \cong \bigoplus_{i=1}^{g}\left(\mathrm{Ind}_{D_1\cap\tau_i H\tau_i^{-1}}^{D_1} W^{\tau_i}\right).$$

The character on the direct sum of spaces is the sum of characters on the components. Moreover, the dimension of the trivial component is not affected by conjugation by elements of the Galois group. Thus, whether v is complex or real, we obtain:

$$L_v(s, V; L/K) = \prod_{i=1}^{g} L_v(s, \mathrm{Ind}_{H_i}^{D_i} W; L/L^{D_i})$$

and again we have reduced the statement to the case $g = 1$. We thus omit the indices i in the next part of the proof.

We have a tower $u|w|v$ of infinite primes. If v, w are both complex or both real, then $H = D$ and the induction is trivial. Thus we may assume that v is real and w is complex, so u is also complex. Since H is trivial, χ is some multiple of the trivial character χ_0, and consequently $V = \mathrm{Ind}_H^G W$ has character $\dim(W)\mathrm{Ind}_H^G \chi_0$. Therefore, $\mathrm{Ind}_H^G \chi = \dim(W)\chi_0 + \dim(W)\chi_-$. We have that:

$$L_v(s, \mathrm{Ind}_H^G \chi; L/K) = \Gamma_{\mathbb{R}}(s)^{\dim(W)} \Gamma_{\mathbb{R}}(s+1)^{\dim(W)} = \Gamma_{\mathbb{C}}(s)^{\dim(W)} = \Gamma_{\mathbb{C}}(s)^{\chi(1)} = L_w(s, \chi; L/M).$$

This shows the desired equality in the case $g = 1$, which completes the proof. □

5.6. **The Artin Conductor.** Recall that in the statement of Theorem 2.11, there was an *exponential factor*, namely

$$(|\Delta| N\mathfrak{f}(\chi)_0)^{s/2}.$$

In order to extend the theorem to Artin L-functions, we first need to generalize this factor to the non-abelian case. Therefore, we need a non-abelian equivalent of a conductor, the Artin conductor. In this section, we follow [Ser79, Chap. VI].

5.6.1. *Higher Ramification Groups.* Recall that in Section 1 we introduced the decomposition and the inertia group which provided a way of describing the ramification of a prime in an extension. We will study ramification further by introducing the so-called higher ramification groups. Since we are interested in how a prime ramifies, we will temporarily restrict our attention to the local case and consider the completion of the field under a valuation.

So let K be a field which is complete under the discrete valuation v_K and let $A = \{x \in K \mid v_K(x) \geq 0\}$ be its valuation ring. Then K is a local field, so let $\mathfrak{p}$ be the unique maximal ideal of K, and $\overline{A} = A/\mathfrak{p}$ be the residue field. Consider a finite Galois extension L of K which is complete under the discrete valuation v_L and let $B = \{x \in L \mid v_L(x) \geq 0\}$, $\mathfrak{P}$ be its maximal ideal, and $\overline{B} = B/\mathfrak{P}$ be the residue field.

Let $G = \mathrm{Gal}(L/K)$. Note that $G = D_{\mathfrak{P}}$, since $\mathfrak{P}$ is the only prime of L and G preserves the set of primes of L above $\mathfrak{p}$.

There exists an $x \in B$ such that x generates B as an A-algebra.

Lemma 5.22. *Let $\sigma \in G$ and $i \geq -1$ be an integer. The following are equivalent:*

(1) *σ acts trivially on $B/\mathfrak{P}^{i+1}$,*
(2) *$v_L(\sigma(b) - b) \geq i + 1$ for all $b \in B$,*
(3) *$v_L(\sigma(x) - x) \geq i + 1$.*

Proof. The equivalence of (1) and (2) is clear: σ acts trivially on $B/\mathfrak{P}^{i+1}$ if an only if $\sigma(b) = b \bmod \mathfrak{P}^{i+1}$ for any $b \in B$ if and only if $\sigma(b) - b$ is divisible by $\mathfrak{P}^{i+1}$, or in other words $v_L(\sigma(b) - b) \geq i + 1$.

To see that (1) and (3) are equivalent, note that the image x_i of x in $B_i = B/\mathfrak{P}^{i+1}$ generates B_i as an A-algebra. Then $\sigma(x_i) = x_i$ if and only if σ acts trivially on all of B_i, so we are done. ☐

Definition 5.23. For $i \geq -1$, the *ith ramification group* is

$$G_i = \{\sigma \in G \mid v_L(\sigma(x) - x) \geq i + 1\}$$

(or the other equivalent conditions from Lemma 5.22).

We note that these groups indeed agree with the decomposition and interia groups for $i = -1$ and 0, respectively, and define a filtration of G.

Proposition 5.24. *The groups G_i form a decreasing sequence of normal subgroups of G with $G_{-1} = G$, $G_0 = I$, the inertia subgroup of G, and $G_i = \{1\}$ for sufficiently large i. Moreover, $G/G_0 \cong \operatorname{Gal}\left(\overline{B}/\overline{A}\right)$.*

Proof. To see that $G = I$, recall that by Lemma 5.22

$$G_0 = \left\{\sigma \in G \mid \sigma \text{ acts trivially on } \overline{B}\right\}$$

and I was defined to be the kernel of the homomorphism

$$\epsilon \colon G \to \operatorname{Gal}\left(\overline{B}/\overline{A}\right).$$

This homomorphism is surjective by Proposition 1.11, and hence $G/G_0 \cong \operatorname{Gal}\left(\overline{B}/\overline{A}\right)$ ☐

We will now define a function which will allow us to restate the definition of a ramification group.

Definition 5.25. We define the function $i_G \colon G \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by

$$i_G(\sigma) = v_L(\sigma(x) - x)$$

for $\sigma \neq 1$ and $i_G(1) = \infty$.

Proposition 5.26.

(1) *$i_G(\sigma) \geq i + 1$ if and only if $\sigma \in G_i$,*
(2) *$i_G(\tau\sigma\tau^{-1}) = i_G(\sigma)$,*
(3) *$i_G(\sigma\tau) \geq \inf(i_G(\sigma), i_G(\tau))$.*

Proof. Properties (1) and (2) are immediate. For (3), note that using (1) we can write

$$i_G(\sigma) = \inf\{i + 1 \mid \sigma \in G_i\}$$

and hence

$$i_G(\sigma\tau) = \inf\{i + 1 \mid \sigma\tau \in G_i\} \geq \inf\{i + 1 \mid \sigma \in G_i \text{ and } \tau \in G_i\} \geq \inf(i_G(\sigma), i_G(\tau))$$

as requested. ☐

Proposition 5.27. *Let H be a subgroup of G and $M = L^H$ so that $\operatorname{Gal}(L/M) = H$. Then for any $\sigma \in H$ we have $i_G(s) = i_H(s)$ or, equivalently, $H_i = G_i \cap H$.*

Proof. Let C be the valuation ring of M and $\mathfrak{q}$ be the maximal ideal of M. Recall that $\sigma \in H_i$ if and only if $\sigma \in H$ and σ acts trivially on $C/\mathfrak{q}^{i+1}$ which is equivalent to $\sigma \in H$ and $\sigma \in G_i$. $\square$

Corollary 5.28. *Let K_r be the largest unramified subextension of L over K and $H = \mathrm{Gal}(L/K_r)$. Then $H = G_0$, the inertia group, and $G_i = H_i$ for $i \geq 0$.*

Proof. We already proved that $H = G_0$ in Proposition 1.13. To finish the proof, we simply apply Proposition 5.27. $\square$

Proposition 5.29. *Suppose furthermore that H is a normal subgroup of G, i.e. M is a Galois extension of K. For any $\sigma \in G$, we have*

$$i_{G/H}(\sigma H) = \frac{1}{e'} \sum_{\tau \in H} i_G(\sigma\tau),$$

where $e' = e_{\mathfrak{P}/\mathfrak{q}}$, the ramification degree of the extension L/M.

Proof. For $\sigma = 1$, both sides of the equation are equal to ∞. So suppose $\sigma \neq 1$ and let x be an A-generator of B, y be an A-generator of C, the valuation ring of M. By definition, we have that

$$i_{G/H}(\sigma H) \cdot e' = v_M(\sigma(y) - y) \cdot e' = v_L(\sigma(y) - y).$$

To complete the proof, we will show that the elements

$$a = \sigma(y) - y,$$
$$b = \prod_{\tau \in H} (\sigma\tau(x) - x)$$

generate the same ideal in B; in that case, we will have proved that

$$v_L(\sigma(y) - y) = \sum_{\tau \in H} i_G(\sigma\tau).$$

First, we will show that a divides b. Let $f \in C[X]$ be the minimal polynomial of x over M,

$$f(x) = \prod_{\tau \in H} (X - \tau(x)),$$

and $\sigma(f) \in B[X]$ be the polynomial obtained by transforming the coefficients of f by σ,

$$\sigma(f)(X) = \prod_{\tau \in H} (X - \sigma\tau(x)).$$

Since f has coefficients in C, generated by y, and $\sigma(f)$ has coefficients in σC, generated by $\sigma(y)$, the coefficients of $\sigma(f) - f$ are divisible by $a = \sigma(y) - y$. So substituting $X = x$, we obtain that a divides $\sigma(f)(x) - f(x) = \sigma(f(x)) = \pm b$.

Now, we will now show that b divides a. Write y as a polynomial in x with coefficients in A, i.e. $y = g(x)$ for $g(X) \in A[X]$. Then $g(X) - y$ has coefficients in C and x as a root, so it is divisible by the minimal polynomial f of x over C:

$$g(X) - y = f(X)h(X)$$

for some $h \in C[X]$. Therefore, since $\sigma(g) = g$:

$$g(X) - \sigma(y) = \sigma(g)(X) - \sigma(y) = \sigma(f)(X)\sigma(h)(X)$$

and, substituting $X = x$, we obtain

$$-a = y - \sigma(y) = g(x) - \sigma(y) = \sigma(f)(x)\sigma(h)(x) = (\pm b)\sigma(h)(x).$$

Hence b divides a, which completes the proof. □

Corollary 5.30. *If furthermore $H = G_j$ for some $j \geq 0$, then $(G/H)_i = G_i/H$ for $i \leq j$ and $(G/H)_i = \{1\}$ for $i \geq j$.*

We apply a similar method to prove the following proposition, which will be useful later.

Proposition 5.31. *If $\mathfrak{d} = \mathfrak{d}_{L/K}$ is the different of L/K, then*

$$v_L(\mathfrak{d}) = \sum_{\sigma \neq 1} i_G(\sigma).$$

Proof. Let x be an A-generator of B and f be its minimal polynomial over K. Then by Proposition 1.24, $\mathfrak{d}$ is generated by $f'(x)$. But

$$f(X) = \prod_{\sigma \in G} (X - \sigma(x)),$$

so

$$f'(x) = \prod_{\sigma \neq 1} (x - \sigma(x)),$$

and hence

$$v_L(\mathfrak{d}) = v_L(f'(x)) = \sum_{\sigma \neq 1} i_G(\sigma)$$

by definition of i_G. □

While the way we numbered the ramification group is intuitive to define, it is not always the most convenient numbering to work with. While it is preserved under taking subgroups (Proposition 5.27), it does not behave well under quotients. We will introduce a different way of numbering the ramification groups that will make working with quotients easier.

For $u \in \mathbb{R}$, $u \geq -1$, we define $G_u = G_i$ where i is the smallest integer larger or equal to u. In other words, $\sigma \in G_u$ if and only if $i_G(\sigma) \geq u + 1$. For $t \geq -1$, we generalize the notion of the index of a subgroup to

$$(G_0 : G_t) = \begin{cases} (\#G_{-1}/G_0)^{-1} = \#G_0/\#G_{-1} & \text{for } t = -1, \\ 1 & \text{for } -1 \leq t \leq 0, \\ \#G_0/G_t & \text{for } t \geq 0. \end{cases}$$

We then set

$$\varphi(u) = \int_0^u \frac{dt}{(G_0 : G_t)}$$

for $u \geq -1$. Note that $\varphi(u) = u$ for $u \in [-1, 0]$. Moreover, if we let $g_i = \#G_i$, then for $m \leq u \leq m + 1$ we can explicitly write

$$\varphi(u) = \frac{1}{g_0}(g_1 + \ldots + g_m + (u - m)g_{m+1}),$$

and in particular

$$\varphi(m) + 1 = \frac{1}{g_0} \sum_{i=0}^{m} g_i.$$

Proposition 5.32.

(1) *φ is continuous, piecewise-linear, increasing, and concave.*
(2) *$\varphi(0) = 0$.*
(3) *If φ'_l, φ'_r denote the left and right derivatives respectively, then*

$$\varphi'_l(u) = \varphi'_r(u) = \frac{1}{(G_0 : G_u)} \quad \text{for } u \notin \mathbb{Z},$$

$$\varphi'_l(u) = \frac{1}{(G_0 : G_u)}, \ \varphi'_r(u) = \frac{1}{(G_0 : G_{u+1})} \quad \text{for } u \in \mathbb{Z}.$$

Proof. All the assertions are clear from the definition of φ. □

In fact, properties (1)–(3) could be used to characterize the function φ. Since φ is increasing, it has an inverse function ψ.

Proposition 5.33. *Let ψ be the inverse function to φ. Then*

(1) *ψ is continuous, piecewise-linear, increasing, and convex.*
(2) *$\psi(0) = 0$.*
(3) *If $v = \varphi(u)$, then $\psi'_l(v) = \frac{1}{\varphi'_l(u)}$ and $\psi'_r(v) = \frac{1}{\varphi'_r(u)}$. In particular, they are integers.*
(4) *If $v \in \mathbb{Z}$, then $u = \psi(v) \in \mathbb{Z}$.*

Proof. The assertion (1)–(3) are clear. For (4), let $m \in \mathbb{Z}$ be such that $m \leq u \leq m + 1$. Then

$$g_0 v = g_1 + g_2 + \cdots + g_m + (u - m) g_{m+1},$$

so

$$u = \frac{1}{g_{m+1}} (m g_{m+1} - g_1 - \cdots - g_m + g_0 v) \in \mathbb{Z},$$

since $g_{m+1} = \#G_{m+1}$ divides $g_0 = \#G_0, g_1 = \#G_1, \ldots, g_m = \#G_m, g_{m+1} = \#G_{m+1}$. □

Definition 5.34. We define the *upper numbering* of the ramification groups by

$$G^v = G_{\psi(v)},$$

or equivalently

$$G^{\varphi(u)} = G_u.$$

We then have the immediate consequences of Proposition 5.24: $G^{-1} = G$, $G^0 = G_0$ and $G^v = \{1\}$ for sufficiently large v.

As we claimed earlier, the upper number behaves well under taking quotients.

Proposition 5.35. *Let H be a normal subgroup of G. Then*

$$(G/H)^v = \frac{G^v H}{H}.$$

Before the proof of the proposition, we state and prove four lemmas.

Lemma 5.36. *For any $u \geq -1$:*

$$\varphi(u) = \frac{1}{g_0} \sum_{\sigma \in G} \inf(i_G(\sigma), u+1) - 1.$$

Proof. The right hand side of the equation is a continuous, piecewise linear function, whose value at $u = 0$ is 0, and its derivative coincides with the derivative of φ. □

Lemma 5.37. *Let $\sigma \in G$ and set $j(\sigma H) = \sup\{i_G(\sigma\tau) \mid \tau \in H\}$. Then*

$$i_{G/H}(\sigma H) - 1 = \varphi_{L/M}(j(\sigma H) - 1),$$

where $\varphi_{L/M}$ is the function φ corresponding to the group $H = \mathrm{Gal}(L/M)$.

Proof. Set $\sigma \in G$ such that $i_G(\sigma) = j(\sigma H)$, and let $m = i_G(\sigma)$. Take any $\tau \in H$. If $\tau \in H_{m-1}$, then $i_G(\tau) \geq m$, so $i_G(\sigma\tau) \geq m$, so $i_G(\sigma\tau) = m$. If $\tau \notin H_{m-1}$, then $i_G(\tau) < m$ and $i_G(\sigma\tau) = i_G(\tau)$. In either case, $i_G(\sigma\tau) = \inf(i_G(\tau), m)$. Thus Proposition 5.29 yields

$$i_{G/H}(\sigma H) = \frac{1}{e'} \sum_{\tau \in H} \inf(i_G(\tau), m).$$

But $e' = \#H_0$ by Proposition 5.24 and $i_G(\tau) = i_H(\tau)$ by Proposition 5.27. Therefore, by Lemma 5.36 we have that

$$i_{G/H}(\sigma H) = 1 + \varphi_{L/M}(m-1),$$

which completes the proof. □

Lemma 5.38 (Herbrand's Theorem)**.** *If $v = \varphi_{L/M}(u)$, then*

$$G_u H/H = (G/H)_v.$$

Proof. Take any $\sigma \in G$. Then $\sigma H \in G_u H/H$ is equivalent to $j(\sigma H) - 1 \geq u$ and taking $\varphi_{L/M}$ of both sides, this is equivalent to

$$\varphi_{L/M}(j(\sigma H) - 1) \geq \varphi_{L/M}(u).$$

We now apply Lemma 5.37 to conclude that this is equivalent to

$$i_{G/H}(\sigma H) - 1 \geq \varphi_{L/M}(u)$$

which means that $\sigma H \in (G/H)_v$. □

Lemma 5.39. *The functions φ and ψ satisfy*

$$\varphi_{L/K} = \varphi_{M/K} \circ \varphi_{L/M},$$
$$\psi_{L/K} = \psi_{L/M} \circ \psi_{M/K}.$$

Proof. Let $u > -1$, $u \notin \mathbb{Z}$, and $v = \varphi_{L/M}$. Then

$$(\varphi_{M/K} \circ \varphi_{L/M})'(u) = \varphi'_{M/K}(v) \cdot \varphi'_{L/M}(u) = \frac{\#(G/H)_v}{e_{M/K}} \cdot \frac{\#H_u}{e_{L/M}}$$

and the result for φ follows from Hebrand's Theorem 5.38 and $e_{M/K} \cdot e_{L/M} = e_{L/K}$. The result for ψ is now clear, since ψ is the inverse of φ. □

We can finally prove the proposition.

Proof of Proposition 5.35. For $x = \psi_{M/K}(v)$, by definition,

$$(G/H)^v = (G/H)_x$$

and hence, by Herbrand's Theorem 5.38,

$$(G/H)^v = (G/H)_x = G_w H/H$$

for $w = \psi_{L/M}(x) = \psi_{L/M}(\psi_{M/K}(v)) = \psi_{L/K}(v)$ by Lemma 5.39. But $G_w = G^v$ by definition, so

$$(G/H)^v = G^v H/H$$

for all v. □

Theorem 5.40 (Hasse–Arf)**.** *Let G be an abelian group. If v is a jump in the filtration G^v, then $v \in \mathbb{Z}$. In other words, $G_i \neq G_{i+1}$ implies that $\varphi(i) \in \mathbb{Z}$.*

The proof can be found in [Ser79, Chap. V §7].

5.6.2. *The Artin Representation.* We first state a version of Frobenius Reciprocity 5.9 for quotients. Let H be a normal subgroup of G. Given any representation of G/H, we can view it as a representation of G. Given any representation of G with character χ, we can define a representation of G/H with character $\chi^\natural$ by taking the average of values of χ on the preimages; explicitely:

$$\chi^\natural(gH) = \frac{1}{\#H}\sum_{h\in H}\chi(gh).$$

The Frobenius Reciprocity 5.9 becomes.

Theorem 5.41 (Frobenius Reciprocity for quotients)**.** *Let H be a normal subgroup of G. If φ is a class function on G and ψ is a class function on G/H, then*

$$(\varphi, \psi)_G = (\varphi^\natural, \psi)_{G/H}$$

We keep the assumptions and notation from the previous section. Let $f = f_{\mathfrak{P}/\mathfrak{p}}$ be the residue field degree. We define the following function:

$$a_G(\sigma) = \begin{cases} -f \cdot i_G(\sigma) & \text{for } \sigma \neq 1 \\ f \sum\limits_{\sigma\neq 1} i_G(\sigma) & \text{for } \sigma = 1 \end{cases}$$

so that $(a_G, 1_G) = 0$.

The rest of this section will be devoted to proving the following theorem.

Theorem 5.42. *The function a_G is a character of a representation of G.*

If we define $f(\varphi) = (\varphi, a_G)$ for any class function φ and since $a_G(\sigma^{-1}) = a_G(\sigma)$, the theorem is equivalent to proving that $f(\chi)$ is a non-negative integer for any character χ. Before doing that, we need to explore some properties of a_G and f.

Proposition 5.43. *For the inertia group G_0, we have that*

$$a_G = \mathrm{Ind}_{G_0}^G a_{G_0}.$$

Proof. Recall that G_0 is a normal subgroup of G and

$$\mathrm{Ind}_{G_0}^G a_{G_0}(\sigma) = \sum_{\tau G_0 \in G/G_0} a_{G_0}(\tau\sigma\tau^{-1})$$

where we set $a_{G_0}(\tau\sigma\tau^{-1}) = 0$ for $\tau\sigma\tau^{-1} \notin G_0$. Therefore, for $\sigma \notin G_0$, we have that

$$\mathrm{Ind}_{G_0}^G a_{G_0}(\sigma) = 0 = -f \cdot i_G(\sigma) = a_G(\sigma).$$

Moreover, for $\sigma \in G_0$, $\sigma \neq 1$, we have that

$$\mathrm{Ind}_{G_0}^G a_{G_0}(\sigma) = \sum_{\tau G_0 \in G/G_0} a_{G_0}(\tau\sigma\tau^{-1}) = -f \cdot \sum_{\tau G_0 \in G/G_0} i_{G_0}(\tau\sigma\tau^{-1}) = -f \cdot i_G(s) = a_G(s)$$

Finally:

$$\mathrm{Ind}_{G_0}^G a_{G_0}(1) = \sum_{\tau G_0 \in G/G_0} a_{G_0}(\tau 1\tau^{-1}) = \sum_{\tau G_0 \in G/G_0} a_{G_0}(1) = \#G/G_0 \cdot \sum_{\sigma \neq 1} i_{G_0}(s) = a_G(1),$$

since $\#G/G_0 = f$. □

Proposition 5.44. *Let G_i be the ith ramification group and u_i be the character of the augmentation representation of G_i, i.e. $u_i = 1 - r_{G_i}$, where r_{G_i} is the character of the regular representation ($r_{G_i}(1) = g_i$, $r_{G_i}(\sigma) = 0$ for $\sigma \neq 1$). Then*

$$a_G = \sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \mathrm{Ind}_{G_i}^G u_i.$$

Proof. Let $g_i = \#G_i$. Then

$$\mathrm{Ind}_{G_i}^G u_i(\sigma) = \begin{cases} 0 & \text{for } \sigma \notin G_i \\ -f \cdot \frac{g_0}{g_i} & \text{for } \sigma \in G_i, \sigma \neq 1 \end{cases}$$

and $\sum_{\sigma \in G} \mathrm{Ind}_{G_i}^G u_i(\sigma) = 0$. Therefore, for $\sigma \in G_k \setminus G_{k+1}$ we have

$$\sum_{i=0}^{\infty} \frac{1}{(G_0 : G_i)} \mathrm{Ind}_{G_i}^G u_i(\sigma) = \sum_{i=0}^{k} \frac{1}{(G_0 : G_i)} \left(-f \cdot \frac{g_0}{g_i}\right) = -(k+1)f = a_G(\sigma)$$

and for $\sigma = 1$ both sides of the equation are 1_G. □

If φ is class function on G, set

$$\varphi(G_i) = \frac{1}{g_i} \sum_{\sigma \in G_i} \varphi(\sigma),$$

the average of φ on G_i.

Corollary 5.45. *If φ is a class function on G, then*

$$f(\varphi) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} (\varphi(1) - \varphi(G_i)).$$

Proof. We have that

$$\begin{aligned} f(\varphi) &= (\varphi, a_G) \\ &= \left(\varphi, \sum_{i=0}^{\infty} \tfrac{1}{(G_0:G_i)} \mathrm{Ind}_{G_i}^{G} u_i\right) && \text{by Proposition 5.44} \\ &= \sum_{i=0}^{\infty} \tfrac{g_i}{g_0} (\varphi, \mathrm{Ind}_{G_i}^{G} u_i) \\ &= \sum_{i=0}^{\infty} \tfrac{g_i}{g_0} (\varphi_{|G_i}, u_i) && \text{by Frobenius Reciprocity 5.9} \\ &= \sum_{i=0}^{\infty} \tfrac{g_i}{g_0} (\varphi(1) - \varphi(G_i)) \end{aligned}$$

which completes the proof. □

Corollary 5.46. *If χ is a character of a representation V of G, then*

$$f(\chi) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \mathrm{codim} V^{G_i}.$$

Proof. We simply recall that $\chi(1) = \dim V$ and $\chi(G_i) = \dim V^{G_i}$, and apply the previous corollary. □

Corollary 5.47. *If χ is a character of G, then $f(\chi)$ is a non-negative rational number.*

Proof. The function $g_0 \cdot a_G$ is a character of a representation, so $g_0 f(\chi)$ is a non-negative integer. □

Proposition 5.48. *Let H be a normal subgroup of G. Then*

$$a_{G/H} = (a_G)^{\natural}.$$

Proof. This follows from Proposition 5.29. □

Corollary 5.49. *If φ is a class function on G/H, and φ' is the corresponding class function on G, then $f(\varphi) = f(\varphi')$.*

Proof. We have that $f(\varphi) = (\varphi, a_G^{\natural}) = (\varphi', a_G) = f(\varphi')$ by Frobenius Reciprocity for quotients 5.41. □

Proposition 5.50. *Let H be a subgroup of G and M/K be the corresponding extension with discriminant $\Delta_{M/K}$. Then*

$$(a_G)_{|H} = \lambda r_H + f_{M/K} a_H$$

where $\lambda = v_K(\Delta_{M/K})$ and r_H the character of the regular representation.

Proof. If $\sigma \neq 1$ in H, then

$$a_G(\sigma) = -f_{L/K} i_G(\sigma),\ a_H(\sigma) = -f_{L/M} i_H(\sigma),\ r_H(\sigma) = 0,$$

and we simply use $i_G(\sigma) = i_H(\sigma)$ by Proposition 5.27 and $f_{L/K} = f_{L/M} f_{M/K}$ to prove the result.

For $\sigma = 1$, we have that $a_G(1) = v_K(\Delta_{L/K})$ by Proposition 5.31. Recall that by Proposition 1.23

$$\Delta_{L/K} = (\Delta_{M/K})^{[L:M]} \cdot N_{M/K}(\Delta_{L/M}),$$

so taking the valuation of both sides, we obtain

$$v_K(\Delta_{L/K}) = [L : M]v_K(\Delta_{M/K}) + f_{M/K}v_M(\Delta_{L/M})$$

which completes the proof. □

Corollary 5.51. *If χ is a character of H, then*

$$f(\mathrm{Ind}_H^G \chi) = v_K(\Delta_{M/K})\chi(1) + f_{M/K}(\chi).$$

Proof. We have that

$$\begin{aligned} f(\mathrm{Ind}_H^G\chi) &= (\mathrm{Ind}_H^G\chi, a_G) \\ &= (\chi, (a_G)_{|H}) && \text{by Frobenius Reciprocity 5.9} \\ &= (\chi, r_H) + f_{M/K}(\chi, a_H) && \text{by Proposition 5.50} \\ &= v_K(\Delta_{M/K})\chi(1) + f_{M/K}f(\chi) \end{aligned}$$

as requested. □

Proposition 5.52. *Let χ be the character of a representation of G of dimension 1. Let c_χ be the largest integer for which χ restricted to G_{c_χ} is not trivial. Then*

$$f(\chi) = \varphi_{L/K}(c_\chi) + 1.$$

Proof. If $i \le c_\chi$, $\chi(G_i) = 0$, so $\chi(1) - \chi(G_i) = 1$. If $i > c_\chi$, $\chi(G_i) = 1$, so $\chi(1) - \chi(G_i) = 0$. Thus

$$\begin{aligned} f(\chi) &= \sum_{i=0}^{\infty} \frac{g_i}{g_0}(\chi(1) - \chi(G_i)) && \text{by Corollary 5.45} \\ &= \sum_{i=0}^{c_\chi} \frac{g_i}{g_0} \\ &= \varphi_{L/K}(c_\chi) + 1 \end{aligned}$$

where the last equality follows from the explicit description of $\varphi_{L/K}$. □

Corollary 5.53. *Let $H = \ker \chi$ and $M = L^H$. Let c'_χ be the largest integer for which $(G/H)_{c'_\chi}$ is non-trivial. Then $f(\chi) = \varphi_{M/K}(c'_\chi) + 1$ and this number is a non-negative integer.*

Proof. By Herbrand's Theorem 5.38, we have that

$$(G/H)_{c'_\chi} = G_{\psi_{L/M}(c'_\chi)}H/H$$

and hence $c'_\chi = \varphi_{L/M}(c_\chi)$. Finally

$$\varphi_{M/K}(c'_\chi) = \varphi_{M/K}(\varphi_{L/M}(c_\chi)) = \varphi_{L/K}(c_\chi)$$

by Proposition 5.39. Then we apply Proposition 5.52 to get the desired result. □

We can finally prove the main theorem of this section, i.e. that a_G is a character of a representation.

Proof of Theorem 5.42. By Corollary 5.47, it is enough to show that $f(\chi) \in \mathbb{Z}$ for any character χ. By Brauer's Theorem 5.13 we have that

$$\chi = \sum_i n_i \mathrm{Ind}_{H_i}^G \chi_i$$

where $n_i \in \mathbb{Z}$ and χ_i are characters of degree 1 of H_i. Therefore, we only need to show that $f(\mathrm{Ind}_H^G \chi)$ is an integer for a character χ of degree 1. But in this case $f(\chi)$ is an integer by Corollary 5.53, so

$$f(\mathrm{Ind}_H^G(\chi)) = v_K(\Delta_{M/K})\chi(1) + f_{M/K} f(\chi) \in \mathbb{Z}$$

by Corollary 5.51. □

Definition 5.54. The representation of G with the character a_G is the *Artin representation* of G attached to L/K.

Definition 5.55. If χ is a character of G, then

$$\mathfrak{f}(\chi) = \mathfrak{p}^{f(\chi)} = \mathfrak{p}^{(\chi, a_G)}$$

is the *Artin conductor* of χ.

5.6.3. *Globalization.* We have defined the Artin conductor in the local case. We will now globalize it to the general case. Suppose L/K is a finite Galois extension with $G = \mathrm{Gal}(L/K)$. Let A be a Dedekind domain with field of fractions K, B be the integral closure of A in L. If $\mathfrak{P}$ is a non-zero prime over $\mathfrak{p}$, then $\overline{B_\mathfrak{P}} = B/\mathfrak{P}$ is separable over $\overline{A_\mathfrak{p}} = A/\mathfrak{p}$. Then the completion $\hat{B_\mathfrak{P}}$ of $\overline{B_\mathfrak{P}}$ with respect to the valuation associated to $\mathfrak{P}$ is Galois over the corresponding completion $\hat{A}_\mathfrak{p}$, and

$$\mathrm{Gal}(\hat{B_\mathfrak{P}}/\hat{A}_\mathfrak{p}) \cong D_\mathfrak{P}.$$

We then apply the construction from the previous section to $\hat{B_\mathfrak{P}}/\hat{A}_\mathfrak{p}$: the Artin character for this extension will be denoted $a_\mathfrak{P}$. (Note that $a_\mathfrak{P}$ is a priori defined on $D_\mathfrak{P}$ but it can be extended to 0 on $G \setminus D_\mathfrak{P}$.)

Definition 5.56. The *Artin representation* of $G = \mathrm{Gal}(L/K)$ attached to $\mathfrak{p}$ is the representation with the character $a_\mathfrak{p} = \sum_{\mathfrak{P}|\mathfrak{p}} a_\mathfrak{P}$.

In fact, for any choice of $\mathfrak{P}|\mathfrak{p}$:

$$\mathrm{Ind}_{D_\mathfrak{P}}^G a_\mathfrak{P}(\sigma) = \sum_{\tau D_\mathfrak{P} \in G/D_\mathfrak{P}} a_\mathfrak{P}(\tau\sigma\tau^{-1}) = \sum_{\tau D_\mathfrak{P} \in G/D_\mathfrak{P}} a_{\tau\mathfrak{P}}(\sigma) = a_\mathfrak{p}(\sigma),$$

since the Galois group G acts transitively on the primes of L above $\mathfrak{p}$. Thus the Artin representation attached to $\mathfrak{p}$ is the representation induced from the Artin representation of any $D_\mathfrak{P}$ for $\mathfrak{P}|\mathfrak{p}$.

Definition 5.57. If χ is a character, let $f(\chi, \mathfrak{p}) = (\chi, a_\mathfrak{p})$. Then the *Artin conductor* of χ is

$$\mathfrak{f}(\chi, L/K) = \mathfrak{f}(\chi) = \prod_\mathfrak{p} \mathfrak{p}^{f(\chi,\mathfrak{p})}.$$

(Since $f(\chi, \mathfrak{p}) = 0$ for $\mathfrak{p}$ unramified, this product is actually a finite product.)

The previously discussed local properties are hence globalized to the following.

Proposition 5.58.

(1) $\mathfrak{f}(\chi+\chi') = \mathfrak{f}(\chi)\mathfrak{f}(\chi')$, $\mathfrak{f}(1) = (1)$.
(2) *If M/K is a subextension of L/K corresponding to the subgroup H of G and χ is a character of H, then*

$$\mathfrak{f}(\mathrm{Ind}_H^G \chi, L/K) = \Delta_{M/K}^{\chi(1)} N_{M/K}(\mathfrak{f}(\chi, L/M)).$$

(3) *If M/K is, moreover, Galois, and χ is a character of G/H, then*

$$\mathfrak{f}(\chi, L/K) = \mathfrak{f}(\chi, M/K).$$

Finally, suppose K is a number field and consider the ideal

$$\mathfrak{c}(\chi; L/K) = \Delta_{K/\mathbb{Q}}^{\chi(1)} \cdot N_{K/\mathbb{Q}}(\mathfrak{f}(\chi; L/K)).$$

Since this is an ideal of $\mathbb{Z}$, it is generated by a positive integer $c(\chi; L/K)$. Then Proposition 5.58 yields the following.

Proposition 5.59.

(1) $c(\chi+\chi'; L/K) = c(\chi; L/K) \cdot c(\chi'; L/K)$, $c(1; L/K) = |d_{K/\mathbb{Q}}|$, *where $d_{K/\mathbb{Q}}$ generates $\Delta_{K/\mathbb{Q}}$.*
(2) $c(\mathrm{Ind}_H^G \chi; L/K) = c(\chi; L/M)$.
(3) $c(\chi; L/K) = c(\chi; M/K)$.

The function $c(\chi, L/K)^{s/2}$ will hence play the role of the exponential factor in the extended Artin L-function.

5.7. **The Functional Equation for the Artin L-function.** We finally come to the main result.

Definition 5.60. The *completed Artin L-function* is

$$\Lambda(s, \chi; L/K) = c(\chi; L/K)^{s/2} \prod_v L_v(s, \chi; L/K),$$

where the product ranges over the finite and infinite primes of K.

Theorem 5.61. *The completed Artin L-function has a meromorphic continuation to the entire complex plane which satisfies the following functional equation*

$$\Lambda(s, \chi; L/K) = \epsilon(\chi)\Lambda(1-s, \overline{\chi}; L/K),$$

where $\epsilon(\chi)$ is a constant with absolute value 1.

Proof. By the results earlier in the chapter, $\log \Lambda$ is additive and preserved by induction. Moreover, the 1-dimensional case follows from Hecke's Theorem 2.11. Therefore, Brauer's Theorem 5.13 yields the desired result. □

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