# MOTIVIC ACTION FOR SIEGEL MODULAR FORMS

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Abstract. We study the coherent cohomology of automorphic sheaves corresponding to Siegel modular forms $f$ of low weight on GSp(4) Shimura varieties. Inspired by the work of Prasanna–Venkatesh on singular cohomology of locally symmetric spaces, we propose a conjecture that explains all the contributions of a Hecke eigensystem to coherent cohomology in terms of the action of a motivic cohomology group. Under some technical conditions, we prove that our conjecture is equivalent to Beilinson's conjecture for the adjoint $L$-function of $f$. We also prove some unconditional results in special cases. For a lift $f$ of a Hilbert modular form $f_0$ to GSp(4), we produce elements in the motivic cohomology group for which the conjecture holds, using the results of Ramakrishnan on the Asai $L$-function of $f_0$. For a lift $f$ of a Bianchi modular form $f_0$ to GSp(4), we show that our conjecture for $f$ is equivalent to the conjecture of Prasanna-Venkatesh for $f_0$, thus establishing a connection between the motivic action conjectures for locally symmetric spaces of non-hermitian type and those for coherent cohomology of Shimura varieties.

## Contents



## 1. Introduction

A recent conjecture of Venkatesh and the second-named author [PV21] proposes a surprising relationship between the singular cohomology of locally symmetric spaces and higher Chow groups. For example, the simplest instance of this conjecture predicts that the Hecke isotypic components of the singular cohomology of Bianchi modular threefolds are related to regulators of elements in $\mathrm{CH}^2(E \times E, 1)$ for elliptic curves $E$ over imaginary quadratic fields. This prediction is rather mysterious since the locally symmetric spaces in this case are only real manifolds and have no underlying algebraic structure.

In this paper, we propose a similar relationship between the coherent cohomology of automorphic sheaves on Siegel modular threefolds and regulators of elements in $\mathrm{CH}^2(A \times A, 1)$ for abelian surfaces $A$ defined

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over $\mathbb{Q}$ (Conjecture A). The setting is at first sight quite different from the case considered in [PV21]: the underlying locally symmetric space $X$ is now hermitian symmetric, but the automophic form is not cohomological, namely it does not contribute to the singular cohomology of $X$. On the other hand, it does contribute to the coherent cohomology of a suitable sheaf on $X$. This situation has also been considered in the recent thesis of Gyujin Oh [Oh21]; we explain the difference between our work and that of Oh in Section 1.6 below.

After stating the conjecture, we prove three main results:

(1) First, in Theorem B below, we prove Conjecture A conditional on Beilinson's conjecture for the adjoint $L$-value of the associated Siegel modular forms and other standard hypotheses. (In this case, Beilinson's conjecture is the statement that a relevant piece of $\mathrm{CH}^2(A \times A, 1)$ is of rank one, generated by a single element $\alpha$ and that the $L$-value in question is rational, modulo the regulator of $\alpha$ and other standard motivic periods.)
(2) Next, for an abelian surface $A = \mathrm{Res}_{F/\mathbb{Q}}(E)$ obtained by restrictions of scalars from a *real quadratic* field $F$, we verify our conjecture for a rank one subspace of $\mathrm{CH}^2(A \times A, 1)$ (see Theorem C below). Assuming the rank prediction of Beilinson, this proves our conjecture.
(3) Finally, for an elliptic curve $E$ over an *imaginary quadratic* field $F$, we prove a compatibility between our conjecture for the abelian surface $A = \mathrm{Res}_{F/\mathbb{Q}}(E)$ and the conjecture of [PV21] for $E/F$ (see Theorem D). Assuming the rank prediction of Beilinson, this proves that the two conjectures are equivalent.

This last result suggests that the motivic action conjectures for nonhermitian locally symmetric spaces should be connected to (and perhaps implied by) the similar conjectures for coherent cohomology of hermitian locally symmetric spaces. It would be of interest to study this phenomenon in more generality.

We will now elaborate on our conjecture and state these results precisely. We will prioritize getting to the precise statements and defer the comments about the proofs to Section 1.5.

1.1. **Siegel modular forms.** Let $f$ be a non-endoscopic cuspidal holomorphic Siegel modular form of weight $(k, 2)$ and paramodular level $N$, with coefficients in the number field $\mathbb{Q}$. If $X_{\mathbb{Q}}$ is the Siegel modular threefold of paramodular level $N$ defined over $\mathbb{Q}$ and $\mathcal{E}_{k,2}$ is the automorphic vector bundle on $X_{\mathbb{Q}}$ of weight $(k, 2)$, then $f$ defines a section:

$$[f] \in H^0(X_{\mathbb{Q}}, \mathcal{E}_{k,2}). \tag{1.1}$$

Let $\pi^h$ be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ generated by $f$ and let $\pi_f$ be its finite part. Then there is a unique generic automorphic representation $\pi^g$ of $\mathrm{GSp}_4(\mathbb{A})$ with the same finite part. We may then consider a Whittaker-normalized vector $f^W \in \pi^g$ (Definition 4.7), and its associated cohomology class:

$$[f^W] \in H^1(X_{\mathbb{Q}}, \mathcal{E}_{k,2}) \otimes \mathbb{C} = H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2}). \tag{1.2}$$

Then $[f]$ and $[f^W]$ span the two dimensional subspace $H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_{\pi_f}$ giving the total contribution of $\pi_f$ to the cohomology of the automorphic sheaf $\mathcal{E}_{k,2}$ (Theorem 2.12).

Conjecturally (as in [PV21, Sec. 4]), there is an adjoint motive $M(\pi_f, \mathrm{Ad})$ associated with $\pi_f$ in the category of Chow motives over $\mathbb{Q}$, whose Galois representation is the adjoint representation of the Galois representation associated with $\pi_f$, and that is well defined up to isomorphism in this category. In the special case when $f$ corresponds to an abelian surface $A$ over $\mathbb{Q}$ (so that $(k, 2) = (2, 2)$), the motive $M(\pi_f, \mathrm{Ad})$ may be realized explicitly as $\mathrm{Sym}^2 H^1(A)(1)$.

Consider the motivic cohomology group

$$H^1_{\mathcal{M}} := H^1_{\mathcal{M}}(M(\pi_f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1)),$$

where as usual, the subscript $\mathbb{Z}$ indicates classes that extend to an integral model. Now, Beilinson's conjecture predicts that the regulator map to Deligne cohomology:

$$r_{\mathcal{D}} \colon H^1_{\mathcal{M}} \otimes \mathbb{R} \to H^1_{\mathcal{D}}(M(\pi_f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) =: H^1_{\mathcal{D}} \tag{1.3}$$

is an isomorphism of one-dimensional real vector spaces. We consider a certain *natural generator* $\delta^\vee \in (H^1_{\mathcal{D}})^\vee$ (see (1.5) below for an explicit description in the case of abelian surfaces) and define an action $\tau$ of $(H^1_{\mathcal{D}})^\vee$ on the $\pi_f$-isotypic part of the cohomology of $\mathcal{E}_{k,2}$ over $\mathbb{C}$ by setting:

$$H^0(X_{\mathbb{C}}, \mathcal{E}_{k,2})_{\pi_f} \xrightarrow{\tau(\cdot)} H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2})_{\pi_f}$$
$$\tau(\delta^\vee) : [f] \mapsto [f^W].$$

Via the regulator isomorphism (1.3), this gives an action of $(H^1_{\mathcal{M}})^\vee$ on the cohomology of $\mathcal{E}_{k,2}$.

**Conjecture A** (Conjecture 4.19)**.** *The action of $(H^1_{\mathcal{M}})^\vee$ preserves the rational structure $H^*(X_{\mathbb{Q}}, \mathcal{E}_{k,2})_{\pi_f}$. Equivalently, given a non-zero element $\alpha \in H^1_{\mathcal{M}}$, we have that:*

$$\frac{[f^W]}{\delta^\vee(r_{\mathcal{D}}(\alpha))} \in H^1(X_{\mathbb{Q}}, \mathcal{E}_{k,2})_{\pi_f} \subseteq H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2})_{\pi_f}.$$

Our first main theorem is the following.

**Theorem B** (Theorem 4.21)**.** *Conjecture A is implied by Beilinson's conjecture for the adjoint $L$-function $L(\pi_f, \mathrm{Ad}, s)$ at $s = 1$, and Deligne's conjecture for some quadratic character twists of the spin $L$-function $L(\pi_f, \psi_\pm, s)$, together with a non-vanishing hypothesis for $L(\pi_f, \psi_\pm, s)$.*

1.2. **Explication for modular abelian surfaces.** The Brumer–Kramer Conjecture 3.8 predicts that a rational abelian surface of conductor $N$ corresponds to a cuspidal Siegel modular form $f$ of weight $(2,2)$ and paramodular level $N$. In this case, as mentioned earlier, the motive $M(\pi_f, \mathrm{Ad})$ equals $\mathrm{Sym}^2 H^1(A)(1)$ which may be realized in (a Tate twist of) the cohomology of $A \times A$. Therefore, a motivic cohomology class $\alpha \in H^1_{\mathcal{M}}$ admits the following interpretation as a higher Chow element: $\alpha = \{(D_i, \varphi_i)\}$ where:

- $D_i$ is an irreducible divisor on $A \times A$,
- $\varphi_i$ is a meromorphic function on $D_i$,
- $\sum_i \mathrm{div}(\varphi_i) = 0$.

For any $\mathbb{Q}$-basis $\omega_1, \omega_2$ of $H^0(A, \Omega^1_A)$, we define a natural generator of $H^1_{\mathcal{D}}$ by:

$$\delta := (2\pi i)\left((\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) - (\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2)\right). \tag{1.4}$$

Here (as in [PV21, Sec. 2]), we use the fundamental exact sequence of Beilinson to identify $H^1_{\mathcal{D}}$ with a subspace of $H^{0,0}(M(\pi_f, \mathrm{Ad})) \subset H^{1,1}(A \times A)(1)$.

We then fix a polarization of the abelian surface $A$, consider the associated pairing:

$$\langle -, - \rangle_{\mathrm{pol}} \colon \mathrm{Sym}^2 H^1(A) \times \mathrm{Sym}^2 H^1(A) \to \mathbb{Q}(-2),$$

and define

$$\delta^\vee := \frac{\pi^4}{\sqrt{\Delta_{\mathrm{Ad}(f)}}} \langle \delta, - \rangle_{\mathrm{pol}} \in (H^1_{\mathcal{D}})^\vee, \tag{1.5}$$

where $\Delta_{\mathrm{Ad}(f)}$ is the adjoint conductor of $f$. Since $H^1_{\mathcal{D}}$ is one-dimensional, $\delta^\vee$ only depends on the choice of polarization up to scalars in $\mathbb{Q}^\times$. (See Remark 4.16.)

Given $\alpha$ as above, this leads to the explicit expression:

$$\delta^\vee(r_{\mathcal{D}}(\alpha)) = \frac{\pi^4}{\sqrt{\Delta_{\mathrm{Ad}(f)}}} \frac{1}{(2\pi i)^3} \sum_i \int_{D_i(\mathbb{C})} \log|\varphi_i| \cdot \delta \cup \omega_{\mathrm{pol}} \in \mathbb{C}^\times, \tag{1.6}$$

where $\eta_{\mathrm{pol}} \in H^{1,1}(A) \cap H^2(A, \mathbb{Z})$ is the cohomology class associated with the polarization on $A$ and $\omega_{\mathrm{pol}} := \eta_{\mathrm{pol}} \boxtimes \eta_{\mathrm{pol}} \in H^{2,2}(A \times A) \cap H^4(A \times A, \mathbb{Z})$.

Concretely, Conjecture A then predicts that the element

$$\frac{[f^W]}{\frac{2\pi i}{\sqrt{\Delta_{\mathrm{Ad}(f)}}}\sum_i \int\limits_{D_i(\mathbb{C})} \log|\varphi_i| \cdot \delta \cup \omega_{\mathrm{pol}}} \in H^1(X_{\mathbb{C}}, \mathcal{E}_{2,2}) \tag{1.7}$$

is rational in coherent cohomology, namely lives in the $\mathbb{Q}$-subspace $H^1(X_{\mathbb{Q}}, \mathcal{E}_{2,2})$. This gives a subtle relationship between the Siegel modular form and the conjectural abelian surface associated with it.

1.3. **Special case: Hilbert modular forms.** Suppose $F$ is a real quadratic field and let $f$ be the (non-endoscopic) Yoshida lift of a Hilbert modular form $f_0$ of weight $(2,2)$. Using a theorem of Ramakrishnan [Ram87] on special values of Asai $L$-functions for Hilbert modular forms, we prove the following theorem.

**Theorem C** (Theorem 6.2)**.** *Suppose $f$ is the Yoshida lift of a Hilbert modular form $f_0$. Then there is an explicit rank one subspace of $(H^1_{\mathcal{M}})^\vee$ which acts rationally on coherent cohomology. Therefore, assuming the rank prediction of Beilinson's conjecture (Hypothesis 3.3), Conjecture A is true in this case.*

This is one of the first unconditional results towards the motivic action conjectures. As far as we know, the only other known case is dihedral weight one modular forms (c.f. [Hor23] over $\mathbb{C}$ and [DHRV22, Lec22, Zha23] mod $p^n$).

1.4. **Special case: Bianchi modular forms.** Suppose $F$ is now an imaginary quadratic field and $f$ is a Yoshida lift of a Bianchi modular form $f_0$ of weight $(2,2)$. In this case, we prove a compatibility between our conjecture for $f$ and the conjecture of [PV21] for $f_0$.

There are explicit Eichler–Shimura maps [TU22, Section 5.1]

$$\begin{aligned} \omega^i \colon S^F_{2,2}(\mathfrak{N}) &\to H^i(X_0, \mathbb{C}) \\ f_0 &\mapsto \omega^i_{f_0} \end{aligned}$$

for $i = 1, 2$. Given a Whittaker-normalized Bianchi modular form $f_0$, we define the period $u^1(f_0) \in \mathbb{C}$ to satisfy $\frac{\omega^1_{f_0}}{u^1(f_0)} \in H^1(X_0, \mathbb{Q})_{f_0}$. Then the action $\tau_{\mathrm{PV}}$ of $H^1_{\mathcal{D}}(M(f_0, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ on the $f_0$-isotypic part of the cohomology can be described by defining a natural dual generator $\eta^\vee \in H^1_{\mathcal{D}}(M(f_0, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ (Definition 7.8) and setting:

$$\begin{aligned} H^1(X_0, \mathbb{C})_{f_0} &\overset{\tau_{\mathrm{PV}}(-)}{\to} H^2(X_0, \mathbb{C})_{f_0} \\ \tau_{\mathrm{PV}}(\eta^\vee) \colon \frac{\omega^1_{f_0}}{u^1(f_0)} &\mapsto \omega^2_{f_0}. \end{aligned}$$

The main conjecture of [PV21] in this case asserts that the resulting action of the dual motivic cohomology group is rational. It has an explicit form similar to (1.7) (see, for example, equation (7.20)).

We may define maps:

$$\begin{aligned} H^1(X_0, \mathbb{Q})_{f_0} &\overset{\theta_1}{\to} H^0(X_{\mathbb{Q}}, \mathcal{E}_{2,2})_f \\ H^2(X_0, \mathbb{Q})_{f_0} &\overset{\theta_2}{\to} H^1(X_{\mathbb{Q}}, \mathcal{E}_{2,2})_f \end{aligned}$$

which are normalized to preserve the rational structures. We then prove the following compatibility of our conjecture with the conjecture of [PV21].

**Theorem D** (Theorem 7.1)**.** *There is a natural isomorphism:*

$$d^\vee \colon H^1_{\mathcal{D}}(M(f, \mathrm{Ad}), \mathbb{R}(1))^\vee \to H^1_{\mathcal{D}}(M(f_0, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$$

*under which the diagram:*

$$\begin{tikzcd} H^1(X_0,\mathbb{Q})_{f_0}\otimes\mathbb{C} \arrow[r, "\theta_1"] \arrow[d, "{[\text{PV21}]}"', "\tau_{\mathrm{PV}}\circ d^\vee(-)"] & H^0(X_{\mathbb{Q}},\mathcal{E}_{2,2})_f\otimes\mathbb{C} \arrow[d, "\text{our action}"', "\tau(-)"] \\ H^2(X_0,\mathbb{Q})_{f_0}\otimes\mathbb{C} \arrow[r, "\theta_2"] & H^1(X_{\mathbb{Q}},\mathcal{E}_{2,2})_f\otimes\mathbb{C} \end{tikzcd}$$

*commutes, up to $\mathbb{Q}^\times$. In particular, assuming the rank prediction of Beilinson's conjecture (Hypothesis 3.3), our Conjecture A is equivalent to the main conjecture of* [PV21] *in this setting.*

**Remark 1.1.** The original construction of Galois representations attached to cohomological Bianchi modular forms is due to Taylor [Tay91] and Harris-Soudry-Taylor [HST93]. It proceeds via first making the functorial transfer to a (low-weight) Siegel modular form that is not cohomological (but that does contribute to coherent cohomology) and then using congruences to establish the existence of a Galois representation, analogous to the Deligne-Serre method for weight one modular forms. Thus it seems a natural question to compare the motivic action conjectures for the Bianchi (cohomological case) and the Siegel (coherent cohomology) case. More generally, it seems a natural question to study how the motivic action conjectures interact with Langlands functoriality.

1.5. **Overview of the proofs.** The difficulty in defining the motivic action is that there is no canonical way to normalize the contributions to cohomology of Siegel modular forms. In the singular cohomology setting [PV21], a single automorphic representation contributes to all the cohomological degrees of singular cohomology via Eichler–Shimura maps. However, as discussed above in the Siegel case, the contributions to different degrees of the cohomology of an automorphic vector bundle come from different members of an archimedean $L$-packet. In order to define a motivic action, one has to normalize the automorphic embeddings for every element of the $L$-packet.

The idea of the present paper is to normalize $[f]\in H^0(X_{\mathbb{C}},\mathcal{E}_{k,2})_{\pi_f}$ to be rational in coherent cohomology and $[f^W]\in H^1(X_{\mathbb{C}},\mathcal{E}_{k,2})_{\pi_f}$ using the Whittaker model. Then, we pick an explicit generator of the dual Deligne cohomology group $\delta^\vee\in H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}},\mathbb{R}(1))^\vee$ which should act by $[f]\mapsto[f^W]$.

The proof of Theorem B relies on three key ingredients:

1. The relationship between $\langle f^W,f^W\rangle$ and the adjoint $L$-value which was proved by Chen–Ichino [CI19] and Chen [Che22].
2. The relationship between the *Whittaker period* $c^W(f)$ and $c^+(M(f))c^-(M(f))$ (a product of Deligne periods) which is proved in Theorem 5.3, assuming Deligne's conjecture. A similar relationship features in [LPSZ21] and [Oh21]. To obtain this relationship, we need to know that some twists of the spin $L$-function of $f$ do not vanish. This was recently proved in [RY23], but to obtain a statement up to $\mathbb{Q}^\times$ instead of up to $(\mathbb{Q}^{\mathrm{ab}})^\times$, we need a slightly stronger result, c.f. Hypothesis 5.2.
3. An explicit version of Beilinson's conjecture which is proved to be equivalent to Beilinson's conjecture in Theorem 5.7. This is the *non-critical* analogue of Yoshida's period relation [Yos01] for $L$-values which are *critical* in the sense of Deligne [Del79].

The special cases in Theorems C and D both rely on the factorization of $L$-functions:

$$L(f,\mathrm{Ad},s)=L(f_0,\mathrm{Ad},s)L(f_0,\mathrm{Asai},s+1). \tag{1.8}$$

The essential ingredient is a dichotomy for real and imaginary quadratic fields which we summarize in Table 1.1 below.

In both cases, we have a good understanding of the critical $L$-values: due to Shimura [Shi78] in the real case, and Cremona–Whitley [CW94] and Ghate [Gha96] in the imaginary case.

In the real case, the proof of Theorem C then relies on the fact that Beilinson's conjecture for the Asai $L$-function was proved by Ramakrishnan [Ram87]. The key point is that in this case:

$$H^3_{\mathcal{M}}(M(f_0,\mathrm{Asai})_{\mathbb{Z}},\mathbb{Q}(2))\cong H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}},\mathbb{Q}(1))$$

and the Asai motive $M(f_0,\mathrm{Asai})$ is realized directly in the cohomology of the Hilbert modular surface. Ramakrishnan [Ram87] constructed explicit classes in $H^3_{\mathcal{M}}(\mathrm{Asai}M(f_0)_{\mathbb{Z}},\mathbb{Q}(2))$ using Hirzebruch–Zagier divisors

| quadratic field $F$ | $L$-function | value at $s=1$ |
|---|---|---|
| real | $L(f_0, \mathrm{Ad}, s)$ | critical |
| | $L(f_0, \mathrm{Asai}, s+1)$ | non-critical |
| imaginary | $L(f_0, \mathrm{Ad}, s)$ | non-critical |
| | $L(f_0, \mathrm{Asai}, s+1)$ | critical |

Table 1.1. Table showing which $L$-values within the factorization (1.8) are critical and non-critical in the sense of Deligne [Del79] for real and imaginary quadratic fields $F$.

and modular units on them, and related their regulators to the value of the Asai $L$-function. As in the case of weight one modular forms (c.f. [Hor23] over $\mathbb{C}$ and [DHRV22, Lec22, Zha23] mod $p^n$), it seems that one can only prove instances of the motivic action conjectures in the presence of explicit motivic cohomology classes.

In the imaginary case, Theorem D is equivalent to an explicit period relationship given in Theorem 7.20. To prove this theorem, we use again the relationship between adjoint $L$-values and Petersson norms to reduce the statement about the non-critical parts of the periods to a statement about the critical parts.

1.6. **Comparison with previous work.** The idea that motivic cohomology should be related to the cohomology of locally symmetric spaces was first introduced in [PV21, Ven19, GV18]. Subsequently, Harris and Venkatesh [HV19] proposed a similar conjecture for coherent cohomology associated with weight one modular forms by defining an action modulo powers of a prime.

As discussed above, the difficulty of generalizing the action of [PV21] over the complex numbers to coherent cohomology is that there seems to be no way to consistently normalize the automorphic realizations of different elements of an $L$-packet. One exception is the case of Hilbert modular forms: there are natural maps between the elements of the $L$-packet for $\mathrm{SL}_2(\mathbb{R})^d$ coming from partial complex conjugation operators [Shi78, Har90c]. These ideas were used by the first-named author [Hor23] to define a motivic action for (partial) weight one Hilbert modular forms. For more general Shimura varieties, the elements of the $L$-packet have different "sizes" (see Figure A.2 for the Siegel case), so it seems difficult to define natural maps between them.

A different approach is taken by Gyujin Oh [Oh21]. Instead of normalizing the contributions to cohomology and defining an action that is conjecturally rational, he chooses *metrics* on the various cohomology groups and conjectures there is an isomorphism of graded metrized complex vector spaces:

$$\bigwedge^{*}(H^1_{\mathcal{M}}(M(\Pi,\mathrm{Ad})_{\mathbb{Z}},\mathbb{Q}(1))\otimes H^{i_{\min}}(X_{\mathbb{C}},\mathcal{E})_{\Pi}\cong\bigoplus_{i=i_{\min}}^{i_{\max}}H^i(X_{\mathbb{C}},\mathcal{E})_{\Pi},$$

where $X$ is a Shimura variety, $\mathcal{E}$ is an automorphic vector bundle, and $\Pi$ is an automorphic representation satisfying some technical assumptions.

The advantage of Oh's approach is that his conjecture applies to any automorphic vector bundle on any Shimura variety. Oh proves that his conjecture is implied by Beilinson's conjecture in two cases: the case of Siegel threefolds and Picard modular surfaces, under similar hypothesis to ours. However, it seems difficult to extract an explicit rationality statement from it; indeed, rescaling an automorphic form by a complex number of norm one does not change its volume, but it does change its rationality properties.

1.7. **Generalizations.** We end the paper by discussing a generalization of our conjecture to Hilbert–Siegel modular forms for a totally real field $F$ of degree $d$ in Section 8. In that case, the $L$-packet $\Pi$ contains a generic member $\pi^g$ contributing to top degree $d$ in cohomology and one can define an action of a dual canonical generator of the determinant $\bigwedge^d H^1_{\mathcal{D}}(M(\Pi,\mathrm{Ad})_{\mathbb{R}},\mathbb{R}(1))^\vee$ by:

$$\begin{aligned} H^0(X,\mathcal{E})_\Pi &\to H^d(X,\mathcal{E})_\Pi,\\ [f] &\mapsto [f^W]. \end{aligned}$$

However, $\Pi$ will also make intermediate contributions in degrees $0 \leq * \leq d$ and we do not know how to normalize these contributions to obtain the intermediate motivic actions. It would be interesting to explore other cases in where this strategy to define a map between various cohomological degrees using the Whittaker model can be carried out.

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## 2. Preliminaries: cohomology of automorphic vector bundles

Let $G = \mathrm{GSp}_4$ be the symplectic group defined with respect to the matrix $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, i.e.

$$\mathrm{GSp}_4(R) = \{g \in \mathrm{GL}_4(R) \mid {}^t gJg = \mu(g) \cdot J \text{ for some } \mu(g) \in R^\times\}.$$

2.1. **Siegel modular threefolds and automorphic vector bundles.** Let $K_f$ be a neat open subgroup of $\mathrm{GSp}_4(\mathbb{A}_f)$. The open Shimura variety $Y_{G,\mathbb{Q}}$ associated to $G$, level $K_f$ has a canonical model over $\mathbb{Q}$ such that

$$Y_{G,\mathbb{Q}}(\mathbb{C}) \cong G(\mathbb{Q})_+ \backslash [\mathcal{H}_2 \times G(\mathbb{A}_f)]/K_f \cong \coprod_i \Gamma_i \backslash \mathcal{H}_2$$

where the subgroups $\Gamma_i \subseteq \mathrm{Sp}_4(\mathbb{Z})$ corresponds to $K_f$. The Siegel modular threefold $Y_{G,\mathbb{Q}}$ can be identified with the moduli space of abelian surfaces with principal polarization and $K_f$-level structure.

According to [FC90], $Y_{G,\mathbb{Q}}$ has a toroidal compactification $X_{G,\mathbb{Q}}$ defined over $\mathbb{Q}$ associated to a choice of rational polyhedral cone decomposition $\Sigma$. We may choose $\Sigma$ so that $X_{G,\mathbb{Q}}$ is smooth and the boundary $D = \partial X_{G,\mathbb{Q}}$ is a simple normal crossings divisor. We fix this choice once and for all and suppress it from notation.

We follow the exposition in [FC90, Section VI.4] to define automorphic vector bundles on $Y_{G,\mathbb{Q}}$ and $X_{G,\mathbb{Q}}$. Recall that $P \subseteq G$ is the Siegel parabolic with Levi $M_P \cong \mathrm{GL}_2 \times \mathrm{GL}_1$. We write $P^\circ \subseteq G^\circ = \mathrm{Sp}_4$ for the kernels of the similitude character $\nu$. The maximal compact subgroup $K^\circ \subseteq G^\circ(\mathbb{R})$ has complexification $K^\circ_{\mathbb{C}} \subseteq G^\circ(\mathbb{C})$ and there exists $g \in G^\circ(\mathbb{C})$ such that $gM_{P^\circ,\mathbb{C}}g^{-1} = K^\circ_{\mathbb{C}}$. Note that $g \notin G^\circ(\mathbb{R})$; explicitly, we may take $g = \begin{pmatrix} -iI_2 & I_2 \\ I & I \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C})$ (the significance of this detail is explained in A.4).

The symmetric space $\mathcal{H}_2 \cong G^\circ(\mathbb{R})/K^\circ \cong G(\mathbb{R})^+/K$ is naturally embedded in the compact dual $\mathcal{H}_2^\vee = (G/P)(\mathbb{C})$ as a $G(\mathbb{R})^+$-invariant open subset of the $G(\mathbb{C})$-homogeneous space $\mathcal{H}_2^\vee$.

Consider an arithmetic subgroup $\Gamma$ of $G(\mathbb{Q})$ and let $Y_\Gamma$ be the locally symmetric space $\Gamma \backslash \mathcal{H}_2$. We may construct an automorphic vector bundle on $Y_\Gamma$ as follows. For a finite-dimensional rational representation $\varrho \colon P \to \mathrm{GL}(V_\varrho)$, we define a $G(\mathbb{C})$-equivariant vector bundle

$$G(\mathbb{C}) \times_{P(\mathbb{C}),\varrho} V_\varrho \text{ on } \mathcal{H}_2^\vee = (G/P)(\mathbb{C}).$$

Restricting it to $\mathcal{H}_2$ and quotienting by $\Gamma$ defines a vector bundle

$$\mathcal{E}_\varrho \text{ on } \Gamma \backslash \mathcal{H}_2.$$

Harris [Har85, Har86] and Milne [Mil88] showed that these bundles are in fact defined over the canonical model $Y_{G,\mathbb{Q}}$ of the Shimura variety for $G$.

These vector bundles have two natural extensions to the toroidal boundary [Har90b], the canonical and subcanonical extension. We summarize this discussion in a theorem.

**Theorem 2.1** ([FC90, Theorem VI.4.2]).

*(1) The automorphic vector bundle $\mathcal{E}$ on $Y_{G,\mathbb{Q}}$ has a canonical extension $\mathcal{E}^{\mathrm{can}}$ to the toroidal compactification $X_{G,\mathbb{Q}}$.*
*(2) This defines an exact, fully faithful functor*

$$\left\{ \begin{array}{c} \textit{finite-dimensional} \\ P(\mathbb{C})\textit{-representations} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{vector bundles} \\ \textit{over } X_{G,\mathbb{Q}} \end{array} \right\}$$

*which commutes with tensor products and duals.*

If $\mathcal{I}(D)$ is the invertible sheaf on $X_{G,\mathbb{Q}}$ defining the divisor $D$, then we define the subcanonical extension $\mathcal{E}^{\mathrm{sub}}$ of $\mathcal{E}$ to $X_{G,\mathbb{Q}}$ to be $\mathcal{E}^{\mathrm{can}} \otimes \mathcal{I}(D)$.

Finally, given a finite-dimensional complex representation of $K = K^\circ \mathbb{R}_{>0}$, we may extend it to $M_P(\mathbb{C})$ by analytic continuation and then inflate it to $P(\mathbb{C})$. Therefore, we may also associate vector bundles to representations of $K$. In fact, any holomorphic $G(\mathbb{R})^+$-equivariant vector bundle is $C^\infty$-isomorphic to one associated to a representation of $K^\circ$.

**Definition 2.2.** Let $\varrho$ be a representation of $M_P$ of highest weight $(k_1, k_2; m)$ for integers $k_1 \geq k_2 \geq 0$ and $m$. We then write:

$$V_{k_1,k_2;m} = V_\varrho, \tag{2.1}$$

$$\mathcal{E}^*_{k_1,k_2;m} = \mathcal{E}^*_\varrho \qquad \text{for } * \in \{\mathrm{can}, \mathrm{sub}\}. \tag{2.2}$$

We fix the standard choice of basis for $V_{k_1,k_2;m}$, following [Mor04, pp. 902].

Recall that $M_P = \mathrm{GL}_2 \times \mathrm{GL}_1$. Let St be the standard representation of $\mathrm{GL}_2$, $\mathrm{Sym}^k$ the $k$th symmetric power of the standard representation, det the determinant representation of $\mathrm{GL}_2$, and $\mu$ be the identity representation of $\mathrm{GL}_1$. Then:

$$(\mathrm{Sym}^k \otimes \det{}^j) \boxtimes \mu^m \cong V_{k+j,j;m}.$$

For example, $\dim V_{k_1,k_2;m} = 1$ if and only if $k_1 = k_2$.

**Remark 2.3.** Due to an unfortunate clash in standard notation, the highest weight of the restriction of $V_{k_1,k_2;m}$ to $K\mathbb{R}_{>0}$ is $(-k_2, -k_1; m)$ and not $(k_1, k_2; m)$.

**Examples 2.4.** We describe several of the automorphic vector bundles in terms of the Siegel modular threefold $Y = Y_{G,\mathbb{Q}}$ and its toroidal compactification $X = X_{G,\mathbb{Q}}$, and the universal abelian surface $A$ over $Y$ and its semi-abelian extension to $X$. We write $D$ for the boundary divisor $\partial X$.

(1) When $\varrho = \mathrm{St}$ is the standard representation of $M^\circ = \mathrm{GL}_2$, then

$$\mathcal{E}_{\mathrm{St}} \cong \mathcal{T}^*_{A/Y} \cong \Omega^1_{A/Y},$$
$$\mathcal{E}^{\mathrm{can}}_{\mathrm{St}} \cong \mathcal{T}^*_{A/X} \cong \Omega^1_{A/X},$$

(2) When $\varrho = \mathrm{Sym}^2 \mathrm{St}$,

$$\mathcal{E}_{\mathrm{Sym}^2 \mathrm{St}} \cong \Omega^1_Y,$$
$$\mathcal{E}^{\mathrm{can}}_{\mathrm{Sym}^2 \mathrm{St}} \cong \Omega^1_X(\log D),$$

(3) When $\varrho = \det$ is the determinant of the standard representation, then

$$\mathcal{E}_{\det} \cong \omega_{A/Y} \cong \det \Omega_{A/Y},$$
$$\mathcal{E}^{\mathrm{can}}_{\det} \cong \omega_{A/X}(\log D) \cong \det \Omega_{A/X}(\log D),$$
$$\mathcal{E}^{\mathrm{sub}}_{\det} \cong \omega_{A/X} \cong \det \Omega_{A/X}.$$

(4) When $\varrho = \det^3$,

$$\begin{aligned} \mathcal{E}_{\det^3} &= \mathcal{K}_Y \cong \omega_Y^3 && \text{the canonical bundle on } Y, \\ \mathcal{E}^{\mathrm{can}}_{\det^3} &\cong \mathcal{K}_X(\log D) \cong \omega^3_{A/X}, \\ \mathcal{E}^{\mathrm{sub}}_{\det^3} &= \mathcal{K}_X, && \text{[Har90b, Prop. (2.2.6)].} \end{aligned}$$

Henceforth, we omit $G$ from the notation and write $Y = Y_{G,\mathbb{Q}}$ and $X = X_{G,\mathbb{Q}}$. Although $X$ is defined over $\mathbb{Q}$, we sometimes write $X_{\mathbb{Q}}$ to emphasize that we mean the variety over $\mathbb{Q}$ instead of $\mathbb{C}$.

By definition, a Siegel modular form $f$ of level $\Gamma$ and weight $\varrho$ is a section of the vector bundle $\mathcal{E}_\varrho$ over $Y_{\mathbb{C}}$, i.e. $f \in H^0(Y_{\mathbb{C}}, \mathcal{E}_\varrho)$. When $\varrho = \det^k$, $f$ is scalar-valued and $V_\varrho = V_{k,k;0}$.

Let $N \geq 1$ be an integer. The *principal level subgroup of level* $N$ is:

$$\Gamma(N) = \begin{pmatrix} 1_2 + NM_2(\mathbb{Z}) & NM_2(\mathbb{Z}) \\ NM_2(\mathbb{Z}) & 1 + NM_2(\mathbb{Z}) \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}).$$

However, we will be more interested in the following level structures.

**Definition 2.5.** Let $N \geq 1$ be an integer. The *paramodular group* $K(N)$ is defined as:

$$K(N) = \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Q}).$$

The reader can consult [RS06] for a detailed discussion of Siegel modular forms with this level structure. They develop a theory of newforms for these level structures.

A Siegel modular form admits a $q$-expansion by pulling back to the formal completion along boundary strata [Lan13, §7.1.2]. Explicitly in the case of Siegel modular forms, see [FC90, Section V.1] for full level and $\varrho = \det^k$, [Ich14] for principal level $N$ and general $\varrho$, and [FZ21] for paramodular level $N$ and general $\varrho$.

**Theorem 2.6.** *Let $X$ be a Siegel modular variety and let $A$ be a $\mathbb{Q}$-algebra. Then the following properties hold:*

*(1) (Koecher principle) The restriction map*

$$H^0(X_A, \mathcal{E}^{\mathrm{can}}) \to H^0(Y_A, \mathcal{E})$$

*is an isomorphism.*

*(2) (Higher Koecher principle) The restriction map*

$$H^1(X_A, \mathcal{E}^{\mathrm{can}}) \to H^1(Y_A, \mathcal{E})$$

*is injective.*

*(3) (q-expansion principle) Let $f$ be a cuspidal Siegel modular form level $\Gamma(N)$ with Fourier coefficients in a $\mathbb{Q}$-algebra $A$ and $B \subseteq A$ be a subalgebra. Then $f \in H^0(Y_B, \mathcal{E}_\varrho)$ if and only if the Fourier coefficients of $f$ at all cusps are in $B$.*

*(4) There is an isomorphism between the space of cuspidal Siegel modular forms with coefficients in $A$ and the image of the natural map*

$$H^0(X_A, \mathcal{E}^{\mathrm{sub}}) \to H^0(X_A, \mathcal{E}^{\mathrm{can}}).$$

*Proof.* The Koecher principle is due to Faltings–Chai [FC90, Lemma V.1.5] (for full level and $\varrho = \det^k$), which was generalized to vector bundle and to higher degree cohomology by Lan [Lan16]. For the $q$-expansion principle, see more generally [Lan13, Prop. 7.1.2.14]. Finally, (4) is [Har90b, Prop. (5.4.2)]. □

2.2. **Higher cohomology of automorphic vector bundles.** We are interested in the coherent cohomology of the vector bundles $\mathcal{E}^{\mathrm{sub}}$ and $\mathcal{E}^{\mathrm{can}}$. We first note that the cohomology groups are independent of the choice of toroidal compactification and, as a consequence, the Hecke algebra acts on them.

**Theorem 2.7** (Harris [Har90b],[BHR94])**.**

*(1) The cohomology groups $H^i(X, \mathcal{E}^{\mathrm{can}})$ and $H^i(X, \mathcal{E}^{\mathrm{sub}})$ are independent of the choice of toroidal compactification, up to canonical isomorphism.*
*(2) There is a natural action of the Hecke algebra $\mathcal{H} = \mathcal{H}(G(\mathbb{A}_f), K_f)$ relative to level $K_f$ on $H^i(X, \mathcal{E}^{\mathrm{can}})$ and on $H^i(X, \mathcal{E}^{\mathrm{sub}})$.*

Harris [Har90b] and Su [Su18] expressed the coherent cohomology of the vector bundle $\mathcal{E}^{\mathrm{can}}$ in terms of Dolbeault classes. More specifically, we have the following result.

**Theorem 2.8** (Harris, Su)**.** *Let $\mathcal{E}_\varrho$ be an automorphic vector bundle over $Y$ associated to the representation $(\varrho, V_\varrho)$ of $K$ and $\mathcal{E}^{\mathrm{can}}$ be its canonical extension to $X$. Then there is a natural Hecke-equivariant isomorphism*

$$H^i(X_{\mathbb{C}}, \mathcal{E}_\varrho^{\mathrm{can}}) \cong H^i(\mathfrak{P}, K; \mathcal{A}(G)^{K_f} \otimes V_\varrho), \tag{2.3}$$

*where $\mathfrak{P} = \mathrm{Lie}(P)$ is the Lie algebra of the Siegel parabolic and $K = K^\circ \mathbb{R}_{>0} \subseteq G(\mathbb{R})$.*

We now wish to describe the contributions of a holomorphic Siegel modular form of weight $(k_1, k_2; m)$ and level $\Gamma$ to coherent cohomology of automorphic sheaves. Let $\pi$ be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ generated by $f$, which is non-CAP and cuspidal. Its infinite component $\pi_\infty$ is a holomorphic (limit of) discrete series $X^1_{\lambda;m}$ with Harish–Chandra parameter $(\lambda_1, \lambda_2; m) := (k_1 - 1, k_2 - 2; m)$. The $L$-packet at infinite also contains the generic (large) discrete series $X^2_{\overline{\lambda};m}$ with Harish–Chandra parameter $(\overline{\lambda}, m) := (\lambda_1, -\lambda_2; m)$. See Appendix A for the precise definitions and more details about the representation theory of $\mathrm{GSp}_4(\mathbb{R})$.

Writing $\pi_f$ for the finite part of $\pi$, the following lemma describes the $L$-packet $\Pi$ associated to such $\pi_f$ and the automorphic multiplicities.

**Lemma 2.9.** *Let $\Pi$ be the $L$-packet of $\pi_f$ and consider the finite $L$-packet:*

$$\Pi_f = \{\pi'_f \text{ nearly equivalent to } \pi_f \mid \pi'_f \otimes \pi_\infty \text{ is automorphic for some } \pi_\infty\}.$$

*(1) If $\Pi$ is non-endoscopic, then*

$$\Pi = \{\pi'_f \otimes X^1_{\lambda;m}, \pi'_f \otimes X^2_{\overline{\lambda};m} \mid \pi'_f \in \Pi_f\}$$

*consists of two elements for each $\pi'_f \in \Pi_f$:*

$$\pi^h = \pi'_f \otimes X^1_{\lambda;m} \qquad \textit{holomorphic representation}, \tag{2.4}$$

$$\pi^g = \pi'_f \otimes X^2_{\overline{\lambda};m} \qquad \textit{generic representation}, \tag{2.5}$$

*and all of them occur in the automorphic spectrum with multiplicity one.*

*(2) If $\Pi$ is weakly endoscopic* (Yoshida type)*, associated to an automorphic representation $\sigma$ of $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, then there is a bijection:*

$$\{S \subseteq \{v \text{ finite place} \mid \sigma_v \text{ is discrete series}\}\} \to \Pi \tag{2.6}$$

$$S \mapsto \pi_S = \bigotimes (\pi_S)_v$$

*where $(\pi_S)_v$ is generic if and only if $v \notin S$ and*

$$(\pi_S)_\infty = \begin{cases} X^1_{\lambda;m} & |S| \text{ is odd}, \\ X^2_{\overline{\lambda};m} & |S| \text{ is even}. \end{cases} \tag{2.7}$$

*Each $\pi_S$ occurs in the automorphic spectrum with multiplicity one.*

*More specifically, $\pi_S$ is constructed as a theta lift from a definite or indefinite quaternion algebra over $\mathbb{Q}$ ramified at the finite places $S$.*

*Proof.* This follows from Arthur's classification [Art04, GT19]. More specifically, see [LSZ17, Theorem 10.1.3] for (1) and [Rob01] or [Wei09, Corollary 5.4] for (2). □

**Remark 2.10.** The local Langlands correspondence for GSp(4) [GT11] together with the Arthur multiplicity formula give a complete description of the automorphic contributions of $\Pi_f$, but we decided not to spell this out here. In the case of Yoshida lifts from Hilbert modular surfaces, see also results of Roberts [Rob01].

Later, we will focus on the unique element $\pi'_f \in \Pi_f$ which is generic at all places, because we will be interested in representations of paramodular level.

Based on these results and Theorem 2.8, we can describe the contributions of this $L$-packet $\Pi$ to coherent cohomology. Following [PV21], we consider the $\Pi$-isotypic component of the cohomology groups.

**Definition 2.11.** Let $\pi_f$ be a representation of $\mathrm{GSp}_4(\mathbb{A})$ such that $\pi_f^{K_f} \neq 0$ and let $\chi \colon \mathcal{H}^{K_f} \to E$ be the associated character of the spherical Hecke algebra, away from the places where $K_{f,v}$ is not hyperspecial. Then the $\Pi$-*isotypic component* of $H^*(X, \mathcal{E}^\bullet)$ for $\bullet \in \{\mathrm{sub}, \mathrm{can}\}$ is:

$$H^*(X, \mathcal{E}^\bullet)_\Pi = \{\eta \in H^*(X, \mathcal{E}^\bullet) \mid T\eta = \chi(T)\eta \text{ for all } T \in \mathcal{H}^{K_f}\}.$$

Assuming the elements of $\Pi$ are cuspidal, we consider:

$$H^*(X, \mathcal{E})_\Pi = \mathrm{Im}(H^*(X, \mathcal{E}^{\mathrm{sub}})_\Pi \to H^*(X, \mathcal{E}^{\mathrm{can}})_\Pi).$$

We will use the same normalization of contributions to cohomology as as [LPSZ21], i.e. we set $m = k_1+k_2-6$.

The next goal is to compute $H^*(X, \mathcal{E})_\Pi$ for the $L$-packets $\Pi$ described in Lemma 2.9.

**Theorem 2.12** ([LPSZ21, Theorem 5.2])**.** *Let $\pi$ be a cuspidal, non-CAP automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ whose component at $\infty$ is a (limit of) discrete series representation associated to the parameter $(\lambda; m)$. Let $\Pi$ be the $L$-packet associated to $\pi_f$.*

*Consider the four vector bundles associated to the following four representations:*

$$\mathcal{E}_0 = \mathcal{E}_{(k_1,k_2;m)} = \mathcal{E}_{(\lambda_1+1,\lambda_2+2;m)}, \qquad V_0 = V_{k_1,k_2;m}, \tag{2.8}$$

$$\mathcal{E}_1 = \mathcal{E}_{(k_1,4-k_2;m)} = \mathcal{E}_{(\lambda_1+1,-\lambda_2+2;m)}, \qquad V_1 = V_{k_1,4-k_2;m}, \tag{2.9}$$

$$\mathcal{E}_2 = \mathcal{E}_{(k_2-1,3-k_1;m)} = \mathcal{E}_{(\lambda_2+1,-\lambda_1+2;m)}, \qquad V_2 = V_{k_2-1,3-k_1;m}, \tag{2.10}$$

$$\mathcal{E}_3 = \mathcal{E}_{(3-k_2,3-k_1;m)} = \mathcal{E}_{(-\lambda_2+1,-\lambda_1+2;m)}, \qquad V_4 = V_{3-k_2,3-k_1;m}. \tag{2.11}$$

*with notation as in equation (2.1) and $\lambda = (\lambda_1, \lambda_2) = (k_1 - 1, k_2 - 2)$.*

*(1) Suppose first that $\Pi$ is non-endoscopic. Then:*

$$H^i(X_{\mathbb{C}}, \mathcal{E}_i)_\Pi \cong \bigoplus_{\pi'_f \in \Pi_f} \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, X^1_{\lambda;m}\right) \otimes (\pi'_f)^{K_f} \qquad i = 0, 3, \tag{2.12}$$

$$H^i(X_{\mathbb{C}}, \mathcal{E}_i)_\Pi \cong \bigoplus_{\pi'_f \in \Pi_f} \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, X^2_{\lambda;m}\right) \otimes (\pi'_f)^{K_f} \qquad i = 1, 2. \tag{2.13}$$

*(2) Suppose next that $\Pi$ is endoscopic associated to $\sigma$. Using the notation of Lemma 2.9, we have that:*

$$H^i(X_{\mathbb{C}}, \mathcal{E}_i)_\Pi \cong \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, X^1_{\lambda;m}\right) \otimes \bigoplus_{\substack{S \subseteq S(\sigma) \\ |S| \text{ odd}}} \pi_{S,f}^{K_f} \qquad i = 0, 3, \tag{2.14}$$

$$H^i(X_{\mathbb{C}}, \mathcal{E}_i)_\Pi \cong \mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, X^2_{\lambda;m}\right) \otimes \bigoplus_{\substack{S \subseteq S(\sigma) \\ |S| \text{ even}}} \pi_{S,f}^{K_f} \qquad i = 1, 2. \tag{2.15}$$

*In particular, in this case, a finite representation $\pi_f \in \Pi_f$ contributes to either $H^0$ and $H^3$ or $H^1$ and $H^2$ but not both.*

*Finally, each of the spaces* $\mathrm{Hom}_K(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, X^j_{\mu;m})$ *above is 1-dimensional. Moreover, for all* $i$ *and all automorphic sheaves* $\mathcal{E} = \mathcal{E}_\varrho^{\mathrm{can}}$, *we have:*

$$H^i(X_{\mathbb{C}}, \mathcal{E})_\Pi = 0 \qquad \text{if } \mathcal{E} \neq \mathcal{E}_i.$$

Figure 2.1 gives a graphical interpretation of these result.

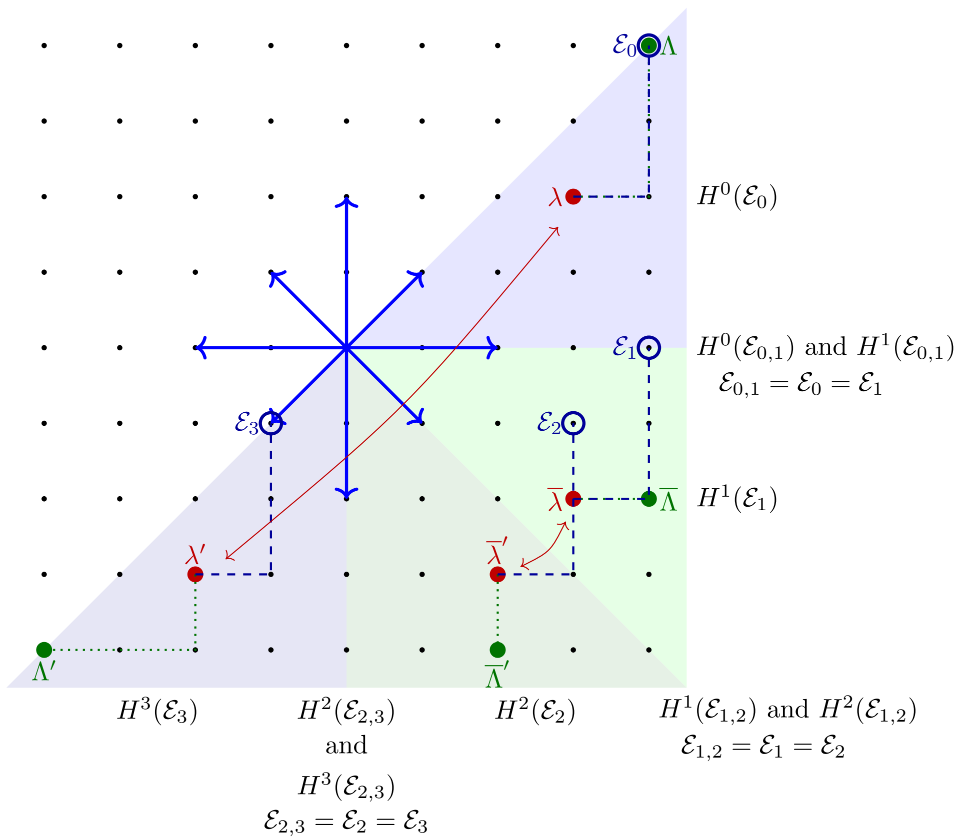


FIGURE 2.1. Contributions to coherent cohomology according to the Harish–Chandra parameter $\lambda$. In this example, $\lambda = (3, 2)$. The Harish–Chandra $\lambda$ parameters are labeled by red dots, the Blattner parameters $\Lambda$, given by Table A.1, are labeled by green dots, and the $K$-types for the vector bundles $\mathcal{E}_i$ are labeled by blue circles. Serre duality corresponds to reflection about the line $\lambda_1 = -\lambda_2$.

Before giving a proof of the theorem, we give a lemma. Recall that $(\mathfrak{P}, K)$-cohomology is the cohomology of the complex with $j$-th term

$$\mathrm{Hom}_K\left(\bigwedge\nolimits^j \mathfrak{p}_- \otimes V_\varrho^\vee, \mathcal{A}(G)^{K_f}\right).$$

We observe that this complex degenerates as follows.

**Lemma 2.13.** *For* $k_1 \geq k_2 \geq 2$ *with* $m \equiv k_1 + k_2 \mod 2$, *let* $(\lambda_1, \lambda_2) = (k_1 - 1, k_2 - 2)$ *and consider* $V_0, V_1, V_2, V_3$ *as in equations* (2.8)–(2.11). *Then:*

$$\dim \mathrm{Hom}_K\left(\bigwedge\nolimits^j \mathfrak{p}_- \otimes V_i^\vee, X^1_{\lambda;m}\right) = \begin{cases} 1 & \text{if } i = j = 0 \text{ or } i = j = 3, \\ 0 & \text{otherwise,} \end{cases} \tag{2.16}$$

$$\dim \mathrm{Hom}_K\left(\bigwedge\nolimits^j \mathfrak{p}_- \otimes V_i^\vee, X^2_{\overline{\lambda};m}\right) = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = j = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

*Proof.* The central characters of the two representations agree and hence we may restrict to $K^\circ$. By Remark 2.3, the highest weight in the $K^\circ$-representation $V^\vee_{r_1,r_2;m}|_{K^\circ}$ is $(r_1,r_2)$. Finally, the highest weight of $\bigwedge^j \mathfrak{p}_-$ is $(0,0)$ when $j=0$, $(0,-2)$ when $j=1$, $(-1,-3)$ when $j=2$, $(-3,-3)$ when $j=3$.

The lemma then follows by recalling the $K^\circ$-types occurring in $X^1_\lambda$ and $X^2_{\overline{\lambda}}$, as described in [Sch17, Sec. 2.2] (see also [Mui09, Lemma 6.1] and Figure A.2). This is summarized in Figure 2.2. ☐

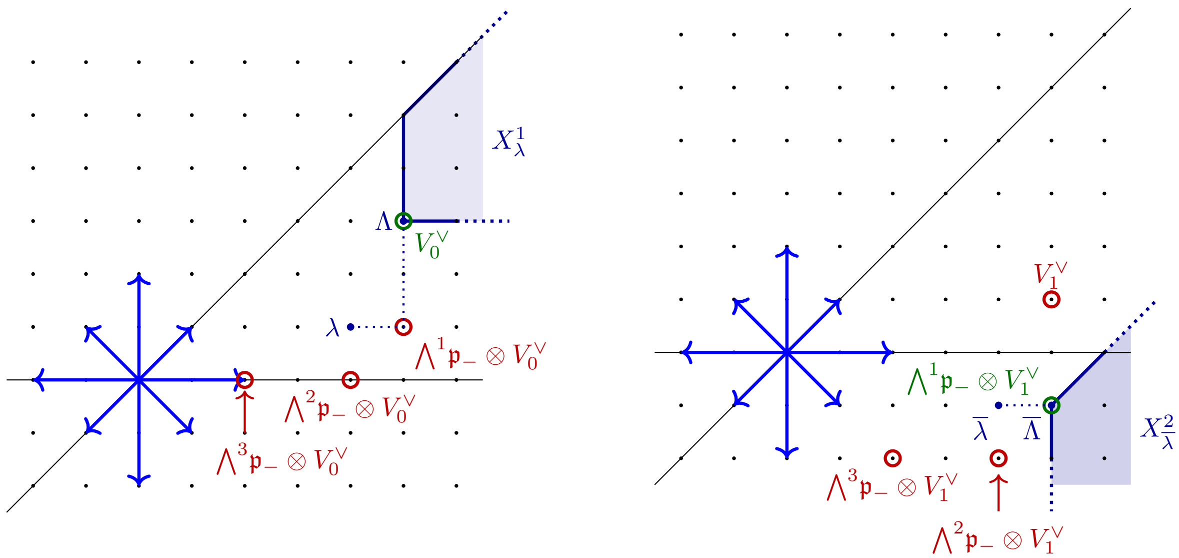


FIGURE 2.2. This diagram indicates the proof of Lemma 2.13 for the representation $V_0$ on the left hand side and $V_1$ on the right hand side when $\lambda=(4,1)$. The shaded regions represent the $K^\circ$-types occurring in $X^1_\lambda$ and $X^2_{\overline{\lambda}}$ and their minimal $K^\circ$-types are indicated by $\Lambda$ and $\overline{\Lambda}$. We also label the representations $\bigwedge^j\mathfrak{p}_-\otimes V_i^\vee$ for $j=0,1,2,3$ and $i=0,1$. One obtains similar diagrams for $i=2,3$ by reflecting over the $\lambda_2=-\lambda_1$ axis.

*Proof of Theorem 2.12.* The proof is a computation based on Theorem 2.8. Taking $\Pi$-isotypic components of both sides of (2.3), we obtain:

$$H^i(X_{\mathbb{C}},\mathcal{E}^{\mathrm{can}}_\varrho)_\Pi = \bigoplus_{\pi_f\otimes\pi_\infty\subseteq\mathcal{A}(G)} H^i(\mathfrak{P},K;\pi_\infty\otimes V_\varrho)\otimes\pi_f^{K_f} \tag{2.18}$$

$$= \bigoplus_{\pi_f\in\Pi_f}(H^i(\mathfrak{P},K;X^1_{\lambda;m}\otimes V_\varrho)\otimes\pi_f^{K_f})^{\oplus m_1(\pi_f)}\oplus(H^i(\mathfrak{P},K;X^2_{\overline{\lambda};m}\otimes V_\varrho)\otimes\pi_f^{K_f})^{\oplus m_2(\pi_f)} \tag{2.19}$$

where constants $m_1(\pi_f)$ and $m_2(\pi_f)$ are appropriate multiplicities in the automorphic spectrum. Then the multiplicity results of Lemma 2.9 together with Lemma 2.13 complete the proof. ☐

The theorem allows us to discuss the contributions to cohomology of vectors in automorphic representations. We fix once and for all the standard bases of $\bigwedge^i\mathfrak{p}_-$ and the representations $V_i$ as in [Mor04, §1.2]. This gives a choice of a highest weight vector in $\bigwedge^i\mathfrak{p}_-\otimes V_i^\vee$ which we denote by $v_i$.

**Definition 2.14.** Let $\pi$ be a non-CAP cuspidal automorphic representation of $G$ such that $\pi_\infty$ is a (non-degenerate limits of) discrete series representation. Let $f=\bigotimes f_v\in\pi$ be a factorizable vector such that:

(1) $f_v\in\pi_v^{K_{f,v}}$ for each finite $v$,

(2) $f_\infty$ is a highest weight vector in a minimal $K$-type of $\pi_\infty$.

The *contribution of $f$ to cohomology* is the associated element $[f] \in H^i(X_\mathbb{C}, \mathcal{E}_i)_\Pi$ defined by the isomorphism from Theorem 2.12:

$$\mathrm{Hom}_K\left(\bigwedge^i \mathfrak{p}_- \otimes V_i^\vee, \pi_\infty\right) \otimes \pi_f^{K_f} \xrightarrow{\cong} H^i(X_\mathbb{C}, \mathcal{E}_i)_\Pi$$
$$\left([v_i \mapsto f_\infty] \otimes \left[\bigotimes_{v<\infty} f_v\right]\right) \mapsto [f],$$

where $i$ is determined by $f_\infty$ and $\pi_\infty$.

**Remark 2.15.** In other words, a vector in an automorphic representation will give a contribution to cohomology once we choose an automorphic embedding of the representation. The difficulty of defining a motivic action for coherent cohomology is that the contributions in different degrees are associated to vectors in different elements of the $L$-packet and at the moment, there seems to be no canonical way to rigidify the choice of automorphic embedding across an $L$-packet.

The only exception seems to be Hilbert modular forms where partial complex conjugation operators give a way to go between different elements of the $L$-packet; this idea goes back to Shimura [Shi78] and Harris [Har90a, Har90c]. This gives a way to define the motivic action in this case [Hor23].

We record the parameters $\lambda$ which can lead to packets $\Pi$ making contributions to multiple degrees of cohomology of the same sheaf.

**Corollary 2.16.** *If $(k_1, k_2) = (k, 2)$ for $k \geq 2$, i.e. $\lambda = (k-1, 0)$, then*

$$H^i(X, \mathcal{E}_{k,2;m})_\Pi \neq 0 \text{ for } i = 0, 1,$$

*and*

$$H^i(X, \mathcal{E}_{1,3-k;m})_\Pi \neq 0 \text{ for } i = 2, 3.$$

The goal of the paper is to study the rationality of these multiple contributions from the point of view of algebraic cycles, using a certain motivic cohomology group.

**Remark 2.17.** There is also a different class of automorphic representations for which there is a different degeneracy of contributions to coherent cohomology. Consider a Harish–Chandra parameter $\lambda = (p, -p)$ and the associated limit of discrete series $X^\times_{\lambda;m}$ for some $m$, described in Appendix A. Then the archimedean $L$-packet of $X^\times_{\lambda;m}$ is the singleton $\{X^\times_{\lambda;m}\}$.

Let $\pi$ be a non-CAP cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ whose component at $\infty$ is the limit of discrete series $X^\times_{\lambda;m}$. Then the $L$-packet of $\Pi$ of $\pi$ contains no representations whose components at infinity are holomorphic (limits of) discrete series, i.e. the $L$-packet is not associated with any holomorphic Siegel modular form.

In this case, all the contributions of $\Pi$ to cohomology are given by:

$$H^i(X, \mathcal{E}_{p+1,2-p;m})_\Pi \neq 0 \text{ for } i = 1, 2.$$

It would be interesting to understand if there is a motivic action $H^1(X, \mathcal{E}_{p+1,2-p;m})_\Pi \to H^2(X, \mathcal{E}_{p+1,2-p;m})_\Pi$ in this case, but we decided not to pursue this here.

As in [LPSZ21, (5.1)], we may use Serre duality (2.22) to define a $\mathrm{GSp}_4(\mathbb{A}_f)$-equivariant pairing:

$$\langle -, - \rangle_{\mathrm{SD}} \colon H^i(X, \mathcal{E}_i) \times H^{3-i}(X, \mathcal{E}_{3-i}) \to \mathbb{Q}\{m\}, \tag{2.20}$$

where $\mathbb{Q}\{m\}$ is the $m$th power of the similitude character of $\mathrm{GSp}_4(\mathbb{A}_f)$. Explicitly:

$$\langle [f], [f'] \rangle_{\mathrm{SD}} = \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K} f(g) f'(g)\, dg \tag{2.21}$$

$$\sim_{\mathbb{Q}^\times} \frac{1}{\pi^3} \langle f, f' \rangle, \tag{2.22}$$

where we use the automorphic normalization of Petersson norm

$$\langle f, f' \rangle = \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} f(g) f'(g)\, dg, \tag{2.23}$$

and $dg$ is the Tamagawa measure as in [CI19].

## 3. Motives and Beilinson's conjecture

We refer to [PV21, Section 2] and [Jan88] for a detailed summary of motivic cohomology, Deligne's conjecture, and Beilinson's conjecture. We only offer a brief introduction of notation below.

For any subring $R \subseteq \mathbb{C}$, we write $R(j) = (2\pi i)^j R \subseteq \mathbb{C}$. We have a natural isomorphism $\mathbb{C} \cong \mathbb{R}(n) \oplus \mathbb{R}(n-1)$ and we write $\pi_{n-1} \colon \mathbb{C} \to \mathbb{R}(n-1)$ for the projection to $\mathbb{R}(n-1)$.

For a Chow motive $M$ over $\mathbb{Q}$, we will be interested in the Beilinson short exact sequences [PV21, (2.1.10), (2.10.11)]:

$$0 \to F^n H^i_{\mathrm{dR}}(M_{\mathbb{R}}) \overset{\widetilde{\pi}_{n-1}}{\to} H^i_B(M_{\mathbb{R}}, \mathbb{R}(n-1)) \to H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)) \to 0, \tag{3.1}$$

$$0 \to H^i_B(M_{\mathbb{R}}, \mathbb{R}(n)) \to H^i_{\mathrm{dR}}(M_{\mathbb{R}})/F^n H^i_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)) \to 0. \tag{3.2}$$

**Definition 3.1.** We define two *Hodge rational structures* on $\det H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n))$:

- $\mathcal{R}(M, i, n)$ is obtained from the short exact sequence (3.1) and the rational structures $F^n H^i_{\mathrm{dR}}(M)$ and $H^i_B(M_{\mathbb{R}}, \mathbb{Q}(n-1))$,
- $\mathcal{DR}(M, i, n)$ is obtained from the short exact sequence (3.2) and the rational structures $H^i_B(M_{\mathbb{R}}, \mathbb{Q}(n))$ and $H^i_{\mathrm{dR}}(M)/F^n H^i_{\mathrm{dR}}(M)$.

The two rational structures are related by the equations:

$$\mathcal{DR}(M, i, n) = (2\pi i)^{-d^-(M,i,n)} \cdot \delta(M, i, n) \cdot \mathcal{R}(M, i, n), \tag{3.3}$$

$$d^-(M, i, n) = \dim H^i_B(M_{\mathbb{C}}, \mathbb{Q}(n))^-, \tag{3.4}$$

$$\delta(M, i, n) = \det \left( H^i_B(M_{\mathbb{C}}, \mathbb{Q}(n)) \otimes \mathbb{C} \overset{\cong}{\to} H^i_{\mathrm{dR}}(M, n) \otimes \mathbb{C} \right). \tag{3.5}$$

Beilinson [Bei85] defined a regulator map:

$$r_{\mathcal{D}} \colon H^i_{\mathcal{M}}(M, \mathbb{Q}(j)) \otimes \mathbb{R} \to H^i_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(j)) \tag{3.6}$$

which gives a different rational structure

$$\det r_{\mathcal{D}}(H^{i+1}_{\mathcal{M}}(M, \mathbb{Q}(n))) \subseteq \det H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)).$$

Scholl [Sch00] showed that there is a unique $\mathbb{Q}$-subspace $H^{i+1}_{\mathcal{M}}(M_{\mathbb{Z}}, \mathbb{Q}(n)) \subseteq H^i_{\mathcal{M}}(M, \mathbb{Q}(j))$ such that if $X$ is a variety which has a regular integral model $\mathcal{X}$, then:

$$H^{i+1}_{\mathcal{M}}(h(X)_{\mathbb{Z}}, \mathbb{Q}(n)) = \mathrm{Im}(H^i_{\mathcal{M}}(h(\mathcal{X}), \mathbb{Q}(n) \to H^i_{\mathcal{M}}(h(X), \mathbb{Q}(n)).$$

Beilinson's conjecture predicts that the difference between the two rational structures is given by an appropriate $L$-value.

**Conjecture 3.2** (Beilinson)**.** *Suppose $n \geq \frac{i}{2} + 1$ and if $n = \frac{i}{2} + 1$, that $H^{2j}_{\text{ét}}(M_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(j))^{G_{\mathbb{Q}}} = 0$. Then:*

*(1) the Beilinson regulator $r_{\mathcal{D}} \colon H^{i+1}_{\mathcal{M}}(M, \mathbb{Q}(n)) \otimes \mathbb{R} \to H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n))$ is an isomorphism,*

*(2) we have, equivalently:*

$$r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(M_{\mathbb{Z}}, \mathbb{Q}(n))) = L^{-i}(M^\vee, 1-n)^* \cdot \mathcal{R}(M, i, n),$$
$$r_{\mathcal{D}}(\det H^{i+1}_{\mathcal{M}}(M_{\mathbb{Z}}, \mathbb{Q}(n))) = L^i(M, n) \cdot \mathcal{DR}(M, i, n).$$

We make the following hypothesis which we keep throughout the paper.

**Hypothesis 3.3.** *We assume that Beilinson's regulator is an isomorphism, i.e. part (1) of Beilinson's Conjecture 3.2.*

Deligne [Del79] defines the notion of a *critical value* $j \in \mathbb{Z}$ and defines periods $c^{\pm}(H^i(M(j))) \in \mathbb{C}^\times/\mathbb{Q}^\times$ such that $L^i(M,j) \sim_{\mathbb{Q}} c^+(H^i(M)(j))$. The $L$-values $L^i(M,n)$ and $L^{-i}(M^\vee, 1-n)^*$ are critical if and only if $H^{i+1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(n)) = 0$. In this case, the first maps in the short exact sequence (3.1) is an isomorphism and we can express Deligne's period in terms of its determinant:

$$c^+(H^i(M)(i+1-n)) = \det(F^n H^i_{\mathrm{d}R}(M_{\mathbb{R}}) \overset{\widetilde{\pi}_{n-1}}{\to} H^i_B(M_{\mathbb{R}}, \mathbb{R}(n-1))). \tag{3.7}$$

3.1. **Motives associated to Siegel modular forms.** Let $f$ be a holomorphic cuspidal Siegel modular form of weight $(k_1,k_2)$ for $k_2 \geq k_2 \geq 2$. For simplicity, assume that the central character of $f$ is trivial and the $f$ has coefficients in $\mathbb{Q}$. Recall that $(k_1,k_2) = (\lambda_1+1, \lambda_2+2)$, where $(\lambda_1,\lambda_2)$ is the Harish–Chandra parameter for the holomorphic (limit of) discrete series representation.

3.1.1. *The motive $M(f)$.* Conjecturally, there exists a pure Chow motive $M(f)$ of rank 4 and weight

$$w = \lambda_1 + \lambda_2 = k_1 + k_2 - 3$$

associated with $f$. Its Betti realization has a Hodge decomposition:

$$H_B^{\lambda_1+\lambda_2}(M(f)) \otimes \mathbb{C} \cong H^{\lambda_1+\lambda_2,0} \oplus H^{\lambda_1,\lambda_2} \oplus H^{\lambda_2,\lambda_1} \oplus H^{0,\lambda_1+\lambda_2}. \tag{3.8}$$

When $\lambda_2 > 0$, the associated Galois representation can be found in the étale cohomology of the Siegel threefold, and hence one can construct a Grothendieck motive associated to $f$ this way. However, for $\lambda_2 = 0$, the form $f$ is not cohomological and hence the motive cannot be constructed this way. In fact, the Galois representation is constructed using congruences with higher weight forms [Tay91].

Let $\psi$ be a Hecke character of finite order and consider the spin $L$-function $L(f,\psi,s) = L(M(f)(\psi), s)$.

**Lemma 3.4.** *The critical values of $L(f,\psi,s)$ are:*

$$\{n \in \mathbb{Z} \mid \lambda_2 + 1 \leq n \leq \lambda_1\}.$$

*In particular, since $\lambda_1 > \lambda_2$, $L(f,\chi,s)$ always has a critical value. For these $n$, according to Deligne's conjecture:*

$$L(f,\psi,n)/(2\pi i)^{2n} g(\psi)^2 c^{\pm}(H^w(M(f))) \in \mathbb{Q}(f,\psi)$$

*where $\pm 1 = (-1)^{n+e(\psi_\infty)}$ and $e(\psi_\infty) = 0$ if $\psi_\infty$ is trivial and 1 otherwise.*

*Proof.* This is a standard computation with $\Gamma$-factors which can be described using the Hodge decomposition (3.8). For example, see [Har04]. □

The Hodge filtration on the de Rham realization $H^w_{\mathrm{dR}}(M(f))$ has four steps:

$$H^w_{\mathrm{dR}}(M(f)) = F^0 \supseteq F^{\lambda_2} \supseteq F^{\lambda_1} \supseteq F^{\lambda_1+\lambda_2} = H^{\lambda_1+\lambda_2,0}. \tag{3.9}$$

These containments are all strict unless $\lambda_2 = 0$.

We choose a basis $\{\omega_i\}$ for $H^w_{\mathrm{dR}}(M(f))$ compatible with the filtration, i.e.

$$F^{\lambda_1+\lambda_2} = \mathrm{span}\{\omega_1\}, \tag{3.10}$$

$$F^{\lambda_1} = \mathrm{span}\{\omega_1,\omega_2\}, \tag{3.11}$$

$$F^{\lambda_2} = \mathrm{span}\{\omega_1,\omega_2,\omega_3\}, \tag{3.12}$$

$$F^{0} = \mathrm{span}\{\omega_1,\omega_2,\omega_3,\omega_4\} \tag{3.13}$$

In the degenerate case $\lambda_2 = 0$, the basis only satisfies equations (3.11) and (3.13).

Moreover, we a decomposition into 2-dimensional eigenspaces for the action of the involution $F_\infty$:

$$H^w_B(M(f)_{\mathbb{C}}, \mathbb{Q}) = H^w_B(M(f)_{\mathbb{C}}, \mathbb{Q})^+ \oplus H^w_B(M(f)_{\mathbb{C}}, \mathbb{Q})^-.$$

We choose bases $\{v_1^\pm, v_2^\pm\}$ of $H_B^w(M(f)_\mathbb{C}, \mathbb{Q})^\pm$. Suppose that under the de Rham-Betti comparison $c_{\mathrm{dR},B}$:

$$c_{\mathrm{dR,B}}(\omega_i) = c_{i1}^+ v_1^+ + c_{i2}^+ v_2^+ + c_{i1}^- v_1^- + c_{i2}^- v_2^- \qquad \text{for } i = 1, 2 \text{ and some } c_{ij}^\pm \in \mathbb{C}. \tag{3.14}$$

**Lemma 3.5.** *In the notation of equation (3.14), we have that:*

$$c^{\pm(-1)^{\lambda_2}}(M(f)(\lambda_1)) = (2\pi i)^{-2\lambda_2} \det(c_{ij}^\pm)_{1\le i,j\le 2} = (2\pi i)^{-2\lambda_2}(c_{11}^\pm c_{22}^\pm - c_{12}^\pm c_{21}^\pm).$$

*Proof.* From equation (3.7), we have that:

$$c^\pm(M(f)(\lambda_1)) = \det(F^{\lambda_2+1} H_{\mathrm{dR}}^w(M(f))_\mathbb{R} \to H_B^w(M(f)_\mathbb{R}, \mathbb{R}(\lambda_2))^\pm)$$

and a rational basis of the de Rham cohomology group is $\omega_1, \omega_2$, while a rational basis of the Betti cohomology group is $(2\pi i)^{\lambda_2} v_i^\pm$. □

## 3.2. The motives $\mathrm{Sym}^2 M(f)$ and $M(f, \mathrm{Ad})$.

We refer to [PV21, Definition 4.2.1] for the definition of the adjoint motive associated to an automorphic representation $\pi$. In the case where $\pi$ is associated with a Siegel modular form $f$, we define $M(f, \mathrm{Ad}) = \mathrm{Ad}\, M(\pi)$ to be the conjectural adjoint motive of $\pi$ of rank 10 and pure weight 0.

It turns out that the motive $M(f, \mathrm{Ad})$ has a simple description in term of the motive $M(f)$ of $f$ described above. We consider the symmetric square motive $\mathrm{Sym}^2 M(f)$ which has rank 10 and weight $2w = 2(\lambda_1+\lambda_2)$. Then:

$$L(\mathrm{Sym}^2 M(f), s) = L(M(f, \mathrm{Ad}), s - w)$$

and hence $M(f, \mathrm{Ad}) \cong \mathrm{Sym}^2 M(f)(w)$. Therefore, it is enough to consider $H^{2w}(\mathrm{Sym}^2 M(f))$. The Hodge numbers of $\mathrm{Sym}^2 M(f)$ are:

$$\{\{(p, 2w - p), (2w - p, p)\} \mid p = 2(\lambda_1 + \lambda_2), 2\lambda_1 + \lambda_2, 2\lambda_1, \lambda_1 + 2\lambda_2, \lambda_1 + \lambda_2\}.$$

**Lemma 3.6.** *The critical values for the symmetric square $L$-function $L(\mathrm{Sym}^2 M(f), s)$ are:*

$$\{n \text{ odd} \mid \lambda_1 + 1 \le n \le \lambda_1 + \lambda_2\} \cup \{n \text{ even} \mid \lambda_1 + \lambda_2 + 1 \le n \le \lambda_1 + 2\lambda_2\}.$$

*In particular, the $L$-function $L(\mathrm{Sym}^2 M(f), s)$ has a critical value if and only if $\lambda_2 > 0$.*

*Proof.* The $\Gamma$-factors are:

$$\Gamma_\mathbb{C}(s)\Gamma_\mathbb{C}(s-\lambda_2)\Gamma_\mathbb{C}(s-2\lambda_2)\Gamma_\mathbb{C}(s-\lambda_1)\Gamma_\mathbb{R}(s+1-(\lambda_1+\lambda_2))^2$$

and the proof is a standard computation. □

**Lemma 3.7** (Yoshida [Yos01, (4.15, 4.16)])**.** *Suppose $f$ is a holomorphic scalar-valued Siegel modular forms of weight $k \ge 3$, i.e. $(\lambda_1, \lambda_2) = (k-1, k-2)$ and $\lambda_2 > 0$ . For simplicity, assume again that $f$ has rational Fourier coefficients. Then:*

$$c^+(\mathrm{Sym}^2(M(f))) = (2\pi i)^{12-6k} c^+(M(f)) c^-(M(f)) \langle f, f\rangle, \tag{3.15}$$

$$c^-(\mathrm{Sym}^2(M(f))) = (2\pi i)^{6-2k} c^+(M(f)) c^-(M(f)) \langle f, f\rangle. \tag{3.16}$$

*Moreover, the Petersson inner products $\langle f, f\rangle$ are explicitly related to a critical value of the standard $L$-function of $f$.*

A similar formula for general weights $(\lambda_1, \lambda_2)$ with $\lambda_2 > 0$ can be derived from Yoshida's formula [Yos01, (3.4)] via the results of Kozima [Koz00].

When $\lambda_2 = 0$, we will derive an analogous formula for the non-critical $L$-value $L(\mathrm{Sym}^2 M(f), 2k-2)$, or equivalently the derivative $L'(\mathrm{Sym}^2 M(f), 1)$, in Theorem 5.7 from Beilinson's conjecture.

3.3. **Motives associated to abelian surfaces.** Let $A$ be an abelian surface over $\mathbb{Q}$. A precise statement of the modularity conjecture for abelian surfaces was stated by Brumer and Kramer [BK14].

**Conjecture 3.8** (Brumer–Kramer [BK14])**.** *Let $A$ be an abelian surface over $\mathbb{Q}$ of conductor $N$ with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. There exists a holomorphic cuspidal Siegel modular form $f$ for $\mathrm{GSp}_4$ with Harish–Chandra parameter $\lambda = (1,0)$ and paramodular level $K(N)$ such that:*

$$L(A,s) = L(f,s).$$

See [BPP+19, CCG20] for evidence for the conjecture.

**Remark 3.9.** In this extended remark, we elaborate on this conjecture and explain a few special cases.

First, note that we only stated one direction of the conjecture in [BK14]. The converse requires two modifications:

- one excludes Siegel modular forms which are *Gritsenko lifts* (a variant of the Saito–Kurokawa lift; see loc. cit. and the references therein for details)
- as pointed out by Frank Calegari, to get a bijection, one also needs to account for abelian fourfolds with quaternionic multiplication: a modified version of the conjecture that account for this was given in [BK19].

Next, we explain how the following examples fit in the framework of the conjecture:

(1) $A = E \times E$ for an elliptic curve $E$ over $\mathbb{Q}$,
(2) $A = E_1 \times E_2$ for two non-isogenous elliptic curves $E_1$ and $E_2$ over $\mathbb{Q}$,
(3) $A = R_{F/\mathbb{Q}}E$ for an elliptic curve $E$ over a quadratic field $F$.

The first example is excluded by the assumption that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$; indeed, the abelian surface $A = E \times E$ has an extra endomorphism $(P,Q) \mapsto (Q,P)$. On the side of Siegel modular forms, one could consider the weight two modular form $f_0$ associated with $E$. Then there is two ways to obtain a Siegel modular form from $f_0$:

- let $f$ be the Yoshida lift of $f_0 \times f_0$; however, then $f$ is not cuspidal,
- let $f$ be the Saito–Kurokawa lift of $f_0$; however, then $f$ is presumably a Gritsenko lift (although we have not verified this).

Therefore, case (1) is ruled out on both sides of the conjecture.

Case (2) is also ruled out by the assumption that $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. Indeed, $\mathrm{End}(E_1 \times E_2) \supseteq \mathbb{Z} \times \mathbb{Z}$. Nonetheless, one can consider the associated weight two modular forms $f_1$, $f_2$ and the Yoshida lift of $f_1$, $f_2$. As explained in Lemma 2.9, one always has the automorphic representation $\pi_\emptyset$ of $\mathrm{GSp}_4(\mathbb{A})$ which is generic at all places and hence paramodular. This means that there is a *generic* paramodular Siegel modular form $f$ with Harish–Chandra parameter $(1,0)$ associated with $f_1$ and $f_2$.

However, we have excluded this case on the side of Siegel modular forms by requiring the $f$ is holomorphic and paramodular. Indeed, if $S$ is an odd finite set of finite places such that the local representations associated with $f_1$ and $f_2$ are discrete series at these places (such a set may not exist in general), then the associated automorphic representation $\pi_S$ is not generic at $v \in S$ and hence not paramodular. In particular, there is no *holomorphic* paramodular Siegel modular form $f$ associated with $f_1$ and $f_2$. In fact, this is the reason we assume that the Siegel modular form is non-endoscopic in our work.

Finally, we consider $A = R_{F/\mathbb{Q}}E$. In this case, $E$ corresponds to an automorphic form for the group $\mathrm{GL}_{2,F}$, i.e. a Hilbert or Bianchi modular form. The conjecture has been verified in both cases [JLR12, BDPŞ15] using non-endoscopic Yoshida lifts from $\mathrm{GL}_{2,F}$: there is both a holomorphic and a generic paramodular Siegel modular form $f$ associated with $A$. Sections 6 and 7 of the present work are devoted to these special cases.

The key difference between case (2) and case (3) is that the $L$-packet on $\mathrm{GSp}_4$ in the latter case is stable, even though the automorphic representation can be constructed as a lift. This is explained by the following observation. For a real quadratic field $F$, there is a quaternion algebra $B$ over $F$ ramified at the two infinite places of $F$ and no finite places. However, when $F = \mathbb{Q} \oplus \mathbb{Q}$, a quaternion algebra $B$ over $F = \mathbb{Q} \oplus \mathbb{Q}$ is $B = B_1 \times B_2$ for quaternion algebras $B_i$ over $\mathbb{Q}$, and hence if we want $B$ to be ramified at the two infinite places, we need $B$ to also be ramified at some finite place.

In summary, one could extend Conjecture 3.8 as follows:

(1) if $A = E_1 \times E_2$ for non-isogenous elliptic curves $E_1$, $E_2$ over $\mathbb{Q}$, then there exists a *generic* Siegel modular form $f$ with Harish–Chandra parameter $\lambda = (1, 0)$ and paramodular level $K(N)$ such that $L(A, s) = L(f, s)$,

(2) if $A$ is a simple abelian surface with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$, then there exists both a *generic* and a *holomorphic* Siegel modular form $f$ with Harish–Chandra parameter $\lambda = (1, 0)$ and paramodular level $K(N)$ such that $L(A, s) = L(f, s)$.

In fact, case (1) is settled: $f$ is the generic Yoshida lift of $f_1 \times f_2$.

**Remark 3.10.** Potential modularity of abelian surfaces was recently proved [BCGP21]. The proof uses the realization of $f$ in the higher coherent cohomology group $H^1(X, \mathcal{E}_{2,2})$ and the modularity lifting results of Calegari–Geraghty [CG18]. It would be interesting to use the Calegari–Geraghty method to give a $p$-adic version of our conjecture, similar to [Ven19, HV19], and understand the connection to the potential modularity theorem.

Together with the Tate conjecture, Conjecture 3.8 implies that the motive $H^1(A)$ is the motive $M(f)$ for an appropriate Siegel modular form $f$ of weight $(2, 2)$. While the $L$-function $L(f, s)$ has a critical value in this case (Lemma 3.5), we see that $L(\mathrm{Sym}^2 M(f), s)$ does not (Lemma 3.6).

Finally, the constants $c^{\pm}_{i,j}$ in Lemma 3.5 may be interpreted in terms of a dual basis under the tautological pairing

$$\langle -, - \rangle \colon H^1(A(\mathbb{C}), \mathbb{Q}) \times H_1(A(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q} \tag{3.17}$$

given by $\langle \omega, \gamma \rangle = \int_\gamma \omega$. We write $\gamma_i^{\pm}$ for the basis of $H_1(A(\mathbb{C}), \mathbb{Q})^{\pm}$ dual to $v_i^{\pm}$, i.e.

$$\langle v_i^{\pm}, \gamma_j^{\pm} \rangle = \delta_{ij}.$$

Then:

$$c^{\pm}_{i,j} = \langle \omega_i, \gamma_j^{\pm} \rangle = \int_{\gamma_j^{\pm}} \omega_i. \tag{3.18}$$

Moreover, Poincaré duality defines a pairing

$$\langle -, - \rangle_{\mathrm{PD}} \colon H^1(A(\mathbb{C}), \mathbb{Q}) \times H^3(A(\mathbb{C}), \mathbb{Q}(2)) \to \mathbb{Q} \tag{3.19}$$

explicitly given after extending coefficients to $\mathbb{R}$ by:

$$\langle \eta_1, \eta_2 \rangle_{\mathrm{PD}} = \frac{1}{(2\pi i)^2} \int_{A(\mathbb{C})} \eta_1 \wedge \eta_2. \tag{3.20}$$

If $w_i^{\pm}$ is a basis of $H^3(A(\mathbb{C}), \mathbb{Q}(2))$ dual to $v_i^{\pm}$, then we also record that:

$$\langle \omega_i, w_j^{\pm} \rangle_{\mathrm{PD}} = \left\langle \sum_{i,k} c^{\pm}_{i,k} v_k^{\pm}, w_j^{\pm} \right\rangle_{\mathrm{PD}} = c^{\pm}_{i,j}. \tag{3.21}$$

## 4. Definition of the motivic action

Let $f$ be a non-endoscopic holomorphic Siegel modular form of weight $(k, 2)$ for $k \geq 2$ even, of paramodular level $K(N)$, and trivial character. The assumption that the character of $f$ is trivial is necessary to use the theory of newforms of [RS07], and this implies that the weight $k$ is even.

For example, if $A$ be an abelian surface over $\mathbb{Q}$ with $\mathrm{End}(A) = \mathbb{Z}$ and conductor $N$, we can let $f$ be the Siegel modular form of weight 2 associated with $A$ via the Brumer–Kramer Conjecture 3.8. The assumption that $f$ is non-endoscopic amounts to the assumption that $A$ is simple (i.e not the product of two elliptic curves).

Let $\pi$ be the automorphic representation associated with $f$, $\pi_f$ be its finite component, and $\Pi$ be the associated $L$-packet. Consider the Siegel modular variety $X$ of paramodular level $K(N)$. Then, according to Corollary 2.16, the $\Pi$-isotypic component of $H^*(X, \mathcal{E}_{k,2})$ is:

$$H^*(X, \mathcal{E}_{k,2})_\Pi \cong H^0(X, \mathcal{E}_{k,2})_\Pi \oplus H^1(X, \mathcal{E}_{k,2})_\Pi, \tag{4.1}$$

where the dimension of each summand is equal to $\sum_{\pi'_f \in \Pi_f} \dim((\pi'_f)^{K(N)})$. Moreover, we will see in the next section that $(\pi'_f)^{K(N)}$ only for the unique generic element $\pi_f$ of the $L$-packet, and in this case $\dim(\pi'_f)^{K(N)} = 1$. Therefore, under our assumptions, each of the summands in equation (4.1) is 1-dimension. The goal of this section is to explain this degeneracy in contributions using a motivic action.

### 4.1. Theory of local newforms and the Whittaker rational structure on $\pi_f$.

Roberts and Schmidt [RS07] developed a theory of local newforms for $\mathrm{GSp}_4$ for paramodular level structures. We summarize these results in this section.

Let $F$ be a non-archimedean local field of characteristic 0, and let $\mathfrak{p}$ be the prime ideal of the ring of integers of $F$. Let $K(\mathfrak{p}^n)$ be the paramodular group of level $\mathfrak{p}^n$:

$$K(\mathfrak{p}^n) = \begin{pmatrix} \mathcal{O} & \mathfrak{p}^n & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-n} \\ \mathcal{O} & \mathfrak{p}^n & \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathcal{O} \end{pmatrix}. \tag{4.2}$$

**Remark 4.1.** Roberts and Schmidt [RS07] use $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$ instead of $J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}$ which accounts for the discrepancy in the definitions of the paramodular groups.

**Definition 4.2.** An irreducible, admissible representation $\pi$ of $\mathrm{GSp}(4, F)$ is *paramodular* if $\pi^{K(\mathfrak{p}^n)} \neq 0$ for some $n \gg 0$. If $n \geq 0$ is minimal with the property, then $K(\mathfrak{p}^n)$ is called the *minimal paramodular level.*

In particular, if $f$ is a Siegel modular form of paramodular level, then by definition any local component $\pi_v$ for finite $v$ of the associated automorphic representation is paramodular.

**Theorem 4.3** (Roberts–Schmidt [RS07, Theorem 7.5.1])**.** *Let $(\pi, V)$ be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. If $\pi$ is paramodular and $K(\mathfrak{p}^n)$ is its minimal paramodular level, then $\dim V^{K(\mathfrak{p}^n)} = 1$.*

This allows us to put a rational structure on the representation $\pi_f$ of $G(\mathbb{A}_f)$ associated to a Siegel modular form with paramodular level structure.

**Corollary 4.4.** *Suppose $(\pi_f, V)$ be an irreducible, admissible representation of $G(\mathbb{A}_f)$ such that $\pi_f = \bigotimes_v \pi_v$ and each $\pi_v$ is paramodular with trivial central character. Then there is a rational structure on $(\pi_f, V)$, i.e. a non-zero $G(\mathbb{A}_f)$-stable subspace $V_0 \subseteq V$ defined over $\mathbb{Q}$.*

*Moreover, the choice of $V_0$ is equivalent to the choice of a vector in the 1-dimensional vector space $V_v^{K(\mathfrak{p}^n)}$ for each finite place $v$ and corresponding prime $\mathfrak{p}$, where $K(\mathfrak{p}^n)$ is the minimal paramodular level of $(\pi_v, V_v)$.*

*Proof.* The result follows Theorem 4.3 and [Wal85, Lemma I.1] applied to $H = K(\mathfrak{p}^n)$, $\chi = 1$, because we have assumed that $\mathbb{Q}(\pi_f) = \mathbb{Q}$. □

**Remark 4.5.** This rational structure is denoted by $\pi'_f$ in [LPSZ21].

**Remark 4.6.** Let $(\pi, V)$ be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Robert–Schmidt [RS07, Theorems 7.5.1, 7.5.8] also prove that:

- if $\pi$ is paramodular, then $\pi$ is generic,
- if $\pi$ is tempered, then $\pi$ is paramodular if and only if it is generic.

Therefore, our paper deals exactly with the automorphic representations whose local components are tempered and generic.

4.2. **The rational structure on $\pi_\infty$.** Let $\pi$ be an automorphic representation of $\mathrm{GSp}_4(\mathbb{R})$ associated to a Siegel modular form with paramodular level structure. Then $\pi = \pi_f \otimes \pi_\infty$ for a discrete series representation $\pi_\infty$, and we know that $\pi_\infty$ is one of the following representations:

(1) $X^1_{\lambda;m}$, holomorphic (limit of) discrete series,
(2) $X^2_{\lambda;m}$, generic (limit of) discrete series,

(Lemma 2.9). Moreover, each such $\pi_f \otimes \pi_\infty$ occurs in the automorphic spectrum with multiplicity one. The goal of this section is to put a rational structure on $\pi_\infty$ in each case.

Let $f = \bigotimes_v f_v \in \pi$ be a non-zero cusp form satisfying:

(1) for each finite $v$, let $K(v^n)$ be the minimal paramodular level of $\pi_v$, and let $f_v \in \pi_v^{K(v^n)}$,
(2) if $\pi_\infty = X^1_\lambda$, $f_\infty$ is any vector in the minimal $K_\infty$-type $V_{(\lambda_1+1,\lambda_2+2)}$,
(3) if $\pi_\infty = X^2_\lambda$, $f_\infty$ is any vector in the minimal $K_\infty$-type $V_{(\lambda_1+1,-\lambda_2)}$.

(See Appendix A for a more detailed discussion of the representation theory of $\mathrm{GSp}_4(\mathbb{R})$ and Figure 2.1 for the location of the minimal $K$-types of $\pi_\infty$). By multiplicity one, such a vector $f$ is characterized by these properties up to a constant.

4.2.1. *The generic representation $\pi_\infty = X^2_\lambda$.* For the generic representation $\pi_\infty = X^2_\lambda$, we note that $\pi$ is globally generic, and hence we can use a global Whittaker model to normalize the choice of $f$.

Let $W$ be the Whittaker functional associated to $f$ and the choice of character $\psi_U$ of the unipotent radical $U$ of the standard Borel (chosen as in [CI19, Section 2.1]):

$$W(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\overline{\psi_U(u)}\,du. \tag{4.3}$$

We decompose $W = \prod_v W_v$ as a product of local Whittaker functionals $W_v$ for $\pi_v$ with respect to $\psi_{U,v}$.

**Definition 4.7.** A vector $f = \bigotimes_v f_v \in \pi$ is *Whittaker-normalized* if:

(1) for $v$ finite, $W_v$ is chosen so that $W_v(1) = 1$,
(2) for $v = \infty$, we normalize $W_\infty$ so that $W_\infty(w_\infty)$ is given by the expression [CI19, (1.2)], where $w_\infty = \mathrm{diag}(1, 1, -1, -1)$ *in* $\mathrm{GSp}_4(\mathbb{R})$.

We write $f^W = \bigotimes f^W_v$ for a Whittaker-normalized vector in $\pi$.

**Remark 4.8.** Comparing our Whittaker-normalized vector $f^W$ to the one used by Chen–Ichino [CI19], we see that $f^W = w_\infty(f^{\mathrm{CI}})$, since we chose a vector with $f_\infty \in V_{\lambda_1+1,-\lambda_2}$ while they choose a vector with $f_\infty \in V_{\lambda_2,-\lambda_1-1}$. See also [Mor04] for a discussion of these normalizations. With these choices, the resulting cohomology class $[w_\infty(f^W)] \in H^2(X_\mathbb{C}, \mathcal{E}_2)_f$ is called *Whittaker-$\mathbb{Q}$-rational* in the terminology of [LPSZ21].

**Remark 4.9.** We use the normalization of the Haar measures in Chen–Ichino [CI19], i.e. the Haar measure on $F_v^\times$ is chosen so that $\mathcal{O}_v^\times$ volume 1. This differs from [RS07] who use $d^\times x = dx/|\cdot|$. In any case, the normalization $W_v(1) = 1$ in our notation agrees with the normalization $Z(W_v, s) = L(\pi_v, s)$ used by Roberts–Schmidt [RS07].

Finally, we compare the Whittaker normalized vector to the rational structure on coherent cohomology. Recall that in Definition 2.14, given a factorizable vector $f = \bigotimes f_v \in \pi$ with $f_v \in \pi_v^{K(\mathfrak{p}^n)}$ and $f_\infty \in \pi_\infty$ a highest weight vector in a minimal $K$-type, we defined its *contribution to cohomology* $[f] \in H^i(X_\mathbb{C}, \mathcal{E}_i)$ with $i$ determined by $f_\infty \in \pi_\infty$ via Theorem 2.12 (see Figure 2.1). In particular, the assumption that $f_\infty \in V_{(\lambda_1+1,-\lambda_2)}$ implies that:

$$[f^W] \in H^1(X_\mathbb{C}, \mathcal{E}_1)_\Pi. \tag{4.4}$$

Next, note that $w_\infty(f_\infty) \in V_{(\lambda_2,-\lambda_1-1)}$, and hence:

$$[w_\infty(f^W)] \in H^2(X_\mathbb{C}, \mathcal{E}_2)_\Pi. \tag{4.5}$$

Since $\dim H^2(X_\mathbb{C}, \mathcal{E}_2)_\Pi = 1$, we can rescale this cohomology class to be rational.

**Definition 4.10.** The *Whittaker period* $c^W(\pi_f)$ associated with $\pi_f$ is a complex number, well-defined up to $\mathbb{Q}^\times$, such that

$$\frac{[w_\infty(f^W)]}{c^W(\pi_f)} \in H^2(X_\mathbb{Q}, \mathcal{E}_2)_\Pi$$

is rational in coherent cohomology. We will also write $c^W(f)$ for $c^W(\pi_f)$.

**Remark 4.11.** This agrees with the Whittaker period $\Omega^W(f)$ in [LPSZ21, Section 10.1]. The idea to define periods using contributions to higher coherent cohomology goes back to Harris [Har04], who used Bessel models instead of Whittaker models.

4.2.2. *The holomorphic representation* $\pi_\infty = X_\lambda^1$. When $\pi_\infty = X_\lambda^1$, the representation $\pi$ is not globally generic. We instead give a rational structure on $\pi_\infty$ using coherent cohomology. This idea goes back to [BHR94] who used it to prove that there is a rational structure on $\pi_f$. Since we already put a rational structure on $\pi_f$, we will instead use it to define a rational structure on $\pi_\infty$.

Let $f = \bigotimes f_v \in \pi$ and assume that for every finite $v$, $f_v$ is Whittaker-normalized according to Definition 4.7. To be more precise, by Lemma 2.9, the $L$-packet of $\pi_f$ consists of $\pi^h = \pi_f \otimes X_\lambda^1$ and $\pi^g = \pi_f \otimes X_\lambda^2$ and we choose $f_v$ for finite $v$ so that the resulting vector in $\pi^g$ is Whittaker-normalized. We then consider the associated cohomology class $[f] \in H^0(X, \mathcal{E}_0)_\Pi$ for $f$ as above.

**Definition 4.12.** A vector $f_\infty \in X_\lambda^1$ is *rational* if the resulting vector $f = \bigotimes f_v \in \pi$ gives an $\mathbb{Q}$-rational section $[f] \in H^0(X, \mathcal{E}_0)_\Pi$.

In other words, we get a rational structure on $X_\lambda^1$ by using the isomorphism:

$$H^0(X_\mathbb{C}, \mathcal{E}_0)_\Pi \cong \mathrm{Hom}_K(V_0^\vee, X_{\lambda;m}^1) \otimes \pi_f^{K_f},$$

and the rational structures on $\pi_f^{K_f}$ defined in Corollary 4.4 and $H^0(X_\mathbb{C}, \mathcal{E}_0)_\Pi$ given by the rational coherent cohomology $H^0(X, \mathcal{E}_0)_\Pi$.

4.3. **The motivic action.** We are finally ready to define the motivic action and state our main theorem. As above, $f$ is a holomorphic cuspidal Siegel modular form of weight $(k, 2)$ for $k \geq 2$ even, paramodular level $N$, with trivial central character, and defined over $\mathbb{Q}$.

We will be interested in the adjoint motive $M(f, \mathrm{Ad})$ and we recall that $M(f, \mathrm{Ad}) = \mathrm{Sym}^2 M(f)(k-1)$. We record the appropriate Beilinson short exact sequence (3.1):

$$0 \to \underbrace{F^k H_{\mathrm{dR}}^{2k-2}(\mathrm{Sym}^2 M(f)_\mathbb{R})}_{\dim_\mathbb{R}=3} \xrightarrow{\tilde{\pi}_{k-1}} \underbrace{H_B^{2k-2}(\mathrm{Sym}^2 M(f)_\mathbb{R}, \mathbb{R}(k-1))}_{\dim_\mathbb{R}=4} \to \underbrace{H_\mathcal{D}^{2k-1}(\mathrm{Sym}^2 M(f)_\mathbb{R}, \mathbb{R}(k))}_{\dim_\mathbb{R}=1} \to 0, \tag{4.6}$$

where $H^{2k-1}_{\mathcal{D}}(\mathrm{Sym}^2 M(f)_{\mathbb{R}}, \mathbb{R}(k)) \cong H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$, etc.

Because $\dim H^{2k-1}_{\mathcal{D}}(\mathrm{Sym}^2 M(f)_{\mathbb{R}}, \mathbb{R}(k)) = 1$, the Deligne cohomology group is naturally identified with its determinant.

As in [PV21, Hor23] , we want to define the action of the dual Deligne cohomology group $H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi$ and conjecture that the resulting action of the rational structure coming from motivic cohomology descends to the rational structure on coherent cohomology.

We will define a natural generator of $H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$. First, note that:

$$\begin{aligned} H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) &\cong H^{2k-1}_{\mathcal{D}}(\mathrm{Sym}^2 M(f)_{\mathbb{R}}, \mathbb{R}(k)) \\ &\cong H^{2k-2}_{B}(\mathrm{Sym}^2 M(f)_{\mathbb{R}}, \mathbb{R}(k-1))/\widetilde{\pi}_{k-1} && \text{by (4.6)} \\ &\cong (H^{k-1,k-1})^{c_B=(-1)^{k-1}, F_\infty=(-1)^{k-1}} && \text{[PV21, Sec. 2.2.4]} \end{aligned}$$

Given a basis $\omega_1, \omega_2 \in F^{k-1}H^{k-1}_{\mathrm{dR}}(M(f))$, a natural basis of $F^k H_{\mathrm{dR}}(\mathrm{Sym}^2 M(f)_{\mathbb{R}})$ is given by

$$\omega_1 \otimes \omega_1, \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1, \omega_2 \otimes \omega_2$$

and we define a natural basis of $(H^{k-1,k-1})^{c_B=(-1)^{k-1}, F_\infty=(-1)^{k-1}}$ below.

**Definition 4.13.** A *natural generator* of the Deligne cohomology group $H^{2k-1}_{\mathcal{D}}(\mathrm{Sym}^2 M(f)_{\mathbb{R}}, \mathbb{R}(k))$ is

$$\delta' = (\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) + (-1)^{k-1}(\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2) \in (H^{k-1,k-1}(\mathrm{Sym}^2 M(f)))^{c_B=(-1)^{k-1}, F_\infty=(-1)^{k-1}}.$$

Therefore, a *natural generator* of the Deligne cohomology group $H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ is:

$$\delta = \delta(\omega_1, \omega_2) = (2\pi i)^{k-1}\delta' \in (H^{0,0}(M(f,\mathrm{Ad})))^{c_B=1, F_\infty=1}.$$

Both $\delta$ and $\delta'$ are well-defined up to a rational constant; specifically,

$$\delta(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = (ad - bc)\delta(\omega_1, \omega_2)$$

and similarly for $\delta'$.

As in [PV21], it is the dual to $H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ that should act on cohomology. We will use a polarization to identify $H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ with its dual, following [PV21, §2.2].

**Definition 4.14.** Let $\langle -, - \rangle_{\mathrm{pol}} \colon M(f) \otimes M(f) \to \mathbb{Q}(-(k-1))$ be a polarization of the motive $M(f)$ of weight $k-1$, i.e. a perfect $(-1)^{k-1}$-symmetric pairing whose Betti realization $H_B(s)$ gives a polarization of the rational Hodge structure $H_B(M(f))$.

Associated with $\langle -, - \rangle_{\mathrm{pol}}$, there is a symmetric perfect pairing

$$\langle -, - \rangle_{\mathrm{pol}} \colon M(f, \mathrm{Sym}^2) \otimes M(f, \mathrm{Sym}^2) \to \mathbb{Q}(-2(k-1)),$$

inducing a polarization of the rational Hodge structure $H_B(M(f,\mathrm{Sym}^2)_{\mathbb{C}}, \mathbb{Q})$. Using $M(f,\mathrm{Sym}^2)(k-1) = M(f,\mathrm{Ad})$, we also get an associated polarization

$$\langle -, - \rangle_{\mathrm{pol}} \colon M(f,\mathrm{Ad}) \otimes M(f,\mathrm{Ad}) \to \mathbb{Q},$$

**Definition 4.15.** The *dual natural generator* $\delta^\vee$ of $H^1_{\mathcal{D}}(M(f,\mathrm{Ad}), \mathbb{R}(1))^\vee$ associated with a natural generator $\delta$ and a polarization is defined by

$$\delta^\vee := \frac{\pi^{2k}}{\sqrt{\Delta_{\mathrm{Ad}(f)}}}\langle \delta, - \rangle_{\mathrm{pol}}.$$

**Remark 4.16.** Even though we have chosen a polarization in order to define the dual natural generator, it is well-defined up to $\mathbb{Q}^\times$. Indeed, $\delta^\vee$ only depends on the restriction of the polarization to a pairing on the one-dimensional vector space $H^1_{\mathcal{D}}(M(f,\mathrm{Ad}), \mathbb{R}(1))$, which is unique up to $\mathbb{Q}^\times$.

When $(k,2) = (2,2)$, $f$ corresponds to an abelian surface $A$ with $\mathrm{End}(A)$ according to the Brumer–Kramer conjecture 3.8. Then the polarization $\langle -, - \rangle_{\mathrm{pol}}$ on $M(f) = H^1(A)$ is associated with a polarization of $A$, which is unique up to rational scalars. Indeed, if $i \colon A \to A^\vee$ and $j \colon A \to A^\vee$ are two polarizations, then $j^\vee \circ i \colon A \to A$ is an isogeny, i.e. an element of $\mathrm{End}(A) \otimes \mathbb{Q} = \mathbb{Q}$. This shows that $i$ is a scalar multiple of $j$.

**Remark 4.17.** While the appearance of the factor $\frac{\pi^{2k}}{\sqrt{\Delta_{\mathrm{Ad}(f)}}}$ may be surprising at this stage, it seems to be needed for the eventual rationality statement. More specifically, it is related to factors appearing in the functional equation for the adjoint $L$-function. Perhaps there is a different phrasing of the conjecture which avoids using the functional equation, but we decided not to pursue this here and stuck to our original approach for computational convenience.

**Definition 4.18.** Let $f^W \in \pi_f \otimes X^2_\lambda$ be a Whittaker-normalized vector and $f \in \pi_f \otimes X^1_\lambda$ be the associated $\mathbb{Q}$-rational vector. We define an action of $H^1_{\mathcal{D}}(M(f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi$ by letting a dual natural generator $\delta^\vee \in H^1_{\mathcal{D}}(M(f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ act by:

$$\begin{gathered} H^0(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi \to H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi \\ [f] \mapsto [f^W]. \end{gathered}$$

Note that this action is well-defined up to $\mathbb{Q}^\times$.

Recall that we have a Beilinson regulator (3.6) map:

$$r_{\mathcal{D}} \colon H^1_{\mathcal{M}}(M(f, \mathrm{Ad}), \mathbb{Q}(1)) \otimes \mathbb{R} \to H^1_{\mathcal{D}}(M(f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) \tag{4.7}$$

which is an isomorphism by Hypothesis 3.3 (i.e. part (1) of Beilinson's Conjecture 3.2). Under this hypothesis, we define a degree-shifting action of $H^1_{\mathcal{M}}(M(f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee \otimes \mathbb{R}$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi$ using the Beilinson regulator 4.7 and Definition 4.18. Our main conjecture is that the resulting motivic action preserves the rational structure on coherent cohomology.

**Conjecture 4.19.** *The action of $H^1_{\mathcal{M}}(M(f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee \otimes \mathbb{R}$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi$ descends to rational structures, i.e. the action of $H^1_{\mathcal{M}}(M(f, \mathrm{Ad}), \mathbb{Q}(1))^\vee$ preserves the rational structure $H^*(X, \mathcal{E}_{k,2})_\Pi \subseteq H^*(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi$.*

We can make this explicit in the following way. Given a non-zero motivic cohomology class $\alpha \in H^1_{\mathcal{M}}(M(f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))$, we have that:

$$\frac{\delta^\vee}{\delta^\vee(r_{\mathcal{D}}(\alpha))} \in H^1_{\mathcal{M}}(M(f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee.$$

According to Definition 4.18, this element acts by:

$$\begin{gathered} H^0(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi \to H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2})_\Pi \\ [f] \mapsto \frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta\rangle_{\mathrm{pol}}}. \end{gathered} \tag{4.8}$$

Therefore, Conjecture 4.19 is equivalent to the following rationality statement.

**Conjecture 4.20.** *For a non-zero motivic cohomology class $\alpha \in H^1_{\mathcal{M}}(M(f, \mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))$, the coherent cohomology class*

$$\frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta\rangle_{\mathrm{pol}}} \in H^1(X_{\mathbb{C}}, \mathcal{E}_{k-2})_\Pi$$

*is rational, i.e. belongs to $H^1(X, \mathcal{E}_{k-2})_\Pi \subseteq H^1(X_{\mathbb{C}}, \mathcal{E}_{k-2})_\Pi$.*

**Theorem 4.21.** *Assume:*

*(1) Hypothesis 5.2 about the non-vanishing of some quadratic twists $L(M(f), \psi_\pm, s)$ at $s = k-1$,*
*(2) Deligne's conjecture for $L(M(f), \psi_\pm, s)$ at the critical point $s = k-1$,*
*(3) Beilinson's conjecture for $L(M(f, \mathrm{Ad}), s)$ at the point $s = 1$.*

*Then Conjecture 4.20 holds (and hence so does Conjecture 4.19).*

**Remark 4.22.** Without assumption (3), i.e. Hypothesis 5.2, Conjecture 4.20 is true up to an unknown factor in $(\mathbb{Q}^{\mathrm{ab}})^\times$. Indeed, recent results of Radziwiłł–Yang [RY23] give non-vanishing of some twists of $L(M(f), s)$, but unless these twists are quadratic, they may contribute a factor in $(\mathbb{Q}^{\mathrm{ab}})^\times$.

The proof of Theorem 4.21 will occupy the next section. For the reader's convenience, we give an outline here. The goal is to prove the rationality of the cohomology class

$$\omega = \frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta\rangle_{\mathrm{pol}}} \in H^1(X_{\mathbb{C}}, \mathcal{E}_{k,2})_{\Pi}.$$

We consider a rational class $\eta \in H^2(X, \mathcal{E}_{1,3-k})_{\Pi}$ and compute that they pair to a non-zero rational number under the Serre duality pairing $\langle \omega, \eta\rangle_{\mathrm{SD}} \in \mathbb{Q}^{\times}$. Since the space $H^1(X, \mathcal{E}_{k,2})_{\Pi}$ is 1-dimensional, this will prove the theorem.

We temporarily write $a \approx b$ if $a = \pi^n\sqrt{c}b$ for $n \in \mathbb{Z}$ and $c \in \mathbb{Q}^{\times}$, and present the main steps of the argument up to $\approx$. More precise statements are given in the following sections.

(1) Recall the Whittaker-normalized vector $w_\infty(f^W)$, which has an associated cohomology class $[w_\infty(f^W)] \in H^2(X_{\mathbb{C}}, \mathcal{E}_{1,3-k})_f$, and the Whittaker period $c^W(f)$, which was defined so that

$$\eta = \frac{[w_\infty(f^W)]}{c^W(f)} \in H^2(X, \mathcal{E}_{1,3-k})_f$$

is rational (Definition 4.10).

(2) In Theorem 5.3, we will check that the period $c^W(f)$ is related to Beilinson periods for $M(f)$ as follows:

$$c^W(f) \approx c^+(M(f)) \cdot c^-(M(f)).$$

(3) Chen–Ichino [CI19] prove that:

$$\langle f^W, f^W\rangle \approx L(M(f, \mathrm{Ad}), 1).$$

(4) In Theorem 5.7, we check that Beilinson's conjecture is equivalent to the statement:

$$L(M(f, \mathrm{Ad}), 1) \approx c^+(M(f)) \cdot c^-(M(f)) \cdot \langle r_{\mathcal{D}}(\alpha), \delta^\vee\rangle_{\mathrm{PD}}$$

for $\alpha \in H^1_{\mathcal{M}}(\mathrm{Ad}(M), \mathbb{Q}(1))$.

(5) Finally, we compute that:

$$\begin{aligned}
\langle \omega, \eta\rangle_{\mathrm{SD}} &\approx \frac{\langle f^W, f^W\rangle}{\langle r_{\mathcal{D}}(\alpha), \delta^\vee\rangle_{\mathrm{PD}} \cdot c^W(f)} \\
&\approx \frac{L(M(f, \mathrm{Ad}), 1)}{\langle r_{\mathcal{D}}(\alpha), \delta^\vee\rangle_{\mathrm{PD}} \cdot c^W(f)} && \text{by (3)} \\
&\approx \frac{c^+(M(f))c^-(M(f))}{c^W(f)} && \text{by (4)} \\
&\approx 1 && \text{by (2)}
\end{aligned}$$

which completes the proof up to $\approx$.

In Section 6, we also prove Conjecture 4.19, unconditionally on the rationality statement of Beilinson's conjecture, in the case of Yoshida lifts associated to real quadratic fields by using results of Ramakrishnan [Ram87]. In Section 7, we consider the case of elliptic curves over imaginary quadratic fields and show that our conjecture implies the conjecture of Prasanna–Venkatesh [PV21] in this case. In both of these special cases, we only consider the case $k = 2$.

## 5. Proof of Theorem 4.21

5.1. **The Whittaker period.** Recall that we defined the Whittaker period $c^W(\pi_f)$ associated to the Whittaker-normalized vector $f^W$ (Definition 4.10). Inspired by the work of Loeffler–Pilloni–Skinner–Zerbes [LPSZ21], we relate it to Deligne's conjecture for spin $L$-functions.

**Theorem 5.1** (Loeffler, Pilloni, Skinner, Zerbes)**.** *Let $f$ be a holomorphic Siegel modular form of weight $(k_1, k_2)$, paramodular level $N$, and coefficients in $E$. Then for Dirichlet characters $\psi_+, \psi_-$ such that $\psi_\pm(-1) = \pm 1$, we have that:*

$$c^W(f) \sim_{E(\psi_+,\psi_-)^\times} \Lambda(f, \psi_+, k_1 - 1)\,\Lambda(f, \psi_-, k_1 - 1)\,.$$

*Proof.* This is Proposition 10.3 in [LPSZ21] (stated in the case $k_1 \geq k_2 \geq 3$; for generalization to the case $k_2 = 2$, see [LZ20]). Note that we use the motivic normalization of $L$-functions, instead of the automorphic one. □

We stated the theorem for the right-most critical value, according to Lemma 3.4. There are also analogous statements for the other critical values.

Next, we want to express $c^W(f)$ in terms of Deligne periods, so we need to make sure that there exist non-vanishing twists of the spin $L$-function.

**Hypothesis 5.2.** *There exist quadratic Dirichlet characters $\psi_+$, $\psi_-$ such that $\psi_\pm(-1) = \pm 1$ such that*

$$L(f, \psi_+, k_1 - 1) \cdot L(f, \psi_-, k_1 - 1) \neq 0.$$

**Theorem 5.3.** *Let $f$ be a holomorphic Siegel modular form of weight $(k_1, k_2)$, paramodular level $N$, and rational coefficients. Assuming Deligne's conjecture, we have that:*

$$c^W(f) \sim_{(\mathbb{Q}^{\mathrm{ab}})^\times} \pi^{-4(k_1-1)+2(k_2-2)} c^+(M(f)(k_1-1)) c^-(M(f)(k_1-1)),$$

*where $c^\pm(M(f)(k_1))$ are the Deligne periods associated to the motive $M(f)(k_1)$ of $f$. Moreover, under Hypothesis 5.2,*

$$c^W(f) \sim_{\mathbb{Q}^\times} \pi^{-4(k_1-1)+2(k_2-2)} c^+(M(f)(k_1-1)) c^-(M(f)(k_1-1)).$$

*Proof.* We have that:

$$\Lambda(f, \psi_\pm, s) = L(f, \psi_\pm, s) \cdot L_\infty(f, \psi_\pm, s)$$

where

$$L_\infty(f, \psi_\pm, s) = \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{C}}(s + 2 - k_2),$$

and hence

$$L_\infty(f, \psi_\pm, k_1 - 1) = \Gamma_{\mathbb{C}}(k_1 - 1) \Gamma_{\mathbb{C}}(s + k_1 + 1 - k_2)) \sim_{\mathbb{Q}^\times} \pi^{-2(k_1-1)+(k_2-2)}.$$

Altogether, Theorem 5.1 shows that according to Beilinson's conjecture:

$$c^W(f) \sim_{E(\psi_+,\psi_-)^\times} \pi^{-2(k_1-1)+(k_2-2)} g(\psi_+)^2 g(\psi_-)^2 c^+(M(f)(k_1-1)) c^-(M(f)(k_1-1)),$$

as long as there exists $\psi_+$, $\psi_-$ such that $L(f, \psi_\pm, k_2 - 1) \neq 0$.

Finally, consider the automorphic representation $\Pi$ of $\mathrm{GL}_4(\mathbb{A})$ associated with $f$ so that $L(f, s) = L(\Pi, s)$. If $f$ is non-endoscopic, then $\Pi$ is cuspidal, so such $\psi_+$ and $\psi_-$ exist by the recent result of [RY23]. On the other hand, if $f$ is endoscopic, then $\Pi = \pi_1 \boxplus \pi_2$ for automorphic representations $\pi_1$, $\pi_2$ of $\mathrm{GL}_2(\mathbb{A})$, so such $\psi_+$ and $\psi_-$ exist by [FH95]. □

**Remark 5.4.** The strategy behind the proof goes back to Harris' occult periods [Har04] who uses Bessel periods instead of Whittaker periods; see [LPSZ21, Remark 6.7] for the difference between the two approaches.

## 5.2. The adjoint $L$-value and Petersson inner products.

We recall a theorem of Chen and Ichino which relates the square of the Petersson norm of $f^W$ to the adjoint $L$-value, and hence verifies the conjecture of Lapid–Mao [LM15] in this case. Note that our notation is slightly different; the precise relationship is $(\lambda_1 + 1, -\lambda_2) = (\lambda_1^{\mathrm{CI}}, \lambda_2^{\mathrm{CI}})$.

**Theorem 5.5** (Chen–Ichino [CI19], Chen [Che22])**.** *Let $\pi$ be a globally generic cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ of square-free paramodular level $N$ such that the Harish–Chandra parameter of $\pi_\infty$ is $(\lambda_1, -\lambda_2)$. Let $f^W \in \pi$ be a Whittaker-normalized vector. Then:*

$$\langle f^W, f^W \rangle = 2^c \cdot \frac{\Lambda(1, \pi, \mathrm{Ad})}{\Lambda(2)\Lambda(4)} \cdot \prod_{v \mid N\infty} C_v$$

*where $c = 2$ if $\pi$ is endoscopic and $c = 1$ otherwise, $\Lambda(s)$ is the completed Riemann $\zeta$-function, and $C_v$ are explicitly described constants, satisfying:*

$$\prod_{v \mid N\infty} C_v \sim_{\mathbb{Q}^\times} \pi^{3\lambda_1 + \lambda_2 + 8}.$$

*When the level is not square-free, then the same formula holds up to an unknown factor in $E^\times$, where $E$ is the field of definition of $\pi_f$.*

**Corollary 5.6.** *If $f$ is a holomorphic Siegel modular form of weight $(k,2)$ with coefficients in $E$, and $f^W$ is the associated Whittaker-normalized generic Siegel modular form, then:*

$$\langle f^W, f^W\rangle \sim_{E^\times} \pi^{3(k-1)+5}\cdot \Lambda(f,\mathrm{Ad},1).$$

*Proof.* Recall that $(\lambda_1,\lambda_2)=(k-1,0)$, so $\pi^{3\lambda_1+\lambda_2+8}=\pi^{3(k-1)+8}$. Also, we have that:

$$\Lambda(2)=\frac{\pi^2}{6}\cdot\pi^{-2/2}\Gamma(2/2),$$
$$\Lambda(4)=\frac{\pi^4}{90}\cdot\pi^{-4/2}\Gamma(4/2).$$

This gives the result. □

5.3. **Beilinson's conjecture for the adjoint $L$-value.** Next, we give an explicit version of Beilinson's conjecture for the symmetric square $L$-function in terms of a non-zero motivic cohomology class $\alpha\in H^{2k-1}_{\mathcal{M}}(M(f,\mathrm{Sym}^2)_{\mathbb{Z}},\mathbb{Q}(k))$ and a natural generator $\delta'$, as in Definition 4.13.

**Theorem 5.7.** *Let $f$ be a holomorphic Siegel modular form of even weight $(k,2)$ with trivial central character, defined over $\mathbb{Q}$, and let $M(f)$ be the associated motive over $\mathbb{Q}$. Fix a polarization pairing on $M(f)$ and write $\langle -,-\rangle_{\mathrm{pol}}$ for the induced pairing on $M(f,\mathrm{Sym}^2)$. For $\alpha\in H^{2k-1}_{\mathcal{M}}(M(f,\mathrm{Sym}^2)_{\mathbb{Z}},\mathbb{Q}(k))$, Beilinson's conjecture for the adjoint $L$-function is equivalent to the equation:*

$$L'(f,\mathrm{Sym}^2,k-1)\sim_{\mathbb{Q}^\times}\pi^{-2(k-1)}\cdot c^+(M(f)(k-1))\cdot c^-(M(f)(k-1))\cdot\langle\delta',r_{\mathcal{D}}(\alpha)\rangle_{\mathrm{pol}}$$

*for a dual natural generator $\delta'^\vee$ (Definition 4.15).*

*Proof.* We first introduce the notation relevant to the motive $M(f)$ and recall Lemma 3.5 which computed $c^\pm(M(f))$. Recall that:

$$H^{k-1}_B(M(f),\mathbb{Q})\cong H^{k-1}_B(M(f),\mathbb{Q})^+\oplus H^{k-1}_B(M(f),\mathbb{Q})^-$$

and we fix a basis $v_i^\pm$. In other words, $F_\infty$ acts by $v_i^+\mapsto v_i^+$ and $v_i^-\mapsto -v_i^-$ for $i=1,2$. Note that the polarization pairing descends to a $(-1)^{k-1}$-symmetric pairing on $H^{k-1}_B(M(f),\mathbb{Q})^\pm$ and we may assume that that basis $v_1^+,v_2^+,v_1^-,v_2^-$ is chosen so that its matrix is

$$(2\pi i)^{-(k-1)}\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1\\ & & -1 & \end{pmatrix}.$$

For a basis $\omega_1,\omega_2$ of $F^1H^{k-1}_{\mathrm{dR}}(M(f))$, the comparison map is given by:

$$\begin{aligned} F^1H^{k-1}_{\mathrm{dR}}(M(f)) &\overset{\tilde{\pi}_0}{\to} H^{k-1}_B(M(f)_{\mathbb{C}},\mathbb{R})\\ \omega_1 &\mapsto c^+_{1,1}v_1^+ + c^+_{1,2}v_2^+ + c^-_{1,1}v_1^- + c^-_{1,2}v_2^-,\\ \omega_2 &\mapsto c^+_{2,1}v_1^+ + c^+_{2,2}v_2^+ + c^-_{2,1}v_1^- + c^-_{2,2}v_2^-.\end{aligned}$$

Then:

$$c^\pm := \det(c^\pm_{i,j}) = c^\pm(M(f)(k-1)) \tag{5.1}$$

as in Lemma 3.5.

Throughout the rest of the proof, we will use the shorthand

$$M := M(f,\mathrm{Sym}^2).$$

Consider the Beilinson short exact sequence associated to the motive $M$:

$$0\to F^kH^{2k-2}_{\mathrm{dR}}(M_{\mathbb{R}})\overset{\tilde{\pi}_{k-1}}{\to} H^{2k-2}_B(M_{\mathbb{R}},\mathbb{R}(k-1))\to H^{2k-1}_{\mathcal{D}}(M_{\mathbb{R}},\mathbb{R}(k))\to 0. \tag{5.2}$$

Beilinson's conjecture 3.2 says that

$$r_{\mathcal{D}}(H^{2k-1}_{\mathcal{M}}(M_{\mathbb{Z}}, \mathbb{Q}(k))) = L'(M, k-1) \cdot \frac{\det H_B^{2k-2}(M_{\mathbb{R}}, \mathbb{Q}(k-1))}{\det \tilde{\pi}_{k-1}(F^k H^{2k-2}_{\mathrm{dR}}(M_{\mathbb{Q}}))} \quad \text{in } H^{2k-1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(k)). \tag{5.3}$$

To make this explicit, we choose bases for the various spaces.

We have that:

$$F^k H^{2k-2}_{\mathrm{dR}}(M) = \mathbb{Q}\omega_{1,1} \oplus \mathbb{Q}\omega_{2,2} \oplus \mathbb{Q}\omega_{1,2} \tag{5.4}$$

where

$$\omega_{i,i} = \omega_i \otimes \omega_i, \qquad i = 1, 2, \tag{5.5}$$

$$\omega_{1,2} = \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1. \tag{5.6}$$

We identify the 10-dimensional space $H^{2k-2}_B(M_{\mathbb{C}}, \mathbb{Q})$ with $\mathrm{Sym}^2 H^{k-1}_B(M(f)_{\mathbb{C}}, \mathbb{Q})$, a quotient of $H^{k-1}_B(M(f)_{\mathbb{C}}, \mathbb{Q}) \otimes H^{k-1}_B(M(f)_{\mathbb{C}}, \mathbb{Q})$. The space $H^{2k-2}_B(M_{\mathbb{C}}, \mathbb{Q})^{(-1)^{k-1}}$ is 4-dimensional, spanned by

$$u_{1,1} = v_1^+ \otimes v_1^- + v_1^- \otimes v_1^+, \tag{5.7}$$

$$u_{1,2} = v_1^+ \otimes v_2^- + v_2^- \otimes v_1^+, \tag{5.8}$$

$$u_{2,1} = v_2^+ \otimes v_1^- + v_1^- \otimes v_2^+, \tag{5.9}$$

$$u_{2,2} = v_2^+ \otimes v_2^- + v_2^- \otimes v_2^+, \tag{5.10}$$

and hence $u_{i,j}(k-1) = (2\pi i)^{k-1} u_{i,j}$ is a basis of $H^{2k-2}_B(M_{\mathbb{C}}, \mathbb{Q}(k-1))^+$. We record the polarization pairing in this basis:

$$(\langle u_{i,j}, u_{k,\ell}\rangle_{\mathrm{pol}})_{i,j,k,\ell} = 2(2\pi i)^{-2(k-1)} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \tag{5.11}$$

Finally, $H^{2k-1}_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(k))$ is 1-dimensional, spanned by $r_{\mathcal{D}}(\alpha)$ for an element $\alpha \in H^{2k-1}_{\mathcal{M}}(M, \mathbb{Q}(k))$.

In these bases, the map $\widetilde{\pi}_{k-1}$ can be described as follows:

$$F^k H^{2k-2}_{\mathrm{dR}}(M_{\mathbb{Q}}) \to H^{2k-2}_B(M_{\mathbb{R}}, \mathbb{R}(k-1))$$

$$\omega_{1,1} = \omega_1 \otimes \omega_1 \mapsto c^+_{1,1}c^-_{1,1}u_{1,1} + c^+_{1,1}c^-_{1,2}u_{1,2} + c^+_{1,2}c^-_{1,1}u_{2,1} + c^+_{1,2}c^-_{1,2}u_{2,2}, \tag{5.12}$$

$$\omega_{2,2} = \omega_2 \otimes \omega_2 \mapsto c^+_{2,1}c^-_{2,1}u_{1,1} + c^+_{2,1}c^-_{2,2}u_{1,2} + c^+_{2,2}c^-_{2,1}u_{2,1} + c^+_{2,2}c^-_{2,2}u_{2,2}, \tag{5.13}$$

$$\begin{aligned}\omega_{1,2} = \omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1 \mapsto\ & (c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1})u_{1,1} + (c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2})u_{1,2} \\ & + (c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})u_{2,1} + (c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2})u_{2,2}.\end{aligned} \tag{5.14}$$

In other words, in the chosen bases, the matrix of this transformation is:

$$\begin{pmatrix} c^+_{1,1}c^-_{1,1} & c^+_{2,1}c^-_{2,1} & c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1} \\ c^+_{1,1}c^-_{1,2} & c^+_{2,1}c^-_{2,2} & c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2} \\ c^+_{1,2}c^-_{1,1} & c^+_{2,2}c^-_{2,1} & c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1} \\ c^+_{1,2}c^-_{1,2} & c^+_{2,2}c^-_{2,2} & c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2} \end{pmatrix}.$$

Let $v_1 = \widetilde{\pi}_{k-1}(\omega_{1,1}), v_2 = \widetilde{\pi}_{k-1}(\omega_{1,2}), v_3 = \widetilde{\pi}_{k-1}(\omega_{2,2})$, i.e. the columns of the above matrix. We complete it to a basis by choosing $v_4 = (2\pi i)^{4(k-1)} \dfrac{u_{1,2}}{c}$, where we choose $c \in \mathbb{R}$ to satisfy:

$$v_1 \wedge v_2 \wedge v_3 \wedge v_4 = u_{1,1}(k-1) \wedge u_{1,2}(k-1) \wedge u_{2,1}(k-1) \wedge u_{2,2}(k-1)$$

so that $v_1 \wedge v_2 \wedge v_3 \wedge v_4$ is a basis for $\det H_B^{2k-2}(M_{\mathbb{R}}, \mathbb{Q}(k-1))$. A computation of the determinant shows that:

$$c = \det \begin{pmatrix} c^+_{1,1}c^-_{1,1} & c^+_{2,1}c^-_{2,1} & c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1} & 0 \\ c^+_{1,1}c^-_{1,2} & c^+_{2,1}c^-_{2,2} & c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2} & 1 \\ c^+_{1,2}c^-_{1,1} & c^+_{2,2}c^-_{2,1} & c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1} & 0 \\ c^+_{1,2}c^-_{1,2} & c^+_{2,2}c^-_{2,2} & c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2} & 0 \end{pmatrix} = c^+c^-(c^+_{2,2}c^-_{1,1} - c^+_{1,2}c^-_{2,1}). \tag{5.15}$$

Finally, suppose that $r_{\mathcal{D}}(\alpha) \in H_{\mathcal{D}}^{2k-1}(M_{\mathbb{R}}, \mathbb{R}(k))$ lifts to an element

$$a_1v_1 + \cdots + a_4v_4 \in H_B^{2k-2}(M_{\mathbb{R}}, \mathbb{R}(k-1))$$

Then Beilinson's conjecture (5.3) amounts to the equation:

$$a_4 \sim_{\mathbb{Q}^\times} L'(M, k-1). \tag{5.16}$$

In order to pick out $a_4$, we consider a natural generator:

$$\delta' = \delta'(\omega_1, \omega_2) = (\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) + (-1)^{k-1}(\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2) \in H_{\mathcal{D}}^{2k-1}(M_{\mathbb{R}}, \mathbb{R}(k))$$

as in Definition 4.13.

We first compute $\delta'$ in terms of the basis $u_{i,j}$:

$$\begin{aligned} \omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1 &\mapsto (c^+_{1,1}v^+_1 + c^+_{1,2}v^+_2 + c^-_{1,1}v^-_1 + c^-_{1,2}v^-_2) \otimes (c^+_{2,1}v^+_1 + c^+_{2,2}v^+_2 - c^-_{2,1}v^-_1 - c^-_{2,2}v^-_2) \\ &\quad + (c^+_{2,1}v^+_1 + c^+_{2,2}v^+_2 - c^-_{2,1}v^-_1 - c^-_{2,2}v^-_2) \otimes (c^+_{1,1}v^+_1 + c^+_{1,2}v^+_2 + c^-_{1,1}v^-_1 + c^-_{1,2}v^-_2) \\ \omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2 &\mapsto (c^+_{2,1}v^+_1 + c^+_{2,2}v^+_2 + c^-_{2,1}v^-_1 + c^-_{2,2}v^-_2) \otimes (c^+_{1,1}v^+_1 + c^+_{1,2}v^+_2 - c^-_{1,1}v^-_1 - c^-_{1,2}v^-_2) \\ &\quad + (c^+_{1,1}v^+_1 + c^+_{1,2}v^+_2 - c^-_{1,1}v^-_1 - c^-_{1,2}v^-_2) \otimes (c^+_{2,1}v^+_1 + c^+_{2,2}v^+_2 + c^-_{2,1}v^-_1 + c^-_{2,2}v^-_2) \end{aligned}$$

$$\begin{aligned} \delta' \mapsto\ & 2(-c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1})u_{1,1} + 2(-c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2})u_{1,2} \\ & + 2(-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})u_{2,1} + 2(-c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2})u_{2,2} \end{aligned} \tag{5.17}$$

We will check that:

$$\langle \delta', v_i \rangle_{\mathrm{pol}} = 0 \qquad \text{for } i = 1, 2, 3, \tag{5.18}$$

$$\langle \delta', v_4 \rangle_{\mathrm{pol}} = 4\frac{(2\pi i)^{2(k-1)}}{c^+c^-} \tag{5.19}$$

because $v_i \in H^{2(k-1),0} \oplus H^{0,2(k-1)}$ as they are images of elements $F^kH^{2k-2}_{\mathrm{dR}}(M_{\mathbb{R}})$. We may verify equation (5.18) by direct computation; for example, for $i = 1$, we have:

$$\begin{aligned} \langle \delta', v_1 \rangle &= \langle \delta', c^+_{1,1}c^-_{1,1}u_{1,1} + c^+_{1,1}c^-_{1,2}u_{1,2} + c^+_{1,2}c^-_{1,1}u_{2,1} + c^+_{1,2}c^-_{1,2}u_{2,2} \rangle && (5.12) \\ &= 2\Big\langle (-c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1})u_{1,1} + (-c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2})u_{1,2} \\ &\qquad + (-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})u_{2,1} + (-c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2})u_{2,2}, \\ &\qquad\quad c^+_{1,1}c^-_{1,1}u_{1,1} + c^+_{1,1}c^-_{1,2}u_{1,2} + c^+_{1,2}c^-_{1,1}u_{2,1} + c^+_{1,2}c^-_{1,2}u_{2,2} \Big\rangle && (5.17) \\ &= 4(2\pi i)^{-2(k-1)}\Big((-c^+_{1,1}c^-_{2,1} + c^+_{2,1}c^-_{1,1})(c^+_{1,2}c^-_{1,2}) - (-c^+_{1,1}c^-_{2,2} + c^+_{2,1}c^-_{1,2})(c^+_{1,2}c^-_{1,1}) \\ &\qquad - (-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})(c^+_{1,1}c^-_{1,2}) + (-c^+_{1,2}c^-_{2,2} + c^+_{2,2}c^-_{1,2})(c^+_{1,1}c^-_{1,1})\Big) && (5.11) \\ &= 0 \end{aligned}$$

The computations for $i = 2, 3$ are similar.

Next, equations (5.17) and (5.11) give:

$$\langle \delta', u_{1,2} \rangle_{\mathrm{pol}} = -4(2\pi i)^{-2(k-1)}(-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1}). \tag{5.20}$$

Therefore:

$$\begin{aligned}\langle \delta', v_4\rangle_{\mathrm{pol}} &= \frac{-4(-2\pi i)^{-2(k-1)}(-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})}{(2\pi i)^{-4(k-1)}c} \\ &= \frac{-4(2\pi i)^{2(k-1)}(-c^+_{1,2}c^-_{2,1} + c^+_{2,2}c^-_{1,1})}{c^+c^-(c^+_{2,2}c^-_{1,1} - c^+_{1,2}c^-_{2,1})} \\ &= 4\frac{(2\pi i)^{2(k-1)}}{c^+c^-},\end{aligned}$$

proving equation (5.19).

Therefore:

$$\begin{aligned}\langle \delta', r_{\mathcal{D}}(\alpha)\rangle_{\mathrm{pol}} &= \left\langle \delta', \sum_{i=1}^4 a_i v_i \right\rangle_{\mathrm{pol}} \\ &= a_4\langle \delta', v_4\rangle && (5.18) \\ &= \frac{4(2\pi i)^{2(k-1)}a_4}{c^+c^-} && (5.19)\end{aligned}$$

Therefore, Beilinson's conjecture is equivalent to:

$$\begin{aligned}L'(M, k-1) &\sim_{\mathbb{Q}^\times} a_4 && (5.16) \\ &\sim_{\mathbb{Q}^\times} \pi^{-2(k-1)}c^+c^-\langle \delta', r_{\mathcal{D}}(\alpha)\rangle_{\mathrm{pol}} \\ &\sim_{\mathbb{Q}^\times} \pi^{-2(k-1)}c^+(M(f)(k-1))c^-(M(f)(k-1))\langle \delta', r_{\mathcal{D}}(\alpha)\rangle_{\mathrm{pol}} && (5.1)\end{aligned}$$

This completes the proof. □

**Remark 5.8.** Theorem 5.7 is the non-critical analogue of Yoshida's formulas (3.15), (3.16).

The formulation of Beilinson's conjecture via a Poincaré duality pairing on Deligne cohomology is standard; another example for non-critical values of spin $L$-functions can be found in [CLJ19, CLJ22].

**Example 5.9.** In this extended example, we given an alternative proof of Theorem 5.7 in the endoscopic case, i.e. a Siegel modular form $f$ of weight $(k, 2)$ associated with a pair $f_1$, $f_2$ of even weight $k$ modular forms with trivial central characters and rational Fourier coefficients. This also serves as a useful check and a prelude to the results of the next two sections.

The alternative proof is based on the following factorization of motives:

$$M(f) = M(f_1) \oplus M(f_2),$$
$$M(f, \mathrm{Sym}^2) = M(f_1, \mathrm{Sym}^2) \oplus M(f_2, \mathrm{Sym}^2) \oplus M(f_1) \otimes M(f_2),$$

associated with the factorization of $L$-functions

$$L'(f, \mathrm{Sym}^2, k-1) = L(f_1, \mathrm{Sym}^2, k-1)L(f_2, \mathrm{Sym}^2, k-1)L'(f_1 \times f_2, k-1).$$

We give explicit forms of Beilinson's Conjecture for $M(f_i)$, $M(f_i, \mathrm{Sym}^2)$ and $M(f_1) \otimes M(f_2)$.

(1) For $i = 1, 2$ we pick a basis $v_i^\pm$ of the 1-dimensional space $H^{k-1}_B(M(f_i)_{\mathbb{C}}, \mathbb{Q})^\pm$ and a basis $\omega_i$ of $F^{k-1}H^{k-1}_{\mathrm{dR}}M(f_i)$. Note that this agrees with the notation in the proof of Theorem 5.7. Then:

$$F^{k-1}H^{k-1}_{\mathrm{dR}}(M(f_i)_{\mathbb{R}}) \overset{\widetilde{\pi}_{k-2}}{\to} H^{k-1}_B(M(f_i)_{\mathbb{R}}, \mathbb{R}(k-2))$$
$$\omega_i \mapsto c_i^+v_i^+ + c_i^-v_i^-,$$

i.e. $c^\pm_{i,j} = \delta_{i,j}c_i^\pm$. A rational basis of $H^{k-1}_B(M(f_i)_{\mathbb{R}}, \mathbb{R}(k-2))$ is given by $v_i^+(2\pi i)^{k-2}$, and hence

$$c^\pm(M(f_i)(1)) = (2\pi i)^{-(k-2)}c_i^\pm \tag{5.21}$$
$$c^\pm(M(f_i)(k-1)) = c_i^\mp = c. \tag{5.22}$$

(2) The short exact sequence (3.1) for $\mathrm{Sym}^2 M(f_i)$ is:

$$0 \to F^k H^{2k-2}_{\mathrm{dR}}(\mathrm{Sym}^2 M(f_i)) \to H^{2k-2}_B(M_{\mathbb{R}}, \mathbb{R}(k-1)) \to 0$$
$$\omega_i \otimes \omega_i \mapsto c_i^+ c_i^- (v_i^+ \otimes v_i^- + v_i^- \otimes v_i^+),$$

where we note that $H^{2k-2}_B(M_{\mathbb{R}}, \mathbb{Q}(k-1)) \cong H^{2k-2}_B(M_{\mathbb{R}}, \mathbb{Q})^-$ via multiplication by $(2\pi i)^{-(k-1)}$, and the latter space is spanned by $u_{i,i} = (v_i^+ \otimes v_i^- + v_i^- \otimes v_i^+)$. Deligne's conjecture hence amounts to:

$$L(M(f_i, \mathrm{Sym}^2), k-1) = (2\pi i)^{-(k-1)} c_i^+ c_i^-. \tag{5.23}$$

(3) The short exact sequence (3.1) for $M^{1,2} := M(f_1) \otimes M(f_2)$ is:

$$\begin{array}{ccccccc} 0 \longrightarrow & F^k H^{2k-2}_{\mathrm{dR}}(M^{1,2}_{\mathbb{R}}) & \xrightarrow{\tilde{\pi}_{k-1}} & H^{2k-2}_B(M^{1,2}_{\mathbb{R}}, \mathbb{R}(k-1)) & \longrightarrow & H^{2k-1}_{\mathcal{D}}(M^{1,2}_{\mathbb{R}}, \mathbb{R}(k)) & \longrightarrow 0 \\ & \omega_1 \otimes \omega_2 & & u_{1,2} = v_1^+ \otimes v_2^- & & \delta' := \omega_1 \otimes \overline{\omega_2} - \overline{\omega_1} \otimes \omega_2 & \\ & & & u_{2,1} = v_1^- \otimes v_2^+ & & & \\ & \omega_1 \otimes \omega_2 & \longrightarrow & c_1^+ c_2^- u_{1,2} + c_1^- c_2^+ u_{2,1} & & & \\ & & & 2(c_1^+ c_2^- u_{1,2} - c_1^- c_2^+ u_{2,1}) & \longrightarrow & \delta' & \end{array}$$

We let $w_1 = c_1^+ c_2^- u_{1,2} + c_1^- c_2^+ u_{2,1}$ and $w_2 = -\frac{(2\pi i)^{2(k-1)}}{c_1^- c_2^+} u_{1,2}$ so that

$$w_1 \wedge w_2 = -\frac{(2\pi i)^{2(k-1)}}{c_1^- c_2^+} c_1^- c_2^+ u_{2,1} \wedge u_{1,2} = u_{1,2}(k-1) \wedge u_{2,1}(k-1) \in \wedge^2 H^{2k-2}_B(M_{\mathbb{R}}, \mathbb{Q}(k-1)).$$

Finally, let $\alpha \in H^{2k-1}_{\mathcal{M}}(M^{1,2}, \mathbb{Q}(k))$ and suppose that $r_{\mathcal{D}}(\alpha) \in H^{2k-1}_{\mathcal{D}}(M^{1,2}_{\mathbb{R}}, \mathbb{R}(k))$ lifts to $a_1 w_1 + a_2 w_2 \in H^{2k-2}_B(M^{1,2}_{\mathbb{R}}, \mathbb{R}(k-1))$. Then Beilinson's conjecture 3.2 predicts that $a_4 \sim_{\mathbb{Q}^\times} L'(f_1 \times f_2, k-1)$. On the other hand,

$$\begin{aligned} \langle \delta', r_{\mathcal{D}}(\alpha) \rangle_{\mathrm{pol}} &= 2\langle c_1^+ c_2^- u_{1,2} - c_1^- c_2^+ u_{2,1}, a_1 v_1 + a_2 v_2 \rangle_{\mathrm{pol}} \\ &= 2a_2 \langle c_1^+ c_2^- u_{1,2} - c_1^- c_2^+ u_{2,1}, v_2 \rangle_{\mathrm{pol}} \\ &= -2\frac{(2\pi i)^{2(k-1)}}{c_1^- c_2^+} a_2 \langle c_1^+ c_2^- u_{1,2} - c_1^- c_2^+ u_{2,1}, u_{1,2} \rangle_{\mathrm{pol}} \\ &\sim_{\mathbb{Q}^\times} a_2. \end{aligned}$$

Altogether, this shows that:

$$L'(f_1 \times f_2, k-1) \sim_{\mathbb{Q}^\times} \langle \delta', r_{\mathcal{D}}(\alpha) \rangle_{\mathrm{pol}}. \tag{5.24}$$

The results of (1)–(3) altogether give the formula:

$$\begin{aligned} L'(f, \mathrm{Sym}^2, k-1) &= L(f_1, \mathrm{Sym}^2, k-1) L(f_2, \mathrm{Sym}^2, k-1) L'(f_1 \times f_2, k-1) \\ &\sim_{\mathbb{Q}^\times} (2\pi i)^{-2(k-1)} c_1^+ c_1^- c_2^+ c_2^- \langle \delta', r_{\mathcal{D}}(\alpha) \rangle_{\mathrm{pol}} \\ &\sim_{\mathbb{Q}^\times} \pi^{-2(k-1)} c^+(M(f)(k-1)) c^-(M(f)(k-1)) \langle \delta', r_{\mathcal{D}}(\alpha) \rangle_{\mathrm{pol}}, \end{aligned}$$

recovering the result of Theorem 5.7.

Although Theorem 5.7 is still true, the definition of the motivic action required the form $f$ to be non-endoscopic, see Remark 6.1) for a detailed discussion.

5.4. **Completing the proof.** We are now ready to complete the proof of Theorem 4.21.

*Proof of Theorem 4.21.* We just need to check that

$$\frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}}} \in H^2(X_{\mathbb{C}}, \mathcal{E}_{k,2})_f$$

is a $\mathbb{Q}$-rational cohomology class. Recall from Definition 4.10 that

$$\frac{[w_\infty f^W]}{c^W(f)} \in H^1(X, \mathcal{E}_{1,3-k})_f$$

is an $\mathbb{Q}$-rational cohomology class. Since $H^2(X_{\mathbb{C}}, \mathcal{E}_{k,2})_f$ is 1-dimensional, it is enough to check that:

$$q := \frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \left\langle \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta^\vee \rangle_{\mathrm{pol}}}, \frac{[w_\infty f^W]}{c^W(f)} \right\rangle_{\mathrm{SD}} \in \mathbb{Q}^\times. \tag{5.25}$$

Recall the functional equation for the adjoint $L$-function gives:

$$L^*_\infty(f, \mathrm{Ad}, 0) L'(f, \mathrm{Ad}, 0) = \sqrt{\Delta_{\mathrm{Ad}(f)}} L_\infty(f, \mathrm{Ad}, 1) L(f, \mathrm{Ad}, 1) \tag{5.26}$$

where $\Delta_{\mathrm{Ad}(f)} \in \mathbb{Q}^\times$ is the adjoint conductor. Recalling that:

$$L_\infty(f, \mathrm{Ad}, s) = \Gamma_{\mathbb{C}}(s + (k-1))^3 \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{R}}(s+1)^2,$$

we have that:

$$\pi^{-3(k-1)} L'(f, \mathrm{Ad}, 0) \sim_{\mathbb{Q}^\times} \sqrt{\Delta_{\mathrm{Ad}(f)}} \cdot \Lambda(f, \mathrm{Ad}, 1) \tag{5.27}$$

We observe that if $\alpha \in H^1_{\mathcal{M}}(M(f, \mathrm{Ad}), \mathbb{Q}(1))$, then $\alpha' = (2\pi i)^{k-1} \alpha \in H^{2k-1}_{\mathcal{M}}(\mathrm{Sym}^2 M(f), \mathbb{Q}(k))$, and similarly we observe that $\delta' = (2\pi i)^{(k-1)} \delta$, so

$$\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}} = \langle r_{\mathcal{D}}(\alpha'), \delta' \rangle_{\mathrm{pol}}. \tag{5.28}$$

We finally compute $q := \frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \left\langle \frac{[f^W]}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}}}, \frac{[w_\infty f^W]}{c^W(f)} \right\rangle_{\mathrm{SD}}$ up to rational factors:

$$\begin{aligned}
q &\sim_{\mathbb{Q}^\times} \pi^{-3} \frac{\sqrt{\Delta_{\mathrm{Ad}(f)}}}{\pi^{2k}} \frac{\langle f^W, f^W \rangle}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{PD}} c^W(f)} && (2.22) \\
&\sim_{\mathbb{Q}^\times} \frac{\pi^{(k-1)} \sqrt{\Delta_{\mathrm{Ad}(f)}} \cdot \Lambda(1, \mathrm{Ad}, f)}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}} c^W(f)} && \text{Corollary 5.6} \\
&\sim_{\mathbb{Q}^\times} \pi^{-2(k-1)} \frac{L'(f, \mathrm{Ad}, 0)}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}} c^W(f)} && (5.27) \\
&\sim_{\mathbb{Q}^\times} \frac{\pi^{-4(k-1)} c^+(M(f)(k-1)) c^-(M(f)(k-1)) \langle r_{\mathcal{D}}(\alpha'), \delta' \rangle_{\mathrm{pol}}}{\langle r_{\mathcal{D}}(\alpha), \delta \rangle_{\mathrm{pol}} c^W(f)} && \text{Theorem 5.7} \\
&\sim_{\mathbb{Q}^\times} \frac{(2\pi i)^{-4(k-1)} c^+(M(f)(k-1)) c^-(M(f)(k-1))}{c^W(f)} && (5.28) \\
&\sim_{\mathbb{Q}^\times} 1. && \text{Theorem 5.3}
\end{aligned}$$

This proves equation (5.25) and hence the theorem. □

## 6. Yoshida lifts from real quadratic fields

Let $E$ be an elliptic curve over a quadratic field $F$ such that $E^\sigma$ is not isogenous to $E$, where $\langle \sigma \rangle = \mathrm{Gal}(F/\mathbb{Q})$, and consider the associated abelian surface $A = R_{F/\mathbb{Q}} E$. We explicate our conjecture in this case, both when $F$ is a real (this section) and an imaginary quadratic field (next section).

Formally, the motive of $A$ is the restriction of scalars of the motive of $E$:

$$M := H^1(A) \cong R_{F/\mathbb{Q}} H^1(E). \tag{6.1}$$

We introduce some notation relevant to the various realizations of the motive $M$. Let $F = \mathbb{Q}(\alpha)$ where $\alpha^2 = \pm D$ for the discriminant $D > 0$. We fix an embedding $\sigma\colon F \hookrightarrow \mathbb{C}$ and assume $\sigma(\alpha) = \sqrt{\pm D} \in \mathbb{C}$. Writing $\langle c \rangle = \mathrm{Gal}(F/\mathbb{Q})$, the other embedding is $\sigma^c\colon F \hookrightarrow \mathbb{C}$. Let $\omega$ be a Néron differential on $E$ defined over $F$, and write $\omega^\sigma$, $\omega^{\sigma^c}$ for the associated differential forms on $E^\sigma(\mathbb{C})$ and $E^{\sigma^c}(\mathbb{C})$, respectively. Pick a

$\mathbb{Z}$-basis for $H_1(E^\sigma(\mathbb{C}),\mathbb{Z})$ and let $\widetilde{\gamma}_1,\widetilde{\gamma}_2$ be the dual basis of $H^1(E^\sigma(\mathbb{C}),\mathbb{Z})$ via the tautological pairing. The Galois automorphism $c$ gives a diffeomorphism:

$$E^\sigma(\mathbb{C}) \to E^{\sigma^c}(\mathbb{C})$$
$$\omega^\sigma \mapsto \overline{\omega^{\sigma^c}}.$$

Then:

(1) $F^1 H^1_{\mathrm{dR}}(M_{\mathbb{Q}}) = F\omega$,
(2) $H^1_B(M_{\mathbb{C}},\mathbb{R}) = H^1_B(E^\sigma(\mathbb{C}),\mathbb{R}) \oplus H^1_B(E^{\sigma^c}(\mathbb{C}),\mathbb{R})$ is spanned by $(\widetilde{\gamma}_i,\widetilde{\gamma}_j^c)$ for $i,j=1,2$.

In terms of the symmetric square motive, we have that:

$$\operatorname{Sym}^2 H^1(A) \cong R_{F/\mathbb{Q}} \operatorname{Sym}^2 H^1(E) \oplus \mathrm{Asai}_{F/\mathbb{Q}} H^1(E); \tag{6.2}$$

see, for example, Ghate [Gha96] for a definition of the Asai motive.

At this point, the cases of real and imaginary quadratic fields diverge:

(1) for $F$ real quadratic, $R_{F/\mathbb{Q}} \operatorname{Sym}^2 H^1(A)$ is critical and $\mathrm{Asai}_{F/\mathbb{Q}} H^1(A)$ is non-critical,
(2) for $F$ imaginary quadratic, $R_{F/\mathbb{Q}} \operatorname{Sym}^2 H^1(A)$ is non-critical and $\mathrm{Asai}_{F/\mathbb{Q}} H^1(A)$ is critical.

Interestingly, even though these two setups are quite different, our conjecture covers both cases simultaneously. We treat the former case in this section and the latter in the next section.

Henceforth, suppose $E$ is an elliptic curve over a real quadratic field $F$. Then $E$ corresponds to a Hilbert modular form $f_0$ of parallel weight 2 by [FLHS15], and the assumption that $E^\sigma$ is not isogenous to $E$ amounts to $f_0^\sigma \neq f_0$. Therefore, we can identify $M(f_0)$ with $M = R_{F/\mathbb{Q}} H^1(E)$:

$$M(f_0) = R_{F/\mathbb{Q}} H^1(E). \tag{6.3}$$

Poincaré duality on $H^1(E)$ gives a pairing

$$\langle -,-\rangle_{\mathrm{PD}} \colon H^1(E)(1) \times H^1(E) \to \mathbb{Q},$$

and hence there is a canonical polarization pairing on $M(f_0)$ given by:

$$\langle x,y\rangle_{\mathrm{pol}} = \langle x(1),y\rangle_{\mathrm{PD}}(-1).$$

The associated abelian surface $A = R_{F/\mathbb{Q}} E$ should correspond to a Siegel modular form $f$ of paramodular level according to Conjecture 3.8. This Siegel modular form was constructed in [JLR12], building on the ideas of Yoshida [Yos80, Yos84]. More precisely, Yoshida constructs an explicit Siegel modular form for the Siegel congruence subgroup, while Johnson–Schmidt construct the desired modular form of paramodular level. They proved the desired equality of $L$-functions:

$$L(M(f),s) = L(f,s) = L(f_0,s) = L(M(f_0),s). \tag{6.4}$$

According to the Tate conjecture, this gives the identification:

$$M(f) = M(f_0). \tag{6.5}$$

We remark that starting a Hilbert modular form $f_0$ of weight $(2,2)$ with rational Fourier coefficients, we only know how to construct the associated elliptic curve $E$ over $F$ when $f_0$ transfers to a quaternion algebra over $F$ split at a unique infinite place. For higher weight forms, the motive $M(f_0)$ was constructed by Blasius–Rogawski [BR93] using other methods. On the other hand, the Asai motive $M(f_0,\mathrm{Asai}) = \mathrm{Asai}_{F/\mathbb{Q}} H^1(E)$ appears directly in the cohomology of the Hilbert modular surface and was constructed by Kings [Kin98].

**Remark 6.1.** Yoshida also considers the split case $F = \mathbb{Q}\oplus\mathbb{Q}$, i.e. lifts a pair of classical modular forms to a Siegel modular form. However, as explained in Lemma 2.9, there is no holomorphic paramodular level Siegel modular form associated with a pair of classical modular forms. More precisely, if we fix any level structure $K_f$, then lifts of a pair of modular forms will contribute to either $H^0$ or $H^1$ but not both. Therefore, any purported motivic action in these cases would need to not only change the representation at infinity but would also need to change the representation at some finite places. Since this seems to have a different nature than other motivic actions [PV21, Hor23], we decided not to pursue this case further here.

The factorization (6.2) gives the following equality of $L$-functions:

$$L(f, \mathrm{Sym}^2, s) = L(f_0, \mathrm{Sym}^2, s) \cdot L(f_0, \mathrm{Asai}, s), \tag{6.6}$$

$$L(f, \mathrm{Ad}, s) = L(f_0, \mathrm{Ad}, s) \cdot L(f_0, \mathrm{Asai}, s+1). \tag{6.7}$$

Using this factorization and Ramakrishnan's results [Ram87], we can prove that the motivic action is rational without assuming part (2) of Beilinson's Conjecture 3.2.

**Theorem 6.2.** *Let $f$ be a Siegel modular form associated with a weight $(2,2)$ Hilbert modular form $f_0$. Then there is an explicit 1-dimensional subspace $R^\vee \subseteq H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ which acts rationally. In particular, Conjecture 4.19 is true under Hypothesis 3.3 that the Beilinson regulator is an isomorphism.*

For the rest of the section, we will build up to the proof of this theorem by summarizing Ramakrishnan's results for completeness, following [Kay16, Ram87]. Let $X_0$ be a toroidal compactification of the Hilbert modular surface over $\mathbb{Q}$ of level $\mathfrak{N}$. Recall that:

$$H^3_{\mathcal{M}}(X_0, \mathbb{Q}(2)) \cong \mathrm{CH}^{2,1}(X_0)$$

where the higher Chow group $\mathrm{CH}^{2,1}(X_0)$ is generated by formal $\mathbb{Q}$-rational sums $\sum_i a_i(C_i, \phi_i)$ where $C_i$ are closed irreducible curves on $X_{\overline{\mathbb{Q}}}$ and $\phi_i \in \mathcal{O}(C_i)^\times$ satisfy $\sum_i a_i \mathrm{div}(\phi_i) = 0$, up to equivalence [Kay16, Def. III.4]. We also have an explicit definition of the Deligne cohomology group, as above:

$$H^3_{\mathcal{D}}(X_0, \mathbb{R}(2)) \cong H^{1,1}(X_0, \mathbb{C}) \cap H^2_B(X_{0,\mathbb{C}}, \mathbb{R}(1))^+.$$

Because $X_0$ is a surface, we have a natural pairing on $H^{1,1}(X_{0,\mathbb{C}}, \mathbb{C})$ defined by

$$\langle \omega_1, \omega_2 \rangle = \int_{X(\mathbb{C})} \omega_1 \wedge \omega_2. \tag{6.8}$$

Then Beilinson's regulator (3.6) is the map

$$r_{\mathcal{D}} \colon H^3_{\mathcal{M}}(X_0, \mathbb{Q}(2)) \otimes \mathbb{R} \to H^3_{\mathcal{D}}(X_0, \mathbb{R}(2))$$

defined by the property: for $\alpha = \sum a_i(C_i, \phi_i)$

$$\langle r_{\mathcal{D}}(\alpha), \omega \rangle_{\mathrm{PD}} = \frac{1}{2\pi i} \sum_i a_i \int_{C_i(\mathbb{C})} \log|\phi_i| \cdot \omega|_{C_i}.$$

Compare this to the explicit form of Theorem 5.7.

**Remark 6.3.** In fact, we use $H^i_{\mathcal{M}}$ to denote the *integral subspace* of motivic cohomology group, and accordingly, we should be considering an integral version of the Chow groups above, but we omit this in the exposition.

When $F = \mathbb{Q} \oplus \mathbb{Q}$, the Hilbert modular surface $X_0$ is a product of two modular curves, and the Asai $L$-function recovers the Rankin–Selberg $L$-function of a pair of weight two forms. Beilinson proved his conjecture in this case by using modular units attached to the product of their product [Bei85] (c.f. [Kay16, Theorem III.8]). The motivic cohomology classes are called *Beilinson–Flach elements*. However, as explained in Remark 6.1, this case does not fall under the scope of our conjecture.

Instead, we consider the case where $F/\mathbb{Q}$ is real quadratic and use Ramakrishnan's results. He defined a $\mathbb{Q}$-subspace $R$ of $\mathrm{CH}^{2,1}(X_0) \otimes \mathbb{Q}$ generated by sums $\sum a_i(C_i, \phi_i)$ where $C_i$ are Hirzebruch–Zagier divisors on $X_0$ and $\phi_i$ are modular units on the associated modular curves. It is natural to conjecture that these cycles generate the Chow group, but as far as we know this is not currently known. We write $R_{f_0}$ for the $f_0$-isotypic component of $R$; then $R_{f_0} \subseteq H^3_{\mathcal{M}}(M(f_0, \mathrm{Asai}), \mathbb{Q}(2))$.

**Theorem 6.4** (Ramakrishnan)**.** *Let $f_0$ be a cuspidal Hilbert modular form form of weight $(2,2)$ which is not a base change form from $\mathbb{Q}$. Then:*

*(1) The subspace* $R_{f_0}$ *belongs to* $H^3_{\mathcal{M}}(M(f_0, \mathrm{Asai})_{\mathbb{Z}}, \mathbb{Q}(2))$.
*(2)* $r_{\mathcal{D}}(R_{f_0}) = L'_{\mathrm{Asai}}(f_0, 1) \cdot \mathcal{R}(M(f_0, \mathrm{Asai}), 2, 2) \subseteq H^3_{\mathcal{D}}(M(f_0, \mathrm{Asai})_{\mathbb{R}}, \mathbb{R}(2))$.

*Proof.* Part (1) can be proved by picking an integral model of the Hilbert modular variety $X_0$ following Rapoport [Rap78] and verifying that both the Hirzebruch–Zagier divisors and the modular units on them extend to this integral model. For part (2), see [Ram87, Prop. 12.30]. □

As in Theorem 5.7, we express this conjecture in terms of a canonical generator.

**Definition 6.5.** A *natural generator* of $H^3_{\mathcal{D}}(M(f_0, \mathrm{Asai}), \mathbb{R}(2))$ is given by:

$$\eta = \frac{(\omega^{\sigma} \otimes \overline{\omega^{\sigma^c}} - \overline{\omega^{\sigma}} \otimes \omega^{\sigma^c})}{\sqrt{D}} \in H^{1,1}(E^{\sigma}(\mathbb{C}) \times E^{\sigma^c}(\mathbb{C}), \mathbb{C})^{-}.$$

**Corollary 6.6.** *Let* $\eta$ *be a natural generator of* $H^3_{\mathcal{D}}(M(f_0, \mathrm{Asai})_{\mathbb{R}}, \mathbb{R}(2))$*. Then for some* $\alpha = \sum_i a_i(C_i, \phi_i) \in H^3_{\mathcal{M}}(M(f_0, \mathrm{Asai})_{\mathbb{Z}}, \mathbb{Q}(2))$ *where* $C_i$ *are Hirzebruch–Zagier divisors and* $\phi_i$ *are modular units, we have that:*

$$L'_{\mathrm{Asai}}(f_0, 1) = \langle r_{\mathcal{D}}(\alpha), \eta \rangle_{\mathrm{PD}} = \frac{1}{2\pi i} \sum_i a_i \int_{C_i(\mathbb{C})} \log|\phi_i| \cdot \eta|_{C_i}.$$

Let $f$ be the paramodular level $D^2 N^F_{\mathbb{Q}}(\mathfrak{N})$ Siegel modular of weight $(2, 2)$ associated with $f_0$ by Johnson-Lueng–Roberts [JLR12]

We compare the natural generators of the Deligne cohomology groups.

**Lemma 6.7.** *Under the natural isomorphism:*

$$d\colon H^3_{\mathcal{D}}(M(f_0, \mathrm{Asai}), \mathbb{R}(2)) \to H^3_{\mathcal{D}}(\mathrm{Sym}^2(M(f)), \mathbb{R}(2))$$

*a natural generator* $\eta \in H^3_{\mathcal{D}}(M(f_0, \mathrm{Asai}), \mathbb{R}(2))$ *maps to*

$$d(\eta) = 2\sqrt{D}^{-1}\delta$$

*where* $\delta \in H^1_{\mathcal{D}}(M(f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ *is a natural generator (Definition 4.13).*

*Proof.* Recall that under the identification $M(f_0) = M(f)$, a choice of basis $\omega_1, \omega_2$ of $F^1 H^1_{\mathrm{dR}}(M(f)_{\mathbb{Q}})$ is:

$$\omega_1 = \omega,$$
$$\omega_2 = \sqrt{D}\omega.$$

The assertion follows by direct computation. □

We finally deduce Theorem 6.2.

*Proof of Theorem 6.2.* We have shown in Theorem 4.21 that Conjecture 4.19 is true under three assumptions:

(1) Deligne's conjecture for $L(M(f), s)$ at the central point $s = 1$,
(2) Beilinson's conjecture for $L(M(f, \mathrm{Ad}), s)$ at the point $s = 1$,
(3) Hypothesis 5.2.

We need to check these three conditions for $M(f) = M(f_0)$ where $f$ is the paramodular Yoshida lift associated with $f_0$.

Parts (1) and (3) are classical: see [Shi78, Har90c] and [FH95, Theorem A], respectively. Therefore, it is enough to prove part (2). Thanks to the factorization (6.7), it is enough to prove Beilinson's conjecture for the adjoint $L$-function of $f_0$ and the Asai $L$-function of $f_0$:

- The $L$-value at $s = 1$ of $L_{\mathrm{ad}}(f_0, s)$ is critical and explicitly related to the Petersson inner product $\langle f_0, f_0 \rangle$ (e.g. [IP21, Prop. 6.6]).

- The $L$-value at $s = 2$ of $L_{\mathrm{Asai}}(f_0, s)$ is non-critical and Beilinson's conjecture for this $L$-function was proved by Ramakrishnan [Ram87]; see Theorem 6.4 above.

This completes the proof. $\square$

Together with the factorization (6.7), it is now clear that Corollary 6.6 is equivalent to our explicit form of Beilinson's conjecture in Theorem 5.7. The "critical part" of the period is given by the Petersson norm of $f_0$:

$$c^W(f) \approx c^+(M(f))c^-(M(f)) \approx \langle f_0, f_0\rangle \tag{6.9}$$

where $\approx$ means up rational numbers and powers of $\pi$.

## 7. Yoshida lifts from imaginary quadratic fields

Suppose $E$ is a modular elliptic curve over an imaginary quadratic field. Under technical assumptions, Caraiani–Newton [CN23] prove that $E$ is modular, i.e. there is an associated Bianchi cusp form $f_0$ of weight 2, building on the potential modularity result in [ACC$^+$23].

As in the previous section, we assume that $E^\sigma$ is not isogenous to $E$ where $\langle\sigma\rangle = \mathrm{Gal}(F/\mathbb{Q})$, i.e. $f_0^\sigma \not\cong f_0$. The associated abelian surface $A = R_{F/\mathbb{Q}}E$ should correspond to a Siegel modular form $f$ of paramodular level according to Conjecture 3.8; these may be constructed explicitly using a Yoshida-type lifts from $O(3,1)$ to $\mathrm{Sp}(4)$ which we will discuss in the next section.

The factorization (6.2) may be written as:

$$M(f, \mathrm{Ad}) \cong M(f_0, \mathrm{Ad}) \oplus M(f_0, \mathrm{Asai})(1), \tag{7.1}$$

which gives a commutative diagram:

$$\begin{array}{ccc} H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1)) & \xrightarrow{\cong} & H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1)) \oplus H^3_{\mathcal{M}}(M(f_0,\mathrm{Asai})_{\mathbb{Z}}, \mathbb{Q}(2)) \\ \downarrow{\scriptstyle r_{\mathcal{D}}} & & \downarrow{\scriptstyle r_{\mathcal{D}}\oplus r_{\mathcal{D}}} \\ H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) & \xrightarrow{\cong} & H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) \oplus H^3_{\mathcal{D}}(M(f_0,\mathrm{Asai})_{\mathbb{R}}, \mathbb{R}(2)). \end{array}$$

A simple computation shows that:

$$H^3_{\mathcal{D}}(M(f_0,\mathrm{Asai})_{\mathbb{R}}, \mathbb{R}(2)) = 0. \tag{7.2}$$

Therefore, we get an isomorphism:

$$d\colon H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) \xrightarrow{\cong} H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)). \tag{7.3}$$

Assuming Hypothesis 3.3 that the regulator map is an isomorphism, equation 7.2 implies that

$$H^3_{\mathcal{M}}(M(f_0,\mathrm{Asai})_{\mathbb{Z}}, \mathbb{Q}(2)) = 0. \tag{7.4}$$

Under this assumption, we hence get an isomorphism

$$m\colon H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1)) \xrightarrow{\cong} H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1)). \tag{7.5}$$

Our target theorem is the following.

**Theorem 7.1.** *Let $f_0$ be a Bianchi modular form of weight $(2,2)$ with trivial character and rational Fourier coefficients, and let $f$ be the associated Siegel modular form. Under Hypothesis 3.3, our Conjecture 4.19 implies Conjecture 7.11 which is an explicit form of the conjecture of Prasanna–Venkatesh [PV21]. More precisely, we have the following two statements.*

*(1) For $i = 1, 2$ there is a rational map*

$$\theta_i\colon H^i(X_0, \mathbb{Q})_{f_0} \to H^{i-1}(X, \mathcal{E}_{2,2})_f \tag{7.6}$$

*such that, under the natural isomorphism* (7.3) *of dual Deligne cohomology groups:*

$$d^\vee \colon H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad}),\mathbb{R}(1))^\vee \to H^1_{\mathcal{D}}(M(f,\mathrm{Ad}),\mathbb{R}(1))^\vee, \tag{7.7}$$

*the diagram*

$$\begin{tikzcd} H^1(X_0,\mathbb{Q})_{f_0}\otimes\mathbb{C} \arrow[r, "\theta_1"] \arrow[d, "\eta*"] & H^0(X,\mathcal{E}_{2,2})_f\otimes\mathbb{C} \arrow[d, "d^\vee(\eta)*"] \\ H^2(X_0,\mathbb{Q})_{f_0}\otimes\mathbb{C} \arrow[r, "\theta_2"] & H^1(X,\mathcal{E}_{2,2})_f\otimes\mathbb{C} \end{tikzcd}$$

*commutes for any* $\eta \in H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad}),\mathbb{R}(1))$.

(2) *Assuming Hypothesis 3.3, under the natural isomorphism* (7.5) *of dual motivic cohomology groups:*

$$m^\vee \colon H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad}),\mathbb{Q}(1)) \to H^1_{\mathcal{M}}(M(f,\mathrm{Ad}),\mathbb{Q}(1)), \tag{7.8}$$

*we have a commutative diagram:*

$$\begin{tikzcd} H^1(X_0,\mathbb{Q})_{f_0} \arrow[r, "\theta_1"] \arrow[d, "\alpha*"] & H^0(X,\mathcal{E}_{2,2})_f \arrow[d, "m^\vee(\alpha)*"] \\ H^2(X_0,\mathbb{Q})_{f_0}\otimes\mathbb{C} \arrow[r, "\theta_2"] & H^1(X,\mathcal{E}_{2,2})_f\otimes\mathbb{C} \end{tikzcd}$$

*for any* $\alpha \in H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad}),\mathbb{Q}(1))^\vee$. *Therefore, the rationality of the action of* $\alpha$ *is equivalent to the rationality of the action of* $m^\vee(\alpha)$.

**Remark 7.2.** The key point is that Theorem 7.1 is proved without assuming Beilinson's Conjecture 3.2. Part (1) is a statement only about the action of the Deligne cohomology groups, so it makes sense to formulate it without even assuming that the motivic cohomology groups have rank 1 (i.e. without Hypothesis 3.3). Part (2) then follows from part (1) after imposing this hypothesis, but still without assuming the rationality statement of Beilinson's Conjecture 3.2 (2). While both conjectures are implied by the rationality statement in Beilinson's conjecture for $L(f,\mathrm{Ad},1) = L(f_0,\mathrm{Ad},1)L(f_0,\mathrm{Asai},2)$, we instead give a direct relationship between the conjectures.

**Remark 7.3.** A similar theorem should be true for Bianchi modular forms $f_0$ of any parallel weight $(k,k)$. We decided to not pursue this here, because the conjecture in [PV21] is not stated for cohomology of local systems.

7.1. **Theta lifts from** $\mathrm{GO}(3,1)$ **to** $\mathrm{GSp}_4$**.** We follow [HST93, BDPŞ15] to construct both a holomorphic Siegel modular form $f$ and a Whittaker normalized Siegel modular form $f^W$ of paramodular level associated with a Bianchi cusp form $f_0$.

**Theorem 7.4** (Harris–Soudry–Taylor [HST93], Berger–Dembélé–Pacetti–Şengün [BDPŞ15, Theorem 4.1])**.** *Let* $F/\mathbb{Q}$ *be an imaginary quadratic field of discriminant* $D$ *and let* $\mathfrak{N}$ *be an ideal of* $\mathcal{O}_F$*. ideal. Let* $f_0$ *be a Bianchi modular form of level* $\mathfrak{N}$ *and weight* $(k,k)$ *for some* $k \geq 2$ *even, which is not Galois invariant. Then there exists a holomorphic Siegel modular form* $f$ *of weight* $(k,2)$ *and paramodular level* $N = D^2 N_{F/\mathbb{Q}}\mathfrak{N}$ *with Hecke eigenvalues, epsilon factor, and spinor L-function determined explicitly by* $f_0$.

Let $\Pi$ be the $L$-packet containing the automorphic representation generated by $f$. Then $\Pi$ also contains a generic representation, generated by a Whittaker-normalized vector $f^W$. In fact, both $f$ and $f^W$ can be constructed in the following uniform way using theta lifting, following [BDPŞ15, Section 4] for details.

Let $\sigma_F$ be an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ associated with a weight $(k,k)$ Bianchi modular form $f_0$ with trivial Nebentypus. There is a quadratic space $X$ such that:

$$\mathrm{GSO}(X) \cong \mathrm{GL}_{2,F} \times_{\mathrm{GL}_{1,F}} \mathrm{GL}_{1,\mathbb{Q}}$$

and hence a representation of $\mathrm{GSO}(X)$ corresponds to a representation of $\mathrm{GL}_{2,F}$ together with an extension $\widetilde{\omega}$ of its central character. By choosing the trivial extension of the central character, we get an automorphic representation $\widetilde{\sigma_F}$ of $\mathrm{GSO}(X)$ associated with $\sigma_F$. Since the theta correspondence is defined for representations of $\mathrm{GO}(X)$, we further to extend the representation there. We consider two extensions:

- $\hat{\sigma}_F^+ = \bigotimes_{v \text{ finite}} \sigma_v^+ \otimes \sigma_\infty^+$,
- $\hat{\sigma}_F^- = \bigotimes_{v \text{ finite}} \sigma_v^+ \otimes \sigma_\infty^-$.

(see [BDPŞ15, pp. 361] for the notation). We apply the theta correspondence to these representations. We know that:

- for finite $v$, $\theta(\sigma_v^+)$ is the unique generic representation of $\mathrm{GSp}_4(\mathbb{Q}_v)$ with the given $L$-parameter [Rob01],
- $\theta(\sigma_\infty^-) = X_\lambda^1$ for $\lambda = (k-1, 0)$,
- $\theta(\sigma_\infty^+) = X_\lambda^2$ for $\lambda = (k-1, 0)$.

Moreover, we know that $\Theta(\hat{\sigma}_F^\pm)$ is non-vanishing by the local non-vanishing together with [Tak09, Theorem 1.2], and cuspidal by [Tak09, Theorem 1.3(b)]. By choosing vectors (see [BDPŞ15] and the references therein), we obtain Theorem 7.4.

We now discuss the contributions of the above modular forms to cohomology and hence maps induced on cohomology via the theta lifts. Given a Bianchi modular form $f_0$ of weight $(k, k)$, there are two natural contributions to the singular cohomology of a local system $\mathcal{V}_{k,k}^F$ on the Bianchi modular threefold $X_0$:

$$\omega^i(f_0) \in H_!^i(X_0, \mathcal{V}_{k,k}^F(\mathbb{C}))_{f_0};$$

see for example the explicit description in [TU22, Section 5.1]. Because the right hand side is one-dimensional, we may use the natural rational structure on the right hand side to define periods $u^i(f_0) \in \mathbb{C}^\times$ such that:

$$\frac{\omega^i(f_0)}{u^i(f_0)} \in H_!^i(X_0, \mathcal{V}_{k,k}^F(\mathbb{Q}))_{f_0} \qquad i = 1, 2. \tag{7.9}$$

Moreover, define the period $d^W(f) \in \mathbb{C}^\times$ by normalizing $[f^W]$ to be rational:

$$\frac{[f^W]}{d^W(f)} \in H^1(X, \mathcal{E}_{k,2})_f. \tag{7.10}$$

**Remark 7.5.** When $k = 2$, the conjecture of Prasanna-Venkatesh [PV21] amounts to a relationship between $u^1(f_0)$, $u^2(f_0)$ and a Beilinson regulator; see Proposition 7.13 and its corollary for details. Similarly, our Conjecture 4.19 is equivalent to $d^W(f) \sim \langle r_{\mathcal{D}}(\alpha), \delta\rangle_{\mathrm{pol}}$ for a natural generator $\delta$, which we proved is equivalent to Beilinson's Conjecture in Theorem 5.7 (under some assumptions).

**Definition 7.6.** We define the *cohomologically-normalized theta lifts* to be the maps:

$$\theta \colon H^*(X_0, \mathcal{V}_{k,k}^F(\mathbb{Q}))_{f_0} \to H^{*-1}(X, \mathcal{E}_{k,2})_f \tag{7.11}$$

induced by Theorem 7.4 and normalized rationally, i.e. explicitly:

$$\theta\left(\frac{\omega^1(f_0)}{u^1(f_0)}\right) = [f], \tag{7.12}$$

$$\theta\left(\frac{\omega^2(f_0)}{u^2(f_0)}\right) = \frac{[f^W]}{d^W(f)}. \tag{7.13}$$

They are well-defined up to $\mathbb{Q}^\times$ (or more generally $E^\times$ if both $f$ and $f_0$ have coefficients in $E$.)

**Remark 7.7.** Note that the domain of our map $\theta$ in (7.11) is the singular cohomology of the Bianchi threefold, while the codomain is the (coherent) sheaf cohomology of $\mathcal{E}_{k,2}$ on the Shimura variety $X$. Therefore, it is difficult to interpret $\theta$ geometrically.

### 7.2. **Explication of the conjecture of Prasanna–Venkatesh** [PV21]**.**

In order to prove Theorem 7.1, we explicate the conjecture of Prasanna–Venkatesh [PV21] in the case of motives associated to elliptic curves over imaginary quadratic fields.

**Definition 7.8.** A *natural generator* of $H^3_{\mathcal{D}}(M(f_0, \mathrm{Sym}^2), \mathbb{R}(2))$ is:

$$\eta' = i(\omega^\sigma \otimes \overline{\omega^\sigma} + \overline{\omega^\sigma} \otimes \omega^\sigma).$$

Therefore, a *natural generator* of $H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad}),\mathbb{R}(1))$ is given by:

$$\eta = -(2\pi)(\omega^\sigma \otimes \overline{\omega^\sigma} + \overline{\omega^\sigma} \otimes \omega^\sigma).$$

Similarly to Definition 4.15, we also define the dual natural generator $\eta^\vee$ of $H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad}),\mathbb{R}(1))$ by:

$$\eta^\vee := \frac{\pi^2}{\sqrt{\Delta_{\mathrm{Ad}(f_0)}}}\langle \eta, - \rangle_{\mathrm{PD}}$$

**Definition 7.9.** The action of the dual natural generator $\eta^\vee \in H^1_{\mathcal{D}}(M(f_0,\mathrm{Ad})_{\mathbb{R}},\mathbb{R}(1))^\vee$ on singular cohomology by:

$$H^1(X_0,\mathbb{C})_{f_0} \mapsto H^2(X_0,\mathbb{C})_{f_0}$$
$$\frac{\omega^1(f_0)}{u^1(f_0)} \mapsto \omega^2(f_0).$$

**Remark 7.10.** As explained in [Hor23, Remark 5.10], we expect that the adjoint conductor $\Delta_{\mathrm{Ad}(f_0)}$ is a square. This is proved in [Hor23, Proposition 5.8] except when the local component of the automorphic representation of $f_0$ at a place dividing 2 is the theta lift from a ramified quadratic extension. In particular, it is at least true that $\sqrt{\Delta_{\mathrm{Ad}(f_0)}} \in \sqrt{2}\mathbb{Q}^\times$. In particular, the factor of $\sqrt{\Delta_{\mathrm{Ad}(f_0)}}$ is not strictly necessary when defining the natural generator, but we keep it here for consistency with Definition 4.15.

**Conjecture 7.11** (Prasanna–Venkatesh [PV21])**.** *Via the dual Beilinson regulator, the resulting action of $H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad}),\mathbb{Q}(1))^\vee$ preserves the rational structure $H^*(X_0,\mathbb{Q})_{f_0}$.*

**Remark 7.12.** As stated, this conjecture is actually an amalgamation of the conjecture of Prasanna–Venkatesh [PV21] and the conjecture and Cremona–Whitley [CW94].

Indeed, the original phrasing of Prasanna–Venkatesh is slightly different. Rather than considering a dual natural generator $\eta$ in Definition 7.8, they consider the normalized element:

$$\widetilde{\eta}^\vee = \frac{\eta^\vee}{h(\omega^\sigma)},$$

where $h(\omega) = \langle \omega^\sigma, \omega^\sigma \rangle_{\mathrm{PD}}$ is Faltings' height. Cremona–Whitley [CW94] conjectured that Faltings' height is explicitly related to $u^1(f_0)$ and provided numerical evidence. Granting this relationship, the action in Definition 7.9 has the more familiar form

$$\widetilde{\eta}^\vee(\omega^1(f_0)) = \omega^2(f_0). \tag{7.14}$$

The advantage of this phrasing is that $\widetilde{\eta}^\vee$ does not depend on the choice of Néron differential $\omega^\sigma$.

We offer the alternative phrasing above here to bring it closer in line with our conjecture in the case of Siegel modular forms. The reason we cannot phrase our motivic action similarly to (7.14) is that the holomorphic Siegel modular form does not have a Whittaker model, and hence we are forced to normalize it using coherent cohomology directly. Therefore, the analogue of $[f]$ is actually $\frac{\omega^1(f_0)}{u^1(f_0)}$ and not just $\omega^1(f_0)$. Both Conjectures 4.19 and 7.11 then take the form:

$$(\text{natural generator}) * (\text{rational class}) = (\text{Whittaker class}).$$

In Section 8, we discuss a generalization of this to Hilbert–Siegel modular forms.

We next check that this conjecture is consistent with Beilinson's conjecture for $M(f_0,\mathrm{Ad})$.

**Proposition 7.13.** *Beilinson's conjecture for $M(f_0,\mathrm{Sym}^2)$ is equivalent to:*

$$L'(f_0,\mathrm{Sym}^2,0) \sim_{\mathbb{Q}^\times} \pi \cdot \langle r_{\mathcal{D}}(\alpha'), \eta' \rangle_{\mathrm{PD}} \cdot u^1(f_0)$$

*for $\alpha' \in H^3_{\mathcal{M}}(M(f_0,\mathrm{Sym}^2),\mathbb{Q}(2))$ and a natural generator $\eta'$ from Definition 7.8.*

The proof is a formal computation with Beilinson's conjecture, similar to the proof of Theorem 5.7, and will occupy the rest of this subsection. Before that, we state a corollary which is implicit in [Urb95, PV21].

**Corollary 7.14.** *Beilinson's conjecture for $M(f_0,\mathrm{Ad})$ implies Conjecture 7.11.*

*Proof.* To check Conjecture 7.11, we must check that for $\alpha \in H^1_{\mathcal{M}}(M(f_0,\mathrm{Ad}),\mathbb{Q}(1))$, the cohomology class

$$\frac{\sqrt{\Delta_{\mathrm{Ad}(f_0)}}}{\pi^2}\frac{\omega^2(f_0)}{\langle r_{\mathcal{D}}(\alpha),\eta\rangle_{\mathrm{PD}}} \in H^2(X_0,\mathbb{C})$$

is rational. By pairing the above cohomology class with rational cohomology class $\frac{\omega^1(f_0)}{u^1(f_0)}$ under Poincaré duality, it is enough to verify that

$$\langle f_0,f_0\rangle \sim_{\mathbb{Q}^\times} \sqrt{\Delta_{\mathrm{Ad}(f_0)}}^{-1}\pi^2 u^1(f_0)\cdot\langle r_{\mathcal{D}}(\alpha),\eta\rangle_{\mathrm{PD}}.$$

We have that:

$$\begin{aligned}
\langle f_0,f_0\rangle &\sim_{\mathbb{Q}^\times} \pi^{-2}L(f_0,\mathrm{Ad},1) && \text{[Urb95, Prop. 7.1]}\\
&\sim_{\mathbb{Q}^\times} \sqrt{\Delta_{\mathrm{Ad}(f_0)}}^{-1}\pi L'(f_0,\mathrm{Ad},0) && \text{functional equation}\\
&\sim_{\mathbb{Q}^\times} \sqrt{\Delta_{\mathrm{Ad}(f_0)}}^{-1}\pi^2\langle r_{\mathcal{D}}(\alpha'),\eta'^\vee\rangle_{\mathrm{PD}}\cdot u^1(f_0) && \text{Proposition 7.13,}\\
&\sim_{\mathbb{Q}^\times} \sqrt{\Delta_{\mathrm{Ad}(f_0)}}^{-1}\pi^2\langle r_{\mathcal{D}}(\alpha),\eta^\vee\rangle_{\mathrm{pol}}\cdot u^1(f_0)
\end{aligned}$$

as claimed. □

In the next two sections, we will prove Proposition 7.13.

7.2.1. *The motive* $M=\mathrm{Res}_{F/\mathbb{Q}}H^1(E)$. We consider $M=\mathrm{Res}_{F/\mathbb{Q}}H^1(E)$ and $n=1$, where $E$ is an elliptic curve over an imaginary quadratic field $F=\mathbb{Q}(\sqrt{-D})$.

We keep the notation of the previous section and write $\bar\sigma$ for $\sigma^c$ Let $\{\gamma_1^c,\gamma_2^c\}$ denote the basis of $H_1(E^{\bar\sigma}(\mathbb{C}),\mathbb{Z})$ obtained by applying $c$ to the basis $\{\gamma_1,\gamma_2\}$, and let $\{\widetilde\gamma_1^c,\widetilde\gamma_2^c\}$ denote the dual basis of $H^1(E^{\bar\sigma}(\mathbb{C}),\mathbb{Z})$. Thus

$$\langle\gamma_i^c,\omega^{\bar\sigma}\rangle=\overline{\langle\gamma_i,\omega^\sigma\rangle},$$

for $i=1,2$.

The point $n=1$ is critical and the relevant map is:

$$F^1H^1_{\mathrm{dR}}(M_{\mathbb{R}})\to H^1_B(M_{\mathbb{R}},\mathbb{R}(0))=H^1_B(M_{\mathbb{C}},\mathbb{R})^+.$$

Now,

$$F^1H^1_{\mathrm{dR}}(M_{\mathbb{Q}})=F\omega,$$

while

$$H^1_B(M_{\mathbb{C}},\mathbb{R})=H^1_B(E^\sigma(\mathbb{C}),\mathbb{R})\oplus H^1_B(E^{\bar\sigma}(\mathbb{C}),\mathbb{R}).$$

Suppose that

$$\langle\omega^\sigma,\gamma_1\rangle=a=a_1+a_2i,\quad \langle\omega^\sigma,\gamma_2\rangle=b=b_1+b_2i,$$

where the $a_i,b_i$ are in $\mathbb{R}$. Via the comparison isomorphisms:

$$\begin{aligned}
\omega^\sigma &= (a_1+a_2i)\widetilde\gamma_1+(b_1+b_2i)\widetilde\gamma_2,\\
\omega^{\bar\sigma} &= (a_1-a_2i)\widetilde\gamma_1^c+(b_1-b_2i)\widetilde\gamma_2^c.\\
(\sqrt{-D}\omega)^\sigma &= \sqrt{D}\left[(a_1i-a_2)\widetilde\gamma_1+(b_1i-b_2)\widetilde\gamma_2\right].\\
(\sqrt{-D}\omega)^{\bar\sigma} &= -\sqrt{D}\left[(a_1i+a_2)\widetilde\gamma_1^c+(b_1i+b_2)\widetilde\gamma_2^c\right].
\end{aligned}$$

Thus the map

$$F^1H^1_{\mathrm{dR}}(M_{\mathbb{R}})\to H^1_B(M_{\mathbb{C}},\mathbb{C})$$

sends

$$\begin{aligned}
\omega &\mapsto a_1(\widetilde\gamma_1,\widetilde\gamma_1^c)+a_2i(\widetilde\gamma_1,-\widetilde\gamma_1^c)+b_1(\widetilde\gamma_2,\widetilde\gamma_2^c)+b_2i(\widetilde\gamma_2,-\widetilde\gamma_2^c),\\
\sqrt{-D}\omega &\mapsto \sqrt{D}\,[a_1i(\widetilde\gamma_1,-\widetilde\gamma_1^c)-a_2(\widetilde\gamma_1,\widetilde\gamma_1^c)+b_1i(\widetilde\gamma_2,-\widetilde\gamma_2^c)-b_2(\widetilde\gamma_2,\widetilde\gamma_2^c).]
\end{aligned}$$

Consequently, the map

$$F^1 H^1_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^1_B(M_{\mathbb{C}}, \mathbb{R})$$

sends

$$\omega \mapsto a_1(\widetilde{\gamma}_1, \widetilde{\gamma}_1^c) + b_1(\widetilde{\gamma}_2, \widetilde{\gamma}_2^c),$$
$$\sqrt{-D}\omega \mapsto -\sqrt{D}\,[a_2(\widetilde{\gamma}_1, \widetilde{\gamma}_1^c) + b_2(\widetilde{\gamma}_2, \widetilde{\gamma}_2^c).]$$

Clearly this lands in $H^1_B(M_{\mathbb{C}}, \mathbb{R})^+$ and taking the determinant with respect to the bases $\{\omega, \sqrt{-D}\omega\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ where

$$\mathbf{e}_1 := (\widetilde{\gamma}_1, \widetilde{\gamma}_1^c), \quad \mathbf{e}_2 := (\widetilde{\gamma}_2, \widetilde{\gamma}_2^c)$$

gives:

$$c^+(M(1)) = -\sqrt{D}(a_1 b_2 - a_2 b_1). \tag{7.15}$$

One can easily also check that

$$c^-(M(1)) = -\sqrt{D}(a_1 b_2 - a_2 b_1) = c^+(M(1)) \tag{7.16}$$

by considering the map

$$F^1 H^1_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^1_B(M_{\mathbb{C}}, \mathbb{R})^-$$

which sends

$$\omega \mapsto a_2 i(\widetilde{\gamma}_1, -\widetilde{\gamma}_1^c) + b_2 i(\widetilde{\gamma}_2, -\widetilde{\gamma}_2^c),$$
$$\sqrt{-D}\omega \mapsto \sqrt{D}\,[a_1 i(\widetilde{\gamma}_1, -\widetilde{\gamma}_1^c) + b_1 i(\widetilde{\gamma}_2, -\widetilde{\gamma}_2^c).]$$

Deligne's conjecture for Bianchi modular forms was studied by Cremona–Whitley and Hida.

**Theorem 7.15** ([CW94, (2.4)], [Hid94])**.** *Let $f_0$ be a Bianchi modular form of weight $(2,2)$, rational coefficients, and trivial central character. Then for $\chi = 1$ and for some quadratic characters $\chi$:*

$$L(f_0, \chi, 1) \sim_{\mathbb{Q}^\times} \pi^2 u^1(f_0).$$

*Therefore, Deligne's conjecture for $M(f_0)$ is equivalent to:*

$$c^\pm(M(f_0)(1)) \sim_{\mathbb{Q}^\times} \pi^2 u^1(f_0).$$

**Remark 7.16.** In our notation, $\int_{E(\mathbb{C})} \omega \wedge \overline{\omega}$ is a totally imaginary number, because:

$$\overline{\int_{E(\mathbb{C})} \omega \wedge \overline{\omega}} = \int_{E(\mathbb{C})} \overline{\omega \wedge \overline{\omega}} = -\int_{E(\mathbb{C})} \omega \wedge \overline{\omega}.$$

We caution the read that Cremona–Whitley [CW94] use the notation $\int_{E(\mathbb{C})} \omega \wedge \overline{\omega}$ to denote its imaginary part which accounts for the factor of $i$ in our phrasing of their conjecture.

7.2.2. *The motive $M = M(f_0, \mathrm{Asai})$.* We consider the Asai motive $M = M(f_0, \mathrm{Asai})$ and the critical value $s = 2$. We refer to Ghate [Gha96] for a detailed exposition of Deligne's conjecture for Asai motives of Bianchi modular forms. We record that:

$$c^+(M(2)) \sim_{\mathbb{Q}^\times} c^+(M) \cdot (2\pi i)^6 \qquad \text{[Gha96, (11)]} \tag{7.17}$$

$$c^+(M) \sim_{\mathbb{Q}^\times} c^+(M(f_0)) \cdot (2\pi i)^{-2} \qquad \text{[Gha96, Prop. 3 and Remark 3]} \tag{7.18}$$

$$c^-(M) \sim_{\mathbb{Q}^\times} \frac{-4}{\sqrt{-D}} c^+(M(f_0)) \qquad \text{[Gha96, Prop. 3]} \tag{7.19}$$

In particular, the $c^\pm(M(2))$ periods are determined in terms of the $c^+(M(f))$ period discussed above.

The main result of Ghate's thesis, strengthened by Loeffler–Williams is the following.

**Theorem 7.17** (Ghate [Gha96, Theorem 1], Loeffler–Williams [LW20, Corollary A.10])**.** *Let $f_0$ be a Bianchi modular form of weight $(2,2)$, trivial central character. Then:*

$$L(f_0, \mathrm{Asai}, 2) \sim_{\mathbb{Q}^\times} (2\pi i)^4 u^1(f_0).$$

*Therefore, Deligne's conjecture for the Asai motive is equivalent to:*

$$c^+(M(f_0, \mathrm{Asai})(2)) \sim_{\mathbb{Q}^\times} (2\pi i)^4 u^1(f_0).$$

**Remark 7.18.** Note that this is consistent with Theorem 7.15 and equations (7.17) (7.18):

$$u^1(f_0) = c^+(M(f_0)(1))\pi^{-2} = c^+(M(f_0)) = \pi^2 c^+(M(f_0, \mathrm{Asai})) = \pi^{-4} c^+(M(f_0, \mathrm{Asai})(2)).$$

7.2.3. *The motive $M = \mathrm{Sym}^2 H^1(E)$.* We consider $M = \mathrm{Sym}^2 H^1(E)$, $n = 1$, where $E$ is an elliptic curve over an imaginary quadratic field $F$. We keep the notation of the previous examples. The Beilinson exact sequence (3.1) in this case is:

$$0 \to F^2 H^2_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^2_B(M_{\mathbb{R}}, \mathbb{R}(1)) \to H^3_{\mathcal{D}}(M_{\mathbb{R}}, \mathbb{R}(2)) \to 0.$$

Now,

$$H^2_B(M_{\mathbb{R}}, \mathbb{R}(1)) = H^2_B(M_{\mathbb{C}}, \mathbb{R}(1))^+ \overset{\cdot(2\pi i)^{-1}}{\simeq} H^2_B(M_{\mathbb{C}}, \mathbb{R})^-.$$

We can realize $M$ as a submotive of $\mathrm{Res}_{F/\mathbb{Q}}(H^1(E) \otimes H^1(E))$. Then

$$F^2 H^2_{\mathrm{dR}}(M) = F \cdot \omega \otimes \omega,$$

while

$$H^2_B(M_{\mathbb{C}}, \mathbb{R}) = \mathrm{Sym}^2 H^1_B(E^\sigma(\mathbb{C}), \mathbb{R}) \oplus \mathrm{Sym}^2 H^1_B(E^{\bar\sigma}(\mathbb{C}), \mathbb{R}).$$

Via the comparison isomorphisms:

$$\begin{aligned}(\omega \otimes \omega)^\sigma &= ((a_1 + a_2 i)\widetilde{\gamma}_1 + (b_1 + b_2 i)\widetilde{\gamma}_2) \otimes ((a_1 + a_2 i)\widetilde{\gamma}_1 + (b_1 + b_2 i)\widetilde{\gamma}_2) \\ &= (a_1 + a_2 i)^2 \widetilde{\gamma}_1 \otimes \widetilde{\gamma}_1 + (a_1 + a_2 i)(b_1 + b_2 i)(\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 + \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_1) + (b_1 + b_2 i)^2 \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_2.\end{aligned}$$

Likewise,

$$\begin{aligned}(\omega \otimes \omega)^{\bar\sigma} &= (a_1 - a_2 i)^2 \widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_1^c + (a_1 - a_2 i)(b_1 - b_2 i)(\widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_2^c + \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_1^c) + \\ &\qquad (b_1 - b_2 i)^2 \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_2^c. \\ (\sqrt{-D} \cdot \omega \otimes \omega)^\sigma &= \sqrt{D} i \cdot [(a_1 + a_2 i)^2 \widetilde{\gamma}_1 \otimes \widetilde{\gamma}_1 + (a_1 + a_2 i)(b_1 + b_2 i)(\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 + \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_1) + \\ &\qquad (b_1 + b_2 i)^2 \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_2]. \\ (\sqrt{-D} \cdot \omega \otimes \omega)^{\bar\sigma} &= -\sqrt{D} i \cdot [(a_1 - a_2 i)^2 \widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_1^c + (a_1 - a_2 i)(b_1 - b_2 i)(\widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_2^c + \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_1^c) \\ &\qquad (b_1 - b_2 i)^2 \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_2^c.]\end{aligned}$$

Let

$$\begin{aligned}\mathbf{e}_{11} &= (\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_1, \widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_1^c) & \mathbf{f}_{11} &= (\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_1, -\widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_1^c) \\ \mathbf{e}_{12} &= \frac{1}{2}(\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 + \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_1, \widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_2^c + \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_1^c) & \mathbf{f}_{12} &= \frac{1}{2}(\widetilde{\gamma}_1 \otimes \widetilde{\gamma}_2 + \widetilde{\gamma}_2 \otimes \widetilde{\gamma}_1, -\widetilde{\gamma}_1^c \otimes \widetilde{\gamma}_2^c - \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_1^c) \\ \mathbf{e}_{22} &= (\widetilde{\gamma}_2 \otimes \widetilde{\gamma}_2, \widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_2^c) & \mathbf{f}_{22} &= (\widetilde{\gamma}_2 \otimes \widetilde{\gamma}_2, -\widetilde{\gamma}_2^c \otimes \widetilde{\gamma}_2^c).\end{aligned}$$

Then the map

$$F^2 H^2_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^2_B(M_{\mathbb{C}}, \mathbb{C})$$

is given by:

$$\begin{aligned}\omega \otimes \omega &\mapsto (a_1^2 - a_2^2) \cdot \mathbf{e}_{11} + 2a_1 a_2 i \cdot \mathbf{f}_{11} + 2(a_1 b_1 - a_2 b_2) \cdot \mathbf{e}_{12} \\ &\qquad + 2(a_1 b_2 + a_2 b_1) i \cdot \mathbf{f}_{12} + (b_1^2 - b_2^2)\mathbf{e}_{22} + 2 b_1 b_2 i \cdot \mathbf{f}_{22}. \\ \sqrt{-D} \cdot \omega \otimes \omega &\mapsto \sqrt{D} \cdot \left[(a_1^2 - a_2^2) i \cdot \mathbf{f}_{11} - 2a_1 a_2 \mathbf{e}_{11} + 2(a_1 b_1 - a_2 b_2) i \cdot \mathbf{f}_{12} \right. \\ &\qquad \left. -2(a_1 b_2 + a_2 b_1) \cdot \mathbf{e}_{12} + (b_1^2 - b_2^2) i \cdot \mathbf{f}_{22} - 2 b_1 b_2 \cdot \mathbf{e}_{22}\right].\end{aligned}$$

Thus the map

$$F^2 H^2_{\mathrm{dR}}(M_{\mathbb{R}}) \to H^2_B(M_{\mathbb{C}}, \mathbb{R}(1))$$

is given by:

$$\omega\otimes\omega \mapsto \mathbf{v}_1 := 2a_1a_2i\cdot\mathbf{f}_{11} + 2(a_1b_2+a_2b_1)i\cdot\mathbf{f}_{12} + 2b_1b_2i\cdot\mathbf{f}_{22},$$
$$\sqrt{-D}\cdot\omega\otimes\omega \mapsto \mathbf{v}_2 := \sqrt{D}\left[(a_1^2-a_2^2)i\cdot\mathbf{f}_{11} + 2(a_1b_1-a_2b_2)i\cdot\mathbf{f}_{12} + (b_1^2-b_2^2)i\cdot\mathbf{f}_{22}\right].$$

Clearly this lands in $H^2_B(M_{\mathbb{R}},\mathbb{R}(1)) = H^2_B(M_{\mathbb{C}},\mathbb{R}(1))^+$, an $\mathbb{R}$-basis for this space being $\{\mathbf{f}_{11}(1),\mathbf{f}_{12}(1),\mathbf{f}_{22}(1)\}$. In fact, this is a $\mathbb{Q}$-basis for $H^2_B(M_{\mathbb{R}},\mathbb{Q}(1))$.

We now pick $\mathbf{v}_3$ in $H^2_B(M_{\mathbb{C}},\mathbb{R}(1))^+$ such that

$$\mathbf{v}_1\wedge\mathbf{v}_2\wedge\mathbf{v}_3 = \mathbf{f}_{11}(1)\wedge\mathbf{f}_{12}(1)\wedge\mathbf{f}_{22}(1).$$

There are two obvious good choices of such a $\mathbf{v}_3$: we could take $\mathbf{v}_3 = \frac{1}{\alpha}\mathbf{f}_{11}$ or $\mathbf{v}_3 = \frac{1}{\beta}\mathbf{f}_{22}$, with $\alpha,\beta\in\mathbb{R}(1)$.

Let us work through the case $\mathbf{v}_3 = \frac{1}{\alpha}\mathbf{f}_{11}$ for instance. (The other case is similar.) Then

$$\begin{aligned}\alpha &= \frac{1}{2\pi i}\cdot\frac{\sqrt{D}}{(2\pi)^2}\cdot\big((a_1b_2+a_2b_1)(b_1^2-b_2^2) - b_1b_2\cdot 2(a_1b_1-a_2b_2)\big)\\ &= -\frac{1}{2\pi i}\cdot\frac{\sqrt{D}}{(2\pi)^2}\cdot(b_1^2+b_2^2)(a_1b_2-a_2b_1).\end{aligned}$$

The statement of the conjecture is then simply that $H^3_{\mathcal{M}}(M_{\mathbb{Z}},\mathbb{Q}(2))$ is rank one, generated by $\boldsymbol{\alpha}$ say, and if

$$\frac{1}{L'(M,1)}r_{\mathcal{D}}(\boldsymbol{\alpha})$$

is lifted to an element of $H^2_B(M_{\mathbb{R}},\mathbb{R}(1))$ and expanded in the basis $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$, then the coefficient of $\mathbf{v}_3$ lies in $\mathbb{Q}$.

As before, we make this explicit using Poincaré duality. We may assume that $\gamma_1,\gamma_2$ have been picked so that

$$\langle\gamma_1,\gamma_2\rangle_{\mathrm{PD}} = 1.$$

Then

$$\langle\widetilde{\gamma}_1,\widetilde{\gamma}_1\rangle_{\mathrm{PD}} = \langle\widetilde{\gamma}_2,\widetilde{\gamma}_2\rangle_{\mathrm{PD}} = 0, \quad \text{while} \qquad \langle\widetilde{\gamma}_1,\widetilde{\gamma}_2\rangle_{\mathrm{PD}} = \frac{1}{2\pi i}.$$

Let

$$\eta' = i(\omega^\sigma\otimes\overline{\omega^\sigma} + \overline{\omega^\sigma}\otimes\omega^\sigma)\in\mathrm{Sym}^2 H^1(E^\sigma(\mathbb{C})).$$

Since $\eta$ is totally imaginary and of type $(1,1)$, we must have

$$\langle P_1(\mathbf{v}_1),\eta'\rangle_{\mathrm{PD}} = \langle P_1(\mathbf{v}_2),\eta'\rangle_{\mathrm{PD}} = 0,$$

where $P_1$ is just the projection onto the first component. As a check, we verify this explicitly now.

First note that

$$\begin{aligned}\omega^\sigma\otimes\overline{\omega^\sigma} &= ((a_1+a_2i)\widetilde{\gamma}_1 + (b_1+b_2i)\widetilde{\gamma}_2)\otimes((a_1-a_2i)\widetilde{\gamma}_1 + (b_1-b_2i)\widetilde{\gamma}_2)\\ &= (a_1^2+a_2^2)\widetilde{\gamma}_1\otimes\widetilde{\gamma}_1 + (a_1+a_2i)(b_1-b_2i)\widetilde{\gamma}_1\otimes\widetilde{\gamma}_2 +\\ &\qquad (a_1-a_2i)(b_1+b_2i)\widetilde{\gamma}_2\otimes\widetilde{\gamma}_1 + (b_1^2+b_2^2)\widetilde{\gamma}_2\otimes\widetilde{\gamma}_2.\end{aligned}$$

Hence

$$\eta' = 2i[(a_1^2+a_2^2)\widetilde{\gamma}_1\otimes\widetilde{\gamma}_1 + (a_1b_1+a_2b_2)(\widetilde{\gamma}_1\otimes\widetilde{\gamma}_2 + \widetilde{\gamma}_2\otimes\widetilde{\gamma}_1) + (b_1^2+b_2^2)\widetilde{\gamma}_2\otimes\widetilde{\gamma}_2].$$

Thus (using $\langle\widetilde{\gamma}_1\otimes\widetilde{\gamma}_2,\widetilde{\gamma}_2\otimes\widetilde{\gamma}_1\rangle_{\mathrm{PD}} = -1/(2\pi i)^2$),

$$\langle P_1(\mathbf{v}_1),\eta'\rangle_{\mathrm{PD}} = \frac{-2}{(2\pi i)^2}\cdot\left[b_1b_2\cdot(a_1^2+a_2^2) - (a_1b_2+a_2b_1)\cdot(a_1b_1+a_2b_2) + a_1a_2\cdot(b_1^2+b_2^2)\right] = 0,$$

and

$$\langle P_1(\mathbf{v}_2),\eta'\rangle_{\mathrm{PD}} = \frac{-\sqrt{D}}{(2\pi i)^2}\cdot\left[(a_1^2-a_2^2)(b_1^2+b_2^2) + 2(a_1b_1-a_2b_2)(a_1b_1+a_2b_2) + (b_1^2-b_2^2)(a_1^2+a_2^2)\right] = 0,$$

as expected. On the other hand,

$$\langle P_1(\mathbf{v}_3), \eta' \rangle_{\mathrm{PD}} = \frac{i}{(2\pi i)^2} \cdot \frac{1}{\alpha}(b_1^2 + b_2^2) = \frac{-2\pi}{\sqrt{D}(a_1 b_2 - a_2 b_1)}.$$

By equation (7.15), we have that $-\sqrt{D}(a_1b_2 - a_2b_1) = c^+(H^1(E)(1))$, and hence Beilinson's conjecture is equivalent to:

$$\langle r_{\mathcal{D}}(\boldsymbol{\alpha}), \eta' \rangle_{\mathrm{PD}} \sim_{\mathbb{Q}^\times} L'(\mathrm{Sym}^2 H^1(E), 1) \frac{2\pi}{c^+(H^1(E)(1))},$$

i.e.

$$L'(\mathrm{Sym}^2 H^1(E), 1) = \frac{1}{2\pi} \cdot \langle r_{\mathcal{D}}(\boldsymbol{\alpha}), \eta' \rangle_{\mathrm{PD}} \cdot c^+(H^1(E)(1)).$$

This proves Proposition 7.13.

Explicitly, if $\boldsymbol{\alpha}$ is represented by $(C_i, f_i)$ on $E \times E$, Beilinson's conjecture is equivalent to

$$\sum_i \int_{C_{i,\mathbb{C}}} \log|f_i| \cdot \frac{1}{2} \left(p_1^*\omega \wedge p_2^*\overline{\omega} + p_1^*\overline{\omega} \wedge p_2^*\omega\right) \in \mathbb{Q} \cdot \frac{4\pi^2}{\sqrt{D}(a_1b_2 - a_2b_1)} \cdot L'(\mathrm{Sym}^2(E), 1). \tag{7.20}$$

Here we write $\omega$ instead of $\omega^\sigma$ etc.

7.3. **Completing the proof of Theorem 7.1.** We start by relating the natural generators of the two Deligne cohomology groups.

**Lemma 7.19.** *Under the natural isomorphism*

$$d\colon H^1_{\mathcal{D}}(M(f_0, \mathrm{Ad}), \mathbb{R}(1)) \to H^1_{\mathcal{D}}(M(f, \mathrm{Ad}), \mathbb{R}(1)),$$

*a natural generator* $\eta \in H^1_{\mathcal{D}}(M(f_0, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ *maps to*

$$d(\eta) = -2\sqrt{D}^{-1}\delta,$$

*where* $\delta \in H^1_{\mathcal{D}}(M(f, \mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))$ *is a natural generator (Definition 4.13).*

*Therefore, the dual generator* $\delta^\vee$ *from Definition 4.15 is identified with* $-2\pi^2\eta^\vee$.

*Proof.* Recall that under the identification $M(f_0) = M(f)$, a choice of basis $\omega_1, \omega_2$ of $F^1 H^1_{\mathrm{dR}}(M(f)_{\mathbb{Q}})$ is:

$$\begin{aligned} \omega_1 &= \omega^\sigma \\ \omega_2 &= \sqrt{-D}\omega^\sigma \end{aligned}$$

We then compute that:

$$\begin{aligned} \delta' &= (\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) - (\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2) \\ &= (\omega^\sigma \otimes \overline{\sqrt{-D}\omega^\sigma} + \overline{\sqrt{-D}\omega^\sigma} \otimes \omega^\sigma) - (\sqrt{-D}\omega^\sigma \otimes \overline{\omega^\sigma} + \overline{\omega^\sigma} \otimes \sqrt{-D}\omega^\sigma) \\ &= -\sqrt{-D}(\omega^\sigma \otimes \overline{\omega^\sigma} - \overline{\omega^\sigma} \otimes \omega^\sigma) + \sqrt{-D}(\omega^\sigma \otimes \overline{\omega^\sigma} + \overline{\omega^\sigma} \otimes \omega^\sigma) \\ &= -2\sqrt{D}\eta'. \end{aligned}$$

Therefore,

$$\begin{aligned} \delta &= (2\pi i)\delta' \\ &= -2\sqrt{D}(2\pi i)\eta' \\ &= -2\sqrt{D}\eta \end{aligned}$$

Finally, for dual generators, we have that:

$$\begin{aligned}
\delta^\vee &= \frac{\pi^4}{\sqrt{\Delta_{\mathrm{Ad}(f)}}}\langle \delta, - \rangle_{\mathrm{pol}} \\
&= -2\frac{\pi^4\sqrt{D}}{\sqrt{\Delta_{\mathrm{Ad}(f)}}}\langle \eta, - \rangle_{\mathrm{pol}} \\
&= -2\frac{\pi^4}{\sqrt{\Delta_{\mathrm{Ad}(f_0)}}}\langle \eta, - \rangle_{\mathrm{pol}} && \Delta_{\mathrm{Ad}(f)} = \Delta_{\mathrm{Ad}(f_0)} \cdot D \\
&= -2\pi^2\eta^\vee,
\end{aligned}$$

as claimed. □

We now return to the proof of Theorem 7.1. It suffices to prove that the diagram

$$\begin{tikzcd}
H^1(X_0, \mathbb{Q})_{f_0} \otimes \mathbb{C} \arrow[r, "\theta_1"] \arrow[d, "d^\vee(\delta^\vee)*"] & H^0(X, \mathcal{E}_{2,2})_f \otimes \mathbb{C} \arrow[d, "\delta^\vee *"] \\
H^2(X_0, \mathbb{Q})_{f_0} \otimes \mathbb{C} \arrow[r, "\theta_2"] & H^1(X, \mathcal{E}_{2,2})_f \otimes \mathbb{C}
\end{tikzcd}$$

commutes, up to $\mathbb{Q}^\times$. By Lemma 7.19 together with $\Delta(\mathrm{Ad}(f)) = \Delta(\mathrm{Ad}(f_0)) \cdot D$, we just need to check that:

$$\delta^\vee * \theta_1(\omega^1(f_0)/u^1(f_0)) \sim_{\mathbb{Q}^\times} \pi^2\theta_2(\eta^\vee * \omega^1(f_0)/u^1(f_0)).$$

We compute both sides:

$$\begin{aligned}
\delta * \theta_1(\omega^1(f_0)/u^1(f_0)) &= \delta * [f] && (7.12) \\
&= [f^W] && \text{Definition 4.18}
\end{aligned}$$

and

$$\begin{aligned}
\theta_2(\eta * \omega^1(f_0)/u^1(f_0)) &= \theta_2(\omega^2(f_0)) \\
&= \frac{u^2(f_0)}{d^W(f)} \cdot [f^W] && (7.13).
\end{aligned}$$

Therefore, Theorem 7.1 is equivalent to the following period identity.

**Theorem 7.20.** *We have that:*

$$d^W(f) \sim_{\mathbb{Q}^\times} \pi^2 u^2(f_0).$$

*Proof.* First, we describe $d^W(f)$ and $u^2(f_0)$ in terms of the more accessible periods $c^W(f)$ and $u^1(f_0)$ and Petersson inner products.

As in the proof of Theorem 4.21, we have that:

$$\begin{aligned}
L(f, \mathrm{Ad}, 1) &\sim_{\mathbb{Q}^\times} \pi^9 \cdot \Lambda(f, \mathrm{Ad}, 1) \\
&\sim_{\mathbb{Q}^\times} \pi^9\pi^{-8} \cdot \langle f^W, f^W \rangle && \text{Theorem 5.6} \\
&\sim_{\mathbb{Q}^\times} \pi^9\pi^{-8}\pi^3 \cdot c^W(f) \cdot d^W(f) && (2.22)
\end{aligned}$$

Therefore:

$$L(f, \mathrm{Ad}, 1) \sim_{\mathbb{Q}^\times} \pi^4 \cdot c^W(f_0) \cdot d^W(f) \tag{7.21}$$

$$L(f_0, \mathrm{Ad}, 1) \sim_{\mathbb{Q}^\times} \pi^2 \cdot u^1(f_0) \cdot u^2(f_0) \qquad \text{[Urb95, Prop. 7.1]} \tag{7.22}$$

by relating the adjoint $L$-value to the Petersson norm, and using Serre duality and Poincaré duality, respectively.

Next, we use the factorization:

$$L(f, \mathrm{Ad}, 1) = L(f_0, \mathrm{Ad}, 1)L(f_0, \mathrm{As}, 2)$$

and Ghate–Loeffler–Williams Theorem 7.17:

$$L(f_0, \mathrm{As}, 2) \sim_{\mathbb{Q}^\times} (2\pi i)^4 u^1(f_0)$$

to conclude that

$$L(f, \mathrm{Ad}, 1) \sim_{\mathbb{Q}^\times} \pi^4 \cdot L(f_0, \mathrm{Ad}, 1) \cdot u^1(f_0).$$

Together with equations (7.21), (7.22), this shows that:

$$\pi^4 \cdot c^W(f) \cdot d^W(f) \sim \pi^6 \cdot u^1(f_0)^2 \cdot u^2(f_0).$$

Finally, we have that:

$$\begin{aligned} c^W(f) &\sim_{\mathbb{Q}^\times} \Lambda(f, \psi_+, 1)\Lambda(f, \psi_-, 1) && \text{Theorem 5.1} \\ &\sim_{\mathbb{Q}^\times} \pi^{-4} L(f, \psi_+, 1)L(f, \psi_-, 1) \\ &\sim_{\mathbb{Q}^\times} \pi^{-4} L(f_0, \psi_+, 1)L(f_0, \psi_-, 1) \\ &\sim_{\mathbb{Q}^\times} u^1(f_0)^2 && \text{Cremona–Whitley–Hida Theorem 7.15} \end{aligned}$$

This shows that $d^W(f_0) \sim_{\mathbb{Q}^\times} \pi^2 u^2(f_0)$, as claimed. □

## 8. Hilbert–Siegel modular forms

There are many potential generalizations of the above results to other Shimura varieties. Suppose $X$ is a Shimura variety for an algebraic group $G$ and $\pi_H$ is a cuspidal automorphic representation of $G$ satisfying:

(1) $\pi_{H,\infty}$ is a non-degenerate holomorphic limit of discrete series,
(2) the archimedean $L$-packet $\Pi_\infty$ of $\pi_\infty$ contains a generic representation $\pi_{\infty,G}$,
(3) the global $L$-packet $\Pi$ contains a globally generic representation $\pi_G = \pi_f \otimes \pi_{\infty,G}$, contributing to the highest degree of coherent cohomology.

Let $r = \mathrm{rank}(H^1_{\mathcal{D}}(\mathrm{Ad}\, M(\Pi)_{\mathbb{R}}, \mathbb{R}(1)))$. We expect that there is an action of an appropriate element of $\bigwedge^r H^1_{\mathcal{D}}(\mathrm{Ad}\, M(\Pi)_{\mathbb{R}}, \mathbb{R}(1))^\vee$ given by:

$$H^0(X_{\mathbb{C}}, \mathcal{E})_{\pi_f} \to H^r(X_{\mathbb{C}}, \mathcal{E})_{\pi_f}, \tag{8.1}$$

$$[f] \to [f^W], \tag{8.2}$$

where $f$ is normalized to be rational in $H^0$ and $f^W \in \pi_G$ is normalized using the Whittaker model. The conjecture is then that the resulting action of $\bigwedge^r H^1_{\mathcal{M}}(\mathrm{Ad}\, M(\Pi), \mathbb{Q}(1)))^\vee$ via the Beilinson regulator is rational. The strategy for proving Theorem 4.21, replacing the results of Chen–Ichino [CI19] with the more general Lapid–Mao Conjecture [LM15] is likely enough to prove that the above conjecture is equivalent to Beilinson's conjecture.

The key drawback of this approach is the the intermediate contributions to $H^*(X_{\mathbb{C}}, \pi_F)$ for $0 < * < r$ will likely neither be holomorphic nor generic, so we do not know how to define the action of the full Deligne cohomology group $H^1_{\mathcal{D}}(\mathrm{Ad}\, M(\Pi)_{\mathbb{R}}, \mathbb{R}(1))^\vee$ as in [PV21, Hor23].

We illustrate this general strategy the example of Hilbert–Siegel modular forms. It would be interesting to work out the details for higher-dimensional Siegel modular varieties and unitary Shimura varieties; see related work of Oh [Oh21] (Picard modular surfaces) and Atanasov's thesis [Ata22].

Let $F/\mathbb{Q}$ be a totally real field of degree $d$ and $\sigma_1, \ldots, \sigma_d \colon F \hookrightarrow \mathbb{R}$ be the real embeddings of $F$. We consider the restriction of scalars $G = R_{F/\mathbb{Q}}\, \mathrm{GSp}_{4,F}$.

Let $f$ be a Hilbert–Siegel modular form for $F$ of weight $\underline{k}$ where $\underline{k}_j = (k_{j,1}, k_{j,2})$ is the Blattner parameter of the corresponding (limit of) discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$. The $L$-packet $\Pi_\infty$ at $\infty$ associated to weight $\lambda$ is then:

$$\Pi_\infty = \{\pi_I \mid I \subseteq \{1, \dots, d\}\}, \tag{8.3}$$

$$(\pi_J)_i = \begin{cases} X^1_\lambda, & i \notin I, \\ X^2_\lambda, & i \in I \end{cases} \tag{8.4}$$

(Lemma A.1).

Once again, let us assume that $f$ is not endoscopic and has paramodular level $\mathfrak{N}$. Then the analogue of Lemma 2.9 describes the $L$-packet $\Pi$ associated with $f$:

$$\Pi = \{\pi_f \otimes \pi_I \mid I \subseteq \{1, \dots, d\}, \pi_f \in \Pi_f\},$$

each occurring with multiplicity one in the automorphic spectrum.

Associated to a weight $(\underline{k}, m)$ is a sheaf $\mathcal{E}_{\underline{k},m}$ over the $3d$-dimensional Hilbert–Siegel modular variety $X$. To each partition $J = (J_0, J_1, J_2, J_3)$:

$$\{1, \dots, d\} = J_0 \cup J_1 \cup J_2 \cup J_3,$$

we associate a weight $\underline{k}(\underline{J})$ given by

$$\underline{k}(J)_j = \begin{cases} (k_{j,1}, k_{j,2}) & j \in J_0, \\ (k_{j,1}, 4 - k_{j,2}) & j \in J_1, \\ (k_{j,2} - 1, 3 - k_{j,1}) & j \in J_2, \\ (3 - k_{j,2}, 3 - k_{j,1}) & j \in J_3. \end{cases}$$

We write $\mathcal{E}_{\underline{k}(J)}$ for the associated sheaf.

The analogue of Theorem 2.12 is then the following. We make it slightly less explicit in order to avoid overcomplicating the notation.

**Theorem 8.1** (Harris, Su)**.** *For each partition $J = (J_0, \dots, J_3)$, there is a unique contribution of $\Pi$ to cohomology of the automorphic sheaf $\mathcal{E}_{\underline{k}(J)}$ in degree $j = |J_1| + 2|J_2| + 3|J_3|$.*

**Corollary 8.2.** *Let $f$ be a Hilbert–Siegel modular form of weight $\underline{k}_j := (k_j, 2)$ for $k_j \geq 2$. Then*

$$\dim H^i(X, \mathcal{E}_{(k_j,2)_j})_\Pi = \binom{d}{i}.$$

Under the paramodularity assumption, the representation:

$$\pi = \pi_f \otimes \pi_{\{1,\dots,d\}} \in \Pi$$

is globally generic, and we may define a Whittaker-normalized vector

$$f^W = \bigotimes f_v^W \in \pi_f \otimes \pi_{\{1,\dots,d\}} \tag{8.5}$$

analogously to Definition 4.7. In particular, $f_v^W$ is Whittaker-normalized at both the finite and the infinite places. Similarly, we consider

$$f = \bigotimes f_v \in \pi_f \otimes \pi_\emptyset \tag{8.6}$$

which is normalized to give a rational contribution to cohomology

$$[f] \in H^0(X, \mathcal{E}_{\underline{k}})_\Pi. \tag{8.7}$$

We assume that there is a motive $M(f)$ over $F$ associated with $f$, and we consider the restriction of scalars motive $R_{F/\mathbb{Q}}M(f)$. We are interested in

$$R_{F/\mathbb{Q}}M(f, \mathrm{Sym}^2)(k-1) \cong R_{F/\mathbb{Q}}M(f, \mathrm{Ad}).$$

Since the motives are considering are restrictions of scalars, the Beilinson short exact sequence (3.1) in this case is a direct sum of short exact sequences:

(8.8)
$$0 \to F^{2k-2}H_{\mathrm{dR}}(M(f,\mathrm{Sym}^2)_{\mathbb{R},\sigma_j} \xrightarrow{\tilde{\pi}_1} H_B^{2k-2}(M(f,\mathrm{Sym}^2)_{\mathbb{R},\sigma_j}, \mathbb{R}(1)) \to H_{\mathcal{D}}^{2k-1}(M(f,\mathrm{Sym}^2)_{\mathbb{R},\sigma_j}, \mathbb{R}(k)) \to 0$$

for each $j = 1, \ldots, d$. Moreover, each $H_{\mathcal{D}}^{2k-1}(M(f,\mathrm{Sym}^2)_{\mathbb{R},\sigma_j}, \mathbb{R}(k))$ is 1-dimensional.

**Definition 8.3.** We define a *natural generator* of $\delta_j \in H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R},\sigma_j}, \mathbb{R}(1))$ analogously to Definition 4.13. This defines a *natural generator*:

$$\delta = (\delta_1, \ldots, \delta_d) \in H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)) = \bigoplus_j H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R},\sigma_j}, \mathbb{R}(1))$$

and its *determinant*:

$$\wedge\delta = \delta_1 \wedge \cdots \wedge \delta_d \in \bigwedge^d H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1)).$$

Moreover, define a *dual natural generator* $\wedge\delta^\vee$ as in Definition 4.15.

This allows us to define a motivic action of the coherent cohomology of Siegel modular varieties.

**Definition 8.4.** We define the action of $\wedge\delta^\vee \in \bigwedge^d H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R}}, \mathbb{R}(1))^\vee$ by:

$$H^0(X_{\mathbb{Q}}, \mathcal{E}_{(k_j,2)_j})_\Pi \to H^d(X_{\mathbb{C}}, \mathcal{E}_{(k_j,2)_j})_\Pi$$
$$[f] \mapsto [f^W].$$

Assuming that the Beilinson's regulator map (3.6):

$$r_{\mathcal{D}} \colon H^1_{\mathcal{M}}(M(f,\mathrm{Ad}), \mathbb{Q}(1)) \otimes \mathbb{R} \to \bigoplus_{j=1}^d H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R},\sigma_j}, \mathbb{R}(1)) \tag{8.9}$$

is an isomorphism, this defines an action of $\bigwedge^d H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{\underline{2}})_{\pi_f}$.

**Remark 8.5.** It would be natural to define the action of $\delta_j^\vee \in H^1_{\mathcal{D}}(M(f,\mathrm{Ad})_{\mathbb{R},\sigma_j}, \mathbb{R}(1))^\vee$ by:

$$H^0(X_{\mathbb{Q}}, \mathcal{E}_{(k_j,2)_j})_{\pi_f} \to H^1(X_{\mathbb{C}}, \mathcal{E}_{(k_j,2)_j})_{\pi_f}$$
$$[f] \to [f^{\sigma_j}]$$

for some element $f^{\sigma_j} \in \pi_j = \pi_f \otimes \pi_{\{j\}}$, i.e. the representation of $\mathrm{GSp}_4(\mathbb{R})^d$ which is generic exactly at the $j$th place. However, there seems to be no natural normalization of $f^{\sigma_j}$; indeed, the representation $\pi_j$ is not generic so we cannot use a Whittaker model and $\dim H^1(X_{\mathbb{C}}, \mathcal{E}_{\underline{2}})_{\pi_f} = d$, so we cannot normalize it cohomologically.

This leads to the analogue of Conjecture 4.19 for totally real fields.

**Conjecture 8.6.** *The action of $\bigwedge^d H^1_{\mathcal{M}}(M(f,\mathrm{Ad})_{\mathbb{Z}}, \mathbb{Q}(1))^\vee$ on $H^*(X_{\mathbb{C}}, \mathcal{E}_{\underline{2}})_{\pi_f}$ from Definition 8.4 descends to rational structures on coherent cohomology, i.e. it defines an action on $H^*(X, \mathcal{E}_{\underline{2}})_{\pi_f} \subseteq H^*(X_{\mathbb{C}}, \mathcal{E}_{\underline{2}})_{\pi_f}$.*

**Theorem 8.7.** *Assuming Beilinson's conjecture for $M(f,\mathrm{Ad})$, Deligne's conjecture for appropriate twists of $M(f)$, and the analogue of Hypothesis 5.2, Conjecture 8.6 is true.*

*Proof.* This follows from the same argument as Theorem 4.21, so we only point out how each of the ingredients generalizes to this case.

- The analogue of Theorem 5.3 holds for Hilbert–Siegel modular forms.
- The results of Chen–Ichino [CI19] apply to Hilbert–Siegel modular forms.
- The explicit form of Beilinson's conjecture given in Theorem 5.7 has an analogous statement for Hilbert–Siegel modular forms.

We omit the details. □

## Appendix A. Representation theory of $\mathrm{GSp}_4(\mathbb{R})$

For completeness, we include a summary of results from the representation theory of $\mathrm{GSp}_4(\mathbb{R})$ in this appendix. We follow Schmidt [Sch17] and use the notation therein. We also include some results of [Mui09].

A.1. **(Limits of) discrete series for** $\mathrm{Sp}_4(\mathbb{R})$ **and** $\mathrm{GSp}_4(\mathbb{R})$**.** Let $K^\circ \cong U(2)$ be the maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$ and $\mathfrak{g}^\circ$ be its Lie algebra. A *representation of* $\mathrm{Sp}_4(\mathbb{R})$ is a $(\mathfrak{g}^\circ, K^\circ)$-module.

The *roots* are elements of $(\mathfrak{h}_C)' = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}, \mathbb{C})$ where $\mathfrak{h}_{\mathbb{C}}$ is the Cartan subgroup of the complexified Lie algebra $\mathfrak{g}^\circ_{\mathbb{C}}$ of $\mathrm{Sp}_4(\mathbb{R})$. The root space has a natural real subspace $E \cong \mathbb{R}^2$ and the *analytically integral elements* are $\mathbb{Z}^2 \subseteq E$. Explicitly, we have a maximal torus $S \subseteq U(2) \subseteq \mathrm{Sp}_{4,\mathbb{R}}$, explicitly given by:

$$S(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta_1 & & \sin\theta_1 & \\ & \cos\theta_2 & & \sin\theta_2 \\ -\sin\theta_1 & & \cos\theta_1 & \\ & -\sin\theta_2 & & \cos\theta_2 \end{pmatrix} \right\} \tag{A.1}$$

and $(n_1, n_2) \in \mathbb{Z}^2$ corresponds to the map sending the above matrix to $e^{i\theta_1 n_1} e^{i\theta_2 n_2}$.

The *walls* are spanned by the roots and the *Weyl group* $W$ is generated by the reflections over the walls. The *compact Weyl group* $W_K$ is generated by the reflection about the orange line labeled $\ell$ in Figure A.1.

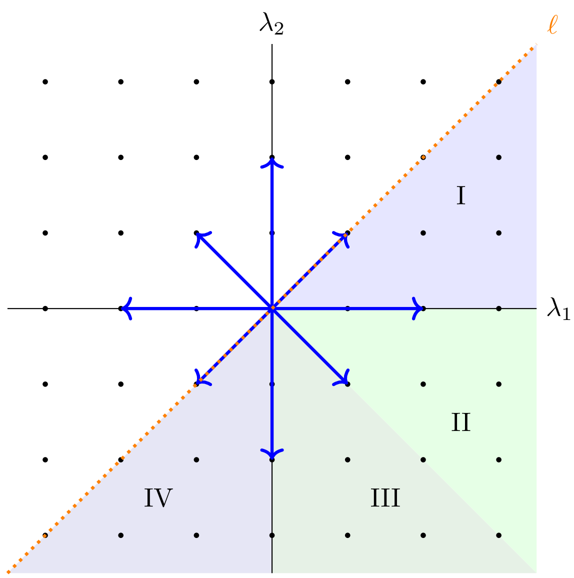


Figure A.1. A graph of $E \cong \mathbb{R}^2 \subseteq \mathbb{Z}^2$ and the root system for $\mathrm{Sp}_4(\mathbb{R})$.

Each $\lambda = (\lambda_1, \lambda_2) \in E$ with $\lambda_1 \geq \lambda_2$ corresponds to a $K^\circ$-type $V_\lambda$ with highest weight $\lambda$. For each $\lambda$ inside the regions I, II, III, IV indicated in Figure A.1, there is a discrete series representation of $\mathrm{Sp}_4(\mathbb{R})$ associated to it:

- (I) $X^1_\lambda$ *holomorphic discrete series* for $\lambda$ in region I
- (II) $X^2_\lambda$ *generic discrete series* for $\lambda$ in region II,
- (III) $X^3_\lambda$ *generic discrete series* for $\lambda$ in region III,
- (IV) $X^4_\lambda$ *antiholomorphic discrete series* for $\lambda$ in region IV.

The element $\lambda$ is called the *Harish–Chandra parameter* of $X_\lambda$. In each case, the associated *Blattner parameter* $\Lambda = \lambda + \delta$ where $\delta$ is described in Table A.1 is the highest weight of the minimal $K^\circ$-type occuring in $X_\lambda$.

The limits of discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ lie on the border of the regions:

| Region | $\delta$ | $\Lambda$ |
|---|---|---|
| I | $(1,2)$ | $(\lambda_1+1,\lambda_2+2)$ |
| II | $(1,0)$ | $(\lambda_1+1,\lambda_2)$ |
| III | $(0,-1)$ | $(\lambda_1,\lambda_2-1)$ |
| IV | $(-2,-1)$ | $(\lambda_1-2,\lambda_2-1)$ |

TABLE A.1. The invariant $\delta=\delta_\lambda^{\mathrm{nc}}-\delta_\lambda^{\mathrm{c}}$ for $\lambda$ in various regions of the root space.

(I, II) For $(\lambda_1,\lambda_2)=(p,0)$, we have
- $X_\lambda^1$: *holomorphic limit of discrete series*,
- $X_\lambda^2$: *generic limit of discrete series*.

(II, III) For $(\lambda_1,\lambda_2)=(p,-p)$, we have
- $X_\lambda^2$, $X_\lambda^3$: two *generic limits of discrete series*

(III, IV) For $(\lambda_1,\lambda_2)=(0,-p)$, we have
- $X_\lambda^3$: *generic limit of discrete series*.
- $X_\lambda^4$: *antiholomorphic limit of discrete series*,

We finally consider representations of $\mathrm{GSp}_4(\mathbb{R})$. First, write

$$\mathrm{Sp}_4(\mathbb{R})^\pm=\{g\in\mathrm{GSp}_4(\mathbb{R})\mid \nu(g)=\pm1\}=\mathrm{Sp}_4(\mathbb{R})\ltimes\mathrm{diag}(1,1,-1,-1),$$
$$K^\pm=K\ltimes\mathrm{diag}(1,1,-1,-1).$$

A *representation of* $\mathrm{Sp}_4(\mathbb{R})^\pm$ is a $(\mathfrak{g}^\circ,K^\pm)$-module. Then:

$$\mathrm{GSp}_4(\mathbb{R})=\mathbb{R}_{>0}\times\mathrm{Sp}_4(\mathbb{R})^\pm \text{ and } \mathfrak{g}=\mathrm{Lie}(\mathrm{GSp}_4(\mathbb{R}))=\mathbb{R}\oplus\mathfrak{g}^\circ.$$

and a *representation of* $\mathrm{GSp}_4(\mathbb{R})$ is a $(\mathfrak{g},K^\pm)$-module, i.e. a representation of $(\mathfrak{g}^\circ,K^\pm)$ together with an integer $m\in\mathbb{Z}$ such that $m\equiv\lambda_1+\lambda_2+1$ (2) that accounts for the central character.

Conjugation by the element

$$w_\infty=\mathrm{diag}(1,1,-1,-1)\in\mathrm{Sp}_4(\mathbb{R})^\pm$$

corresponds to the reflection $\lambda=(\lambda_1,\lambda_2)\mapsto\lambda'=(-\lambda_2,-\lambda_1)$. Therefore, for $\mathrm{GSp}_4(\mathbb{R})$:

- $X_\lambda^1$ and $X_{\lambda'}^4$ combine into a single representation, denoted $X_\lambda^1$; we call it a *holomorphic discrete series*,
- $X_\lambda^2$ and $X_{\lambda'}^3$ combine into a single representation, denoted $X_\lambda^2$; we call it a *generic discrete series*,
- there is an extra parameter $m\in\mathbb{Z}$ such that $m\equiv\lambda_1+\lambda_2+1$ (2) corresponding the central character of the representation.

We write $X_{\lambda;m}^i$ when we want to indicate the integer $m$ corresponding to the central character.

Consequently, the analogous statement is true for limits of discrete series:

- for $\lambda=(p,0)$, there is a *holomorphic limit of discrete series* $X_\lambda^1$ and a *generic discrete series* $X_\lambda^2$,
- for $\lambda=(p,-p)$, there is one *generic discrete series* $X_\lambda^\times$.

A.2. **Archimedean $L$-packets.** We identify the dual group of $\mathrm{GSp}_4$ with $\mathrm{GSp}_4(\mathbb{R})$ as in [Sch17, Section 3]. Langlands parameters are hence continuous homomorphisms

$$W_\mathbb{R}\to\mathrm{GSp}_4(\mathbb{C}).$$

For $\lambda = (\lambda_1, \lambda_2)$ inside region I, let $\overline{\lambda} = (\lambda_1, -\lambda_2)$ which is in region II, we have discrete series representation $X^1_\lambda$ (holomorphic) and $X^2_{\overline{\lambda}}$ (generic). Their common $L$-parameter $\varphi$ is:

$$re^{i\theta} \mapsto \mathrm{diag}(e^{i(\lambda_1+\lambda_2)\theta}, e^{i(\lambda_1-\lambda_2)\theta}, e^{-i(\lambda_1+\lambda_2)\theta}, e^{-i(\lambda_1-\lambda_2)\theta})$$
$$j \mapsto \begin{pmatrix} & & (-1)^\epsilon & \\ & & & (-1)^\epsilon \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

where $\epsilon = \lambda_1 + \lambda_2$.

The component group of the $L$-parameter $\varphi$ has two elements, represented by the identity and $\mathrm{diag}(1, -1, 1, -1)$. This agrees with the size of the $L$-packet above.

The limits of discrete series $X^1_\lambda$ (holomorphic) and $X^2_{\overline{\lambda}}$ (generic) for $\lambda = \overline{\lambda} = (p, 0)$ with $p > 0$ have the common $L$-parameter:

$$re^{i\theta} \mapsto \mathrm{diag}(e^{ip\theta}, e^{ip\theta}, e^{-ip\theta}, e^{-ip\theta})$$
$$j \mapsto \begin{pmatrix} & & (-1)^p & \\ & & & (-1)^p \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

The component group of the $L$-parameter has two elements, corresponding to the fact that we have a two-element $L$-packet $\{X^1_\lambda, X^2_{\overline{\lambda}}\}$.

Finally, for $\lambda = (p, -p)$ with $p > 0$, we have one limit of discrete series representation. The component group is trivial in this case and $X^\times_\lambda$ is the only element of the $L$-packet.

We summarize the above results in the following lemma.

**Lemma A.1.** *Given $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ with $\lambda_1 \geq \lambda_2 \geq 0$ (in region I), let*

$$\begin{aligned} \overline{\lambda} &= (\lambda_1, -\lambda_2) && \text{in region II}, \\ \overline{\lambda}' &= (\lambda_2, -\lambda_1) && \text{in region III}, \\ \lambda' &= (-\lambda_2, -\lambda_1) && \text{in region IV}. \end{aligned}$$

*The $L$-packets for $\mathrm{GSp}_4(\mathbb{R})$ associated to $\lambda$ contains the holomorphic discrete series $X^1_\lambda$ and the generic discrete series $X^2_{\overline{\lambda}}$. Moreover,*

$$\begin{aligned} X^1_\lambda|_{\mathrm{Sp}_4(\mathbb{R})} &= X^1_\lambda \oplus X^4_{\lambda'} \\ X^2_{\overline{\lambda}}|_{\mathrm{Sp}_4(\mathbb{R})} &= X^2_{\overline{\lambda}} \oplus X^3_{\overline{\lambda}'}. \end{aligned}$$

See Figure 2.1 for the location of these Harish–Chandra parameters and the associated Blattner parameters in the root space.

A.3. **Construction of the archimedean $L$-packets.** We discuss the construction of (limits of) discrete series using parabolic induction, following [Mui09]. The group $\mathrm{GSp}_4(\mathbb{R})$ has three conjugacy classes of parabolic subgroups: *Borel parabolic $B$*, *Siegel parabolic $P$*, *Klingen parabolic $Q$*:

$$B = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}, \qquad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \qquad Q = \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix} \tag{A.2}$$

with Levi subgroups:

$$M_B = \begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} \qquad M_P = \begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix} \qquad M_Q = \begin{bmatrix} * & & * & \\ & * & & \\ * & & * & \\ & & & * \end{bmatrix} \tag{A.3}$$

$$\cong \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1, \qquad \cong \mathrm{GL}_2 \times \mathrm{GL}_1, \qquad \cong \mathrm{GL}_1 \times \mathrm{GL}_2\,. \tag{A.4}$$

Given a representation of each Levi subgroup, inflated to the parabolics, we obtain a representation of $\mathrm{GSp}_4(\mathbb{R})$ by normalized parabolic induction.

- Given characters $\chi_1, \chi_2$, and $\sigma$ of $\mathbb{R}^\times$, we denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $\mathrm{GSp}_4$ obtained by normalized parabolic induction from the character
$$\begin{bmatrix} b & & * & * \\ * & a & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{bmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c).$$
- Given an admissible representation $\pi$ of $\mathrm{GL}_2(\mathbb{R})$ and a character $\sigma$ of $\mathbb{R}^\times$, we denote by $\pi \rtimes \sigma$ the representation of $\mathrm{GSp}_4(\mathbb{R})$ obtained by normalized parabolic induction from the representation:
$$\begin{bmatrix} A & * \\ & c\,{}^tA^{-1} \end{bmatrix} \mapsto \sigma(c)\pi(A)$$
of the Siegel parabolic $P$.
- Given an admissible representation $\pi$ of $\mathrm{GL}_2(\mathbb{R})$ and a character $\chi$ of $\mathbb{R}^\times$, we denote by $\chi \rtimes \pi$ the representation of $\mathrm{GSp}_4(\mathbb{R})$ obtained by normalized parabolic induction from the representation:
$$\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \mapsto \chi(t)\pi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$
of the Klingen parabolic $Q$.

Muić describes the $K$-types occurring in each of the parabolic inductions [Mui09, Lemma 6.1] and uses them to describe the composition series [Mui09, Sections 9, 10, 11] in terms of the Langlands classification of irreducible representations (see, for example [Kna01]). We recall the necessary facts below.

For $k \geq 1$, let $D_k^+$ (resp. $D_k^-$) be the holomorphic (resp. antiholomorphic) discrete series for $\mathrm{SL}_2(\mathbb{R})$ of lowest (resp. highest) $K$-type $(k+1)$ (resp. $-(k+1)$). When $k = 1$, we still use the notation $D_1^+$ and $D_1^-$ for the holomorphic and antiholomorphic limits of discrete series.

Moreover, for any $\ell$, we write $\sigma_\ell$ for the character $\sigma_\ell(c) = |c|^\ell \operatorname{sgn}(c)^\ell$ of $\mathbb{R}^\times$.

For $\lambda = (\lambda_1, \lambda_2)$ for integers $\lambda_1 \geq \lambda_2 \geq 0$, we write $\lambda'$, $\overline{\lambda}$, $\overline{\lambda}'$ as in Lemma A.1.

**Theorem A.2** (Muić). *Let $\lambda = (\lambda_1, \lambda_2)$ for integers $\lambda_1 \geq \lambda_2 \geq 0$. Parabolic induction from the Klingen parabolic $Q \cap \mathrm{Sp}_4(\mathbb{R})$ to $\mathrm{Sp}_4(\mathbb{R})$ have the following composition series:*

$$0 \longrightarrow X_\lambda^1 \oplus X_{\overline{\lambda}}^2 \longrightarrow \sigma_{\lambda_2} \rtimes D_{\lambda_1}^+ \longrightarrow \mathrm{Lang}(\sigma_{\lambda_2} \rtimes D_{\lambda_1}^+) \longrightarrow 0,$$

$$0 \longrightarrow X_{\overline{\lambda}'}^3 \oplus X_{\lambda'}^4 \longrightarrow \sigma_{\lambda_2} \rtimes D_{\lambda_1}^- \longrightarrow \mathrm{Lang}(\sigma_{\lambda_2} \rtimes D_{\lambda_1}^-) \longrightarrow 0,$$

*where $\mathrm{Lang}(-)$ are Langlands quotient representations.*

The above results rely on the description of $K^\circ$-types occuring in the parabolic induction [Mui09, Lemma 6.1]. We indicate the $K^\circ$-types and these short exact sequences in Figure A.2.

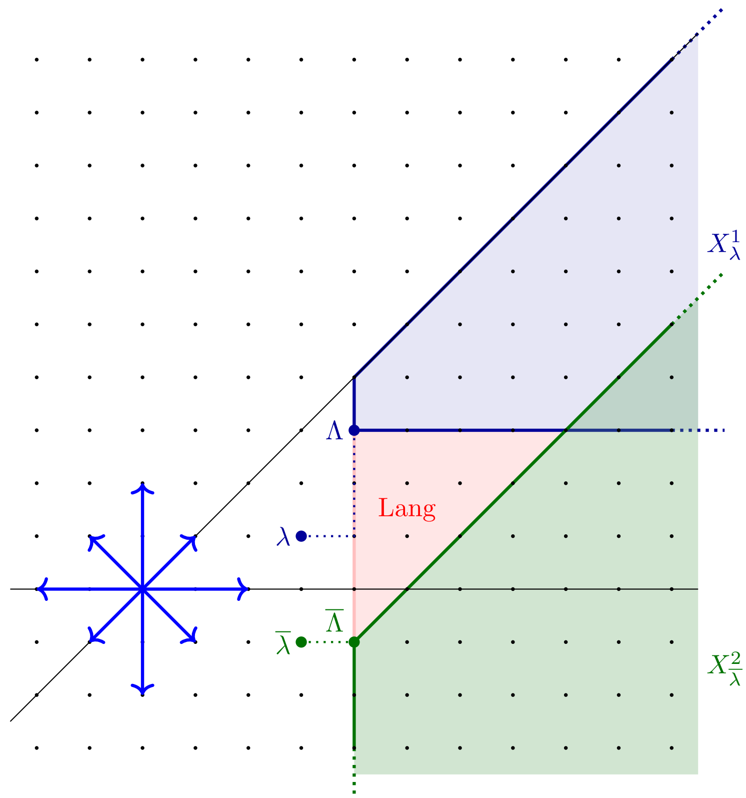


FIGURE A.2. This shaded region shows the $K$-types occuring in the parabolic induction $\sigma_{\lambda_2} \rtimes D^+_{\lambda_1}$ from the Klingen parabolic $Q$, according to [Mui09, Lemma 6.1]. The central character determines the parity of the occuring $K$-types and we do not indicate this here. We also do not indicate the multiplicities. The subrepresentations $X^1_\lambda$ and $X^2_{\overline{\lambda}}$ are shown in blue and green, respectively, and the Langlands quotient is shown in pink.

We now want to write down the analogous statement for representations of $\mathrm{GSp}_4(\mathbb{R})$. Recall that for any character $\eta = |\cdot|^s \operatorname{sgn}^\epsilon$ for $s \in \mathbb{C}$, $\epsilon \in \{0,1\}$ and any $k \in \mathbb{Z} \setminus \{0\}$, we have a (limit of) discrete series representation $\delta(\eta, k)$ fitting in the short exact sequences:

$$0 \longrightarrow \delta(\eta,k) \longrightarrow \eta|\cdot|^{k/2}\operatorname{sgn}^{k+1} \times \eta|\cdot|^{-k/2} \longrightarrow \zeta(\eta,k) \longrightarrow 0$$

$$0 \longrightarrow \zeta(\eta,k) \longrightarrow \eta|\cdot|^{-k/2} \times \eta|\cdot|^{k/2}\operatorname{sgn}^{k+1} \longrightarrow \delta(\eta,k) \longrightarrow 0$$

where $\chi_1 \times \chi_2$ denotes parabolic induction from the standard Borel of $\mathrm{GL}_2(\mathbb{R})$ and $\zeta(\eta,k)$ is finite-dimensional (of dimension $k$ when $k > 0$). Moreover,

$$\delta(\eta,k)|_{\mathrm{SL}_2(\mathbb{R})} \cong D^+_k \oplus D^-_k.$$

**Corollary A.3.** *Let $\lambda = (\lambda_1, \lambda_2)$ for integers $\lambda_1 \geq \lambda_2 \geq 0$. Parabolic induction from the Klingen parabolic $Q$ of $\mathrm{GSp}_4(\mathbb{R})$ has the following composition series*

$$0 \longrightarrow X^1_\lambda \oplus X^2_{\overline{\lambda}} \longrightarrow \sigma_{\lambda_2} \rtimes \delta(\eta,\lambda_1) \longrightarrow \mathrm{Lang}(\sigma_{\lambda_2} \rtimes \delta(\eta,\lambda_1)) \longrightarrow 0,$$

*where $\eta$ determines the central characters of $X^1_\lambda$ and $X^2_{\overline{\lambda}}$ and $\mathrm{Lang}(\sigma_{\lambda_2} \rtimes \delta(\eta,\lambda_1))$ is a Langlands quotient representation.*

The picture is analogous to Figure A.2 but also contains the $K$-types obtained by reflection about the $\lambda_2 = -\lambda_1$ diagonal.

Finally, we describe the representations occuring in the parabolic induction from the Siegel parabolic for completeness.

**Theorem A.4** (Muić). *Let $\lambda = (\lambda_1, \lambda_2)$ for integers $\lambda_1 \geq \lambda_2 \geq 0$. Parabolic induction from the Siegel parabolic $P \cap \mathrm{Sp}_4(\mathbb{R})$ of $\mathrm{Sp}_4(\mathbb{R})$ has the composition series:*

$$0 \longrightarrow X^2_\lambda \oplus X^3_{\lambda'} \longrightarrow \delta(|\cdot|^{(\lambda_1-\lambda_2)/2}\,\mathrm{sgn}^{\lambda_2}, \lambda_1+\lambda_2) \rtimes \mathbb{1} \longrightarrow \mathrm{Lang} \longrightarrow 0$$

*where* Lang *is a Langlands quotient.*

This leads to an analogous statement for $\mathrm{GSp}_4(\mathbb{R})$ which we omit here.

A.4. **Weyl group action.** Let $G = \mathrm{Sp}_{4,\mathbb{R}}$ in this subsection; in the main body of the article, this group is denoted $G^\circ$. Throughout this appendix, we have considered a non-split maximal torus $S \subseteq K^\circ = U(2) \subseteq G$ given by equation (A.1). A natural candidate for the map

$$H^0(X_{\mathbb{C}}, \mathcal{E}_0) \to H^1(X_{\mathbb{C}}, \mathcal{E}_1)$$

is the Weyl operator $w_0 \in W_G(S)(\mathbb{R})$ which acts on $K^\circ$-types by $(n_1, n_2) \mapsto (n_1, -n_2)$. In this section, we explain that this Weyl element does not lift to an element in $N_G(S)(\mathbb{R})$ , and hence this naïve construction does not produce the desired operator. We thank Tasho Kaletha for explaining this to us.

Let $T = \{\mathrm{diag}(t_1, t_2, t_1^{-1}, t_2^{-1}) \mid t_1, t_2 \in \mathbb{R}\}$ be the split maximal torus of $\mathrm{Sp}_{4,\mathbb{R}}$. We then have two short exact sequences:

$$1 \to T \to N_G(T) \to W_G(T) \to 1 \tag{A.5}$$

$$1 \to S \to N_G(S) \to W_G(S) \to 1 \tag{A.6}$$

and both the Weyl groups $W_G(T)$ and $W_G(S)$ are isomorphic to the dihedral group of order 8 as algebraic groups over $\mathbb{R}$.

The key observation is that the short exact sequence (A.5) remains exact on $\mathbb{R}$-points, i.e. every element of $W_G(T)(\mathbb{R})$ has a representative in $N_G(T)(\mathbb{R})$. Explicitly, the Weyl element $w_0 \in W_G(T)(\mathbb{R})$ which sends $(n_1, n_2) \mapsto (n_1, -n_2)$ has the following representative:

$$g_0 = \begin{pmatrix} 1 & & & \\ & & & -1 \\ & & 1 & \\ & 1 & & \end{pmatrix} \in N_G(T)(\mathbb{R}).$$

However, the short exact sequence (A.5) fails to be exact on $\mathbb{R}$-points. In fact, the elements of $W_G(S)(\mathbb{R})$ that lift to $N_G(S)(\mathbb{R})$ are exactly the elements that can be realized in $N_{K^\circ}(S)(\mathbb{R})$. In other words:

$$\mathrm{Im}(N_G(S)(\mathbb{R}) \to W_G(S)(\mathbb{R})) = \mathrm{Im}(W_{K^\circ}(S)(\mathbb{R}) \to W_G(S)(\mathbb{R})).$$

Therefore, the element $w \in W_G(S)(\mathbb{R})$ which sends $(n_1, n_2) \mapsto (n_1, -n_2)$ does not lift to $N_G(S)(\mathbb{R})$, and hence does not allow us to define the desired operator $H^0(X_{\mathbb{C}}, \mathcal{E}_0) \to H^1(X_{\mathbb{C}}, \mathcal{E}_1)$.

Explicitly, the point is that if $P^\circ \subseteq G^\circ = \mathrm{Sp}_4$ is the Siegel parabolic and $M_{P^\circ}$ is its Levi with torus $T$, then $gM_{P^\circ,\mathbb{C}}g^{-1} = K^\circ_{\mathbb{C}}$, but $g = \begin{pmatrix} -iI_2 & I_2 \\ I_2 & I_2 \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C}) \setminus \mathrm{GSp}_4(\mathbb{R})$. Therefore, the representative over $\mathbb{C}$ of $w$ is given by:

$$gg_0g^{-1} = \begin{pmatrix} 1 & & & \\ & i & & \\ & & 1 & \\ & -1+i & & -i \end{pmatrix} \notin \mathrm{GSp}_4(\mathbb{R}).$$

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