BALANCED TRIPLE PRODUCT p-ADIC L-FUNCTIONS AND STARK POINTS

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ABSTRACT. Let E be an elliptic curve over $\mathbb{Q}$ and $\varrho_1, \varrho_2\colon \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(L)$ be two odd Artin representations. We use p-adic methods to investigate the part of the Mordell–Weil group $E(H) \otimes L$ on which the Galois group acts via $\varrho_1 \otimes \varrho_2$. When the rank of the group is two, Darmon–Lauder–Rotger used a dominant triple product p-adic L-function to study this group, and gave an *Elliptic Stark Conjecture* which relates its value outside of the interpolation range to two Stark points and one Stark unit. Our paper achieves a similar goal in the rank one setting. We first generalize Hsieh's construction of a 3-variable balanced triple product p-adic L-function in order to allow Hida families with classical weight one specializations. We then give an *Elliptic Stark Conjecture* relating its value outside of the interpolation range to a Stark point and two Stark units. As a consequence, we give an explicit p-adic formula for a point which should conjecturally lie in $E(H) \otimes L$. We prove our conjecture for dihedral representations associated with the same imaginary quadratic field. This requires a generalization of the results of Bertolini–Darmon–Prasanna which we prove in the appendix.

CONTENTS

1. INTRODUCTION

Let E be an elliptic curve over $\mathbb{Q}$ and $\varrho_1, \varrho_2\colon \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}_2(L)$ be two odd Artin representations such that $\det \varrho_1 \cdot \det \varrho_2 = 1$. We can consider the part of the Mordell-Weil group $E(H)$ on which the Galois group $\mathrm{Gal}(H/\mathbb{Q})$ acts via $\varrho_1 \otimes \varrho_2$; formally:

$$E[\varrho_1 \otimes \varrho_2] = \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(\varrho_1 \otimes \varrho_2, E(H)_L).$$

An equivariant version of the Birch–Swinnerton-Dyer conjecture then predicts that the rank of $E[\varrho_1 \otimes \varrho_2]$ is equal to the order of vanishing of the L-function $L(\varrho_E \otimes \varrho_1 \otimes \varrho_2, s)$ at $s = 1$, and the leading term of its Taylor expansion is explicitly related to the elements of $E[\varrho_1 \otimes \varrho_2]$. The goal of this paper is to investigate the group $E[\varrho_1 \otimes \varrho_2]$ using p-adic methods when its rank is 1. We start with a brief summary before explaining the details.

When $E[\varrho_1 \otimes \varrho_2]$ has rank 2, the analogous question was considered by Darmon, Lauder, and Rotger:

(1) They prove that there is a p-adic L-function $\mathcal{L}_p^{\varrho_1}(\ell)$ with an interpolation property which holds for integers $\ell \geq 3$.[1] It is associated with the triple $(E, \varrho_1, \varrho_2)$, but it is *unbalanced* — ϱ_1 plays a special role in the construction. [DR14]

(2) They conjecture a special value formula outside of the interpolation range, the *rank two Elliptic Stark Conjecture*:

$$(1.1) \qquad \mathcal{L}_p^{\varrho_1}(1) \sim_{L^\times} \frac{\det \begin{pmatrix} \log_{E,p}(P_{1,1}) & \log_{E,p}(P_{1,2}) \\ \log_{E,p}(P_{2,1}) & \log_{E,p}(P_{2,2}) \end{pmatrix}}{\log_p(u_1)},$$

where $P_{i,j} \in E(H)_L$ are points in the $(\varrho_1 \otimes \varrho_2)$-isotypic component and $u_1 \in \mathcal{O}_H^\times$ is a unit in the $\mathrm{Ad}^0 \, \varrho_1$-isotypic component. [DLR15, Conjecture ES]

(3) They prove (1.1) when ϱ_1 and ϱ_2 are both induced from the same imaginary quadratic field in which p splits. [DLR15, Theorem 3.3]

When $E[\varrho_1 \otimes \varrho_2]$ has rank 1, we achieve the same goals in the present paper:

(1) We prove that there is a p-adic L-function $\mathcal{L}_p^{\mathrm{bal}}(\ell)$ with an interpolation property which holds for integers $\ell \geq 2$.[2] It is associated with the triple $(E, \varrho_1, \varrho_2)$, but in our case it is *balanced* — ϱ_1 and ϱ_2 play equivalent roles. [Theorem A]

(2) We conjecture a special value formula outside of the interpolation range, the *rank one Elliptic Stark Conjecture*:

$$(1.2) \qquad \mathcal{L}_p^{\mathrm{bal}}(1) \sim_{\sqrt{L^\times}} \frac{\log_{E,p}(P)}{\log_p(u_1)^{1/2} \log_p(u_2)^{1/2}}$$

where $P \in E(H)_L$ is a point in the $(\varrho_1 \otimes \varrho_2)$-isotypic component and $u_i \in \mathcal{O}_H^\times$ is a unit in the $\mathrm{Ad}^0 \, \varrho_i$-isotypic component, for $i = 1, 2$. [Conjecture C]

(3) We prove (1.2) when ϱ_1 and ϱ_2 are both induced from the same imaginary quadratic field in which p splits. [Theorem D]

As opposed to the rank two setting, our formula (2) can be rewritten to give a p-adic analytic formula for a p-adic point:

$$(1.3) \qquad \exp_{E,p}(\mathcal{L}_p^{\mathrm{bal}}(1) \cdot \log_p(u_1)^{1/2} \cdot \log_p(u_2)^{1/2}) \in E(H_p)_L$$

conjecturally lies in $E(H)_L$.

Bertolini, Seveso, and Venerucci have also been studying the rank one group $E[\varrho_1 \otimes \varrho_2]$ using a balanced p-adic L-function[3]. Their Oberwolfach report [BSV20] conjectures that there is a canonical multiple L_p^{can} of a p-adic L-function L_p whose value at $(2, 1, 1)$ is equal to $\log_{E,p}(P)$. They explain that special cases of this conjecture have been verified in the CM setting. They give a different expression for this p-adic L-function using an endoscopic lift to GSp_4, and relate its non-vanishing at $\ell = 1$ to the non-vanishing of an appropriate Selmer class. This is the subject of their forthcoming work with Andreatta [ABSV].

In the rest of the introduction, we will state our results precisely and explain the new ideas which allow us to achieve these goals.

1.1. Construction of the balanced p-adic L-function. Let f be the normalized weight two modular form associated with E by the Modularity Theorem, and g, h be the normalized weight one modular

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[1]In general, there is a 3-variable p-adic L-function $\mathcal{L}_p^{\varrho_1}(k, \ell, m)$ with an interpolation property when ℓ is *dominant*. Here, we take $(k, \ell, m) = (2, \ell, 1)$ for simplicity.

[2]Again, in general, there is a 3-variable p-adic L-function $\mathcal{L}_p^{\mathrm{bal}}(k, \ell, m)$ with an interpolation property when the weights are *balanced*. Here, we take $(k, \ell, m) = (2, \ell, \ell)$ for simplicity.

[3]An algorithmic study of this p-adic L-function, motivated by their work, has been carried out in the PhD thesis of the first-named author [Dal21].

forms associated with ϱ_1, ϱ_2, respectively, by the work of Khare and Wintenberger [KW09a, KW09b]. The analytic properties of the L-function we are interested in are understood in terms of the Garrett triple product L-function associated with f, g, and h:

$$L(\varrho_E \otimes \varrho_1 \otimes \varrho_2, s) = L(f \times g \times h, s).$$

Then the completed L-function $\Lambda(f \times g \times h, s)$ satisfies a functional equation relating the value at s to the value at $2 - s$. The assumption that $\det \varrho_1 \det \varrho_2 = 1$ implies that the global root number $\epsilon(f \times g \times h)$ is ± 1. Since we are interested in the rank one setting, we will assume that:

$$\epsilon(f \times g \times h) = -1$$

(see Hypothesis A). Since the global root number is the product of local root numbers

$$\epsilon(f \times g \times h) = \prod_v \epsilon_v(f \times g \times h),$$

and $\epsilon_\infty(f \times g \times h) = 1$, the assumption amounts to the fact that there is an odd number of primes v such that $\epsilon_v(f \times g \times h) = -1$.

Let $p \geq 5$ be a prime number. We assume that f, g, h are ordinary and regular at p (see Hypothesis B), and consider Hida families $\mathbf{g}$, $\mathbf{h}$ whose weight one specializations are fixed p-stabilizations g_α, h_α of g and h, respectively. Letting g_ℓ, h_m be the weight ℓ, m specializations of $\mathbf{g}$, $\mathbf{h}$, respectively, we note that

$$\epsilon(f \times g_\ell \times h_m) = \prod_v \epsilon(f \times g_\ell \times h_m) = \prod_{v < \infty} \epsilon_v(f \times g_\ell \times h_m) \cdot \epsilon_\infty(f \times g_\ell \times h_m) = -\epsilon_\infty(f \times g_\ell \times h_m)$$

by rigidity of automorphic types. In particular, if the weights $(2, \ell, m)$ are *balanced*, i.e. $2 \leq \ell < 2 + m$, $2 \leq m < 2 + \ell$, and $\ell + m \equiv 0 \pmod 2$, we have that

$$\epsilon(f \times g_\ell \times h_m) = +1,$$

and hence we expect that generically the central L-value is non-vanishing:

$$L(f \times g_\ell \times h_m, (\ell + m)/2) \neq 0,$$

and there should be a p-adic L-function interpolating these values divided by appropriate periods. In comparison, Darmon, Lauder, and Rotger start with $\epsilon(f \times g \times h) = +1$ in the f-dominant region and hence obtain $\epsilon(f \times g_\ell \times h_m) = +1$ in the g-dominant region, i.e. for $\ell > m + 2$.

There are two known constructions of p-adic L-functions interpolating the central critical L-values in the balanced region for three Hida families $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$. The first construction was due to Greenberg and Seveso [GS20], and the second more explicit construction was given by Hsieh [Hsi21]. The difference between the constructions was outlined in the latter paper and it seems that Hsieh's approach is more suitable for arithmetic applications, such as the one in the present paper.

However, as observed by the first-named author in [Dal23a, Section 4.4.4], the ramification assumption (3) in [Hsi21, Theorem B] implies that the Hida families $\mathbf{g}$ and $\mathbf{h}$ cannot both have classical weight one specializations. In particular, as far as we know, there is no suitable balanced triple product p-adic L-function to study classical points with weights $(2, 1, 1)$.

Therefore, the first goal of our paper is to extend Hsieh's construction to a setting where the Hida families $\mathbf{g}$ and $\mathbf{h}$ do have classical weight one specializations. The final assumption we make (see Hypothesis C) is:

$$\epsilon_q(f \times g \times h) = -1 \text{ implies that } v_q(N_i) \leq 2,$$

where N_1, N_2, N_3 are the levels of f, g, h, respectively. This weakens Hsieh's assumption that $v_q(N_i) = 1$.

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and field embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$.

Theorem A (Corollary 6.4). *Under the above hypotheses, there exist open admissible neighborhoods $\mathcal{U}_g$ and $\mathcal{U}_h$ of the classical weight 1 in the weight space, and a (square root) balanced triple product p-adic L-function $\mathcal{L}_p^{\mathrm{bal}}\colon \mathcal{U}_g \times \mathcal{U}_h \to \mathbb{C}_p$, associated with f and the Hida families $\mathbf{g}$ and $\mathbf{h}$. It has the following interpolation property for integers $\ell \in \mathcal{U}_g$ and $m \in \mathcal{U}_h$ such that $\ell, m \geq 2$ and $2 - \ell \leq m \leq 2 + \ell$:*

$$\mathcal{L}_p^{\mathrm{bal}}(\ell, m)^2 = \frac{\Lambda(f \times g_\ell \times h_m, c)}{\langle f, f\rangle \langle g_\ell, g_\ell \rangle \langle h_m, h_m \rangle} \cdot \frac{\mathcal{E}_p^{\mathrm{bal}}(f \times g_\ell \times h_m)}{\mathcal{E}_p(g_\ell, \mathrm{Ad}) \cdot \mathcal{E}_p(h_m, \mathrm{Ad})} \in \overline{\mathbb{Q}},$$

where $c = (\ell + m)/2$ is the center of the functional equation, and

$$\mathcal{E}_p^{\mathrm{bal}}(f \times g_\ell \times h_m) = (1 - \alpha_f \beta_g \beta_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c}),$$

$$\mathcal{E}_p(F_k, \mathrm{Ad}) = (1 - \alpha_F^{-2}\chi_{F_k}(p)p^{k-1})(1 - \alpha_F^{-2}\chi_{F_k}(p)p^{k-2}) \qquad\qquad F_k \in \{g_\ell, h_m\}$$

are as in [Hsi21, (1.2)] *and* [Hsi21, p. 416], *respectively.*

A more general square root balanced triple product p-adic L-function for three Hida families $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ is constructed in Theorem 5.5.

Note that the interpolation property for our triple product p-adic L-function differs from Hsieh [Hsi21, Theorem B]. The three key differences (which we will expand on momentarily) are:

- Hsieh's p-adic L-function is defined on the entire weight space.
- Our interpolation property does not involve Gross periods, but rather just the Petersson norms of the three forms f, g_ℓ, and h_m.
- The Euler factor at p in Hsieh's p-adic L-function is only $\mathcal{E}_p^{\mathrm{bal}}(f \times g_\ell \times h_m)$ instead of the above quotient. Indeed, the adjoint Euler factors above are absorbed by the Gross periods.

However, it seems that our p-adic L-function is the correct one for the eventual arithmetic application in the present work. For example, both the periods and the Euler factor at p closely resemble those in [DR14] in the unbalanced setting. Moreover, the proof of Theorem D below also only seems to work with this interpolation property. We expand on the differences in Remark 5.6.

Next, we explain the technical novelty which allows us to loosen the ramification assumption in Hsieh. Ichino's formula [Ich08] for the central value of the triple product p-adic L-function is on the quaternion algebra D ramified at v such that $\epsilon_v(f \times g_\ell \times h_m) = -1$, and utilizes the Jacquet–Langlands transfers f^D, g_ℓ^D, h_m^D of the three modular forms f, g_ℓ, h_m. Under our assumptions, D is ramified at $v = \infty$ and at an odd number of finite primes q.

Let q be an odd prime. As observed by Pizer [Piz80a], if D is ramified at q and f is a (twist-minimal) new cusp form of weight $k \geq 2$, level q^2, character χ of conductor at most q, then:

$$\dim S_k^D(q^2, \chi)[f] = 2,$$

i.e. there is a two-dimensional space of quaternionic modular forms for $D^\times$ which transfer to the same modular form f (cf. Proposition 3.10). This raises two questions:

(1) For each (ℓ, m), can we choose vectors f^D, g_ℓ^D, h_m^D associated with f, g, h and compute the associated local integrals in Ichino's integral representation?

(2) Do there exist Hida families $\mathbf{g}^D$, $\mathbf{h}^D$ associated with the Hida families $\mathbf{g}$, $\mathbf{h}$ such that the specializations g_ℓ^D, h_m^D recover the choices in question (1)?

We give a positive answer to these questions by introducing "extra Hecke operators" which recover multiplicity one in the Jacquet–Langlands correspondence.

Proposition B (Proposition 3.14). *There exists an operator $\langle \varpi_{D_q} \rangle$ on $\dim S_k^D(q^2, \chi)$ associated with a choice of local uniformizer ϖ_{D_q} of D_q, commuting with the Hecke operators. For each twist-minimal*

new cusp form f, of weight $k \geq 2$, level q^2, and character χ of $\mathrm{cond}(\chi) \leq q$:

$$S_k^D(q^2, \chi)[f] \cong \underbrace{S_k^D(q^2, \chi)[f]^{\langle \varpi_{D_q} \rangle = +\sqrt{\chi_q(q)}}}_{\dim = 1} \oplus \underbrace{S_k^D(q^2, \chi)[f]^{\langle \varpi_{D_q} \rangle = -\sqrt{\chi_q(q)}}}_{\dim = 1}.$$

Here, χ_q is the local component at q of the adèlization of χ.

To answer question (1), we can choose a vector in either of the one-dimensional eigenspaces. Section 2 computes the necessary local integrals in Ichino's formula. Answering question (2) is then also based on Proposition B, although it is technical and occupies Sections 3 and 4. We show that for each choice of sign $+1$ or -1, there is a one-dimensional space of Hida families on $D^\times$ associated with the Hida families on GL_2, strengthening the results in [Dal23b]. Theorem 4.23 proves this statement also in the case of Coleman families. From it, we deduce a control theorem à la Hida, which is the content of Corollary 4.28. The proof of these results relies on the construction of auxiliary quaternionic eigenvarieties endowed with closed immersions in the classical cuspidal one. In order to construct an explicit p-adic L-function, we need to identify quaternionic Hida families with (a slight generalization of the) Λ-adic forms in [Hsi21]; this is done in Proposition 4.26.

Note that our choice of vectors on the quaternion algebra $D^\times$ is only well-defined up to scalars. We also restrict ourselves to working with an admissible affinoid in the weight space, instead of proving a control theorem over the whole weight space. It would be interesting to generalize our control theorem to the whole weight space and answer the natural questions about Gross periods for the different Hida families associated with the choice of eigenvalues of the operators $\langle \varpi_{D_q} \rangle$. This could lead to a definition of an integral p-adic L-function analogous to Hsieh's.

1.2. The Elliptic Stark Conjecture.

Having defined the balanced triple product p-adic L-function $\mathcal{L}_p^{\mathrm{bal}}$, we turn to studying its value at the *BSD point* $(\ell, m) = (1, 1)$ which lies outside of the interpolation range. We start by introducing the relevant arithmetic objects.

Let us briefly recall our setup: the triple (f, g, h) corresponds to a triple $(E, \varrho_1, \varrho_2)$ of an elliptic curve E over $\mathbb{Q}$, and two Artin representation $\varrho_i \colon \mathrm{Gal}(H/\mathbb{Q}) \to \mathrm{GL}(V_i)$ for two-dimensional L-vector spaces V_1 and V_2. Recall that:

$$E[V_1 \otimes V_2] = \mathrm{Hom}_{L[\mathrm{Gal}(H/\mathbb{Q})]}(V_1 \otimes V_2, E(H) \otimes_{\mathbb{Q}} L),$$

and under the rank one assumption, we may choose a basis:

$$\Phi \colon V_1 \otimes V_2 \to E(H) \otimes_{\mathbb{Q}} L,$$

well-defined up to $L^\times$.

We fix an embedding $H \hookrightarrow \mathbb{Q}_p^{\mathrm{ur}}$ and let $\sigma_p \in \mathrm{Gal}(H/\mathbb{Q})$ be the associated Frobenius element. Under our classicality and regularity assumptions, we have that $\varrho_i(\sigma_p) = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}$ with $\alpha_i \neq \beta_i$. Letting V_i^α be the α_i-eigenspace of V_i, we consider a non-zero element in the one-dimensional subspace $v_{\alpha\alpha} \in V^{\alpha\alpha} = V_1^\alpha \otimes V_2^\alpha \subseteq V_1 \otimes V_2$. Finally, we let

$$P_{\alpha\alpha} = \Phi(v_{\alpha\alpha}) \in E(H) \otimes L,$$

which is well-defined up to $L^\times$, but depends on the choice of Frobenius eigenvalues α_1 and α_2.

Next, we consider the adjoint representations $\mathrm{Ad}^0(\varrho_i) = \mathrm{Hom}^0(V_i, V_i)$. By [DLR15, Proposition 1.5], the Stark unit group

$$U_H[\mathrm{Ad}^0 \varrho_i] = \mathrm{Hom}_{L[\mathrm{Gal}(H/\mathbb{Q})]}(\mathrm{Ad}^0 V_i, \mathcal{O}_H^\times \otimes L)$$

is also of rank one, and we may choose a basis:

$$\Psi_i \colon \mathrm{Ad}^0 V_i \to \mathcal{O}_H^\times \otimes L,$$

well-defined up to $L^\times$.

The eigenvalues of $\mathrm{Ad}^0 \varrho_i(\sigma_p)$ are 1, $\frac{\alpha_i}{\beta_i}$, $\frac{\beta_i}{\alpha_i}$, and we fix eigenvectors in the α_i/β_i-eigenspaces:

$$v_i^{\alpha/\beta} \in (\mathrm{Ad}^0 V_i)^{\alpha_i/\beta_i}.$$

Finally, this defines

$$u_{i,\alpha} = \Psi_i(v_i^{\alpha/\beta}) \in \mathcal{O}_H^\times \otimes L,$$

which is well-defined up to $L^\times$, but depends on the choice of Frobenius eigenvalue α_i.

Write $r(E, \varrho_{gh})$ for the rank of $E[\varrho_g \otimes \varrho_h]$ which is, conjecturally on BSD, equal to $\mathrm{ord}_{s=1} L(f \times g \times h, s)$.

Conjecture C (Rank one Elliptic Stark Conjecture 6.5). *If $r(E, \varrho_{gh}) > 1$, then $\mathcal{L}_p^{\mathrm{bal}}(1,1) = 0$. If $r(E, \varrho_{gh}) = 1$, then:*

$$\mathcal{L}_p^{\mathrm{bal}}(1,1)^2 \sim_{L^\times} \frac{\log_{E,p}(P_{\alpha\alpha})^2}{\log_p(u_{1,\alpha}) \log_p(u_{2,\alpha})},$$

where

$$\log_p \colon (\mathcal{O}_H^\times)_L \to H_p \otimes L \qquad\qquad\qquad p\text{-adic logarithm},$$
$$\log_{E,p} \colon E(H)_L \to H_p \otimes L \qquad\qquad p\text{-adic formal group logarithm for } E.$$

Both sides of the equality depend on the same choice of α_1 and α_2. At first glance, it seems that the right hand side depends on H and the choices associated with it, but a simple computation reveals that it does not — see Remark 6.6.

Finally, we prove that this conjecture is true if ϱ_1, ϱ_2 are both induced from characters of the same imaginary quadratic field. Let K be an an imaginary quadratic field and p be a prime which splits in K. Let ψ_g, ψ_h be finite order Hecke characters of K such that $(\psi_g \psi_h) \circ N_{K/\mathbb{Q}} = 1$, and consider the theta series $g = \theta_{\psi_g}$, $h = \theta_{\psi_h}$. Let $\psi_1 = \psi_g \psi_h$ and $\psi_2 = \psi_g \psi_h^\sigma$. We introduce some explicit assumptions under which our running hypotheses hold:

(1) $\epsilon_q(f_K, \psi_2) = +1$ for all finite primes q of K,
(2) g and h are ordinary and regular,
(3) if $\epsilon_q(f \times g \times h) = -1$, then $v_q(N_1) = 1$ and $v_q(N_2) = v_q(N_3) = 2$;

Theorem D (Theorem 6.9). *Under assumptions (1)–(3) above, Conjecture C is true.*

The idea of the proof of this theorem is to factor the triple product p-adic L-function into Rankin–Selberg and Katz p-adic L-functions (Theorem 6.14). Under our ramification hypotheses, we need a certain generalization of the result of Bertolini–Darmon–Prasanna [BDP13] on Rankin–Selberg p-adic L-functions. Indeed, their result holds under the Heegner hypothesis, which is not satisfied under assumption (3) above. We loosen the Heegner hypothesis slightly to include the case of interest to us. We prove this result in Appendix A.

In Appendix B, we provide a digest of examples to which our conjecture and theorem apply. In future work, we hope to give numerical evidence for Conjecture C, based on the algorithms in [Dal23a].

1.3. **Organization of the paper.** The main technical innovation of the paper is the study of extra operators on the space of quaternionic modular forms, which facilitate our choice of test vectors for the p-adic L-function. This occupies the first three sections, which consider the local (Section 2), global (Section 3), and p-adic (Section 4) Jacquet–Langlands transfers. A reader interested in the construction of the p-adic L-function may proceed directly to Section 5, where we use this choice of test vectors to generalize Hsieh's construction. The arithmetic applications are discussed in Section 6, which may also be read independently of all the previous sections. Appendix A gives generalizations of results of Bertolini–Darmon–Prasanna which are used the proof of Theorem D, while Appendix B contains examples.

2. Local JL correspondence and test vectors

In this section, we consider a finite extension F of $\mathbb{Q}_\ell$ for $\ell \neq 2$ (which will eventually be $\mathbb{Q}_\ell$) and three local representations π_1, π_2, π_3 of $\mathrm{GL}_2(F)$ such that the product of their central characters is trivial.

We classify when $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$ and in all cases when $c(\pi_1), c(\pi_2), c(\pi_3) \leq 2$, we compute the relevant local triple product integrals on a definite quaternion algebra over F. The new contribution of this section is the computations of these local integrals when $c(\pi_i) = 2$ for some i.

Throughout this section, we often work with Weil–Deligne representations $\sigma_i = \sigma(\pi_i)$ associated with π_i via the Local Langlands Correspondence [BH06]. Their triple product ϵ- and L-factors agree with the automorphic ones [HK04, Proposition 2.1].

For completeness, we briefly recall the explicit Local Langlands Correspondence [BH06, Chapter 8] for $\ell \neq 2$. For a character χ of $K^\times$, for a local field K, we write $\xi = \xi(\chi)$ for the associated character of W_K via Class Field Theory.

(1) If $\pi = \pi(\chi_1, \chi_2)$ is a principal series representation, then $\sigma(\pi) = \xi(\chi_1) \oplus \xi(\chi_2)$ is reducible.
(2) If $\pi = \mathrm{St} \otimes \chi$ is a twist of the Steinberg representation, then $\sigma(\pi) = \mathrm{Sp}(2) \otimes \xi(\chi)$ is a twist of the special representation.
(3) If $\pi = \pi_\psi$ is supercuspidal, associated with an *admissible pair* (K, ψ), where K/F is a quadratic extension and ψ is a character of $K^\times$, then $\sigma(\pi) = \mathrm{Ind}_{W_K}^{W_F}(\xi(\psi)\Delta_\psi^{-1})$ for a character Δ_ψ defined in [BH06, Section 34.4]. Then $c(\pi) = 2$ if and only if π has *depth 0* (or *level 0*), i.e. K is unramified over F and ψ has conductor 1 (*level 0*). In this case Δ_ψ is the unramified quadratic character of $K^\times$.

The representations in (2) and (3) are called *discrete series*.

2.1. **Local ϵ-factors and L-factors.** The following proposition classifies all cases when $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$.

Proposition 2.1 (Prasad)**.** *Let $\sigma_1, \sigma_2, \sigma_3$ be three Weil–Deligne representations of W_F such that the product of their determinants is trivial. Then $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ if and only if there is a reordering $\{\sigma_1', \sigma_2', \sigma_3'\} = \{\sigma_1, \sigma_2, \sigma_3\}$ such that one of the following holds:*

(1) $\sigma_i' \cong \mathrm{Sp}(2) \otimes \det(\sigma_i')$ for $i = 1, 2, 3$;
(2) $\sigma_1' \cong \mathrm{Sp}(2) \otimes \det(\sigma_1')$, σ_2' is irreducible and $\sigma_3' \cong (\sigma_2')^\vee \otimes \det(\sigma_1')^{-1}$;
(3) there is a quadratic extension K/F and characters ξ_1, ξ_2, ξ_3 such that $\sigma_i' = \mathrm{Ind}_{W_K}^{W_F} \xi_i$ is irreducible (so $\xi_i^\sigma \neq \xi_i$) and either $\xi_1 \xi_2 \xi_3 = 1$ or $\xi_1 \xi_2 \xi_3^\sigma = 1$ or $\xi_1 \xi_2^\sigma \xi_3 = 1$ or $\xi_1 \xi_2^\sigma \xi_3^\sigma = 1$.

Proof. These results are contained in [Pra90, Section 8]. See [Pra90, Proposition 8.6] for (1) and [Pra90, Proposition 8.5] (2). For completeness, we compute $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3)$ for $\sigma_i = \mathrm{Ind}_{W_K}^{W_F} \xi_i$; the other possibilities are dealt with similarly. First, note that:

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \cong \left(\mathrm{Ind}_{W_K}^{W_F}(\xi_1\xi_2) \otimes \sigma_3 \right) \oplus \left(\mathrm{Ind}_{W_K}^{W_F}(\xi_1\xi_2^\sigma) \otimes \sigma_3 \right)$$

and hence by standard properties of ϵ-factors:

$$\begin{aligned}
\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) &= \epsilon\left(\mathrm{Ind}_{W_K}^{W_F}(\xi_1\xi_2) \otimes \sigma_3 \right) \epsilon\left(\mathrm{Ind}_{W_K}^{W_F}(\xi_1\xi_2^\sigma) \otimes \sigma_3 \right) \\
&= \xi_{K/F}(-1)\epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2) \cdot \xi_{K/F}(-1)\epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2^\sigma) \\
&= \epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2) \cdot \epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2^\sigma) \\
&= \det(\sigma_3)(-1)\epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2) \cdot \det(\sigma_3)(-1)\epsilon(\sigma_3|_{W_K} \otimes \xi_1\xi_2^\sigma),
\end{aligned}$$

where $\xi_{K/F}$ is the quadratic character of W_F associated with K/F. Tunnell's Theorem [Pra90, Theorem 8.2] shows that $\det(\sigma_3)(-1)\epsilon(\sigma_3|_{W_K}\otimes\xi_1\xi_2) = -1$ if and only if $(\xi_1\xi_2)^{-1} = \xi_3$ or $(\xi_1\xi_2)^{-1} = \xi_3^\sigma$. Therefore, $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ if and only if condition (3) holds after reordering. ∎

We make the following symplifying hypothesis.

Hypothesis 2.2. For all i, the conductor $c(\pi_i)$ of π_i is at most 2.

In particular, if $\sigma = \mathrm{Ind}_{W_K}^{W_F} \chi$, then K/F is unramified, the conductor $c(\chi)$ of χ is 1, and $\chi^\sigma \neq \chi$.

Next, we record the L-factors in each of the cases when $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$.

Proposition 2.3. *We have that:*

$$L(\sigma_1 \otimes \sigma_2 \otimes \sigma_3, s) = \begin{cases} \zeta_F(s+3)\zeta_F(s+2)^2 & \text{case (1)} \\ \zeta_F(2s+2) & \text{case (2)} \\ \zeta_F(2s) & \text{case (3)} \end{cases}$$

Proof. In case (1), note that:

$$\mathrm{Sp}(2) \otimes \mathrm{Sp}(2) \otimes \mathrm{Sp}(2) \cong \mathrm{Sp}(4) \oplus \mathrm{Sp}(2)|\cdot| \oplus \mathrm{Sp}(2)|\cdot|$$

and $L(\mathrm{Sp}(m), s) = \zeta_\ell(s+m-1)$. Part (2) follows from

$$L(\mathrm{Sp}(2) \otimes \sigma \otimes \sigma^\vee, s) = L(\sigma \otimes \sigma^\vee, s+1)$$

and [GJ78, Corollary 1.3], because $\sigma \cong \sigma \otimes \eta$ for an unramified quadratic character η. For (3), note that:

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \cong \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2\chi_3) \oplus \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2\chi_3^\sigma) \oplus \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2^\sigma\chi_3) \oplus \mathrm{Ind}_{W_K}^{W_F}(\chi_1\chi_2\chi_3^\sigma).$$

Under the assumption that $\epsilon(\sigma_1 \otimes \sigma_3 \otimes \sigma_3) = -1$, exactly one of these characters is trivial and the other characters are ramified. This shows that:

$$L(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \zeta_K(s) = \zeta_F(2s),$$

because K/F is unramified. ∎

We will also need the adjoint L-factors.

Proposition 2.4. *We have that:*

$$\mathrm{Ad}(\mathrm{Sp}(2) \otimes \xi) \cong \mathrm{Sp}(3)|\cdot|,$$

$$\mathrm{Ad}(\mathrm{Ind}_{W_K}^{W_F}(\chi)) \cong \mathrm{Ind}_{W_K}^{W_F}(\chi/\chi^\sigma) \oplus \chi_{K/F},$$

and hence:

$$L(\mathrm{Ad}(\mathrm{Sp}(2) \otimes \xi), s) = \zeta_F(s+1),$$

$$L(\mathrm{Ad}(\mathrm{Ind}_{W_K}^{W_F}(\xi))) = L(\chi_{K/F}, s) = \frac{\zeta_F(2s)}{\zeta_F(s)}.$$

2.2. **Local Jacquet–Langlands correspondence.** In cases when $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$, the local integral should be non-vanishing on the non-split quaternion algebra D over F (cf. [Pra90, Theorem 1.2]). In this section, we briefly summarize the local Jacquet–Langlands correspondence following [BH06, Chapter 13].

The local Jacquet–Langlands correspondence is a bijection

$$\mathrm{JL}\colon \mathrm{Rep}^{\mathrm{ds}}(\mathrm{GL}_2(F)) \to \mathrm{Rep}(D^\times)$$

between discrete series representations of $\mathrm{GL}_2(F)$ and irreducible smooth admissible representations of $D^\times$. For $\pi = \mathrm{St} \otimes \chi$, we have that:

$$\mathrm{JL}(\mathrm{St}(2) \otimes \chi) = \chi \circ \nu,$$

where $\nu\colon D^\times \to F^\times$ is the reduced norm.

For supercuspidal representations, an explicit description of the correspondence is given in [BH06, Section 56]. We only describe it here for supercuspidal representations of conductor two, i.e. $\pi = \pi_\psi$ for the unramified quadratic extension K of F and character ψ of $K^\times$ of conductor 1 such that $\psi^\sigma \neq \psi$.

Recall that Δ_ψ is the unramified quadratic character of $K^\times$ in this case. We then have that:

- $\pi_\psi \otimes \Delta_\psi \cong \pi_\psi$,
- under the local Langlands correspondence, the corresponding representation of W_F is $\mathrm{Ind}_{W_K}^{W_F}(\xi(\psi)\Delta_\psi)$.

We describe the representation

$$\pi_\psi^D = \mathrm{JL}(\pi_\psi)$$

explicitly. Let $\mathcal{O}_D$ be a maximal order of D and $\varpi_D \in \mathcal{O}_D$ be a uniformizer. We will assume that $\varpi_D^2 = \varpi_F$ for a uniformizer ϖ_F of F. There is a filtration on $\mathcal{O}_D^\times$

$$(2.1) \qquad U_D^a = \begin{cases} \mathcal{O}_D^\times & a = 0 \\ 1 + \varpi_D^a \mathcal{O}_D & a \geq 1. \end{cases}$$

Given an unramified character ψ of $K^\times$, we may extend it to a character Ψ of $K^\times U_D^1$ by letting U_D^1 act trivially. Then:

$$\pi_\psi^D \cong \text{c-}\mathrm{Ind}_{F^\times U_D^1}^{D^\times} \Psi.$$

Let π be a smooth irreducible representation of $D^\times$, which is automatically finite-dimensional because $D^\times/F^\times$ is compact. Thus $\pi|_{U_D^a} = 1$ for $a \gg 0$. We define the *conductor* $c(\pi)$ of π to be $a + 1$ where a is the smallest integer such that $\pi|_{U_D^a} = 1$. Note that under the assumption that ψ has conductor 1, π_ψ^D has conductor two. For completess, we verify that all representations of $D^\times$ of conductor two are obtained this way, following [Car84].

Note that $\varpi_K = \varpi_F$ because K/F is an unramfied quadratic extension, and $k = \mathcal{O}_K/\varpi_K\mathcal{O}_K$ is a quadratic extension of $f = \mathcal{O}_F/\varpi_F\mathcal{O}_F$. Finally, since $d = \mathcal{O}_D/\varpi\mathcal{O}_D$ is also a quadratic extensions of f, it is isomorphic to k. By definition, a representation of conductor two factors through

$$(2.2) \qquad D^\times/U_D^1 \cong d^\times \rtimes \langle \varpi_D \rangle.$$

Note that $\varpi_D \mathcal{O}_D \varpi_D^{-1} = \mathcal{O}_D$ because the maximal order $\mathcal{O}_D \subseteq D$ is unique, and hence conjugation by ϖ_D preserves $d^\times$. Moreover, $\varpi_D^2 = \varpi_F$ is in the center $F^\times$ of $D^\times$ and hence acts trivially. In particular, we have a subgroup: $d^\times \times \langle \varpi_F \rangle \subseteq d^\times \rtimes \langle \varpi_D \rangle$.

Next, observe that a character ψ of $K^\times$ which is trivial on U_D^1 corresponds precisely to a character of $k^\times \times \langle \varpi_F \rangle$. Therefore, we may identify π_ψ^D with the inflation of the induction $\mathrm{Ind}_{d^\times \times \langle \varpi_F \rangle}^{d^\times \rtimes \langle \pi_D \rangle} \psi$. This representation is reducible unless $\psi^{\varpi_D} \neq \psi$. Altogether, we get the following result.

Proposition 2.5. *Suppose π is an smooth irreducible representation of $D^\times$ and that $c(\pi) = 2$. Then there is a character ψ of $d^\times \times \langle \varpi_F \rangle$ with $\psi^{\varpi_D} \neq \psi$ such that:*

$$\pi = \mathrm{Inf}_{D^\times/U_D^1}^{D^\times} \mathrm{Ind}_{d^\times \times \langle \varpi_F \rangle}^{d^\times \rtimes \langle \varpi_D \rangle} \psi.$$

In particular, π is two-dimensional and in the basis corresponding to the decomposition above, we have:

$$\pi(x) = \begin{pmatrix} \psi(x) & \\ & \psi^{\varpi_D}(x) \end{pmatrix}, \qquad x \in d^\times \times \langle \varpi_F \rangle,$$

$$\pi(\varpi_D) = \begin{pmatrix} & 1 \\ \psi(\varpi_F) & \end{pmatrix}.$$

Finally, if ω is the central character of π, then $\psi|_{F^\times} = \omega$ and the last equality may be written:

$$\pi(\varpi_D) = \begin{pmatrix} & 1 \\ \omega(\varpi_F) & \end{pmatrix}.$$

Proof. The first claim then follows from [Car84, Section 5.1]. The rest of the proposition is immediate. ∎

We identify the representation π_ψ^D with the one described explicitly in Proposition 2.5.

2.3. **Local integral for zero, two, and three supercuspidal representations of level ℓ^2.** Let π_1, π_2, π_3 be irreducible admissible representations of $\mathrm{GL}_2(F)$ with central characters ω_i which satisfy $\omega_1 \omega_2 \omega_3 = 1$. Let $\sigma_1, \sigma_2, \sigma_3$ be corresponding representations of W_F such that $\det(\sigma_1) \det(\sigma_2) \det(\sigma_3) = 1$. As above, let D be the non-split quaternion algebra over F, and π_i^D be the Jacquet–Langlands transfer of π_i to $D^\times$.

Prasad [Pra90, Theorem 1.4] proves that there exists a non-zero trilinear form on $\pi_1^D \otimes \pi_2^D \otimes \pi_3^D$ if and only if $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$. There is a natural trilinar form:

$$(2.3) \qquad I_v'(\phi) = \int\limits_{F^\times \backslash D^\times} \langle \pi(g)\phi, \widetilde{\phi} \rangle \, dg \qquad \phi \in \pi_1^D \otimes \pi_2^D \otimes \pi_3^D$$

and the goal of this section is to choose vectors ϕ and compute $I_v'(\phi) \neq 0$ explicitly when $\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = -1$ and $c(\sigma_i) \leq 2$. Here, $\langle -, - \rangle$ is a pairing between $\pi_1^D \otimes \pi_2^D \otimes \pi_3^D$ and its contragredient representation, and $\widetilde{\phi}$ is the vector dual to ϕ under this pairing. Recall that there were three cases (1)–(3) outlined in Proposition 2.1 and we will treat each of them separately.

Remark 2.6. As far as we know, these are the first such results when one of the components of $\pi_1 \times \pi_2 \times \pi_3$ is supercuspidal and $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$ so the trilinear form is on the quaternion algebra $D^\times$. When $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = +1$, the trilinear form is on GL_2 and Dimitrov–Nyssen [DN10] show how to choose vectors in $\pi_1 \times \pi_2 \times \pi_3$ when at least one component is not supercuspidal.

2.3.1. *Case (1): zero supercuspidal representations.* Suppose $\pi_i = \mathrm{St}(2) \otimes \omega_i$ for $i = 1, 2, 3$. Then:

$$\pi_i^D = \omega_i \circ \nu \colon D^\times \to \mathbb{C}^\times$$

is one-dimensional and we choose any non-zero vectors $\phi_i \in \pi_i^D$.

Proposition 2.7. *For characters $\omega_1, \omega_2, \omega_3$ of $F^\times$ such that $\omega_1 \omega_2 \omega_3 = 1$ and any non-zero vectors $\phi_i \in \pi_i^D$, we have that*

$$\frac{I_v'(\phi)}{\langle \phi, \widetilde{\phi} \rangle} = 2\mu(\mathcal{O}_D^\times)$$

for $\phi = \phi_1 \times \phi_2 \times \phi_3$.

Proof. See the proof of [Woo12, Proposition 4.5]. ∎

2.3.2. *Case (2): two supercuspidal representations.* Consider representations π_1, π_2, π_3 of $\mathrm{GL}_2(F)$ such that:

(1) π_1, π_2 are supercuspidal, $\pi_3 = \mathrm{St} \otimes \omega_3$ is a twist of the Steinberg representation,

(2) the product $\omega_1 \cdot \omega_2 \cdot \omega_3$ of their central characters it trivial,

(3) $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$, i.e. $\pi_1^\vee = \pi_2 \otimes \omega_3^{-1}$.

We assume that π_1, π_2 have conductor 2 and let π_i^D be the representation of $D^\times$ corresponding to π_i for $i = 1, 2, 3$ under the the Jacquet–Langlands correspondence. Then:

(1) $\pi_1^D = \pi_\psi$ for some character ψ of $d^\times \times \langle \varpi_F \rangle$ such that $\psi|_{F^\times} = \omega_1$,

(2) $\pi_2^D \cong \pi_{\psi^{-1}} \otimes \omega_3^{-1}$,

(3) $\pi_3^D \cong \omega_3 \circ \nu$.

Proposition 2.8. *Let π_1, π_2, π_3 be as above, $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2$, and*

(1) $\phi_i^{\epsilon_i} \in \pi_i^D$ *non-zero such that* $\pi_i^D(\varpi_D)\phi_i^{\epsilon_i} = \epsilon_i \sqrt{\omega_i(\ell)}\phi_i^{\epsilon_i}$ *(cf. Proposition 2.5),*

(2) $\phi_3 \in \pi_3^D$ *nonzero.*

Then for $\phi^\epsilon = \phi_1^{\epsilon_1} \times \phi_2^{\epsilon_2} \times \phi_3$, we have that:

$$\frac{I_v'(\phi^\epsilon)}{\langle \phi^\epsilon, \widetilde{\phi^\epsilon} \rangle} = (1 + \epsilon_1 \epsilon_2 \sqrt{\omega_3(\ell)}) \frac{\mu(\mathcal{O}_D^\times)}{2}.$$

Proof. We simplify the notation throughout the proof and write $\pi_i = \pi_i^D$. We compute, using the above description of the local representations:

$$\begin{aligned}
I_v'(\phi^\epsilon) &= \int_{F^\times \backslash D^\times} \langle \pi(g)\phi^\epsilon, \widetilde{\phi^\epsilon} \rangle \, dg \\
&= \int_{\mathcal{O}_D^\times} \langle \pi(g)\phi^\epsilon, \widetilde{\phi^\epsilon} \rangle \, dg + \int_{\mathcal{O}_D^\times} \langle \pi(g)\pi(\varpi_D)\phi^\epsilon, \widetilde{\phi^\epsilon} \rangle \, dg \qquad D^\times = F^\times \mathcal{O}_D^\times \cup \varpi_D F^\times \mathcal{O}_D^\times \\
&= (1 + \epsilon_1 \epsilon_2 \sqrt{\omega_1(\ell)\omega_2(\ell)}\omega_3(\ell)) \sum_{x \in d^\times} \langle \pi(x)\phi^\epsilon, \widetilde{\phi^\epsilon} \rangle \mu(U_D^1) \\
&= (1 + \epsilon_1 \epsilon_2 \sqrt{\omega_3(\ell)})\mu(U_D^1) \sum_{x \in d^\times} \langle \pi(x)\phi^\epsilon, \widetilde{\phi^\epsilon} \rangle
\end{aligned}$$

Finally, for $x \in d^\times$, we have that:

$$\pi_1(x)\phi_1^{\epsilon_1} = \frac{\psi(x) + \epsilon_1 \psi^{\varpi_D}(x)}{2}\phi_1^+ + \frac{\psi(x) - \epsilon_1 \psi^{\varpi_D}(x)}{2}\phi_1^-,$$

$$\pi_2(x)\phi_2^{\epsilon_2} = \left(\frac{\psi^{-1}(x) + \epsilon_2 \psi^{-1,\varpi_D}(x)}{2}\phi_2^+ + \frac{\psi^{-1}(x) - \epsilon_2 \psi^{-1,\varpi_D}(x)}{2}\phi_2^- \right) \omega_3(N_{d/k}x)^{-1},$$

$$\pi_3(x)\phi_3 = \omega_3(N_{d/k}x)\phi_3.$$

We hence obtain

$$\sum_{x \in d^\times} \pi(x)\phi^\epsilon = S_{++}\phi^{++} + S_{+-}\phi^{+-} + S_{-+}\phi^{-+} + S_{--}\phi^{--}$$

for:

$$\begin{aligned}
S_{++} &= \frac{1}{4} \sum_{x \in d^\times} (\psi(x) + \epsilon_1 \psi^{\varpi_D}(x))(\psi^{-1}(x) + \epsilon_2 \psi^{-1,\varpi_D}(x)), \\
&= \frac{1}{4}|d^\times|(1 + \epsilon_1 \epsilon_2) \\
S_{+-} &= \frac{1}{4} \sum_{x \in d^\times} (\psi(x) + \epsilon_1 \psi^{\varpi_D}(x))(\psi^{-1}(x) - \epsilon_2 \psi^{-1,\varpi_D}(x)) \\
&= \frac{1}{4}|d^\times|(1 - \epsilon_1 \epsilon_2),
\end{aligned}$$

$$S_{-+} = \frac{1}{4} \sum_{x \in d^{\times}} (\psi(x) - \epsilon_1 \psi^{\varpi_D}(x))(\psi^{-1}(x) + \epsilon_2 \psi^{-1,\varpi_D}(x))$$

$$= \frac{1}{4} |d^{\times}|(1 - \epsilon_1 \epsilon_2),$$

$$S_{--} = \frac{1}{4} \sum_{x \in d^{\times}} (\psi(x) - \epsilon_1 \psi^{\varpi_D}(x))(\psi^{-1}(x) - \epsilon_2 \psi^{-1,\varpi_D}(x))$$

$$= \frac{1}{4} |d^{\times}|(1 + \epsilon_1 \epsilon_2).$$

This shows that:

$$\sum_{x \in d^{\times}} \pi(x)\phi^{\epsilon} = \frac{1}{2} |d^{\times}|(\phi^{\epsilon} + \phi^{-\epsilon})$$

and hence:

$$I_v'(\phi^{\epsilon}) = (1 + \epsilon_1 \epsilon_2 \sqrt{\omega_3(\ell)}) \frac{\mu(\mathcal{O}_D^{\times})}{2} \langle \phi^{\epsilon} + \phi^{-\epsilon}, \widetilde{\phi^{\epsilon}} \rangle = (1 + \epsilon_1 \epsilon_2 \sqrt{\omega_3(\ell)}) \frac{\mu(\mathcal{O}_D^{\times})}{2} \langle \phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle.$$

The rest of the results follow.　　　　　　　　　　　　　　　　　　　　　　　　　　　　∎

2.3.3. *Case (3): three supercuspidal representations.* We next consider the case of three twist-minimal supercuspidal representations π_1, π_2, π_3 of $\mathrm{GL}_2(F)$ conductor 2 such that $\omega_1 \cdot \omega_2 \cdot \omega_3 = 1$; according to Proposition 2.5:

$$\pi_i^D = \psi_{\psi_i}^D \tag{2.4}$$

where $\psi_i^{\sigma} \neq \psi_i$ and we note that

$$\psi_1 \psi_1^{\varpi_D} \psi_2 \psi_2^{\varpi_D} \psi_3 \psi_3^{\varpi_D} = 1. \tag{2.5}$$

Proposition 2.1 (3) classifies when $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$. We compute the local integrals associated with all possible choices of vectors $\phi_i^{\pm} \in \pi_i^D = \pi_{\psi_i}^D$ as in Proposition 2.8 and check that $I_v'(\phi) \neq 0$ for some ϕ exactly when $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3) = -1$.

Proposition 2.9. *Let π_1, π_2, π_3 be as above. For $\epsilon \in \{\pm 1\}^3$ let $\phi^{\epsilon} = \phi_1^{\epsilon_1} \times \phi_2^{\epsilon_2} \times \phi_3^{\epsilon_3} \in \pi^D = \pi_1^D \times \pi_2^D \times \pi_3^D$. Then:*

$$\frac{I_v'(\phi^{\epsilon})}{\langle \phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle} = (1 + \epsilon_1 \epsilon_2 \epsilon_3) \frac{\mu(\mathcal{O}_D^{\times})}{4}.$$

Proof. Once again, we simplify the notation to write $\pi_i = \pi_i^D$ etc. We proceed as in the proof of Proposition 2.8:

$$I_v'(\phi^{\epsilon}) = \int_{F^{\times} \backslash D^{\times}} \langle \pi(g)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle \, dg$$

$$= \int_{\mathcal{O}_D^{\times}} \langle \pi(g)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle \, dg + \int_{\mathcal{O}_D^{\times}} \langle \pi(g)\pi(\varpi_D)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle \, dg \qquad D^{\times} = F^{\times}\mathcal{O}_D^{\times} \cup \varpi_D F^{\times}\mathcal{O}_D^{\times}$$

$$= (1 + \epsilon_1 \epsilon_2 \epsilon_3 \sqrt{\omega_1(\ell)\omega_2(\ell)\omega_3(\ell)}) \sum_{x \in d^{\times}} \langle \pi(x)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle \mu(U_D^1)$$

$$= (1 + \epsilon_1 \epsilon_2 \epsilon_3) \mu(U_D^1) \sum_{x \in d^{\times}} \langle \pi(x)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle.$$

Next, we need to compute $\sum_{x \in d^{\times}} \langle \pi(x)\phi^{\epsilon}, \widetilde{\phi^{\epsilon}} \rangle$. For $x \in d^{\times}$ and $i = 1, 2, 3$:

$$\pi_i(x)\phi_i^{\epsilon_i} = \frac{\psi_i(x) + \epsilon_i \psi_i^{\varpi_D}(x)}{2} \phi_i^+ + \frac{\psi_i(x) - \epsilon_i \psi_i^{\varpi_D}(x)}{2} \phi_i^-.$$

Therefore:

$$\pi(x)\phi^\epsilon = \sum_{\eta \in \{\pm 1\}^3} s_{\eta\epsilon} \phi^\eta,$$

$$s_{\eta\epsilon} = \frac{1}{8} \prod_{i=1}^3 (\psi_i(x) + \epsilon_i \eta_i \psi_i^{\varpi_D}(x))$$

$$= \frac{1}{8} \sum_{\delta \in \{0,1\}^3} (\epsilon_1\eta_1)^{\delta_1} (\epsilon_2\eta_2)^{\delta_2} (\epsilon_3\eta_3)^{\delta_3} \psi_1^{\varpi_D^{\delta_1}}(x) \psi_2^{\varpi_D^{\delta_2}}(x) \psi_3^{\varpi_D^{\delta_3}}(x).$$

and hence:

$$\langle \pi(x)\phi^\epsilon, \widetilde{\phi}^\epsilon \rangle = s_{\epsilon\epsilon} \langle \phi^\epsilon, \widetilde{\phi}^\epsilon \rangle = \frac{1}{8} \sum_{\delta \in \{0,1\}^3} \psi_1^{\varpi_D^{\delta_1}}(x) \psi_2^{\varpi_D^{\delta_2}}(x) \psi_3^{\varpi_D^{\delta_3}}(x) \langle \phi^\epsilon, \widetilde{\phi}^\epsilon \rangle.$$

Altogether, we have that:

$$\frac{\sum_{x \in d^\times} \langle \pi(x)\phi^\epsilon, \widetilde{\phi}^\epsilon \rangle}{\langle \phi^\epsilon, \widetilde{\phi}^\epsilon \rangle} = \frac{1}{8} \cdot |d^\times| \cdot \left| \left\{ \delta \in \{0,1\}^3 \ \middle| \ \psi_1^{\varpi_D^{\delta_1}} \psi_2^{\varpi_D^{\delta_2}} \psi_3^{\varpi_D^{\delta_3}} = 1 \right\} \right|.$$

Without loss of generality, suppose that $\psi_1\psi_2\psi_3 = 1$. We claim that then:

$$S = \left\{ \delta \in \{0,1\}^3 \ \middle| \ \psi_1^{\varpi_D^{\delta_1}} \psi_2^{\varpi_D^{\delta_2}} \psi_3^{\varpi_D^{\delta_3}} = 1 \right\} = \{(0,0,0),(1,1,1)\}.$$

By assumption, $(0,0,0) \in S$ and by equation (2.5) also $(1,1,1) \in S$. To verify that S cannot be larger, suppose without loss of generality that $(1,0,0) \in S$, i.e. $\psi_1^{\varpi_1}\psi_2\psi_3 = 1$. Then $\psi_1^{\varpi_1} = \psi_1$, but this contradicts the admissibility of the pair (K, ψ_1). ∎

Remark 2.10. It would be interesting to treat the case of supercuspidal representations of higher conductor as well, but this would take us too far afield from the ultimate arithmetic goals of the paper.

3. Global JL correspondence and test vectors

We are ready to study the global consequences of Section 2. We focus our attention on quaternionic modular forms with level structure given by orders which are *residually inert* at the primes where the quaternion algebra ramifies; the local theory considered in the previous section allows a precise understanding of such forms.

From now on, D denotes a quaternion algebra over $\mathbb{Q}$ (and not a local quaternion algebra as in the previous section). For simplicity of exposition, until Section 3.6, we restrict ourselves to the case where the quaternion algebra is ramified exactly at one odd prime ℓ and at infinity. However, everything we state in this section generalizes to any definite quaternion $\mathbb{Q}$-algebra (see also Remark 3.15); in particular, the results in Section 3.6 and Sections 4–6 deal with the general situation.

For any place v of $\mathbb{Q}$, we denote $D_v = D \otimes_{\mathbb{Q}} \mathbb{Q}_v$, where we understand $\mathbb{Q}_\infty$ to be $\mathbb{R}$. Similarly, for any order $R \subset D$ and any finite place v, we denote $R_v = R \otimes_{\mathbb{Z}} \mathbb{Z}_v$. As a last piece of notation, we set $D(\mathbb{A}) = D \otimes_{\mathbb{Q}} \mathbb{A}$, for $\mathbb{A}$ the adèles of $\mathbb{Q}$, $\widehat{D} = D \otimes_{\mathbb{Q}} \mathbb{A}_f$, for $\mathbb{A}_f$ the finite adèles, and $\widehat{R} = R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, for $\widehat{\mathbb{Z}}$ the profinite completion of the integers.

3.1. **A remark on the structure of quaternion algebras at ramified primes.** We begin by recalling a general formalism to deal with definite quaternion algebras over a local field, which will make our exposition clearer and independent of the choices of the uniformizers. Most of the content of this section can be found in [Voi21, Section 13]. Let D_ℓ be a quaternion division algebra over $\mathbb{Q}_\ell$ and let $\mathbb{Q}_\ell(\varpi_\ell)$ and $\mathbb{Q}_{\ell^2}$ be, respectively, one of the two ramified quadratic extension of $\mathbb{Q}_\ell$ and the unique unramified quadratic one; as D_ℓ is division, there exists embeddings of these two fields in D_ℓ. Consider $\nu_\ell : D_\ell \longrightarrow \mathbb{Q}_\ell$ to be the reduced norm map at ℓ. We extend the ℓ-adic valuation v_ℓ of $\mathbb{Q}_\ell$, to the division

algebra: $w_\ell = \frac{1}{2} \cdot v_\ell \circ \nu_\ell$. We fix
$$\mathcal{O}_{D_\ell} = \{x \in D_\ell \mid w_\ell(x) \in \mathbb{Z}_\ell\}$$
and denote by $\varpi_{D_\ell} \in \mathcal{O}_{D_\ell}$ a uniformizer of D_ℓ, namely an element in $\mathcal{O}_{D_\ell}$ with valuation $w_\ell(\varpi_{D_\ell}) = 1/2$;
it is not difficult to notice that one can take $\varpi_{D_\ell} = \varpi_\ell$. Therefore, we can decompose the division algebra D_ℓ as

$$(3.1) \qquad\qquad\qquad\qquad D_\ell = \mathbb{Q}_{\ell^2} \oplus \varpi_\ell \mathbb{Q}_{\ell^2}.$$

Denoting by $\varpi_{\mathbb{Q}_{\ell^2}}$ the uniformizer of $\mathbb{Q}_{\ell^2}$, we can further write

$$(3.2) \qquad\qquad D_\ell = \mathbb{Q}_\ell \oplus \varpi_{\mathbb{Q}_{\ell^2}} \mathbb{Q}_\ell \oplus \varpi_\ell \mathbb{Q}_\ell \oplus \varpi_{\mathbb{Q}_{\ell^2}} \varpi_\ell \mathbb{Q}_\ell,$$

with the condition $\varpi_{\mathbb{Q}_{\ell^2}} \varpi_\ell = \varpi_\ell \overline{\varpi_{\mathbb{Q}_{\ell^2}}} = -\varpi_\ell \varpi_{\mathbb{Q}_{\ell^2}}$.

3.2. **Residually inert orders.** We briefly recall the definition of residually inert (at ℓ) orders in D. The
interested reader may consult [Voi21, Section 24.3], [HPS89a] and [Piz80b] for a more detailed exposition.

Definition 3.1. Let N be a positive integer prime to ℓ. We say that an order $R \subset D$ of level $N\ell^2$ is

- *residually split* at the prime $q \mid N$ if R_q is an Eichler order of level $q^{\mathrm{val}_q(N)}$;
- *residually inert* at ℓ (also known as special or Pizer order) if there exists a ramified quadratic extension $\mathbb{Q}_\ell(\varpi_\ell)/\mathbb{Q}_\ell$, such that R_ℓ is conjugate to

$$\mathcal{O}_\ell + \{x \in D_\ell \mid \nu_\ell(x) \in \ell\mathbb{Z}_\ell\} = \mathcal{O}_\ell + \varpi_{D_\ell} \mathcal{O}_{D_\ell},$$

  where $\mathcal{O}_\ell$ is the ring of integers of $\mathbb{Q}_\ell(\varpi_\ell)$.

In order to shorten the notation, we call such a global order a *Pizer order of level $N\ell^2$*, however, we
remark that these orders are a type of *basic* orders and we point to [Voi21, Remark 24.5.7] for a complete
discussion on the different terminologies for such orders.

Remark 3.2.

- If the order has level $N\ell$, then the local order at ℓ is no longer residually inert, but it is the unique maximal order in D_ℓ. In particular, an order R of level $N\ell^2$ is contained in an Eichler order of level $N\ell$.
- Adding a subscript ℓ at the notation of Section 2, we notice that, by equation (2.1), $R_\ell^\times \supseteq U_{D_\ell}^1$ with quotient $\mathbb{F}_\ell^\times$. The quotient in equation (2.2) recovers the observations in [Piz80b, proofs of Propositions 1.8 and 9.26].

For any prime $q \neq \ell$, we fix a $\mathbb{Q}_q$-linear isomorphism $\iota_q \colon D_q \cong M_2(\mathbb{Q}_q)$; up to changing this isomorphism, we may assume that

$$(3.3) \qquad\qquad \iota_q(R_q) = \left\{ \gamma \in M_2(\mathbb{Z}_q) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N M_2(\mathbb{Z}_q)} \right\}.$$

At the ramified prime ℓ, we assume that $R_\ell = \mathcal{O}_\ell + \varpi_{D_\ell} \mathcal{O}_{D_\ell}$. For the rest of this section, we fix R to be
a Pizer order of level $N\ell^2$, and advise the reader that every time we pick a Pizer order, we are implicitly
assuming the above identifications. We also introduce the following notation

$$(3.4) \quad U_1(R) = \left\{ r = (r_q) \in \widehat{R}^\times \;\middle|\; \iota_q(r_q) \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N M_2(\mathbb{Z}_q)}, \text{ for } q \mid N \text{ and } r_\ell \in 1 + \varpi_{D_\ell} R_\ell \right\}.$$

It is not difficult to notice that it is an open compact subgroup of $\widehat{B}^\times$ (cf. also [Dal23b, Lemma 2.1.3]).

3.3. **Lifting characters.** Let χ be a Dirichlet character of conductor C, for $C \mid N\ell$. Every such
character can be lifted to a character of $\widehat{R}^\times$; for simplicity we consider the case of ℓ odd. A similar

construction works for $\ell = 2$ and a precise recipe is provided in [HPS89a, Section 7.2] as we focus on the odd case. Let q be a prime and let χ_q be the q-component of the character χ. We define the lift $\widetilde{\chi}_q$ of χ_q to $R_q^\times$ as follows:

(1)   If $q \mid N\ell$, but $q \nmid C$, we set $\widetilde{\chi}_q(r) = 1$ for any $r \in R_q^\times$.

(2)   If $q \mid N$ and $q \mid C$, we set $\widetilde{\chi}_q(r) = \chi_q(d)$ for any $r \in R_q^\times$ such that $\iota_q(r) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$;

(3)   If $q = \ell > 2$ and $\ell \mid C$, we fix, once and for all, an odd character ε_ℓ with $\mathrm{cond}(\varepsilon_\ell) = \ell$. For every even character ϕ_ℓ of $\mathrm{cond}(\phi_\ell) = \ell$, we fix, once and for all, a character γ_ℓ with $\mathrm{cond}(\gamma_\ell) = \ell$ and such that $\gamma_\ell^2 = \phi_\ell$. As remarked in [HPS89a, Section 7.2], the particular choice of ε_ℓ and γ_ℓ is not important, but the fact that a certain choice is fixed once and for all is crucial.

    (a)   We first extend ε_ℓ to $R_\ell^\times$ via the composition:

$$
\begin{array}{ccccc}
\mathcal{O}_\ell^\times & \longrightarrow & (\mathcal{O}_\ell/\varpi_{D_\ell}\mathcal{O}_\ell)^\times & \xrightarrow{\;\cong\;} & (\mathbb{Z}_\ell/\ell\mathbb{Z}_\ell)^\times \\
\downarrow & & & & \downarrow{\scriptstyle\varepsilon_\ell} \\
R_\ell^\times = (\mathcal{O}_\ell + \varpi_{D_\ell}\mathcal{O}_{D_\ell})^\times & & \xrightarrow{\quad\widetilde{\varepsilon}_\ell\quad} & & \overline{\mathbb{Q}}_p^\times.
\end{array}
$$

    (b)   If χ_ℓ is even, let γ_ℓ be the *square root* character associated with it, as above. We define $\widetilde{\chi}_\ell(r) = \gamma_\ell(\nu_\ell(r))$, for any $r \in R_\ell^\times$. Clearly, $\widetilde{\chi}_\ell(r) = \widetilde{\chi}_\ell(r')$ if $r \equiv r' \pmod{\varpi_{D_\ell}\mathcal{O}_{D_\ell}}$.

    (c)   If χ_ℓ is odd, then $\chi_\ell = \varepsilon_\ell \cdot \chi'_\ell$, with χ'_ℓ even. We then define $\widetilde{\chi}_\ell = \widetilde{\varepsilon}_\ell \widetilde{\chi'_\ell}$.

Definition 3.3. We denote the lift of χ to $\widehat{R}^\times$ by $\widetilde{\chi}$, defined as $\widetilde{\chi} = \prod_{q \mid C} \widetilde{\chi}_q$.

Recall that we can define the adèlization of χ,

$$
(3.5) \qquad\qquad \chi_{\mathbb{A}} : \mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_+ (1 + N\widehat{\mathbb{Z}})^\times \longrightarrow \mathbb{C}^\times,
$$

as the unique finite order Hecke character such that $\chi_{\mathbb{A}}((1,\ldots,1,q,1,\ldots)) = \chi(q)^{-1}$; then $\chi_q(q) = \chi(q)^{-1}$ if $q \nmid N$.

Remark 3.4.

(1) The restriction of $\widetilde{\chi}$ to $\mathbb{Z}$ recovers the starting character χ. At any prime q away from ℓ, the lifting process of each local component χ_q consists exactly in the adèlization of its inverse, $(\chi_q^{-1})_{\mathbb{A}}$.

(2) The lifting process can be constructed compatibly with the inclusion of Pizer orders $R \supseteq R'$, with R' Pizer order of level $N'\ell^2$, $N \mid N'$.

(3) By construction, the character $\widetilde{\chi}$ is trivial on $U_1(R)$. More precisely, it is a character of $(\widehat{R}/\delta(R))^\times$, for $\delta(R)$ the different ideal of $\widehat{R}$ (cf. [HPS89a, Sections 7.1-7.2]).

3.4. Forms on definite quaternion algebras.

In this section we recall the various notions of quaternionic forms and their explicit relations. We fix an odd prime $p \neq \ell$ and an absolute closure of $\mathbb{Q}_p$, which we denote by $\overline{\mathbb{Q}_p}$.

3.4.1. *Quaternionic modular forms.* Let A be a commutative ring and consider the space of polynomials in two variables, $A[X,Y]$. We endow this module with the action of invertible matrices $GL_2(A)$ given by

$$
(3.6) \qquad\qquad \gamma \cdot P(X,Y) = P((X,Y) \cdot \gamma),
$$

for any $\gamma \in GL_2(A)$ and $P \in A[X,Y]$. For any $m \in \mathbb{Z}_{\geq 0}$ we define the submodule of homogeneous polynomials of degree m and denote it by $L_m(A)$; the $GL_2(A)$-action descends to an action on $L_m(A)$. In the following, the ring A will be an algebra over the ring of integers of a finite extension of $\mathbb{Q}_p$, for example, $\overline{\mathbb{Q}_p}$.

Definition 3.5 (Quaternionic modular forms)**.** Let R be a Pizer order of level $N\ell^2$, and assume that $p \mid N$. Let A be a $\mathbb{Z}_p$-algebra and fix an A-valued Dirichlet character χ with conductor $C \mid N\ell$. A

quaternionic modular form for $D^\times$, of weight $k \in \mathbb{Z}_{\geq 2}$, level structure R, and character χ, is a continuous function $\varphi : \widehat{D}^\times \longrightarrow L_{k-2}(A)$, such that

$$\varphi(d\hat{d}zr) = (\chi_\mathbb{A})^{-1}(z)\,\widetilde{\chi}(r)\,z_p^{2-k}\,(r_p^{-1} \cdot \varphi(\hat{d})),$$

for $d \in D^\times$, $\hat{d} \in \widehat{D}^\times$, $z \in \mathbb{A}_f^\times$, and $r \in \widehat{R}^\times$. Here $\widetilde{\chi}$ is the lifting constructed in Definition 3.3. We denote the space of quaternionic modular forms by $S_k^D(R, \chi, A)$.

Note that, for $z \in \mathbb{A}_f^\times \cap \widehat{R}^\times$, $\widetilde{\chi}(z) = (\chi_\mathbb{A})^{-1}(z)$, and that φ is right-invariant under $U_1(R)^{(p)}$, i.e. away from p. The space $S_k^D(R, \chi, A)$ inherits a right-action of $GL_2(\mathbb{Z}_p)$ given by $\varphi_{|\gamma}(\hat{d}) = \gamma^{-1} \cdot \varphi(\hat{d})$.

As defined, quaternionic modular forms are not the *classical* quaternionic modular forms considered in [HPS89a], which are (non-unitarized) automorphic forms on D. In order to clarify the situation, as well as making explicit certain actions on the space of quaternionic modular forms (see Section 3.5), we recall the definition of automorphic forms and algebraic quaternionic modular forms, highlighting their relations with quaternionic modular forms.

3.4.2. *Automorphic forms on $D^\times$.* Fix an isomorphism $\iota : \overline{\mathbb{Q}_p} \cong \mathbb{C}$ and let $\Psi_\infty : D_\infty = D \otimes_\mathbb{Q} \mathbb{R} \hookrightarrow M_2(\mathbb{C})$ be an embedding such that $\iota(\iota_p(d)) = \Psi_\infty(d)$, for any $d \in D^\times \subset D_p^\times \overset{\iota_p}{\cong} GL_2(\mathbb{Q}_p)$. For any $k \geq 2$, let

$$(3.7) \qquad\qquad \rho_{\infty,k}^u : D_\infty^\times \longrightarrow \mathrm{Aut}(L_{k-2}(\mathbb{C}))$$

be the unitarized representation defined by

$$(3.8) \qquad\qquad \rho_{\infty,k}^u(d_\infty)(P(X,Y)) = |\nu_{D_\infty}(d_\infty)|^{\frac{k-2}{2}}\,\Psi_\infty(d_\infty) \cdot P(X,Y).$$

Definition 3.6 (Automorphic forms on $D^\times$). Let R and χ be as above. We define the space $\mathcal{A}_k^D(R, \chi)$ of automorphic forms on $D^\times$, of weight k, level R, and character χ, as the vector space of continuous functions $\phi : D(\mathbb{A})^\times \longrightarrow L_{k-2}(\mathbb{C})$, such that

$$\phi(dxd_\infty rz) = (\chi_\mathbb{A})^{-1}(z)\,\widetilde{\chi}(r)\,\rho_{\infty,k}^u(d_\infty^{-1})\,(\phi(x)),$$

for $d \in D^\times$, $d_\infty \in D_\infty^\times$, $z \in \mathbb{A}^\times$, and $r \in \widehat{R}^\times$.

There is a correspondence between quaternionic modular forms and automorphic forms. Precisely, to each $\varphi \in S_k^D(R, \chi)$ we associate the automorphic form $\Phi(\varphi) \in \mathcal{A}_k^D(R, \chi)$ defined as

$$(3.9) \qquad\qquad \Phi(\varphi)(d) = \rho_{\infty,k}^u(d_\infty^{-1})\Big(\iota\big(d_p \cdot \varphi(d_f)\big)\Big),$$

for any $d \in D(\mathbb{A})^\times$, with finite part $d_f \in \widehat{D}^\times$, and components d_p and d_∞, respectively at p and infinity.

3.4.3. *Algebraic quaternionic modular forms.* In order to construct the p-adic L-function in the following Section 5, we need Hida families of quaternionic modular forms. We define them in Section 4.2, similarly to [Hsi21, Definition 4.1], and for this purpose we should introduce algebraic quaternionic modular forms.

Consider the Iwasawa algebra $\Lambda = \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$. For any $z \in 1 + p\mathbb{Z}_p$, we denote by $[z]_\Lambda$ the group-like element of Λ determined by z. For $r \geq 1$, let $R_{N\ell^2 p^r}$ be a sequence of nested Pizer orders on level $N\ell^2 p^r$ contained in the nested sequence of Eichler order R_{Np^r}. We consider the finite set

$$(3.10) \qquad\qquad X_r = D^\times \backslash \widehat{D}^\times / U_1(R_{N\ell^2 p^r}),$$

and let $\mathcal{O}[X_r]$ be the finitely generated $\mathcal{O}$-module spanned by divisors on X_r. The Iwasawa algebra Λ acts $\mathbb{Z}_p$-linearly on $\mathcal{O}[X_r]$ via

$$(3.11) \qquad\qquad [z]_\Lambda \cdot x = x\,(1,\dots,1,(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}),1,\dots),$$

for $x \in X_r$, $z \in 1 + p\mathbb{Z}_p$ and $(1,\dots,1,(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}),1,\dots)$ the adèle with p-th component $(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix})$.

Definition 3.7 (Algebraic quaternionic modular forms)**.** Let χ be a Dirichlet character of conductor dividing $N\ell p^r$. We define the space of *algebraic quaternionic modular forms* of level $R_{N\ell^2 p^r}$, weight k and character χ as the finite Λ-module

$$\mathbb{S}_k^D(R_{N\ell^2 p^r}, \chi) = \left\{ f \in \mathrm{Hom}_{\mathrm{cts}}(\mathcal{O}[X_r], L_{k-2}(\overline{\mathbb{Q}_p})) \;\middle|\; \begin{array}{c} f(xzr) = (\chi_{\mathbb{A}})^{-1}(z)\widetilde{\chi}(r)f(x) \\ \text{for all } z \in \mathbb{A}_f^\times,\, r \in \widehat{R}_{N\ell^2 p^r}^\times \end{array} \right\}.$$

Recall that any $\varphi \in S_k^D(R_{N\ell^2 p^r}, \chi)$ is not $U_1(R_{N\ell^2 p^r})$-invariant but only $U_1(R_{N\ell^2 p^r})^{(p)}$-invariant. The relation between $\mathbb{S}_k^D(R_{N\ell^2 p^r}, \chi)$ and $S_k^D(R_{N\ell^2 p^r}, \chi)$ is made clear via the association

$$(3.12) \qquad S_k^D(R_{N\ell^2 p^r}, \chi) \ni \varphi(x) \longmapsto f(x) = x_p \cdot \varphi(x) \in \mathbb{S}_k^D(R_{N\ell^2 p^r}, \chi).$$

Remark 3.8. We will define Hecke operators on $\mathbb{S}_k^D(R_{N\ell^2 p^r}, \chi)$ momentarily (cf. Section 3.5.1). For now, let us mention that the Λ-action on the space of quaternionic modular forms $S_k^D(R_{N\ell^2 p^r}, \chi)$ takes the form of

$$(3.13) \qquad [z]_\Lambda \cdot \varphi(x) = z \cdot \varphi(x(1,\ldots,1,(\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}),1,\ldots)) = (\chi_{\mathbb{A},p})^{-1}(z)\,\varphi(x),$$

In other words, we can think of the Λ-action as a diamond operator at the prime p.

3.5. Operators on quaternionic modular forms.

3.5.1. *Hecke operators.* As in the usual setting of classical modular forms, one can define Hecke operators acting on the space of quaternionic modular forms. These are easily defined taking into account the adèlic description of these forms; for any prime $q \neq \ell$, let $\varsigma_q \in \mathbb{A}_f^\times$ be the element characterized by $\varsigma_{q,q} = q$ and 1 at other places. Let A be again a $\mathbb{Z}_p$-algebra and let $\varphi \in S_k^D(R_{N\ell^2}, \chi, A)$. For any $\hat{d} \in \widehat{D}^\times$, the Hecke operator T_q is determined by

$$(3.14) \qquad T_q \varphi(\hat{d}) = \begin{cases} \varphi\left(\hat{d}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varsigma_q \end{smallmatrix}\right)\right) + \displaystyle\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi\left(\hat{d}\left(\begin{smallmatrix} \varsigma_q & a \\ 0 & 1 \end{smallmatrix}\right)\right) & \text{for each prime } q \nmid N\ell^2, \\[3ex] \varphi|_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)}\left(\hat{d}\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varsigma_p \end{smallmatrix}\right)\right) + \displaystyle\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi|_{\left(\begin{smallmatrix} p & a \\ 0 & 1 \end{smallmatrix}\right)}\left(\hat{d}\left(\begin{smallmatrix} \varsigma_p & a \\ 0 & 1 \end{smallmatrix}\right)\right) & \text{for } q = p \text{ and } p \nmid N, \end{cases}$$

while the Hecke operator U_q is

$$(3.15) \qquad U_q \varphi(\hat{d}) = \begin{cases} \displaystyle\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \varphi\left(\hat{d}\left(\begin{smallmatrix} \varsigma_q & a \\ 0 & 1 \end{smallmatrix}\right)\right) & \text{for } q \mid N, \\[3ex] \displaystyle\sum_{a \in \mathbb{Z}/p\mathbb{Z}} \varphi|_{\left(\begin{smallmatrix} p & a \\ 0 & 1 \end{smallmatrix}\right)}\left(\hat{d}\left(\begin{smallmatrix} \varsigma_p & a \\ 0 & 1 \end{smallmatrix}\right)\right) & \text{for } q = p \text{ and } p \mid N. \end{cases}$$

Remark 3.9.

(1) In the case of a definite quaternion algebra, one can express the action of quaternionic Hecke operators via Brandt matrices. Via these matrices we can also define Hecke operators for the primes of ramification for the quaternion algebra. As Hecke operators are compatible with the Jacquet–Langlands correspondence (up to a sign at ℓ), applied to forms which are new at ℓ, U_ℓ will be the zero-operator. We refer the interested reader to [Eic73], [Piz77], [Piz80b] and [HPS89b] for their precise definition.

(2) We define an Atkin–Lehner operator in Definition 3.18, following [Piz80b]. One can find quaternionic analogues of the local Atkin–Lehner operators in Section 9 of loc. cit. and [Piz77]. For what concerns diamond operators, they can be defined as in [Hsi21, Section 4.4].

(3) The first-named author wishes to remark that the definition of the U_ℓ operator provided in [Dal23b] is not correctly stated. As above, one should use Brandt matrices to define it and not the involution obtained by multiplication with an element of norm ℓ in $D_\ell^\times$. This definition is correct for Eichler orders, as it coincides (up to a sign) with the Atkin–Lehner operator. The proofs and results contained in loc. cit. are unfazed by this misstated definition.

3.5.2. *Multiplicity two.* A peculiar property of Pizer orders of level $N\ell^2$ is the failure of multiplicity one for Hecke-eigenspaces of quaternionic modular forms. This phenomenon has been extensively studied in both [Piz80b] and [HPS89b], and it can be summarized in the following proposition.

Proposition 3.10 (Pizer, Hijikata–Pizer–Shemanske)**.** *Let $f \in S_k(\Gamma_1(N\ell^2), \chi)^{\mathrm{new}}$ be a Hecke-eigenform, twist-minimal at ℓ. Suppose that the ℓ-component of χ has conductor $\mathrm{cond}(\chi_\ell) \leq \ell$. There is an isomorphism of Hecke-eigenspaces*

$$S_k^D(R_{N\ell^2}, \chi, \mathbb{Q}_p(f))[f] = S_k(N\ell^2, \chi)[f]^{\oplus 2}.$$

In particular, the Hecke-eigenspace $S_k^D(R_{N\ell^2}, \chi, \mathbb{Q}_p(f))[f]$ is two-dimensional.

Remark 3.11. The Jacquet–Langlands correspondence associates to any automorphic representation π_D of $D^\times(\mathbb{A})$ an automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. By Strong Multiplicity One for GL_2 and its inner forms, there cannot be two different automorphic representations π_D and π_D' of $D^\times(\mathbb{A})$ that correspond to the same automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. Proposition 3.10 is therefore about automorphic forms, i.e. elements of π_D and π.

For $\mathrm{GL}_2(\mathbb{A})$, there is a well-known theory of newforms: there is a compact open subgroup $K_\pi \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ such that $\dim \pi_f^{K_\pi} = 1$. This is no longer the case for $D^\times(\mathbb{A})$; in fact, underlying Proposition 3.10 is the fact that $\dim \pi_{D,f}^{R_{N\ell^2}^\times} = 2$, which was observed locally at the prime ℓ in Proposition 2.5.

3.5.3. *An extra operator.* Using the methods of Section 2, we can define an extra operator acting on the space of A-valued quaternionic modular forms $S_k^D(R_{N\ell^2}, \chi, A)$; its nature is local, arising from the structure of the order at the prime of ramification ℓ.

With a slight change of notation from Section 2, let ϖ_D be the element in $\widehat{D}^\times$ defined as

$$(3.16) \qquad\qquad \varpi_D = (1, \ldots, 1, \varpi_{D_\ell}, 1, \ldots),$$

for ϖ_{D_ℓ} a quaternionic uniformizer of $D_\ell/\mathbb{Q}_\ell$; note that $N_{\mathbb{Q}_\ell(\varpi_{D_\ell})/\mathbb{Q}_\ell}(\varpi_{D_\ell}) = \varpi_{\mathbb{Q}_\ell}$ for a uniformizer of $\mathbb{Q}_\ell$. We will choose ϖ_{D_ℓ} so that $\varpi_{D_\ell}^2 = \ell$. Equivalently, we are fixing our chosen ramified quadratic extension $\mathbb{Q}_\ell(\varpi_{D_\ell})$ to be $\mathbb{Q}_\ell(\sqrt{\ell})$. Notice that one can fix another choice of uniformizer and, mutatis mutandis, all the results in this section and in Section 4 are similarly obtained.

Definition 3.12. We define the operator $\langle \varpi_{D_\ell} \rangle$ on the space $S_k^D(R_{N\ell^2}, \chi, A)$ as

$$\langle \varpi_{D_\ell} \rangle \varphi(\hat{d}) = \varphi\left(\hat{d}\varpi_D \right).$$

Proposition 3.13.

(1) *Iterating $\langle \varpi_{D_\ell} \rangle$ twice, one has*

$$\langle \varpi_{D_\ell} \rangle^2 \varphi(\hat{d}) = \varphi\left(\hat{d}\left(1, \ldots, 1, \varpi_{D,\ell}^2, 1, \ldots\right) \right) = \chi_N(\ell)^{-1}\varphi(\hat{d}).$$

(2) *The operator $\langle \varpi_{D_\ell} \rangle$ commutes with the Hecke-operators.*

Proof. We notice that $(1, \ldots, 1, \ell, 1, \ldots) \in \mathbb{A}_f^\times$, and

$$\chi_{\mathbb{A}}^{-1}(\ell) = 1 = \chi_{\ell,\mathbb{A}}^{-1}\left((1, \ldots, 1, \ell, 1, \ldots) \right) \chi_{N,\mathbb{A}}^{-1}\left((\ell, \ldots, \ell, 1, \ell, \ldots) \right),$$

hence

$$\chi_{\ell,\mathbb{A}}^{-1}\left((1, \ldots, 1, \ell, 1, \ldots) \right) = \chi_{N,\mathbb{A}}\left((\ell, \ldots, \ell, 1, \ell, \ldots) \right) = \chi_N^{-1}(\ell).$$

Both statements follow now from the definitions of $\langle \varpi_{D_\ell} \rangle$, quaternionic modular forms, and Hecke operators away from ℓ. ∎

The choice of the notation $\langle \varpi_{D_\ell} \rangle$ is justified by the above proposition; this extra operator is closer to a diamond operator than a replacement for the Hecke operator at ℓ. Let us remark that χ_N^{-1} evaluated at ℓ coincides with the evaluation at the same integer of the adèlization of the character χ_ℓ; this follows by the definition of the adèlization as the rational elements not prime to the conductor are mapped to 1.

Proposition 3.14. *Let $f \in S_k(\Gamma_1(N\ell^2), \chi)^{\mathrm{new}}$ be a Hecke-eigenform, twist-minimal at ℓ. As above, suppose that the ℓ-component of χ has conductor $\mathrm{cond}(\chi_\ell) \leq \ell$. The two-dimensional Hecke-eigenspace $S_k^D(R_{N\ell^2}, \chi, A)[f]$ decomposes into one-dimensional eigenspaces under the action of $\langle \varpi_{D_\ell} \rangle$,*

$$S_k^D(R_{N\ell^2}, \chi, A)[f] = S_k^D(R_{N\ell^2}, \chi, A)[f]^{\langle \varpi_{D_\ell} \rangle = +\sqrt{\chi_N^{-1}(\ell)}} \oplus S_k^D(R_{N\ell^2}, \chi, A)[f]^{\langle \varpi_{D_\ell} \rangle = -\sqrt{\chi_N^{-1}(\ell)}}.$$

Proof. Let π_f be the automorphic representation associated with f and consider $\pi_{f,\ell}$ its local component at ℓ. By definition of automorphic forms, the quaternionic modular forms in $S_k^D(R_{N\ell^2}, \chi, A)[f]$ correspond to (semi-)invariant vectors in the automorphic representation π_f^D, obtained as the Jacquet–Langlands transfer of π_f. In other words, we can identify $S_k^D(R_{N\ell^2}, \chi, A)[f]$ with the space $(\pi_f^D)^{R^\times}$ of (semi-)invariant vectors. By definition, the action of the operator $\langle \varpi_{D_\ell} \rangle$ is local on the representation $\pi_{f,\ell}^D$ and it acts as the local uniformizer ϖ_{D_ℓ}. Since the representation $\pi_{f,\ell}^D$ is a twist-minimal smooth representation of $D_\ell^\times$ of conductor 2, we can apply Proposition 2.5 and conclude the proof. Notice that the central character of π_f is (up to the archimedean component) the adèlization of the character χ, thus, the central character of $\pi_{f,\ell}$ at ℓ coincides with $\chi_{\mathbb{A},\ell}((1,\ldots,1,\ell,1,\ldots)) = \chi_N^{-1}(\ell)$. ∎

Remark 3.15.

(1) The result of Lemma 3.14 depends only on the local representation at ℓ, more precisely, on its automorphic type and its *minimal* conductor.

(2) The local automorphic type, the conductor and the twist-minimality property are rigid in p-adic Hida families of modular forms (see [FO12, Lemma 2.14] and [Hsi21, Remark 3.1]).

(3) If one considers more general quaternion algebras, the multiplicity will be given by

$$\prod_{\substack{\ell \in \mathrm{Ram}(D) - \{\infty\}: \\ \mathrm{cond}_\ell(\pi) = \ell^2}} 2.$$

Therefore, considering a $\langle \varpi_{D_\ell} \rangle$-operators for each $\ell \in \mathrm{Ram}(D) - \{\infty\}$, one can recover multiplicity one for quaternionic modular forms.

(4) In the case $\chi = 1$, one can define a whole set of operators, defining a dihedral group of order $2(\ell + 1)$, as in [Piz80b, Section 9]. Among these, by our analysis in Section 2, one can recognize the operator $\langle \varpi_{D_\ell} \rangle$, the Atkin–Lehner operator, and a few mysterious others arising from the local structure of the order. The indefinite setting has been studied in [dVP13].

Our analysis is partially motivated by the explicit work of Pizer [Piz80b], where the study of Pizer orders is addressed with a more elementary flavor; in particular, the above lemma provides a more explicit take on the conductor ℓ^2 case of the statements contained in [HPS89b, Section 9]. Moreover, if the central character is trivial, we recover the setting studied by Pizer. Even though the authors of loc. cit. observed and studied the higher multiplicity phenomenon without addressing how to recover one-dimensional eigenspaces, their explicit work has been a key input for our results. It also helped us to implement the operator $\langle \varpi_{D_\ell} \rangle$ in Pizer's setting in magma [BCP97]. Our implementation can be found at [1] and we use it to compute a first example below.

Example 3.16. With the above notations, take $\ell = 7$ and let $f \in S_2(\Gamma_0(7^2))^{\text{new}}$, be the unique newform of level of level 7^2 and trivial character. In [HPS89b, Example 10.5], the authors compute that the f-eigenspace of quaternionic modular forms of weight 2, level 7^2 and trivial character, is generated by two forms, namely φ_1 and φ_2, whose values on the ideal classes are

$$\varphi_1 \longleftrightarrow (1, -1, 0, 0)^t, \qquad\qquad \varphi_2 \longleftrightarrow (0, 0, 1, -1)^t.$$

Since the character is trivial, the eigenvalues of the operator $\langle \varpi_{D_\ell} \rangle$ are ± 1. Our implementation in `magma` yields the action of the $\langle \varpi_{D_\ell} \rangle$-operator as

$$\langle \varpi_{D_7} \rangle \varphi_1 = \varphi_1, \qquad\qquad \langle \varpi_{D_7} \rangle \varphi_2 = -\varphi_2.$$

Thus, we decompose

$$S_2^D(R_{7^2})^{\text{new}}[f] = \left(S_2^D(R_{7^2})^{\text{new}}[f]\right)^{\langle \varpi_{D_7} \rangle = +1} \oplus \left(S_2^D(R_{7^2})^{\text{new}}[f]\right)^{\langle \varpi_{D_7} \rangle = -1},$$

for $\left(S_2^D(R_{7^2})^{\text{new}}[f]\right)^{\langle \varpi_{D_7} \rangle = +1} = \langle \varphi_1 \rangle$ and $\left(S_2^D(R_{7^2})^{\text{new}}[f]\right)^{\langle \varpi_{D_7} \rangle = -1} = \langle \varphi_2 \rangle$.

For completeness, we conclude this section with an observation on the action of the operator $\langle \varpi_{D_\ell} \rangle$ on forms which are ℓ-new but *not* twist-minimal at the same prime.

Lemma 3.17. *The $\langle \varpi_{D_\ell} \rangle$-eigenvalue on the space $S_k^D(Np^n\ell^2, \chi)$ is $\pm\sqrt{\chi_{\mathbb{A}, Np^n}(\ell)^{-1}}$.*

Proof. Once again, looking at the local description. If π_ℓ is an admissible irreducible representation of $D_\ell^\times$ of conductor 1, then it is (inflated from) a character $\pi_\ell : \langle \varpi_{D_\ell} \rangle \to \mathbb{C}^\times$. In particular, it is determined by $\pi_\ell(\varpi_{D_\ell})$ and once again $\pi_\ell(\varpi_\ell^2) = \pi(\varpi_{\mathbb{Q}_\ell})^2 = \chi(\varpi_{\mathbb{Q}_\ell})^2$. ∎

3.5.4. Λ-*action and operators.* Recall the notation introduced in Section 3.4.3; equation (3.13) shows that the Λ-action commutes with Hecke operators and also with the $\langle \varpi_{D_\ell} \rangle$-operators. Considering the Λ-action on quaternionic modular forms is critical for ensuring the necessary properties of the morphisms between eigenvarieties we construct in Section 4.1. More precisely, we make use of the relation

$$(3.17) \qquad\qquad \langle \varpi_{D_\ell} \rangle^2 \varphi = \chi_{\mathbb{A}, \ell}(\ell) \left[\langle \ell \rangle\right]_\Lambda \cdot \varphi,$$

for $\langle \ell \rangle \in 1 + p\mathbb{Z}_p$ the image of ℓ under the projection $\langle \cdot \rangle : \mathbb{Z}_p^\times \longrightarrow 1 + p\mathbb{Z}_p$.

3.6. **Pairings.** We introduce here the general setting we deal with in Sections 4 and 5. Let D be a definite quaternion algebra over $\mathbb{Q}$; it is ramified at an odd number of primes $\ell \,\|\, \text{disc}(D)$, for $\text{disc}(D)$ the discriminant of D. We denote by R_{N_+, N_-} a fixed Pizer order of level $N_+ N_-$, with $N_- = N_-^{\text{sp}} N_-^{\text{sc}}$, such that

$$(3.18) \qquad (N_+, \text{disc}(D)) = 1, \qquad (N_-^{\text{sp}}, N_-^{\text{sc}}) = 1, \qquad N_-^{\text{sp}} \,\|\, \text{disc}(D), \qquad N_-^{\text{sc}} \,\|\, \text{disc}(D)^2.$$

We will occasionally simplify this notation to $R_N = R_{N_+, N_-}$ where $N = N_+ N_-$. The notation refers to the local factors of the corresponding representations of $\text{GL}_2(\mathbb{A})$: they are *special* at primes dividing N_-^{sp} and *supercuspidal* at primes dividing N_-^{sc}. As discussed above, we need the extra operator $\langle \varpi_{D_\ell} \rangle$ at primes $\ell | N_-^{\text{sc}}$. We remark that $(N_-^{\text{sp}})^2 N_-^{\text{sc}} = \text{disc}\, D^2$. Moreover, we fix χ to be a Dirichlet character of conductor $c \mid N_+ N_-$, and a weight $k \in \mathbb{Z}_{\geq 2}$.

3.6.1. *The quaternionic Petersson product.* We recall here the definition of the Petersson product between quaternionic modular forms; our definition extends the one provided in [Hsi21, Section 4.2]. Following [GS20, Section 2.1], we consider the *p-adic norm form* $\mathrm{N}_p : D^\times \backslash \widehat{D}^\times \longrightarrow \mathbb{Z}_p^\times$ given by

$$(3.19) \qquad\qquad \mathrm{N}_p(-) = \left(|\nu_{\mathbb{A}_f}(-)|_{\mathbb{A}_f} \nu_{\mathbb{A}_f}(-)\right)_p,$$

for $(-)_p : \mathbb{A}_f^\times \longrightarrow \mathbb{Q}_p^\times$, the projection to the p-component. In other words, N_p is defined as the p-component of the normalization of the reduced norm by the adèlic absolute value. With the notation of

[Hsi21, Section 4.2], we denote the *p-adic cyclotomic character* by ε_{cyc}; this is the character

$$\varepsilon_{cyc} : \mathbb{Q}_+ \backslash \mathbb{A}_f^\times \longrightarrow \mathbb{Z}_p^\times$$

(3.20)
$$a \mapsto \left(|a|_{\mathbb{A}_f} \, a \right)_p = |a|_{\mathbb{A}_f} \, a_p.$$

Therefore, $N_p = \varepsilon_{cyc} \circ \nu_{\mathbb{A}_f}$, as the quaternion algebra D is definite.

We consider the pairing

(3.21)
$$(-,-)_{N_+,N_-} : S_k^D(R_{N_+,N_-},\chi) \times S_k^D(R_{N_+,N_-},\chi^{-1}) \longrightarrow \overline{\mathbb{Q}_p},$$

given by

(3.22)
$$(\varphi_1,\varphi_2)_{N_+,N_-} = \sum_{[x] \in D^\times \backslash \widehat{D}^\times / \widehat{R}_{N_+,N_-}^\times} \frac{N_p(x)^{k-2}}{\#\Gamma_{N_+,N_-}(x)} \langle \varphi_1(x),\varphi_2(x) \rangle_{k-2},$$

where $\Gamma_{N_+,N_-}(x) = (D^\times \cap x \widehat{R}_{N_+,N_-}^\times x) \, \mathbb{Q}^\times / \mathbb{Q}^\times$ and

(3.23)
$$\langle -,- \rangle_{k-2} : L_{k-2}(\overline{\mathbb{Q}_p}) \times L_{k-2}(\overline{\mathbb{Q}_p}) \longrightarrow \overline{\mathbb{Q}_p}$$

is the paring defined by

(3.24)
$$\langle X^i Y^{k-2-i}, X^j Y^{k-2-j} \rangle_{k-2} = \begin{cases} (-1)^i \binom{k-2}{i}^{-1} & \text{if } i+j=k-2, \\ 0 & \text{if } i+j \neq k-2. \end{cases}$$

For any $g \in \mathrm{GL}_2(\overline{\mathbb{Q}_p})$, let $\bar{g} = \det(g)g^{-1}$ be its image under the canonical involution on $\mathrm{GL}_2(\overline{\mathbb{Q}_p})$. Given such g and any $P, Q \in L_{k-2}(\overline{\mathbb{Q}_p})$, we have:

(3.25)
$$\langle g \cdot P, Q \rangle_{k-2} = \langle P, \bar{g} \cdot Q \rangle_{k-2}.$$

That implies that the pairing $(-,-)_{N_+,N_-}$ is well defined, as

(3.26)
$$\begin{aligned} N_p(xr)^{k-2} \langle \varphi_1(xr),\varphi_2(xr) \rangle_{k-2} &= N_p(x)^{k-2} \nu_p(r_p)^{k-2} \langle r_p^{-1} \cdot \varphi_1(x), r_p^{-1} \cdot \varphi_2(x) \rangle_{k-2} \\ &= \nu_p(r_p)^{k-2} \nu_p(r_p^{-1})^{k-2} N_p(x)^{k-2} \langle \varphi_1(x),\varphi_2(x) \rangle_{k-2} \\ &= N_p(x)^{k-2} \langle \varphi_1(x),\varphi_2(x) \rangle_{k-2}, \end{aligned}$$

for $r \in \widehat{R}_{N_+,N_-}^\times$ and $[x] \in D^\times \backslash \widehat{D}^\times / \widehat{R}_{N_+,N_-}^\times$.

Definition 3.18. With the notation of Section 3.1 (cf. also [Piz80b, Section 9]), let $\tau_{N_+,N_-}^D \in \widehat{D}^\times$ the element determined by

$$\tau_{N_+,N_-,q}^D = \begin{cases} 1 & \text{if } q \nmid N_+N_-, \\ \iota_q^{-1}\left(\begin{pmatrix} 0 & 1 \\ -q^{v_q(N_+N_-)} & 0 \end{pmatrix} \right) & \text{if } q \mid N_+, \\ 1 & \text{if } q \mid N_-^{\mathrm{sp}}, \\ \varpi_{\mathbb{Q}_{\ell^2}} & \text{if } q = \ell \mid N_-^{\mathrm{sc}}. \end{cases}$$

We define the Atkin–Lehner operator

$$[\tau_{N_+,N_-}^D] : S_k^D(R_{N_+,N_-},\chi) \longrightarrow S_k^D(R_{N_+,N_-},\chi^{-1}),$$

as

$$[\tau_{N_+,N_-}^D] \varphi(x) = \tau_{N_+,N_-,p}^D \cdot \varphi(x\tau_{N_+,N_-}^D) \, \chi_{\mathbb{A}}(\nu(x)).$$

Lemma 3.19. *The Atkin-Lehner operator is a well-defined involution.*

Proof. It is easy to notice that the level of $[\tau^D_{N_+,N_-}]\varphi$ is still R_{N_+,N_-}, but we need to check that its character is χ^{-1}. Given $r \in \widehat{R}^\times_{N_+,N_-}$, we compute

$$([\tau^D_{N_+,N_-}]\varphi)(xr) = \tau^D_{N_+,N_-,p} \cdot \varphi(x\,r\,\tau^D_{N_+,N_-})\,\chi_\mathbb{A}(\nu(x))\,\chi_\mathbb{A}(\nu(r)).$$

We must only check the primes $\ell \mid N^{\mathrm{sc}}_-$, as the local behavior at the other primes is the one of the usual Atkin–Lehner operator. Let ℓ be such a prime. With the formalism of Section 3.1, we note that, for $r = x + \varpi_{\mathbb{Q}_{\ell^2}}y + \varpi_{D_\ell}z \in R^\times_\ell$, with $x,y \in \mathbb{Z}_\ell$ and $z \in \mathcal{O}_{D_\ell}$,

$$r\varpi_{\mathbb{Q}_{\ell^2}} = x\varpi_{\mathbb{Q}_{\ell^2}} + \varpi_{\mathbb{Q}_{\ell^2}}y\varpi_{\mathbb{Q}_{\ell^2}} + \varpi_{D_\ell}z\varpi_{\mathbb{Q}_{\ell^2}} = \varpi_{\mathbb{Q}_{\ell^2}}(x - y\varpi_{\mathbb{Q}_{\ell^2}}) + \varpi_{D_\ell}z\varpi_{\mathbb{Q}_{\ell^2}}.$$

Therefore, $r\varpi_{\mathbb{Q}_{\ell^2}} \equiv \varpi_{\mathbb{Q}_{\ell^2}}r \pmod{\varpi_{D_\ell}\mathcal{O}_{D_\ell}}$, hence $\widetilde{\chi}_\ell(r\,\varpi_{\mathbb{Q}_{\ell^2}}) = \widetilde{\chi}_\ell(\varpi_{\mathbb{Q}_{\ell^2}}r)$. As

$$\varpi_{\mathbb{Q}_{\ell^2}}R_{N_+,N_-,\ell} = R_{N_+,N_-,\ell}\,\varpi_{\mathbb{Q}_{\ell^2}}$$

we deduce that the operator is well defined, and inverting the character. We conclude noticing that $[\tau^D_{N_+,N_-}][\tau^D_{N_+,N_-}]\,\varphi(x) = (\tau^D_{N_+,N_-,p})^2 \cdot \varphi\left(x(\tau^D_{N_+,N_-})^2\right)$, and

$$(\tau^D_{N_+,N_-,q})^2 = \begin{cases} 1 & \text{if } q \nmid N_+N_-, \\ -q^{v_q(N_+N_-)} & \text{if } q \mid N_+, \\ 1 & \text{if } q \mid N^{\mathrm{sp}}_-, \\ \varpi^2_{\mathbb{Q}_{\ell^2}} & \text{if } q = \ell \mid N^{\mathrm{sc}}_-. \end{cases}$$

Therefore, $(\tau^D_{N_+,N_-,q})^2 \in \mathbb{Z}_q$, for each q.   ∎

Proposition 3.20. *The pairing*

$$\langle-,-\rangle_{N_+,N_-} : S^D_k(R_{N_+,N_-},\chi) \times S^D_k(R_{N_+,N_-},\chi) \longrightarrow \overline{\mathbb{Q}_p}$$

defined by $\langle\varphi_1,\varphi_2\rangle_{N_+,N_-} = (\varphi_1,[\tau^D_{N_+,N_-}]\varphi_2)_{N_+,N_-}$ is perfect and equivariant for both the action of the Hecke operators and of $\langle\varpi_{D_\ell}\rangle$.

Proof. The Hecke-equivariance follows exactly as in [Hsi21, Section 4.2] from [Hid06, Lemma 3.5]. Since $\ell^2 \| \mathrm{disc}(D)$, it is not difficult to compute that $\Gamma_{N_+,N_-}(x) = \Gamma_{N_+,N_-}(x\,\varpi_{D_\ell})$. Writing temporarily $X = D^\times\backslash\widehat{D}^\times/\widehat{R}^\times_{N_+,N_-}$, we have that:

$$(\langle\varpi_{D_\ell}\rangle\,\varphi_1,\varphi_2)_{N_+,N_-} = \sum_{[x]\in X}\frac{\mathrm{N}_p(x)^{k-2}}{\#\Gamma_{N_+,N_-}(x)}\langle\varphi_1(x), \tau^D_{N_+,N_-,p}\cdot\varphi(x\tau^D_{N_+,N_-})\chi_\mathbb{A}(\nu(x))_{N_+,N_-}\rangle_{k-2}$$

$$= \sum_{[x]\in X}\frac{\chi^{-1}_\mathbb{A}(\nu(\varpi_{D_\ell}))\,\mathrm{N}_p(x)^{k-2}}{\mathrm{N}_p(\varpi_{D_\ell})^{k-2}\,\#\Gamma_{N_+,N_-}(x)}\langle\varphi_1(x), \tau^D_{N_+,N_-,p}\cdot\varphi(x\varpi^{-1}_{D_\ell}\tau^D_{N_+,N_-})\chi_\mathbb{A}(\nu(x))_{N_+,N_-}\rangle_{k-2}$$

$$= \chi^{-1}_\mathbb{A}(\nu(\varpi_{D_\ell}))\sum_{[x]\in X}\frac{\mathrm{N}_p(x)^{k-2}}{\#\Gamma_{N_+,N_-}(x)}\langle\varphi_1(x), \tau^D_{N_+,N_-,p}\cdot\varphi(x\tau^D_{N_+,N_-,p}\varpi_{D_\ell}\nu(\varpi_{D_\ell})^{-1})\,\chi_\mathbb{A}(\nu(x))_{N_+,N_-}\rangle_{k-2}$$

$$= (\varphi_1,\langle\varpi_{D_\ell}\rangle\,\varphi_2)_{N_+,N_-}$$

where the second equality is obtained by the substitution $x \mapsto x\,\varpi^{-1}_{D_\ell}$, and we have used $\mathrm{N}_p(\varpi_{D_\ell})^{k-2} = 1$ and $\varpi^{-1}_{D_\ell}\tau^D_{N_+,N_-,p} = \tau^D_{N_+,N_-,p}\varpi_{D_\ell}(-\varpi^{-2}_{D_\ell}) = \tau^D_{N_+,N_-,p}\varpi_{D_\ell}\nu(\varpi_{D_\ell})^{-1}$.   ∎

3.6.2. *Automorphic and quaternionic pairings.* Let $\chi_\mathbb{A}$ be the adèlization of χ and R_{N_+,N_-} a Pizer order in D of level N_+N_-, we denote by $\mathcal{A}^D_2(R_{N_+,N_-},\chi_\mathbb{A})$ the space of scalar-valued (i.e. weight 2) automorphic forms of level $\widehat{R}_{N_+,N_-}$ and character $\chi_\mathbb{A}$. For any two such forms $f \in \mathcal{A}^D_2(R_{N_+,N_-},\chi_\mathbb{A})$ and $f' \in \mathcal{A}^D_2(R_{N_+,N_-},\chi^{-1}_\mathbb{A})$, we define the pairing

$$(3.27)\qquad\qquad\qquad \langle f,f'\rangle = \int_{\mathbb{A}^\times D^\times\backslash D(\mathbb{A})^\times} f(x)\,f'(x)\,dx$$

for dx the Tamagawa measure on $\mathbb{A}^\times \backslash D(\mathbb{A})^\times$.

Let $\varphi \in S_k^D(R_{N_+,N_-}, \chi, \overline{\mathbb{Q}}_p)$ be a quaternionic modular form of weight k and character χ, and let u be a polynomial in $L_{k-2}(\mathbb{C})$. We can define the matrix coefficient $\Phi_u(\varphi) \in \mathcal{A}_2^D(R_{N_+,N_-}, \chi_{\mathbb{A}}^{-1})$ as $\Phi_u(\varphi)(x) = \langle \Phi(\phi)(x), u \rangle_{k-2}$, for any $x \in D(\mathbb{A})^\times$ (recall equation (3.9)). The above pairing is related to the one defined in Proposition 3.20 via the following lemma.

Lemma 3.21. *Let φ be as above and consider u and v in $L_{k-2}(\mathbb{C})$. Hence,*

$$\langle [\tau_{N_+,N_-}^D] \, \Phi_u(\varphi), \Phi_v(\varphi) \rangle = \frac{\mathrm{vol}(R_{N_+,N_-}^\times)}{\left(N_+ N_-^{\mathrm{sc}} \right)^{(k-2)/2} (k-1)} \, \langle \varphi, \varphi \rangle_{N_+,N_-} \, \langle u, v \rangle_{k-2}.$$

Before proving the lemma, we remark that if R is an Eichler order of level N_+, the Atkin–Lehner operator becomes $\tau_{N_+,1}^D$, and we recover the formula in [Hsi21, equation (4.5)].

Proof of Lemma 3.21. This is an application of Schur orthogonality relations for linear pairings once we split the integration over $\mathbb{A}^\times D^\times \backslash D(\mathbb{A})^\times = \mathbb{A}_f^\times D^\times \backslash \widehat{D}^\times \times \mathbb{R}^\times \backslash D_\infty^\times$. More precisely, one can proceed as in the proof of [GS20, Lemma 3.2], obtaining

$$\langle \chi_{\mathbb{A}} \, \Phi_u(\varphi), \Phi_v(\varphi) \rangle = \frac{\mathrm{vol}(R_{N_+,N_-}^\times)}{(k-1)} \, \langle \chi_{\mathbb{A}} \, \varphi, \varphi \rangle_{N_+,N_-} \, \langle u, v \rangle_{k-2}.$$

The factor $\nu \left(\tau_{N_+,N_-}^D \right)^{(k-2)/2}$ arises from the definition of the isomorphism f in [GS20, equation (16)]; one must notice that they work with automorphic forms valued in the dual space $(L_{k-2}(\mathbb{C}))^\vee$, so their action at ∞ is obtained by inverting x_∞. Expressing everything under our definitions and our choices of normalizations we obtain the stated equality. We conclude noticing that $\nu \left(\tau_{N_+,N_-}^D \right) = N_+ N_-^{\mathrm{sc}}$. ∎

4. THE JL CORRESPONDENCE IN FAMILIES

As in Section 3.6, we take D and $R_{p^n N_+,N_-}$ for $n \in \mathbb{Z}_{\geq 0}$ and $p \geq 5$ a prime with $(p, N_+ N_-) = 1$, satisfying the conditions in equation (3.18). We also fix a Dirichlet character χ of conductor $c \mid N_+ N_-$ and assume that, for all primes $q \mid N_-$,

$$(4.1) \qquad v_q(c) = v_q(N_-^{\mathrm{sp}}) - 1 = 0, \ \text{if} \ q \mid N_-^{\mathrm{sp}}, \qquad \text{and} \qquad v_q(c) \leq v_q(N_-^{\mathrm{sc}}) - 1 = 1 \ \text{if} \ q \mid N_-^{\mathrm{sc}}.$$

This section is devoted to the study of quaternionic p-adic Hida families, with level structure given by such Pizer orders. Our eventual goal is to study a Jacquet–Langlands transfer in families for Coleman/Hida families on GL_2 of level N. We show that for any collection of signs ϵ_ℓ for each $\ell | N_-^{\mathrm{sc}}$ corresponding to eigenvalues of $\langle \varpi_{D_\ell} \rangle$, there is a Coleman/Hida family on $D^\times$ that transfers to a given Coleman/Hida family on GL_2. To that effect, we study eigenvarieties for the Hecke algebra augmented by the extra operators $\langle \varpi_{D_\ell} \rangle$, and obtain an open immersion of eigenvarieties á la Chenevier [Che04]. The main tool is therefore Chenevier's interpolation technique, adjusted for dealing with this more general setup. We conclude with a control theorem in the spirit of Hida (see Corollary 4.28).

The main reference for this section is Bellaiche's book [Bel21], together with [Buz07], [Che04], and [Lud17]. We try to keep the discussion as brief as possible; we recall only the necessary notions and definitions, and provide references where details can be found.

4.1. **Eigenvarieties.** Let $\mathcal{W}$ denote the weight space, i.e. the rigid analytic space over $\mathbb{Q}_p$ whose $\overline{\mathbb{Q}}_p$-points are

$$(4.2) \qquad \qquad \mathcal{W}(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \overline{\mathbb{Q}}_p^\times).$$

We freely identify any $\overline{\mathbb{Q}}_p$-point $\kappa \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ with the corresponding character of $\mathbb{Z}_p^\times$. The space $\mathcal{W}$ is endowed with an admissible covering $\mathscr{C}$ by admissible affinoid open subsets, as constructed in [Buz07, Section 6]; in loc. cit. the covering is considered on the Fredholm variety, while here we prefer to deal

only with the weight space, as in [Bel21], out of convenience of exposition. For any $X = \mathrm{Sp}(A) \in \mathscr{C}$, we consider the space of cuspidal overconvergent p-adic modular forms as defined in [Buz07, Section 6] and denote it by $\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(X)$ (in loc. cit. $\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(X_n)$ is denoted by M_n, for $X_n \in \mathscr{C}$). This is a Banach A-module and satisfies *Property* (Pr) as in [Bel21, Section 3.1.6]. Similarly, we consider the space of cuspidal overconvergent p-adic quaternionic modular forms as defined in [Buz07, Section 9], and denote it by $\mathbb{S}^{D,\dagger}_{N_+ N_-, \chi}(X)$ (in loc. cit. $\mathbb{S}^{D,\dagger}_{N_+ N_-, \chi}(X_n)$ is denoted by $\mathbf{S}^D_\kappa(U; r)$, for κ and r associated with $X_n \in \mathscr{C}$); once again, this space is a Banach A-module and satisfies *Property* (Pr). In order to keep the discussion of this section brief, we opt to not recall the precise definition of overconvergent automorphic forms and instead refer the interested reader to [Buz07] and [Che04]. Let us remark that this decision will not compromise the proofs of the main results as they rely only on properties and operators on classical automorphic forms.

We are interested in the eigenvectors of $\langle \varpi_{D_\ell} \rangle$, for each $\ell \mid N^{\mathrm{sc}}_-$; therefore, we fix once and for all a square root

$$\sqrt{\chi_{\mathbb{A}, N_+ N_- / \ell^2}(\ell)^{-1}},$$

(4.3)

which we call the *positive* square root of $\chi_{\mathbb{A}, N_+ N_- / \ell^2}(\ell)^{-1}$. We consider the field extension

$$\mathcal{K} = \mathbb{Q}_p \left(\left\{ \sqrt{\chi_{\mathbb{A}, N_+ N_- / \ell^2}(\ell)^{-1}} \right\}_{\ell \mid N^{\mathrm{sc}}_-} \right)$$

(4.4)

and let $\mathcal{O}$ be its ring of integers. Moreover, let $\mathbf{I}' = \mathcal{O}[\![1 + p\mathbb{Z}_p]\!]$ be the corresponding finite extension of the Iwasawa algebra Λ. As we will make use of equation (3.17), we need to extend $\mathbf{I}'$, adjoining the square roots of $[\langle \ell \rangle]_{\mathbf{I}'}$; we set

$$\mathbf{I} = \mathbf{I}' \left[\left\{ \sqrt{[\langle \ell \rangle]_{\mathbf{I}'}} \right\}_{\ell \mid N^{\mathrm{sc}}_-} \right]$$

(4.5)

for the corresponding finite flat extension of Λ. Taking its normal closure in $\mathrm{Frac}(\mathbf{I})$ if needed, we can assume that $\mathbf{I}$ is a normal domain. As above, we fix a compatible choice of square roots

$$\sqrt{[\langle \ell \rangle]_{\mathbf{I}'}}, \text{ for } \ell \mid N^{\mathrm{sc}}_-,$$

(4.6)

which we refer to as the *positive* square roots; we shorten the notation and set

$$\langle \sqrt{\ell} \rangle_{\mathbf{I}} = \sqrt{[\langle \ell \rangle]_{\mathbf{I}'}}, \text{ for } \ell \mid N^{\mathrm{sc}}_-.$$

(4.7)

Let $\mathbb{T}_{\mathbf{I}}$ be the polynomial algebra over $\mathbf{I}$ generated by the Hecke operators T_q for $q \nmid pN_+ N_-$, U_l for $l \mid pN_+ N^{\mathrm{sp}}_-$ and the Diamond operators away form $pN_+ N_-$; it is the Hecke algebra away from $\ell \mid N^{\mathrm{sc}}_-$, and we remark that $\mathbb{T}_{\mathbf{I}}$ is a commutative algebra over $\mathbf{I}$ with a distinguished element, the U_p operator. We extend the Λ-action defined in Section 3.5.4 to an $\mathbf{I}$-action via the group-like elements. Similarly, we consider corresponding $\mathbf{I}$-action on the space of p-adic modular forms and automorphic forms (cf. [Hsi21, Section 2.3]).

Getting back to Banach modules, for any $X = \mathrm{Sp}(A)$ as above, there exist ring homomorphisms,

$$\psi_X : \mathbb{T}_{\mathbf{I}} \longrightarrow \mathrm{End}_A(\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(X)) \qquad \text{and} \qquad \psi^D_X : \mathbb{T}_{\mathbf{I}} \longrightarrow \mathrm{End}_A(\mathbb{S}^{D,\dagger}_{N_+ N_-, \chi}(X)),$$

(4.8)

such that the image of the U_p-operator under each map defines a compact operator (see [Buz07, Section 6, Lemma 12.2]). Moreover, for any $X' = \mathrm{Sp}(A') \in \mathscr{C}$, the A'-modules $\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(X) \widehat{\otimes}_A A'$ and $\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(X')$ (resp. $\mathbb{S}^{D,\dagger}_{N_+ N_-, \chi}(X) \widehat{\otimes}_A A'$ and $\mathbb{S}^{D,\dagger}_{N_+ N_-, \chi}(X')$) are *linked* (see [Bel21, Definition 3.5.1]), as the $\mathbf{I}$-action commutes with Hecke operators (cf. Section 3.5.4).

4.1.1. *Classical eigenvarieties.* By [Bel21, Theorem 3.6.3] (cf. [Che04, Theoreme 6.3.6]), there exist two cuspidal eigenvarieties $\mathscr{E}_{\mathrm{GL}_2}$ and $\mathscr{E}_D$ associated, respectively, with the eigenvariety data

$$(4.9) \quad \mathscr{D}^{\mathrm{GL}_2}_{N_+N_-,\chi} = (\mathcal{W}, \mathbb{T}_{\mathbf{I}}, U_p, \mathbb{S}^{\dagger}_{N_+N_-,\chi}, \psi_{\mathrm{GL}_2}) \quad \text{and} \quad \mathscr{D}^{D}_{N_+N_-,\chi} = (\mathcal{W}, \mathbb{T}_{\mathbf{I}}, U_p, \mathbb{S}^{D,\dagger}_{N_+N_-,\chi}, \psi_D).$$

We recall that the eigenvariety $\mathscr{E}_{\mathrm{GL}_2}$ (resp. $\mathscr{E}_D$) is a rigid analytic space over $\mathcal{K}$ endowed with

- a locally finite morphism $\omega_{\mathrm{GL}_2} : \mathscr{E}_{\mathrm{GL}_2} \longrightarrow \mathcal{W}$ (resp. $\omega_D : \mathscr{E}_D \longrightarrow \mathcal{W}$), called the *weight map*;
- a morphism of rings $\psi_{\mathrm{GL}_2} : \mathbb{T}_{\mathbf{I}} \longrightarrow \mathcal{O}(\mathscr{E}_{\mathrm{GL}_2})$ (resp. $\psi_D : \mathbb{T}_{\mathbf{I}} \longrightarrow \mathcal{O}(\mathscr{E}_D)$), which sends U_p to an invertible function;
- the morphism $\omega_{\mathrm{GL}_2} \times \psi_{\mathrm{GL}_2}(U_p)^{-1} : \mathscr{E}_{\mathrm{GL}_2} \longrightarrow \mathcal{W} \times \mathbb{A}^1_{\mathrm{rig}}$ (resp. $\omega_D \times \psi_D(U_p)^{-1} : \mathscr{E}_D \longrightarrow \mathcal{W} \times \mathbb{A}^1_{\mathrm{rig}}$) is finite ([Bel21, Proposition 3.7.7]).

The general definition of eigenvariety satisfies analogous properties and it can be found in [Bel21, Definition 3.6.2].

Remark 4.1.

(1) By [Bel21, Theorem 3.6.3], there exists an eigenvariety for each eigenvariety data and it is unique up to unique isomorphism.

(2) The eigenvariety construction of $\mathscr{E}_D$ provided in [Buz07] applies in our case, as the adèlization $\widehat{R}^{\times}$ is a compact open subset of $\widehat{D}^{\times}$; cf. [Dal23b, Lemma 2.3].

(3) By [Bel21, Proposition 3.7.7], all the eigenvarieties over $\mathcal{W}$ are separated; moreover, as in [Lud17], it is enough for us to work with the reduced rigid analytic space associated with each eigenvariety.

Hypothesis 4.2. From now on, we assume all the eigenvarieties we consider throughout this note are reduced and separated.

Lemma 4.3 ([Buz07, Lemma 5.9]). *Let $(\mathscr{E}, \psi, \omega)$ be either $(\mathscr{E}_{\mathrm{GL}_2}, \psi_{\mathrm{GL}_2}, \omega_{\mathrm{GL}_2})$ or $(\mathscr{E}_D, \psi_D, \omega_D)$. For any discretely valued extension $\mathcal{K}'/\mathcal{K}$, the association*

$$\mathscr{E}(\mathcal{K}') \longrightarrow \mathrm{Hom}_{\mathrm{ring}}(\mathbb{T}_{\mathbf{I}}, \mathcal{K}') \times \mathcal{W}(\mathcal{K}') \qquad \text{mapping} \qquad z \longmapsto (\,[h \mapsto \psi(h)(z)], \omega(z)),$$

defines a bijection between the $\mathcal{K}'$-valued points of $\mathscr{E}$ and the set of $\mathcal{K}'$-valued systems of eigenvalues with non-zero U_p-eigenvalue.

Remark 4.4. Lemma 4.3 holds for any general eigenvariety ([Bel21, Theorem 3.7.1]).

Proposition 4.5 (p-adic extension of the JL correspondence). *The Jacquet–Langlands correspondence extends uniquely to a closed immersion*

$$\mathscr{E}_D \xrightarrow{\ JL_p\ } \mathscr{E}_{\mathrm{GL}_2}$$

compatible with the eigenvariety structure, that is, $\omega_D = \omega_{\mathrm{GL}_2} \circ JL_p$ and $\psi_D = JL_p^* \circ \psi_{\mathrm{GL}_2}$.

Before proving the proposition, we need to recall the notion of arithmetic points on $\mathcal{W}(\overline{\mathbb{Q}_p})$.

Definition 4.6. We say that $\kappa \in \mathcal{W}(\overline{\mathbb{Q}_p})$ is classical if there exists a point $z \in \mathscr{E}_{\mathrm{GL}_2}(\overline{\mathbb{Q}_p})$ such that $\omega_{\mathrm{GL}_2}(z) = \kappa$ and z corresponds to a classical modular form; we denote by $\mathcal{W}^{\mathrm{cl}}(\overline{\mathbb{Q}_p})$ the set of classical points. We say that $\kappa \in \mathcal{W}(\overline{\mathbb{Q}_p})$ is *arithmetic* if there exists a point $z \in \mathscr{E}_{\mathrm{GL}_2}(\overline{\mathbb{Q}_p})$ such that $\omega_{\mathrm{GL}_2}(z) = \kappa$ and z corresponds to a form of weight-character $\kappa = (k, \varepsilon) \in \mathcal{W}(\overline{\mathbb{Q}_p}) = \mathrm{Hom}(\mathbb{Z}_p^{\times}, \overline{\mathbb{Q}_p}^{\times})$ with $k \in \mathbb{Z}_{\geq 2}$ and ε a p-adic character of finite order. We denote by $\mathcal{W}^{\mathrm{arith}}(\overline{\mathbb{Q}_p})$ the set of arithmetic points. By Coleman classicality's theorem [Col97, Theorem 1.1], we know that arithmetic points of small slope are classical.

Proof of Proposition 4.5. We want to apply [Lud17, Proposition 2.10]. We define the *very Zariski-dense* (see [Bel21, Definition 3.8.1]) subset $Z \subset \mathscr{E}_D(\overline{\mathbb{Q}_p})$, as the set of arithmetic $\overline{\mathbb{Q}_p}$-points z such that $\omega_D(z) \in \mathcal{W}^{\mathrm{arith}}(\overline{\mathbb{Q}_p})$. *Coleman's classicality theorem* combined with [HPS89b, Theorem 7.17] (cf. also

Proposition 3.10) implies that both $\mathscr{E}_D$ and $\mathscr{E}_{\mathrm{GL}_2}$ are endowed with a *classical structure* (see [Bel21, Definition 3.8.5]) given by classical forms over $\mathcal{W}^{\mathrm{arith}}(\overline{\mathbb{Q}_p})$. [Bel21, Proposition 3.8.6] tells us that the subset of all classical points $\mathscr{Z} \subset \mathscr{E}_D$ is *very Zariski-dense* in $\mathscr{E}_D$, thus, setting $Z = \mathscr{Z}(\overline{\mathbb{Q}_p})$ we can assume it to be very Zariski-dense in $\mathscr{E}_D(\overline{\mathbb{Q}_p})$; Z embeds in $\mathscr{E}_{\mathrm{GL}_2}(\overline{\mathbb{Q}_p})$ by the classical *Jacquet–Langlands correspondence* and *Strong Multiplicity One* for automorphic representations. More precisely, Lemma 4.3 defines an injective association

$$JL_p : \mathscr{E}_D^{\mathrm{cl}}(\overline{\mathbb{Q}_p}) \ni z \longleftrightarrow \pi_z^D \longmapsto \pi_z \longleftrightarrow JL_p(z) \in \mathscr{E}_{\mathrm{GL}_2}^{\mathrm{cl}}(\overline{\mathbb{Q}_p}),$$

where π_z^D is the automorphic representation over $D^\times(\mathbb{A})$ associated with the classical point z and $\pi_z = JL(\pi_z^D)$. This inclusion is compatible with the structure of the two eigenvarieties by definition of the map. The uniqueness is ensured by the reducedness of the eigenvarieties together with [BC09, Lemma 7.2.7] and Lemma 4.3.   ∎

4.1.2. *The extended eigenvariety.* The Hecke algebra $\mathbb{T}_{\mathbf{I}}$ acts on quaternionic modular forms with $R_{p^n N_+, N_-}$-level structure and, for each $\ell \mid N_-^{\mathrm{sc}}$, we can extend $\mathbb{T}_{\mathbf{I}}$ by adjoining the operator $\langle \varpi_{D_\ell} \rangle$.

Definition 4.7. We define the $\mathcal{K}$-algebra

$$\widetilde{\mathbb{T}}_{\mathbf{I}} = \mathbb{T}_{\mathbf{I}}\Big[\langle \varpi_{D_\ell} \rangle \,\Big|\, \ell \mid N_-^{\mathrm{sc}} \Big]$$

endowed with the canonical injection $\mathbb{T}_{\mathbf{I}} \hookrightarrow \widetilde{\mathbb{T}}_{\mathbf{I}}$. We refer to it as the *extended Hecke algebra*. By Proposition 3.13 and equation (3.17), $\widetilde{\mathbb{T}}_{\mathbf{I}}$ is commutative.

While the algebra $\mathbb{T}_{\mathbf{I}}$ acts on both $\mathbb{S}_{N_+N_-,\chi}$ and $\mathbb{S}_{N_+N_-,\chi}^D$, $\widetilde{\mathbb{T}}_{\mathbf{I}}$ acts only on $\mathbb{S}_{N_+N_-,\chi}^D$. We can consider the tuple

$$(4.10) \qquad\qquad \widetilde{\mathscr{D}}_{N_+N_-,\chi}^D = (\mathcal{W}, \widetilde{\mathbb{T}}_{\mathbf{I}}, U_p, \mathbb{S}_{N_+N_-,\chi}^{D,\dagger}, \psi_D),$$

which defines an eigenvariety datum, as one only needs to prove the following lemma.

Lemma 4.8. *For any pair of affinoid subdomains,* $X' = \mathrm{Sp}(A') \subseteq X = \mathrm{Sp}(A) \subset \mathcal{W}$ *with* $X, X' \in \mathscr{C}$, *the* A'-*modules* $\mathbb{S}_{N_+N_-,\chi}^{D,\dagger}(X)\widehat{\otimes}_A A'$ *and* $\mathbb{S}_{N_+N_-,\chi}^{D,\dagger}(X')$ *are linked (see* [Bel21, *Definition 3.5.1]) under* $\widetilde{\mathbb{T}}_{\mathbf{I}}$.

Proof. We apply [Bel21, Lemma 3.5.2] to [Buz07, Lemma 12.2]; we must prove that the two morphisms are $\widetilde{\mathbb{T}}_{\mathbf{I}}$-equivariant. The first inclusion morphism is $\mathbb{T}_{\mathbf{I}}$-equivariant (recall that we are changing the structure only at the ramified places). Thus, the properties of $\langle \varpi_{D_\ell} \rangle$ in Proposition 3.13 together with the relation in Section 3.5.4 imply the $\widetilde{\mathbb{T}}_{\mathbf{I}}$-equivariance of both morphisms.   ∎

We deduce the following proposition.

Proposition 4.9. *There exists a reduced eigenvariety associated with the datum* $\widetilde{\mathscr{D}}_{N_+N_-,\chi}^D$, *which we denote by* $(\widetilde{\mathscr{E}}_D, \widetilde{\omega}_D, \widetilde{\psi}_D)$. *We will refer to it as the* extended eigenvariety for D.

This variety plays the role of an auxiliary eigenvariety, whose points parameterize system of Hecke eigenvalues with finite U_p-slope together with a collection of additional eigenvalues associated with the $\langle \varpi_{D_\ell} \rangle$-operators. In particular, applying Lemma 4.3 to the inclusion $\mathbb{T}_{\mathbf{I}} \hookrightarrow \widetilde{\mathbb{T}}_{\mathbf{I}}$, we obtain a morphism on the $\mathbb{C}_p$-points of the eigenvarieties. This canonical map extends to a morphism of eigenvarieties; in fact, more is true.

Proposition 4.10. *The canonical inclusion* $\mathbb{T}_{\mathbf{I}} \hookrightarrow \widetilde{\mathbb{T}}_{\mathbf{I}}$ *defines a unique finite morphism,*

$$\widetilde{\pi} : \widetilde{\mathscr{E}}_D \longrightarrow \mathscr{E}_D,$$

compatible with the eigenvariety structure. Moreover, it sends classical points to classical points.

Proof. We follow the proof of [Lud17, Proposition 3.15] and construct the map on the local pieces of the eigenvariety. More precisely, we take $\widetilde{\mathscr{E}}_{D,X} = \widetilde{\omega}_D^{-1}(X) \subset \widetilde{\mathscr{E}}_D$, for any $X = \mathrm{Sp}(A)$ in the admissible covering $\mathscr{C}$. We consider the admissibile covering $\{\widetilde{\mathscr{E}}_{D,X}(U)\}_U$ indexed over connected open affinoids $U \subseteq X$; by construction, each $\widetilde{\mathscr{E}}_{D,X}(U)$ is an affinoid $\mathrm{Sp}(\widetilde{\mathbb{T}}_\mathbf{I}(U))$, for

$$\widetilde{\mathbb{T}}_\mathbf{I}(U) = \mathrm{Im}\left(\widetilde{\mathbb{T}}_\mathbf{I} \otimes_{\mathcal{K}} \mathcal{O}(U) \longrightarrow \mathrm{End}_{\mathcal{O}(U)}\left(\left(\mathbb{S}_{N_+ N_-, \chi}^{D,\dagger}(X)\widehat{\otimes}_{\mathcal{O}(X)}\mathcal{O}(U)\right)^{\text{finite slope}}\right)\right)^{\text{red}}$$

the reduced image of $\widetilde{\mathbb{T}}_\mathbf{I} \otimes_{\mathcal{K}} \mathcal{O}(U)$. For any U as above, $\mathbb{T}_\mathbf{I} \hookrightarrow \widetilde{\mathbb{T}}_\mathbf{I}$ induces the inclusion $\mathbb{T}_\mathbf{I}(U) \hookrightarrow \widetilde{\mathbb{T}}_\mathbf{I}(U)$; therefore, we obtain

$$\iota_U^* : \widetilde{\mathscr{E}}_{D,X}(U) \longrightarrow \mathscr{E}_{D,X}(U).$$

Note that the $\mathscr{E}_{D,X}(U)$ form an admissible covering of $\mathscr{E}_{D,X}$. The composition of ι_U^* with the natural inclusion $\mathscr{E}_{D,X}(U) \hookrightarrow \mathscr{E}_{D,X}$ determines the morphism

$$\widetilde{\pi}_U : \widetilde{\mathscr{E}}_{D,X}(U) \longrightarrow \mathscr{E}_{D,X}.$$

Let X' be an admissible open in $\mathscr{C}$ and take $U \subset X$ and $U' \subset X'$, connected open affinoids. Up to refining the covering, we can assume that $X \subseteq X'$. Invoking [Buz07, Lemma 5.2] as in [Lud17, Proposition 3.15], we deduce that $\widetilde{\pi}_U|_{\widetilde{\mathscr{E}}_{D,X}(U\cap U')} = \widetilde{\pi}_{U\cap U'}$, therefore the morphisms $\widetilde{\pi}_U$ glue compatibly. Again, the uniqueness is ensured by the reducedness of the eigenvarieties together with [BC09, Lemma 7.2.7] and Lemma 4.3. Let now z be a classical point in $\widetilde{\mathscr{E}}_D(\overline{\mathbb{Q}_p})$. Such z corresponds to a classical eigenform φ_z identified by a system of Hecke and $\langle \varpi_{D_\ell}\rangle$-eigenvalues for $\ell \mid N_-^{\mathrm{sc}}$, say $\widetilde{\lambda}_z$. The morphism $\widetilde{\pi}$, by construction, maps $\widetilde{\lambda}_z$ to the eigensystem λ_z obtained by forgetting the $\langle\varpi_{D_\ell}\rangle$-eigenvalues. Lemma 4.3 shows that $\pi(z)$ is classical. Finiteness follows from the relation

$$\langle\varpi_{D_\ell}\rangle^2 - \chi_{N_+ N_-/\ell^2}(\ell)^{-1} \cdot [\langle\ell\rangle]_\mathbf{I} = 0$$

obtained from equation (3.17). $\blacksquare$

4.1.3. *Idempotents.* For any $\ell \mid N_-^{\mathrm{sc}}$, recall the square root operators $\langle\sqrt{\ell}\rangle_\mathbf{I}$ defined in equation (4.7). They are invertible elements of the algebra $\mathbf{I}$, as the $[\langle\ell\rangle]_\mathbf{I}$ are invertible. Therefore, they determine invertible operators. We can hence define idempotents in $\mathbf{I}$ and on $\mathbb{S}_{N_+ N_-, \chi}^D$.

Definition 4.11. For each prime $\ell \mid N_-^{\mathrm{sc}}$, we set

$$e_\pm^{D_\ell} = \frac{1}{2}\left(\mathbb{1} \mp \frac{1}{\langle\sqrt{\ell}\rangle_\mathbf{I}\sqrt{\chi_{\mathbb{A}, N_+ N_-/\ell^2}(\ell)^{-1}}}\langle\varpi_{D_\ell}\rangle\right).$$

Moreover, for each tuple of signs $\epsilon = (\epsilon_\ell)_{\ell\mid N_-^{\mathrm{sc}}} \in \{\pm 1\}^{\#\{\ell\mid N_-^{\mathrm{sc}}\}}$, we define a canonical idempotent

$$e_\epsilon^D = \prod_{\ell\mid N_-^{\mathrm{sc}}} e_{\epsilon_\ell}^{D_\ell}.$$

Remark 4.12.

(1) It is immediate from the definition that the sum $e_+^{D_\ell} + e_-^{D_\ell}$ is the identity element $\mathbb{1}$ in $\widetilde{\mathbb{T}}_\mathbf{I}$. Therefore, the sum of all these idempotents is

$$\sum_{\epsilon\in\{\pm 1\}^{\#\{\ell\mid N_-^{\mathrm{sc}}\}}} e_\epsilon^D = \mathbb{1}.$$

(2) The operators $\langle\sqrt{\ell}\rangle_\mathbf{I}$ take care of the p-component of the square root as explained by equation (3.13).

Proposition 4.13.

(1) Let $\kappa = (k, \kappa_p) \in \mathcal{W}^{\mathrm{cl}}(\overline{\mathbb{Q}}_p)$ be any weight-character and let $n = \max\{1, v_p(\mathrm{cond}(\kappa_p))\}$. For any $\ell \mid N_-^{\mathrm{sc}}$, let $S_k^{D,\pm}(p^n N_+ N_-, \chi \kappa_p) = e_{\pm}^{D_\ell} S_k^D p^n N_+ N_-, \chi \kappa_p)$. There is a canonical decomposition

$$S_k^D(p^n N_+ N_-, \chi \kappa_p) = S_k^{D,+}(p^n N_+ N_-, \chi \kappa_p) \oplus S_k^{D,-}(p^n N_+ N_-, \chi \kappa_p).$$

(2) The projectors $e_{\pm}^{D_\ell}$ are orthogonal with respect to the quaternionic Petersson inner product defined in Section 3.6.1.

Proof. It is clear that the two idempotents $e_{\pm}^{D_\ell}$ give rise to projectors on $S_k^D(p^n N \ell^2, \chi \kappa_p)$, satisfying the identity $\mathbb{1} = e_+^{D_\ell} + e_-^{D_\ell}$, i.e. $e_-^{D_\ell} = \mathbb{1} - e_+^{D_\ell}$. Therefore, for any $\varphi \in S_k^D(p^n N \ell^2, \chi \kappa_p)$, we compute $e_{\pm}^{D_\ell} e_{\mp}^{D_\ell} \varphi = 0$, in fact, $e_{\pm}^{D_\ell} e_{\mp}^{D_\ell} \varphi = e_{\pm}^{D_\ell} \varphi - e_{\pm}^{D_\ell} e_{\pm}^{D_\ell} \varphi = e_{\pm}^{D_\ell} \varphi - e_{\pm}^{D_\ell} \varphi = 0$. This proves (1). Part (2) follows from Proposition 3.20 and equation (3.13). ∎

4.1.4. *The extended eigenvarieties of idempotent type.* We fix one of the idempotents in Definition 4.11, say e_ϵ^D, and consider the tuple

$$(4.11) \qquad\qquad e_\epsilon^D \widetilde{\mathscr{D}}_{N_+ N_-, \chi}^D = (\mathcal{W}, \widetilde{\mathbb{T}}_{\mathbf{I}}, U_p, e_\epsilon^D \mathbb{S}_{N_+ N_-, \chi}^{D,\dagger}, \psi_D).$$

Proposition 4.14. *There exists a unique reduced eigenvariety associated with the datum $e_\epsilon^D \widetilde{\mathscr{D}}_{N_+ N_-, \chi}^D$, which we denote by $(\widetilde{\mathscr{E}}_D^\epsilon, \widetilde{\omega}_D^\epsilon = \widetilde{\omega}_{D|_{\widetilde{\mathscr{E}}_D^\epsilon}}, \widetilde{\psi}_D)$ and we will refer to it as the* extended eigenvariety for D.

Proof. The above Proposition 4.13 guarantees that the projection by e_ϵ^D (which is linear and continuous) produces Banach modules satisfying *Property* (Pr) (see [Bel21, Exercise 3.1.23]). Since e_ϵ^D commutes with all Hecke operators, Lemma 4.8 shows that the modules are still linked. ∎

Proposition 4.15. *The natural map*

$$\widetilde{\iota}^\epsilon : \widetilde{\mathscr{E}}_D^\epsilon \longrightarrow \widetilde{\mathscr{E}}_D,$$

is the unique closed immersion compatible with the eigenvariety structure. Moreover, $\widetilde{\mathscr{E}}_D$ is the disjoint union of the images of $\widetilde{\mathscr{E}}_D^\epsilon$, for all the $\epsilon \in \{\pm 1\}^{\#\{\ell \mid N_-^{\mathrm{sc}}\}}$.

Proof. This is just [Bel21, Exercise 3.7.2 and Exercise 3.6.4], together with the direct sum decomposition of Proposition 4.13. Alternatively, one can also consider an approach similar to the proof of [Lud17, Proposition 2.10]. ∎

Proposition 4.16. *The morphism $\widetilde{JL}_p^\epsilon$ of eigenvarieties, obtained by composition,*

$$\widetilde{JL}_p^\epsilon = JL_p \circ \widetilde{\pi} \circ \widetilde{\iota}^\epsilon : \widetilde{\mathscr{E}}_D^\epsilon \longrightarrow \mathscr{E}_{\mathrm{GL}_2}$$

is a closed immersion.

Proof. The morphism preserves the eigenvariety structures and it is finite, by composition of finite morphisms of eigenvarieties; recall that closed immersions are finite by definition, see e.g. [FvdP04, Definition 4.5.7 and Definition 4.10.1]. As in the proof of [Lud17, Lemma 2.9], we consider an affinoid $V \subset \mathcal{W} \times \mathbb{A}_{\mathrm{rig}}^1$, its preimage $X = (\widetilde{\omega}_D^\epsilon \times \widetilde{\psi}_D(U_p)^{-1})^{-1}(V)$ in $\widetilde{\mathscr{E}}_D^\epsilon$, and the natural *surjective* map

$$\widetilde{\psi}_D \otimes (\widetilde{\omega}_D^\epsilon \times \widetilde{\psi}_D(U_p)^{-1})^* : \widetilde{\mathbb{T}}_{\mathbf{I}} \otimes_K \mathcal{O}(V) \longrightarrow \mathcal{O}(X).$$

To prove that $\widetilde{JL}_p^\epsilon$ is a closed immersion, it is enough to show that the restriction

$$\widetilde{\psi}_D \otimes (\widetilde{\omega}_D^\epsilon \times \widetilde{\psi}_D(U_p)^{-1})^* : \mathbb{T}_{\mathbf{I}} \otimes_K \mathcal{O}(V) \longrightarrow \mathcal{O}(X).$$

is still surjective. This follows by the finiteness of $\mathcal{O}(X)$ as a $\mathcal{O}(V)$-module (see Section 4.1.1) and from the relation

$$\widetilde{\psi}_D(\langle \varpi_{D_\ell} \rangle) = \epsilon_\ell \sqrt{\chi_{\mathbb{A}, \ell}(\ell)} \, \widetilde{\psi}_D(\langle \sqrt{\ell} \rangle_{\mathbf{I}}), \quad \text{for all } \ell \mid N_-^{\mathrm{sc}}.$$

We can hence follow through the proof of [Lud17, Proposition 2.10], considering finite subsets $I_V \subset \mathbb{T}_{\mathbf{I}}$ containing the operators $\langle \sqrt{\ell} \rangle_{\mathbf{I}}$ for $\ell \mid N_-^{\mathrm{sc}}$. ∎

We can summarize the above eigenvarieties and their morphisms in the following commutative diagrams:

$$(4.12)$$

$$
\begin{array}{c}
\widetilde{\mathscr{E}}_D^{\epsilon} \xhookrightarrow{\ \widetilde{\iota}^{\epsilon}\ } \widetilde{\mathscr{E}}_D \xrightarrow{\ \widetilde{JL}_p^{\epsilon}\ } \\[2mm]
\widetilde{\omega}_D^{\epsilon}\times\widetilde{\psi}_D^{\epsilon}(U_p)^{-1} \quad \widetilde{\omega}_D\times\widetilde{\psi}_D(U_p)^{-1} \quad \widetilde{\pi} \\[2mm]
\mathscr{E}_D \xrightarrow{\ JL_p\ } \mathscr{E}_{\mathrm{GL}_2} \\[2mm]
\mathcal{W}\times\mathbb{A}^1_{\mathrm{rig}} \quad \omega_D\times\psi_D(U_p)^{-1} \\[2mm]
\omega_{\mathrm{GL}_2}\times\psi_{\mathrm{GL}_2}(U_p)^{-1} \\[2mm]
\widetilde{\mathbb{T}}_{\mathbf{I}} \\
\mathbb{T}_{\mathbf{I}},
\end{array}
$$

$$(4.13)$$

$$
\begin{array}{c}
\mathcal{O}(\widetilde{\mathscr{E}}_D^{\epsilon}) \xleftarrow{\ (\widetilde{\iota}^{\epsilon})^*\ } \mathcal{O}(\widetilde{\mathscr{E}}_D) \xleftarrow{\ (\widetilde{JL}_p^{\epsilon})^*\ } \\[2mm]
\widetilde{\psi}_D^{\epsilon} \quad \widetilde{\psi}_D \quad (\widetilde{\pi})^* \\[2mm]
\widetilde{\mathbb{T}}_{\mathbf{I}} \\[2mm]
\mathcal{O}(\mathscr{E}_D) \xleftarrow{\ (JL_p)^*\ } \mathcal{O}(\mathscr{E}_{\mathrm{GL}_2}). \\[2mm]
\psi_D \quad \psi_{\mathrm{GL}_2} \\[2mm]
\mathbb{T}_{\mathbf{I}}
\end{array}
$$

4.1.5. *The image of* $\widetilde{JL}_p^{\epsilon}$. We conclude this section with a couple of interesting observations which strengthen the comparison between our immersion and Chenevier's one. As explained in [Che05, Section 2.1.1, Section 4.6, and Proposition 4.7.(b)], one can construct a reduced closed rigid analytic subvariety $\mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0} \subseteq \mathscr{E}_{\mathrm{GL}_2}$ whose classical points correspond to forms which are new at the primes dividing the discriminant of D.

Proposition 4.17. *The morphism* JL_p *is an isomorphism onto* $\mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0}$.

Proof. By Proposition 4.5 and the above discussion, we know that JL_p is a closed immersion. Note that classical points are dense both in $\mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0}$ and in $\mathscr{E}_D$, and hence it is enough to show that

$$JL_p(\mathscr{E}_D^{\mathrm{cl}}) \supseteq \mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0,\mathrm{cl}}$$

in order to prove that the morphism is surjective. Each point in $\mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0,\mathrm{cl}}$ corresponds to a Hecke eigenform which, under our assumptions, can be transferred to the quaternion algebra. ∎

Proposition 4.18. *The morphism* $\widetilde{\pi}_{|\widetilde{\mathscr{E}}_D^{\epsilon}}$ *is an isomorphism onto its image.*

Proof. We would like to apply [BC09, Proposition 7.2.8], but $\mathrm{Im}(\widetilde{\pi}_{|\widetilde{\mathscr{E}}_D^{\epsilon}})$ is not necessarily an eigenvariety. However, modifying the sets I_V as in the proof of Proposition 4.16, the proof of [BC09, Proposition 7.2.8] applies verbatim to our setup. ∎

Corollary 4.19. *The composition* $\widetilde{\pi} \circ \widetilde{\iota}^{\epsilon}$ *is an open and closed immersion. Therefore,* $\widetilde{JL}_p^{\epsilon}$ *is an open and closed immersion into* $\mathscr{E}_{\mathrm{GL}_2}^{\mathrm{tr}=0}$.

Proof. The first assertion follows by combining Proposition 4.18 with Proposition 4.15. The second one follows from Proposition 4.17. ∎

Remark 4.20. For any two distinct choices of signs, ϵ and ϵ', the images of the eigenvarieties $\widetilde{\mathscr{E}}_D^\epsilon$ and $\widetilde{\mathscr{E}}_D^{\epsilon'}$ are not disjoint in $\mathscr{E}_{\mathrm{GL}_2}$, since their intersection contains the twist-minimal forms.

4.2. **Families of quaternionic modular forms.** This section and the next one contain the main results we employ in the construction of the triple product p-adic L-function, namely the existence of families of quaternionic modular forms of finite slope. We follow the definition given in [Che04, Section 6.2.6], and restrict our attention to the two eigenvarieties

$$(4.14) \qquad \mathscr{E}_{\mathrm{GL}_2} \sim (\mathcal{W}, \mathbb{T}_{\mathbf{I}}, U_p, \mathbb{S}_{N_+N_-,\chi}^\dagger, \psi_{\mathrm{GL}_2}) \qquad \text{and} \qquad \widetilde{\mathscr{E}}_D^\epsilon \sim (\mathcal{W}, \widetilde{\mathbb{T}}_{\mathbf{I}}, U_p, e_\epsilon^D \mathbb{S}_{N_+N_-,\chi}^{D,\dagger}, \widetilde{\psi}_D^\epsilon).$$

To $\mathbb{S}_{N_+N_-,\chi}^\dagger$ (resp. $e_\epsilon^D \mathbb{S}_{N_+N_-,\chi}^{D,\dagger}$), one can associate a *sheaf of Banach modules* on $\mathcal{W}$, which we denote by $\mathscr{S}_{N_+N_-,\chi}^\dagger$ (resp. $\mathscr{S}_{N_+N_-,\chi}^{D,\epsilon,\dagger}$); this geometric object is roughly obtained glueing the Banach modules of overconvergent forms compatibly. As we are only interested in its sheaf nature, we do not recall the precise definition here, but refer the reader to [Che04, Section 3] and [Che05, Section 1] for a thorough discussion. Let $\mathscr{T}_{\mathbf{I}}^{\mathrm{GL}_2}$ be the closure of $\mathbb{T}_{\mathbf{I}}$ in

$$(4.15) \qquad \mathcal{E}_{\mathbf{I}}^{\mathrm{GL}_2} = \left\{ h \in \mathrm{End}(\mathscr{S}_{N_+N_-,\chi}^\dagger) \,\middle|\, h \text{ is integral and rational} \right\},$$

where the topology is the coarsest topology such that, for any open affinoid $X \in \mathcal{W}$, the restriction map

$$(4.16) \qquad \mathcal{E}_{\mathbf{I}}^{\mathrm{GL}_2} \longrightarrow \mathrm{End}_{\mathcal{O}_X(X)}^{\mathrm{cts}}(\mathbb{S}_{N_+N_-,\chi}^\dagger(X))$$

is continuous; here $\mathrm{End}_{\mathcal{O}_X(X)}^{\mathrm{cts}}(\mathbb{S}_{N_+N_-,\chi}^\dagger(X))$ is endowed with the topology induced by the supremum norm. Analogously (cf. [Che04, Proposition 4.5.4] and equation (3.17)), we define $\widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon}$ as the closure of $\widetilde{\mathbb{T}}_{\mathbf{I}}$ in

$$(4.17) \qquad \mathcal{E}_{\mathbf{I}}^{D,\epsilon} = \left\{ h \in \mathrm{End}(\mathscr{S}_{N_+N_-,\chi}^{D,\epsilon,\dagger}) \,\middle|\, h \text{ is integral and rational} \right\}.$$

Let $(\mathscr{E}, \mathcal{W}, \mathbb{S}^\dagger, \psi, \omega, \mathscr{T})$ be either the tuple

$$(4.18) \qquad (\mathscr{E}_{\mathrm{GL}_2}, \mathcal{W}, \mathbb{S}_{N_+N_-,\chi}^\dagger, \psi_{\mathrm{GL}_2}, \omega_{\mathrm{GL}_2}, \mathscr{T}_{\mathbf{I}}^{\mathrm{GL}_2}) \qquad \text{or} \qquad (\widetilde{\mathscr{E}}_D^\epsilon, \mathcal{W}, e_\epsilon^D \mathbb{S}_{N_+N_-,\chi}^{D,\dagger}, \widetilde{\psi}_D^\epsilon, \widetilde{\omega}_D^\epsilon, \widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon}).$$

For any open affinoid $\mathcal{U}$ in $\mathcal{W}$, we denote the submodule of *power-bounded elements* by

$$(4.19) \qquad \mathcal{O}(\mathcal{U})^0 = \left\{ s \in \mathcal{O}(\mathcal{U}) \,\middle|\, |s(u)| \le 1 \text{ for all } u \in \mathcal{U} \right\}.$$

If $\mathcal{U}$ is reduced, which is always the case if $\mathcal{U}$ is small enough, $\mathcal{O}(\mathcal{U})^0$ is compact in $\mathcal{O}(\mathcal{U})$ (see [BC09, Lemma 7.2.11]). The same definition applies for any affinoid $\mathcal{X} \subset \mathscr{E}$.

Definition 4.21 (Families of quaternionic cusp forms). Let $\kappa \in \mathcal{W}(\mathbb{C}_p)$ and let $\varphi \in \omega^{-1}(\kappa)$ be a p-adic overconvergent cuspidal form. We define a *family of quaternionic cuspidal modular forms passing through φ* as the collection of

- an affinoid open $\mathcal{U} \subseteq \mathcal{W}$, $\kappa \in \mathcal{U}(\mathbb{C}_p)$;
- an affinoid $\mathcal{X} \subset \mathscr{E}$, endowed with a finite morphism $\boldsymbol{\varphi} \colon \mathcal{X} \longrightarrow \mathcal{U}$, surjective when restricted to any irreducible component of $\mathcal{X}$;
- a $\mathbb{C}_p$-point $x_0 \in \mathcal{X}(\mathbb{C}_p)$ such that $\boldsymbol{\varphi}(x_0) = \kappa$;
- a continuous ring homomorphism $\lambda \colon \mathscr{T} \longrightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{X})^0$;

satisfying:

- for all $x \in \mathcal{X}(\mathbb{C}_p)$, there exists a form $\varphi_x \in \mathcal{S}^\dagger(\mathcal{X}) \cap \omega^{-1}(\omega(x))$, such that, for all $h \in \mathscr{T}$,

$$h(\varphi_x) = \lambda(h)(x)\,\varphi_x;$$

- the form φ is such that one can take $\varphi_{x_0} = \varphi$.

We say that the family is parameterized by $\mathcal{X}$, and that it has *slope* α (resp. *finite slope*) if every form φ_x in the family has slope $v_p(\lambda(U_p)(x)) = \alpha$ (resp. $v_p(\lambda(U_p)(x))$ is finite). We call a family of slope 0 a *Hida family*, while we refer to families of finite slope bigger equal than 0 as *Coleman families*.

Remark 4.22.

   (1) By [Bel21, Proposition 6.2.7], if there exists a classical point in a family, then the classical points are dense in the family.
   (2) All the eigenvarieties we consider are equidimensional of dimension one ([Bel21, Proposition 3.7.5]).

Theorem 4.23. *For any Coleman family* $\mathbf{f}$ *on* $\mathscr{E}_{\mathrm{GL}_2}$, *there exist at most a* unique *Coleman family on* $\widetilde{\mathscr{E}}_D^\epsilon$, *of the same slope, lifting* $\mathbf{f}$. *Moreover, there exist a family* $\boldsymbol{\varphi}$ *on* $\widetilde{\mathscr{E}}_D^\epsilon$, *for each choice of signs* ϵ, *corresponding to* $\mathbf{f}$, *if* $\mathbf{f}$ *passes through a classical point whose corresponding form is supercuspidal at each prime* $\ell \mid N_-^{\mathrm{sc}}$.

Proof. Let $(\mathcal{U}, \kappa, \mathcal{X}, x_0, \mathbf{f} \colon \mathcal{X} \to \mathcal{U})$ be a Coleman family on $\mathscr{E}_{\mathrm{GL}_2}$ and $(\widetilde{JL_p}^\epsilon)^{-1}(\{x_0\})$ be the preimage of x_0. It is either empty or it contains a unique point $x_0^{D,\epsilon} \in \widetilde{\mathscr{E}}_D^\epsilon$. If $(\widetilde{JL_p}^\epsilon)^{-1}(\{x_0\}) = \emptyset$, up to shrinking $\mathcal{X}$, and hence $\mathcal{U}$, we can assume that $\mathcal{X} \cap \mathrm{Im}(\widetilde{JL_p}^\epsilon) = \emptyset$ (by Proposition 4.16). Therefore, there is no family on $\widetilde{\mathscr{E}}_D^\epsilon$ lifting $(\mathcal{U}, \kappa, \mathcal{X}, x_0, \mathbf{f} \colon \mathcal{X} \to \mathcal{U})$. Suppose now that $x_0^{D,\epsilon} = (\widetilde{JL_p}^\epsilon)^{-1}(x_0)$. Since $\mathcal{X}$ can be taken to be an open affinoid and closed immersions are finite, $\mathcal{X}_D^\epsilon = (\widetilde{JL_p}^\epsilon)^{-1}(\mathcal{X})$ is an open affinoid (see [FvdP04, discussion after Definition 4.5.7]). We hence define $\boldsymbol{\varphi}^\epsilon = \mathbf{f} \circ \widetilde{JL_p}^\epsilon$. As above, up to shrinking $\mathcal{X}$, and hence $\mathcal{U}$, we can assume that $\mathcal{X}$ is contained in $\mathrm{Im}(\widetilde{JL_p}^\epsilon)$, therefore $\boldsymbol{\varphi}^\epsilon$ is a finite morphism, surjective when restricted to any irreducible component. As $\widetilde{JL_p}^\epsilon$ is an eigenvariety morphism, we obtain $\boldsymbol{\varphi}^\epsilon(x_0^{D,\epsilon}) = \kappa$. It remains to lift the ring homomorphism λ. Consider now the quotient map

$$\widetilde{\mathbb{T}}_{\mathbf{I}} \cong \frac{\mathbb{T}_{\mathbf{I}}[X_\ell]_{\ell \mid N^{\mathrm{sc}}}}{(X_\ell^2 - \chi_{\mathbb{A},\ell}(\ell)[\ell]_{\mathbf{I}})_{\ell \mid N_-^{\mathrm{sc}}}} \longrightarrow \mathbb{T}_{\mathbf{I}},$$

determined by the choice of signs ϵ, obtained by sending

$$X_\ell \longmapsto \epsilon_\ell \sqrt{\chi_{\mathbb{A},\ell}(\ell)} \langle \sqrt{\ell} \rangle_{\mathbf{I}}, \text{ for each } \ell \mid N_-^{\mathrm{sc}}.$$

This map defines, for any such affinoid open $\mathcal{V} \subseteq \mathcal{W}$, a diagram

$$
\begin{array}{ccccc}
\widetilde{\mathbb{T}}_{\mathbf{I}} & \longrightarrow & \mathrm{Im}(\widetilde{\mathbb{T}}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{D,\epsilon,\dagger}_{N_+ N_-, \chi}(\mathcal{V}))) & \overset{\widetilde{\lambda}^D_\mathcal{V}}{\dashrightarrow} & \mathcal{O}_{\mathcal{X}_D^\epsilon}(\mathcal{X}_D^\epsilon)^0 \\
\downarrow & & \downarrow & & \uparrow \scriptstyle{(\widetilde{JL_p}^\epsilon)^*} \\
\mathbb{T}_{\mathbf{I}} & \longrightarrow & \mathrm{Im}(\mathbb{T}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{\dagger}_{N_+ N_-, \chi}(\mathcal{V}))) & \overset{\lambda_\mathcal{V}}{\longrightarrow} & \mathcal{O}_\mathcal{X}(\mathcal{X})^0,
\end{array}
$$

where the first square is a commutative square of continuous algebra homomorphisms, with the second vertical map obtained from the Jacquet–Langlands correspondence (cf. Section 3.5.1) and the chosen quotient map. We can define $\widetilde{\lambda}^D_\mathcal{V}$ as the composition of morphisms in the right hand side square, where $\lambda_\mathcal{V}$ is the restriction of λ. Let us remark that the function $\widetilde{\lambda}^D_\mathcal{V}(\langle \varpi_{D_\ell} \rangle) \in \mathcal{O}_\mathcal{V}(\mathcal{V})^0$ obtained by composition is the constant function $\widetilde{\lambda}^D(\langle \varpi_{D_\ell} \rangle)(x) = \epsilon_\ell \sqrt{\chi_{\mathbb{A},\ell}(\ell)} \langle \sqrt{\ell} \rangle_{\mathbf{I}}$, for all $x \in \mathcal{X}$ (cf. Lemma 3.17). The morphism $\widetilde{\lambda}^D_\mathcal{V}$ is a continuous ring homomorphism. It remains to extend $\widetilde{\lambda}^D_\mathcal{V}$ to $\widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon}$. We start noticing that the defined morphisms $\widetilde{\lambda}^D_\mathcal{V}$ glue compatibly, as the $\lambda_\mathcal{V}$ do. Moreover, $\widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon}$ is a commutative $\mathbf{I}$-algebra, hence the translates of $\mathrm{Im}(\widetilde{\mathbb{T}}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{D,\epsilon,\dagger}_{N_+ N_-, \chi}(\mathcal{V})))$ define an open covering. The uniqueness of the lift, as well as the invariance of the slope, follow now from the construction of $\boldsymbol{\varphi}^\epsilon$. ∎

Remark 4.24.

(1) In the above proof, we note that fixing the eigenvalue of each $\langle \varpi_{D_\ell} \rangle$, shows that

$$\mathrm{Im}(\widetilde{\mathbb{T}}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{D,\epsilon,\dagger}_{N_+N_-,\chi}(\mathcal{V}))) = \mathrm{Im}(\mathbb{T}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{D,\epsilon,\dagger}_{N_+N_-,\chi}(\mathcal{V}))).$$

(2) We must stress the fact that the sequence $\{U_p^{n!}\}_n$ still converges in $\widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon}$, as it does so in each closure $\overline{\mathrm{Im}(\widetilde{\mathbb{T}}_{\mathbf{I}} \to \mathrm{End}^{\mathrm{cts}}_{\mathcal{O}_\mathcal{V}(\mathcal{V})}(\mathbb{S}^{D,\epsilon,\dagger}_{N_+N_-,\chi}(\mathcal{V})))}$.

4.3. Λ-adic quaternionic forms and Hida families.
Even though Theorem 4.23 guarantees the existence of Coleman families, we need a more explicit way to describe them. In particular we restrict our attention to Hida families.

4.3.1. Λ-adic quaternionic forms.
We keep the notation introduced in Section 3.4.3 and fix a chain of inclusions of orders

$$(4.20) \qquad R_{N_+,N_-} \supset R_{pN_+,N_-} \supset R_{p^2 N_+,N_-} \supset \cdots \supset R_{p^n N_+,N_-} \supset \cdots \supset R_{p^\infty N_+,N_-},$$

for

$$(4.21) \qquad R_{p^\infty N_+,N_-} = \left\{ r \in \widehat{R}_{N_+,N_-} \,\middle|\, \iota_p(r_p) = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) a, d \in \mathbb{Z}_p^\times, \, b \in \mathbb{Z}_p \right\}.$$

Considering the set

$$(4.22) \qquad X_\infty = D^\times \backslash \widehat{D}^\times / U_1(R_{p^\infty N_+,N_-}),$$

for

$$(4.23) \qquad U_1(R_{p^\infty N_+,N_-}) = \left\{ r \in U_1(R_{N_+,N_-}) \,\middle|\, \iota_p(r_p) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} a \in \mathbb{Z}_p^\times, \, b \in \mathbb{Z}_p \right\},$$

we have the natural quotient maps

$$(4.24) \qquad X_\infty \longrightarrow X_m \longrightarrow X_n,$$

for any $m > n$. We also take $P_n = \left((1+T)^{p^n} - 1\right)$, a height one prime ideal in $\Lambda = \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!]$, for $T = \langle 1+p \rangle_\Lambda - 1$. Recall that we define the diamond operators as in [Hsi21, Section 4.4]. We extend the notion of Λ-adic forms provided in loc. cit. as follows.

Definition 4.25. Let $\mathbf{S}^D(R_{N_+,N_-}, \Lambda)$ be the space of functions $\mathbf{f} \colon X_\infty \longrightarrow \Lambda$, such that, for any $z \in 1 + p\mathbb{Z}_p$,

$$\mathbf{f}(xz) = \mathbf{f}(x)\langle z \rangle^2 \langle z \rangle_\Lambda^{-1},$$

and, for any n sufficiently large,

$$\mathbf{f} \ (\mathrm{mod} \ P_n) \colon X_\infty \longrightarrow \Lambda/P_n$$

factors through X_n. We call it the space of Λ-*adic quaternionic modular forms* of level R_{N_+,N_-}.

By construction,

$$(4.25) \qquad \mathbf{S}^D(R_{N_+,N_-}, \Lambda) = \varprojlim_n \mathrm{Hom}_\Lambda(\mathbb{Z}_p[X_n], \Lambda/P_n) \otimes_{\Lambda,\iota_2} \Lambda,$$

for $\iota_2 \colon \Lambda \longrightarrow \Lambda$ the $\mathbb{Z}_p$-algebra morphism (twisting the action at p) defined by

$$(4.26) \qquad \iota_2 \colon T \longmapsto (1+T)^{-2}(1+p)^2 - 1.$$

Therefore, the Λ-module $\mathbf{S}^D(R_{N_+,N_-}, \Lambda)$ is compact and endowed with the Hecke action defined by

$$(4.27) \qquad t \cdot \mathbf{f}(x) = \mathbf{f}(t \cdot x), \quad \text{for any } t \in \widetilde{\mathbb{T}}_{\mathbf{I}} \text{ and } x \in X_\infty.$$

Recall the notation of Section 3.5.4. For any Dirichlet character χ modulo N_+N_-p, valued in $\mathbb{Z}_p$, we define

(4.28)
$$\mathbf{S}^D(R_{N_+,N_-},\chi,\Lambda) = \left\{ \mathbf{f} \in \mathbf{S}^D(R_{N_+,N_-},\Lambda) \right\} \left| \begin{array}{c} \mathbf{f}(xzr) = (\chi_{\mathbb{A}})^{-1}(z)\widetilde{\chi}(r)f(x) \left\langle \varepsilon_{cyc}(z)\right\rangle^2 \left[\left\langle \varepsilon_{cyc}(z)\right\rangle\right]_\Lambda \\ \text{for all } z \in \mathbb{A}_f^\times, \, r \in \widehat{R}_{p^\infty N_+,N_-}^\times \end{array} \right. \right\},$$

and, for any finite flat extension Λ'/Λ and any Λ'-valued Dirichlet character χ, we set

(4.29)
$$\mathbf{S}^D(R_{N_+,N_-},\Lambda') = \mathbf{S}^D(R_{N_+,N_-},\Lambda) \otimes_\Lambda \Lambda' \supseteq \mathbf{S}^D(R_{N_+,N_-},\chi,\Lambda').$$

From equation (4.25), we deduce the compactness of $\mathbf{S}^D(R_{N_+,N_-},\Lambda)$, hence the ordinary projector $e^{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ converges in $\mathrm{End}_\Lambda(\mathbf{S}^D(R_{N_+,N_-},\Lambda))$, as it converges in each $\mathrm{End}_\Lambda(\mathbb{Z}_p[X_n])$. We define

(4.30)
$$e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+,N_-},\chi,\Lambda')$$

as the space of ordinary Λ-adic quaternionic forms.

4.3.2. *Hida families.* The ordinary projector e^{ord} determines an idempotent in endomorphism ring of quaternionic modular forms and this implies a decomposition into the ordinary and non-ordinary components of the eigenvarieties, which we will denote by the corresponding superscript. We obtain the following diagram of eigenvarieties:

(4.31)

All the results obtained in Section 4.1 descend to the ordinary eigenvarieties.

Proposition 4.26. *Let $(\mathcal{U},\mathcal{X},\boldsymbol{\varphi},\lambda)$ be a Hida family on $\widetilde{\mathscr{E}}_D^\epsilon$ (that is, a family in $\widetilde{\mathscr{E}}_D^{\epsilon,\mathrm{ord}}$), and suppose that it contains a classical point. Up to shrinking it, we can identify the family $(\mathcal{U},\mathcal{X},\boldsymbol{\varphi},\lambda)$ with the $\mathcal{O}(\mathcal{U})^0$-adic module*

$$(e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+,N_-},\mathcal{O}(\mathcal{U})^0))[\boldsymbol{\varphi}] = \left\{ \mathbf{f} \in \mathbf{S}^D(R_{N_+,N_-},\Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0 \;\middle|\; t \cdot \mathbf{f} = \lambda_{\boldsymbol{\varphi}}(t)\,\mathbf{f}, \; \text{for } t \in \widetilde{\mathbb{T}}_{\mathbf{I}} \right\},$$

where $\lambda_{\boldsymbol{\varphi}} = \omega_{\mathrm{GL}_2}^* \circ \lambda \colon \widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon} \longrightarrow \mathcal{O}(\mathcal{U})^0$.

Proof. The weight map ω_{GL_2} is étale at cuspidal ordinary classical points ([Bel21, Theorem 7.6.4 and Remark 7.6.6], where the weight is shifted by 2) and, by [Che04, Proposition 6.2.7], these points are dense in the family. We deduce that $\mathcal{O}(\mathcal{X}) \cong \mathcal{O}(\widetilde{JL}_p^\epsilon(\mathcal{X}))$ is a finite flat algebra over $\mathcal{O}(\mathcal{U})$, hence finite flat over $\mathbf{I}$. Up to shrinking $\mathcal{U}$ to a subaffinoid in the admissible covering $\mathscr{C}$, and intersecting its preimage with $\mathcal{X}$, we can identify $\widetilde{JL}_p^\epsilon(\mathcal{X}) = \mathcal{U}$ by [FvdP04, Lemma 8.1.3] (restrict ω_{GL_2} to a suitable wide open affinoid neighborhood in $\mathcal{X}$, hence shrink $\mathcal{U}$ to a be contained in the wide open neighborhood of the target affinoid). Therefore, ω_{GL_2} becomes an isomorphism. Let now $\mathbf{f}$ be an element in $(e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+,N_-},\Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0)[\boldsymbol{\varphi}]$; by equation (4.25), it is uniquely characterized by the sequence

$$\left\{ \mathbf{f} \;(\mathrm{mod}\; P_n) \colon X_n \longrightarrow \mathcal{O}(\mathcal{U})^0 \otimes_\Lambda \Lambda/P_n \right\}_{n \gg 1}.$$

For n big enough we consider the specialization map $\lambda_{\boldsymbol{\varphi},n}$ defined as

$$\lambda_{\boldsymbol{\varphi},n} \colon \widetilde{\mathbb{T}}_{\mathbf{I}} \longrightarrow \widetilde{\mathscr{T}}_{\mathbf{I}}^{D,\epsilon} \xrightarrow{\lambda_{\boldsymbol{\varphi}}} \mathcal{O}(\mathcal{U})^0 \longrightarrow \mathcal{O}(\mathcal{U})^0 \otimes_\Lambda \Lambda/P_n.$$

Up to $\mathcal{O}(\mathcal{U})^0$-constants, we can then associate each $\mathbf{f} \;(\mathrm{mod}\; P_n)$ to

$$x_n = (\lambda_{\boldsymbol{\varphi},n}(t))_{t \in \widetilde{\mathbb{T}}_{\mathbf{I}}} \in \mathcal{X}(\mathcal{O}(\mathcal{U})^0 \otimes_\Lambda \Lambda/P_n),$$

where x_n corresponds to the system of eigenvalues associated with the quaternionic modular form $\mathbf{f}$ (mod P_n). Note that here we are using the fact that, up to further shrinking $\mathcal{U}$, $\mathcal{O}(\mathcal{U})^0$ is a finitely generated compact Λ-module. We have then proved the sought-for identification. ∎

Let now $(\mathcal{U}, \mathcal{X}, \boldsymbol{\varphi}, \lambda)$ be a Hida family on $\widetilde{\mathscr{E}}_D^\epsilon$ and let $\boldsymbol{\varphi}^{\mathrm{GL}_2}$ be the ordinary family corresponding to it. We consider the space of Λ-adic classical Hida families and its eigenspace

$$(4.32) \quad (e^{\mathrm{ord}}\mathbf{S}^{\mathrm{GL}_2}(\Gamma_1(N_+N_-), \Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0)[\boldsymbol{\varphi}^{\mathrm{GL}_2}] = \left\{ \mathbf{f} \in \mathcal{O}(\mathcal{U})^0[\![T]\!] \,\middle|\, t \cdot \mathbf{f} = \lambda_{\boldsymbol{\varphi}^{\mathrm{GL}_2}}(t)\mathbf{f}, \text{ for } t \in \mathbb{T}_\mathbf{I} \right\}.$$

Theorem 4.27. *Suppose that $\boldsymbol{\varphi}^{\mathrm{GL}_2}$ contains a classical point. Up to shrinking $\mathcal{U}$, there exists an isomorphism of rank one $\mathcal{O}(\mathcal{U})^0$-modules*

$$(e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+, N_-}, \Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0)[\boldsymbol{\varphi}] \cong (e^{\mathrm{ord}}\mathbf{S}^{\mathrm{GL}_2}(\Gamma_1(N_+N_-), \Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0)[\boldsymbol{\varphi}^{\mathrm{GL}_2}].$$

Proof. Proceeding as in the proof of Proposition 4.26, up to shrink the family, we can assume it to be étale over the weight space and identify it with the corresponding neighborhood $\mathcal{U}$. Moreover, we can assume that all the points are minimal [Bel21, Lemma 7.4.8], therefore the corresponding eigespaces are one-dimensional. The isomorphism in the statement is then obtained keeping track of the constants in $\mathcal{O}(\mathcal{U})^0$. By Strong Multiplicity One on GL_2 and finite flatness of $\mathcal{O}(\mathcal{U})^0$ over $\mathbf{I}$, we deduce that

$$(e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+, N_-}, \Lambda) \otimes_\Lambda \mathcal{O}(\mathcal{U})^0)[\boldsymbol{\varphi}] \otimes_{\mathcal{O}(\mathcal{U})^0} \mathrm{Frac}(\mathcal{O}(\mathcal{U})^0) \cong \mathrm{Frac}(\mathcal{O}(\mathcal{U})^0).$$

The rank-1 statement follows now from the GL_2-case. An alternative approach can be obtained combining [Bel21, Theorem 7.6.4, Theorem 8.1.5 and Lemma 8.1.1], keeping in mind that we are considering cuspidal eigenvarieties and the closed immersions of eigenvarieties of [Bel21, Theorem 7.2.3] hold true. ∎

We are now ready to state the control theorem generalizing [Dal23b, Theorem 4.10]. Notice that the specialization morphism can be made explicit as in [Hsi21, Theorem 4.2].

Corollary 4.28 (Hida's Control Theorem). *For any arithmetic weight $(k, \varepsilon_n) \in \mathcal{W}(\mathbb{C}_p) \cap \mathcal{U}$,*

$$(e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+, N_-}, \mathcal{O}(\mathcal{U})^0))[\boldsymbol{\varphi}] \otimes_\Lambda \Lambda/P_{(k, \varepsilon_n)} \cong (e^{\mathrm{ord}}S_k^D(R_{p^n N_+, N_-}))[\varphi_{(k, \varepsilon_n)}].$$

Proof. The above Theorem 4.27 together with Strong Multiplicity One and Proposition 3.14 imply the isomorphism between the rank 1 modules in the statement. ∎

Remark 4.29. In the introduction (Section 1) we already pointed out that the condition for having weight one classical specializations in a family $\boldsymbol{\varphi}^{\mathrm{GL}_2}$ are rather strict. In the setting of this section, this condition can be read easily from the level N_-: a family will not contain classical weight one specializations unless $N_-^{\mathrm{sp}} = 1$, i.e. $N_- = N_-^{\mathrm{sc}}$. Therefore, throughout the rest of the paper, we will work under this assumption.

5. Balanced triple product p-adic L-function

In this section, we prove our main theorem about the existence of balanced p-adic L-functions and their interpolation property.

5.1. Definition of the p-adic L-function.
Let $\mathbf{F} = (\mathbf{f}, \mathbf{g}, \mathbf{h})$ be the triple product of primitive Hida families of tame conductors $(N_1, N_2, N_3) \in \mathbb{N}^3$ and characters (χ_1, χ_2, χ_3). Let $N = \mathrm{lcm}(N_1, N_2, N_3)$. For any classical weight (k_1, k_2, k_3), we write $(f_{k_1}, g_{k_2}, h_{k_3}) = \mathbf{F}(k_1, k_2, k_3)$; moreover, we let:

$$\Sigma^- = \{\ell \text{ finite} \mid \epsilon_\ell(f_{k_1} \times g_{k_2} \times h_{k_3}) = -1\},$$

which is independent of the choice of (k_1, k_2, k_3) by the rigidity of automorphic types.

We make the following assumptions:

- $|\Sigma^-|$ is odd; then there exists a definite quaternion algebra D over $\mathbb{Q}$ ramified exactly at the places in Σ^- (and infinity),
- if $\ell \in \Sigma^-$, then $v_\ell(N) \le 2$ (this is an improvement on Hsieh's [Hsi21] assumption that $v_\ell(N) = 1$).

We now choose test vectors on the quaternionic group $D^\times(\mathbb{A})$. First, we write

$$(5.1) \qquad \Sigma_*^{-,\mathrm{sc}} = \{\ell \in \Sigma^- \mid * \text{ is supercuspidal at } \ell\} \qquad\qquad \text{for } * \in \{f, g, h\},$$

$$(5.2) \qquad \Sigma^{-,\mathrm{sc}} = \Sigma_f^{-,\mathrm{sc}} \cup \Sigma_g^{-,\mathrm{sc}} \cup \Sigma_h^{-,\mathrm{sc}} \subseteq \Sigma^-,$$

$$(5.3) \qquad \Sigma^{-,n} = \{\ell \in \Sigma^- \mid n \text{ of } \pi_{f,\ell},\, \pi_{g,\ell},\, \pi_{h,\ell} \text{ are supercuspidal}\} \qquad \text{for } n \in \{0, 1, 2, 3\}.$$

Once again, these sets only depend on $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$.

Recall from Proposition 2.1 that $\Sigma^- = \Sigma^{-,0} \sqcup \Sigma^{-,2} \sqcup \Sigma^{-,3}$. The computations of the local integrals at $\ell \in \Sigma^-$ are in Propositions 2.7, 2.8, 2.9, respectively.

Next, we choose the signs which determine the Hida families on $D^\times$:

$$\epsilon_* \in \{\pm 1\}^{\Sigma_*^{-,\mathrm{sc}}} \qquad\qquad \text{for } * \in \{f, g, h\},$$

$$\epsilon = (\epsilon_f, \epsilon_g, \epsilon_h).$$

To shorten the notation from Section 4.3 we will write $eS^{D,\epsilon}(N, \psi, \mathcal{U})$ for the space of Λ-adic quaternionic forms $e^{\mathrm{ord}}\mathbf{S}^D(R_{N_+, N_-}, \psi_\mathbb{A}^{-1}, \mathcal{O}(\mathcal{U})^0)$ where the extra Hecke operators act accordingly to the choice of signs ϵ. Then by Theorem 4.23 and Proposition 4.26, for any classical point of the weight space there exist an open admissible affinoid neighborhood $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3$ of this point, and elements:

$$(\mathbf{f}^{D,\epsilon_f}, \mathbf{g}^{D,\epsilon_g}, \mathbf{h}^{D,\epsilon_h}) \in eS^{D,\epsilon_f}(N_1, \psi_1, \mathcal{U}_1)[\mathbf{f}] \times eS^{D,\epsilon_g}(N_2, \psi_2, \mathcal{U}_2)[\mathbf{g}] \times eS^{D,\epsilon_h}(N_2, \psi_2, \mathcal{U}_3)[\mathbf{h}].$$

Note that these choices are only well-defined up to elements in $\mathcal{O}(\mathcal{U})^\times = \mathcal{O}(\mathcal{U}_1)^\times \hat{\otimes} \mathcal{O}(\mathcal{U}_2)^\times \hat{\otimes} \mathcal{O}(\mathcal{U}_3)^\times$.

Finally, we bring these forms to the common level N, following [Hsi21, Definition 4.8].

Definition 5.1.

(1) Define the adjustments of levels $\mathbf{d}_f$, $\mathbf{d}_g$, $\mathbf{d}_h$ as in [Hsi21, Section 3.4]. At $\ell \in \Sigma^{-,\mathrm{sc}}$, we make no additional adjustment.

(2) Consider the sets $\Sigma_{*,0}^{\mathrm{IIb}}$ as in [Hsi21, Section 3.4] and define:

$$(\mathbf{f}^{D\star,\epsilon_f}, \mathbf{g}^{D\star,\epsilon_g}, \mathbf{h}^{D\star,\epsilon_h}) \in eS^{D,\epsilon_f}(N_1, \psi_1, \mathcal{U}_1)[\mathbf{f}] \times eS^{D,\epsilon_g}(N_2, \psi_2, \mathcal{U}_2)[\mathbf{g}] \times eS^{D,\epsilon_h}(N_3, \psi_3, \mathcal{U}_2)[\mathbf{h}]$$

by

$$\mathbf{f}^{D\star,\epsilon_f} = \sum_{I \subseteq \Sigma_{f,0}^{\mathrm{IIb}}} (-1)^{|I|} \beta_I(f)^{-1} V_{\mathbf{d}_f/n_f} \mathbf{f}^{D,\epsilon_f},$$

$$\mathbf{g}^{D\star,\epsilon_f} = \sum_{I \subseteq \Sigma_{g,0}^{\mathrm{IIb}}} (-1)^{|I|} \beta_I(g)^{-1} V_{\mathbf{d}_g/n_g} \mathbf{g}^{D,\epsilon_g},$$

$$\mathbf{h}^{D\star,\epsilon_h} = \sum_{I \subseteq \Sigma_{h,0}^{\mathrm{IIb}}} (-1)^{|I|} \beta_I(h)^{-1} V_{\mathbf{d}_h/n_h} \mathbf{f}^{D,\epsilon_h}.$$

We now define an unnormalized version of the triple product p-adic L-function.

Definition 5.2. Let $\mathcal{R} = \mathcal{O}(\mathcal{U}) = \mathcal{O}(\mathcal{U}_1) \hat{\otimes}_\mathcal{O} \mathcal{O}(\mathcal{U}_2) \hat{\otimes}_\mathcal{O} \mathcal{O}(\mathcal{U}_3)$.

(1) Define the *triple product* $\mathbf{F}^{D\star,\epsilon} \colon (D^\times \backslash \widehat{D}^\times)^3 \to \mathcal{R}$ by $\mathbf{F}^{D\star,\epsilon} = \mathbf{f}^{D\star,\epsilon_f} \boxtimes \mathbf{g}^{D\star,\epsilon_g} \boxtimes \mathbf{h}^{D\star,\epsilon_h}$.

(2) The associated *theta element* is:

$$\Theta_{\mathbf{F}^{D\star,\epsilon}} = (\mathbf{F}^{D\star,\epsilon})^*(\Delta_\infty^\dagger) \in \mathcal{R},$$

where $\Delta_\infty^\dagger$ is the *regularized diagonal cycle* from [Hsi21, Definition 4.6].

Since $\mathbf{F}^{D\star,\epsilon}$ is only well-defined up to scalars, so is $\Theta_{\mathbf{F}^{D\star,\epsilon}}$. To define the genuine p-adic L-function, we will divide $\Theta_{\mathbf{F}^{D\star,\epsilon}}$ by the Petersson norm of $\mathbf{F}^{D\star,\epsilon}$.

Recall the quaternionic Petersson product introduced in Section 3.6.1. Considering such pairing, for $\mathbf{f}$ and $\mathbf{f}' \in \mathbf{S}^D(N_+N_-, \chi, \mathbf{I})$ we define

$$(5.4) \qquad \mathbf{B}_{N_+, N_-, \alpha}(\mathbf{f}, \mathbf{f}') = \sum_{[x] \in D^\times \backslash \widehat{D}^\times / \widehat{R}^\times_{N_+p^\alpha, N_-}} \frac{\chi_{\mathbb{A}}(\nu_{\mathbb{A}_f}(x)) \langle \mathrm{N}_p(x) \rangle [\langle \mathrm{N}_p(x) \rangle]_{\mathbf{I}}}{\#\Gamma_{N_+p^\alpha, N_-}(x)} \mathbf{f}\left(x\, \tau^D_{N_+p^\alpha, N_-}\right) \mathbf{f}'(x).$$

As in [Hsi21, Definition 4.3], we obtain a Hecke-equivariant $\mathbf{I}$-bilinear pairing

$$(5.5) \qquad\qquad\qquad \mathbf{B}_N \colon e\mathbf{S}^D(N, \chi, \mathbf{I}) \times e\mathbf{S}^D(N, \chi, \mathbf{I}) \to \mathbf{I},$$

$$(5.6) \qquad\qquad\qquad \mathbf{B}_N(\mathbf{f}, \mathbf{f}') = \varprojlim_\alpha \mathbf{B}_{N,\alpha}(\mathbf{f}, \mathbf{f}') \in \varprojlim_\alpha \mathbf{I}/P_\alpha = \mathbf{I}$$

such that for every arithmetic point $\kappa = (k, \epsilon)$ and integer $\alpha \geq \max\{1, v_p(\mathrm{cond}(\epsilon))\}$, we have (cf. [Hsi21, Proof of Lemma 4.4])

$$(5.7) \qquad\qquad\qquad \mathbf{B}_N(\mathbf{f}, \mathbf{f}')(\kappa) = (-1)^k \langle \mathbf{U}_p^{-\alpha} \mathbf{f}_\kappa, \mathbf{f}'_\kappa \rangle_{N_+p^\alpha, N_-}.$$

Definition 5.3. For $\mathbf{f}^D \in e\mathbf{S}^D(N, \chi, \mathcal{O}(\mathcal{U}))$, the *Petersson inner product* of $\mathbf{f}^D$ is

$$\eta_{\mathbf{f}^D} = \mathbf{B}_N(\mathbf{f}^D, \mathbf{f}^D) \in \mathcal{O}(\mathcal{U})$$

and the *Petersson norm* of $\mathbf{f}^D$ is:

$$\|\mathbf{f}^D\| = \eta_{\mathbf{f}^D}^{1/2} \in \mathrm{Frac}\,\mathcal{O}(\mathcal{U}).$$

Similarly, for a triple product $\mathbf{F}^D = \mathbf{f}^D \boxtimes \mathbf{g}^D \boxtimes \mathbf{h}^D$, its *Petersson norm* is:

$$\|\mathbf{F}^D\| = \|\mathbf{f}^D\| \cdot \|\mathbf{g}^D\| \cdot \|\mathbf{h}^D\| \in \mathrm{Frac}\,\mathcal{O}(\mathcal{U}).$$

Before defining the p-adic L-function we recall two more technical points from [Hsi21], related to the choice of test vectors at primes dividing N^+:

- There is a twist $\mathbf{F}' = (\mathbf{f} \otimes \chi_1, \mathbf{g} \otimes \chi_2, \mathbf{h} \otimes \chi_3)$ by Dirichlet characters χ_1, χ_2, χ_3 modulo M with $M^2 | N^+$ such that $\chi_1\chi_2\chi_3 = 1$ and $\mathbf{F}'$ satisfies Hypothesis 6.1 of loc. cit. (see also Remark 6.2).
- There is a *fudge factor* $\mathfrak{f}_{\mathbf{F}'} = \prod_{q|N^+} f_{\mathbf{F}',q} \in \mathcal{R}^\times$ defined in Proposition 6.12 of loc. cit., and, enlarging $\mathcal{O}$ if necessary, we have that $\sqrt{\mathfrak{f}_{\mathbf{F}'}} \in \mathcal{R}^\times$.

By [Hsi21, Lemma 6.11], there exists $\epsilon^{\Sigma^-}(\mathbf{F}) \in \mathcal{R}^\times$ such that

$$\epsilon^{\Sigma^-}(\mathbf{F})(\kappa) = \epsilon^{\Sigma^-}(f_{\kappa_1}) \epsilon^{\Sigma^-}(g_{\kappa_2}) \epsilon^{\Sigma^-}(h_{\kappa_3})$$

is the product of the away-from-Σ^- parts of the root numbers.

Definition 5.4. The *(genuine) square root balanced triple product p-adic L-function* associated with $\mathbf{F}$ and ϵ is:

$$\mathcal{L}^{\mathrm{bal}}_{\mathbf{F},\epsilon}(\kappa) = \frac{\Theta_{\mathbf{F}'^{D\star,\epsilon}}}{\|\mathbf{F}^{D\star,\epsilon}\|} \cdot 2^{-\frac{|\Sigma^-|+1-k_1-k_2-k_3}{2}} \cdot (N^-)^{-1/2} \cdot \epsilon^{\Sigma^-}(\mathbf{F})^{-1/2} \cdot \sqrt{\mathfrak{f}_{\mathbf{F}'}}^{-1} \prod_{\ell \in \Sigma^{-,\mathrm{sc}}} \frac{\ell^{k_1+k_2+k_3-13/2}}{\sqrt{\zeta_\ell(2)}} \in \mathrm{Frac}\,\mathcal{R}.$$

5.2. The interpolation property.

Consider the subset of *arithmetic points* in $\mathcal{U}$:

$$(5.8) \qquad \mathcal{U}^{\mathrm{arith}} = \left\{ \kappa = (k_1, k_2, k_3, \chi_1, \chi_2, \chi_3) \in \mathcal{U}_1^{\mathrm{arith}} \times \mathcal{U}_2^{\mathrm{arith}} \times \mathcal{U}_3^{\mathrm{arith}} \mid k_1 + k_2 + k_3 \equiv 0 \pmod 2 \right\},$$

and let $\mathcal{U}^{\mathrm{bal}}$ be the subset of *balanced arithmetic points*:

$$(5.9) \qquad \mathcal{U}^{\mathrm{bal}} = \left\{ \kappa = (k_1, k_2, k_3, \chi_1, \chi_2, \chi_3) \in \mathcal{U}^{\mathrm{arith}} \mid k_1 + k_2 + k_3 > 2k_i \text{ for all } i = 1, 2, 3 \right\}.$$

Theorem 5.5. *For an arithmetic point $\kappa = (k_1, k_2, k_3, \chi_1, \chi_2, \chi_3) \in \mathcal{U}^{\mathrm{bal}}$ in the balanced range we have:*

$$(\mathcal{L}_{\mathbf{F},\epsilon}^{\mathrm{bal}}(\kappa))^2 = \frac{\Gamma_{\mathbf{V}_\kappa^\dagger}(0) \cdot L(\mathbf{V}_\kappa^\dagger, 0)}{(\sqrt{-1})^{k_1+k_2+k_3-1} \langle \mathbf{F}_\kappa, \mathbf{F}_\kappa \rangle} \cdot \mathcal{E}_p^{\mathrm{bal}}(\mathbf{V}_\kappa) \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} (1-q^{-1})^2 \cdot \prod_{\ell \in \Sigma^{-,2}} \frac{1 + \epsilon_{\ell,1}\epsilon_{\ell,2}\sqrt{\omega_3(\ell)}}{2} \prod_{\ell \in \Sigma^{-,3}} \frac{1 + \epsilon_{\ell,1}\epsilon_{\ell,2}\epsilon_{\ell,3}}{4},$$

where $\mathcal{E}_p(V_\kappa)$ is an Euler factor defined by:

$$\mathcal{E}_p^{\mathrm{bal}}(\mathbf{V}_\kappa) = \frac{\mathcal{E}_p(\mathrm{Fil}_{\mathrm{bal}}^+ \mathbf{V}_\kappa)}{\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})}$$

and $\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})$ is a modified adjoint Euler factor (5.12).

In particular, $\mathcal{L}_{\mathbf{F},\epsilon}^{\mathrm{bal}} \neq 0$ only if ϵ satisfies:

(1) *if $\ell \in \Sigma^{-,2}$, then $\epsilon_\ell = (\epsilon_{\ell,1}, \epsilon_{\ell,2})$: if the special representation is a twist of Steinberg by ω_3 such that $\omega_3(\ell) = 1$, then $\epsilon_{\ell,1} \cdot \epsilon_{\ell,2} = 1$,*

(2) *if $\ell \in \Sigma^{-,3}$, then $\epsilon_{\ell,1} \cdot \epsilon_{\ell,2} \cdot \epsilon_{\ell,3} = 1$.*

Remark 5.6. In this extended remark, we compare our p-adic L-function to the one defined by Hsieh [Hsi21, Theorem B]. The main difference is that his p-adic L-function is defined without normalizing by the Petersson norm $\|\mathbf{F}^{D*,\epsilon}\|$ on the quaternion algebra. As a result, note that Gross periods ([Hsi21, Definition 4.12]) do not feature in the interpolation formula for our p-adic L-function.

We instead use the above definition for two reasons:

(1) It seems that this is the correct p-adic L-function to state an Elliptic Stark Conjecture 6.5 for. For example, we will soon see in Theorem 6.9 that it is this p-adic L-function that admits a natural factorization in the CM case.

(2) We did not prove the analogue of Theorem 4.5 in loc. cit.: that the $\mathbf{I}$-modules $eS^{D,\pm}(N, \mathbf{I})[\lambda_\mathbf{f}^D]$ are free of rank one over the whole algebra $\mathbf{I}$. Instead, we satisfy ourselves with the local statement in Theorems 4.23 and 4.27, because it is enough for our arithmetic applications. However, this means that our choices of vectors $\mathbf{F}^{D\star,\epsilon}$ are only well-defined up to scalars, and hence only the quotient by the Petersson norm is well-defined.

The terminology *genuine p-adic L-function* is inspired by the discussion after [Hsi21, Theorem A]. The advantage is that it is independent of choices, but the disadvantage is that it is only an element of Frac $\mathcal{R}$ and not $\mathcal{R}$.

The denominators of $\mathcal{L}_{\mathbf{F},\epsilon}^{\mathrm{bal}}$ should be captured by the congruence module of $\mathbf{F}^{D\star,\epsilon}$, which in turn should be related to the congruence module for $\mathbf{F}$ (see [Hsi21, Remark 7.8]). Indeed, if one could choose vectors $\mathbf{f}^{D,\pm} \in eS^{D,\pm}(N, \mathbf{I})[\lambda_\mathbf{f}^D]$ as in [Hsi21, Theorem 4.5], then one could also define a p-adic L-function in $\mathbf{I}\hat{\otimes}\mathbf{I}\hat{\otimes}\mathbf{I}$, generalizing the one constructed in loc. cit. It would then differ from our genuine p-adic L-function by $\sqrt{\eta_{\mathbf{F}'^{D\star,\epsilon}}}$ and one could presumably show that $\eta_{\mathbf{F}'^{D\star,\epsilon}}$ is a generator for the congruence module as in [Hsi21, Section 7.2]. We decided to defer these questions to future work.

Remark 5.7. Note that the Euler factor at p:

$$\mathcal{E}_p(\mathbf{V}_\kappa) = \frac{\mathcal{E}_p(\mathrm{Fil}_{\mathrm{bal}}^+ \mathbf{V}_\kappa)}{\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})}$$

is analogous to the Euler factor for the unbalanced p-adic L-function constructed by Darmon–Rotger [DR14, Theorem 1.3]:

$$\frac{\mathcal{E}(f, g, h)}{\mathcal{E}_0(f)\mathcal{E}_1(f)} = \frac{(1 - \beta_f \alpha_g \alpha_h p^{-c})(1 - \beta_f \alpha_g \beta_h p^{-c})(1 - \beta_f \beta_g \alpha_h p^{-c})(1 - \beta_f \beta_g \beta_h p^{-c})}{(1 - \beta_f^2 \chi_f^{-1}(p)p^{1-k})(1 - \beta_f^2 \chi_f^{-1}(p)p^{-k})}.$$

The proof of Theorem 5.5 will occupy the rest of this section and amounts to generalizing the results of [Hsi21, Section 4]. We split it into two parts:

- an intermediate interpolation property obtained from Ichino's formula [Ich08], with factors coming from certain normalized local zeta integrals,

  • the evaluation of the local zeta integrals.

5.3. An intermediate interpolation property.

We start by proving the analogue of [Hsi21, Corollary 4.13], which gives an interpolation property up to certain local factors.

Proposition 5.8. *Let:*

$$
(5.10) \qquad \mathscr{I}^{\star}_{\Pi_{\kappa,q}} = \begin{cases} I_q(\phi_q^{\star} \otimes \widetilde{\phi}_q^{\star}) \cdot B_{\Pi_q} \cdot \dfrac{\zeta_q(1)^2}{|N|_q^2 \zeta_q(2)^2} \cdot \omega_{F,q}^{-1}(\mathbf{d}_F)|\mathbf{d}_F^{k-2}|_q & \text{for } q|N^+, \\[2ex] I_q(\phi_q \otimes \widetilde{\phi}_q) \cdot B_{\Pi_q} \cdot q\dfrac{\zeta_q(1)^5}{\zeta_q(2)^3} \cdot \omega_{F,q}^{-1}(\mathbf{d}_F)|\mathbf{d}_F^{k-2}|_q & \text{for } q \in \Sigma^{-,2}, \\[2ex] I_q(\phi_q \otimes \widetilde{\phi}_q) \cdot B_{\Pi_q} \cdot q\dfrac{\zeta_q(1)^4}{\zeta_q(2)^2} \cdot \omega_{F,q}^{-1}(\mathbf{d}_F)|\mathbf{d}_F^{k-2}|_q & \text{for } q \in \Sigma^{-,3} \end{cases}
$$

be the normalized local zeta integral at $q|(N/d^-)$, with $I_q(\phi_q^{\star} \otimes \widetilde{\phi}_q^{\star})$ defined in equation (5.13), and let:

$$
(5.11) \qquad \mathscr{I}^{\mathrm{bal}}_{\Pi_{\kappa,q}} = I_p^{\mathrm{ord}}(\phi_p, \breve{\mathbf{t}}_n) \cdot B_{\Pi_p^{\mathrm{ord}}}^{[n]} \frac{\omega_{F,p}^{1/2}(-p^{2n})|p|_p^{-n(k_1+k_2+k_3)}}{\alpha_p(F)^{2n}\zeta_p(2)^2}.
$$

be the normalized p-adic zeta integral defined in [Hsi21, (4.21)] (see loc. cit. for details).

  Then for $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathcal{U}^{\mathrm{bal}}$ in the balanced range, we have the interpolation formula:

$$
\frac{\Theta_{\mathbf{F}^{D\star}}(\kappa)^2}{\langle F^D, F^D \rangle} = 2^{\#\Sigma^{-,0}+1-k_1-k_2-k_3} \cdot N^- \cdot \frac{L(1/2, \Pi_\kappa) \cdot \epsilon^{\Sigma^-}(f_{k_1})\epsilon^{\Sigma^-}(g_{k_2})\epsilon^{\Sigma^-}(h_{k_3})}{\langle f_{k_1}, f_{k_1}\rangle\langle g_{k_2}, g_{k_2}\rangle\langle h_{k_3}, h_{k_3}\rangle} \cdot \frac{\mathscr{I}^{\mathrm{bal}}_{\Pi_{\kappa,p}}}{\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})} \cdot \prod_{q|(N/d^-)} \mathscr{I}^{\star}_{\Pi_{\kappa,q}}.
$$

where $f_{k_1}, g_{k_2}, h_{k_3}$ are the newforms associated with the specializations of $\mathbf{F}$ at κ,

$$
\epsilon^{\Sigma^-}(F) = \prod_{\ell|N/d^-} \epsilon(1/2, \pi_{F,\ell})|N_F|_\ell^{(2-k_i)/2} \in \widehat{\mathbb{Z}}_{(p)}^{\times}
$$

is the away-from-Σ^- part of the root number of $F \in \{f_{k_1}, g_{k_2}, h_{k_3}\}$, and

$$
(5.12) \qquad \mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad}) = \mathcal{E}_p(f_{k_1}\,\mathrm{Ad})\mathcal{E}_p(g_{k_2}, \mathrm{Ad})\mathcal{E}_p(h_{k_3}, \mathrm{Ad}).
$$

is the modified Euler factor in [Hsi21, (3.10)].

Proof. Let $\kappa = (\kappa_1, \kappa_2, \kappa_3)$ for $\kappa_i = (k_i, \chi_i)$. We write

$$
(5.13) \qquad I_\ell(\phi_\ell \otimes \widetilde{\phi}_\ell) = \frac{L(1, \Pi_v, \mathrm{Ad})}{\zeta_\ell(2)^2 L(1/2, \Pi_v)} \cdot \frac{I'_\ell(\phi_\ell \otimes \widetilde{\phi}_\ell)}{\langle \phi_\ell, \widetilde{\phi}_\ell \rangle}
$$

for the normalized local integrals. Then Ichino's Formula [Ich08, Theorem 1.1, Remark 1.3] gives the following expression (by combining [Hsi21, p. 473, proof of Proposition 4.10] with Lemma 3.21):

$$
(5.14) \qquad \frac{I(\varrho(\breve{\mathbf{t}}_n)\phi_F^{D\star})^2}{\langle F^D, F^D \rangle} = \frac{\mathrm{vol}(\widehat{\mathcal{O}}_D^{\times})}{8} \cdot \frac{\zeta_{\mathbb{Q}}(2)^2 L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})}
$$
$$
\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q \in \Sigma^- \cup \{\infty\}} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q \notin p \cup \Sigma^-} I_q(\phi_q^{\star} \otimes \widetilde{\phi}_q^{\star})
$$
$$
\cdot \omega_F^{-1/2}(\hat{N}_1^+)\omega_{F,p}^{-1}(p^n)\alpha_p(F)^{2n} \prod_{i=1}^3 \frac{\mathrm{vol}(\widehat{R}_{N_i p^{2n}}^{\times})}{(N_i^+ N_i^{-,\mathrm{sc}} p^{2n})^{(k_i-2)/2}(k_i-1)},
$$

where

$$
\langle F^D, F^D \rangle = \langle U_p^{-n} f^D, f^D \rangle_{N_1 p^n} \langle U_p^{-n} g^D, g^D \rangle_{N_2 p^n} \langle U_p^{-n} h^D, h^D \rangle_{N_3 p^n}.
$$

Next, we use the volume formula from [Piz80b, Theorem 3.4]:

$$
\mathrm{vol}(\widehat{R}_N^{\times}) = \frac{48}{N} \prod_{q||N^-} \zeta_q(1) \prod_{q^2||N^-} \zeta_q(2) \prod_{q|N^+} \frac{\zeta_q(2)}{\zeta_q(1)}
$$

$$= \frac{48}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \prod_{q||N^-} \frac{1 + q^{-1}}{1 - q^{-1}} \prod_{q^2||N^-} \frac{1 + q^{-1}}{1 - q^{-2}}$$

$$= \frac{48}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \prod_{q||N^-} \frac{\zeta_q(1)^2}{\zeta_q(2)} \prod_{q^2||N^-} \zeta_q(1).$$

Using this, the final factor in equation (5.14) becomes:

$$(5.15) \qquad \prod_{i=1}^{3} \frac{48}{(N_i^+ N_i^{-,\mathrm{sc}} p^{2n})^{(k_i-2)/2} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_i p^{2n})](k_i - 1)} \prod_{q||N_i^-} \frac{\zeta_q(1)^2}{\zeta_q(2)} \prod_{q^2||N_i^-} \zeta_q(1).$$

As we will need it shortly, we also record that:

$$(5.16) \qquad \frac{\mathrm{vol}(\widehat{\mathcal{O}}_D^\times)}{\mathrm{vol}(\widehat{R}_N^\times)^2} = \frac{1}{48} \prod_{q||N^-} q \frac{1}{\zeta_q(1)} \prod_{q^2||N^-} q^3 \frac{\zeta_q(1)}{\zeta_q(2)^2} \prod_{q|N^+} \frac{\zeta_q(1)^2}{|N|_q^2 \zeta_q(2)^2}.$$

Next, we recall the relationship between the Petersson norm of a newform f and the adjoint L-value [Hsi21, (2.18)]:

$$(5.17) \qquad \langle f, f \rangle_{\Gamma_0(N)} = \frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]}{2^k w(f)} \cdot L(1, \pi, \mathrm{Ad}) \cdot \prod_{q|N} B_{\pi_q},$$

where B_{π_q} are given by [Hsi21, (2.18)]. In particular:

$$(5.18) \qquad L(1, \Pi, \mathrm{Ad}) = \prod_{i=1}^{3} \langle f_i^\circ, f_i^\circ \rangle \frac{2^{k_i} w(f_i^\circ)}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_i p^{c_i})] \prod_{q|N} B_{(\pi_i)_q}},$$

where f_i° is the newform associated with f_i and its level is $N p^{c_i}$. We will write $B_{\Pi_{F,q}} = B_{\pi_{1,q}} B_{\pi_{2,q}} B_{\pi_{3,q}}$, as in loc. cit..

Finally, Hsieh [Hsi21, Proposition 4.9] gives a relationship between $I(\varrho(\breve{\mathbf{t}}_n)\phi_F^{D\star})$ and $\Theta_{\mathbf{F}^{D\star}}(\underline{Q})$:

$$(5.19) \qquad \Theta_{\mathbf{F}^{D\star}}(\kappa) = \frac{1}{\mathrm{vol}(\widehat{R}_N^\times)} I(\varrho(\breve{\mathbf{t}}_n)\phi_F^{D\star}) \cdot \frac{\omega_{F,p}^{1/2}(p^n)|p|^{-\frac{k_1+k_2+k_3}{2}}}{\alpha_p(F)^n \zeta_p(2)} \frac{1}{\omega_F^{1/2}(\hat{\mathbf{d}}_f) \mathbf{d}_F^{(k-2)/2}}.$$

Next, we put all of these facts together to get:

$$\frac{\Theta_{\mathbf{F}^{D\star}}(\kappa)^2}{\langle F^D, F^D \rangle} = \frac{1}{\mathrm{vol}(\widehat{R}_N^\times)^2} \frac{I(\varrho(\breve{\mathbf{t}}_n)\phi_F^{D\star})^2}{\langle F^D, F^D \rangle} \cdot \frac{\omega_{F,p}(p^n)|p^n|^{-(k_1+k_2+k_3)}}{\alpha_p(F)^{2n} \zeta_p(2)^2} \frac{1}{\omega_F(\hat{\mathbf{d}}_f) \mathbf{d}_F^{k-2}} \tag{5.19}$$

$$= \frac{\mathrm{vol}(\hat{\mathcal{O}}_D^\times)}{8 \mathrm{vol}(\widehat{R}_N^\times)^2} \cdot \frac{\zeta_{\mathbb{Q}}(2)^2 L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \tag{5.14}$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q \in \Sigma^- \cup \{\infty\}} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q \notin p \cup \Sigma^-} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)$$

$$\cdot \omega_F^{-1/2}(\hat{N}_1^+) \omega_{F,p}^{-1}(p^n) \alpha_p(F)^{2n} \prod_{i=1}^{3} \frac{\mathrm{vol}(\widehat{R}_{N_i p^{2n}}^\times)}{(N_i^+ N_i^{-,\mathrm{sc}} p^{2n})^{k_i - 2/2}(k_i - 1)}$$

$$\cdot \frac{\omega_{F,p}(p^n)|p^n|^{-(k_1+k_2+k_3)}}{\alpha_p(F)^{2n} \zeta_p(2)^2} \frac{1}{\omega_F(\hat{\mathbf{d}}_f) \mathbf{d}_F^{k-2}}$$

$$= \frac{\mathrm{vol}(\hat{\mathcal{O}}_D^\times)}{8 \mathrm{vol}(\widehat{R}_N^\times)^2} \cdot \frac{\zeta_{\mathbb{Q}}(2)^2 L(1/2, \Pi)}{L(1, \Pi, \mathrm{Ad})} \tag{5.15}$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q \in \Sigma^- \cup \{\infty\}} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q \notin p \cup \Sigma^-} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)$$

$$\cdot \left(\prod_{i=1}^{3} \frac{48}{(N_i^+ N_i^{-,\mathrm{sc}} p^{2n})^{(k_i-2)/2} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_i p^{2n})](k_i - 1)} \prod_{q||N_i^-} \frac{\zeta_q(1)^2}{\zeta_q(2)} \prod_{q^2||N_i^-} \zeta_q(1) \right)$$

$$\cdot \frac{p^{6n}}{\zeta_p(2)^2} \frac{\omega_F^{-1/2}(\hat{N}_1^+)}{\omega_F(\hat{\mathbf{d}}_f)\mathbf{d}_F^{\frac{k-2}{2}}}$$

$$= \frac{48^3 \operatorname{vol}(\hat{\mathcal{O}}_D^\times)}{8\operatorname{vol}(\widehat{R}_N^\times)^2} \cdot \frac{\zeta_{\mathbb{Q}}(2)^2 L(1/2,\Pi)}{\langle f^\circ, f^\circ \rangle \langle g^\circ, g^\circ \rangle \langle h^\circ, h^\circ \rangle} \cdot \left(\prod_{q|N} B_{\Pi_{F,q}} \right) \cdot 2^{-k_1-k_2-k_3} w(f^\circ)w(g^\circ)w(h^\circ) \qquad (5.18)$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q\in\Sigma^-\cup\{\infty\}} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q\notin p\cup\Sigma^-} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)$$

$$\cdot \left(\prod_{i=1}^3 \frac{1}{(N_i^+ N_i^{-,\mathrm{sc}})^{\frac{k_i-2}{2}}(k_i-1)} \right) \left(\prod_{q||N^-} \frac{\zeta_q(1)^6}{\zeta_q(2)^3} \right) \left(\prod_{q\in\Sigma^{-,2}} \frac{\zeta_q(1)^4}{\zeta_q(2)} \right) \left(\prod_{q\in\Sigma^{-,3}} \zeta_q(1)^3 \right)$$

$$\cdot \left(\prod_{i=1}^3 [\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(p^{c_i})]p^{-2n}(1+p^{-1})^{-1} \right) \frac{p^{6n}}{\zeta_p(2)^2} \frac{\omega_F^{-1/2}(\hat{N}_1^+)}{\omega_F(\hat{\mathbf{d}}_f)\mathbf{d}_F^{\frac{k-2}{2}}}$$

$$= 2^5 3^2 \cdot \frac{\zeta_{\mathbb{Q}}(2)^2 L(1/2,\Pi)}{\langle f^\circ, f^\circ \rangle \langle g^\circ, g^\circ \rangle \langle h^\circ, h^\circ \rangle} \cdot \left(\prod_{q|N} B_{\Pi_{F,q}} \right) \cdot 2^{-k_1-k_2-k_3} w(f^\circ)w(g^\circ)w(h^\circ) \qquad (5.16)$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q\in\Sigma^-\cup\{\infty\}} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q\notin p\cup\Sigma^-} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)$$

$$\cdot \left(\prod_{i=1}^3 \frac{1}{(N_i^+ N_i^{-,\mathrm{sc}})^{\frac{k_i-2}{2}}(k_i-1)} \right) \left(\prod_{q||N^-} q\frac{\zeta_q(1)^5}{\zeta_q(2)^3} \right)$$

$$\cdot \left(\prod_{q\in\Sigma^{-,2}} q^3\frac{\zeta_q(1)^5}{\zeta_q(2)^3} \right) \left(\prod_{q\in\Sigma^{-,3}} q^3\frac{\zeta_q(1)^4}{\zeta_q(2)^2} \right) \left(\prod_{q|N^+} \frac{\zeta_q(1)^2}{|N|_q^2\zeta_q(2)^2} \right)$$

$$\cdot \left(\prod_{i=1}^3 [\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(p^{c_i})]p^{-2n}(1+p^{-1})^{-1} \right) \frac{p^{6n}}{\zeta_p(2)^2} \frac{\omega_F^{-1/2}(\hat{N}_1^+)}{\omega_F(\hat{\mathbf{d}}_f)\mathbf{d}_F^{\frac{k-2}{2}}}.$$

Next, we recall the computation of some of factors from [Hsi21, p. 474, p. 478]:

$$(5.20) \qquad I_\infty(\phi_\infty \otimes \widetilde{\phi}_\infty) = (4\pi^2)^{-1}(k_1-1)(k_2-1)(k_3-1)$$

$$(5.21) \qquad I_q(\phi_q \otimes \widetilde{\phi}_q) = 2\zeta_q(1)^{-2} \qquad\qquad\qquad\qquad\qquad\qquad \text{for } q||N^-,$$

$$(5.22) \qquad B_{\Pi_q} = (-1)\frac{\zeta_q(2)^3}{\zeta_q(1)^3} \qquad\qquad\qquad\qquad\qquad\qquad \text{for } q||N^-.$$

Plugging these into the above equations and recalling that $\zeta_{\mathbb{Q}}(2) = \frac{\pi}{6}$ and $\omega_{F,q}^{1/2}(N_f^+)B_{\Pi_q} = B_{\Pi_{F,q}}$ for $q \neq p$ by definition ([Hsi21, p. 477]), we can simplify the final expression in the chain of equalities to:

$$\frac{\Theta_{\mathbf{F}^{D\star}}(\kappa)^2}{\langle F^D, F^D \rangle} = (-2)^{\#\Sigma^{-,0}} 2^{1-k_1-k_2-k_3} \cdot N^- \cdot \frac{L(1/2,\Pi)}{\langle f^\circ, f^\circ \rangle \langle g^\circ, g^\circ \rangle \langle h^\circ, h^\circ \rangle} \cdot \left(\prod_{q|N/d^-} B_{\Pi_q} \right) \cdot w(f^\circ)w(g^\circ)w(h^\circ)$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi}_p, \breve{\mathbf{t}}_n) \prod_{q^2||N^-} I_q(\phi_q \otimes \widetilde{\phi}_q) \prod_{q|N^+} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)$$

$$\cdot \left(\prod_{i=1}^3 \frac{1}{(N_i^+ N_i^{-,\mathrm{sc}})^{\frac{k_i-2}{2}}} \right) \left(\prod_{q\in\Sigma^{-,2}} q^3\frac{\zeta_q(1)^5}{\zeta_q(2)^3} \right) \left(\prod_{q\in\Sigma^{-,3}} q^3\frac{\zeta_q(1)^4}{\zeta_q(2)^2} \right) \left(\prod_{q|N^+} \frac{\zeta_q(1)^2}{|N|_q^2\zeta_q(2)^2} \right)$$

$$\cdot \left(\frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(p^{c_i})]}{1+p^{-1}} \right) \zeta_p(2)^{-2} \frac{\omega_F^{-1/2}(\hat{N}^-)}{\omega_F(\hat{\mathbf{d}}_f)\mathbf{d}_F^{\frac{k-2}{2}}}.$$

Next, we recall that for $F \in \{f, g, h\}$:

$$w(F) = \prod_{\ell < \infty} \epsilon(1/2, \pi_{F,\ell})$$

$$= \epsilon(1/2, \pi_{F,p}) \prod_{q||N^-} \epsilon(1/2, \pi_{F,q}) \prod_{q|N/d^-} \epsilon(1/2, \pi_{F,q})$$

and for $q||N^-$:

$$-1 = \epsilon(1/2, \Pi_q) = \omega_{F,q}^{-1/2}(q)\epsilon(1/2, \pi_{f_1,q})\epsilon(1/2, \pi_{f_2,q})\epsilon(1/2, \pi_{f_3,q}).$$

We define the away-from-Σ^- part of the conductor to be:

$$\epsilon^{\Sigma^-}(F) = \prod_{q|N/d^-} \epsilon(1/2, \pi_{F,q})|N_F|_q^{(2-\underline{k})/2} \in \widehat{\mathbb{Z}}_{(p)}^\times.$$

This gives:

$$\frac{\Theta_{\mathbf{F}^{D\star}}(\kappa)^2}{\langle F^D, F^D\rangle} = 2^{\#\Sigma^{-,0}+1-k_1-k_2-k_3} \cdot N^- \cdot \frac{L(1/2, \Pi)}{\langle f^\circ, f^\circ\rangle\langle g^\circ, g^\circ\rangle\langle h^\circ, h^\circ\rangle} \cdot \epsilon^{\Sigma^-}(f^\circ) \cdot \epsilon^{\Sigma^-}(g^\circ) \cdot \epsilon^{\Sigma^-}(h^\circ)$$

$$\cdot I_p^{\mathrm{ord}}(\phi_p \otimes \widetilde{\phi_p}, \breve{\mathbf{t}}_n)B_{\Pi_{F,p}}\left(\prod_{i=1}^3 \frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(p^{c_i})]\epsilon(1/2, \pi_{f_i,p})}{1+p^{-1}}\right) \cdot \zeta_p(2)^{-2}$$

$$\cdot \prod_{q\in\Sigma^{-,2}} I_q(\phi_q \otimes \widetilde{\phi}_q)B_{\Pi_q}q\frac{\zeta_q(1)^5}{\zeta_q(2)^3}\omega_{F,q}^{-1}(\hat{\mathbf{d}}_f)|\mathbf{d}_F^{\frac{k-2}{2}}|_q$$

$$\cdot \prod_{q\in\Sigma^{-,3}} I_q(\phi_q \otimes \widetilde{\phi}_q)B_{\Pi_q}q\frac{\zeta_q(1)^4}{\zeta_q(2)^2}\omega_{F,q}^{-1}(\hat{\mathbf{d}}_f)|\mathbf{d}_F^{\frac{k-2}{2}}|_q$$

$$\cdot \prod_{q|N^+} I_q(\phi_q^\star \otimes \widetilde{\phi}_q^\star)B_{\Pi_q}\frac{\zeta_q(1)^2}{|N|_q^2\zeta_q(2)^2}\omega_{F,q}^{-1}(\hat{\mathbf{d}}_f)|\mathbf{d}_F^{\frac{k-2}{2}}|_q.$$

To finish the proof, we recall from [Hsi21, p. 477] that:

$$(5.23) \qquad \frac{B_{\Pi_p^{\mathrm{ord}}}^{[n]}}{B_{\Pi_{F,p}}} = \omega_{f,p}^{1/2}(-p^{-2n})\prod_{i=1}^3 \frac{\alpha_{f_i,p}| \cdot |_p^{1/2}(p^{2n})}{\epsilon(1/2, \pi_{f_i,p})} \cdot \frac{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(p^{c_i})]}{1+p^{-1}} \cdot \mathcal{E}_p(f_i, \mathrm{Ad}). \qquad \blacksquare$$

5.4. **Computation of local factors.** By Proposition 5.8, proving Theorem 5.5 amounts to computing the local factors. For p and $q|N^+$, they were already computed by Hsieh.

Proposition 5.9 (Hsieh)**.**

  (1) We have that:
$$\mathscr{I}_{\Pi_{\kappa,p}}^{\mathrm{bal}} = \mathcal{E}_{\mathrm{bal}}(\Pi_{\kappa,p}) \cdot \frac{1}{L(1/2, \Pi_{\kappa,p})}.$$

  (2) For $q|N^+$:
$$\mathscr{I}_{\Pi_{\kappa,q}}^\star = \mathfrak{f}_{\mathbf{F},q}(\kappa) \cdot \begin{cases} (1+q^{-1})^2 & \text{if } q \in \Sigma_{\mathrm{exc}}, \\ 1 & \text{otherwise} \end{cases}$$

Proof. Part (1) is [Hsi21, Proposition 5.6] and part (2) is [Hsi21, Proposition 6.12]. $\blacksquare$

Therefore, it remains to compute $\mathscr{I}_{\Pi_{\kappa,\ell}}$ for $\ell \in \Sigma^-$ such that $\ell^2||N$, i.e. for $\ell \in \Sigma^{-,2}$ and $\ell \in \Sigma^{-,3}$.

Proposition 5.10. *Suppose $\ell \in \Sigma^{-,2}$ and $\omega_{F,\ell} = 1$. Then:*
$$\mathscr{I}_{\Pi_{\kappa,\ell}}^\star = -\frac{1+\epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2}\frac{\ell^{-2(k_1+k_2+k_2)+13}}{\zeta_\ell(2)}.$$

Proof. We use:

  • Proposition 2.8:
$$\frac{I_\ell'(\phi_\ell^{\epsilon_\ell} \otimes \widetilde{\phi}_\ell^{\epsilon_\ell})}{\langle \phi_\ell^{\epsilon_\ell}, \widetilde{\phi}_\ell^{\epsilon_\ell}\rangle} = \frac{1+\epsilon_1\epsilon_2(\sqrt{\omega_3(\ell)})}{2},$$

  • Propositions 2.3, 2.4:
$$L(s, \Pi_\ell, \mathrm{Ad}) = \zeta_\ell(s+1)(\zeta_\ell(2s)/\zeta_\ell(s))^2, \qquad L(s, \Pi_\ell) = \zeta_\ell(s+1/2)^2.$$

Therefore:

$$I_\ell(\phi_\ell^{\epsilon_\ell} \otimes \widetilde{\phi_\ell^{\epsilon_\ell}}) = \frac{\zeta_\ell(2)\zeta_\ell(2)^2}{\zeta_\ell(2)^2\zeta_\ell(1)^2\zeta_\ell(1)^2} \frac{1 + \epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2} = \frac{\zeta_\ell(2)}{\zeta_\ell(1)^4} \frac{1 + \epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2}.$$

Note that $B_{\pi_\ell} = 1$ if π_ℓ is supercuspidal and $\pi_\ell \otimes \xi \cong \pi_\ell$ for the unramified quadratic character ξ of $\mathbb{Q}_\ell^\times$ (cf. [Col18, Section 4.2]). Therefore, $B_{\Pi_\ell} = B_{\pi_1}B_{\pi_2}B_{\pi_3} = (-1)\frac{\zeta_\ell(2)}{\zeta_\ell(1)}$ (similarly to [Hsi21, p. 478]). Altoghether, we have that:

$$\begin{aligned}
\mathscr{I}_{\Pi_{\kappa,\ell}}^\star &= -\frac{\zeta_\ell(2)}{\zeta_\ell(1)^4} \frac{1 + \epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2} \frac{\zeta_\ell(2)}{\zeta_\ell(1)} \ell \frac{\zeta_\ell(1)^5}{\zeta_\ell(2)^3} |\mathbf{d}_F^{k-2}|_\ell \\
&= -\frac{1 + \epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2} \frac{\ell|\ell|_\ell^{2(k_1+k_2+k_2-6)}}{\zeta_\ell(2)},
\end{aligned}$$

as claimed.   ∎

Proposition 5.11. *Suppose* $\ell \in \Sigma^{-,3}$ *and* $\omega_{F,\ell} = 1$. *Then:*

$$\mathscr{I}_{\Pi_{\kappa,\ell}}^\star = \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4} \frac{\ell^{-2(k_1+k_2+k_3)+13}}{\zeta_\ell(2)}.$$

Proof. We use:

- Proposition 2.9:
$$\frac{I'_\ell(\phi_\ell^{\epsilon_\ell} \otimes \widetilde{\phi_\ell^{\epsilon_\ell}})}{\langle \phi_\ell^{\epsilon_\ell}, \widetilde{\phi_\ell^{\epsilon_\ell}} \rangle} = \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4},$$

- Propositions 2.3, 2.4:
$$L(s, \Pi_\ell, \mathrm{Ad}) = (\zeta_\ell(2s)/\zeta_\ell(s))^3, \qquad L(s, \Pi_\ell) = \zeta_\ell(2s).$$

Therefore:

$$I_\ell(\phi_\ell^{\epsilon_\ell} \otimes \widetilde{\phi_\ell^{\epsilon_\ell}}) = \frac{\zeta_\ell(2)^3}{\zeta_\ell(2)^2\zeta_\ell(1)^3\zeta_\ell(1)} \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4} = \frac{\zeta_\ell(2)}{\zeta_\ell(1)^4} \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4}.$$

Moreover, $B_{\Pi_\ell} = 1$. Altogether, we have that:

$$\begin{aligned}
\mathscr{I}_{\Pi_{\kappa,\ell}}^\star &= \frac{\zeta_\ell(2)}{\zeta_\ell(1)^4} \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4} \cdot \ell \frac{\zeta_\ell(1)^4}{\zeta_\ell(2)^2} \omega_{F,\ell}^{-1}(\mathbf{d}_f^{k-2}) |\mathbf{d}_F^{k-2}|_\ell \\
&= \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4} \frac{\ell|\ell|_\ell^{2(k_1+k_2+k_3-6)}}{\zeta_\ell(2)},
\end{aligned}$$

as claimed.   ∎

5.5. **Finishing the proof of Theorem 5.5.** Putting everything together gives the interpolation property.

Proof of Theorem 5.5. We compute using the definition of $\mathcal{L}_{\mathbf{F},\epsilon}^{\mathrm{bal}}$:

$$\begin{aligned}
(\mathcal{L}_{\mathbf{F},\epsilon}^{\mathrm{bal}}(\kappa))^2 &= \frac{\Theta_{\mathbf{F}'^{D\star,\epsilon}}^2}{\|\mathbf{F}^{D\star,\epsilon}\|^2} \cdot 2^{-(|\Sigma^-|+1-k_1-k_2-k_3)} \cdot \epsilon^{\Sigma^-}(\mathbf{F})(\kappa)^{-1} \cdot \mathfrak{f}_{\mathbf{F}'}(\kappa)^{-1} \prod_{\ell \in \Sigma^{-,\mathrm{sc}}} \frac{\ell^{2(k_1+k_2+k_3)-13}}{\zeta_\ell(2)} \\
&= \frac{L(1/2, \Pi_\kappa)}{\langle \mathbf{F}_\kappa, \mathbf{F}_\kappa \rangle} \cdot \frac{\mathscr{I}_{\Pi_{\kappa,p}}^{\mathrm{bal}}}{\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})} \cdot \prod_{\ell^2||N^-} \mathscr{I}_{\Pi_{\kappa,\ell}}^\star \frac{\ell^{2(k_1+k_2+k_3)-13}}{\zeta_\ell(2)} \cdot \prod_{q|N^+} I_{\Pi_{\kappa,q}}^\star \cdot \mathfrak{f}_{\mathbf{F}'}(\kappa)^{-1} \\
&= \frac{L(1/2, \Pi_\kappa)}{\langle \mathbf{F}_\kappa, \mathbf{F}_\kappa \rangle} \cdot \frac{\mathcal{E}_{\mathrm{bal}}(\Pi_{\kappa,p})}{L(1/2, \Pi_{\kappa,p})\mathcal{E}_p(\mathbf{F}_\kappa, \mathrm{Ad})} \cdot \prod_{\ell \in \Sigma^{-,2}} \frac{1 + \epsilon_1\epsilon_2\sqrt{\omega_3(\ell)}}{2} \prod_{\ell \in \Sigma^{-,3}} \frac{1 + \epsilon_1\epsilon_2\epsilon_3}{4} \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} \frac{1}{(1 + q^{-1})^2}
\end{aligned}$$

using Proposition 5.8 for the second equality and Propositions 5.9, 5.10, 5.11 for the final one. This proves the theorem.   ∎

6. Application: Elliptic Stark Conjecture in rank one

We are now ready to discuss the analogue of the Elliptic Stark Conjecture of Darmon–Lauder–Rotger [DLR15] in this setting. We will consider the restriction of $\mathcal{L}_p^{\mathrm{bal}}$ to the weights $(k_1, k_2, k_3) = (2, \ell, m)$ with the goal of studying the point $(2, 1, 1)$ which is not balanced and hence lies outside of the interpolation range.[4] We state a conjecture expressing this value in terms of arithmetic data and prove it in some special cases.

6.1. **Statement of the conjecture.** Let E be an elliptic curve over $\mathbb{Q}$ and let $V_p(E)$ be the associated p-adic Galois representation. Let $\varrho\colon G_{\mathbb{Q}} \to \mathrm{GL}(V_\varrho)$ be an Artin representation, i.e. a complex continuous Galois representation, which factors through a finite quotient $\mathrm{Gal}(H/\mathbb{Q})$ of $G_{\mathbb{Q}}$, with coefficients in a finite extension L of $\mathbb{Q}$. We consider the Hasse–Weil–Artin L-series $L(E, \varrho, s)$ associated with the p-adic Galois repersentation $V_p(E) \otimes V_\varrho$ of $G_{\mathbb{Q}}$. The equivariant version of the Birch–Swinnerton-Dyer Conjecture asserts that the analytic rank of $L(E, \varrho, s)$ at $s = 1$ is equal to the rank of the V_ϱ-isotypic component of $E(H) \otimes L$:

$$\mathrm{ord}_{s=1} L(E, \varrho, s) = \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes L).$$

Note that the value $L(E, \varrho, s)$ at $s = 1$ is outside of the convergence region of the Euler product defining the L-function. We put ourselves in a situation where the analytic continuation and functional equation for the L-function $L(E, \varrho, s)$ is known:

- let f be the modular form of level N_1 associated with the elliptic curve E of conductor N_1,
- suppose that $\varrho = \varrho_{gh} = \varrho_g \otimes \varrho_h$ for two odd irreducible two-dimensional Artin representations ϱ_g and ϱ_h; let g and h be the modular forms of conductors N_2 and N_3 corresponding to ϱ_g and ϱ_h, respectively.

Then

$$L(E, \varrho_{gh}, s) = L(f \times g \times h, s)$$

is the triple product L-function studied by Garret. We assume that:

$$\det \varrho_g \det \varrho_h = 1,$$

i.e. if χ is the character of g, then χ^{-1} is the character of h. Then the representation $V_E \otimes V_g \otimes V_h$ is self-dual, and hence there is a functional equation with the root number $\epsilon(E, \varrho_{gh}) = \pm 1$. Moreover, the global root number is a product of finite local root numbers:

$$\epsilon(E, \varrho_{gh}) = \prod_{v | \mathrm{lcm}(N_1, N_2, N_3)} \epsilon_v(E, \varrho_{gh}),$$

because $\epsilon_\infty(E, \varrho_{gh}) = 1$.

In the seminal work [DLR15], Darmon–Lauder–Rotger studied the L-value $L(E, \varrho_{gh}, s)$ when the analytic rank is even and at least two, using p-adic analytic method. Therefore, they assume that $\epsilon(E, \varrho_{gh}) = 1$, and in fact that $\epsilon_v(E, \varrho_{gh}) = 1$ for all v.

In rank two, they proposed the Elliptic Stark Conjecture: a formula relating the value of a triple product p-adic L-function associated with f, g, h to a regulator of p-adic logarithms of points on $E(H) \otimes L$ and a p-adic logarithm of a Stark unit.

The main motivation for our work is to develop a rank one version of this conjecture. Our first hypothesis is therefore.

Hypothesis A. The global root number $\epsilon(E, \varrho_{gh})$ is -1. Therefore, there is an odd number of finite places v such that

$$\epsilon_v(E, \varrho_{gh}) = -1.$$

---

[4]We hope that using ℓ for the weight instead of the prime of supersingular type will cause no confusion to the reader.

Next, we want to construct a balanced p-adic L-function associated with the L-function $L(E, \varrho_{gh}, s) = L(f \times g \times h, s)$. Under Hypothesis A, this L-function vanishes at $s = 1$. However, if we consider Hida families associated with g and h, and the L-value $L(f \times g_\ell \times h_\ell, s)$ for $\ell \geq 2$, then

$$\epsilon_\infty(f \times g_\ell \times h_\ell) = -1, \qquad \epsilon_v(f \times g_\ell \times h_\ell) = \epsilon_v(E, \varrho_{gh}) \text{ for } v \text{ finite},$$

$$\epsilon(f \times g_\ell \times h_\ell) = +1.$$

In particular, we expect that the central L-value for $L(f \times g_\ell \times h_\ell, s)$ is generically non-vanishing and hence there should be a non-vanishing p-adic L-function interpolating these values.

To define the Hida families, assume that g and h are ordinary at p and consider p-stabilizations of g and h. For $F \in \{g, h\}$, $F \in S_1(N, \chi_F)$, suppose the Hecke polynomial is:

$$x^2 - a_p(F)x + \chi_F(p) = (x - \alpha_p(F))(x - \beta_p(F))$$

with roots $\alpha_p(F)$ and $\beta_p(F)$. We then have p-stabilizations $F_\alpha, F_\beta \in S_1(Np, \chi_F)$ such that:

$$U_p F_\alpha = \alpha F_\alpha, \quad U_p F_\beta = \beta F_\beta.$$

We will assume that F is *regular*, i.e. $\alpha_p(F) \neq \beta_p(F)$, and hence it has two distinct p-stabilizations.

We then make the following classicality hypothesis.

Hypothesis B. Assume that f is ordinary at p. For $F \in \{g, h\}$, F is ordinary and regular at p (i.e. ϱ_g is irreducible and $\varrho_g(\sigma_p)$ has two distinct eigenvalues for the Frobenius σ_p), and it is not the theta series of a character of a real quadratic field in which p splits.

Under this hypothesis, Darmon–Lauder–Rotger, based on results of Cho and Vatsal and of Bellaïche and Dimitrov, proved that there are no non-classical p-adic modular forms in the generalized eigenspace of F_α.

Proposition 6.1 ([DLR15, Proposition 1.1])**.** *Under Hypothesis B, the natural inclusion:*

$$S_1(Np, \chi_F)_{\mathbb{C}_p}[F_\alpha] \hookrightarrow S_1^{\mathrm{oc,ord}}(N, \chi_F)[\![F_\alpha]\!]$$

is an isomorphism of $\mathbb{C}_p$-vector spaces.

By Proposition 6.1, there exist Hida families $\mathbf{g}_\alpha$ and $\mathbf{h}_\alpha$ whose specializations at weight one are g_α and h_α, respectively.

We want to consider a balanced p-adic L-function $\mathcal{L}_p^{\mathrm{bal}}$ associated with f and the Hida families $\mathbf{g}_\alpha$ and $\mathbf{h}_\alpha$. Therefore, we need to make another hypothesis which will guarantee that Hypothesis 2.2 needed to construct the balanced p-adic L-function is satisfied.

Hypothesis C. For each prime q such that $\epsilon_q(E, \varrho_{gh}) = -1$, $v_q(N_i) \leq 2$ for $i = 1, 2, 3$.

Under this hypothesis, we have constructed a balanced triple product p-adic L-function associated with the triple f, g, h, and the Elliptic Stark Conjecture concerns its value at $(2, 1, 1)$, which is outside of the range of interpolation.

Definition 6.2. Let f, g, h be three modular forms of weights $2, 1, 1$ satisfying Hypotheses A, B, C. Let $f_\alpha, g_\alpha, h_\beta$ be ordinary p-stabilizations of $f, g,$ and h, and consider Hida families $\mathbf{f}, \mathbf{g}, \mathbf{h}$ specializing to p-stabilizations $f_\alpha, g_\alpha, h_\alpha$ of f, g, h, respectively. Define the *2-variable triple product p-adic L-function* by:

$$\mathcal{L}_p^{\mathrm{bal},f} = 2^{-\#\Sigma^{-,3}} \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} (1 - q^{-1})^{-2} \cdot \mathcal{E}_p(f_\alpha, \mathrm{Ad}) \cdot \mathcal{L}_{\mathbf{f} \times \mathbf{g} \times \mathbf{h}, \pm 1}^{\mathrm{bal}}$$

using Definition 5.4 for $\epsilon_q = \begin{cases} (+1, +1) & q \in \Sigma^{-,2}, \\ (+1, +1, +1) & q \in \Sigma^{-,2}. \end{cases}$
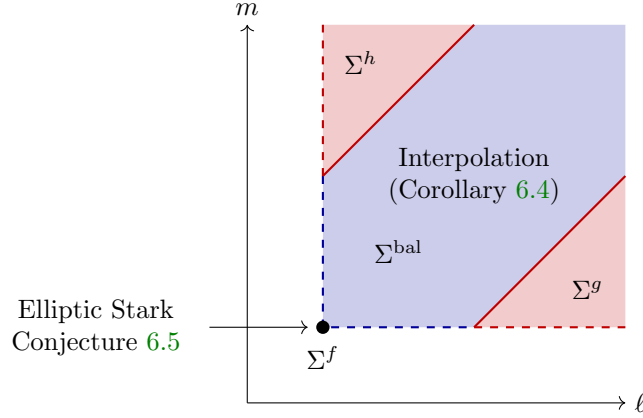
FIGURE 1. We fix $k = 2$, and consider two weights ℓ, m with $\ell + m \equiv 0 \pmod 2$. We indicate the four regions for the weights ℓ, m; Σ^F is the region where F is dominant, and Σ^{bal} is the region where the weights are balanced and $\ell, m \geq 2$. We also indicate the point $(\ell, m) = (1, 1)$ where our Elliptic Stark Conjecture 6.5 applies.

Remark 6.3. One could presumably weaken or even remove the assumption that f is ordinary at p by defining the 2-variable p-adic L-function directly, instead of considering a Hida family through f_α and referring to the 3-variable p-adic L-function. However, we decided not to pursue this point here.

As a corollary to Theorem 5.5, we get the following interpolation property.

Corollary 6.4. *For $\ell, m \geq 2$ such that $2 - \ell \leq m \leq 2 + \ell$ and $\ell + m \equiv 0 \pmod 2$:*

$$\mathcal{L}_p^{\text{bal},f}(\ell,m)^2 = \frac{\Lambda(f \times g_\ell \times h_m, (\ell+m)/2)}{\langle f, f \rangle \langle g_\ell, g_\ell \rangle \langle h_m, h_m \rangle} \cdot \mathcal{E}_p^{\text{bal}}(\mathbf{V}_\kappa) \cdot \mathcal{E}_p(f_\alpha, \text{Ad}),$$

where $\Lambda(f \times g_\ell \times h_m, s)$ is the completed triple product L-function.

The value $\mathcal{L}_p^{\text{bal},f}(1,1) = \mathcal{L}_{\mathbf{f} \times \mathbf{g} \times \mathbf{h}, \epsilon}^{\text{bal}}(2, 1, 1)$ will be expressed in terms of arithmetic data associated with the triple $(E, \varrho_g, \varrho_h)$. We introduce this next.

Let $\sigma_p \in \text{Gal}(H/\mathbb{Q})$ be the Frobenius at p associated with an embedding $H \to \mathbb{Q}_p^{\text{ur}}$. Under Hypothesis B, for $F \in \{g, h\}$, we have that

$$\varrho_F(\sigma_p) = \begin{pmatrix} \alpha_p(F) & 0 \\ 0 & \beta_p(F) \end{pmatrix}$$

and we may hence consider the one-dimensional eigenspace $V_F^\alpha \subseteq V_F$ for σ_p associated with the eigenvalue $\alpha(F)$. This determines a one-dimensional L-subspace:

$$V_{\alpha\alpha} = V_g^\alpha \otimes V_h^\alpha \subseteq V_g \otimes V_h$$

and we fix an element $v_{\alpha\alpha} \in V_{\alpha\alpha}$. On the other hand, under Hypothesis A, we expect that $r(E, \varrho_{gh}) = \dim_L \text{Hom}_{G_\mathbb{Q}}(V_{gh}, E(H) \otimes L) \geq 1$. If $\dim_L \text{Hom}_{G_\mathbb{Q}}(V_{gh}, E(H) \otimes L) = 1$, we choose its basis

$$\Phi \colon V_{gh} \to E(H) \otimes L,$$

and let

$$E(H)_L^{V_{gh}} = \Phi(V_{gh}) \subseteq E(H) \otimes L.$$

We will consider the point:

$$\Phi(v_{\alpha\alpha}) \in E(H)_L^{V_{gh}} \subseteq E(H) \otimes L.$$

Next, associated with $F \in \{g, h\}$, we have an adjoint representation $\mathrm{Ad}_F = \mathrm{Hom}^0(V_F, V_F)$, with Frobenius eigenvalues $1, \frac{\alpha_p(F)}{\beta_p(F)}, \frac{\beta_p(F)}{\alpha_p(F)}$. By [DLR15, Proposition 1.5], we have that

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{Ad}_F, \mathcal{O}_{H_F}^{\times} \otimes L)$$

is one-dimensional. Let φ be its basis and consider

$$(\mathcal{O}_{H_F}^{\times})_L^{\mathrm{Ad}_F} = \varphi(\mathrm{Ad}_F) \subseteq \mathcal{O}_{H_F}^{\times} \otimes L.$$

Under the extra assumption that $\alpha_p(F) \neq -\beta_p(F)$ or ϱ_F is induced from a character of an imaginary quadratic field in Hypothesis B, the subspace

$$U_{F_\alpha} = \left\{ u \in (\mathcal{O}_{H_F}^{\times})_L^{\mathrm{Ad}_F} \;\middle|\; \sigma_p(u) = \frac{\alpha_p(F)}{\beta_p(F)} \right\}$$

is one-dimensional [DLR15, Lemma 1.6]. In any case, we let $u_{F_\alpha} \in U_{F_\alpha}$ be a non-torsion element.

Conjecture 6.5 (Elliptic Stark Conjecture, rank one case). *Assume Hypotheses A, B, C and that* $r(E, \varrho_{gh}) \geq 1$. *If* $r(E, \varrho_{gh}) > 1$, *then* $\mathcal{L}_p^{\mathrm{bal}, f}(1, 1) = 0$. *If* $r(E, \varrho_{gh}) = 1$, *then:*

$$\mathcal{L}_p^{\mathrm{bal}, f}(1, 1) \sim_{\sqrt{L^{\times}}} \frac{\log_{E,p}(\Phi(v_{\alpha\alpha}))}{\log_p(u_{g_\alpha})^{1/2} \log_p(u_{h_\alpha})^{1/2}},$$

where

$$\log_p \colon (\mathcal{O}_H^{\times})_L \to H_p \otimes L \qquad\qquad\qquad p\text{-}adic\ logarithm,$$

$$\log_{E,p} \colon E(H)_L \to H_p \otimes L \qquad\qquad p\text{-}adic\ formal\ group\ logarithm\ for\ E.$$

Remark 6.6. We would like to thank Alan Lauder for suggesting the following sanity check. The left hand side in the conjecture is independent of the number field H, whereas the right hand side seems to depend on the Frobenius $\sigma_p \in \mathrm{Gal}(H/\mathbb{Q})$ and its eigenvalues for the Artin representations ϱ_g and ϱ_h. However, the Frobenius σ_p acts on the right hand side as:

$$\frac{(\alpha_p(g)\alpha_p(h))^2}{\frac{\alpha_p(g)}{\beta_p(g)} \frac{\alpha_p(h)}{\beta_p(h)}} = \alpha_p(g)\alpha_p(h)\beta_p(g)\beta_p(h)$$

$$= \chi_g(p)\chi_h(p)$$

$$= 1.$$

Remark 6.7. Taking the formal group law exponential $\exp_{E,p}$, we get an interesting formula for a p-adic point on the elliptic curve E which is conjecturally in $E(H) \otimes L$:

$$\exp_p(\mathcal{L}_p^{f, \mathrm{bal}}(1, 1) \log_p(u_{g_\alpha})^{1/2} \log_p(u_{h_\alpha})^{1/2}) \stackrel{?}{\in} E(H)_L^{V_{gh}} \subseteq E(H) \otimes L.$$

6.2. **Proof in the CM case.** Let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D_K$ in which p splits. The next goal of the paper is to prove the Elliptic Stark Conjecture 6.5 in the case when ϱ_g and ϱ_h are representations induced from Dirichlet characters of K.

6.2.1. *Statement of the theorem.* Let $\psi \colon G_K \to \mathbb{C}^{\times}$ be a finite order character of conductor $\mathfrak{c} \subseteq \mathcal{O}_K$. We can then consider the associated Artin representation representation

$$V_\psi = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$$

with determinant $\chi = \det V_\psi = \psi \circ \mathrm{Tr}$, where $\mathrm{Tr} \colon G_{\mathbb{Q}}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map. The weight one modular form associated with this Artin representation is explicitly constructed as a theta series for the character of $\mathbb{A}_K^{\times}$ corresponding to ψ:

$$\theta_\psi \in M_1(D_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}), \chi).$$

Moreover, θ_ψ is a cusp form if and only if $\psi^\sigma \neq \psi$ where σ is a generator of $\mathrm{Gal}(K/\mathbb{Q})$. We will only be interested in this case.

We consider two finite order characters ψ_g and ψ_h such that $\chi_h = \chi_g^{-1}$, where $\chi_g = \psi_g \circ \mathrm{Tr}$ and $\chi_h = \psi_h \circ \mathrm{Tr}$, and $\psi_g^\sigma \neq \psi_g$, $\psi_h^\sigma \neq \psi_h$. Then we have associated modular forms

$$g = \theta_{\psi_g} \in S_1(N_g, \chi) \qquad h = \theta_{\psi_g} \in S_1(N_h, \overline{\chi}).$$

Write $\mathfrak{c}_\star$ for the conductor of $\psi_\star$ for $\star \in \{g, h\}$.

The goal is to study the Elliptic Stark Conjecture 6.5 in this example. We have a factorization

$$V_{gh} = V_{\psi_g} \otimes V_{\psi_h} \cong V_{\psi_g \psi_h} \oplus V_{\psi_g \psi_h^\sigma},$$

and we will write

$$\psi_1 = \psi_g \psi_h^\sigma, \ \ \psi_2 = \psi_g \psi_h.$$

This corresponds to the factorization of L-functions

$$L(E, \varrho_{gh}, s) = L(E/K, \psi_1, s) L(E/K, \psi_2, s).$$

Hypothesis A is hence equivalent to the following hypothesis.

Hypothesis A'. We have that:

$$\epsilon(E/K, \psi_1) \cdot \epsilon(E/K, \psi_2) = -1,$$

i.e. without loss of generality, $\epsilon(E/K, \psi_1) = +1$, $\epsilon(E/K, \psi_2) = -1$.

Next, we want to work out when Hypothesis B holds in this case. The values α_p, β_p depend on the splitting of p in K:

$$\{\alpha_p, \beta_p\} = \begin{cases} \{\psi(\sigma_{\mathfrak{p}}), \psi(\sigma_{\overline{\mathfrak{p}}})\} & \text{if } (p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits,} \\ \{\sqrt{\psi(\sigma_p)}, -\sqrt{\psi(\sigma_p)}\} & \text{if } (p) \text{ is inert.} \end{cases}$$

Therefore, Hypothesis B amounts to the following.

Hypothesis B'. For $\psi \in \{\psi_g, \psi_h\}$:

- θ_ψ is ordinary at p,
- θ_ψ is not also the theta series of a character of a real quadratic field in which p splits,
- when p splits in K, $\psi(\sigma_{\mathfrak{p}}) \neq \psi(\sigma_{\overline{\mathfrak{p}}})$.

Finally, Hypothesis C is that for each prime ℓ such that $\epsilon_\ell(E_K, \psi_1) \cdot \epsilon_\ell(E_K, \psi_2) = -1$, we have that $v_\ell(N_f), v_\ell(N_{K/\mathbb{Q}} \mathfrak{c}_g), v_\ell(N_{K/\mathbb{Q}} \mathfrak{c}_h) \leq 2$. For simplicity, we make a slightly stronger assumption.

Hypothesis C'. For each prime ℓ of K such that $\epsilon_\ell(E_K, \psi_1) \cdot \epsilon_\ell(E_K, \psi_2) = -1$, we have that:

- $\epsilon_\ell(E_K, \psi_2) = +1$,
- $v_\ell(N_f) = 1$, $v_\ell(N_{K/\mathbb{Q}} \mathfrak{c}_g) = 2$, $v_\ell(N_{K/\mathbb{Q}} \mathfrak{c}_h) = 2$.

Moreover, for each prime q of K such that $\epsilon_q(E_K, \psi_1) \cdot \epsilon_q(E_K, \psi_2) = +1$, we have that $\epsilon_q(E_K, \psi_1) = +1$ and $\epsilon_q(E_K, \psi_2) = +1$.

Remark 6.8. We expect that the results of the section still hold under the weaker assumption, by replacing of Appendix A by a generalization of the results of Brooks [HB15], and by calculating the local integrals when all three representations are supercuspidal.

We consider three arithmetic quantities associated with our data of f, ψ_g, ψ_h:

- following [Gro87, Section 11] (more generally, [CST14]),

$$(6.1) \qquad c_{f,\psi_1} = \pi_f \left(\sum_{\sigma \in \mathrm{Gal}(H/\mathbb{Q})} \psi_1^{-1}(\sigma) s^\sigma \right),$$

where π_f is the projection onto the f-isotypic component of an appropriate Shimura set S and $s \in S$ is a Heegner point for $H_{c(\psi_1)}/K$;

- following [DLR15, p. 37] (cf. [LZZ18, Definition 3.3.1]),

$$(6.2) \qquad P_{\psi_2} = \pi_f \left(\sum_{\sigma \in \mathrm{Gal}(H/K)} \psi_2^{-1}(\sigma) t^\sigma \right) \in E(H)_L^{\psi_2}$$

where π_f is the modular parametrization by an appropriate Shimura curve X and $t \in X$ is a Heegner point for $H_{c(\psi_2)}/K$;

- following [DLR15, p. 38], for $\psi_0 \in \{\psi_g/\psi_g^\sigma, \psi_h/\psi_h^\sigma\}$, we let

$$(6.3) \qquad u_{\psi_0} = \sum_{\sigma \in \mathrm{Gal}(H/K)} \psi_0^{-1}(\sigma) u^\sigma \in (\mathcal{O}_H^\times)_L^{\psi_0}$$

where u is an elliptic unit for $H_{c(\psi_0)}/K$ and the unit group is written additively.

Theorem 6.9. *Let K be an imaginary quadratic field and p be a prime which splits in K. Then, under Hypothesis A', B', C', the rank one Elliptic Stark Conjecture 6.5 holds. Explicitly, if $r(E_K, \psi_1) = 0$ and $r(E_K, \psi_2) = 1$, then there is an explicit constant $\lambda \in \sqrt{L^\times}$ such that:*

$$\mathcal{L}_p^{f,\mathrm{bal}}(1,1) = \lambda \cdot \overbrace{\langle c_{f,\psi_1}, c_{f,\psi_1} \rangle^{1/2}}^{\in \sqrt{L^\times}} \cdot \frac{\log_{E,p}(P_{\psi_2})}{\log_p(u_{\psi_g/\psi_g^\sigma})^{1/2} \log_p(u_{\psi_h/\psi_h^\sigma})^{1/2}},$$

where $\langle -, - \rangle$ is the height pairing on S.

Remark 6.10. The factor $\langle c_{f,\psi_1}, c_{f,\psi_1} \rangle^{1/2} \in L^\times$ could, of course, be combined with $\lambda \in \sqrt{L^\times}$ in the formula. However, we include it here, because it seems to describe the "arithmetic" contribution to the special value associated with the character ψ_1. It is quite interesting that this factor has a similar form to the other (non-algebraic) factors, and may therefore be relevant to an integral version of the result or to similar results in other settings.

The proof of this theorem is based on a factorization of the p-adic L-function corresponding to the factorization

$$L(E, \varrho_{gh}, s) = L(E/K, \psi_1, s) \cdot L(E/K, \psi_2, s).$$

Following [DLR15, Section 3.2], we recall two related p-adic L-functions.

6.2.2. *The Katz p-adic L-function for K* [Kat76]. Let Σ be the set of characters of K of conductor dividing a fixed integral ideal $\mathfrak{c} \subseteq \mathcal{O}_K$. We then define $\Sigma_K = \Sigma_K^{(2)} \cup \Sigma_K^{(2')}$ where:

$$\Sigma_K^{(2)} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2) \text{ with } \kappa_1 \geq 1, \kappa_2 \leq 0\},$$

$$\Sigma_K^{(2')} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1, \kappa_2) \text{ with } \kappa_1 \leq 0, \kappa_2 \geq 1\}.$$

Katz defined a p-adic L-function

$$\mathcal{L}_p(K) \colon \widehat{\Sigma}_K \to \mathbb{C}_p$$

defined on the p-adic completion $\widehat{\Sigma}_K$ of Σ_K with the interpolation property

$$(6.4) \qquad \mathcal{L}_p(K)(\psi) = \mathfrak{a}(\psi) \cdot \mathfrak{e}(\psi) \cdot \mathfrak{f}(\psi) \cdot \frac{\Omega_p^{\kappa_1 - \kappa_2}}{\Omega^{\kappa_1 - \kappa_2}} \cdot L_{\mathfrak{c}}(\psi^{-1}, 0) \qquad \text{for } \psi \in \Sigma_K^{(2)},$$

where:

- $\mathfrak{a}(\psi) = (\kappa_1 - 1) \pi^{-\kappa_2}$, $\mathfrak{e}(\psi) = (1 - \psi(\mathfrak{p})p^{-1})(1 - \psi^{-1}(\bar{\mathfrak{p}}))$ and $\mathfrak{f}(\psi) = D_K^{\kappa_2/2} 2^{-\kappa_2}$,
- $\Omega \in \mathbb{C}^\times$ and $\Omega_p \in \mathbb{C}_p^\times$ are CM periods attached to K, cf. [BDP12, (2-15), (2-17)],
- $L_{\mathfrak{c}}(\psi^{-1}, s)$ is the $\mathfrak{c}$-depleted Hecke L-function associated with ψ^{-1}.

It satisfies a functional equation of the form

$$\mathcal{L}_p(K)(\psi) = \mathcal{L}_p(K)((\psi^\sigma)^{-1} N_{K/\mathbb{Q}}), \tag{6.5}$$

relating the values in $\Sigma_K^{(2)}$ to $\Sigma_K^{(2')}$.

A finite order Hecke character ψ has trivial infinity type and hence lies outside Σ_K, i.e. outside the interpolation range (6.4); however it still belongs to $\widehat{\Sigma}_K$. Katz proves a p-adic Kroenecker limit formula [DLR15, (47)]:

$$\mathcal{L}_p(\psi) = \frac{-1}{24 N_{K/\mathbb{Q}}\mathfrak{c}}\mathfrak{e}(\psi)\log_p(u_\psi), \qquad \text{for } \psi \text{ of finite order,} \tag{6.6}$$
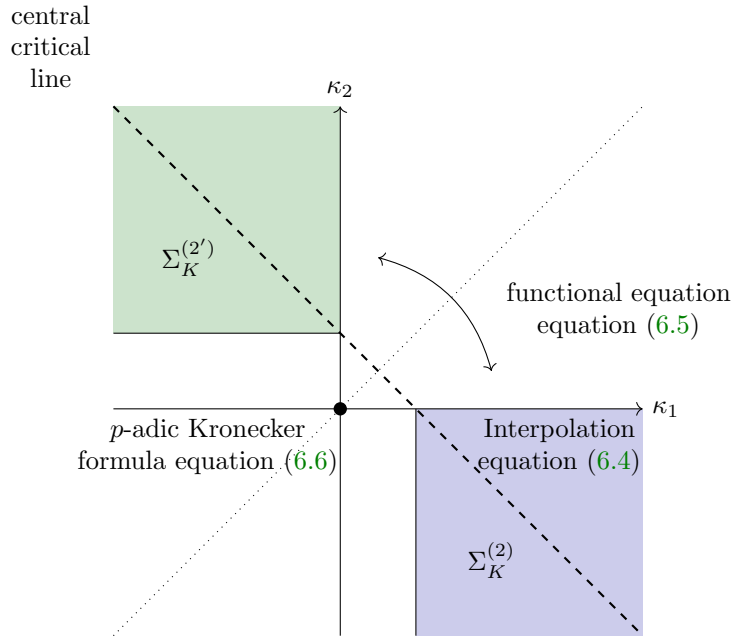
for u_ψ defined in (6.3).

FIGURE 2. The following diagram shows the diagram of infinity types (κ_1, κ_2) for the characters in Σ. We indicate interpolation region $\Sigma_K^{(2)}$ for the Katz p-adic L-function in blue (6.4), the functional equation (6.5) with the axis of symmetry given by the dotted line, and the point where the p-adic Kroenecker limit formula (6.6) is valid. The dashed line is the central critical line

6.2.3. *The BDP p-adic Rankin L-function for f and K.* For any character ψ of K with infinity type (κ_1, κ_2), we consider the Rankin–Selberg L-function associated with f and θ_ψ:

$$L(f, \psi, s) = L(f \times \theta_\psi, s - (\kappa_1 + \kappa_2 + 1)/2).$$

When the conductor of ψ is coprime to the level N_f of f, a p-adic L-function was constructed in [BDP13] and a special value formula outside of the interpolation range was proved.

Assume Hypothesis A.1, i.e. $N_f = N_+ N_-$ and N_- is a square-free product of finite primes. Let $c \geq 1$ be a positive integer relatively prime to pN^+ and divisible by N_-. Bertolini–Darmon–Prasanna assume that $N_- = 1$, but we prove the analogue of their results in Appendix A.

Let $\Sigma_{f,c}$ be the set of characters defined in Definition A.4, so that for $\psi \in \Sigma_{f,c}$, $L(f, \psi, s)$ is self-dual and has $s = 0$ as its central critical point. Note that $\Sigma_{f,c}$ naturally decomposes as

$$\Sigma_{f,c} = \Sigma_{f,c}^{(1)} \cup \Sigma_{f,c}^{(2)} \cup \Sigma_{f,c}^{(2')},$$

$$\Sigma_{f,c}^{(1)} = \{\psi \text{ of infinity type } (1,1) \mid \psi \circ \mathrm{Tr} = 1\},$$

$$\Sigma_{f,c}^{(2)} = \{\psi \text{ of infinity type } (\kappa + 2, -\kappa) \text{ for } \kappa \geq 1\},$$

$$\Sigma_{f,c}^{(2')} = \{\psi \text{ of infinity type } (-\kappa, \kappa + 2) \text{ for } \kappa \geq 1\}.$$

We next summarize Theorem A.5 which is a generalization of the main results of [BDP13].

There is a p-adic L-function

$$\mathcal{L}_p(f, K) \colon \widehat{\Sigma}_{f,c} \to \mathbb{C}_p^\times$$

defined on the p-adic completion $\widehat{\Sigma}_{f,c}$ of $\Sigma_{f,c}$ with the interpolation property

$$(6.7) \qquad \mathcal{L}_p(f, K)(\psi) = \mathfrak{a}(f, \psi) \cdot \mathfrak{e}(f, \psi)^2 \cdot \mathfrak{f}(f, \psi) \cdot \frac{\Omega_p^{4\kappa+4}}{\Omega^{4\kappa+4}} \cdot L(f, \psi^{-1}, 0) \qquad\qquad \psi \in \Sigma_{f,c}^{(2)},$$

where

- $\mathfrak{a}(f, \psi) = \kappa!(\kappa+1)!\pi^{2\kappa+1}$, $\mathfrak{e}(f, \psi) = 1 - a_p(f)\psi^{-1}(\bar{\mathfrak{p}}) + \psi^{-2}(\bar{\mathfrak{p}})p$,
- $\mathfrak{f}(f, \psi) = (2/c\sqrt{D_K})^{2\kappa+1} \cdot \prod_{q|c_0} \frac{q - \chi_\kappa(q)}{q-1} \cdot \prod_{q|c_-} \frac{q^2}{1-q^2} \cdot \omega(f, \psi)^{-1}$, and $\omega(f, \psi) \in \overline{\mathbb{Q}}$ is defined in [BDP13, (5.1.11)].

Next, for a finite order character ψ of K, we have that $\psi N_K \in \Sigma_{f,c}^{(1)}$ which is outside of the interpolation range (6.7) and Bertolini–Darmon–Prasanna prove the following p-adic Gross–Zagier formula:

$$(6.8) \qquad\qquad \mathcal{L}_p(f, K) = \left(1 - \frac{a_p(f)}{\psi(\bar{\mathfrak{p}})p} + \frac{1}{\psi^2(\bar{\mathfrak{p}})p}\right)^2 \cdot \log_p(P_\psi)^2,$$

for P_ψ defined in (6.2).
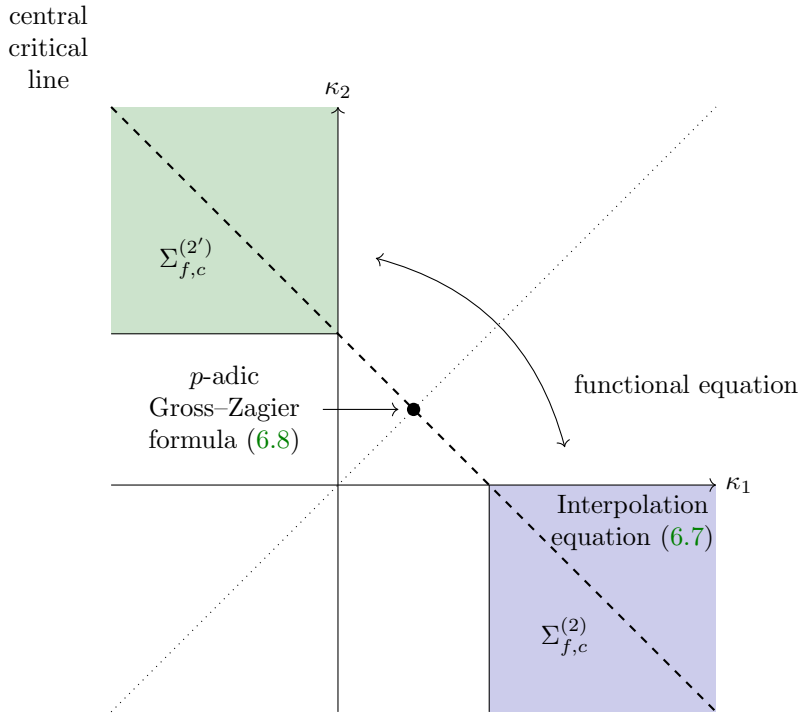

FIGURE 3. The following diagram shows the diagram of infinity types (κ_1, κ_2) for the character in Σ. We indicate interpolation region $\Sigma_{f,c}^{(2)}$ for the BDP p-adic L-function in blue (6.4), the expected functional equation with the axis of symmetry given by the dotted line, and the point where the p-adic Gross–Zagier formula (6.8) is valid.

6.2.4. *Gross' formula.* The following formula was originally discovered by Gross [Gro87, Proposition 11.2] in a special case and proved in full generality in [CST14, Theorem 1.10]:

$$(6.9) \qquad \frac{L^{(\Sigma)}(f,\psi,1)}{\pi^2 \langle f,f \rangle} = 2^3 \frac{\langle c_{f,\psi}, c_{f,\psi} \rangle}{[\mathcal{O}_c^\times : \mathbb{Z}^\times]^2 c \sqrt{D_K}},$$

for $c_{f,\chi}$ defined in (6.1) and $\Sigma = \{v \text{ places dividing } (N,c)\}$. In particular, $\frac{\sqrt{D_K} L(f,\psi,1)}{\pi^2 \langle f,f \rangle}$ lies in the number field $\mathbb{Q}(f,\psi)$.

6.2.5. *Explicit CM Hida families.* We follow [DLR15, Section 3.2] to recall Hida's construction of an explicit CM Hida family $\mathbf{F}$ specializing in weight one to $F_\alpha \in \{g_\alpha, h_\alpha\}$. Recall that for a finite order character ψ of K such that $\psi^\sigma \neq \psi$, we have an associated cusp form:

$$\theta_\psi \in S_1(D_K N_{K/\mathbb{Q}}(\mathfrak{c}), \chi).$$

We fix a character λ of infinity type $(0,1)$ and conductor $\overline{\mathfrak{p}}$ and valued in $\mathbb{Q}(\lambda)$. Let $\mathbb{Q}_p(\lambda)$ be the p-adic completion of $\mathbb{Q}(\lambda)$ determined by a fixed embedding $\mathbb{Q}(\lambda) \hookrightarrow \overline{\mathbb{Q}_p}$. We have that

$$\mathcal{O}_{\mathbb{Q}_p(\lambda)}^\times = \mu \times W$$

where μ is finite and W is a free $\mathbb{Z}_p$-module, and we write

$$\langle - \rangle \colon \mathcal{O}_{\mathbb{Q}_p(\lambda)}^\times \to W$$

for the projection.

For $\psi \in \{\psi_g, \psi_h\}$ and every integer $k \geq 1$, define

$$(6.10) \qquad \psi_{k-1}^{(p)} = \psi \langle \lambda \rangle^{k-1}$$

which is independent on the choice of λ, and let

$$(6.11) \qquad \psi_{k-1}(\mathfrak{q}) = \begin{cases} \psi_{k-1}^{(p)}(\mathfrak{q}) & \mathfrak{q} \neq \overline{\mathfrak{p}}, \\ \chi_F(p) p^{k-1} / \psi_{k-1}^{(p)}(\mathfrak{p}) & \mathfrak{q} = \overline{\mathfrak{p}}. \end{cases}$$

Then the ordinary p-stabilization of the associated theta series:

$$(6.12) \qquad F_k = \theta_{\psi_{k-1}} \in S_k(D_K N_{K/\mathbb{Q}}(\mathfrak{c}(\psi_F)), \chi_F)$$

is the weight k specialization of the Hida family $\mathcal{F}$ and, by definition, $F_1 = F$.

6.2.6. *Factorization of the p-adic L-function.* We are now ready prove the factorization of the triple product balanced p-adic L-function. We start by recalling the factorization of the classical L-functions, then analyze the periods, and finally deal with the auxillary factors. The resulting statement is Theorem 6.14 below.

Note that the norm N_K from K to $\mathbb{Q}$ can be regarded as a Hecke character of K of infinity type $(1,1)$. Since $\langle \lambda \rangle$ has infinity type $(0,1)$, $\langle \lambda \lambda^\sigma \rangle$ has infinity type $(1,1)$ and we may identify it with N_K.

Note that:

$$V_{g_\ell} \otimes V_{h_\ell} = V_{\psi_{g,\ell-1}^{(p)}} \otimes V_{\psi_{h,\ell-1}^{(p)}} \cong V_{\psi_{g,\ell-1}^{(p)} \psi_{h,\ell-1}^{(p)}} \oplus V_{\psi_{g,\ell-1}^{(p)} \psi_{h,\ell-1}^{(p),\sigma}}$$

and

$$\psi_{g,\ell-1}^{(p)} \cdot \psi_{h,\ell-1}^{(p)} = \psi_g \cdot \psi_h \cdot \langle \lambda \rangle^{2\ell-2} = \psi_2 \langle \lambda \rangle^{2\ell-2}$$

$$\psi_{g,\ell-1}^{(p)} \cdot \psi_{h,\ell-1}^{(p),\sigma} = \psi_g \cdot \psi_h^\sigma \cdot \langle \lambda \lambda^\sigma \rangle^{\ell-1} = \psi_g \psi_h^\sigma N_K^{\ell-1} = \psi_1 N_K^{\ell-1}.$$

Via the Artin formalism, the above results in a factorization of the triple product L-function:

$$L(f \times g_\ell \times h_\ell, s) = L(f_K \times \psi_2 \langle \lambda \rangle^{2\ell-2}, s) \cdot L(f_K \times \psi_1, s - (\ell-1)),$$

where $\theta_{\psi_1\langle\lambda\rangle^{2k-2}}$ has weight $2k-1$ and θ_{ψ_2} has weight 1.

By Corollary 6.4, we know that $\mathcal{L}_p^{\mathrm{bal},f}(\ell,\ell)$ is related to:

$$L(f \times g_\ell \times h_\ell, \ell) = L(f_K \times \psi_2\langle\lambda\rangle^{2\ell-2}, \ell) \cdot L(f_K \times \psi_1, 1)$$
$$= L(f_K \times \psi_2\langle\lambda\rangle^{2\ell-2}N_K^{-\ell}, 0) \cdot L(f_K \times \psi_1, 1).$$

More specifically, dividing by the period $\langle f, f\rangle\langle g_\ell, g_\ell\rangle\langle h_\ell, h_\ell\rangle$, we have:

$$(6.13) \qquad \frac{L(f \times g_\ell \times h_\ell, \ell)}{\langle f, f\rangle\langle g_\ell, g_\ell\rangle\langle h_\ell, h_\ell\rangle} = \frac{1}{\langle g_\ell, g_\ell\rangle\langle h_\ell, h_\ell\rangle} \cdot L(f_K \times \psi_2\langle\lambda\rangle^{2\ell-2}N_K^{-\ell}, 0) \cdot \frac{L(f_K \times \psi_1, 1)}{\langle f, f\rangle}.$$

Note that

$$\Psi_{gh}(\ell) = \psi_2^{-1}\langle\lambda\rangle^{-2\ell+2}N_K^\ell \in \Sigma_K^{(2)}$$

has infinity type $(\ell, -\ell + 2)$, so by equation (6.7):

$$(6.14) \quad L(f_K \times \Psi_{gh}(\ell)^{-1}, 0) = \mathfrak{a}(\Psi_{gh}(\ell))^{-1} \cdot \mathfrak{e}(f, \Psi_{gh}(\ell))^{-1} \cdot \mathfrak{f}(\Psi_{gh}(\ell))^{-1} \cdot \frac{\Omega^{4\ell-4}}{\Omega_p^{4\ell-4}} \cdot \mathcal{L}_p(f, K)(\Psi_{gh}(\ell)).$$

Next, define for $F \in \{g, h\}$

$$\Psi_F(\ell) = \psi_{F,\ell-1}^{-2}\chi_F N_K^\ell$$

and recall fact from [DLR15, Lemma 3.8]: for some K-admissible functions $\mathfrak{f}_2(k)$, $\mathfrak{f}_3(k)$

$$(6.15) \qquad \mathcal{L}_p(K)(\Psi_F(\ell)) = \frac{\mathfrak{e}(\Psi_F(\ell))\mathfrak{f}(\Psi_F(\ell))}{\mathfrak{f}_2(\ell)\cdot\mathfrak{f}_3(\ell)} \cdot \langle F_\ell, F_\ell\rangle \cdot \left(\frac{\pi\Omega_p}{\Omega}\right)^{2\ell-2}.$$

Thus equation (6.13) becomes

$$\frac{\Lambda(f \times g_\ell \times h_\ell, \ell)}{\langle f, f\rangle\langle g_\ell, g_\ell\rangle\langle h_\ell, h_\ell\rangle} \cdot \mathcal{L}_p(K)(\psi_g(\ell)) \cdot \mathcal{L}_p(K)(\psi_h(\ell)) = \frac{\mathfrak{e}(\Psi_g(\ell))\mathfrak{f}(\Psi_g(\ell))\mathfrak{e}(\Psi_h(\ell))\mathfrak{f}(\Psi_h(\ell))}{\mathfrak{f}_2(\ell)^2\mathfrak{f}_3(\ell)^2\mathfrak{e}(f, \Psi_{gh}(\ell))\mathfrak{f}(\Psi_{gh}(\ell))}$$
$$\cdot \mathcal{L}_p(f, K)(\Psi_{gh}(\ell)) \cdot \frac{L(f_K \times \psi_1, 1)}{\pi^2\langle f, f\rangle},$$

where we have canceled the factor $\mathfrak{a}(\Psi_{gh}(\ell))$ with the other powers of π and the Γ-factors for the triple product L-function.

Next, using Corollary 6.4, we get that:

$$\mathcal{L}_p^{\mathrm{bal},f}(\ell,\ell)^2 \cdot \mathcal{L}_p(K)(\psi_g(\ell)) \cdot \mathcal{L}_p(K)(\psi_h(\ell)) = \mathcal{E}(\ell) \cdot \mathcal{F}(\ell) \cdot \mathcal{L}_p(f, K)(\Psi_{gh}(\ell)) \cdot \frac{L(f_K \times \psi_1, 1)}{\pi^2\langle f, f\rangle},$$

where

$$(6.16) \qquad \mathcal{E}(\ell) = \frac{\mathcal{E}_p^{\mathrm{bal}}(\mathbf{V}_\kappa) \cdot \mathcal{E}_p(f_\alpha, \mathrm{Ad}) \cdot \mathfrak{e}(\Psi_g(\ell)) \cdot \mathfrak{e}(\Psi_h(\ell))}{\mathfrak{e}(f, \Psi_{gh}(\ell))},$$

$$(6.17) \qquad \mathcal{F}(\ell) = 2^{3-2\ell} \frac{\mathfrak{f}(\Psi_g(\ell))\mathfrak{f}(\Psi_h(\ell))}{\mathfrak{f}_2(\ell)^2\mathfrak{f}_3(\ell)^2\mathfrak{f}(\Psi_{gh}(\ell))}.$$

Lemma 6.11. *For any $\ell \geq 0$,*

$$\mathcal{E}(\ell) = (1 - \beta_f\psi_1(\mathfrak{p})p^{-1})(1 - \beta_f\psi_1(\overline{\mathfrak{p}})p^{-1}).$$

Proof. We recall that

$$L_p(\theta_{\psi_{\ell-1}}, s) = L_\mathfrak{p}(\psi_{\ell-1}, s) \cdot L_{\overline{\mathfrak{p}}}(\psi_{\ell-1}, s)$$

and hence for $F \in \{g, h\}$

$$\alpha_F = \psi_F(\mathfrak{p})\langle\lambda(\mathfrak{p})\rangle^{\ell-1}$$
$$\beta_F = \chi_F(p)p^{\ell-1}/\alpha_F = p^{\ell-1}\psi_F(\overline{\mathfrak{p}})\langle\lambda(\mathfrak{p})\rangle^{1-\ell}.$$

We first check that for $F \in \{g, h\}$:

$$(6.18) \qquad \mathcal{E}_p(F_{\alpha,\ell}, \mathrm{Ad}) = \mathfrak{e}(\Psi_F(\ell)),$$

Indeed, according to [Hsi21, p. 416]

$$\mathcal{E}_p(F_{\alpha,\ell}, \mathrm{Ad}) = (1 - \alpha_F^{-2}\chi_F(p)p^{\ell-1})(1 - \alpha_F^{-2}\chi_F(p)p^{\ell-2})$$
$$= (1 - \psi_F(\mathfrak{p})^{-2}\langle\lambda(\mathfrak{p})\rangle^{-2\ell+2}\chi_F(p)p^{\ell-1})(1 - \psi_F(\mathfrak{p})^{-2}\langle\lambda(\mathfrak{p})\rangle^{-2\ell+2}\chi_F(p)p^{\ell-2}),$$

while

$$\mathfrak{e}(\Psi_F(\ell)) = (1 - \Psi_F(\ell)(\mathfrak{p})p^{-1})(1 - \Psi_F(\ell)^{-1}(\bar{\mathfrak{p}}))$$
$$= (1 - \psi_{F,\ell-1}^{-2}(\mathfrak{p})\chi_F(p)p^{\ell}p^{-1})(1 - (\psi_{F,\ell-1}^{-2}(\bar{\mathfrak{p}})\chi_F(p)p^{\ell})^{-1})$$
$$= (1 - \psi_F(\mathfrak{p})^{-2}\langle\lambda(\mathfrak{p})\rangle^{-2\ell+2}\chi_F(p)p^{\ell-1})(1 - (\chi_F(p)p^{\ell-1}/(\psi_F(\mathfrak{p})\langle\lambda(\mathfrak{p})\rangle)^{\ell-1})^2\chi_F(p)^{-1}p^{-\ell})$$
$$= (1 - \psi_F(\mathfrak{p})^{-2}\langle\lambda(\mathfrak{p})\rangle^{-2\ell+2}\chi_F(p)p^{\ell-1})(1 - \psi_F(\mathfrak{p})^{-2}\langle\lambda(\mathfrak{p})\rangle^{-2\ell+2}\chi_F(p)p^{\ell-2}),$$

verifying equation (6.18).

Therefore, we have:

$$\mathcal{E}_p^{\mathrm{bal}}(V_\kappa) \cdot \mathcal{E}_p(f_\alpha, \mathrm{Ad}) \cdot \mathfrak{e}(\Psi_g(\ell))\mathfrak{e}(\Psi_h(\ell)) = (1 - \alpha_f\beta_g\beta_h p^{-\ell})(1 - \beta_f\alpha_g\beta_h p^{-\ell})(1 - \beta_f\beta_g\alpha_h p^{-\ell})(1 - \beta_f\beta_g\beta_h p^{-\ell})$$
$$= (1 - \alpha_f(\psi_g\psi_h)(\bar{\mathfrak{p}})\langle\lambda(\mathfrak{p})\rangle^{2-2\ell}p^{\ell-2})(1 - \beta_f(\psi_g\psi_h^\sigma)(\mathfrak{p})p^{-1})(1 - \beta_f(\psi_g\psi_h^\sigma)(\bar{\mathfrak{p}})p^{-1})(1 - \beta_f(\psi_g\psi_h)(\bar{\mathfrak{p}})\langle\lambda(\mathfrak{p})\rangle^{2-2\ell}p^{\ell-2})$$

It remains to observe that:

$$\mathfrak{e}(f, \Psi_{gh}(\ell)) = (1 - \alpha_f\Psi_{gh}^{-1}(\bar{\mathfrak{p}}))(1 - \beta_f\Psi_{gh}^{-1}(\bar{\mathfrak{p}}))$$
$$= (1 - \alpha_f\psi_2(\bar{\mathfrak{p}})\langle\lambda(\bar{\mathfrak{p}})\rangle^{2\ell-2}p^{-\ell})(1 - \beta_f\psi_2(\bar{\mathfrak{p}})\langle\lambda(\bar{\mathfrak{p}})\rangle^{2\ell-2}p^{-\ell})$$
$$= (1 - \alpha_f(\psi_g\psi_h)(\bar{\mathfrak{p}})\langle\lambda(\bar{\mathfrak{p}})\rangle^{2\ell-2}p^{-\ell})(1 - \beta_f(\psi_g\psi_h)(\bar{\mathfrak{p}})\langle\lambda(\bar{\mathfrak{p}})\rangle^{2\ell-2}p^{-\ell}),$$

and use $\mathfrak{p}\bar{\mathfrak{p}} = (p)$. ∎

Remark 6.12. Note that:

$$L_p(f_K \times \psi_1, s)^{-1} = (1 - \alpha_f\psi_1(\bar{\mathfrak{p}})p^{-s})(1 - \beta_f\psi_1(\bar{\mathfrak{p}})p^{-s})(1 - \alpha_f\psi_1(\mathfrak{p})p^{-s})(1 - \beta_f\psi_1(\mathfrak{p})p^{-s})$$

and hence the factor $\mathcal{E}(\ell)$ is part of the local L-factor at p of $L(f_K \times \psi_1, 1)$.

Lemma 6.13. *The function $\mathcal{F}(\ell)$ is $K(\psi_g, \psi_h)$-admissible according to* [DLR15, Definition 3.5], *i.e. it extends to an element of* $\mathrm{Frac}\,\mathcal{O}(\mathcal{U})$ *and* $\mathcal{F}(1) \in K(\psi_g, \psi_h)^\times$.

Proof. This follows from the same argument as the proof of [DLR15, Lemma 3.6]. ∎

Altogether, this gives the following result.

Theorem 6.14. *There is a factorization:*

$$\mathcal{L}_p^{\mathrm{bal},f}(\ell,\ell)^2 \cdot \mathcal{L}_p(K)(\psi_g(\ell)) \cdot \mathcal{L}_p(K)(\psi_h(\ell)) = \mathcal{F}(\ell) \cdot \mathcal{L}_p(f, K)(\Psi_{gh}(\ell)) \cdot \frac{L(f_K \times \psi_1, 1)}{\langle f, f\rangle} \cdot \mathcal{E}(\ell).$$

Then Theorem 6.9 follows by evaluating the above factorization at $\ell = 1$ and using equations (6.6), (6.8), and (6.9).

APPENDIX A. A GENERALIZATION OF RESULTS OF BERTOLINI–DARMON–PRASANNA

A.1. **Statement of the results.** Let f be a weight two cuspidal eigenform for $\Gamma_0(N)$, which we assume to have trivial Nebentypus for simplicity. Let K be an imaginary quadratic field of discriminant $-D_K$ and p be a prime which splits in K. For any Hecke character ψ of K with infinity type (κ_1, κ_2), we consider the Rankin–Selberg L-function associated with f and θ_ψ:

$$L(f, \psi, s) = L(f \times \theta_\psi, s - (\kappa_1 + \kappa_2 + 1)/2).$$

We will assume the condition:

(A.1) $$\psi|_{\mathbb{A}^\times} = |\cdot|^2,$$

where $|\cdot|$ is the norm character of $\mathbb{Q}$, which ensures the L-function $L(f, \psi, s)$ is self-dual with central critical value at $s = 0$.

Under the Heegner hypothesis (if $q|N$, then q is either split of ramified in K and if $q^2|N$, then q is split in K) and the assumption that the conductor of ψ is coprime to the level N of f, Bertolini, Darmon, and Prasanna [BDP13] constructed a p-adic L-function interpolating the critical values $L(f, \psi^{-1}, 0)$ when $\kappa_1 \geq 1$ and $\kappa_2 \leq 0$, and proved a special value formula for $\kappa_1 = \kappa_2 = 0$.

The goal of this appendix is to reprove their result, allowing a squarefree product of inert primes to divide N.

Hypothesis A.1. If $q^2|N$, then q is split in K.

Instead, we make the following assumption on the conductor of ψ under which $\epsilon_q(f, \psi) = +1$ at the inert primes q dividing N.

Hypothesis A.2. Let N_0 be the product of primes dividing N which are inert in K. Then the conductor of ψ is divisible exactly by N_0 and coprime to N/N_0.

Remark A.3. The *weak Heegner hypothesis* allows a product N_- of an even number of inert primes to divide N. Assuming that the conductor of ψ is coprime to N — the opposite to Hypothesis A.2 — we have that $\epsilon_q(f, \psi) = -1$ for all primes $q|N_-$. The integral representation for the Rankin–Selberg L-function is then on a Shimura curve associated with a quaternion algebra of discriminant N_-. A generalization of the results of Bertolini–Darmon–Prasanna to this setting was obtained by Brooks [HB15].

As mentioned above, our Hypothesis A.2 instead forces $\epsilon_q(f, \psi) = +1$ at all primes $q|N_0$, and in fact we will assume that $\epsilon_q(f, \psi) = +1$ for all q, so that the integral representation for $L(f, \psi, s)$ is still on the modular curve. This is why we do not require the parity assumption.

Finally, combining the results of this appendix with the results of Brooks, one could presumably allow $N_0 \cdot N_-|N$ where N_- is a product of an even number of inert primes, and the conductor of ψ is divisible exactly by N_0, and coprime to N/N_0. We decided not to pursue this generality here.

Under Hypothesis A.1, there is an ideal $\mathfrak{N}$ of $\mathcal{O}_K$ of norm NN_0; we fix such an ideal. Given an integer $c \geq 1$ divisible exactly by N_0 and coprime to $(N/N_0)D_K$, we consider an order $\mathcal{O}_c$ of $\mathcal{O}_K$ of conductor c. Setting $\mathfrak{N}_c = \mathfrak{N} \cap \mathcal{O}_c$, we have:

$$\mathcal{O}_c/\mathfrak{N}_c \cong \mathbb{Z}/N\mathbb{Z}.$$

Indeed, for $q|N$ coprime to c, if $\mathfrak{q}|q$ then

$$\mathcal{O}_{c,\mathfrak{q}}/\mathfrak{N}_{c,\mathfrak{q}} \cong \mathcal{O}_{K,\mathfrak{q}}/\mathfrak{N}_{\mathfrak{q}} \cong \mathbb{Z}/(q^{v_q(N)})\mathbb{Z}.$$

Moreover, for $q|N_0$, $\mathcal{O}_{c,q} = \mathbb{Z} + q\mathcal{O}_{K_q}$ and:

$$\mathcal{O}_{c,q}/\mathfrak{N}_{c,q} = (\mathbb{Z} + q\mathcal{O}_{K_q})/q\mathcal{O}_{K_q} \cong \mathbb{Z}/q\mathbb{Z}.$$

Note that for $q|N_0$, $1 + q\mathcal{O}_{K_q} \subseteq \mathcal{O}_{c,q}^\times \subseteq \mathcal{O}_{K_q}^\times$ and

$$\begin{array}{ccccc}
\mathcal{O}_{c,q}^\times/(1+q\mathcal{O}_{K_q}) & \hookrightarrow & \mathcal{O}_{K_q}^\times/(1+q\mathcal{O}_{K_q}) & \twoheadrightarrow & \mathcal{O}_{K_q}^\times/\mathcal{O}_{c,q}^\times \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathbb{F}_q^\times & \longrightarrow & \mathbb{F}_{q^2}^\times & \longrightarrow & \mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times
\end{array}$$

In particular, a character of $\mathcal{O}_{K_q}^\times$ which is trivial on $\mathcal{O}_{c,q}^\times$ has conductor at most 1 and is trivial on $\mathbb{F}_q^\times \subseteq \mathbb{F}_{q^2}^\times$.

Definition A.4. Let $\Sigma_{f,c}$ denote the set of Hecke characters ψ of K such that:

(1) $\psi|_{\widehat{\mathcal{O}}_c^\times} = 1$, but for $q|N_0$, $\psi_q|_{\mathcal{O}_{K,q}^\times} \neq 1$,
(2) the infinity type (κ_1, κ_2) of ψ satisfies $\kappa_1 + \kappa_2 = 2$,
(3) $\psi\psi^\sigma = |\cdot|^2$,
(4) $\epsilon_q(f, \psi^{-1}) = +1$ for all finite primes q.

By the above discussion, if $\psi \in \Sigma_{f,c}$, then ψ_q for $q|N_0$ has conductor 1 and $\psi|_{\mathbb{Z}_q^\times} = 1$. In particular, assumption (1) in the definition implies Hypothesis A.2. If $N_0 = 1$, this agrees with $\Sigma_{f,cc}$ in [BDP13]. The final condition is automatic except possibly for the primes in the set:

$$S(f) = \{q \mid q \text{ divides } N \text{ and } D_K\}.$$

For example, if N is coprime to D_K, $S(f) = \emptyset$.

For $\psi \in \Sigma_{f,c}$, $L(f, \psi, s)$ is self-dual and has $s = 0$ as its central critical point. Moreover, $\Sigma_{f,c}$ naturally decomposes as

$$\Sigma_{f,c} = \Sigma_{f,c}^{(1)} \cup \Sigma_{f,c}^{(2)} \cup \Sigma_{f,c}^{(2')},$$

$$\Sigma_{f,c}^{(1)} = \{\psi \text{ of infinity type } (1,1)\},$$

$$\Sigma_{f,c}^{(2)} = \{\psi \text{ of infinity type } (\kappa+2, -\kappa) \text{ for } \kappa \geq 1\},$$

$$\Sigma_{f,c}^{(2')} = \{\psi \text{ of infinity type } (-\kappa, \kappa+2) \text{ for } \kappa \geq 1\}.$$

Theorem A.5.

(1) *There is a p-adic L-function $\mathcal{L}_p(f, K)\colon \widehat{\Sigma}_{f,c} \to \mathbb{C}_p^\times$ defined on the p-adic completion $\widehat{\Sigma}_{f,c}$ of $\Sigma_{f,c}$ with the following interpolation property:*

$$\frac{\mathcal{L}_p(f, K)(\psi)}{\Omega_p^{4\kappa+4}} = \mathfrak{a}(f, \psi) \cdot \mathfrak{e}(f, \psi)^2 \cdot \mathfrak{f}(f, \psi) \cdot \frac{L(f, \psi^{-1}, 0)}{\Omega^{4\kappa+4}} \qquad \psi \in \Sigma_{f,c}^{(2)},$$

where

- $\mathfrak{a}(f, \psi) = \kappa!(\kappa+1)!\pi^{2\kappa+1}$, $\mathfrak{e}(f, \psi) = 1 - a_p(f)\psi^{-1}(\bar{\mathfrak{p}}) + \psi^{-2}(\bar{\mathfrak{p}})p$,
- $\mathfrak{f}(f, \psi) = (2/c\sqrt{D_K})^{2\kappa+1} \prod_{q|c/N_0} \frac{q - \chi_K(q)}{q-1} \prod_{q|N_0} \frac{q^2-1}{q^2} \cdot \omega(f, \psi)^{-1}$, *and $\omega(f, \psi) \in \overline{\mathbb{Q}}$ is defined in* [BDP13, (5.1.11)].

(2) *For a finite order character ψ of K, $\psi N_K \in \Sigma_{f,c}^{(1)}$ which is outside of the interpolation range (6.7), there is a p-adic Gross–Zagier formula:*

$$\mathcal{L}_p(f, K) = \left(1 - \frac{a_p(f)}{\psi(\bar{\mathfrak{p}})p} + \frac{1}{\psi^2(\bar{\mathfrak{p}})p}\right)^2 \cdot \log_p(P_\psi)^2,$$

for P_ψ defined by:

$$P_\psi = \pi_f\left(\sum_{\sigma \in \mathrm{Gal}(H_c/K)} \psi^{-1}(\sigma)t^\sigma\right) \in E(H_c)_L^{\psi_2}$$

for a Heegner point t for H_c/K.

Remark A.6. We remark that Liu–Zhang–Zhang [LZZ18] proved a formula similar to part (2) of the above theorem for any abelian variety parameterized by a Shimura curve over a totally real field. In particular, in the case of Shimura curves over $\mathbb{Q}$, they removed all the ramification hypothesis in [BDP13, HB15].

However, their construction of the p-adic L-function does not have an interpolation property as explicit as (1). More specifically, (1) can be interpreted as an equality in $\overline{\mathbb{Q}}$ (or even an explicit finite extension of $\mathbb{Q}$), having fixed embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \mathbb{C}_p$. Crucially, the CM periods Ω and Ω_p make both sides algebraic.

The interpolation property of Liu–Zhang–Zhang is not an equality of algebraic numbers and relies on an identification $\mathbb{C} \cong \mathbb{C}_p$. In particular, the CM periods do not show up in their interpolation property, and we have not find a direct way to deduce the above result from their work.

A.2. **An explicit Waldspurger formula.** The crux of the proof of Theorem A.5 is to give an explicit version of Waldspurger's formula, generalizing [BDP13, Theorem 4.6].

Theorem A.7. *Let f be a normalized eigenform in $S_2(N)$ and let $\psi \in \Sigma_{f,c}^{(2)}$ be a Hecke character of K of infinity type $(2 + j, -j)$. Suppose also that c and D_K are odd, and let w_K denote the number of roots of unity in K. Then:*

$$C(f, \psi, c) \cdot L(f, \psi^{-1}, 0) = \left| \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a}) N\mathfrak{a}^{-j} \cdot (\delta_k^j f)(\mathfrak{a}^{-1}, t_\mathfrak{a}) \right|,$$

where the representatives of the ideal classes in $\mathrm{Pic}(\mathcal{O}_c)$ are chosen to be coprime to the $\mathfrak{N}_c$ and the constant $C(f, \psi, c) \in \mathbb{C}$ is given by:

$$C(f, \chi, c) = \frac{1}{4} \pi^{k+2j-1} \Gamma(j+1) \Gamma(k+j) w_K |D_K|^{1/2} \cdot c \, \mathrm{vol}(\mathcal{O}_c)^{-\ell} \cdot 2^{\#S_f} \cdot \prod_{q|c/N_0} \frac{q - \chi_K(q)}{q-1} \cdot \prod_{q|N_0} \frac{q^2 - 1}{q^2}.$$

Proof. The proof in [BDP13, Section 4] can be applied verbatim to our case, until the local calculations in Section 4.6. The extra local integral is then computed in the next section, resulting in Proposition A.8. ∎

A.3. **The key computation.** The assumption in [BDP13] that the conductor of ψ is coprime to the level N of f is used in the explicit computations of the local zeta integrals in Section 4.6. We use the same notation as loc. cit. but consider the following extra setting:

- let $\mathbb{Q}_{q^2} = \mathbb{Q}_q[\varpi]$ where $\varpi^2 \in \mathbb{Q}_q^\times$ is a unit is the unramified quadratic extension of $\mathbb{Q}_q$, and let σ be the non-trivial automorphism of $\mathbb{Q}_{q^2}$ over $\mathbb{Q}_q$,
- ψ is an anticyclomotic character of $\mathbb{Q}_{q^2}^\times$ of conductor $r \geq 1$ such that $\psi \neq \psi^\sigma$ and $\psi|_{\mathbb{Q}_q^\times} = 1$, (here, anticyclotomic means that $\psi\psi^\sigma = 1$; in particular, note that $\psi^2 \neq 1$),
- π is the Steinberg representation St of $\mathrm{GL}_2(\mathbb{Q}_q)$.

Analogously to p. 1118 of loc. cit. we consider:

$$J(\varsigma, \vartheta) = \int_{N_{\mathbb{Q}_{q^2}/\mathbb{Q}_q}(\mathbb{Q}_{q^2}^\times)} W_F(d(a)) W_{\Theta,\varsigma}(d(a)) \Phi_\vartheta^s(d(a)) |a|^{-1} d^\times a.$$

As written, the equation in [BDP13] involves the integral of $\mathbb{Q}_q^\times$, but tracing back the reference [Pra06, Section 3] reveals this is the correct generalization at inert places.

We also recall from Proposition 4.24 of loc. cit. that:

$$W_{\Theta,\varsigma}(d(a)) = \int_{\mathbb{Q}_{q^2}^{(1)}} \varsigma(h^{-1}(h')^\sigma) \psi(hh') \, dh$$

for any h' such that $N(h') = a$, where

$$\mathbb{Q}_{q^2}^{(1)} = \{x \in \mathbb{Q}_{q^2} \mid N(x) = 1\} \subseteq \mathbb{Z}_{p^2}^\times$$

and the Haar measure is normalized so that $\mathrm{vol}(\mathbb{Q}_{q^2}^{(1)}) = 1$.

We recall that the group of norm one elements in the unramified quadratic extension of $\mathbb{Q}_p$ has the following description:

$$(A.2) \qquad\qquad \mathbb{Q}_{q^2}^{(1)} = \left\{ \frac{t}{t^\sigma} \; \middle| \; t \in \mathbb{Z}_{q^2}^\times \right\}.$$

Therefore,

$$W_{\Theta,\varsigma}(d(a)) = \int_{\mathbb{Z}_{q^2}^\times} \varsigma((th')^\sigma/t)\psi(th'/t^\sigma)\, dt.$$

We choose:

- $\varsigma = \mathbf{I}_{\mathbb{Z}_q + q^r \varpi \mathbb{Z}_q}$, the indicator function of $\mathbb{Z}_q + q^r \varpi \mathbb{Z}_q \subseteq \mathbb{Z}_q + \varpi \mathbb{Z}_q = \mathbb{Z}_{q^2}$,
- $\vartheta = \mathbf{I}_{(\mathbb{Z}_q + q^r \mathbb{Z}_q \varpi)\mathsf{j}}$ where $\mathsf{j} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $W_F(d(a)) = |a|\mathbf{I}_{\mathbb{Z}_q}(a)$, since $\pi = \mathrm{St}$ (cf. Proposition 4.23 of loc. cit.).

Then $\varsigma((th')^\sigma/t) = 1$ if and only if

$$(th')^\sigma/t \in \mathbb{Z}_q + q^r \varpi \mathbb{Z}_q.$$

For $x \in \mathbb{Z}_{q^2}$, the condition $x \in \mathbb{Z}_q + q^r \varpi \mathbb{Z}_q$ is equivalent to $v_q(x - x^\sigma) \geq r$. Therefore, the above condition is simply:

$$v_q((t^\sigma)^2 (h')^\sigma - t^2 h') \geq r.$$

Next, we observe that for $v(h') \geq r$, this condition is automatic, and in this case:

$$W_{\Theta,\varsigma}(d(a)) = \psi(h') \int_{\mathbb{Z}_{q^2}^\times} (\psi/\psi^\sigma)(t)\, dt$$

$$= 0 \qquad\qquad\qquad\qquad \psi/\psi^\sigma \neq 1.$$

Since $W_F(d(a)) = |a|\mathbf{I}_{\mathbb{Z}_q}(a)$ and $h'(h')^\sigma = a$, the remaining part of the integral $J(\varsigma, \vartheta)$ has $0 \leq v_q(h') \leq r-1$. Fix $m = v(h')$ and write $h' = q^m \cdot u$ for $u \in \mathbb{Z}_{q^2}^\times$. Then:

$$(t^\sigma)^2 (h')^\sigma - t^2 h' = q^m((t^\sigma)^2 u^\sigma - t^2 u),$$

and the condition on the valuation can be concisely written as

$$t^2 u \equiv (t^2 u)^\sigma \mod q^{r-m}.$$

Dividing both sides by the unit tt^σ gives the equivalent condition:

$$tu/t^\sigma \equiv (tu/t^\sigma)^\sigma \mod q^{r-m}.$$

Finally, this shows that:

$$\int_{v_q(a)=2m} W_F(d(a))W_{\Theta,\varsigma}(d(a))|a|^{s-1}d^\times a = \iint_{\substack{u,t \in \mathbb{Z}_{q^2}^\times \\ tu/t^\sigma \equiv (tu/t^\sigma)^\sigma \bmod q^{r-m}}} \psi(tu/t^\sigma)\, d^\times u\, d^\times t$$

$$= \iint_{\substack{v,t \in \mathbb{Z}_{q^2}^\times \\ v \equiv v^\sigma \bmod q^{r-m}}} \psi(v)\, d^\times v \qquad\qquad v = t/t^\sigma u.$$

Now, if $m \geq 1$, this final integral is 0, because there exists $v_0 \in \mathbb{Z}_{q^2}^\times$ such that $v_0 \equiv 1 \mod q^{r-m}$ and $\psi(v_0) \neq 1$.

For $v \in \mathbb{Z}_{q^2}^\times$ such that $v \equiv v^\sigma \mod q^r$, there exists $v_0 \in \mathbb{Z}_q^\times$ such that $v_0 \equiv v \mod q^r$, and hence

$$\psi(v) = \psi(v_0) = 1,$$

since $\psi|_{\mathbb{Z}_q^\times} = 1$. Altogether, this shows that:

$$J(\varsigma, \vartheta) = \mathrm{vol}(\mathbb{Z}_{q^2}^\times) \cdot \mathrm{vol}\{v \in \mathbb{Z}_{q^\times}^\times \mid v \equiv v^\sigma \mod q^r\}.$$

Next, recall that the measure is normalized so that $\mathrm{vol}(\mathbb{Q}_{q^2}^{(1)}) = 1$ and hence by equation (A.2), we also have $\mathrm{vol}(\mathbb{Z}_{q^2}^\times) = 1$. To compute $J(\varsigma, \vartheta)$, it remains to compute the index of

$$S = \{v \in \mathbb{Z}_{q^2}^\times \mid v \equiv v^\sigma \bmod q^r\}$$

in $\mathbb{Z}_{q^2}^\times$. We consider the intersection of S with the filtration $U_n = 1 + q^n\mathbb{Z}_{q^2}^\times$ on $U_0 = \mathbb{Z}_{q^2}^\times$. Clearly, for $n \geq r$, $S_n = S \cap U_n = U_n$ because the condition $v^\sigma \equiv v \bmod p^r$ is automatically satisfied. The successive quotients of the filtration are:

$$U_0/U_1 \cong \mathbb{F}_{q^2}^\times$$
$$U_n/U_{n+1} \cong \mathbb{F}_{q^2}$$

and for $n < r$ the condition $v \equiv v^\sigma \bmod q^r$ gives:

$$S_0/S_1 \cong \mathbb{F}_q^\times,$$
$$S_n/S_{n+1} \cong \mathbb{F}_q.$$

Overall, this shows that:

$$[\mathbb{Z}_{q^2}^\times : S] = \frac{q^2 - 1}{q - 1} \cdot q^{r-1}.$$

Therefore, finally:

$$J(\varsigma, \vartheta) = \frac{q - 1}{q^{r-1}(q^2 - 1)}.$$

We rephrase this result as in Section 4.6 of loc. cit.

Proposition A.8. *For the above choices of ς and ϑ, we have*

$$J(\varsigma, \vartheta) = \frac{1}{q^r}\frac{q - 1}{q + 1} \cdot L_q(\pi_f, \pi_{\overline{\eta'}}, s)L_q(2s, \epsilon_K)^{-1}|_{s=1/2}.$$

Proof. We recall that:

- $L_q(1, \epsilon_K) = \frac{1}{1+1/q} = \frac{q}{q+1}$,
- by [Jac72, Theorem 15.1], since $\pi = \mathrm{St}$, we have

$$L(s, \pi \times \pi_\psi) = L(s, \pi_\psi \otimes |\cdot|^{1/2}) = L(s + 1/2, \pi_\psi) = L(s + 1/2, \psi),$$

  and hence:

$$L(1/2, \pi \times \pi_\psi) = L(1, \psi) = \frac{1}{1 - \psi(q)q^{-2}} = \frac{1}{1 - q^{-2}} = \frac{q^2}{q^2 - 1}.$$

Thus:

$$\frac{I(\varsigma, \vartheta)}{L_q(\pi_f, \pi_{\overline{\eta'}}, s)L_q(2s, \epsilon_K)^{-1}|_{s=1/2}} = \frac{q - 1}{q^{r-1}(q^2 - 1)}\frac{q^2 - 1}{q^2}\frac{q}{q + 1}$$
$$= \frac{1}{q^r} \cdot \frac{q - 1}{q + 1}. \qquad\blacksquare$$

A.4. **Finishing the proof.** Theorem A.5 is again proved by following [BDP13, Section 5] word-for-word, and replacing Theorem 4.6 of loc. cit, with Theorem A.7 above.

Appendix B. Examples

The hypotheses of Conjecture 6.5 and Theorem 6.9 may seem quite restrictive at first glance, so in this section we collect many examples where they are satisfied. Even though we have only provided evidence for the conjecture in the CM case so far, we still wish to give instances where the conjecture could be verified numerically by adapting the algorithms in [Dal23a]. We hope to carry out this numerical verification in future work.

We divide the examples into four categories, namely:

(1) $h = g^*$, weight 1 forms with CM by the same field K;
(2) $h \neq g^*$, weight 1 forms with CM by the same field K;
(3) $h = g^*$, weight 1 forms without CM;
(4) $h \neq g^*$, weight 1 forms without CM.

Recall that we denote the form obtained by complex conjugation from g by g^* and that its corresponding representation is the contragredient of π_g. In the CM cases, K represents the imaginary quadratic field. As in the previous sections, we denote the various analytic ranks by $r(E_K) = \mathrm{ord}_{s=1}(L(E_K, s))$, and $r(E_K, \psi) = \mathrm{ord}_{s=1}(L(E_K, \psi, s))$. We always consider elliptic curves with analytic (and algebraic) rank 0 over $\mathbb{Q}$. In the first two categories of examples, we keep track of which among the various Hypotheses A, B, C, A', B', and C' is satisfied. All our computations are made with the help of `magma` [BCP97] and we make extensive use of the LMFDB Database [LMF24]; all the labels refer to items listed there. We developed a simple procedure for computing the Hecke characters ψ_g and ψ_h and hence studying order of vanishing of the L-functions; these routines can be found at [2]. In the following tables, we highlight the primes of supercuspidal type in blue.

B.1. **CM case, $h = g^*$.** We consider here the case of $h = g^*$ with CM by K. In this situation, the two characters ψ_1 and ψ_2 of Section 6 are, respectively, $\mathbb{1}$ and $\psi = \psi_g \psi_h^\sigma = \psi_g / \psi_g^\sigma$, for σ the generator of $\mathrm{Gal}(K/\mathbb{Q})$. For each example below, we checked that Hypotheses A', B', and C are satisfied, so the Elliptic Stark Conjecture 6.5 applies. The stronger hypothesis C', under which Theorem 6.9 applies, is only sometimes satisfied so we separate the examples into Tables 1 and 2, accordingly.

TABLE 1. Examples where $h = g^*$, both with CM by the same field K. Hypotheses A', B', and C' are satisfied; Theorem 6.9 applies.

E	Level	$g = h^*$	Level	$r(E_K) + r(E_K, \psi)$	K	p
14.a2	$2 \cdot 7$	1911.1.h.c	$3 \cdot 7^2 \cdot 13$	$0 + 1$	$\mathbb{Q}(\sqrt{-39})$	41
14.a.2	$2 \cdot 7$	1911.1.h.d	$3 \cdot 7^2 \cdot 13$	$0 + 1$	$\mathbb{Q}(\sqrt{-39})$	5
17.a1	17	2023.1.c.a	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	23
17.a1	17	2023.1.c.b	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	23
17.a1	17	2023.1.c.c	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
17.a1	17	2023.1.c.d	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
19.a.1	19	2527.1.d.c	$7 \cdot 19^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
21.a4	$3 \cdot 7$	2695.1.l.a	$5 \cdot 7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	37
21.a4	$3 \cdot 7$	2695.1.g.i	$5 \cdot 7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	67
26.a1	$2 \cdot 13$	1183.1.d.a	$7 \cdot 13^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
26.b1	$2 \cdot 13$	1183.1.d.a	$7 \cdot 13^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
34.a1	$2 \cdot 17$	2023.1.c.a	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
34.a1	$2 \cdot 17$	2023.1.c.b	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
34.a1	$2 \cdot 17$	2023.1.c.d	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
52.a2	$2^2 \cdot 13$	1183.1.d.a	$7 \cdot 13^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
55.a1	$5 \cdot 11$	175.1.d.a	$5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	23
187.b1	$11 \cdot 17$	2023.1.c.c	$7 \cdot 17^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11

TABLE 2. Examples where $h = g^*$, both with CM by K. Hypotheses A', B', and C are satisfied, so Conjecture 6.5 applies. However, Theorem 6.9 does not apply as Hypothesis C' is not satisfied.

E	Level	g	Level	$r(E_K) + r(E_K, \psi)$	K	p
15.a3	$3 \cdot 5$	525.1.be.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
15.a3	$3 \cdot 5$	525.1.k.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
15.a3	$3 \cdot 5$	693.1.h.a	$3^2 \cdot 7 \cdot 11$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	11
30.a7	$2 \cdot 3 \cdot 5$	693.1.bp.a	$3^2 \cdot 7 \cdot 11$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	11
35.a1	$5 \cdot 7$	525.1.p.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
35.a1	$5 \cdot 7$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
35.a1	$5 \cdot 7$	525.1.u.b	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
39.a4	$3 \cdot 13$	1183.1.d.a	$7 \cdot 13^2$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
49.a4	7^2	539.1.c.b	$7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	5
51.a.1	$3 \cdot 17$	2023.1.c.d	$7 \cdot 17^2$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	11
65.a2	$5 \cdot 13$	1183.1.d.a	$7 \cdot 13^2$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	11
85.a.2	$5 \cdot 17$	2023.1.c.a	$7 \cdot 17^2$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	23
85.a.2	$5 \cdot 17$	2023.1.c.b	$7 \cdot 17^2$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	23
85.a.2	$5 \cdot 17$	2023.1.c.d	$7 \cdot 17^2$	$1 + 0$	$\mathbb{Q}(\sqrt{-7})$	23
195.a1	$3 \cdot 5 \cdot 13$	175.1.d.a	$5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	23

B.2. **CM case, $h \neq g^*$.** We consider here the case of $h \neq g^*$ with CM by the same field K and we collect the examples into two tables. Note that, differently from the previous section (Section B.1), we have $\psi_1, \psi_2 \neq \mathbb{1}$, hence it becomes difficult to check Hypothesis C' for the examples in Table 3 and Table 4. This last condition can be checked by explicitly recovering the Hecke characters ψ_1 and ψ_2 via [2] and computing the local ϵ-factors. We list a few forms satisfying all the Hypotheses A', B', and C in Table 3. In these examples, Hypothesis C' represents a rather strict assumption but, as observed in Remark A.3, it can be relaxed by extending [HB15]. Some pairs are related by a twist of a Dirichler character, while others are not. The latter represents the most interesting situation and one can notice that the Artin representation $V_g \otimes V_h$ decomposes as direct sum of two irreducible two-dimensional Artin representations. The second table of this section, that is, Table 4 presents a few examples not satisfying Hypothesis C', i.e. ones where Theorem 6.9 does not apply, but Conjecture 6.5 does.

TABLE 3. Examples where $h \neq g^*$ both with CM by the same field K. Hypotheses A', B', and C' are satisfied; Theorem 6.9 applies.

E	Level	g	Level	$r(E_K, \psi_1) + r(E_K, \psi_2)$	K	p
		h	Level			
14.a1	$2 \cdot 7$	539.1.c.a	$7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	5
		2156.1.h.b	$2^2 \cdot 7^2 \cdot 5$			
15.a1	$3 \cdot 5$	175.1.d.a	$5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-7})$	11
		1575.1.h.d	$3^2 \cdot 5^2 \cdot 7$			
70.a1	$2 \cdot 5 \cdot 7$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
77.b1	$7 \cdot 11$	539.1.c.a	$7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	5
		539.1.c.b	$7^2 \cdot 11$			

TABLE 4. Examples where $h \neq g^*$ both with CM by the same field K. Hypotheses A', B', and C are satisfied; Conjecture 6.5 applies.

E	Level	g	Level	$r(E_K, \psi_1) + r(E_K, \psi_2)$	K	p
		h	Level			
15.a1	$3 \cdot 5$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
20.a1	$2^2 \cdot 5$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		525.1.u.b	$3 \cdot 5^2 \cdot 7$			
42.a1	$2 \cdot 3 \cdot 7$	539.1.c.a	$7^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{-11})$	11
		2156.1.h.b	$2^2 \cdot 7^2 \cdot 5$			
55.a1	$5 \cdot 11$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		525.1.u.b	$3 \cdot 5^2 \cdot 7$			
155.a2	$5 \cdot 31$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
155.a2	$5 \cdot 31$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.b	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
210.a1	$2 \cdot 3 \cdot 5 \cdot 7$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
210.d1	$2 \cdot 3 \cdot 5 \cdot 7$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			
490.a1	$2 \cdot 3 \cdot 5 \cdot 7$	525.1.u.a	$3 \cdot 5^2 \cdot 7$	$0 + 1$	$\mathbb{Q}(\sqrt{-3})$	13
		2100.1.bn.a	$2^2 \cdot 3 \cdot 5^2 \cdot 7$			

Remark B.1. It is not automatic that finding the right level and character produces a situation where our work applies. By Proposition 2.1 (2) ([Pra90, Proposition 8.5]), in order to obtain local sign $\varepsilon_\ell(E, \varrho_{gh}) = -1$, the supercuspidal representations $\pi_{g,\ell}$ and $\pi_{h,\ell}$ need to satisfy $\pi_{g,\ell} \cong \pi_{h,\ell}^* = \pi_{h^*,\ell}$. There are several examples which fail this final condition:

- g: 2023.1.c.a, 2023.1.c.b and h: 2023.1.c.c, 2023.1.c.d;
- g: 2527.1.d.c, 2527.1.d.d and h: 2527.1.d.f.

B.3. **Non-CM case,** $h = g^*$. In this section we report a few examples to which our Conjecture 6.5 applies, but the form g does not have CM, so Theorem 6.9 does not. We divide the examples depending on the projective image of the Artin representation associated with g.

B.3.1. *RM but no CM:*. We begin by giving examples of some weight 1 modular forms with RM by a field $F/\mathbb{Q}$ but without CM. When $h = g^*$, we once again have

$$(B.1) \qquad V_g \otimes V_{g^*} = \text{Ind}_{\mathbb{Q}}^F(\mathbb{1}) \oplus V_\psi,$$

for $\psi = \psi_g/\psi_g^\sigma$; here σ is a generator of $\mathrm{Gal}(F/\mathbb{Q})$.

TABLE 5. Examples where $h = g^*$, both with RM by F; $r(E_F)$ and $r(E_F, \psi)$ are computed.

E	Level	g	Level	$r(E_F) + r(E_F, \psi)$	F	p
15.a1	$3 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
15.a1	$3 \cdot 5$	1025.1.f.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
20.a1	$2^2 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
20.a1	$2^2 \cdot 5$	1025.1.f.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
21.a1	$3 \cdot 7$	396.1.d.a	$2^2 \cdot 3^2 \cdot 11$	$1 + 0$	$\mathbb{Q}(\sqrt{11})$	5
30.a1	$2 \cdot 3 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
30.a1	$2 \cdot 3 \cdot 5$	1025.1.f.a	$5^2 \cdot 41$	$0 + 1$	$\mathbb{Q}(\sqrt{5})$	11
39.a1	$3 \cdot 7$	396.1.d.a	$2^2 \cdot 3^2 \cdot 11$	$0 + 1$	$\mathbb{Q}(\sqrt{11})$	5

B.3.2. A_4: To compute these examples, we make wide use of the technical remarks in [DLR15, Section 5.1]. In this situation, $V_g \otimes V_{g^*}$ decomposes as $\mathbb{1} \oplus \mathrm{Ad}_g$, for Ad_g the adjoint representation of g. Therefore it is enough to compute the order of vanishing $\mathrm{ord}_{s=1} L(E, \mathrm{Ad}_g, s)$. Let H be the Galois field defined by the *projective* Artin representation associated with g, let $h(x)$ be a degree 4 polynomial whose splitting field is H. Denoting by M the field generated by a single root of $h(x)$, by [DLR15, Section 5.1.1], we have

$$(B.2) \qquad r(E, \mathrm{Ad}_g) = \mathrm{ord}_{s=1} L(E, \mathrm{Ad}_g, s) = r(E_M) - r(E).$$

Hence, $\mathrm{ord}_{s=1} L(f \times g \times h, 1) = r(E_M)$.

TABLE 6. Examples where $h = g^*$ is an exotic form of projective type A_4; $r(E_M)$ and $r(E_H)$ are computed.

E	Level	g	Level	$r(E)$	$r(E_M)$	$r(E_H)$	Polynomial of M	p
15.a1	$3 \cdot 5$	325.1.u.a	$5^2 \cdot 13$	0	1	3	$x^4 - x^3 - 3 \cdot x + 4$	11
26.a1	$2 \cdot 13$	1183.1.x.a	$7 \cdot 13^2$	0	1	3	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11
26.a1	$2 \cdot 13$	1183.1.z.a	$7 \cdot 13^2$	0	1	3	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11
26.a1	$2 \cdot 13$	1183.1.bd.a	$7 \cdot 13^2$	0	1	3	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11
39.a1	$3 \cdot 13$	1183.1.x.a	$7 \cdot 13^2$	0	1	5	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11
39.a1	$3 \cdot 13$	1183.1.z.a	$7 \cdot 13^2$	0	1	5	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11
39.a1	$3 \cdot 13$	1183.1.bd.a	$7 \cdot 13^2$	0	1	5	$x^4 - x^3 + 5 \cdot x^2 - 4 \cdot x + 3$	11

B.3.3. S_4: As above, $V_g \otimes V_h$ decomposes as $\mathbb{1} \oplus \mathrm{Ad}_g$.

TABLE 7. Examples where $h = g^*$ is an exotic form of projective type S_4; $r(E)$ and $r(E, \mathrm{Ad}_g)$ are computed.

E	Level	g	Level	$r(E) + r(E, \mathrm{Ad}_g)$	p
11.a1	11	968.1.h.a	$2^3 \cdot 11^2$	$0 + 1$	5
33.a1	$3 \cdot 11$	968.1.h.a	$2^3 \cdot 11^2$	$0 + 1$	5
55.a1	$5 \cdot 11$	968.1.h.a	$2^3 \cdot 11^2$	$0 + 1$	23

We report here a few more modular forms to which our setting applies, but computing the corresponding analytic ranks seems computationally demanding: 1224.1.m.a, 1224.1.m.b, 1800.1.l.a, and 1800.1.l.b.

B.3.4. A_5: Unfortunately, computing examples with forms of projective image A_5 seems out of reach at the present moment, due to the difficulty of computing Artin representations over degree 60 number fields. Even trying to address the problem as in [DLR15, Section 5.1.2], as the computation of the unique subfield of degree 2 contained in the Artin field is demanding. However, we list here a few modular forms to which our setting should apply: 1825.1.y.a, 2079.1.dd.a, 3069.1.cd.a, 3069.1.cd.b, and 3168.1.cb.a.

B.4. **Non-CM case, $h \neq g^*$.** Computing examples in this setting is rather challenging, but we present a few examples of dihedral forms with RM and no CM, and exotic ones with S_4 projective image. In both cases, we only consider forms g and h with are related by a twist by Dirichlet character.

B.4.1. *RM but no CM:*. In this situation, we consider g and h with RM by the same field $\mathbb{Q}(\sqrt{5})$. We write $V_g \otimes V_h = \mathrm{Ind}_{\mathbb{Q}(\sqrt{5})}^{\mathbb{Q}}(\psi_g) \otimes \mathrm{Ind}_{\mathbb{Q}(\sqrt{5})}^{\mathbb{Q}}(\psi_h) \, \mathrm{Ind}_{\mathbb{Q}(\sqrt{5})}^{\mathbb{Q}}(\psi_g \psi_h) \oplus \mathrm{Ind}_{\mathbb{Q}(\sqrt{5})}^{\mathbb{Q}}(\psi_g \psi_h^\sigma)$.

TABLE 8. Examples where $h \neq g^*$ are dihedral forms, both with RM by $\mathbb{Q}(\sqrt{5})$ and without CM.

E	Level	g	Level	$r(E_F, \psi_g \psi_h) + r(E_F, \psi_g \psi_h^\sigma)$	p
		h	Level		
15.a1	$3 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$0 + 1$	11
		1025.1.f.a	$5^2 \cdot 41$		
20.a1	$2^2 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$1 + 0$	11
		1025.1.f.a	$5^2 \cdot 41$		
30.a1	$2 \cdot 3 \cdot 5$	1025.1.i.a	$5^2 \cdot 41$	$0 + 1$	11
		1025.1.f.a	$5^2 \cdot 41$		

B.4.2. S_4: We consider the case of $g \neq h^*$ but $h = g \otimes \chi$, for χ a Dirichlet character. In this situation, the representation $V_g \otimes V_h$ decomposes as $\rho_1 \oplus \rho_3$ for ρ_d a d-dimensional Artin representation, $d = 1, 3$.

TABLE 9. Examples where $h \neq g^*$, $h = g \otimes \chi$, exotic forms of projective type S_4; $r(E, \rho_1)$ and $r(E, \rho_3)$ are computed.

E	Level	g	Level	$r(E, \rho_1) + r(E, \rho_3)$	p
		h	Level		
15.a1	$3 \cdot 5$	981.1.d.a	$3^2 \cdot 109$	$0 + 1$	7
		981.1.d.b	$3^2 \cdot 109$		
24.a1	$2^3 \cdot 3$	981.1.d.a	$3^2 \cdot 109$	$1 + 0$	7
		981.1.d.b	$3^2 \cdot 109$		
30.a1	$2 \cdot 3 \cdot 5$	981.1.d.a	$3^2 \cdot 109$	$1 + 0$	7
		981.1.d.b	$3^2 \cdot 109$		
33.a1	$3 \cdot 11$	981.1.d.a	$3^2 \cdot 109$	$1 + 0$	7
		981.1.d.b	$3^2 \cdot 109$		

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