Algebraic cycles & the Langlands program

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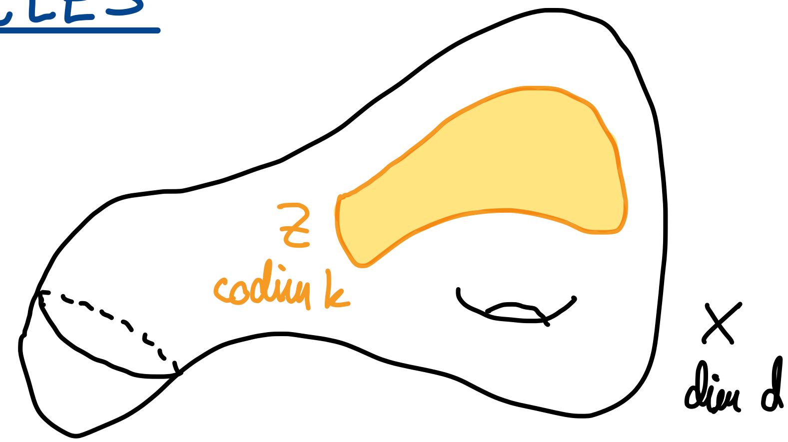
September 6, 2023

ALGEBRAIC CYCLES



Singular cohomology. $H^n(X(\mathbb{C}),\mathbb{Z})$

$$\omega \longmapsto \int_Z \omega|_Z \in H^n(X(\mathbb{C}),\mathbb{Z})^\vee \underset{\uparrow}{\cong} H^{2d-n}(X(\mathbb{C}),\mathbb{Z})$$

Poincaré duality

$$\rightsquigarrow [Z]_{sing} \in H^{2d-n}(X(\mathbb{C}),\mathbb{Z})$$

When is this class non-zero?

Hodge decomposition: $H^n_{sing}(X(\mathbb{C}),\mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}$, $\quad \omega \in H^n_{sing}(X(\mathbb{C}),\mathbb{C})$

$H^{p,q}$: locally spanned by $dz_{i_1}\wedge\dots\wedge dz_{i_p}\wedge d\bar z_{j_1}\wedge\dots\wedge d\bar z_{j_q}$

$\rightsquigarrow$ If Z is locally given by $z_{d-k+1}=\dots=z_d=0$, then:

$$\int_Z \omega|_Z \neq 0 \iff (p,q)=(d-k,d-k) \qquad [\operatorname{codim} X = d]$$

$$\boxed{\textbf{Hodge conjecture.}\quad H^{2k}(X(\mathbb{C}),\mathbb{Z})\cap H^{k,k} = \operatorname{span}\{[Z] : Z\subseteq X,\ \operatorname{codim} Z = k\}}$$

Known for $d=2$: Lefschetz $(1,1)$ theorem.

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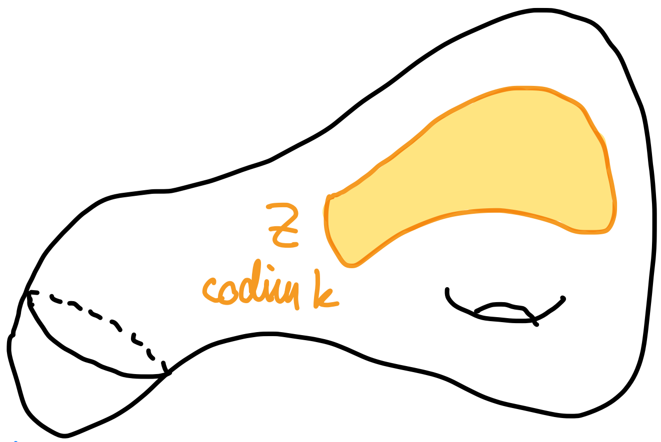

X
dim d

Singular cohomology.

Hodge conjecture: $= ?$

$$[Z]_{\text{sing}} \in \underbrace{H^{2k}(X(\mathbb{C}), \mathbb{Z}) \cap H^{k,k}}_{\text{Hodge classes}}$$

(comparison thm)

$$[Z]_{\text{sing}} \longleftrightarrow [Z]_\ell$$

Étale cohomology.

ℓ = prime

Tate conjecture: $= ?$

$$\underbrace{H^{2k}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(k))^{G_{\mathbb{Q}}}}_{\text{Tate classes}} \ni [Z]_\ell$$

MAPS BETWEEN COHOMOLOGY

X_1, X_2 varieties, $\dim X_i = d_i$

$$H^{i_1}(X_1) \longrightarrow H^{i_2}(X_2) \quad \leftrightsquigarrow \quad H^{i_1}(X_1) \otimes H^{i_2}(X_2)^\vee \underset{\text{Poincaré duality}}{\cong} H^{i_1}(X_1) \otimes H^{2d_2 - i_2}(X_2)$$

$$\underset{\text{Künneth formula}}{\cong} H^{i_1 + 2d_2 - i_2}(X_1 \times X_2)$$

Questions.
- Hodge class ?
- Tate class ?
- Given by an algebraic cycle ?

... & THE LANGLANDS PROGRAM

$$\left\{\begin{array}{c}\text{weight } 2\\ \text{modular forms}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{2-dim'l reps}\\ \text{of } G_{\mathbb{Q}}\end{array}\right\} \leftarrow\cdots \left\{\begin{array}{c}\text{elliptic}\\ \text{curves}/\mathbb{Q}\end{array}\right\}$$

$$f \longmapsto H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \cong H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longleftarrow E$$

X = modular curve $/\mathbb{Q}$

B = quaternion algebra $/\mathbb{Q}$

$B \otimes \mathbb{R} \cong M_2(\mathbb{R})$

$\to$ X^B = Shimura curve ass. to quaternion algebra B

$\to$ f^B = Jacquet-Langlands transfer of f to $B^\times$

<u>Fact.</u> $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \cong H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \cong H^1_{\text{ét}}(X^B_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_{f^B}$

$\Rightarrow$ Tate conj. implies $\exists$ alg. cycle $Z \subset X \times X^B$ inducing the Jacquet-Langlands correspondence!

<u>General question.</u> Are instances of Langlands functoriality induced by algebraic cycles?

… & THE LANGLANDS PROGRAM

$F = \mathbb{Q}(\sqrt{D})$ real quadratic field

$$\left\{\begin{matrix}\text{weight } (2,2) \text{ Hilbert} \\ \text{modular forms}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{2-dim'l reps} \\ \text{of } G_F\end{matrix}\right\} \leftarrow \cdots \left\{\begin{matrix}\text{elliptic} \\ \text{curves } /F\end{matrix}\right\}$$

$$f \longmapsto ??? \qquad \cong H^1_{et}(E_{\bar F}, \mathbb{Q}_\ell) \longleftrightarrow E$$

$$\circlearrowleft G_F$$

(Oda's conjecture)

X = Hilbert modular surface$/\mathbb{Q}$ $\rightsquigarrow$ $H^2_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \cong H^1_{et}(E_{\bar F}, \mathbb{Q}_\ell) \otimes H^1_{et}(E^\sigma_{\bar F}, \mathbb{Q}_\ell)$

$\circlearrowleft G_{\mathbb{Q}}$ $\qquad \sigma \in Gal(F/\mathbb{Q})$

[Side note: sometimes, $\exists B/F$ s.t. $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus \mathbb{H}$

$\rightsquigarrow X^B$ Shimura curve $/F$ s.t. $H^1_{et}(X^B_{\bar F}, \mathbb{Q}_\ell)_{f^B} \cong H^1_{et}(E_{\bar F}, \mathbb{Q}_\ell)$.]

❗ No canonical choice of B! ❗

… & THE LANGLANDS PROGRAM

$F = \mathbb{Q}(\sqrt{D})$ real quadratic field

$$\left\{\begin{array}{c}\text{weight (2,2) Hilbert} \\ \text{modular forms}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{2-dim'l reps} \\ \text{of } G_F\end{array}\right\} \dashleftarrow \left\{\begin{array}{c}\text{elliptic} \\ \text{curves } /F\end{array}\right\}$$

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$\circlearrowleft G_{\mathbb{Q}}$ $\qquad \sigma \in Gal(F/\mathbb{Q})$

B = quaternion algebra $/F$ $\longrightarrow$ X^B = surface ass. to quaternion algebra B

$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$ $\longrightarrow$ f^B = Jacquet-Langlands transfer of f to $B^\times$

$\Rightarrow H^2_{et}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \underset{(*_\ell)}{\cong} H^2_{et}(X^B_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)_{f^B} \rightsquigarrow$ Given by algebraic cycle? Hard!

> **Thm (Ichino-Prasanna).** $\exists$ Hodge class on $X \times X^B$ inducing:
> - isom. of Hodge structures,
> - isom. $(*_\ell)$ for all ℓ, via comparison theorem

Related work: Naomi Sweeting for Yoshida lifts.

ABELIAN SURFACES

$$\left\{\begin{array}{c}\text{weight } (2,2) \text{ Siegel}\\ \text{modular forms}\end{array}\right\} \xrightarrow{???} \left\{\begin{array}{c}\text{4-dim'l symplectic}\\ G_{\mathbb{Q}}\text{-reps}\end{array}\right\} \overset{\cdots}{\longleftarrow} \left\{\begin{array}{c}\text{abelian}\\ \text{surfaces}/\mathbb{Q}\end{array}\right\}$$

$$f \longmapsto ??? \qquad \cong H^1_{ét}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \longleftrightarrow A$$

$$G_{\mathbb{Q}} \curvearrowright \quad + \text{ Weil pairing}$$

$X =$ Siegel modular threefold$/\mathbb{Q} \rightsquigarrow H^*_{ét}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f = 0$

<u>Instead</u>: $\exists$ line bundle $\mathcal{E}$ over X s.t. $H^i(X,\mathcal{E})_f \neq 0$ for $i = 0, 1$.

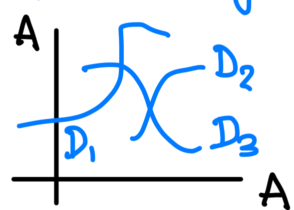
<u>Degenerate case of the above</u>:

$$\begin{array}{ccc} X_1, X_2 & & \mathcal{E}/X \text{ line bundle} \\ H^i(X_1)_{f_1} \cong H^i(X_2)_{f_2} & \rightsquigarrow & H^0(X,\mathcal{E})_f,\ H^1(X,\mathcal{E})_f \end{array}$$

Explained by algebraic cycle on $X_1 \times X_2$?

Explained by "higher algebraic cycle" on $A \times A$?

$\alpha = \{(D_i, f_i)\}$:



- $D_i \subseteq A \times A$ irred. 3-fold
- $f_i =$ function on D_i
- $\sum \operatorname{div}(f_i) = 0$

<u>Conj (H.-Prasanna).</u> $\exists$ action $H^0(X,\mathcal{E})_f \xrightarrow{\alpha_*} H^1(X,\mathcal{E})_f$

$$[f]^0 \longmapsto \frac{[f]^1}{\sum_i \int_{D_i} \log|f_i| \cdot \omega|_{D_i}} \qquad (\omega = \text{explicit cohomology class})$$

<u>Thm.</u> (H.-Prasanna). True for $A = E \times E^\sigma/\mathbb{Q}$ for E/F elliptic curve.

MOTIVIC ACTION CONJECTURES.

COHOMOLOGY THEORY	OVER $\mathbb{C}$	MODULO p^n	OVER $\mathbb{Q}_p$
Singular cohomology (related to, e.g. elliptic curves/$K=\mathbb{Q}(\sqrt{-d})$)	Prasanna-Venkatesh ('16/'22)	Venkatesh ('16/'19)	Venkatesh ('16/'19)
Coherent cohomology on...			
• Modular curves (related to Stark units & Stark's conj.)	H. ('20/'23)	Harris-Venkatesh ('17/'19) Darmon-Harris-Rotger-Venkatesh Zhang, Lecouturier (~'22)	(ongoing)
• Hilbert modular varieties (related to Stark units & Stark's conj.)	H. ('20/'23)	H. ('20/'23)	???
• Siegel modular varieties (related to abelian surfaces)	H.-Prasanna ('23) Oh ('22)	???	???