FINITE HOMOGENEOUS STRUCTURES AND ARITIES OF PERMUTATION GROUPS

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Contents

Introduction

These are notes from an Undergraduate Research Opportunities project at Imperial College London under the supervision of Professor Martin Liebeck. They are an introduction to the theory of relational complexity and arity, focusing mainly on finite structures.

The standard and natural way of describing a mathematical object is specifying how its elements interact (relations between elements) and what operations on them are allowed. For example, a group is a set with *addition* (binary operation), a distinguished neutral element (1-ary relation), and inverses (1-ary operation). Model theory is a branch of mathematics that deals with mathematical objects understood this way. Ever since its conception, it has found broad applications, proving very useful in theoretical computer science, the study of data bases, and algebraic geometry.

To describe an object, we have to choose a (mathematical) *language* to do it in. This means that (depending on that choice), we can have very different presentations of the same structure. A natural question that arises is: which presentation of a structure is best to work with? The theory of homogeneous structures provides one way of answering the question for (purely) *relational structures*, i.e. structures that do not have any operations.

A homogeneous relational structure has the property that every part of the structure looks exactly the same: whichever subset we *zoom in* at, there is no way of telling where we are. For example, the rational numbers with the linear order and the full bipartite graph

($K_{n,n}$) are both homogeneous structures. Homogeneity is the highest degree of symmetry we can require from a structure, making homogeneous structures very well-behaved. Hence if an object is not homogeneous, it is useful to know if we can represent it (perhaps in another language) as a homogeneous structure. The smallest arity of relations required to do this is called the *relational complexity* of a structure and it tells us how homogeneous it is. Therefore, when we have a relational structure, we can say which way of representing it is the most convenient and how *nice* the representation is.

The beauty of homogeneous structures becomes most vivid in their connection to permutation group theory. In [Mac11], Macpherson writes: *The context of homogeneous structures provides a meeting-point of ideas from combinatorics, model theory, permutation group theory, and descriptive set theory, and connections to theoretical computer science and to universal algebra are beginning to emerge.* There is a very natural way of describing relational structures using permutation groups with a lot of properties transferring along in the connection. Relational complexity, for example, corresponds to the arity of permutation groups (a property of orbits of the action on k-sets), which has often been associated with difficulty of computations in a group.

Even though there are a number of classification results for homogeneous structures with some additional properties (including a very general theory due to Lachlan [KL87]), there are still a lot of unanswered questions about relational complexity (especially in the infinite case). Similarly, we still do not know much about the arity of group actions, even in some basic cases. For example, there are no good estimates for the arity of the symmetric group on partitions of a fixed type. Therefore, homogeneity of structures and arity are still a very active area of research.

In these notes, we will study homogeneity of relational structures and describe their relation to permutation groups. We review some classification results and work out some known arities.

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1. Relational structures and homogeneity

Recall that a *relational structure* $\mathcal{X}$ a pair $(X, \mathcal{R})$ consisting of a set X together with a family of relations $\mathcal{R}$ on X. (In model theoretic terminology, this is the same as a model of a purely relational language.)

For instance, a graph Γ is a relational structure $(V\Gamma, \{E\Gamma\})$, where $V\Gamma$ is the vertex set of Γ and $E\Gamma$ is the adjacency relation (i.e. $(x, y) \in E\Gamma$ if and only if x adjacent to y). Similarly, a graph Γ with a colouring $\omega\colon E\Gamma \to C$ of edges, where C is the set of colours, is a relational structure $(X, \{R_c\}_{c\in C})$, where $X = V\Gamma$ and R_c is the relation of adjacency in colour c (i.e. xR_cy if and only if $\omega(x, y) = c$).

The rational numbers with the linear order $(\mathbb{Q}, \{\leq\})$ is a relational structure.

A *substructure* $(X', \mathcal{R})$ of $\mathcal{X} = (X, \mathcal{R})$ is a subset $X' \subseteq X$ with the same set of relations (restricted to X'). We will denote it by $\langle X' \rangle_{\mathcal{X}}$ or, if the underlying structure is clear, simply $\langle X' \rangle$.

For example, $(\mathbb{Z}, \{\leq\})$ is a substructure of $(\mathbb{Q}, \{\leq\})$. So in fact $(\mathbb{Z}, \{\leq\}) = \langle \mathbb{Z} \rangle$ (with the underlying structure $(\mathbb{Q}, \{\leq\})$). But we cannot add another relation to the structure, so $(\mathbb{Z}, \{\leq, \mod n\})$ is not a substructure of $(\mathbb{Q}, \{\leq\})$.

For a graph, a substructure is simply a subgraph, that is a subset of vertices with all edges between them. It is important to note that in choosing a subgraph, we only choose the vertices, not the edges: whenever two vertices are connected in a graph, they are also connected in a subgraph.

Given two relational structures, one can ask when they can be considered the same or equivalent.

Definition 1.1. An *automorphism* of $\mathcal{X} = (X, \mathcal{R})$ is a bijection

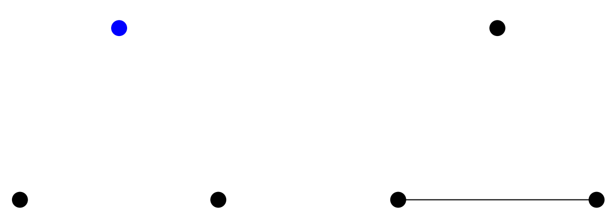
$$f \colon X \to X$$

such that $\mathbf{a} \in R$ if and only if $f(\mathbf{a}) \in R$ for $R \in \mathcal{R}$. We denote the group of automorphisms of $\mathcal{X}$ by $\mathrm{Aut}(\mathcal{X})$.

Definition 1.2. A function $f \colon X \to Y$ is an *isomorphism* of relational structures $\mathcal{X} = (X, \mathcal{R}_X)$ and $\mathcal{Y} = (X, \mathcal{R}_Y)$ if it is an isomorphism of the permutation groups $(\mathrm{Aut}(\mathcal{X}), X)$, $(\mathrm{Aut}(\mathcal{Y}), Y)$. We then say that $\mathcal{X}$ and $\mathcal{Y}$ are *isomorphic* or *equivalent*.

Typically, one defines an automorphism of an object A by saying it is an isomorphism $A \to A$. However, this is impossible in this case. The intuitive idea that an isomorphism should be a structure preserving and reflecting bijection (that is, x and y are related in $\mathcal{X}$ if and only if their images $f(x)$ and $f(y)$ are related in $\mathcal{Y}$) cannot be formalised for the following reasons.

First, the same property of a set can be described by different relations. Consider $(\mathbb{Z}, \{\leq\})$ where $\leq$ is the linear order, and $(\mathbb{Z}, \{s\})$ where s is the binary successor relation. Even though $0 \leq 2$ but 2 is not a successor of 0, the linear order induces the successor relation and vice versa, so the structures are equivalent. Moreover, even the arities of relations that describe the same property can be different. For the set $\{1, 2, 3\}$, the 1-ary relation $R_1 = \{1\}$, and the binary relation $R_2 = \{(1, 2), (2, 1)\}$ are essentially the same:



Finally, we can have equivalent relational structures with different numbers of relations. This is best illustrated by graphs: an empty graph on n vertices (0 relations) is the same as a full graph on n vertices (1 relation). Since all the vertices are connected in the full graph, the adjacency relation carries no information about the graph.

Notice that this definition differs from the notion of a graph isomorphism since it does not distinguish between a graph and its complement (it does not matter which relation we call "edges" and which we call "non-edges").

Definition 1.3. A relational structure $\mathcal{X}$ is *homogeneous* if for any $X_1, X_2 \subseteq X$ and any isomorphism $f\colon X_1 \to X_2$ of $\langle X_1\rangle$ and $\langle X_2\rangle$, there exists an automorphism $\overline{f} \in \mathrm{Aut}(\mathcal{X})$ such that $\overline{f}_{|X_1} = f$.

Examples 1.4. The 5-cycle C_5 is a homogeneous graph.

We define the graph n^d by $V(n^d) = [n]^d$ where $[n] = \{1, 2, \ldots, n\}$ and two d-tuples are adjacent if they differ on exactly one coordinate. The only homogeneous graph n^d with $n, d > 1$ which is homogeneous is 3^2. More about homogeneity of n^d can be found in Subsection 2.1.

The digraph $C_3(\overrightarrow{C_3})$ which is an undirected 3-cycle with each vertex replaced by a directed 3-cycle is homogeneous. Similarly, the digraph $\overrightarrow{C_3}(C_3)$ which is a directed 3-cycle with each vertex replaced by an undirected 3-cycle is homogeneous.

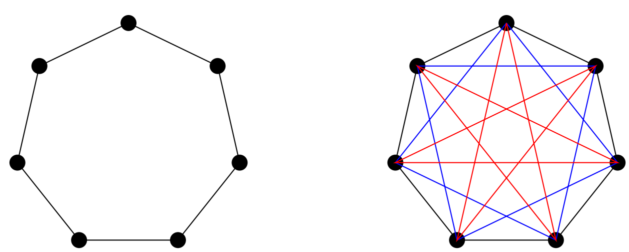
For a list of homogeneous graphs and digraphs, see Section 4.

We already noticed that we can present a relational structure in different but equivalent ways. We introduce the notion of relational complexity to measure how homogeneous a structure is (what is the "smallest" homogeneous presentation of the structure).

Definition 1.5. A relational structure $\mathcal{X}$ is *k-ary* if there exists a homogeneous relational structure $\mathcal{Y}$ with k-ary relations which is equivalent to $\mathcal{X}$.

The *relational complexity* $\kappa(\mathcal{X})$ of $\mathcal{X}$ is the least k such that $\mathcal{X}$ is k-ary.

Example 1.6. The relational complexity of C_7 is 2, because the following structures are equivalent—they both correspond to D_{14} acting on $\{1, 2, \ldots, 7\}$.



Note that the colours on the right correspond to distances in the graph on the left (black is distance 1, blue is distance 2, red is distance 3). We know that the structures are equivalent, because the automorphism group of C_7 is distance-transitive, so any automorphism preserving the graph on the left, necessarily preserves the graph on the right.

2. Arities of permutation groups

In this section we will introduce the arity of a permutation group and review some known results. For an introduction to permutation groups, see [DM96].

We will use the notation of [CMS96]. We will write (G, X) for a permuatation group G acting on X. When the set X and the action is clear, we will abuse the notation and write G instead of (G, X).

Definition 2.1. Consider the natural action of G on X^n. We write $\mathrm{tp}(\mathbf{a})$ for the orbit of $\mathbf{a}$ of the action and we call the orbits *n-types*.

For $I \subseteq \{1, 2, \ldots, n\}$ we have the projection $\pi_I \colon X^n \to X^I$ which induces a map from X^n/G to X^I/G. Identifying I with $\{1, 2, \ldots, r\}$, we may view this as a map from the n-types to the r-types. For $r \leq n$ the *r-type* of an n-tuple $\mathbf{a}$ will be the function which associates to each $I \subseteq \{1, 2, \ldots, n\}$ of cardinality r the r-type of the r-tuple $\pi_I(\mathbf{a})$.

We say that *r-types determine n-types* if any two n-tuples with the same r-type have the same n-type. The *arity* $\mathrm{Ar}(G, X)$ is the least r such that r-types determine n-types for all $n \geq r$.

This definition gives a very formal setting for working with arities of permutation groups. The simplest way to define the arity of (G, X) is to say that $\mathrm{Ar}(G, X)$ is the least r such that two n-tuples $\mathbf{a}, \mathbf{b}$ are in the same orbit, whenever any corresponding subsequences of $\mathbf{a}, \mathbf{b}$ of length r are in the same orbit, and that is the intuition we will follow.

We can also reformulate the notion of arity as follows.

Remark 2.2. The arity $\mathrm{Ar}(G, X)$ is the greatest r such that $(r-1)$-types do not determine r-types.

However, this does not mean that the arity is the least r for which r-types *do* determine $(r+1)$-types. For example, one can show that the arity of $\mathrm{Sym}(4) \wr \mathrm{Sym}(2)$ acting naturally on $\{1, 2, 3, 4\}^2$ is 4, but 2-types do determine 3-types in this permutation group.

2.1. Some arities of permutation groups.

The two extreme examples of arities are the natural actions of $\mathrm{Sym}(n)$ and Alt(n) which are respectively 2 and $n-1$. In this subsection we will explore some other known arities. A less detailed but broader exposition of the results and open problems can be found in [Che00, Sec. 3, 9].

We will use the notation $\left[{n \atop k}\right]$ for the set of k-elements subsets of $\{1, 2, \ldots, n\}$. There is a natural action of $\mathrm{Sym}(n)$ on $\left[{n \atop k}\right]$ extending the action of $\mathrm{Sym}(n)$ on $\{1, 2, \ldots, n\}$. We will first work out the arity of this action following [CMS96]. We start by deducing a lower bound for the arity from an example.

Proposition 2.3. *Let $k \in \{1, 2, \ldots, n/2\}$. Then* $\mathrm{Ar}\left(\mathrm{Sym}(n), \left[{n \atop k}\right]\right) \geq 2 + [\log_2 k]$.

Proof. Let $r = [\log_2 k] + 1$. We will give an example of $(r+1)$-tuples $\mathbf{a}$, $\mathbf{b}$ with the same r-type and distinct $(r+1)$-types.

Let Ω be an n-element set. Assume $A = \{0, 1, \ldots, 2^r - 1\} \subseteq \Omega$ is a subset of cardinality 2^r and let $A' \subseteq \Omega \setminus A$ be a subset of cardinality $k - 2^{r-1}$.

Writing the elements of A in binary, for $1 \leq i \leq r$, we define the following subsets of A:

$$A_i := \{\text{strings whose } i\text{th coordinate is } 1\} \subset A,$$

$$A_e := \{\text{strings with even number of nonzero entries}\} \subset A,$$

$$A_o := \{\text{strings with odd number of nonzero entries}\} \subset A.$$

Clearly, $|A_i| = 2^{r-1}$ because exactly one coordinate is fixed, and $|A_e| = |A_o| = 2^{r-1}$ because $A_e \cup A_o = A$, $A_e \cap A_o = \emptyset$.

Now we let $B_i := A_i \cup A'$, $B_e := A_e \cup A'$, $B_o := A_o \cup A'$. Since $A \cap A' = \emptyset$, $|A'| = k - 2^{r-1}$, we get $B_i, A_e, A_o \in \begin{bmatrix} n \\ k \end{bmatrix}$.

Consider the $(r+1)$-tuples

$$\mathbf{a} = (B_1, B_2, \ldots, B_r, B_e),\ \ \mathbf{b} = (B_1, B_2, \ldots, B_r, B_o).$$

We claim that $\mathbf{a}$ and $\mathbf{b}$ have the same r-type but distinct $(r+1)$-types. First, consider the r-tuples $\mathbf{a}_i$ in $\mathbf{a}$ and $\mathbf{b}_i$ in $\mathbf{b}$ obtained by omitting the ith coordinate, $1 \le i \le r$. Let $\sigma_i \in \mathrm{Sym}(\Omega)$ be the permutation that maps a string in A to the same string with only the ith bit changed and acts like the identity on $\Omega \setminus A$. Then σ_i maps $\mathbf{a}_i$ to $\mathbf{b}_i$. Finally, if σ maps A_i to A_i, then σ fixes the ith coordinate of any string in A, so $\sigma_{|A} = \mathrm{id}_A$. But then $\sigma(A_e) = A_e \not\subseteq B_o$, so $\mathbf{a}$ and $\mathbf{b}$ have distinct $(r+1)$-types. This shows

$$\mathrm{Ar}\left(\mathrm{Sym}(n), \begin{bmatrix} n \\ k \end{bmatrix}\right) \ge r + 1$$

as requested. □

Example 2.4. Consider the permutation group $\left(\mathrm{Sym}(5), \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$. We will show that

$$\mathrm{Ar}\left(\mathrm{Sym}(5), \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) \ge 3$$

We let

$$A = \{0, 1, 2, 3\}$$

and we represent the numbers in A in binary to get

$$A_1 = \{2, 3\}, A_2 = \{1, 3\}, A_o = \{1, 2\}, A_e = \{0, 3\}.$$

In this case $A' = \emptyset$, so we consider the 3-tuples

$$(A_1, A_2, A_e),\ \ (A_1, A_2, A_o).$$

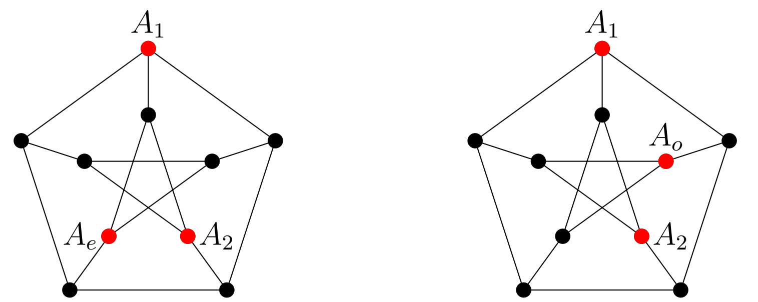
Then

$$\begin{array}{rcll} (A_1, A_o) & \mapsto & (A_1, A_e) & \text{via } (01)(23), \\ (A_2, A_o) & \mapsto & (A_2, A_e) & \text{via } (02)(13), \end{array}$$

but if $\sigma \in \mathrm{Sym}(5)$ fixes A_1 and A_2, then $\sigma(3) = 3$, so $\sigma(A_o) \neq A_e$.

The Petersen graph can be defined as follows: let $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ be the vertex set and declare two 2-sets connected if their intersections are empty. Because the automorphism group of this graph is $\mathrm{Sym}(5)$, the permutation group $\left(\mathrm{Sym}(5), \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$ can be identified with this graph (the strict sense of this will be discussed in Section 3).

The example above actually shows that the Petersen graph is not homogeneous.



The subgraphs $\langle A_1, A_2, A_e\rangle$ and $\langle A_1, A_2, A_o\rangle$ are both isomorphic to $\overline{K_3}$, 3 vertices with no edges. However, A_1, A_2, A_e have a mutual friend (in this case $\{1,5\}$), but A_1, A_2, A_o have no mutual friends. Therefore, there cannot be an automorphism sending one subgraph to the other.

Theorem 2.5. *Let* $k \in \{1, 2, \ldots, n/2\}$. *Then* $\mathrm{Ar}\left(\mathrm{Sym}(n), \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\right) = 2 + [\log_2 k]$.

Before the proof, we will introduce some notation. The purpose of this is to express in a simple way when two r-element sequences of k-element subsets of $\{1, 2, \ldots, n\}$ are in the same orbit. The main point is to look at the cardinalities of all the intersections of the subsets.

Fix r. Let B_r be the free boolean algebra on r generators, c_i. For $c \in B_r$, let $c^1 = c$ and c^{-1} be the complement of c. Then the atoms of B_r correspond to elements of $\{\pm 1\}^r$ via the bijection γ given by

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \mapsto \bigcap_i c_i^{\varepsilon_i} = c_1^{\varepsilon_1} \cap \ldots \cap c_r^{\varepsilon_r}.$$

We define the graph associated to B_r as follows

$$V\Gamma_r = \{\pm 1\}^r,$$

$$\varepsilon - \varepsilon' \text{ if and only if } \varepsilon \text{ and } \varepsilon' \text{ differ in exactly one coordinate.}$$

We label the edge $\varepsilon - \varepsilon'$ with the union of the atoms associated to its vertices. Then if ε and ε' differ exactly at the ith coordinate, then the label on $\varepsilon - \varepsilon'$ is in fact an atom in the algebra generated by $\{c_j\}_{j \neq i}$.

In the proof of the theorem, we will use the following reformulation of two r-tuples $\mathbf{a}$ and $\mathbf{b}$ having the same r-type. Let A and B be boolean algebras generated by the r-tuples (i.e. we take all the intersections, unions, and complements of the sets in $\mathbf{a}$ and $\mathbf{b}$) and let $f \colon B_r \to A$, $g \colon B_r \to B$ be the homomorphisms induced by

$$f(c_i) = a_i, \ \ g(c_i) = b_i.$$

Then we claim that $\mathbf{a}$ and $\mathbf{b}$ realize the same r-type if and only if for any atom $\alpha \in B_r$ we have $|f(\alpha)| = |g(\alpha)|$. The intuition is again that $f(\alpha)$ and $g(\alpha)$ correspond to the intersections of the k-element subsets of $\{1, 2, \ldots, n\}$. To show that, first suppose that $\sigma \in \mathrm{Sym}(n)$ sends $\mathbf{a}$ to $\mathbf{b}$. Then $g = \sigma \circ f$ (because $\sigma(a_i) = b_i$) and hence $|f(\alpha)| = |\sigma(f(\alpha))| = |g(\alpha)|$. For the other implication, suppose for any atom $\alpha \in B_r$ we have $|f(\alpha)| = |g(\alpha)|$. Then we can write

$$c_i = \bigcup_j \alpha_j^i$$

for atoms α_j^i and we get

$$a_i = f(c_i) = \bigcup_j f(\alpha_j^i)$$

$$b_i = g(c_i) = \bigcup_j g(\alpha_j^i)$$

for any i, which are partitions of a_i and b_i into disjoint sets with corresponding cardinalities. Thus we can map a_i to b_i via some $\sigma \in \mathrm{Sym}(n)$.

Proof of Theorem 2.5. By Proposition 2.3, it is enough to show that

$$\mathrm{Ar}\left(\mathrm{Sym}(n), \begin{bmatrix} n \\ k \end{bmatrix}\right) \leq 2 + [\log_2 k].$$

Fix any $r \geq 1 + [\log_2 k]$. We will show that $(r-1)$-types determine r-types. Let $\mathbf{a}, \mathbf{b}$ be two r-tuples with the same $(r-1)$-type and A, B be their corresponding algebras. We label the edges and vertices of the graph Γ_r as follows: Γ_r^A is the graph Γ_r with $\varepsilon \in V\Gamma_r$ labelled $|f(\gamma^{-1}(\varepsilon))|$ and $\varepsilon - \varepsilon'$ labelled $|f(\gamma^{-1}(\varepsilon) \cup \gamma^{-1}(\varepsilon'))|$. The labelled graph Γ_r^B is defined analogously.

We observed before the proof that we can think of the edge labels of Γ_r as atoms in the algebra generated by $\{c_i\}_{i \neq j}$. Therefore, since $\mathbf{a}$ and $\mathbf{b}$ have the same $(r-1)$-type, we conclude that the edge labels of Γ_r^A and Γ_r^B coincide. We will show that the vertex labels coincide as well. Note that the edge label on $\varepsilon - \varepsilon'$ is the sum of the vertex labels on ε and ε'. Since the graph Γ_r is connected, it is enough to show that the vertex labels coincide in one vertex (and the rest will follow).

Consider the 2^{r-1} edges of Γ_r that correspond to atoms in B_r for which $\varepsilon_1 = 1$. The image of these atoms under f are 2^{r-1} disjoint subsets of a_1 (which has k elements). Now since $2^{r-1} \leq k$, at least one of these atoms is empty, so one of the edges in Γ_r^A (and hence also in Γ_r^B) is labelled 0. So both vertices connected by this edge are labelled 0 in Γ_r^A (and Γ_r^B). Therefore the vertex labels of Γ_r^A and Γ_r^B coincide, so the cardinalities of the images of the atoms in B_r under f and g are equal. Hence $\mathbf{a}$ and $\mathbf{b}$ have the same r-type. □

Example 2.6. This shows that the arity of $\left(\mathrm{Sym}(5), \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]\right)$ is in fact 3. As we have seen in Example 2.4, the first inequality showed that the Petersen graph is not homogeneous. The fact that the arity is 3, actually means that there is a structure with 3-ary relations which is equivalent to the Petersen graph and homogeneous. This will be discussed in more detail in Section 3.

As we have seen, the arity of $\mathrm{Sym}(n)$ acting on $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is known precisely. The paper [CMS96] explores the more general arity of $\mathrm{Sym}(n) \wr \mathrm{Sym}(d)$ acting on $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]^d$, presenting some general bounds and precise answers in particular cases (e.g. n^2, 2^d).

The arity of $\mathrm{Sym}(n) \wr \mathrm{Sym}(d)$ acting on n^d has been worked out precisely in [Sar99] and [Sar00].

2.2. **Primitive affine binary permutation groups.** We will now look into the case of primitive permutation groups and introduce non-trivial examples of primitive binary permutation groups. These arise by considering affine permutation groups.

So let $\mathbb{F}$ be a field and V be a d-dimensional vector field over $\mathbb{F}$. We denote by $\mathrm{AGL}(V) := V \ltimes \mathrm{GL}(V)$, the general affine linear group, and by $\mathrm{A\Gamma L}(V) := V \ltimes \Gamma\mathrm{L}(V)$, the general affine semilinear group.

Definition 2.7. An *affine linear group* G of dimension d is a subgroup of $\mathrm{A\Gamma L}(V)$ containing the full translation group as a subgroup, i.e. $G = V \ltimes G_0$, where $G_0 \leq \Gamma\mathrm{L}(V)$, the stabilizer of 0.

We say that G is *strictly linear*, if $G_0 \leq \mathrm{GL}(V)$.

Proposition 2.8. *A primitive 1-dimensional strictly linear affine group G satisfies* $\mathrm{Ar}(G, \mathbb{F}) \leq 3$*, and* $\mathrm{Ar}(G, \mathbb{F}) = 2$ *if and only if G is cyclic or dihedral.*

Proof. The 'if' implication is clear: cyclic and dihedral groups are binary.

For the 'only if' implication, note that the stabilizer of any two points in $\mathbb{F}$ is trivial in this case. To show that $\mathrm{Ar}(G, \mathbb{F}) \leq 3$, let $r \geq 4$ and consider two r-tuples $\mathbf{a}$ and $\mathbf{b}$ with the same $(r-1)$-type. There exists $g \in G$ such that

$$(a_1, \ldots, a_{r-1}) \mapsto (b_1, \ldots, b_{r-1}),$$

so we may assume that $a_i = b_i$ for $i = 1, \ldots, r-1$. If $a_1 = a_2 = \ldots = a_{r-1}$, then clearly $\mathbf{a}$ and $\mathbf{b}$ have the same r-type. Suppose that $a_1 \neq a_2$. There exists $g' \in G$ such that

$$(a_1, a_2 \ldots, a_{r-2}, a_r) \mapsto (a_1, a_2, \ldots, a_{r-2}, b_r).$$

Since $r \geq 4$, $g \in G_{a_1,a_2}$, so $g = 1$, and $b_r = g(a_r) = a_r$. Therefore, $\mathbf{a}$ and $\mathbf{b}$ have the same r-type and we showed that $\mathrm{Ar}(G, \mathbb{F}) \leq 3$.

Now suppose that $\mathrm{Ar}(G, \mathbb{F}) = 2$. We will show that G is cyclic or dihedral, i.e. $G = \mathbb{F} \ltimes G_0$ and $|G_0| \leq 2$. So suppose that $|G_0| > 2$, i.e. there exists $g \in G_0$ such that $g \neq \pm 1$. Since G is 1-dimensional, $g \in \mathbb{F}$. Consider the triples

$$(0, -1, g), (0, -1, g^{-1}).$$

Since the stabilizer of any two points is trivial, these triples lie in distinct orbits. We check that they have the same 2-types:

$$(0, g^{-2})(0, g) = (0, g^{-1}),$$

$$(1-g, g^{-1})(-1, g) = ((-1+1-g)g^{-1}, (g+1-g)g^{-1}) = (-1, g^{-1}),$$

which contradicts binarity. □

Proposition 2.9. *A primitive 1-dimensional affine group G which is not strictly linear satisfies* $\mathrm{Ar}(G, \mathbb{F}) \leq 4$.

Moreover, $\mathrm{Ar}(G, \mathbb{F}) = 2$ *if and only if* $\mathbb{F} = \mathbb{F}_{q^2}$ *and* $G_0 = \langle x^{q-1} \rangle \ltimes \langle \sigma \rangle$*, where x is a primitive element of $\mathbb{F}$ and* $\sigma \in \Gamma\mathrm{L}(\mathbb{F}) \setminus \mathrm{GL}(\mathbb{F})$ *has order 2.*

Proof. We know that $G = \mathbb{F} \ltimes G_0$ and $G_0 = H \ltimes \Gamma$ where $H \leq \mathrm{GL}(\mathbb{F})$, $\Gamma \leq \mathrm{Aut}(\mathbb{F})$ and we can write

$$G = \mathbb{F} \ltimes H \ltimes \Gamma.$$

We will write $t_{c,h,\sigma}$ for the element mapping

$$v \mapsto hv^\sigma + c$$

First, we will show that $\mathrm{Ar}(G, \mathbb{F}) \leq 4$. To do this, we will show that the stabilizer of any two elements of $\mathbb{F}$ is binary. Let $v_1, v_2 \in \mathbb{F}$ be distinct elements and let $g \in G_{v_1,v_2}$. If we write $g = t_{c,h,\sigma}$, then

$$v_1 = hv_1^\sigma + c,$$

$$v_2 = hv_2^\sigma + c,$$

and given σ, we can determine uniquely $c(\sigma)$, $h(\sigma)$ that satisfy these equations. Thus

$$G_{v_1,v_2} = \{t_{c(\sigma),h(\sigma),\sigma \ \mid \ \sigma\in\Gamma}\} \cong \Gamma$$

and we conclude that G_{v_1,v_2} is cyclic, so it is binary.

Consider $r \geq 5$ and suppose $\mathbf{a}$, $\mathbf{b}$ are two r-tuple with the same $(r-1)$-type. We may (as in the proof of Proposition 2.8) assume that

$$\mathbf{a} = (a_1, \ldots, a_{r-1}, a),\ \ \mathbf{b} = (a_1, \ldots, a_{r-1}, b)$$

and $a_1 \neq a_2$. Since $\mathbf{a}$ and $\mathbf{b}$ have the same $(r-1)$-type in $(G, \mathbb{F})$, they have the same $(r-3)$-type in $(G_{a_1,a_2}, \mathbb{F})$. As $r \geq 5$ and by binarity of G_{a_1,a_2}, they have the same r-type in $(G_{a_1,a_2}, \mathbb{F})$ and so also in $(G, \mathbb{F})$. Therefore, $\mathrm{Ar}(G, \mathbb{F}) \leq 4$.

Now suppose that G is binary. We will show that $\mathbb{F} = \mathbb{F}_{q^2}$, $H = \langle x^{q+1} \rangle$ and $|\Gamma| = 2$.

We start by showing that $|\Gamma| = 2$. Similarly as in the proof of Proposition 2.8, if we assume again that $|\Gamma| > 2$, we conclude that for $g_0 = h\sigma \in G_0$, $1^{g_0} = 1^{g_0^{-1}}$, i.e. $h^\sigma = h^{-1}$. Therefore, σ^2 fixes H for any $\sigma \in \Gamma$, and by primitivity we conclude that H^Γ generates $\mathbb{F}$, so $\sigma^2 = 1$. Therefore $|\Gamma| \leq 2$ and since G is not strictly linear, $|\Gamma| = 2$.

Hence $\Gamma = \{1, \sigma\}$ for $\sigma \in \Gamma\mathrm{L}(\mathbb{F}) \setminus \mathrm{GL}(\mathbb{F})$ with order 2 and we can conclude that for a prime power q we have $\mathbb{F} = \mathbb{F}_{q^2}$.

Now we will show that $H = \langle x^{q+1} \rangle$ for a primitive element $x \in \mathbb{F}$. As we noticed before, for any $h \in H$ we have $h^\sigma = h^q = h^{-1}$, so in particular $H \subseteq \langle x^{q+1} \rangle$ for some primitive $x \in \mathbb{F}$. The goal is to show that any element $s \in \langle x^{q+1} \rangle$ is an element of H. Since G is primitive, H contains some $r \neq \pm 1$. For some $b \in \langle x^{q+1} \rangle$ we have $b^\sigma = bs$. Consider the triples

$$(0, b, b/(r+1)), (0, bs, bs/(r+1)).$$

We check that the 3-tuples have the same 2-type:

- $t_{0,1,\sigma}(0, b) = (0, bs)$
- $t_{0,r^{-1},\sigma}(0, b/(r+1)) = (0, bs/(r+1))$, because

$$r^{-1} \frac{b^\sigma}{r^{-1}+1} = \frac{bs}{r(r^{-1}+1)} = \frac{bs}{1+r}$$

- $t_{bs(1-r),r,\sigma}(b, b/(r+1)) = (bs, bs/(r+1))$, because

$$r(b)^\sigma + bs(1-r) = rbs + bs(1-r) = bs$$

$$r\frac{b^\sigma}{r^{-1}+1} + bs(1-r) = bs\left(\frac{r}{1+r^{-1}} + 1 - r\right) = bs\frac{r^{-1}}{1+r^{-1}} = \frac{bs}{1+r}.$$

Therefore, by binarity, there exists $g \in G$ that maps

$$(0, b, b/(r+1)), (0, bs, bs/(r+1)).$$

But g fixes 0, so $g \in G_0$ and $b^g = bs$, so either $g = \sigma$ or $g = s$. But

$$(b/(r+1))^\sigma \neq bs/(r+1),$$

so we get $g = s$.

The remainder of the proof will show that $G = \mathbb{F}_{q^2} \ltimes \langle x^{q-1}\rangle \ltimes \langle\sigma\rangle$ is indeed binary, i.e. $(r-1)$-types determine r-types for $r \geq 3$. As we have seen before, the stabilizer of any two points consists of two elements, in particular

$$G_{0,v} = \{1, t_{0,v^{1-q},\sigma}\}.$$

For $w \in \mathbb{F}$ we have

$$t_{0,v^{1-q},\sigma}(w) = v^{1-q}w^\sigma = v^{1-q}w^q,$$

so the stabilizer of three distinct points in G is trivial. This already shows that $(r-1)$-types determine r-types for $r \geq 4$ and we have reduced the problem two showing that 2-types determine 3-types. Consider two triples

$$(a_1, a_2, a_3), (b_1, b_2, b_3)$$

with the same 2-type. Firstly, we can assume that $a_1 = b_1$, $a_2 = b_2$ and then apply a translation by $-a_1$ to reduce the triples to

$$\mathbf{a} = (0, v, a), \mathbf{b} = (0, v, b).$$

Since the triples have the same 2-type, we know that for some $h_1 \in \langle x^{q-1}\rangle$ and $i \in \{0,1\}$

$$b = h_1 a^{\sigma^i},$$

and for some $h_2 \in \langle x^{q-1}\rangle$ and $j \in \{0,1\}$

$$b = h_2 a^{\sigma^j} + v - h_2 v^{\sigma^j}$$

Therefore

$$h_1 a^{\sigma^i} = h_2 a^{\sigma^j} + v - h_2 v^{\sigma^j}. \tag{1}$$

If $i = j$, then equation (1) yields

$$a^{\sigma^i} = \frac{v - h_2 v^{\sigma^i}}{h_2 - h_1}$$

(unless $h_1 = h_2$, in which case we are done). Then

$$b = h_1 a^{\sigma^i} = h_1 \frac{v - h_2 v^{\sigma^i}}{h_2 - h_1}$$

If $i = 0$, then

$$v^{1-q}a^\sigma = v^{1-q}\left(\frac{v - h_2 v}{h_2 - h_1}\right)^\sigma = v^{1-q}v^q h_1 \frac{1 - h_2}{h_2 - h_1} = b,$$

so $t_{0,v^{1-q},\sigma}$ maps $\mathbf{a}$ to $\mathbf{b}$. If $i = 1$, then a similar argument shows $\mathbf{a}$ and $\mathbf{b}$ have the same 3-type.

Now suppose $i \neq j$. Without loss of generality, assume $i = 1$, $j = 0$. Then equation (1) yields $h_1 a^q - h_2 a = v - h_2 v$ and taking the images of both sides under σ we get a system of equations

$$\begin{cases} h_1 a^q - h_2 a & = v(1-h_2) \\ h_1^{-1} a - h_2^{-1} a^q & = v^q(1-h_2^{-1}) \end{cases}$$

We can use it to get $v(1-h_2) = h_1 h_2 v^q (h_2^{-1} - 1)$, so that $v^{1-q} = h_1 \frac{1-h_2}{1-h_2} = h_1$ (unless $h_2 = 1$, in which case we are done), and

$$b = h_1 a^q = v^{1-q} a^\sigma.$$

Therefore, $t_{0,v^{1-q},\sigma}$ maps $\mathbf{a}$ to $\mathbf{b}$, showing that $\mathbb{F}_{q^2} \ltimes \langle x^{q-1} \rangle \ltimes \langle \sigma \rangle$ is binary. □

In [Che13] Cherlin shows that these are in fact all primitive binary affine permutation groups. He has also conjectured (for example, in [Che00, Ch. 9]) that these are in fact all finite primitive binary permutation groups. In a recent paper [Wis14], Wiscons reduced the conjecture to the almost simple case.

3. Relational structures and permutation groups

One way of describing a relational structure is through its automorphism group, the group of all bijections preserving the relations on the set. On the other hand, given a permutation group acting on a set, one can consider relations which are invariant under the group action. This is a Galois connection between relational structures and permutation groups which we describe in this section.

To a relational structure $\mathcal{X} = (X, \mathcal{R})$, assign the permutation group $(\mathrm{Aut}(\mathcal{X}), X)$.

If $\mathcal{X}_1 = (X, \mathcal{R}_1)$ and $\mathcal{X}_2 = (X, \mathcal{R}_2)$ satisfy $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then

$$\mathrm{Aut}(\mathcal{X}_2) \subseteq \mathrm{Aut}(\mathcal{X}_1).$$

Indeed, if $g \in \mathrm{Aut}(\mathcal{X}_2)$, then $g\colon X \to X$ preserves all the relations in $\mathcal{R}_2$, so it preserves all the relations in $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

Recall that a permutation group (G, X) is *k-closed* if for

$$\mathcal{R} = X^k/G$$

(i.e. two k-tuples are related if they are conjugate under G), we have $G = \mathrm{Aut}(X, \mathcal{R})$. If G has arity r, then, in particular, it is r-closed. This motivates the following definition.

Let (G, X) be a permutation group and $r = \mathrm{Ar}(G, X)$. Consider the relational structure $(X, \mathcal{R}(G, X))$, where

$$\mathcal{R}(G, X) = X^r/G$$

is a set of r-ary relations (subsets of X^r), the r-types in X. When X is known, we will simply write $\mathcal{R}(G)$. Note that this forces $(X, \mathcal{R}(G))$ to be homogeneous and $G = \mathrm{Aut}(X, \mathcal{R}(G))$.

If $G_1 \leq G_2$, then obviously $\mathcal{R}(G_1) \subseteq \mathcal{R}(G_2)$.

Example 3.1. We can identify the undirected n-cycle C_n with the dihedral group D_{2n} and the directed n-cycle $\overrightarrow{C_n}$ with the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Some properties of permutation groups and relational structures are preserved well in the connection, one of which is the relational complexity of a relational structure (arity of a permutation group).

Proposition 3.2. *Suppose $\mathcal{X} = (X, \mathcal{R})$ has relational complexity k. Then $(\mathrm{Aut}(\mathcal{X}), X)$ has arity k.*

Proof. Let $G = \mathrm{Aut}(\mathcal{X})$ and $r = \mathrm{Ar}(G, X)$. Note that $\mathcal{X}$ is equivalent to $(X, \mathcal{R}(G))$, since G is r-closed, so, by homogeneity of $(X, \mathcal{R}(G))$ and the fact that all relations $R \in \mathcal{R}(G, X)$ are r-ary, we get $k \leq r$.

We will now show that k-types determine n-types in X. We assume (possibly changing $\mathcal{X}$ to and equivalent) that $\mathcal{X}$ is homogeneous with k-ary relations. Take two n-tuples $\mathbf{a}, \mathbf{b}$ with the same k-type. Since their k-types are the same and all the relations are k-ary, we know that the substructures $\mathcal{X}$ corresponding to $\mathbf{a}$ and $\mathbf{b}$ are isomorphic (via an isomorphism preserving the order of elements in $\mathbf{a}$ and $\mathbf{b}$). Thus by homogeneity, there exists an automorphism $g \in G$ extending this isomorphism, which sends $\mathbf{a}$ to $\mathbf{b}$. This shows $r \leq k$. ☐

Proposition 3.3. *Suppose (G, X) has arity r. Then $(X, \mathcal{R}(G))$ has relational complexity r.*

Proof. Let $\mathcal{R} = \mathcal{R}(G, X)$, $\mathcal{X} = (X, \mathcal{R})$ and $k = \kappa(\mathcal{X})$. Suppose that $\mathcal{X}' = (X, \mathcal{R}')$ equivalent to $\mathcal{X}$ is homogeneous with k-ary relations. But then $G = \mathrm{Aut}(\mathcal{X}')$ and by Proposition 3.2 the arity of (G, X) is k. ☐

Corollary 3.4. *If we identify permutation groups with relational structures, then the arity and the relational complexity are equivalent.*

Example 3.5. In Examples 2.4 and 2.6, we discussed the arity of $\left(\mathrm{Sym}(5), \left[{5 \atop 2}\right]\right)$ and its connection to the Petersen graph. We can easily see that the automorphism group of the Petersen graph is precisely $\left(\mathrm{Sym}(5), \left[{5 \atop 2}\right]\right)$. While it might be easier for some purposes to think of $\left(\mathrm{Sym}(5), \left[{5 \atop 2}\right]\right)$ as the Petersen graph, we know that the Petersen graph is not homogeneous. Because $\mathrm{Ar}\left(\mathrm{Sym}(5), \left[{5 \atop 2}\right]\right) = 3$, we know that the Petersen graph is equivalent (as a relational structure) to a homogeneous structure with 3-ary relations (its relational complexity is 3). The Petersen graph is not homogeneous as a 2-ary structure, but it is homogeneous as a 3-ary structure.

4. Classification of finite homogeneous graphs and digraphs

The aim of this section is to list all finite homogeneous digraphs. To do this, we first list the finite homogeneous graphs in Subsection 4.1 and then use that as the first step to find all finite homogeneous digraphs in Subsection 4.2.

4.1. **Classification of finite homogeneous graphs.**

This subsection classifies the finite homogeneous graphs following [Gar76]. For clarity of the exposition, we have reorganised the proof, and added some details and pictures.

For a graph Γ, we will denote by $\overline{\Gamma}$ the complement of Γ, we will write $t \cdot \Gamma$ for a disjoint union of t graphs Γ. We will also write $K_{n;t}$ for the full t-partite graph with parts of size n, i.e. $\overline{t \cdot K_n}$, and $K_{n,n}$ for $K_{n;2}$.

Theorem 4.1. *If Γ is a homogeneous graph then it is isomorphic to one of $t \cdot K_n$, $K_{n;t}$, C_5, 3^2.*

Throughout this section, we will use the simplified notation $\langle X \rangle$ for $\langle X \rangle_\Gamma$, where $X \subseteq V\Gamma$. Moreover, we let ϱ be the distance function on Γ.

Lemma 4.2. *Suppose Γ is homogeneous.*

(1) *$\overline{\Gamma}$ is also homogeneous.*
(2) *Γ is regular.*
(3) *If $\langle U \rangle$ is a connected component for $U \subseteq V\Gamma$, then the diameter of $\langle U \rangle$ is at most 2.*

Proof. (1) and (2) are clear. For (3), suppose $\varrho(u, u') = 3$ for $u, u' \in U$ and the shortest path is (u, u_1, u_2, u'). Then $\varrho(u, u_2) = 2$. Now since $\langle \{u, u'\} \rangle \cong \langle \{u, u_2\} \rangle \cong \overline{K_2}$, by homogeneity there exists an automorphism that sends (u, u') to (u, u_2). That contradicts $\varrho(u, u') = 3 \neq 2 = \varrho(u, u_2)$. □

For a graph Γ and $u \in V\Gamma$, we let $\Gamma(u) = \{v \mid \{u, v\} \in E\Gamma\}$ be the set of neighbours of u. We also define Γ' to be $\langle \Gamma(u) \rangle$. Since $\langle \Gamma(u) \rangle \cong \langle \Gamma(u') \rangle$ for any $u, u' \in V\Gamma$, Γ' is well-defined.

Lemma 4.3. *If Γ is homogeneous, then Γ' is homogeneous.*

Proof. Fix Γ homogeneous and $u \in V\Gamma$. Suppose $V_1, V_2 \subseteq \Gamma(u)$ form isomorphic subgraphs $\langle V_1 \rangle$, $\langle V_2 \rangle$ with $f \colon V_1 \to V_2$ isomorphism. We will show that f extends to an automorphism of $\langle \Gamma(u) \rangle$. Consider $V_1' = V_1 \cup \{u\}$, $V_2' = V_2 \cup \{u\}$. Then $\langle V_1' \rangle \cong \langle V_2' \rangle$ via $f' \colon V_1' \to V_2'$ defined by

$$f'(v) = \begin{cases} f(v) & \text{for } v \in V_1 \\ u & \text{for } v = u \end{cases}$$

Then by homogeneity f' extends to an automorphism $\overline{f'}$ of Γ. But $\overline{f'}$ induces an automorphism $\overline{f} = \overline{f'}_{|\Gamma(u)}$ of $\langle \Gamma(u) \rangle$ extending f. □

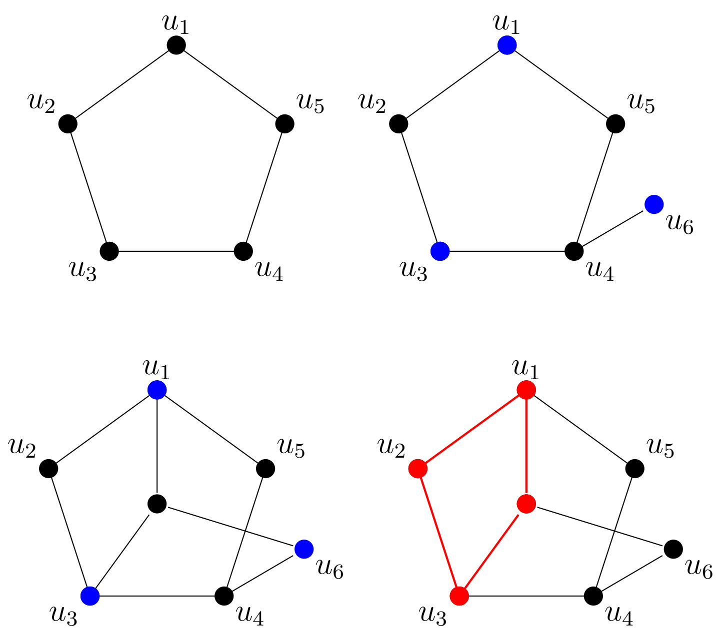
Proposition 4.4. *For a homogeneous graph Γ:*

(1) *if $\Gamma' \cong K_r$, then $\Gamma \cong t \cdot K_{1+r}$, $t \geq 1$;*
(2) *if $\Gamma' \cong k \cdot K_1$, $k \geq 2$, then either $\Gamma \cong C_5$ or $\Gamma \cong K_{k,k}$;*
(3) *if $\Gamma' \cong K_{r;t}$, $t \geq 2$, then $\Gamma \cong K_{1+r;t}$;*
(4) *if $\Gamma' \cong t \cdot K_r$, $r \geq 2$, $t \geq 2$, then $r = t = 2$ and $\Gamma \cong 3^2$;*
(5) *$\Gamma' \ncong C_5$, $\Gamma' \ncong 3^2$.*

Lemma 4.5. (1) *Any homogeneous graph of girth 5 is isomorphic to C_5.*
(2) *Any homogeneous graph of girth 4 is bipartite.*

Proof. (1) Let k be the valency of Γ. If $k \geq 3$, then consider the triples $\{u, v, w\}$ of pairwise non-adjacent vertices (triangle in $\overline{\Gamma}$). Since $k \geq 3$ and there are no triangles in Γ, there exists a triple $\{u, v, w\}$ of pairwise non-adjacent vertices with $\Gamma(u) \cap \Gamma(v) \cap \Gamma(w) \neq \emptyset$.

The rest of the proof is illustrated below. Let $(u_1, u_2, \dots, u_5)$ be a 5-cycle in Γ. Since $k \geq 3$, there exists $u_6 \in V\Gamma \setminus \{u_i\}_{i=1}^{5}$ adjacent to u_4. We consider the triple $\{u_1, u_3, u_6\}$. Since $g = 5$, these are pairwise non-adjacent vertices. By homogeneity, we can map (u, v, w) to (u_1, u_3, u_6) by an automorphism, so $\Gamma(u_1) \cap \Gamma(u_3) \cap \Gamma(u_6) \neq \emptyset$, which contradicts $g = 5$.



Therefore $k = 2$ and $\Gamma \cong C_5$.

(2) It is easy to prove that a homogeneous graph is distance-transitive (the only non-trivial case being $\Gamma' \cong K_{t;r}$ which will be clear from the proof of Proposition 4.4 (3)). In particular, if $u \in V\Gamma$ and $v \in \overline{\Gamma}$, then

$$c_2 := |\Gamma(v) \cap \Gamma(u)|,$$

$$a_2 := |\Gamma(v) \cap \overline{\Gamma}(u)| = k - c_2 \geq 1.$$

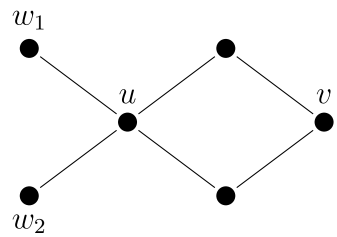
(where k is the valency of Γ) are independent on the choice of u and v. Choose $v' \in \Gamma(v) \cap \overline{\Gamma}(u)$ and set

$$A := \Gamma(v) \cap \Gamma(u), \;\; A' := \Gamma(v') \cap \Gamma(u).$$

Since Γ has no triangles, we have $A \cap A' = \emptyset$, so $k \geq 2c_2$, whence $c_2 \leq a_2$. Since $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex subgraphs of $\langle \Gamma(u) \rangle$ isomorphic to $j \cdot K_1$, so it acts transitively on the $\binom{k}{c_2}$ c_2-subsets of $\Gamma(u)$ and each vertex of $\overline{\Gamma}$ corresponds to a unique such subset. This yields

$$\left|\overline{\Gamma}(u)\right| = k(k-1)/c_2 \geq \binom{k}{c_2} = k(k-1)\dots(k-c_2+1)/c_2!,$$

so $c_2 = 2$ and each of the 2-subsets of $\Gamma(u)$ corresponds to a unique vertex of $\overline{\Gamma}$. But if $u \in V\Gamma$, $v \in \overline{\Gamma}(u)$ and $w_1, w_2 \in \Gamma(u) \cap \overline{\Gamma}(v)$, then $\mathrm{Aut}(\Gamma)$ acts transitively on the subgraphs of Γ of the form

$\langle u, v, \Gamma(u) \cap \Gamma(v), w_1, w_2 \rangle \;\cong$ 

This means $\mathrm{Aut}(\Gamma)_{uv}$ acts transitively on the $\binom{k-2}{2}$ unordered pairs of $\Gamma(u) \cap \overline{\Gamma}(v)$. Hence each such pair corresponds to a unique vertex of $\Gamma(u) \cap \overline{\Gamma}(v)$, so $\binom{k-2}{2} = k - 2$. This gives $k = 2$, $a_2 = 3$, $|V\Gamma| = 16$ and $\langle\overline{\Gamma}\rangle$ is isomorphic to the Petersen graph. But the Petersen graph is not homogeneous, since it contains two distinct kinds of vertex subgraphs isomorphic to $\overline{K_3}$ (see Example 2.4). □

Proof of Proposition 4.4. Take $u \in U$ and consider $\Gamma(u)$.

For (1), if $\langle \Gamma(u)\rangle \cong K_r$ and there exists $v \in \overline{\Gamma}(u)$ in the same connected component as u, i.e. (u, v', v) is a shortest path, then $|\Gamma(v')| \geq r$ which contradicts $\langle \Gamma(v')\rangle \cong K_r$.

For (2), note that since the diameter of Γ is 2, $g \leq 5$, and since $\Gamma' \cong k \cdot K_1$, Γ has no triangles, i.e. $g \geq 4$. Therefore using Lemma 4.5 either $g = 4$ and Γ is bipartite, so $\Gamma \cong K_{k,k}$, or $g = 5$ and $\Gamma \cong C_5$.

For (3), suppose $\langle \Gamma(u)\rangle \cong K_{r;t}$, $r \geq 2$, $t \geq 2$. For $v \in \Gamma(u)$ we have

$$\Gamma(v) \cap \overline{\Gamma}(u) = \{w_1, \dots, w_{r-1}\}$$

and in $\langle\Gamma(v)\rangle$ we know that $\langle u, w_1, \dots, w_{t-1}\rangle$ is edgefree and $\Gamma(v)\cap\Gamma(u) \subseteq \Gamma(w_i)$, $1 \leq i \leq t-1$. Similarly, if $v' \in \Gamma(v)\cap\Gamma(u)$, then $\Gamma(v')\cap\Gamma(u) \subseteq \Gamma(w_i)$, $1 \leq i \leq t-1$. Therefore $\Gamma(u) \subseteq \Gamma(w_i)$ and since $w_i \in \overline{\Gamma}(u)$, we conclude that $\Gamma(u) = \Gamma(w_i)$. Therefore for each $v'' \in \Gamma(u)$ we have $\Gamma(v'') \cap \overline{\Gamma}(u) = \{w_1, \dots, w_{t-1}\}$. This shows $\Gamma \cong K_{1+r;t}$.

For (4), suppose $\langle\Gamma(u)\rangle \cong t \cdot K_r$, $r \geq 2$, $t \geq 2$. Let $(U_i)_{i=1}^t$ be the partition of $V\Gamma$ into $\langle U_i\rangle \cong K_r$.
Fix $v \in \overline{\Gamma}(u)$ and suppose that $v_1, v_2 \in U_i \cap \Gamma(v)$ are two vertices. Then a connected component of $\Gamma(v_1)$ contains the pair u, v with $\varrho(u, v) = 2$ and this contradicts $\Gamma(v_1) \cong t \cdot K_r$. Thus $|\Gamma(v)\cap U_i| \leq 1$ for any i and so $c_2 := |\Gamma(v)\cap\Gamma(u)| \leq t$. Each vertex of $\overline{\Gamma}(u)$ corresponds to a unique c_2-subset of $\Gamma(u)$ and $\mathrm{Aut}(\langle\Gamma(u)\rangle)$ acts transitively on

$$\{C \mid C \subseteq \Gamma(u), \langle C\rangle \cong j \cdot K_1\}$$

for each $j \geq 1$. Any element of $\overline{\Gamma}(u)$ corresponds to a choice of c_2 of the U_i's and one of the r elements in each chosen U_i, which shows the inequality:

$$|\overline{\Gamma}(u)| \geq \binom{t}{c_2} r^{c_2}.$$

Now, each $u' \in \Gamma(u)$ satisfies $\langle\Gamma(u')\rangle \cong t \cdot K_r$, so $\Gamma(u') \cap \overline{\Gamma}(u)$ forms $(t-1) \cdot K_r$. Therefore

$$|\overline{\Gamma}(u)| = tr(tr - r)/c_2.$$

This shows $c_2 \leq 2$. If $c_2 = 1$ with $\Gamma(u) \cap \Gamma(v) = \{u_1\} \subseteq U_1$ for some $v \in \overline{\Gamma}(u)$. But then $\langle\langle\overline{\Gamma}(u)\rangle(v)\rangle = \langle\Gamma(v) \setminus \{u_1\}\rangle$ is not regular, so it cannot be homogeneous, a contradiction.

Hence $c_2 = 2$ and each 2-subset $\{x, x'\} \subseteq \Gamma(u)$ for which $\langle x, x' \rangle \cong 2 \cdot K_1$ corresponds to a unique vertex of $\overline{\Gamma}(u)$. Let $v \in \overline{\Gamma}(u)$ and $\Gamma(u) \cap \Gamma(v) = \{u_1, u_2\}$, $u_i \in U_i$. Choose $v_1 \in \Gamma(u_1) \cap \Gamma(v)$, $v_2 \in \Gamma(u_2) \cap \Gamma(v)$. Since u_1 and u_2 are not adjacent and $\langle \Gamma(v) \rangle \cong t \cdot K_r$, v_1 and v_2 are not adjacent. If $t > 2$, we may choose another vertex $w \in \overline{\Gamma}(u) \cap \Gamma(v)$ such that $\Gamma(u) \cap \Gamma(v) \cap \Gamma(w) = \emptyset$. But $\langle u, v, v_1, v_2 \rangle \cong \langle u, v, v_1, w \rangle$ which contradicts homogeneity. Thus $t = 2$.
We define the graph Δ as follows: each subgraph of Γ isomorphic to K_{r+1} is a vertex and two such vertices are adjacent if and only if the subgraphs have a common vertex in Γ. Then $\Gamma = L(\Delta)$, the line graph of Δ. Δ has girth 4 and valency $r + 1$ and is bipartite of diameter 2, i.e. $\Delta = K_{r+1,r+1}$, $r \geq 2$. Now $\mathrm{Aut}(K_{r+1,r+1}) \cong \mathrm{Aut}(L(K_{r+1,r+1}))$ and if $r \geq 3$, then $K_{r+1,r+1}$ contains inequivalent types of subgraphs isomorphic to $2 \cdot K_{1,2}$, so $\mathrm{Aut}(L(K_{r+1,r+1}))$ does not act transitively on the set of vertex subgraphs isomorphic to $2 \cdot K_2$. Hence $r = 2$ and $\Gamma \cong L(K_{3,3}) \cong 3^2$, as requested.

For (5), note that a distance-transitive graph Δ satisfying $\Delta' \cong C_5$ is isomorphic to the icosahedron, but the icosahedron is not homogeneous.
Finally, suppose for a contradiction that $\langle \Gamma(u) \rangle \cong 3^2$. Choose $v \in \Gamma(u)$, $w \in \Gamma(v) \cap \overline{\Gamma}(u)$. Then obviously

$$\Gamma(v) = \{u\} \cup (\Gamma(v) \cap \Gamma(u)) \cup (\Gamma(v) \cap \overline{\Gamma}(v)).$$

Two nonadjacent vertices in 3^2 can have exactly two mutual neighbours, so applying this to $\{u, w\}$ in $\langle \Gamma(v) \rangle \cong 3^2$ we get

$$|\Gamma(v) \cap \Gamma(u) \cap \Gamma(w)| = 2,$$

so that $c_2 = |\Gamma(u) \cap \Gamma(w)| \geq 3$. By considering $\overline{\Gamma}(u) \cap \Gamma(v)$ in $\langle \Gamma(v) \rangle \cong 3^2$ we get $\langle \overline{\Gamma}(u) \cap \Gamma(v) \rangle \cong C_4$. Hence

$$6 \geq a_2 = |\Gamma(w) \cap \overline{\Gamma}(u)| \geq 2,$$

$$3 \leq c_2 = 9 - a_2 \leq 7.$$

Since $|\overline{\Gamma}(u)| = 9 \cdot 4/c_2$ is an integer, we have two cases to consider. First, if $a_2 = 5$, $c_2 = 4$, then $\langle \overline{\Gamma} \rangle$ is a graph of valency five on nine vertices, a contradiction. Second, if $a_2 = 3$, $c_2 = 6$, then $\langle \overline{\Gamma} \rangle$ is a graph of valency 3 on six vertices, i.e. it is $K_{3,3}$ (by looking at strongly regular graphs with these properties). But then in $\overline{\Gamma}$ the vertex subgraph on $\overline{\Gamma}(u)$, $\langle \overline{\Gamma}(u) \rangle_{\overline{\Gamma}}$, is isomorphic to $2 \cdot K_3$ which contradicts (4). □

To complete the proof of Theorem 4.1, we introduce the notion of rank.

Definition 4.6. We define the *rank of a homogeneous graph* Γ to be

$$\mathrm{rank}(\Gamma) = \begin{cases} 0 & \text{for } V\Gamma = \emptyset \\ \mathrm{rank}(\Gamma') + 1 & \text{otherwise} \end{cases}$$

Proof of Theorem 4.1. If $\mathrm{rank}(\Gamma) = 1$, then $\Gamma \cong k \cdot K_1$.

If $\mathrm{rank}(\Gamma) = 2$, then $\mathrm{rank}(\Gamma') = 1$, so $\Gamma' \cong k \cdot K_1$. By Proposition 4.4 (1, 2): if $k = 1$, then $\Gamma \cong K_2$; if $k \geq 2$, then $\Gamma \cong C_5$ or $\Gamma \cong K_{k,k}$.

If $\mathrm{rank}(\Gamma) = 3$, then $\mathrm{rank}(\Gamma') = 2$, so $\Gamma' \cong K_2$, $\Gamma' \cong C_5$ or $\Gamma' \cong K_{k,k}$ ($k \geq 2$). By Proposition 4.4 (5), $\Gamma' \ncong C_5$. If $\Gamma' \cong K_{k,k} = K_{2;k}$, then by Proposition 4.4 (3), $\Gamma \cong K_{3;k}$. Finally, if $\Gamma' \cong K_2$ and by Proposition 4.4 $\Gamma \cong K_3$.

It remains to show that if $\operatorname{rank}(\Gamma) = r \geq 3$, then $\Gamma \cong t \cdot K_r$ or $\Gamma \cong K_{r+1;t}$, but this is clear by induction on r (the base step is above and the induction step is Proposition 4.4 (1, 3)). □

4.2. **Classification of finite homogeneous digraphs.** In this subsection we state classification of finite digraphs following [Lac82] and show the outline of the proof.

Definition 4.7. A *digraph* Γ is a set $V\Gamma$ of vertices with one binary irreflexive relation $E\Gamma$. The *complement* of Γ is the digraph $\overline{\Gamma}$ with $V\overline{\Gamma} = V\Gamma$ and $E\overline{\Gamma} = V\Gamma^2 \setminus (E\Gamma \cup \{(v,v)\})$.

Definition 4.8. We will call edges $(u,v) \in E\Gamma$ such that $(v,u) \in E\Gamma$ *symmetric.*

We say that u *dominates* v (v *is dominated by* u) if $(u,v) \in E\Gamma$ but $(v,u) \notin E\Gamma$. For any $u \in V\Gamma$, we define

$$\overrightarrow{\Gamma}(u) = \{v \in V\Gamma \mid u \text{ dominates } v\}.$$

If Γ is homogeneous, then we let

$$\Gamma^d = \left\langle \overrightarrow{\Gamma}(u) \right\rangle.$$

To state the classification of homogeneous digraphs, we need to introduce some digraphs.

Definition 4.9. The *composition* of digraphs Γ_1 and Γ_2 is the digraph $\Gamma = \Gamma_1(\Gamma_2)$ with

$$V\Gamma = V\Gamma_1 \times V\Gamma_2$$

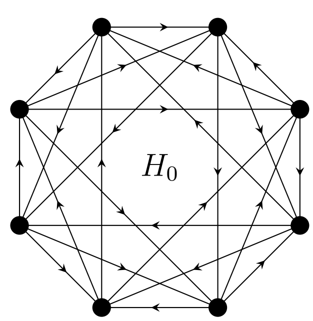
and $((u_1,u_2),(v_1,v_2)) \in E\Gamma$ if either $(u_1,v_1) \in E\Gamma_1$ or $u_1 = v_1$ and $(u_2,v_2) \in E\Gamma_2$.

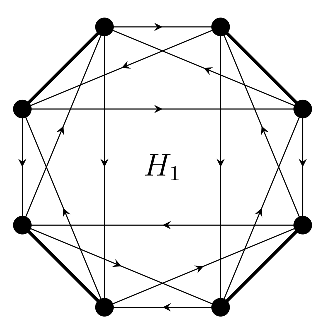
Examples 4.10. For a digraph Γ, the composition $\overline{K_n}(\Gamma)$ is simply $n \cdot \Gamma$, n disjoint copies of Γ.

The graph $K_{n,n}$ is actually the composition $K_2(\overline{K_n})$ and, in general, the complete m-partite graph with parts or cardinality n, $K_{n;m}$, is $K_m(\overline{K_n})$.

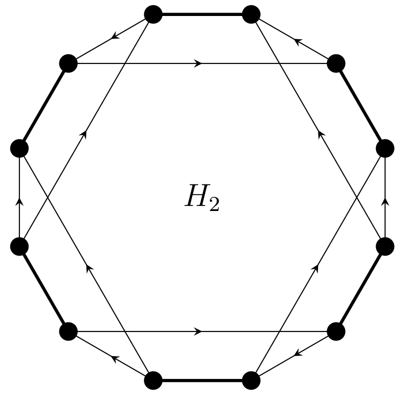
To simplify the notation, we will let $C := \overrightarrow{C_3}$, $D := \overrightarrow{C_4}$.

We also introduce three sporadic homogeneous digraphs H_0, H_1, H_2.





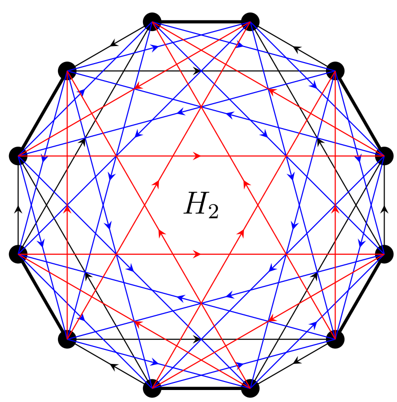
The final digraph is more complicated. We first present it with some omitted edges as it was presented in [Lac82] for clarity.

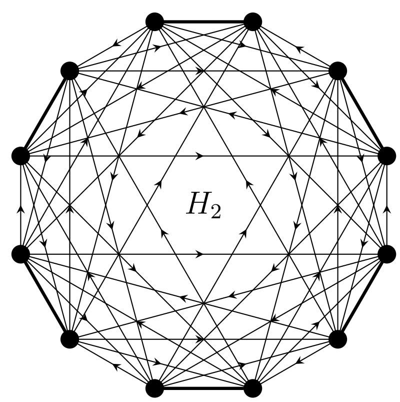


The rules for obtaining the additional edges are as follows. Each vertex v has a unique *mate*, a vertex connected to v by a symmetric edge.

- If v dominates w, then v is dominated by the mate of w.
- If v is dominated by w, then v dominates the mate of w.

The coloured graph on the left shows the edges we obtain using these rules: first 24 edges (blue edges below) and then 12 more edges (red edges below). The picture on the right shows the full graph H_2.





We can now state the classification of finite homogeneous digraphs. Let $\mathcal{S}$ be the set of all homogeneous symmetric digraphs (i.e. homogeneous graphs, classified in Section 4.1), and $\mathcal{A}$ be the set of all homogeneous antisymmetric digraphs.

Theorem 4.11. *Let Γ be a homogeneous digraph. Then:*

- *If $\Gamma \in \mathcal{A}$, then Γ is isomorphic to one of:*
$$D,\ \overline{K_n},\ \overline{K_n}(C),\ C\left(\overline{K_n}\right),\ H_0.$$
- *Either Γ or $\overline{\Gamma}$ is isomorphic to one of:*
$$K_n(A),\ A(K_n),\ S,\ C(S),\ S(C),\ H_1,\ H_2$$
for $S \in \mathcal{S}$, $A \in \mathcal{A}$.

To understand the proof of the theorem, we need a similar notion to the rank of a homogeneous graph.

Definition 4.12. The *(dominating) rank of a homogeneous digraph* Γ is defined as

$$\mathrm{rank}_d(\Gamma) = \begin{cases} 0 & \text{if } V\Gamma = \emptyset \\ \mathrm{rank}_d(\Gamma^d) + 1 & \text{otherwise} \end{cases}$$

The classification of finite homogeneous digraphs is based on the following table.

Table of cases

case	if	then Γ or $\overline{\Gamma}$ is isomorphic to
1	$V\Gamma^d = \emptyset$	$\overline{K_n}$ $(n \geq 1)$ $K_m(\overline{K_n}) = n \cdot K_m$ $(m, n \geq 2)$ C_5 3^2
2	$\Gamma^d \cong \overline{K_1}$	$S(C)$ $(S \in \mathcal{S})$ $K_n(D)$
3	$\Gamma^d \cong \overline{K_2}$	$K_n(C(\overline{K_2}))$ $\overline{D}(\overline{K_2})$ H_1
4	$\Gamma^d \cong \overline{K_n}$ $(n \geq 3)$	$K_m(C(\overline{K_n}))$ $(m \geq 1)$ $\overline{D}(\overline{K_n})$
5	$\Gamma^d \cong K_m(\overline{K_n})$ $(m, n \geq 2)$	$C(K_m(\overline{K_n}))$
6	$\Gamma^d \cong C_5$	$C(C_5)$
7	$\Gamma^d \cong 3^2$	$C(3^2)$
8	$\Gamma^{dd} \cong \overline{K_1}$	$K_n(H_0)$ $(n \geq 1)$ H_2
9	$\Gamma^{dd} \cong \overline{K_n}$ $(n \geq 2)$	$\overline{H_0}(\overline{K_n})$
10	$\Gamma^{dd} \cong K_m(\overline{K_n})$ $(m, n \geq 2)$, C_5, or 3^2	none
11	$\Gamma^{ddd} \cong \overline{K_n}$	none

This is the analog of Proposition 4.4 for digraphs. The proof that all homogeneous digraphs are listed on the right hand side of the table goes the same as the proof of Theorem 4.1 by considering $\mathrm{rank}_d(\Gamma)$ instead of $\mathrm{rank}(\Gamma)$. In this case, row 11 of the table shows that there are no homogeneous graphs Γ with $\mathrm{rank}_d(\Gamma) \geq 3$ and we only have to consider directed ranks 0, 1, 2.

5. Finite binary permutation groups

In general, by Corollary 3.4, classifying permutation groups with arity r is equivalent to classifying finite homogeneous structures with relations of degree r.

When we want to classify finite permutation groups with arity r, it makes sense to set some conditions on X^r/G, e.g. a bound a_r on $|X^r/G|$. These permutation groups will correspond to homogeneous structures over a language L with no operations and a_r relations of arity r. Lachlan's theory [KL87] shows that there is a classification of countable homogenenous structures over a relational language into finitely many types. To quote the introduction of [CMS96]: *The main result of Lachlan's theory resists summary, but it may be expressed very loosely as saying that large finite structures which are homogeneous for a finite relational language can be classified into finitely many types in a manner which is quite satisfactory from a theoretical point of view.* There is however no satisfactory classification of finite homogeneous structures.

In this section we are mainly interested in the finite transitive binary permuatation groups (G, X). These correspond to finite homogeneous relational structures $(X, \mathcal{R}(G))$ with binary, irreflexive relations. We will write $\mathcal{R}(G) = \{\Delta_i\}$. It is worth noting an obvious fact about these structures.

Remark 5.1. Let $\Delta_i, \Delta_j \in \mathcal{R}(G)$. If $(x_0, x_0') \in \Delta_i$ and $(x_0', x_0) \in \Delta_j$, then for all $(x, x') \in \Delta_i$ we have $(x', x) \in \Delta_j$.

If $i = j$, we call Δ_i *symmetric*, and if $i \neq j$, then we call Δ_i and Δ_j *complementary.*

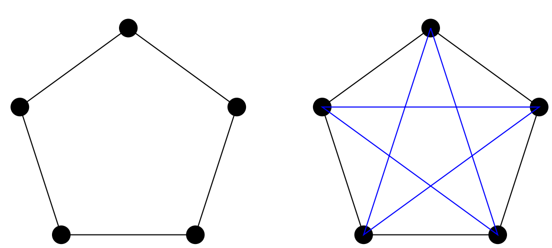
It is often convenient to think of these as coloured digraphs.

Definition 5.2. Let (G, X) be a finite transitive binary permuatation group. The *coloured digraph* Γ *corresponding to* (G, X) has $V\Gamma = X$, $E\Gamma = X^2 \setminus \{(x, x)\}$ and the edge-colouring $\omega \colon E\Gamma \to C$ defined by

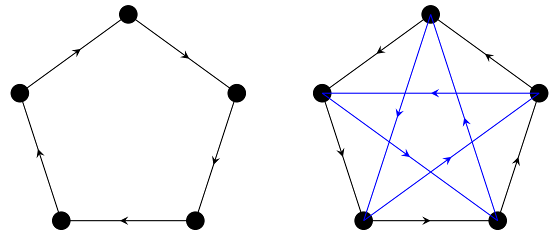
$$\omega(x, x') = c_i \quad \text{if } (x, x') \in \Delta_i.$$

If Δ_i is symmetric, (x, x') and (x', x) are both coloured with c_i, and we treat them as an undirected edge $\{x, x'\}$ coloured in c_i.

Example 5.3. The undirected 5-cycle C_5 (or D_{10} acting naturally on $\{1, 2, 3, 4, 5\}$) and its corresponding graph. Note that if we treat one of the orbits as non-edges, the graphs are the same.



The directed 5-cycle $\overrightarrow{C_5}$ (or $\mathbb{Z}/5\mathbb{Z}$ acting naturally on $\{1, 2, 3, 4, 5\}$) and an example of its corresponding digraph.



By Remark 5.1, we know that there is a complementary orbit to any orbit which is not symmetric, so we can omit one of the two orbits in the digraph. Because the orbits are not symmetric, we may arbitrarily choose the direction of the edges (as long as we are consistent throughout the whole orbit).

We have seen in Proposition 2.9 that primitive binary affine groups which are not strictly linear are of the form $(\mathbb{F}_{q^2} \ltimes \langle x^{q-1}\rangle \ltimes \langle\sigma\rangle, \mathbb{F}_{q^2})$. Let Γ_q be the coloured graph associated to this permutation group.

Proposition 5.4. *The graph Γ_q is a full graph with a colouring in $q-1$ colours, i.e. $|X^2/G| = q$.*

Proof. We claim that

$$(\mathbb{F}^2 \setminus \{(v,v)\})/G = \{[(0,x^i)] \mid i \in \{0,1,\dots,q-2\}\}.$$

First, take any $(v_1,v_2) \in \mathbb{F}$ with $v_1 \neq v_2$. Then $t_{-v_1,1,1}$ maps (v_1,v_2) to $(0,w)$ for $w = v_2 - v_1 \neq 0$. Then $w \in \mathbb{F}^\times$ and since x is primitive, $w = x^j$ for some $j \in \{0,1,\dots,q^2-2\}$. Write $j = k(q-1)+i$ for some $k \in \mathbb{N}$ and $i \in \{0,1,\dots,q-2\}$. Then

$$t_{0,x^{k(1-q)},1}(w) = x^{k(1-q)}x^{k(q-1)+i} = x^i,$$

so (v_1,v_2) is in the orbit $[(0,x^i)]$.

Finally, if $(0,x^j) \in [(0,x^i)]$ for $i,j \in \{0,1,\dots,q-2\}$, then for some $g = t_{0,x^{k(q-1)},\sigma^l} \in G_0$, $l \in \{0,1\}$ we have

$$x^j = g(x^i) = x^{k(q-1)}(x^i)^{q^l}.$$

Then.

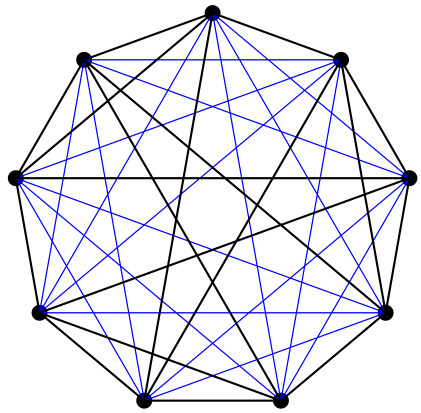
$$j = k(q-1) + i \cdot q^l \in \{0,1,\dots,q-2\}$$

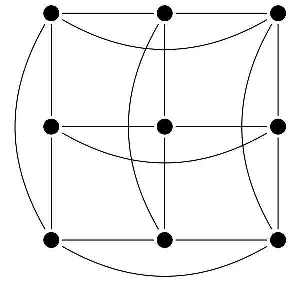
and in particular $k=0$. Either $i=0$ so also $j=0$ or $i \neq 0$ and $l=0$ so $i=j$. In any case $i=j$ □

Examples 5.5. ($q=2$) This case is trivial. As we have seen in Proposition 5.4, Γ_q is just the full graph K_4 and corresponds to $(\mathrm{Sym}(4), \{1,2,3,4\})$. Indeed:

$$G = (\mathbb{F}^4, \langle x\rangle, \langle\sigma\rangle) = (\mathbb{F}_4, \mathbb{F}_4^\times, \langle\sigma\rangle) = \mathrm{A\Gamma L}(\mathbb{F}_4) = \mathrm{Sym}(4).$$

($q=3$) In this case Γ_q has two colours and if we treat one of them as nonedges, we can see that $\Gamma_3 \cong 3^2$.





($q = 4$) This case is more complicated. Recall that we can represent elements of $\mathbb{F}_{16}^{\times}$ as polynomials $a + bx + cx^2 + dx^3$ with operations modulo $x^4 + x + 1$, i.e. assuming that $x^4 = x + 1$. Note that we are working in a field of characteristic 2, so addition and subtraction are the same operation.

We will first show that Γ_4 has no monochromatic triangles. Suppose contrary that (v_1, v_2, v_3) is a triangle, i.e.

$$(v_1, v_2), (v_2, v_3), (v_3, v_1) \in [(0, x^i)]$$

for some $i \in \{0, 1, 2\}$. Then we know that

$$v_1 + v_2, v_2 + v_3, v_3 + v_1 \in x^i\langle x^3\rangle,$$

so $v_1 + v_2 = x^i x^{3k}$ for some k and letting $(v_1', v_2', v_3') = (v_1, v_2, v_3)/(x^{3k})$, another monochromatic traingle. Moreover $v_1' + v_2' = x^i$ so that $v_2' = v_1' + x^i$ and since again $v_2' + v_3', v_3' + v_1' \in x^i\langle x^3\rangle$, we get that

$$v_1' + v_3' + x^i, v_1' + v_3' \in x^i\langle x^3\rangle.$$

Therefore, we have found two elements $w_1, w_2 \in x^i\langle x^3\rangle$ with $w_1 + w_2 = x^i$. We let $w_1' = w_1/x^i$, $w_2' = w_2/x^i$ be elements of $\langle x^3\rangle$ with

$$w_1' + w_2' = 1.$$

It is easy to work out that

$$\langle x^3\rangle = \{1, x^3, x^3 + x^2, x^3 + x\}$$

and for any $w \in \langle x^3\rangle$ we have $1 + w \neq 1$, $x^3 + w \neq 1$ and also

$$(x^3 + x^2) + (x^3 + x) = x + x^2 \neq 1.$$

This contradicts the existence of such w_1', w_2' and shows Γ_4 has no monochromatic triangles.

Therefore Γ_4 is a full graph K_{16} coloured with 3 colours with no monochromatic triangles, an example showing that the Ramsey Number $R(3, 3, 3) \geq 17$. We know that there are exactly two such examples with the colourings $\omega_i \colon \mathbb{F}_{16} \to \mathbb{F}_4$ (where $\omega_i(v, v) = 0$ are the vertices of the graph) given by

$$\omega_0(v, w) = (v + w)^{10}$$

$$\omega_1(v, w) = (v + w)^{10} + (v + v^4 + w + w^4)(v + v^2 + v^4 + v^8)(w + w^2 + w^4 + w^8).$$

A colouring ω_i will make the graph homogeneous, if it is invariant under the action of G. We note that the colouring ω_0 is not G-invariant: $1^\sigma = 1$, $x^\sigma = x^4 = 1 + x$, but:

$$\omega_0(1, x) = 1 + x^4 + x^8 + x^{10} = 1 + x + x^2 + x^{10},$$
$$\omega_0(1, 1 + x) = (1 + 1 + x)^{10} = x^{10}.$$

By elimination, the correct colouring has to be ω_1. One can check that it is indeed G-invariant.

5.1. **Some particular cases.** Let us end by expressing the results of Section 4 in the language of permutation groups. Let (G, X) be a transitive binary permutation group.

First, suppose $\mathcal{R}(G) = \{\Delta_1, \Delta_2\}$. Note that if an orbit is not symmetric, then Δ_1 and Δ_2 are complementary orbits. Then we can think of Δ_2 as non-edges and the graph Γ becomes an uncoloured digraph with no symmetric edges and by the classification of digraphs (Theorem 4.11) it has to be a directed 3-cycle. This corresponds to the natural action of $\mathbb{Z}/3\mathbb{Z}$ on $\{1, 2, 3\}$.

Now, suppose both Δ_1 and Δ_2 are symmetric. Then we can think of Δ_2 as non-edges and the graph Γ is an uncoloured graph. Therefore, we can use Gardiner's classification of homogeneous graphs . The union of m full graphs, $t \cdot K_n$, corresponds to the group $\mathrm{Sym}(n) \wr \mathrm{Sym}(t)$, the 5-cycle corresponds to the natural action of the group $\mathbb{Z}/5\mathbb{Z}$, and the 3^2 graph corresponds to $\mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ acting on pairs of elements of $\{1, 2, 3\}$.

In the next step, we suppose $\mathcal{R}(G) = \{\Delta_1, \Delta_2, \Delta_3\}$ and $\Delta_i \neq \emptyset$. We know that there are two possibilities: either all Δ_i are symmetric, or exactly one Δ_i is symmetric. The latter case is solved by the classification of homogeneous digraphs (Theorem 4.11). We treat one of the antisymmetric orbits as non-edges so that $(X, \mathcal{R}(G))$ is a digraph. The former case, however, already does not fall under any of the classifications. It is equivalent to classifying all homogeneous graphs with an edge-colouring by two colours.

For a summary of results on the classification of finite homogeneous structures, see [Mac11, Ch. 2].

References