MATH 613: HOMOLOGICAL ALGEBRA

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These are notes from the course Math 613: Homological Algebra taught by Prof. Harm Derksen in Winter 2017 at the University of Michigan. They were LaTeX'd by Aleksander Horawa. This version is from July 1, 2017. Please check if a new version is available at my website https://sites.google.com/site/aleksanderhorawa/. If you find a mistake, please let me know at ahorawa@umich.edu.

The textbook for this course was [Wei94], and the notes largely follow this book without specific reference. Citations are made where other resources were used.

Contents

1. Review of category theory

We begin with a short review of the necessary category theory.

Definition 1.1. A *category* $\mathcal{C}$ is

(1) a class of *objects*, $\operatorname{Obj}\mathcal{C}$, and
(2) for all $A, B \in \operatorname{Obj}\mathcal{C}$, a set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* from A to B,
(3) for any $A, B, C \in \operatorname{Obj}\mathcal{C}$, a *composition* map

$$\operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{C}}(B, C) \to \operatorname{Hom}_{\mathcal{C}}(A, C),$$

$$(f, g) \mapsto g \circ f = gf,$$

(4) for any $A \in \operatorname{Obj}\mathcal{C}$, a morphism $\mathrm{id}_A \in \operatorname{Hom}_{\mathcal{C}}(A, A)$,

such that

(a) for any $A, B \in \operatorname{Obj}\mathcal{C}$ and all $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$

$$\mathrm{id}_B f = f = f \mathrm{id}_A,$$

(b) for any $A, B, C, D \in \operatorname{Obj}\mathcal{C}$ and any $f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$, the composition is associative:

$$(hg)f = h(gf).$$

Note that $\operatorname{Obj}\mathcal{C}$ may not be a set: for example, the category of sets cannot have the set of all objects (*Russel paradox*). If $\operatorname{Obj}\mathcal{C}$ is a set, then $\mathcal{C}$ is *small*.

Examples 1.2.

- $\mathcal{C} =$ Sets: objects are sets, morphisms are functions,
- $\mathcal{C} =$ Groups: objects are groups, morphisms are group homomorphisms,
- $\mathcal{C} =$ Ring: rings and ring homomorphisms,
- $\mathcal{C} =$ Top: topological spaces and continuous maps,
- for a ring R, R-mod: left R-modules with R-module homomorphisms, and mod-R: right R-modules with R-module homomorphisms,
- $\mathcal{C} = (A, \leq)$, a poset: $\operatorname{Obj}\mathcal{C} = A$ and

$$\operatorname{Hom}_A(x, y) = \begin{cases} \{1\} & \text{if } x \leq y, \\ \emptyset & \text{otherwise,} \end{cases}$$

- $\mathcal{C} =$ Ab: abelian groups and group homomorphisms.

Definition 1.3. Fix a category $\mathcal{C}$. If $f\colon A \to B$ is a morphism, an *inverse* of f is a morphism $g\colon B \to A$ such that

$$gf = \mathrm{id}_A, \ fg = \mathrm{id}_B.$$

Inverses are unique: if g' is another inverse, then

$$g = \mathrm{id}_A g = (g'f)g = g'(fg) = g'\mathrm{id}_B = g'.$$

Definition 1.4. If f has an inverse, we call it an *isomorphism*, and we write f^{-1} for that inverse.

Definition 1.5. Suppose $\mathcal{C}$ is a category. The *opposite category*, $\mathcal{C}^{\mathrm{op}}$, is defined by

$$\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathcal{C})$$

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \{f^{\mathrm{op}} \mid f \in \mathrm{Hom}_{\mathcal{C}}(B, A)\}$$

and if in $\mathcal{C}$

$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ & C \arrow[u, "g"'] \arrow[ul, "fg"] \end{tikzcd}$$

then in $\mathcal{C}^{\mathrm{op}}$

$$\begin{tikzcd} A \arrow[r, "f^{\mathrm{op}}"] \arrow[dr, "(fg)^{\mathrm{op}}=g^{\mathrm{op}}f^{\mathrm{op}}"'] & B \arrow[d, "g^{\mathrm{op}}"] \\ & C \end{tikzcd}$$

Definition 1.6. A morphism $f\colon B \to C$ is *monic* if for any $A \in \mathrm{Obj}\,\mathcal{C}$ and any $e_1, e_2\colon A \to B$ such that $fe_1 = fe_2$, we have $e_1 = e_2$.

Example 1.7. In Groups, Sets, Top, a morphism is monic if and only if it is injective.

Definition 1.8. A morphism $f\colon A \to B$ is *epi* if for any $C \in \mathrm{Obj}\,\mathcal{C}$ and all $g_1, g_2\colon B \to C$ such that $g_1 f = g_2 f$, we have $g_1 = g_2$.

The notions of monic and epi are *dual*: f is monic in $\mathcal{C}$ if and only if f^{op} is monic in $\mathcal{C}^{\mathrm{op}}$.

Example 1.9. In Sets, an epimorphism is a surjective map.

Let $\mathcal{C}$ be the category of metric (or at least Hausdorff) topological spaces. Then the inclusion $f\colon \mathbb{Q} \to \mathbb{R}$ is not surjective but it is epi in $\mathcal{C}$. Indeed, suppose $g_1, g_2\colon \mathbb{R} \to X$ and $g_1 f = g_2 f$. For any $x \in \mathbb{R}$, there exists a sequence $\{x_n\} \subseteq \mathbb{Q}$ with $\lim\limits_{n\to\infty} x_n = x$. Then

$$g_1(x) = \lim_{n\to\infty} g_1(x_n) = \lim_{n\to\infty} g_1(f(x_n)) = \lim_{n\to\infty} g_2(f(x_n)) = \lim_{n\to\infty} g_2(x_n) = g_2(x)$$

since both g_1 and g_2 are continuous.

Similarly, in Rings, $f\colon \mathbb{Z} \to \mathbb{Q}$ is epi but not surjective.

Definition 1.10. An object $I \in \mathrm{Obj}(\mathcal{C})$ is *initial* if for every $A \in \mathrm{Obj}(\mathcal{C})$ there is a unique morphism $I \to A$.

If I, I' are initial objects, there is a unique morphism $f\colon I \to I'$ and a unique morphism $g\colon I' \to I$. We then get morphisms $fg\colon I' \to I'$ and $gf\colon I \to I$, but $\mathrm{id}_{I'}\colon I' \to I'$ and $\mathrm{id}_I\colon I \to I$ are also such morphisms and hence by uniqueness

$$fg = \mathrm{id}_{I'}, \qquad gf = \mathrm{id}_I.$$

This shows initial objects are unique up to unique isomorphism.

Definition 1.11. An object $T \in \mathrm{Obj}(\mathcal{C})$ is *terminal* if for all $A \in \mathrm{Obj}(\mathcal{C})$, there is a unique morphism $A \to T$.

This is the dual notion to initial object:

$$I \in \mathrm{Obj}(\mathcal{C}) \text{ is initial if and only if } I \in \mathrm{Obj}(C^{\mathrm{op}}) \text{ is terminal.}$$

Examples 1.12. We provide a few examples of initial and terminal objects in a few categories:

	initial	terminal
Sets	$\emptyset$	$\{0\}$
Groups	$\{1\}$	$\{1\}$
Ab	$\{0\}$	$\{0\} = 0$
Rings with 1	$\mathbb{Z}$	0

Definition 1.13. A *zero object* is initial and terminal. We denote it by 0.

If $\mathcal{C}$ has 0, $A, B \in \mathrm{Obj}(\mathbb{C})$, we have maps

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend right, "0"'] & 0 \arrow[r] & B \end{tikzcd}$$

so there is a unique morphism $A \to B$ that factors through $0 \in \mathrm{Obj}(\mathcal{C})$, the *zero morphism.*

Definition 1.14. A monic morphism $f\colon A \to B$ is called a *subobject of* B.

Two subobjects $f\colon A \to B$, $f'\colon A' \to B'$ are isomorphic if there is an isomorphism $g\colon A \to A'$ such that $f = f'g'$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \\ & A' \arrow[u, "f'"'] \end{tikzcd}$$

Example 1.15. In Sets: if $f\colon A \to B$ injective, it is a subobject and we have that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f"'] & B \\ & f(A) \arrow[u] \end{tikzcd}$$

and $A \to B$ and $f(A) \to B$ are isomorphic.

Definition 1.16. Suppose $\mathcal{C}$ has zero object. We say $f\colon A \to B$ is a *kernel* of $g\colon B \to C$ if $gf = 0$ and for every $f'\colon A' \to B$ with $gf' = 0$, there is a unique morphism $h\colon A' \to A$ such that $f' = fh$, i.e. the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "0"] & B \arrow[r, "g"'] & C \\
A' \arrow[u, dotted, "h"] \arrow[ur, "f'"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

commutes. The dual notion is the *cokernel*.

The above is an example of a *universal property*. We can restate it as follows. We define a category $\mathcal{G}$ whose objects are pairs (A, f) with $f\colon A \to B$ and $gf = 0$ and a morphism $(A, f) \to (A', f')$ in $\mathcal{G}$ is a morphism $h\colon A \to A'$ in $\mathcal{C}$ such that $f'h = f$. Then

$$(A, f) \text{ is the kernel of } g \text{ if and only if } (A, f) \text{ is a terminal object in } \mathcal{G}.$$

A kernel is a subobject: indeed, if $e_1, e_2\colon A' \to A$ satisfy $f' = fe_1 = fe_2$, then $e_1 = e_2$ by uniqueness in the universal property

$$\begin{tikzcd}
A' \arrow[d, "e_1"'] \arrow[dr, "f'"] \arrow[drr, bend left, "0"] & & \\
A \arrow[r, "f"'] \arrow[rr, bend left, "0"] & B \arrow[r, "g"'] & C \\
A' \arrow[u, "e_2"] \arrow[ur, "f'"'] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Example 1.17. In Groups, consider $S_2 \to S_n$ sending (12) to (12). This map is not an epimorphism but its cokernel is $\{1\} \to S_n$ (exercise).

Definition 1.18. If $A, B \in \mathrm{Obj}(\mathcal{C})$ then a *product* is an object $A \times B$ together with morphisms $\pi_A\colon A \times B \to A$ and $\pi_B\colon A \times B \to B$ with universal property: if $C \in \mathrm{Obj}(\mathcal{C})$ and $f_A\colon C \to A$, $f_B\colon C \to B$ are morphisms, then there exists a unique morphism $h\colon C \to A \times B$ such that $f_A = \pi_A h$, $f_B = \pi_B h$, i.e. the following diagram

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\
& C \arrow[ul, "f_A"] \arrow[u, dotted, "h"] \arrow[ur, "f_B"'] &
\end{tikzcd}$$

commutes.

Similarly, if A_i, $i \in I$ are objects, their product is an object $\prod_{i\in I} A_i$ together with morphisms $\pi_i\colon \prod_{i\in I} A_i \to A_i$ with analogous universal property.

Definition 1.19. A *coproduct* is an object $A \amalg B$ together with $i_A \colon A \to A \amalg B$ and $i_B \colon B \to A \amalg B$ with the dual universal property, and similarly one can define the coproduct of any family of objects, $\coprod_{i \in I} A_i$.

Examples 1.20. In Sets, the product $A \times B$ is the Cartesian product with $\pi_A(a, b) = a$, $\pi_B(a, b) = b$, and the coproduct $A \amalg B$ is the disjoint union with $i_A \colon A \to A \amalg B$, $i_B \colon B \to A \amalg B$, the inclusion maps.

In Ab, the product and coproduct are the same for finite families of objects. However, for infinite families we have

$$\prod_{i \in I} A_i = \{(a_1, a_2 \ldots) \mid a_i \in A_i\},$$

$$\coprod_{i \in I} A_i = \{(a_1, a_2, \ldots) \mid a_i \in A_i \text{ and } a_i = 0 \text{ for all but finitely many } i\}.$$

In Groups, $G \times H$ is the standard product and $G \amalg H$ is $G * H$, the free group product of G and H. For example,

$$\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle.$$

In the category of rings with 1, Rings_1, $A \times B$ is the standard product and $A \amalg B = A \otimes_{\mathbb{Z}} B$ with

$$i_A \colon A \to A \otimes_{\mathbb{Z}} B,$$
$$a \mapsto a \otimes 1.$$

Note that $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$, so the inclusion map

$$i_{\mathbb{Z}/2} \colon \mathbb{Z}/2 \to \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3$$

is not monic.

Definition 1.21. A *functor* $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is a rule that

(1) assigns to $A \in \operatorname{Obj} \mathcal{C}$ an object $\mathcal{F}A \in \operatorname{Obj} \mathcal{C}$,
(2) assigns to $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ a morphism $\mathcal{F} \in \operatorname{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$

such that

(a) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}A}$,
(b) $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$ if $f \colon A \to B$, $g \colon B \to C$ in $\mathcal{C}$.

Example 1.22. If $A \in \operatorname{Obj} \mathcal{C}$, we have a functor

$$\operatorname{Hom}_{\mathcal{C}}(A, -) \colon \mathcal{C} \to \text{Sets}$$

such that for $B \in \operatorname{Obj} \mathcal{C}$

$$\operatorname{Hom}_{\mathcal{C}}(A, -)(B) = \operatorname{Hom}_{\mathcal{C}}(A, B),$$

and for $f \colon B \to C$

$$\operatorname{Hom}_{\mathcal{C}}(A, f) \colon \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C),$$
$$g \mapsto fg,$$

which can be represented by the following diagram:

$$\begin{tikzcd}
& A \arrow[dl, "g"'] \arrow[dr, dotted, "fg"] & \\
B \arrow[rr, "f"'] & & C
\end{tikzcd}$$

Then $\mathrm{Hom}_{\mathbb{C}}(A, \mathrm{id}_B) = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}$ and

$$\mathrm{Hom}_{\mathcal{C}}(A, hg) = \mathrm{Hom}(A, h) \circ \mathrm{Hom}(A, g),$$

which can be represented by the following diagram:

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[d, "gf"'] \arrow[dr, "h(gf) = (hg)f"] & \\
B \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & C \arrow[r, "h"] & D
\end{tikzcd}$$

Example 1.23. Suppose R is a ring. If M is a left R-module, then we have a functor

$$\mathrm{Hom}_R(M, -)\colon R\text{-mod} \to \mathrm{Ab},$$

and if M is a right R-module then we have a functor

$$M \otimes_R -\colon R\text{-mod} \to \mathrm{Ab}\,.$$

Definition 1.24. A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is *faithful* (resp. *full*) if for any $A, B \in \mathrm{Obj}\,\mathcal{C}$,

$$\mathcal{F}\colon\ \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$$

is injective (resp. surjective).

What we defined above is actually a *covariant functor*.

Definition 1.25. A *contravariant functor* $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to $\mathcal{D}$ is a rule that

(1) assigns to $A \in \mathrm{Obj}\,\mathcal{C}$ an object $\mathcal{F}A \in \mathrm{Obj}\,\mathcal{C}$,
(2) assigns to $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ a morphism $\mathcal{F} \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}B, \mathcal{F}A)$

such that

(a) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}A}$,
(b) $\mathcal{F}(gf) = \mathcal{F}(f)\mathcal{F}(g)$ if $f\colon A \to B$, $g\colon B \to C$ in $\mathcal{C}$.

For a contravariant functor, a commuting triangle maps to a commuting triangle with arrows reversed:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dotted, "gf"'] & B \arrow[d, "g"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}A & \mathcal{F}B \arrow[l, "\mathcal{F}f"'] \\ & \mathcal{F}C \arrow[u, "\mathcal{F}g"'] \arrow[ul, dotted, "\mathcal{F}(gf)"] \end{tikzcd}$$

Example 1.26. There is a contravariant functor $D\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ given by $D(A) = A$ for $A \in \mathrm{Obj}(\mathcal{C})$, $D(f) = f^{\mathrm{op}}$ for f a morphism in $\mathcal{C}$.

Example 1.27. If $A \in \mathrm{Obj}\,\mathcal{C}$ then $\mathrm{Hom}_{\mathcal{C}}(-, A)\colon \mathcal{C} \to \mathrm{Sets}$ is a contravariant functor. So $\mathrm{Hom}_{\mathcal{C}}(A, -)$ is covariant and $\mathrm{Hom}_{\mathcal{C}}(-, A)$ is contravariant.

Definition 1.28. Suppose $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ are functors. A *natural transformation* $\eta\colon \mathcal{F} \to \mathcal{G}$ is a rule that assigns to $A \in \mathrm{Obj}(\mathcal{C})$ a morphism

$$\eta(A)\colon \mathcal{F}(A) \to \mathcal{G}(A)$$

such that for every morphism $f\colon A \to B$ the following diagram

$$\begin{tikzcd} \mathcal{F}A \arrow[r, "\mathcal{F}f"] \arrow[d, "\eta(A)"'] & \mathcal{F}B \arrow[d, "\eta(B)"] \\ \mathcal{G}A \arrow[r, "\mathcal{G}f"'] & \mathcal{G}B \end{tikzcd}$$

commutes.

Example 1.29. Let $\mathcal{F}\colon$ Ab $\to$ Ab be given by

$$\mathcal{F}A = \{a \in A \mid \text{there exists } n \geq 1 \text{ such that } na = 0\}.$$

Then $\eta(A)\colon \mathcal{F}A \hookrightarrow A$ is a natural transformation between $\mathcal{F}$ and the identity functor on Ab.

Example 1.30. Fix a category $\mathcal{C}$. Suppose $e\colon A \to B$ is a morphism. Then

$$\epsilon\colon\ \mathrm{Hom}_{\mathcal{C}}(B, -) \to \mathrm{Hom}_{\mathcal{C}}(A, -)$$

is a natural transformation given by

$$\epsilon(C)\colon\ \mathrm{Hom}_{\mathcal{C}}(B, C) \ni f \mapsto fe \in \mathrm{Hom}_{\mathcal{C}}(A, C),$$

since the following diagram

$$\begin{tikzcd} \mathrm{Hom}(B, C) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(B,f)"] \arrow[d, "\epsilon(C)"'] & \mathrm{Hom}(B, D) \arrow[d, "\epsilon(D)"] \\ \mathrm{Hom}(A, C) \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(A,f)"'] & \mathrm{Hom}(A, D) \end{tikzcd}$$

commutes.

A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ would be an isomorphism if there exists $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F}\mathcal{G} = \mathrm{id}_\mathcal{D}$, $\mathcal{G}\mathcal{F} = \mathrm{id}_\mathcal{C}$. However, such maps seldom exist so we weaken the notion slightly.

Definition 1.31. A natural transformation $\eta\colon \mathcal{F} \to \mathcal{G}$ is a *natural isomorphism* if $\eta(A)\colon \mathcal{F}(A) \to \mathcal{G}(A)$ is an isomorphism for all $A \in \mathrm{Obj}(\mathcal{C})$.

Definition 1.32. A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists a functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G}\mathcal{F}$ is naturally isomorphic to $\mathrm{id}_\mathcal{C}$ and $\mathcal{F}\mathcal{G}$ is naturally isomorphic to $\mathrm{id}_\mathcal{D}$.

Example 1.33. Let $\mathcal{V}$ be the category of finite-dimensional $\mathbb{R}$-vector spaces and $\mathcal{C} \subseteq \mathcal{V}$ be the full subcategory with $\mathrm{Obj}(\mathcal{C}) = \{\mathbb{R}^n \mid n \geq 0\}$. Then $\mathcal{C}$ is small while $\mathcal{V}$ is not small, so $\mathcal{C}$ and $\mathcal{V}$ are not isomorphic. We will show that they are nonetheless equivalent. Define

$$\begin{aligned} \mathcal{F} = \mathrm{id}_{|\mathcal{C}}\colon \mathcal{C} \to \mathcal{V}, &\qquad \text{the inclusion functor,} \\ \mathcal{G}\colon \mathcal{V} \to \mathcal{C}, &\qquad \mathcal{G}(V) = \mathbb{R}^{\dim V}. \end{aligned}$$

We now choose isomorphism

$$\eta(V)\colon \mathbb{R}^{\dim V} \to V$$

for every V (we use the *meta axiom of choice* here). Moreover, there is only one way to define $\mathcal{G}f$ for a linear map $f\colon V \to W$ to make η a natural transformation, i.e. making the square

$$\begin{tikzcd} \mathbb{R}^{\dim V} \arrow[r, "\mathcal{G}f"] \arrow[d, "\eta(V)"'] & \mathbb{R}^{\dim W} \arrow[d, "\eta(W)"] \\ W \arrow[r, "f"'] & V \end{tikzcd}$$

commute. We then have that

$$\mathcal{F}\mathcal{G}\colon \mathcal{V} \to \mathcal{V}$$

and we note that $\eta\colon \mathcal{F}\mathcal{G} \to \mathrm{id}_\mathcal{V}$ is a natural isomorphism

$$\eta(V)\colon \mathbb{R}^{\dim V} = \mathcal{F}\mathcal{G}(V) \to V.$$

There is also a natural isomorphism $\eta_{|\mathcal{C}}\colon \mathcal{G}\mathcal{F} \to \mathrm{id}_\mathcal{C}$.

1.1. **Abelian Categories.**

The reference for this section is [Fre03]. We introduce a general framework where we can develop homological algebra, generalizing categories such as Ab and more generally R-mod.

Definition 1.34. A category $\mathcal{C}$ is an Ab-*category* if for all $A, B \in \mathrm{Obj}\,\mathcal{C}$, $\mathrm{Hom}_\mathcal{C}(A, B)$ is an abelian group, and

$$\begin{aligned} g(f_1 + f_2) = gf_1 + gf_2, &\qquad f, f_1, f_2\colon A \to B, \\ (g_1 + g_2)f = g_1 f + g_2 f, &\qquad g, g_1, g_2\colon B \to C. \end{aligned}$$

This makes $\mathrm{Hom}_\mathcal{C}(A, A)$ a ring with $1 = \mathrm{id}_A$.

Definition 1.35. An *additive category* is an Ab-category with finite products and a zero object.

In additive categories, finite coproducts exist and are the same as product: if $A, B \in \operatorname{Obj} \mathcal{C}$, then we have an object $A \oplus B = A \times B = A \amalg B$ such that the following diagram

$$\begin{tikzcd} & A \oplus B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ A \arrow[ur, bend left, "i_A"] & & B \arrow[ul, bend right, "i_B"'] \end{tikzcd}$$

commutes, so $(A \oplus B, \pi_A, \pi_B)$ is a product, $(A \oplus B, i_A, i_B)$ is a coproduct, and

$$\pi_A i_A = \mathrm{id}_A,\ \pi_B i_B = \mathrm{id}_B,$$

$$\pi_A i_B = 0 = \pi_B i_A,$$

$$i_A \pi_A + i_B \pi_B = \mathrm{id}_{A \oplus B}.$$

Definition 1.36. An additive category is *abelian* if

(1) every morphism has kernel and cokernel,
(2) every monic morphism is kernel of its cokernel,
(3) every epimorphism is cokernel of its kernel.

Let $\mathcal{C}$ be abelian from now on.

Lemma 1.37. *A morphism $f \colon A \to B$ is monic if and only if $\ker f = 0$ (i.e. $0 \to A$ is a kernel of A). Dually, $f \colon A \to B$ is epi if and only if $B \to 0$ is a cokernel of f.*

Proof. Suppose $g \colon K \to A$ is a kernel. If f is monic, then $fg = 0 = f0$, so $g = 0$. Hence g factors through $0 \to A$, so $0 \to A$ is a kernel of f.

Suppose conversely that $0 \to A$ is a kernel of $f \colon A \to B$. If $fg_1 = fg_2$, then

$$0 = fg_1 - fg_2 = f(g_1 - g_2),$$

so $g_1 - g_2$ factors through $0 \to A$, so $g_1 - g_2 = 0$ and $g_1 = g_2$. $\square$

Definition 1.38. If we have maps

$$\begin{tikzcd} & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & B \end{tikzcd}$$

then a pull-back is an object P together with maps $g_1 \colon P \to A_1$, $g_2 \colon P \to A_2$ such that $f_1 g_1 = f_2 g_2$ and (P, g_1, g_2) is universal with this property, i.e. if $u_1 \colon C \to A_1$ and $u_2 \colon C \to A_2$ satisfy $fu_1 = fu_2$ then there exists a unique $h \colon C \to P$ such that $g_1 h = u_1$ and $g_2 h = u_2$, i.e. the following diagram

$$\begin{tikzcd}
C \arrow[rrd, bend left, "u_1"] \arrow[ddr, bend right, "u_2"'] \arrow[dr, dotted] & & \\
& P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\
& A_2 \arrow[r, "f_2"] & B
\end{tikzcd}$$

commutes. The dual notion is called *push-out*.

An abelian category has pull-backs and push-outs. Explicitly, for a map

$$A_1 \oplus A_2 \xrightarrow{(f_1, -f_2)} B$$

we have that

$$P \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} A_1 \oplus A_2$$

is the kernel of $A_1 \oplus A_2 \to B$. The pushout is a cokernel of an analogous map $A_1 \oplus A_2 \to B$.

Lemma 1.39. *Suppose*

$$\begin{tikzcd}
P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\
A_2 \arrow[r, "f_2"'] & B
\end{tikzcd}$$

is a pull-back. Then

(1) *if g_1 is monic, then f_2 is monic,*
(2) *if f_1 is epi, then g_2 is epi.*

Proof. We first prove (1). Indeed, suppose g_1 is monic and take $u\colon C \to A_2$ with $f_2 u = 0$. Then there exists unique $h\colon C \to P$ such that $g_1 h = 0$ and $g_2 h = u$, i.e. the following diagram

$$\begin{tikzcd}
C \arrow[rrd, bend left, "0"] \arrow[ddr, bend right, "u"'] \arrow[dr, dotted, "h"] & & \\
& P \arrow[r, "g_1"] \arrow[d, "g_2"'] & A_1 \arrow[d, "f_1"] \\
& A_2 \arrow[r, "f_2"'] & B
\end{tikzcd}$$

commutes. But g_1 is monic, so this implies $h = 0$, and hence $u = g_2 h = 0$. Hence $0 \to A_2$ is a kernel of f_2, so f_2 is monic.

For (2), suppose f_1 is epi. We recall that $P \to A_1 \oplus A_2$ is kernel of $A_1 \oplus A_2 \to B$, and $A_1 \oplus A_2 \to B$ is epi, because $A_2 \to B$ is epi. Hence $A_1 \oplus A_2 \to B$ is cokernel of

$$P \to A_1 \oplus A_2.$$

Hence the diagram

$$\begin{tikzcd}
P \arrow[r, "g_1"'] \arrow[d, "g_1"] & A_1 \arrow[d, "f_1"] \\
A_2 \arrow[r, "f_2"'] & B
\end{tikzcd}$$

is a pushout. By the dual of (a), $f_1\colon A_1 \to B$ is epi implies that $g_2\colon P \to A_2$ is epi. $\square$

Lemma 1.40. *If $g\colon B \to C$ is a morphism, then there exists a factorization $g = vu$ where u is epi and v is monic*

$$\begin{tikzcd}
B \arrow[r, two heads, "u"] \arrow[rr, bend right, "g"'] & I \arrow[r, hook, "v"] & C
\end{tikzcd}$$

Proof. Let $f\colon A \hookrightarrow B$ be the kernel of g and $u\colon B \twoheadrightarrow I$ be the cokernel of f. Then, because $gf = 0$ and u is the cokernel of f, there exists v such that the following diagram

$$\begin{tikzcd}
A \arrow[r, hook, "f"] & B \arrow[r, two heads, "u"] \arrow[rr, bend right, "g"'] & I \arrow[r, "v"] & C
\end{tikzcd}$$

commutes. We have to show v is monic. Let $w\colon K \to I$ be the kernel of v and let P with $x\colon P \to K$, $y\colon P \to B$ be the pullback. Then, since $f\colon A \to B$ is the kernel of g and

$$gy = (vu)y = v(uy) = v(wx) = (vw)x = 0x = 0,$$

we get a unique map $z\colon P \to A$ such that the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, two heads, "u"] \arrow[rr, bend left, "g"] & I \arrow[r, "v"] & C \\
& P \arrow[u, "y"] \arrow[r, two heads, "x"] \arrow[ul, dotted, "z"] & K \arrow[u, hook, "w"] &
\end{tikzcd}$$

commutes. Since P is the pullback and $u\colon B \to I$ is epi, $x\colon P \to K$ is epi. Now,

$$wx = uy = u(fz) = (uf)z = 0z = 0,$$

and since x is epi, we have that $w = 0$. This shows the kernel of $I \to C$ is $0 \to I$ and hence v is monic. $\square$

The *image* of g is the kernel of the cokernel or equivalently the cokernel of the kernel.

Remark 1.41. For a ring R, R^{op}, the opposite ring is $R^{\text{op}} = R$ as a set with multiplication $*$ in R^{op} defined by $a * b = b \cdot a$. If M is a left R-module, them M is a right R^{op}-module: if $m \in M$, $a \in R^{\text{op}} = R$, then $m * a = a \cdot m$. This indeed gives a right module, for example:

$$(m * a) * b = b \cdot (a \cdot m) = (b \cdot a) \cdot m = m * (b \cdot a) = m * (a * b).$$

The category R-mod is isomorphic to mod-R^{op} and the category R^{op}-mod is isomorphic to mod-R. Moreover, if M is a left R-module, we write ${}_R M$, if M is a right R-module, we write M_R, and if M is a R-S-bimodule, we write ${}_R M_S$.

We work for now in the category mod-R.

Definition 1.42. A sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

is *exact* if $\operatorname{im} f_{i-1} = \ker f_i$ for $i = 1, 2, \ldots, n$.

Then

- $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is injective,
- $B \xrightarrow{g} C \longrightarrow 0$ is exact if and only if g is surjective,
- $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact if and only if f is an isomorphism,
- $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact (we call it a *short exact sequence*) if and only if f is injective, g is surjective, and $C \cong B/A$,
- $0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$ is exact for any f.

Theorem 1.43 (Snake Lemma)**.** *Suppose the diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker p \arrow[d] & \ker q \arrow[d] & \ker r \arrow[d] & \\
 & A' \arrow[r] \arrow[d, "p"] & B' \arrow[r] \arrow[d, "q"] & C' \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker} p \arrow[d] & \operatorname{coker} q \arrow[d] & \operatorname{coker} r \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

has exact rows, exact columns, and commuting squares. Then we have an exact sequence

$$\ker p \to \ker q \to \ker r \to \operatorname{coker} p \to \operatorname{coker} q \to \operatorname{coker} r,$$

i.e. the red sequence in the following diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker p \arrow[r] \arrow[d] & \ker q \arrow[r] \arrow[d] & \ker r \arrow[d] \arrow[dddll, out=-20, in=160] & \\
 & A' \arrow[r] \arrow[d, "p"] & B' \arrow[r] \arrow[d, "q"] & C' \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & \\
 & \operatorname{coker} p \arrow[r] \arrow[d] & \operatorname{coker} q \arrow[r] \arrow[d] & \operatorname{coker} r \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Moreover, if we add zeros at the end of the two middle exact sequences, then we can add zeros at the end of the "snake", i.e. the red sequence in the following diagram exists and is exact

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker p \arrow[r] \arrow[d] & \ker q \arrow[r] \arrow[d] & \ker r \arrow[d] \arrow[dddll, out=-20, in=160] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d, "p"] & B' \arrow[r] \arrow[d, "q"] & C' \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} p \arrow[r] \arrow[d] & \operatorname{coker} q \arrow[r] \arrow[d] & \operatorname{coker} r \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Proof. The proof is diagram chasing, and we omit it here. □

Theorem 1.44 (Five Lemma)**.** *If the diagram*

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}
$$

has exact rows and commuting squares, and f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is also an isomorphism.

Proof. This proof is diagram chasing again, and we omit it here. □

We can generalize both of these lemma to an abelian category $\mathcal{C}$. We present the main ideas below. For more details, see [GM03, Chap. II.5] [Gelfand Manin, II.5 exercises].

Suppose $B \in \operatorname{Obj}\mathcal{C}$. Consider pairs (A, f) where $f\colon A \to B$ is a morphism. Than $(A, f) \sim (A', f')$ if there exists $C \in \operatorname{Obj}\mathcal{C}$ and epimorphisms $g\colon C \twoheadrightarrow A$ and $g'\colon C \twoheadrightarrow A'$ such that $fg = f'g'$:

$$\begin{tikzcd}
C \arrow[r, two heads, "g"] \arrow[d, two heads, "g'"'] & A \arrow[d, "f"] \\
A \arrow[r, "f'"'] & B
\end{tikzcd}$$

Then $\sim$ is an equivalence relation. For example, transitivity is proved as follows: if $(A, f) \sim (A', f')$ and $(A', f') \sim (A'', f'')$, there exist C and C' and $C \twoheadrightarrow A$, $C \twoheadrightarrow A'$, $C' \twoheadrightarrow A'$, $C \twoheadrightarrow A''$ and hence taking the pullback we get the following commutative diagram

$$\begin{tikzcd}
& P \arrow[dl, two heads] \arrow[dr, two heads] \arrow[dddl, dotted, two heads] \arrow[dddr, dotted, two heads] & \\
C \arrow[dr, two heads] \arrow[dd, two heads] & & C' \arrow[dl, two heads] \arrow[dd, two heads] \\
& A' \arrow[dd, "f'"] & \\
A \arrow[dr, "f"'] & & A'' \arrow[dl, "f''"] \\
& B &
\end{tikzcd}$$

and hence $(A, f) \sim (A'', f'')$.

Then we say $a \in B$ if a is a congruence class of (A, f) for some $f\colon A \to B$, i.e. $a = [(A, f)]$. Suppose $g\colon B \to C$. With this definition, if $a = [(A, f)]$ then $g(a) = [(A, gf)]$:

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "gf"] & \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

Then $0 = [(0, f)]$ for the unique map $0 \to B$.

Lemma 1.45. *A morphism $g\colon B \to C$ is monic if and only if for all $a \in B$, $g(a) = 0$ implies that $a = 0$.*

Lemma 1.46. *A morphism $g\colon B \to C$ is epi if and only if for all $c \in C$, there exists $b \in B$ such that $g(b) = c$.*

Lemma 1.47. *A morphism $f\colon A \to B$ is equal to 0 if and only if $f(a) = 0$ for all $a \in A$.*

Lemma 1.48. *If $f\colon A \to B$ is a morphism and $a, a' \in A$ such that $f(a) = f(a')$ then there exists $b \in A$ such that $f(b) = 0$ and for every $g\colon A \to C$ with $g(a) = 0$ we have $g(b) = -g(a')$ and for every $g\colon A \to C$ with $g(a') = 0$ we have $g(b) = g(a)$.*

Proof. If $a = [(D, h)]$ and $a' = [(D', h')]$, then take $b = [D \oplus D', (h, -h')]$. □

These lemmas suffice to proceed with the diagram chasing arguments, so they show that the Snake Lemma 1.43 and the Five Lemma 1.44 hold in an abelian category.

2. Algebraic topology

In this chapter, we review the motivating examples of homology and cohomology from algebraic topology. For a more detailed introduction to the area, see [Hat02].

2.1. **Singular Homology.**

Definition 2.1. A *geometric n-simplex* is

$$\Delta_n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \quad \sum_{i=0}^{n} x_i = 1 \right\}.$$

For a topological space X, a *singular n-simplex* is a continuous function $\sigma\colon \Delta_n \to X$.

We let

$$S_n(X) = \text{free } \mathbb{Z}\text{-module with basis of all singular } n\text{-simplices on } X$$

and define

$$f_i = f_i^n\colon \Delta_{n-1} \to \Delta_n$$

by

$$f_i(x_0, \ldots, x_{n-1}) = (x_0, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-1}).$$

Then we define a map $d_n\colon S_n(X) \to S_{n-1}(X)$ by

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ f_i.$$

Lemma 2.2. *For any n, $d_{n-1} \circ d_n = 0$, so $d^2 = 0$.*

Proof. If $j < i$, then $f_i \circ f_j = f_j \circ f_{i-1}$, so

$$\begin{aligned}
d_{n-1}(d_n(\sigma)) &= \sum_{j=0}^{n-1}\sum_{i=0}^{n}(-1)^{i+j}\sigma \circ f_i^n \circ f_j^{n-1} \\
&= \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ f_i\circ f_j + \sum_{0\le j<i\le n-1}(-1)^{i+j}\sigma\circ f_i\circ f_j \\
&= \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ f_i\circ f_j + \sum_{0\le j<i\le n-1}(-1)^{i+j}\sigma\circ f_j\circ f_{i+1} \\
&= \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ f_i\circ f_j - \sum_{0\le j\le i\le n-1}(-1)^{i+j}\sigma\circ f_j\circ f_i \qquad \text{setting } i \to i+1 \\
&= 0
\end{aligned}$$

showing $d^2 = 0$. □

We hence get a *chain complex* $S_\bullet(X)$:

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{d_{n+1}} S_n(X) \xrightarrow{d_n} S_{n-1}(X) \xrightarrow{d_{n-1}} S_{n-2}(X) \longrightarrow \cdots$$

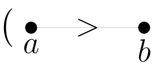
with $d_{n-1}d_n = 0$ for all n. (By convention, $S_n(X) = 0$ for $n < 0$.)

Definition 2.3. We define

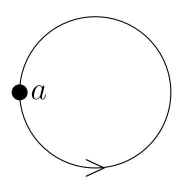
$$\begin{aligned}
Z_n(X) &= \ker d_n, && \text{the } \textit{module of cycles}, \\
B_n(X) &= \operatorname{im} d_{n+1}, && \text{the } \textit{module of boundaries}.
\end{aligned}$$

Note that $B_n(X) \subseteq Z_n(X) \subseteq S_n(X)$ by the lemma above.

Example 2.4. The boundary of a segment from a to b is $b - a$

$\partial($  $) = b - a$

while the boundary of a circle treated as a singular 1-simplex is $a - a = 0$

∂  $= 0$

This justifies the names *cycle* and *boundary*.

Definition 2.5. The *nth singular homology group* is defined as

$$H_n(X) = H_{\mathrm{sing},n}(X) = Z_n(X)/B_n(X) = \ker d_n / \operatorname{im} d_{n+1}.$$

One can show that $H_n(X)$ is a topological invariant: if X is homeomorphic to Y then $H_n(X) \cong H_n(Y)$.

Examples 2.6. We have that $H_0(X) \cong \mathbb{Z}^d$ where d is the number of path-connected components of X.

The homology of a contractible space is trivial, so for example

$$H_n(*) = H_n(\mathbb{R}^n).$$

However, we can distinguish between $\mathbb{R}^n$ and $\mathbb{R}^m$ using homology, because

$$H_j(\mathbb{R}^n \setminus \{*\}) = \begin{cases} \mathbb{Z} & \text{for } j = n-1, \\ 0 & \text{for } 0 < j < n-1. \end{cases}$$

2.2. **Relative homology.** Suppose $Y \subseteq X$ is a subspace. Then $S_n(Y) \subseteq S_n(X)$ and we can define

$$S_n(X, Y) = S_n(X)/S_n(Y)$$

and we get the sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_n(Y) & \longrightarrow & S_n(X) & \longrightarrow & S_n(X,Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} & & \\ 0 & \longrightarrow & S_{n-1}(Y) & \longrightarrow & S_{n-1}(X) & \longrightarrow & S_{n-1}(X,Y) & \longrightarrow & 0 \end{array}$$

We then define

$$Z_n(X, Y) = \ker(d_n\colon S_n(X,Y) \to S_{n-1}(X,Y)),$$

$$B_n(X, Y) = \operatorname{im}(d_{n+1}\colon S_{n+1}(X,Y) \to S_n(X,Y)),$$

$$H_n(X,Y) = Z_n(X,Y)/B_n(X,Y).$$

2.3. **Homology with coefficients.** If M is a $\mathbb{Z}$-module, we can define homology with coefficients in M by setting

$$S_n(X; M) = S_n(X) \otimes_{\mathbb{Z}} M,$$

$$Z_n(X; M) = \ker(d_n\colon S_n(X;M) \to S_{n-1}(X;M)),$$

$$B_n(X; M) = \operatorname{im}(d_{n+1}\colon S_{n+1}(X;M) \to S_n(X;M)),$$

$$H_n(X;M) = Z_n(X;M)/B_n(X;M).$$

Note that $- \otimes_{\mathbb{Z}} M$ and $H_n(-)$ do not commute. Hence taking homology with coefficients is a non-trivial procedure.

We can write concisely

$$H_n(X) = H_n(S_\bullet(X)),$$

$$H_n(X,Y) = H_n(S_\bullet(X,Y)),$$

$$H_n(X;M) = H_n(S_\bullet(X;M)),$$

$$H_n(X,Y;M) = H_n(S_\bullet(X,Y;M)).$$

2.4. **Simplicial homology.**

Definition 2.7. An *abstract simplicial complex* is a pair $K = (V, S)$ where V is a set and S is a set of finite nonempty subsets of V such that

(1) if $v \in V$ then $\{v\} \in S$,
(2) if $\emptyset \neq \tau \subseteq \sigma \in S$ then $\tau \in S$.

If $\sigma \in S$ then $\dim \sigma = |\sigma| - 1$.

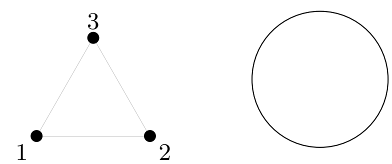
Definition 2.8. A *geometric realization* $|K|$ of $K = (V, S)$ is

$$|K| = \coprod_{\sigma \in S} \Delta_\sigma \Big/ \sim$$

where $\Delta_\sigma = \Delta_{\dim \sigma} = \Delta_{|\sigma|-1}$ with vertices labeled with $\langle v \rangle$, $v \in \sigma$, and if $\tau \subseteq \sigma$ then we have a linear map $f\colon \Delta_\tau \to \Delta_\sigma$ given by $f(\langle v \rangle) = \langle v \rangle$, then

$$\Delta_\tau \ni x \sim f(x) \in \Delta_\sigma.$$

Example 2.9. If $V = \{1, 2, 3\}$ and $S = \{\{1,2\}, \{2,3\}, \{1,3\}, \{1\}, \{2\}, \{3\}\}$, then $|K|$ is a triangle with vertices $1, 2, 3$ which is homeomorphic to a circle



We set

$$C_n(K) = \mathbb{Z}\text{-module with basis of all } n\text{-simplices.}$$

To define $d_n\colon C_n(K) \to C_{n-1}(K)$, we choose a total ordering on V and write

$$C_n(K) \ni \langle v_0 v_1 \dots v_n \rangle \text{ if } v_0 < v_1 < \dots < v_n.$$

Then we define

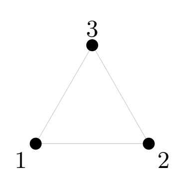
$$d_n(\langle v_0 v_1 \dots v_n \rangle) = \sum_{i=0}^{n} (-1)^n \langle v_0 \dots \hat{v}_i \dots v_n \rangle.$$

Once again, we set

$$Z_n(K) = \ker d_n$$
$$B_n(K) = \operatorname{im} d_{n+1}$$
$$H_{\text{simp},n}(K) = Z_n(K)/B_n(K).$$

Proposition 2.10. *For any n, we have that $H_{simp,n}(K) \cong H_{sing,n}(|K|)$.*

Example 2.11. We compute the homology of



so that $|K| = S^1$. We have that

$$C_0(K) = \mathbb{Z}\langle 1\rangle + \mathbb{Z}\langle 2\rangle + \mathbb{Z}\langle 3\rangle \cong \mathbb{Z}^3$$

$$C_1(K) = \mathbb{Z}\langle 12\rangle + \mathbb{Z}\langle 13\rangle + \mathbb{Z}\langle 23\rangle \cong \mathbb{Z}^3$$

$$d_1 : \begin{cases} \langle 12\rangle & \mapsto & \langle 2\rangle - \langle 1\rangle \\ \langle 13\rangle & \mapsto & \langle 3\rangle - \langle 1\rangle \\ \langle 23\rangle & \mapsto & \langle 3\rangle - \langle 2\rangle \end{cases}$$

so

$$Z_1(K) = \mathbb{Z}(\langle 13\rangle - \langle 23\rangle - \langle 12\rangle)$$

$$B_0(K) = \{\langle a\langle 1\rangle + b\langle 2\rangle + c\langle 3\rangle \mid a+b+c=0\} \cong \mathbb{Z}^2$$

$$B_1(K) = 0$$

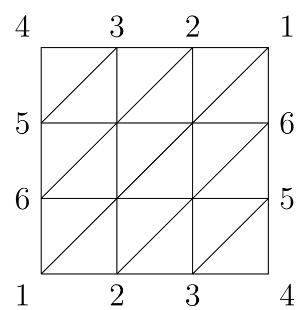
$$Z_0(K) = \mathbb{Z}^3$$

and hence

$$H_1(K) \cong \mathbb{Z},$$

$$H_0(K) \cong \mathbb{Z}.$$

Example 2.12. Recall that $\mathbb{P}^2(\mathbb{R})$ can be thought of as a square with opposite identifications on opposite edges, so $\mathbb{P}^2(\mathbb{R}) = |K|$ where K is the following simplicial complex



Then we have that

$$C_2 = \mathbb{Z}^{18},\ C_1 = \mathbb{Z}^{27},\ C_0 = \mathbb{Z}^{10}$$

and the sequence

$$0 \to \mathbb{Z}^{18} \to \mathbb{Z}^{27} \to \mathbb{Z}^{10} \to 0$$

has $\ker d_2 = 0$ so $H_2(K) = 0$.

However, the map

$$d_2 \colon C_2(K; \mathbb{Z}/2) \to C_1(K, \mathbb{Z}/2)$$

has

$$d_2 \left(\begin{array}{c} \text{sum of all 2-simplices} \\ \text{with right orientation} \end{array} \right) = 2(\langle 12\rangle + \langle 23\rangle + \langle 34\rangle + \langle 45\rangle + \langle 56\rangle - \langle 16\rangle) = 0$$

so

$$H_2(K; \mathbb{Z}/2) = \mathbb{Z}/2.$$

Hence indeed homology with coefficients is a non-trivial construction.

2.5. **Functoriality.** For singular homology, if $f\colon X \to Y$ is continuous , we get an induced map

$$S_n(f) = f_*\colon S_n(X) \to S_n(Y)\colon \sigma \mapsto f \circ \sigma.$$

Moreover, the square

$$\begin{tikzcd} S_n(X) \arrow[r, "f_*"] \arrow[d, "d"] & S_n(Y) \arrow[d, "d"] \\ S_{n-1}(X) \arrow[r, "f_*"] & S_{n-1}(Y) \end{tikzcd}$$

commutes, and hence

$$f_*(B_n(X)) \subseteq B_n(Y),$$
$$f_*(Z_n(X)) \subseteq Z_n(Y),$$

so f_* induces

$$H_n(f) = f_*\colon H_n(X) = Z_n(X)/B_n(X) \to H_n(Y).$$

Altogether, we have a functor

$$H_n\colon \operatorname{Top} \to \operatorname{Ab}.$$

2.6. **Cohomology.** Let M be a $\mathbb{Z}$-module and let

$$S^n(X; M) = \operatorname{Hom}_{\mathbb{Z}}(S_n(X), M).$$

We have the following commutative triangle

$$\begin{tikzcd} S_n(X) \arrow[r] & M \\ S_{n+1}(X) \arrow[u, "d_{n+1}"] \arrow[ur] & \end{tikzcd}$$

which gives a map

$$\delta^n\colon S^n(X; M) \to S^{n+1}(X; M).$$

Concisely, we apply the functor $\operatorname{Hom}_{\mathbb{Z}}(-, M)$ to both $S_n(X)$ and d_n and write

$$\delta^n = \operatorname{Hom}(d_{n+1}, M).$$

We then get the following *cochain complex*

$$\cdots \longrightarrow S^{n-1}(X; M) \xrightarrow{\delta^{n-1}} S^n(X; M) \xrightarrow{\delta^n} S^{n+1}(X; M) \xrightarrow{\delta^{n+1}} \cdots$$

We then define

$$\begin{aligned} Z^n(X; M) &= \ker \delta^n, && \text{the } \textit{cocycles}, \\ B^n(X; M) &= \operatorname{im} \delta^{n-1}, && \text{the } \textit{coboundaries}, \\ H^n(X; M) &= Z^n(X; M)/B^n(X; M), && \text{the cohomology group.} \end{aligned}$$

We admit the convention

$$S^n(X) = S^n(X; \mathbb{Z}), \quad H^n(X) = H^n(X; \mathbb{Z}).$$

Example 2.13. For $X = \mathbb{P}^2(\mathbb{R})$ we have that

$$\begin{array}{ll} H_0 = \mathbb{Z}, & H^0 = \mathbb{Z}, \\ H_1 = \mathbb{Z}/2, & H^1 = 0, \\ H_2 = 0, & H^2 = \mathbb{Z}/2. \end{array}$$

Suppose now X is a smooth manifold and let $\Omega^n(X)$ be the set of n-forms on X, so an element of $\Omega^n(X)$ is $gdf_1 \wedge df_2 \wedge \ldots \wedge df_n$. We have maps

$$d\colon \Omega^n(X) \to \Omega^{n+1}(X)$$

with $d^2 = 0$, which give a cochain complex

$$\cdots \longrightarrow \Omega^{n-1}(X) \xrightarrow{d^{n-1}} \Omega^n(X) \xrightarrow{d^n} \Omega^{n+1}(X) \longrightarrow \cdots$$

called the *de Rham complex*. Its cohomology is called the *de Rham cohomology*

$$H^\bullet_{\mathrm{DR}}(X) = H^\bullet(\Omega^\bullet(X)).$$

Then

$$\Omega^0(X) = \{\text{smooth functions } X \to \mathbb{R}\}.$$

Example 2.14. For $X = S^1$, we have that $2xdx+2ydy = d(x^2+y^2) = d1 = 0$ so $xdx = -ydy$. But then

$$\omega = \frac{dx}{y} = -\frac{dy}{x}$$

but $\omega \neq df$ for any f which shows $H^1_{\mathrm{DR}}(S^1) \neq 0$.

3. Homological algebra

In this chapter, we are in the category of right R-modules, mod-R.

Definition 3.1. A *chain complex* is a diagram

$$C_\bullet: \qquad \cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

with $d_n d_{n+1} = 0$ for all n.

We say $C_\bullet$ is *bounded from below (above)* if there exists a such that for any $n < a$ ($n > a$), $C_n = 0$. Moreover, $C_\bullet$ is *bounded* if it is bounded both from below and from above.

Example 3.2. In the topological examples we have seen for homology, all sequences were bounded from below. For a finite simplicial complex, the chain complex is moreover bounded.

Definition 3.3. We define

$$B_n(C_\bullet) = \operatorname{im} d_{n+1}, \quad Z_n(C_\bullet) = \ker d_n$$

and the nth homology group of $C_\bullet$ to be

$$H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet).$$

Definition 3.4. A *chain complex map* $C_\bullet \to D_\bullet$ is a collection of module homomorphisms $u_n\colon C_n \to D_n$ for all $n \in \mathbb{Z}$ such that $u_{n-1}d_n = d_n u_n$, so the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "u_n"] & C_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. We sometimes write more concisely that $ud = du$.

For a chain complex map $u_n(B_n(C_\bullet)) \subseteq B_n(D_\bullet)$ and $u_n(Z_n(C_\bullet)) \to Z_n(D_\bullet)$ and hence u_n induces a map

$$u_{*,n} \colon H_n(C_\bullet) \to H_n(D_\bullet).$$

Definition 3.5. We define $\mathrm{Ch}(\text{mod-}R)$ to be the category of chain complexes (of right R-modules, but we could apply this construction to more general categories) with chain complex maps as morphisms. This is an abelian category.

We moreover have a homology functor

$$H_n(-) \colon \ \mathrm{Ch}(\text{mod-}R) \to \text{mod-}R.$$

Example 3.6. If X, Y are topological spaces and $f \colon X \to Y$ is continuous, the maps

$$f_{*,n} \colon S_n(X) \to S_n(Y) \colon \sigma \mapsto f \circ \sigma$$

form a chain complex map

$$f_* \colon S_\bullet(X) \to S_\bullet(Y).$$

Then f_* induces

$$f_{*,n} \colon H_n(X) \to H_n(Y)$$

and in fact we have a functor

$$H_n(-) \colon \ \mathrm{Top} \to \mathrm{Ab}\,.$$

Theorem 3.7. *If we have a short exact sequence of chain complexes*

$$0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$$

(or, equivalently, $0 \to C_n \to D_n \to E_n \to 0$ is exact for any n). Then we have a long exact sequence

$$\cdots \longrightarrow H_{n+1}(E_\bullet) \xrightarrow{\partial} H_n(C_\bullet) \longrightarrow H_n(D_\bullet) \longrightarrow H_n(E_\bullet) \xrightarrow{\partial} H_{n-1}(C_\bullet) \longrightarrow \cdots$$

Proof. We apply Snake Lemma 1.43 to get the red maps below

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r, red] & Z_n(C) \arrow[r, red] \arrow[d] & Z_n(D) \arrow[r, red] \arrow[d] & Z_n(E) \arrow[d] \arrow[dddlll, red, rounded corners] & \\
0 \arrow[r] & C_n \arrow[r] \arrow[d, "d"] & D_n \arrow[r] \arrow[d, "d"] & E_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & D_{n-1} \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & 0 \\
 & C_{n-1}/B_{n-1}(C) \arrow[r, red] \arrow[d] & D_{n-1}/B_{n-1}(D) \arrow[r, red] \arrow[d] & E_{n-1}/B_{n-1}(D) \arrow[r, red] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then using the exactness of the red sequence above, we get the following commutative diagram, and apply Snake Lemma 1.43 again to get the red maps

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H_n(C) \arrow[r, red] \arrow[d] & H_n(D) \arrow[r, red] \arrow[d] & D_n(E) \arrow[d] \arrow[ddll, red, rounded corners] & \\
 & C_n/B_n(C) \arrow[r] \arrow[d, "d"] & D_n/B_n(D) \arrow[r] \arrow[d, "d"] & E_n/B_n(E) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(C) \arrow[r] \arrow[d] & Z_{n-1}(D) \arrow[r] \arrow[d] & Z_{n-1}(E) \arrow[d] & \\
 & H_{n-1}(C) \arrow[r, red] \arrow[d] & H_{n-1}(D) \arrow[r, red] \arrow[d] & H_{n-1}(D) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This completes the proof. □

Definition 3.8. A *cochain complex* is a diagram

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \longrightarrow \cdots$$

where $d^{n+1}d^n = 0$ for all n.

If $C_\bullet$ is a chain complex, setting $C^n = C_{-n}$ with $d^n = d_{-n}$ makes $C^\bullet$, $d^\bullet$ a cochain complex.

Definition 3.9. For a chain complex $C_\bullet$ and $p \in \mathbb{Z}$, we define the shift of $C_\bullet$ by p to be the chain complex $C[p]$, $d[p]$ with

$$C[p]_n = C_{p+n}, \quad d[p]_n = (-1)^p d_{p+n}.$$

Analogously, for a cochain complex $C^\bullet$, we define the shift of $C^\bullet$ by p to be the cochain complex

$$C[p]^n = C^{n-p}, \quad d[p]^n = (-1)^p d^{n-p}.$$

This notation is used for example in the following context: instead of saying that for any n, the sequence

$$0 \to Z_n(C_\bullet) \to C_n \to B_{n-1}(C_\bullet) \to 0$$

is exact, we can say that the sequence of chain complexes

$$0 \to Z_\bullet(C_\bullet) \to C_\bullet \to B[-1]_\bullet(C_\bullet) \to 0$$

is exact.

We get the following analogous statement to Theorem 3.7 for cohomology.

Theorem 3.10. *If $0 \to C^\bullet \to D^\bullet \to E^\bullet \to 0$ is exact, then we have a long exact sequence*

$$\cdots \to H^{n-1}(E^\bullet) \to H^n(C^\bullet) \to H^n(D^\bullet) \to H^n(E^\bullet) \to H^{n+1}(C^\bullet) \to \cdots$$

Example 3.11. If X is a topological space and $Y \subseteq X$ is a subset, we have a short exact sequence

$$0 \to S_\bullet(Y) \to S_\bullet(X) \to S_\bullet(X,Y) \to 0$$

and by Theorem 3.7, we get the *long exact sequence of a pair*

$$\cdots \to H_{n+1}(X,Y) \to H_n(Y) \to H_n(X) \to H_n(X,Y) \to \cdots$$

In some cases, this allows us to calculate homology groups of topological spaces.

Example 3.12. Let X be a manifold and $\Omega^n(X)$ be the n-forms on X. If $X = U \cup V$ for $U, V \subseteq X$ open, we have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^n(X) & \longrightarrow & \Omega^n(U) \oplus \Omega^n(V) & \longrightarrow & \Omega^n(U \cap V) & \longrightarrow & 0 \\ & & \omega & \longmapsto & (\omega_{|U}, \omega_{|V}) & & & & \\ & & & & (\omega_1, \omega_2) & \longmapsto & (\omega_1)_{|U\cap V} - (\omega_2)_{|U\cap V} & & \end{array}$$

Applying Theorem 3.10 to this short exact sequence, we get the *Mayer-Vietoris sequence* for de Rham cohomology

$$\cdots \to H^n_{\mathrm{DR}}(X) \to H^n_{\mathrm{DR}}(U) \oplus H^n_{\mathrm{DR}}(V) \to H^n_{\mathrm{DR}}(U \cap V) \to H^{n-1}_{\mathrm{DR}}(X) \to \cdots$$

3.1. **Homotopy.** We will define a homotopy of maps of chain complexes, using the motivation of homotopy from topology.

Let X, Y be topological spaces and $f, g\colon X \to Y$ be continuous maps.

Definition 3.13. A *homotopy* from f to g is a continuous function $h\colon [0,1] \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$. We then say f and g are *homotopic* and being homotopic is an equivalence relation.

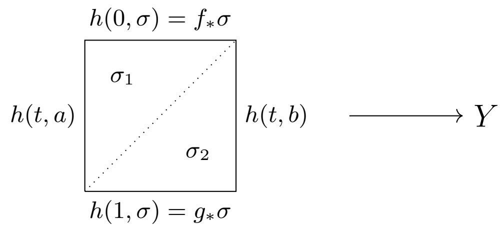
If $\sigma \in X$ is a 0-simplex, then

$$h(t,\sigma)\colon \underbrace{[0,1]}_{\cong\Delta_1} \to Y,$$

so $h(t,\sigma)$ is a 1-simplex in Y. This gives a group homomorphism

$$s_0\colon S_0(X) \to S_1(X).$$

If σ is a 1-simplex in X, then $h(t,\sigma)\colon \Delta_1 \times [0,1] \to Y$, so if $\Delta_1 = [a,b]$ whence $d\Delta_1 = b - a$ we get a map



Then there exist two 2-simplices $\sigma_1 + \sigma_2$ such that

$$d(\sigma_1+\sigma_2) = g_*\sigma - h(t,b) - f_*\sigma + h(t,a) = (g_* - f_*)(\sigma) + s_0(a) - s_0(b) = (g_* - f_*)(\sigma) - s_0(d\sigma).$$

Define $s_1\colon S_1(X) \to S_2(Y)$ by setting

$$s_1(\sigma) = \sigma_1 + \sigma_2 \in S_2(Y).$$

Then $ds_1(\sigma) = (g_* - f_*)(\sigma) - s_0(d\sigma)$, so

$$ds_1 + s_0 d = g_* - f_*.$$

In general, there exists $s_n\colon S_n(X) \to S_{n+1}(Y)$ such that

$$sd + sd = g_* - f_*.$$

Definition 3.14. Topological spaces X and Y are *homotopy equivalent* if there exist continuous $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to id_X and fg is homotopic to id_Y.

Definition 3.15. We say X is *contractible* if X is homotopy equivalent to a point.

Example 3.16. The Euclidean space $\mathbb{R}^n$ is contractible: for

$$f\colon \{0\} \to \mathbb{R}^n,$$
$$g\colon \mathbb{R}^n \to \{0\},$$

we have $gf = \mathrm{id}_{\{0\}}$ and

$$fg\colon \mathbb{R}^n \to \mathbb{R}^n$$

is constant equal to 0 and a homotopy from fg to $\mathrm{id}_{\mathbb{R}^n}$ is

$$h(t,x) = tx.$$

Definition 3.17. Suppose $C_\bullet$, $D_\bullet$ are chain complexes and $f, g\colon C_\bullet \to D_\bullet$ are chain complex maps. A *homotopy* from f to g is a collection of functions $s_n\colon C_n \to D_{n+1}$ for $n \in \mathbb{Z}$ such that

$$d_{n+1}s_n + s_n d_n = g_n - f_n \quad \text{(or, concisely, } ds + sd = g - f\text{)},$$

which can be represented by the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "g_{n+1}"'] \arrow[d, shift left] & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "g_n"'] \arrow[d, shift left, "f_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] \arrow[d, shift right] \arrow[d, shift left, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}"] & D_n \arrow[r, "d_n"] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Recall that f, g induce $f_{*,n}, g_{*,n} \colon H_n(C_\bullet) \to H_n(D_\bullet)$. Explicitly, if $a \in H_n(C_\bullet)$, say $a = x + B_n(C)$ for $x \in Z_n(C)$, then

$$f_*(a) = f(x) + B_n(D) \in H_n(D),$$
$$g_*(a) = g(x) + B_n(D) \in H_n(D).$$

If f, g are homotopic, then have that

$$g_*(a) - f_*(a) = g_n(x) - f_n(x) + B_n(D) = d_{n+1}s_n(x) + s_{n-1}d_n(X) \in B_n(D),$$

so $g_*(a) = f_*(a)$. Hence if f, g are homotpic, then $f_* = g_*$.

Definition 3.18. We say $C_\bullet$, $D_\bullet$ are homotopy equivalent if there exist $f \colon C_\bullet \to D_\bullet$ and $g \colon D_\bullet \to C_\bullet$ such that fg and gf are homotopic to the appropriate identity.

By the above, if $C_\bullet$, $D_\bullet$ are homotopy equivalent, then for any n, $f_{*,n} \colon H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism because

$$g_*f_* = (gf)_* = \mathrm{id}_* = \mathrm{id},$$
$$f_*g_* = (fg)_* = \mathrm{id}_* = \mathrm{id}.$$

Example 3.19. If X is contractible, then

$$H_n(X) = H_n(\{*\}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$H_n(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, $\mathbb{R}^m \setminus \{0\}$ is homotopy equivalent to S^{m-1}, the $(m-1)$-dimensional sphere.

3.2. **Split exact sequences.**

Proposition 3.20. *Suppose we have a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Then the following are equivalent

(1) *there exists* $f' \colon B \to A$ *with* $f'f = \mathrm{id}_A$
(2) *there exists* $g' \colon C \to B$ *with* $gg' = \mathrm{id}_C$
(3) *there exists a submodule* $C' \subseteq B$ *with* $B \cong A \oplus C' \cong f(A) \oplus C'$.

Proof. We only show (1) implies (3). The rest are similar and left as exercises. Take $C' = \ker f'$. For $b \in B$, we have

$$b = \underbrace{ff'(b)}_{\in f(A)} + \underbrace{(b - ff'(b))}_{\in C'},$$

so $f'(b - ff'(b)) = f'(b) - f'ff'(b) = 0$. $\square$

Definition 3.21. A short exact sequence is *split* if any of the above equivalent conditions hold.

Example 3.22. The following short exact sequence is not split

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p^2 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0 \\
 & & 1+(p) & \longrightarrow & p+(p^2) & & & & \\
 & & & & 1+(p^2) & \longrightarrow & 1+(p) & &
\end{array}$$

Definition 3.23. A chain complex $C_\bullet$ is *split* if there exists $s_n \colon C_n \to C_{n+1}$ for any n such that $d_n s_{n-1} d_n = d_n$ for all n (i.e. $dsd = d$).

Suppose $C_\bullet$ is split. Let $b \in B_{n-1}$. Then $b = d(a)$ for some $a \in C_n$ and $ds(b) = dsd(a) = d(a) = b$, so $ds = \mathrm{id}_{B_{n-1}}$.

Then

$$0 \to B_n \to Z_n \to H_n \to 0$$

is also split: if $b \in B_n$ then $b = d(a)$ for $a \in C_{n+1}$ and $ds(b) = dsd(a) = d(a) = b$, so $ds_{|B_n} = \mathrm{id}_{B_n}$. Hence

$$C_n \cong Z_n \oplus B_{n-1} \cong B_n \oplus H_n \oplus B_{n-1}$$

and we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\cong"] & C_n \arrow[r] \arrow[d, "\cong"] & C_{n-1} \arrow[r] \arrow[d, "\cong"] & \cdots \\
 & B_n \arrow[ddr, "\cong"] & B_{n-1} \arrow[ddr, "\cong"] & B_{n-2} & \\
 & \oplus & \oplus & \oplus & \\
 & B_{n+1} & B_n & B_{n-1} & \\
 & \oplus & \oplus & \oplus & \\
 & H_{n+1} \arrow[r, "0"] & H_n \arrow[r, "0"] & H_{n-1} &
\end{tikzcd}$$

The lowest level has *maximal homology* in the sense that all boundary maps are zero.

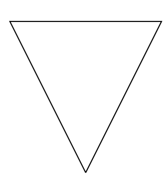
3.3. Mapping cone.

A *cone* $C(X)$ over X is

$$C(X) = [0,1] \times X/\sim,$$

where $(0,x) \sim (0,y)$ for all $x, y \in X$.

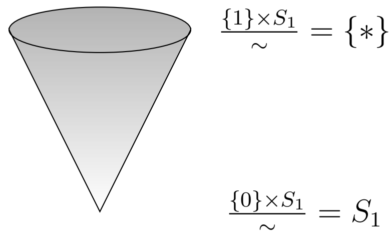
Example 3.24. For $X = [0,1]$, $C(X)$ is a 1-simplex



$$\frac{\{1\}\times[0,1]}{\sim} = \{*\}$$

$$\frac{\{0\}\times[0,1]}{\sim} = [0,1]$$

For $X = S^1$, $C(X)$ is an actual cone, justifying the name

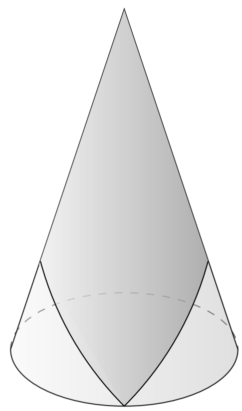


Definition 3.25. Suppose $f\colon X \to Y$ is a continuous map. The *mapping cone* is

$$C(f) = C(X) \amalg Y/\sim$$

where $C(x) \ni (1, x) \sim f(x) \in Y$.

Examples 3.26. Let $f\colon [0, 2\pi] \to S^1$, $f(t) = (\cos t, \sin t)$. Then $C(f)$ is a 1-simplex with a circle attached to $[0, 1]$ via f:



Let $f\colon S^1 \to S^1$ be the map $z \mapsto z^2$. Then $C(f) = \mathbb{P}^2(\mathbb{R})$, because it is a hemisphere with the antipodal identification on the boundary circle.

We generalize the topological notion of a cone to a purely algebraic one.

Definition 3.27. Let $f\colon B_\bullet \to C_\bullet$ be a chain map. We define a new chain complex cone(f) by setting

$$\operatorname{cone}(f)_n = B_{n-1} \oplus C_n$$
$$d(b, c) = (-d(b), d(c) - f(b))$$

or in matrix form

$$d_{\text{cone}} = \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}.$$

This is indeed a chain complex:

$$d^2_{\text{cone}} = \begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} = \begin{pmatrix} d_B^2 & 0 \\ fd_B - d_C f & d_C^2 \end{pmatrix} = 0.$$

We have an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{\begin{pmatrix} 0 \\ \mathrm{id}_C \end{pmatrix}} & \mathrm{cone}(f) & \xrightarrow{(-\mathrm{id}_B\ \ 0)} & B[-1] & \longrightarrow & 0 \\
 & & c & \longrightarrow & (0,c) & & & & \\
 & & & & (b,c) & \longrightarrow & -b & &
\end{array}$$

Since $H_n(B[-1]) = H_{n-1}(B)$, we have the long exact sequence (Theorem 3.7)

$$\cdots \longrightarrow H_n(B) \xrightarrow{\partial} H_n(C) \longrightarrow H_n(\mathrm{cone}(f)) \longrightarrow H_{n-1}(B) \xrightarrow{\partial} H_{n-1}(C) \longrightarrow \cdots$$

Lemma 3.28. *The boundary map ∂ above is f_*.*

Proof. To prove this statement, we trace through the proof of Theorem 3.7. Let $b \in B[-1]_{n+1}$ be a cycle. This lifts to $(-b, 0) \in \mathrm{cone}(f)_{n+1}$. Then

$$d(-b, 0) = (0, f(b)) \in \mathrm{cone}(f)_n.$$

This lifts to $f(b) \in C_n$, and actually $f(b) \in Z_n$. Hence

$$\partial[b] = [f(b)] = f_*[b],$$

completing the proof. $\square$

Definition 3.29. A chain map $f\colon C_\bullet \to D_\bullet$ is a *quasi-isomorphism* if

$$f_*\colon H_n(B_\bullet) \to H_n(C_\bullet)$$

is an isomorphism for all n.

Corollary 3.30. *A chain map $f\colon B_\bullet \to C_\bullet$ is a quasi-isomorphism if and only if* $\mathrm{cone}(f)$ *is exact (i.e. $H_n(\mathrm{cone}(f)) = 0$ for all n).*

Suppose

$$0 \longrightarrow B_\bullet \xrightarrow{f} C_\bullet \longrightarrow D_\bullet \longrightarrow 0$$

is a short exact sequence. Then we have two long exact sequences and in fact they are isomorphic by the Five Lemma 1.44 applied to the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(B) \arrow[r, "f_*"] \arrow[d, "="] & H_n(C) \arrow[r] \arrow[d, "="] & H_n(D) \arrow[r] \arrow[d, dotted, "\cong"] & H_{n-1}(B) \arrow[r, "f_*"] \arrow[d, "="] & H_{n-1}(C) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "f_*"] & H_n(C) \arrow[r] & H_n(\mathrm{cone}(f)) \arrow[r] & H_{n-1}(B) \arrow[r, "f_*"] & H_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

4. Homological δ-functors

Definition 4.1. A function $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is *additive* if $\mathcal{F}\colon \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}B)$ is a group homomorphism.

Definition 4.2. An additive functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is *left-exact* if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$ we have that

$$0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C.$$

One can similarly define *right-exact* and analogous notions for contravariant functors.

Lemma 4.3. *If $\mathcal{F}$ is left exact and $0 \to A \to B \to C$ is exact, then*

$$0 \to \mathcal{F}A \to \mathcal{F}B \to \mathcal{F}C$$

is exact.

Proof. Let D be the image of $g\colon B \to C$. Then the following diagram has an exact row and column

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[dr] & D \arrow[r] \arrow[d] & 0 \\
 & & & C \arrow[d] & \\
 & & & C/D \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

and hence so does the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{F}A \arrow[r] & \mathcal{F}B \arrow[r] \arrow[dr] & \mathcal{F}D \arrow[d] \\
 & & & \mathcal{F}C \arrow[d] \\
 & & & \mathcal{F}C/D
\end{tikzcd}$$

We only have to show exactness at $\mathcal{F}B$. If a is in the kernel of $\mathcal{F} \to \mathcal{F}C$ then since $\mathcal{F}D \to \mathcal{F}C$ is a injective, a is in the kernel of $\mathcal{F}B \to \mathcal{F}D$ and hence a is in the image of $\mathcal{F}A \to \mathcal{F}B$. $\square$

Definition 4.4. A *homological δ-functor* $\mathcal{C} \to \mathcal{D}$ is a collection of additive functors

$$T_n \colon \mathcal{C} \to \mathcal{D}, \quad n \geq 0,$$

together with a morphism $\delta_n \colon T_n(C) \to T_{n-1}(A)$ for every short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that

(1) there exists a long exact sequence

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \xrightarrow{T_n(f)} T_n(B) \xrightarrow{T_n(g)} T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \longrightarrow \cdots$$

where we set $T_n(A) = 0$ for $n < 0$, so T_0 is right exact,

(2) for every morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f_A"] & B' \arrow[r] \arrow[d, "f_B"] & C' \arrow[r] \arrow[d, "f_C"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

we have a commuting square

$$\begin{tikzcd}
T_n(C') \arrow[r, "\delta_n"] \arrow[d, "T_n(f_C)"'] & T_{n-1}(A) \arrow[d, "T_{n-1}(f_A)"] \\
T_n(C) \arrow[r, "\delta_n"] & T_{n-1}(A)
\end{tikzcd}$$

Example 4.5. If $\mathcal{C}$ is an abelian category and $\mathrm{Ch}_{\geq 0}(\mathcal{C})$ is the category of non-negative chain complexes, then the homology functor

$$H_\bullet \colon \mathrm{Ch}_{\geq 0}(\mathcal{C}) \to \mathcal{C}$$

is a homological δ-functor.

Example 4.6. Let R be a ring and $r \in R$. Set

$$T_0 \colon R\text{-mod} \to \mathrm{Ab}, \quad T_0(M) = M/rM,$$

$$T_1(M) = [r]M = \{a \in M \mid ra = 0\},$$

$$T_n(M) = 0 \quad \text{for all } n > 1.$$

Then $T = \{T_n\}$ is a homological δ-functor. Applying Snake Lemma 1.43 to the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[dddll, out=0, in=180] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "r\cdot"] & B \arrow[r] \arrow[d, "r\cdot"] & C \arrow[r] \arrow[d, "r\cdot"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & A/rA \arrow[r] \arrow[d] & B/rB \arrow[r] \arrow[d] & C/rC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we obtain the long exact sequence

$$0 \to T_1(A) \to T_1(B) \to T_1(C) \to T_0(A) \to T_0(B) \to T_0(C) \to 0,$$

showing property (1) holds. Property (2) is trivial.

Definition 4.7. A morphism of homological δ-functors from S to T is a collection of natural transformations $\epsilon_n\colon S_n \to T_n$ for all $n \geq 0$ such that if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then

$$\begin{tikzcd}
S_n(C) \arrow[r, "\delta_n"] \arrow[d, "\epsilon_n(C)"] & S_{n-1}(A) \arrow[d, "\epsilon_{n-1}(A)"] \\
T_n(C) \arrow[r, "\delta_n"] & T_{n-1}(A)
\end{tikzcd}$$

commutes.

Definition 4.8. A homological δ-functor T is *universal* if for every δ-functor S and every natural transformation $\epsilon_0\colon S_0 \to T_0$, there exists a unique natural transformation $\epsilon_n\colon S_n \to T_n$, $n \geq 1$, such that $\epsilon\colon S \to T$ is a morphism of δ-functors.

Example 4.9. The homology functor $H_\bullet\colon \mathrm{Ch}_{\geq 0}(\mathcal{C}) \to \mathcal{C}$ is universal.

5. Projectives and left derived functors

Let $\mathcal{C}$ be an abelian category.

Definition 5.1. An element P is *projective* if for every epimorphism $g\colon B \to C$ and every morphism $\gamma\colon P \to C$, there exists $\beta\colon P \to B$ such that $g \circ \beta = j$

$$\begin{tikzcd} & P \arrow[dl, dotted, "\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, two heads, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

(Equivalently, if $B \to C \to 0$ is exact, then $\mathrm{Hom}_{\mathcal{C}}(P, B) \to \mathrm{Hom}_{\mathcal{C}}(P, C) \to 0$ is exact.)

The functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ is always left exact. For projective modules, this functor is moreover exact.

Lemma 5.2. *An element P is projective if and only if $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is right-exact (and hence exact).*

Proof. The 'if' implication is clear. For the 'only if' implication, suppose $0 \to A \to B \to C \to 0$ is exact. Then

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$$

is exact by left-exactness together with the projective property. $\square$

Example 5.3. In R-mod, free modules are projective.

Lemma 5.4. *If $0 \to A \to B \to P \to 0$ is exact and P is projective, then this short exact sequence splits.*

Proof. For the map $\mathrm{id}\colon P \to P$, there is a unique map $P \to B$ as above, showing that the sequence splits. $\square$

Lemma 5.5. *A direct summand of a projective object is projective.*

Proof. Suppose $P = P_1 \oplus P_2$ and P is projective. Then since

$$\mathrm{Hom}_{\mathcal{C}}(P, -) = \mathrm{Hom}_{\mathcal{C}}(P_1, -) \oplus \mathrm{Hom}_{\mathcal{C}}(P_2, -)$$

is exact, both $\mathrm{Hom}_{\mathcal{C}}(P_1, -)$ and $\mathrm{Hom}_{\mathcal{C}}(P_2, -)$ are both exact. Indeed, if one of them was not, then the counterexample would also work for the direct product. $\square$

Example 5.6. For R equal to $\mathbb{Z}$ or a field or a division ring, an R-module is free if and only if it is projective.

Theorem 5.7 (Quillen–Suslin)**.** *Projective modules over $F[x_1, \ldots, x_n]$ where F is a field are free.*

Examples 5.8. Suppose R_1, R_2 are nonzero rings with 1. If $R = R_1 \times R_2$ then R_1 and R_2 are projective R-modules.

Lemma 5.9. *In R-mod, P is projective if and only if it is a direct summand of a free module.*

Proof. The 'if' implication is clear from Lemma 5.5. For the converse, if P is a projective module, let $F(P)$ be the free module with generators $[a]$ for $a \in P$. Define

$$f\colon F(P) \twoheadrightarrow P$$

by $f([a]) = a$. The sequence

$$0 \to \ker f \to f(P) \to P \to 0$$

is exact, so it splits by Lemma 5.4. Hence $f(P) \cong \ker f \oplus P$. □

Definition 5.10. An R-module M is *indecomposable* if $M \cong M_1 \oplus M_2$ implies $M_1 = 0$ or $M_2 = 0$.

Moreover, M is *simple* if for a submodule $M_1 \subseteq M$ we have $M_1 = 0$ or $M_1 = M$.

Theorem 5.11 (Krull-Schmidt)**.** *Let F be a field and R be a finite-dimensional F-algebra. If M is a finite-dimensional R-module, then $M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_r$, where $M_1, \ldots, M_r$ are indecomposable and if $M \cong M_1' \oplus M_2' \oplus \cdots \oplus M_s'$ with $M_1', \ldots, M_s'$ indecomposable, then $r = s$ and, after reordering, $M_i \cong M_i'$.*

Example 5.12. Taking $M = R$ above, we obtain

$$R = P_1 \oplus P_2 \oplus \cdots \oplus P_r$$

where $P_1, P_2, \ldots, P_r$ are projective indecomposables. If P is any projective finite-dimensional R-modules, then for some M

$$P \oplus M \cong R^d \cong R^d \cong P_1^d \oplus P_2^d \oplus \cdots \oplus P_r^d,$$

so $P_1^{d_1} \oplus P_2^{d_2} \oplus \cdots \oplus P_r^{d_r}$.

We will write R-fdmod for the category of finite-dimensional R-modules.

Example 5.13. Let $R = M_n(F)$ be the ring of $n \times n$ matrices over F. Then

$$R = \underbrace{P \oplus P \oplus \cdots \oplus P}_{n}$$

where

$$P = F^n \text{ are the } i\text{th columns.}$$

The only indecomposables are actually P.

Example 5.14. Let $R \subseteq M_n(F)$ be the subset of upper-triangular matrices. Then

$$R = P_1 \oplus P_2 \oplus \cdots \oplus P_n$$

where

$$P_i = i\text{th columns} = \left\{ \begin{pmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Then the short exact sequence

$$0 \to P_1 \to P_2 \to S_2 \to 0$$

is non-split. Hence S_2 is not projective.

Definition 5.15. An abelian category $\mathcal{C}$ has *enough projectives* if every $M \in \mathrm{Obj}(\mathcal{C})$ there exists an epimorphism $f \colon P \to M$ where P is projective.

Example 5.16. In the category of finite abelian groups, there are no projectives except 0. Indeed, the exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/2n \to \mathbb{Z}/n \to 0$$

is non-split, so $\mathbb{Z}/n$ is not projective. But every other non-zero finite abelian group has direct summand $\mathbb{Z}/n$.

Example 5.17. If R is a ring with 1, then R-mod has enough projectives. If M is a module, then we have a map $F(M) \twoheadrightarrow M$ as described above.

If I_1, I_2 are ideals in a commutative ring with 1, then we have an exact sequence

$$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to I_1 + I_2 \to 0.$$

Example 5.18. Let

$$R = \frac{\mathbb{C}[x, y, z]}{(xy - z^2 - 1)}$$

and $I_1 = (z - 1, x)$, $I_2 = (z + 1, x)$. These are ideals which are not principal but they are indecomposable. Then using the equation $xy = z^2 - 1 = (z - 1)(z + 1)$ we obtain

$$I_1 \cap I_2 = (x).$$

Since the short exact sequence

$$0 \to (x) \to I_1 \oplus I_2 \to R \to 0$$

splits (R is projective), we have that

$$I_1 \oplus I_2 = R \oplus (x) \cong R^2$$

and so I_1, I_2 are projective. Hence we obtained two decompositions of R^2 into indecomposables.

Example 5.19. Take the equation $y^2 = x^3 - x$, so

$$R = \frac{\mathbb{C}[x, y]}{(y^2 - x^3 + x)}.$$

Then the maximal ideals of R are projective.

Definition 5.20. A *resolution* of $M \in \mathrm{Obj}(\mathcal{C})$ is a nonnegative complex $P_\bullet$ together with a morphism $\epsilon\colon P_0 \to M$ such that

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact. If all P are projective, this is a *projective resolution*; if all P are free, this is a *free resolution*, etc.

In this case,

$$H_n(P_\bullet) = \begin{cases} M & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proposition 5.21. *If $\mathcal{C}$ has enough projectives, then every object has a projective resolution.*

Proof. Take any $M \in \mathrm{Obj}(\mathcal{C})$. There is an epi $P_0 \twoheadrightarrow M$ from a projective P_0 and, taking the kernel $K_0 \to P_0$, we have a projective P_1 with an epi $P_1 \to K_0$. We take its kernel $K_1 \to P_1$ and again get a projective $P_2 \twoheadrightarrow K_1$. This way, we get that the diagram

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & & & \\
 & & K_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & P_2 \arrow[ur, two heads] \arrow[rr] & & P_1 \arrow[rr] \arrow[dr, two heads] & & P_0 \arrow[r, two heads] & M \arrow[r, two heads] & 0 \\
 & & & & K_0 \arrow[ur] \arrow[dr] & & & \\
 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

commutes. Continuing, this gives a projective resolution of M. ☐

Let $\mathcal{C}$ be an abelian category with enough projectives, $\mathcal{D}$ be an abelian category, and $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a right exact functor. We present the idea behind derived functors first. Given $M \in \mathrm{Obj}(\mathcal{C})$, choose projective resolution of M:

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Apply $\mathcal{F}$ to P to obtain

$$\cdots \to \mathcal{F}P_3 \to \mathcal{F}P_2 \to \mathcal{F}P_1 \to \mathcal{F}P_0 \to 0.$$

Define

$$L_i(\mathcal{F}M) = H_i(\mathcal{F}(P_\bullet)).$$

Note that $L_0\mathcal{F}(M) = H_0(\mathcal{F}(P_\bullet)) = \mathcal{F}(M)$, since

$$\mathcal{F}(P_1) \to \mathcal{F}(P_0) \to \mathcal{F}(M) \to 0$$

is exact.

Questions: Is $L_i\mathcal{F}(M)$ well-defined? Is $L_i\mathcal{F}$ a functor?

Theorem 5.22 (Comparison Theorem)**.** *Suppose $P_\bullet \to M$ is a projective resolution and $Q_\bullet \to N$ is any resolution, and $f'\colon M \to N$ be a morphism. Then there exists a chain map $f\colon P_\bullet \to Q_\bullet$ such that*

$$\begin{tikzcd}
P_\bullet \arrow[r] \arrow[d, "f"] & M \arrow[d, "f'"] \\
Q_\bullet \arrow[r] & N
\end{tikzcd}$$

commutes. Moreover, f is unique up to homotopy.

Proof. We construct the family $f_0, f_1, \ldots$, making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dotted, "f_2"] \arrow[dr, "\gamma_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dotted, "f_1"] \arrow[dr, "\gamma_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dotted, "f_0"] \arrow[dr, "\gamma_0"] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "d_2"] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, two heads, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

commute, step by step as follows. The map $\gamma_0 = f'\epsilon\colon P_0 \to N$ is surjective, so it lifts to $f_0\colon P_0 \to Q_0$. Next,

$$\gamma_1 = f_0 d_1\colon P_1 \twoheadrightarrow \operatorname{im} d_1 = \ker \eta$$

lifts to $f_1\colon P_1 \to Q_1$, since P_1 is projective. In the next step,

$$\gamma_2 = f_1 d\colon P_2 \twoheadrightarrow \operatorname{im} d_2 = \ker d_1$$

lifts to $f_2\colon P_2 \to Q_2$, since P_2 is projective. We continue this way to construct a chain map $f\colon P_\bullet \to Q_\bullet$ such that the approporiate diagram commutes.

For uniqueness, suppose $g\colon P_\bullet \to Q_\bullet$ is another chain map such that

$$\begin{tikzcd}
P_\bullet \arrow[r] \arrow[d, "g"] & M \arrow[d, "f'"] \\
Q_\bullet \arrow[r] & N
\end{tikzcd}$$

commutes. We can replace the pair f, g by the pair $h = f - g$, 0: if we construct a suitable collection of maps s_i for h and 0, then $ds + sd = h - 0 = f - g$, as required. We let $s_0 = 0\colon M \to Q_0$. The map h_0 factors through $\ker \eta$, since $\eta h_0 = 0$, and since the map $Q_1 \to \ker \eta$ is surjective and P_0 is projective, there is a unique map $s_1\colon P_0 \to Q_1$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[rr] \arrow[d, "h_1"] & & P_0 \arrow[r, "\epsilon"] \arrow[d, "h_0"] \arrow[dl] \arrow[dll, dotted, "s_1"'] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "0"] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & \ker \eta \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

commute. This shows $d_1 s_1 + s_0 \epsilon = d_1 s_1 = h_0$. Now, $d_1(s_1 d_1 - h_1) = h_0 d_1 - d_1 h_1 = 0$, so $s_1 d_1 - h_1$ factors through $\ker d_1$, and as $Q_2 \to \ker d_1$ is surjective and P_1 is projective, we get a unique map $s_2\colon P_1 \to Q_2$ making the triangle in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr] \arrow[d, "h_2"] & & P_1 \arrow[r, "d_1"] \arrow[d, "h_1"] \arrow[dl] \arrow[dll, dotted, "s_2"'] & P_0 \arrow[r, "\epsilon"] \arrow[d, "h_0"] \arrow[dl, "s_1"] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "0"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & \ker d_1 \arrow[r] & Q_1 \arrow[r, "d_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

commute. Continuing this way, we construct the family s such that $ds + sd = f - g$, as required. $\square$

For every object $M \in \operatorname{Obj} \mathcal{C}$, we can choose a projective resolution. Then $L_i\mathcal{F}$ is a functor, called the *left derived functor* of F.

As a consequence of the Comparison Theorem 5.22, if $f' \colon M \to N$, we get a map

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[d, "f'"] \\ \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \end{tikzcd}$$

which is unique up to homotopy, so we get a well-defined map

$$L_i\mathcal{F}(f') = (f_*)_i \colon \underbrace{H_i(\mathcal{F}(P_\bullet))}_{L_i\mathcal{F}M} \to \underbrace{H_i(\mathcal{F}(Q_\bullet))}_{L_i\mathcal{F}N}.$$

Moreover, if $P_\bullet$ and $P'_\bullet$ are two resolutions of M, then $\mathrm{id}\colon M \to M$ gives rise to unique maps (up to homotopy)

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, bend right, "f_1"'] & P_0 \arrow[r] \arrow[d, bend right, "f_0"'] & M \arrow[d, bend right, "\mathrm{id}_M"'] \\ \cdots \arrow[r] & P'_1 \arrow[r] \arrow[u, bend right, "g_1"'] & P'_0 \arrow[r] \arrow[u, bend right, "g_0"'] & M \arrow[u, bend right, "\mathrm{id}_M"'] \end{tikzcd}$$

so we get maps

$$f_* \colon H_\bullet(\mathcal{F}P_\bullet) \to H_*(\mathcal{F}P'_\bullet)$$
$$g_* \colon H_\bullet(\mathcal{F}P'_\bullet) \to H_*(\mathcal{F}P_\bullet)$$

such that $(gf)_* = g_*f_* = \mathrm{id}$

$$(fg)_* = f_*g_* = \mathrm{id}$$

(by uniqueness by to homotopy). Hence the functor is well-defined: for two choices of projective resolutions of objects, the construction yields isomorphic functors.

Example 5.23. In R-mod, for $R = F[x, y]$ where F is a field, let $\mathfrak{m} = (x, y) \subseteq R$ with $M = R/\mathfrak{m} = F$. Then a projective resolution of M is

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow M \longrightarrow 0$$

and we apply $\mathcal{F} = M \otimes_R -$ to this resolution to get

$$0 \longrightarrow F \xrightarrow{0} F^2 \xrightarrow{0} F \longrightarrow 0$$

so

$$H_n(\mathcal{F}(P_\bullet)) = \begin{cases} F, & \text{if } n = 0 \text{ or } 2, \\ F^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.24 (Horseshoe Lemma)**.** *Suppose* $0 \to A' \to A \to A'' \to 0$ *is exact and* $P'_\bullet \xrightarrow{\epsilon'} A'$, $P''_\bullet \xrightarrow{\epsilon''} A''$ *are projective resolutions. Then there exists a unique projective resolution* $P_\bullet \xrightarrow{\epsilon} A$ *and an exact sequence*

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$$

such that

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
P'_\bullet \arrow[r, "\epsilon'"] \arrow[d] & A' \arrow[d] \\
P_\bullet \arrow[r, "\epsilon'"] \arrow[d] & A \arrow[d] \\
P''_\bullet \arrow[r, "\epsilon''"] \arrow[d] & A'' \arrow[d] \\
0 & 0
\end{tikzcd}$$

commutes.

Proof. We recursively construct a projective resolution $P_n = P'_0 \oplus P''_0$, using projectivity and Snake Lemma 1.43. We note that since we have map $P'_0 \to A$ (composition), $P''_0 \to A$ (lift from ϵ''), we get a map $\epsilon \colon P'_0 \oplus P''_0 \to A$ (using the fact that $P'_0 \oplus P''_0$ is both a product and a coproduct) such that the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \epsilon' \arrow[r] & P'_0 \arrow[r, two heads, "\epsilon'"] \arrow[d] \arrow[dr, dotted] & A' \arrow[r] \arrow[d] & 0 \\
 & & P'_0 \oplus P''_0 \arrow[r, dotted, "\epsilon"] \arrow[d] & A \arrow[d, two heads] & \\
0 \arrow[r] & \ker \epsilon'' \arrow[r] & P''_0 \arrow[r, two heads, "\epsilon''"] \arrow[d] \arrow[ur, dotted] \arrow[u, bend right] & A'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

commutes. Note that $P'_0 \oplus P''_0$ is projective as a direct sum of projectives and as both ϵ' and ϵ'' are surjective, so is ϵ. Hence we get the following diagram and we apply the Snake Lemma 1.43 (the cokernels here are all 0) to see that the sequence of kernels is exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker\epsilon' \arrow[r] \arrow[d] & P_0' \arrow[r, two heads, "\epsilon'"] \arrow[d] \arrow[dr] & A' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\epsilon \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r, dotted, "\epsilon"] \arrow[d] & A \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \ker\epsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r, two heads, "\epsilon''"] \arrow[d] \arrow[ur] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We then apply the same procedure to the diagram with kernels to construct $d_1 \colon P_1' \oplus P_1'' \to \ker\epsilon$, where the product is projective and the map is epi onto the kernel

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
P_1' \arrow[r, two heads, "d_1"] \arrow[d] \arrow[dr, dotted] & \ker\epsilon' \arrow[r] \arrow[d] & P_0' \arrow[r, two heads, "\epsilon'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
P_1' \oplus P_1'' \arrow[r, dotted, two heads, "d_1"] \arrow[d] & \ker\epsilon \arrow[r] \arrow[d] & P_0' \oplus P_0'' \arrow[r, "\epsilon"] \arrow[d] & A \arrow[r] \arrow[d, two heads] & 0 \\
P_1'' \arrow[r, two heads, "d_1"] \arrow[d] \arrow[ur, dotted] \arrow[u, bend right] & \ker\epsilon'' \arrow[r] \arrow[d] & P_0'' \arrow[r, two heads, "\epsilon''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

Continuing this way, this constructs the sequence $P_n = P_n' \oplus P_n''$ of projectives with the desired properties. $\square$

Theorem 5.25. *The derived functor $L_i\mathcal{F}$ is a universal homological δ-functor.*

Proof. For an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0,$$

the Horseshoe Lemma 5.24 gives an exact sequence

$$0 \longrightarrow P'_\bullet \longrightarrow P_\bullet \longrightarrow P''_\bullet \longrightarrow 0,$$

and we apply $\mathcal{F}$ to get

$$0 \longrightarrow \mathcal{F}P'_\bullet \longrightarrow \mathcal{F}P_\bullet \longrightarrow \mathcal{F}P''_\bullet \longrightarrow 0.$$

On every level n, we have a splitting

$$0 \longrightarrow P'_n \longrightarrow P_n = P'_n \oplus P''_n \longrightarrow P''_n \longrightarrow 0$$

which shows that there is a map $\mathcal{F}P''_n \to \mathcal{F}P_n$. When we take homology, we get the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_1(\mathcal{F}(P')) \arrow[r] & H_1(\mathcal{F}(P)) \arrow[r] & H_1(\mathcal{F}(P'')) \arrow[dll, "\delta"] \\ & H_0(\mathcal{F}(P')) \arrow[r] & H_0(\mathcal{F}(P)) \arrow[r] & H_0(\mathcal{F}(P'')) \arrow[r] & 0, \end{tikzcd}$$

or, in other words,

$$\cdots \longrightarrow L_1\mathcal{F}A \longrightarrow L_1\mathcal{F}A'' \xrightarrow{\delta} L_0\mathcal{F}A' \longrightarrow L_0\mathcal{F}A \longrightarrow L_0\mathcal{F}A'' \longrightarrow 0.$$

Hence this δ is an approporiate map and we just have to check naturality and universality. We omit this here, but the full proof can be found in [Wei94]. □

6. Injectives and right derived functors

Let $\mathcal{C}$ abelian category. For an object M, $\mathrm{Hom}_{\mathcal{C}}(-, M)$ is a left-exact contravariant functor.

Lemma 6.1. *The following are equivalent*

(1) *for every monic $f\colon A \to B$ and every morphism $g\colon A \to I$, there exists $h\colon B \to I$ such that $hf = g$, i.e. the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

commutes,

(2) $\mathrm{Hom}_{\mathcal{C}}(-, I)$ *is exact,*

(3) *I is projective in $\mathcal{C}^{\mathrm{op}}$.*

Definition 6.2. If any (and hence all) conditions in Lemma 6.1 hold, M is called *injective.*

Proposition 6.3 (Baer's criterion)**.** *In R-mod, I is injective if and only if for every left idea $J \subseteq R$ and every R-module homomorphism $g\colon J \to I$, there exists $\hat{g}\colon R \to I$ such that $hf = g$, i.e. the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "f"] \arrow[d, "g"] & R \arrow[dl, dotted, "\hat{g}"] \\ & I & \end{tikzcd}$$

commutes

Proof. The '*only if*' implication follows directly from the definition. We will use Zorn's lemma to prove the '*if*' implication. Suppose we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B \\ & I & \end{tikzcd}$$

and we want to construct a map $B \to I$. Consider the set

$$\mathcal{S} = \{(C, h) \mid f(A) \subseteq C \subseteq B \text{ submodule, and } hf = g\}$$

and set $(C, h) \leq (C', h')$ if and only if $C \subseteq C'$ and $h'_{|C} = h$. Note that the set is non-empty, because we can choose $C = f(A)$. Suppose $\{(C_x, h_x) \mid x \in X\}$ is a chain, so for any $x, y \in X$

$$(C_x, h_x) \leq (C_y, h_y) \text{ or } (C_x, h_x) \geq (C_y, h_y).$$

Define $C = \bigcup_{x \in X} C_x$ and $h \colon C \to I$ by setting $h(a) = h_x(a)$ if $a \in C_x$ for some $x \in X$. This is well-defined since this is a chain, and $h_{|C_x} = h_x$ for any $x \in X$. Hence $(C_x, h_x) \leq (C, h)$ for any $x \in X$, showing that (C, h) is an upper bound.

By Zorn's lemma, S has a maximal element, call it (C, h). Let $b \in B$. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R \oplus C & \longrightarrow & Rb + C & \longrightarrow & 0 \\ & & a & \longrightarrow & (a, -ab) & & & & \\ & & & & (a, c) & \longrightarrow & (ab + c) & & \end{array}$$

where $J = \{a \in R \mid ab \in C\}$. We let $g \colon J \to I$, $g(a) = h(ab)$ and hence there exists a $\hat{g}$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "f"] \arrow[d, "g"] & R \arrow[dl, dotted, "\hat{g}"] \\ & I & \end{tikzcd}$$

commutes. We then have the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & R \oplus C \arrow[r] \arrow[d, "{(\hat{g}, h)}"'] & \hat{C} = Rb + C \arrow[r] \arrow[dl, dotted, "\hat{h}"] & 0 \\ & & I & & \end{tikzcd}$$

Now, $(C, h) \leq (\hat{C}, \hat{h})$, so $(C, h) = (\hat{C}, \hat{h})$. Hence $b \in C$. This shows that $B \subseteq C$, and hence $B = C$, so we get the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"] & B = C \arrow[dl, "h"] \\ & I & \end{tikzcd}$$

completing the proof. $\square$

Lemma 6.4. *If R is a PID, then I is injective if and only if I is divisible, i.e. for any $a \in I$, $r \in R \setminus \{0\}$, there exists $b \in I$ such that $rb = a$.*

Example 6.5. In Ab $= \mathbb{Z}$-mod, the objects $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, and in fact

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Z}_{p^\infty},$$

where $\mathbb{Z}_{p^\infty} = \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$.

Definition 6.6. An abelian category $\mathcal{C}$ has *enough injectives* if for every $M \in \operatorname{Obj}\mathcal{C}$ there exists a monic $M \to I$ where I is injective. So $\mathcal{C}$ has enough injectives if and only if $\mathcal{C}^{\mathrm{op}}$ has enough projectives.

Example 6.7. In Ab, we can embed $\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Z}/m \to \mathbb{Q}/\mathbb{Z}$ via $1 + (m) \mapsto \frac{1}{m} + \mathbb{Z}$. This does not exactly prove that Ab has enough injectives: it only shows it for finitely generated abelian groups. The general statement takes some more work.

If M is a left R-modules, N is a $\mathbb{Z}$-module, then

$$\operatorname{Hom}_{\mathrm{Ab}}(M, N) \xrightarrow{\cong} \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathrm{Ab}}(R, N))$$

$$f \longrightarrow [(m, r) \mapsto f(rm)]$$

If N is an injective $\mathbb{Z}$-module, then $\operatorname{Hom}_{\mathrm{Ab}}(-, N)$ is exact, and hence

$$\operatorname{Hom}_R(-, \operatorname{Hom}_{\mathrm{Ab}}(R, N))$$

is exact, showing that $\operatorname{Hom}_{\mathrm{Ab}}(R, N)$ is an injective R-module.

Notation. We denote by

$$A^B = \text{ the set of functions } B \to A,$$
$$\mathcal{C}^{\mathcal{D}} = \text{ the category of functors } \mathcal{D} \to \mathcal{C}.$$

By the above, the object $I_0 = \operatorname{Hom}_{\mathrm{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module. The map

$$\Phi\colon M \to I_0^{\operatorname{Hom}_R(M, I_0)},$$
$$\Phi(m)(f) = f(m) \in I_0.$$

is injective (as a set map). Hence the category R-mod has enough injectives.

Definition 6.8. A *coresolution* of $M \in \operatorname{Obj}\mathcal{C}$ is

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

and an injective resolution of M is a coresolution such that I^n are injective objects.

Proposition 6.9. *If $\mathcal{C}$ has enough projectives, then any object has an injective resolution.*

Proof. This is the dual to Proposition 5.21. □

Example 6.10. In Ab, an injective resolution of $\mathbb{Z}$ is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and an injective resolution of $\mathbb{Z}/n$ is

$$0 \longrightarrow \mathbb{Z}/n \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

If $\mathcal{F}$ is a left exact functor and $\mathcal{C}$ has enough injectives, then we define $R^n\mathcal{F}$, *right derived functors*, where for an injective resolution

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

we set

$$R^n\mathcal{F}(M) = H^n(\mathcal{F}(I^\bullet)),$$
$$R^0\mathcal{F}(M) = \mathcal{F}(M).$$

All the results about the left derived functors from Chapter 5 hold dually for right derived functors.

7. Limits

Let $\mathcal{A}$ be an abelian category and $\mathcal{I}$ be another category (often a poset).

Definition 7.1. A *limit* is a functor $\mathcal{F}\colon \mathcal{I} \to \mathcal{A}$ is an object $L \in \operatorname{Obj}\mathcal{A}$ together with morphisms $\pi_i\colon L \to \mathcal{F}(i)$ for all $i \in \operatorname{Obj}\mathcal{I}$ such that

$$\begin{tikzcd} L \arrow[r, "\pi_i"] \arrow[dr, "\pi_j"'] & \mathcal{F}(i) \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(j) \end{tikzcd}$$

commutes for all $f\colon i \to j$, and (L,π) is universal with this property, i.e. if (L',π') satisfy

$$\begin{tikzcd} L' \arrow[r, "\pi'_i"] \arrow[dr, "\pi'_j"'] & \mathcal{F}(i) \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(j) \end{tikzcd}$$

for all $f_i\colon i \to j$, then there exists a unique $h\colon L' \to L$ such that $\pi_i h = \pi'_i$ for all i

$$\begin{tikzcd}
L' \arrow[r, dotted, "h"] \arrow[rr, bend left, "\pi_i'"] \arrow[drr, bend right, "\pi_j'"'] & L \arrow[r, "\pi_i"] \arrow[dr, "\pi_j"] & \mathcal{F}(i) \arrow[d, "\mathcal{F}(f)"] \\
& & \mathcal{F}(j)
\end{tikzcd}$$

We write $(L, \pi) = \lim_{i \in \mathcal{I}} \mathcal{F}(i)$.

Example 7.2. If I is a set with no morphisms, then $L = \prod_{i \in I} \mathcal{F}(i)$ is a product in $\mathcal{A}$.

Definition 7.3. Dually, the *colimit*, $\operatorname{colim}_{i \in \mathcal{I}} \mathcal{F}(i)$, is an object L together with morphisms $i_j \colon \mathcal{F}(j) \to L$ such that the dual universal property holds.

Definition 7.4. A poset $\mathcal{I} = (I, \leq)$ is *directed* (*filtered*) if for all $i, j \in I$, there exists $k \in I$ with $i \leq k$, $j \leq k$ (i.e. $i \to k$, $j \to k$). Then

$$\operatorname{colim}_{i \in I} \mathcal{F}(i) = \operatorname{colim} \mathcal{F}(i) = \varprojlim \mathcal{F}(i)$$

is called the *direct limit*.

Dually, a poset $\mathcal{I}$ is cofiltered if for all $i, j \in I$ there exists k such that $k \leq i$, $k \leq j$ and

$$\lim_{i \in I} \mathcal{F}(i) = \varinjlim \mathcal{F}(i)$$

is called the *inverse limit*.

Example 7.5. We have $I = \mathbb{N}$ is a poset with

$$0 \to 1 \to 2 \to 3 \to \cdots$$

and then for $\mathcal{F}(i) = \mathbb{Z}/2^i$ we get

$$\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/8 \to \cdots$$

with maps $1 \mapsto 2$ everywhere, and hence

$$\varprojlim \mathcal{F}(i) = \varprojlim \mathbb{Z}/2^n = \mathbb{Z}[1/2]/\mathbb{Z}.$$

Direct limits "*feel like unions*".

Conversely, for

$$\cdots \to 3 \to 2 \to 1 \to 0$$

and $\mathcal{F}(i) = \mathbb{Z}/2^i$, we get

$$\cdots \mathbb{Z}/8 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to \mathbb{Z}/1 = \{0\}$$

with the maps $1 \mapsto 1$ everywhere, and hence (in Rings)

$$\varinjlim \mathbb{Z}/2^n = \mathbb{Z}_2,$$

the 2-adic numbers. Inverse limits "*feel like some sort of completion*" and oftentimes one can also define a topology on them.

Definition 7.6. An abelian category $\mathcal{A}$ is *complete* if $\prod_{i \in I} A_i$ exists for every set I. The dual notion is called *cocomplete*.

Lemma 7.7. *Suppose $\mathcal{A}$ is complete, $\mathcal{I}$ is a small category, and $\mathcal{F}\colon \mathcal{I} \to \mathcal{A}$. Then* $\lim \mathcal{F}(i)$ *exists. Similarly, if $\mathcal{A}$ is cocomplete, colimits exist.*

Proof. For a morphism $f\colon j \to k$,

$$\prod_{i \in \operatorname{Obj}\mathcal{I}} \mathcal{F}(i) \xrightarrow{\pi_k - F(f)\pi_j} \mathcal{F}(k)$$

so that

$$\begin{tikzcd} \prod_{i \in \operatorname{Obj}\mathcal{I}} \mathcal{F}(i) \arrow[r, "\pi_j"] \arrow[dr, "\pi_k"'] & \mathcal{F}(j) \arrow[d, "\mathcal{F}(f)"] \\ & \mathcal{F}(k) \end{tikzcd}$$

commutes if the above morphism is 0. Hence we can let K be the kernel to get the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[dr] & \prod_{i \in \operatorname{Obj}\mathcal{I}} \mathcal{F}(i) \arrow[r] \arrow[d, "\pi_i"] \arrow[drr] & & \prod_{f\colon j \to k} \mathcal{F}(k) \arrow[d] \\ & & \mathcal{F}(i) & & \mathcal{F}(k) \end{tikzcd}$$

Then $K = \lim \mathcal{F}(i)$ satisfies the universal property. ◻

8. Sheaves and sheaf cohomology

In this section, we present a brief review the theory of sheaves together with an application of the above theory, sheaf cohomology. For more details, see [Har77, Chap. 2].

We begin with a motivating example.

Example 8.1. Let X be a topological space and, for $U \subseteq X$ open, let

$$\mathcal{F}(U) = \text{set of continuous functions } U \to \mathbb{R}.$$

If $V \subseteq U$, then we have a restriction map $\varrho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ which maps $f \in \mathcal{F}(U)$ to $f_{|V} \in \mathcal{F}(V)$.

We define a category $\operatorname{Top}(X)$ whose objects are open sets $U \subseteq X$ and the partial ordering $\subseteq$ gives morphisms: for $V \subseteq U$, $i_{VU}\colon V \to U$ is the inclusion. Define $\mathcal{F}(i_{VU}) = \varrho_{UV}$. This makes $\mathcal{F}$ into a contravariant functor $\operatorname{Top}(X) \to \mathrm{Ab}$ (or even $\mathbb{R}$-mod).

Definition 8.2. A *pre-sheaf* of abelian groups is a contravariant functor $\mathcal{F}\colon \operatorname{Top}(X) \to \mathrm{Ab}$ with $\mathcal{F}(\emptyset) = 0$. We have a category $\operatorname{Presheaves}(X)$ which is a subcategory of $\mathrm{Ab}^{\operatorname{Top}(X)^{\mathrm{op}}}$. A morphism $\eta\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms

$$\eta(U)\colon \mathcal{F}(U) \to \mathcal{G}(U) \text{ for } U \in \operatorname{Top}(X)$$

such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\eta(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\eta(V)} & \mathcal{G}(V) \end{array}$$

Definition 8.3. A presheaf $\mathcal{F}$ is a *sheaf* if for every $U \subseteq X$ open and open covering U_i, $i \in I$, of U we have

(1) if $f \in \mathcal{F}(U)$ and $f_{|U_i} = 0$ for all i, then $f = 0$,
(2) if $f_i \in \mathcal{F}(U_i)$ for all i and for all i, j we have $(f_i)_{|U_i \cap U_j} = (f_j)_{U_i \cap U_j}$ then there exists $f \in \mathcal{F}(U)$ with $f_{|U_i} = f_i$ for all i.

Note that condition (1) can be restated by requiring uniqueness in condition (2).

If I is totally ordered, then (1) and (2) are equivalent to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \bigoplus_i \mathcal{F}(U_i) & \longrightarrow & \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) \\ & & f & \longrightarrow & (f_{U_i}, i \in I) & & \\ & & & & (g_i, i \in I) & \longrightarrow & (g_i - g_j, i < j) \end{array}$$

being exact.

Definition 8.4. Let $\mathcal{F}$ be a pre-sheaf. The *stalk* of $\mathcal{F}$ at x is the direct limit

$$\mathcal{F}_x = \varinjlim \{\mathcal{F}(U) \mid U \ni x\}$$

or, equivalently,

$$\mathcal{F}_x = \{(U, f) \mid f \in \mathcal{F}(U),\ x \in U\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if there exists W with $x \in W \subseteq U \cap V$ and $f_{|W} = g_{|W}$.

Definition 8.5. If $\mathcal{F}$ is a pre-sheaf, then we define $\mathcal{F}^+$ by

$$\begin{array}{rl} \mathcal{F}^+(U) = & \text{set of all function } U \to \coprod_{x \in U} \mathcal{F}_x \text{ with } f(x) \in \mathcal{F}_x \text{ such that} \\ & \text{for every } y \in U \text{ there exists } V \subseteq U \text{ and } y \in V \text{ and } g \in \mathcal{F}(V) \\ & \text{such that } g \text{ maps to } f(x) \in \mathcal{F}_x \text{ for all } x \in V \end{array} \cdot$$

The universal property of $\mathcal{F}^+$: $\mathcal{F}^+$ is a sheaf and if $\mathcal{G}$ is a sheaf and $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of pre-sheaves, then there exists a unique morphism $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^+ \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

commutes. Then $\mathcal{F}^+$ is called the *sheafification* of $\mathcal{F}$.

Lemma 8.6. *Sheafification is exact.*

Example 8.7. If $X = \mathbb{P}^1(\mathbb{C})$, the Zariski topology on X is: $D \subseteq X$ closed if and only if $D = \mathbb{P}^1(\mathbb{C})$ or D is finite.

Define a sheaf $\mathcal{O}_X$ on $\mathbb{P}^1(\mathbb{C})$ by setting

$$\mathcal{O}_X(U) = \{f \in \mathbb{C}(t) \mid f \text{ regular on } U\},$$

a subring of the function field $\mathbb{C}(t)$.

Write the affine line as $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\} \cong \mathbb{C}$. Then

$$\mathcal{O}_X(\mathbb{P}^1) = \mathbb{C}, \quad \mathcal{O}_X(\mathbb{A}^1) = \mathbb{C}[t].$$

Moveover,

$$\mathcal{O}_X(\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}) = \mathbb{C}\left[\frac{1}{t-a_1}, \dots, \frac{1}{t-a_n}\right]$$

for $a_1, \dots, a_n \in \mathbb{A}^1$.

Define a subsheaf $\mathcal{I}$ of $\mathcal{O}_X$ by setting

$$\mathcal{I}(U) = \{f \in \mathcal{O}_X(U) \mid f_{|U \cap \{0,\infty\}} = 0\}.$$

There is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I} \longrightarrow 0$$

but the last of these, $\mathcal{O}_X/\mathcal{I}$, is not a sheaf. Indeed:

$$(\mathcal{O}_X/\mathcal{I})(\mathbb{A}^1) \cong \mathbb{C}[t]/(t),$$

$$(\mathcal{O}_X/\mathcal{I})(\mathbb{P}^1 \setminus \{0\}) = \mathbb{C}\left[\frac{1}{t}\right] \Big/ \left(\frac{1}{t}\right).$$

Then $0 + (t)$ and $1 + \left(\frac{1}{t}\right)$ agree on $\mathbb{A}^1 \setminus \{0\}$, since

$$(\mathcal{O}_X/\mathcal{I})(\mathbb{A}^1 \setminus \{0\}) = 0,$$

but since $(\mathcal{O}_X/\mathcal{I})(\mathbb{P}^1) = \mathbb{C}$, there is no element there mapping to f_1 and f_2.

Instead, sheafify to get an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow (\mathcal{O}_X/\mathcal{I})^+ \longrightarrow 0$$

where $(O_X/\mathcal{I})^+(\mathbb{P}^1) = \mathbb{C}^2$.

Definition 8.8. The *global sections functor* $\Gamma(X, -)\colon \mathcal{F} \mapsto \mathcal{F}(X)$ is left exact, and

$$R^i\Gamma(X, \mathcal{F}) = H^i(X, \mathcal{F})$$

is called the *sheaf cohomology*.

Example 8.9. For the example above, we have the long exact sequence

$$0 \longrightarrow \underbrace{\Gamma(X, I)}_{=0} \longrightarrow \underbrace{\Gamma(X, \mathcal{O}_X)}_{=\mathbb{C}} \longrightarrow \underbrace{\Gamma(X, (\mathcal{O}_X/I)^+)}_{=\mathbb{C}^2} \longrightarrow \underbrace{H^1(X, I)}_{\neq 0} \longrightarrow \cdots$$

9. Adjoint functors

Definition 9.1. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Then additive functors $L\colon \mathcal{A} \to \mathcal{B}$ and $R\colon \mathcal{B} \to \mathcal{A}$ are *adjoint* if there exists a natural isomorphism

$$T\colon\ \mathrm{Hom}_{\mathcal{B}}(L(A), B) \cong \mathrm{Hom}_{\mathcal{A}}(A, R(B))$$

of groups.

Proposition 9.2 (Yoneda Lemma)**.** *Let $\mathcal{A}$ be an abelian category. A sequence*

$$A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C$$

is exact if for all M,

$$\mathrm{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\ \alpha_*\ } \mathrm{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\ \beta_*\ } \mathrm{Hom}_{\mathcal{A}}(M, C)$$

is exact.

Proposition 9.3. *If L and R are adjoint, then L is right-exact and R is left-exact.*

Proof. Suppose

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$$

is a short exact sequence in $\mathcal{B}$. We apply

$$\mathrm{Hom}_{\mathcal{B}}(L(A), -) \cong \mathrm{Hom}_{\mathcal{A}}(A, R(-))$$

to get

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}}(L(A), B_3) \\
 & \big\downarrow{\cong} & & \big\downarrow{\cong} & & \big\downarrow{\cong} \\
0 \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B_1)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B_2)) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(A, R(B_3))
\end{array}$$

for all $A \in \mathrm{Obj}\,\mathcal{A}$. By Yoneda lemma, we get that

$$0 \to R(B_1) \to R(B_2) \to R(B_3)$$

is exact, and so R is left-exact. Similarly, L is right-exact. □

Example 9.4. Let

(1) A right R-module,
(2) B R-S bimodule,
(3) C right S-module.

Then $A \otimes_R B$ is a right S-module with $(a \otimes b)s = (a \otimes bs)$ and $\mathrm{Hom}_S(B, C)$ is a right R-module with $(fr)(b) = f(rb)$. There is a natural isomorphism

$$T\colon\ \mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

Indeed, if $f\colon A \otimes_R B \to C$, then for we get

$$a \mapsto [f(a \otimes -)\colon B \to C]$$

Conversely, if $A \to \mathrm{Hom}_S(B, C)$ is any map, then we get a bilinear map $A \times B$ to C which factors through a map $A \otimes_R B \to C$.

We have that

$$- \otimes_R B\colon \text{mod-}R \to \text{mod-}S$$
$$\mathrm{Hom}_S(B, -)\colon \text{mod-}S \to \text{mod-}R$$

are an adjoint pair, so $- \otimes_R B$ is right exact and $\mathrm{Hom}_S(B, -)$ is left exact.

Then we set

$$L_i(- \otimes_R B) = \mathrm{Tor}_i(-, B),$$

and we will show that

$$L_i(A \otimes_R -) = \mathrm{Tor}_i(A, -),$$

because

$$L_i(- \otimes_R B) \cong L_i(- \otimes B)(A).$$

Definition 9.5. A *double complex* is a set $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ of objects with horizontal maps

$$d^h\colon C_{p,q} \to C_{p-1,q}$$

and vertical maps

$$d^v\colon C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = 0$, $d^v \circ d^v = 0$, and $d^v d^h + d^h d^v = 0$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & C_{p-1,q+1} \arrow[l] \arrow[d] & C_{p,q+1} \arrow[l] \arrow[d] & C_{p+1,q+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-1,q} \arrow[l] \arrow[d] & C_{p,q} \arrow[l] \arrow[d] & C_{p+1,q} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l] \arrow[d] & C_{p+1,q-1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The map $d^v\colon C_{\bullet q} \to C_{\bullet q-1}$ is almost a chain map: if we set $f_{pq} = (-1)^p d^v_{pq}\colon C_{pq} \to C_{pq-1}$, then $f_{\bullet q}\colon C_{\bullet q} \to C_{\bullet q-1}$ is a chain map. Hence $f_{\bullet q}$ is in $\mathrm{Ch}(\mathcal{C})$, and $f_{\bullet\bullet}$ is in $\mathrm{Ch}(\mathrm{Ch}(\mathcal{C}))$.

Definition 9.6. Assume $\mathcal{C}$ is cocomplete. We set define the total complex

$$\mathrm{Tot}^{\oplus}(C_{\bullet\bullet})_n = \bigoplus_{p+q=n} C_{p,q}$$

with $d = d^v + d^h$, whence

$$d^2 = (d^v + d^h)^2 = (d^v)^2 + (d^v d^h + d^h d^v) + (d^h)^2 = 0 + 0 + 0 = 0.$$

If $\mathcal{C}$ is complete,

$$\mathrm{Tot}^{\pi}(C_{\bullet\bullet})_n = \prod_{p+q=n} C_{p,q}$$

with $d = d^v + d^h$.

Proposition 9.7 (Acyclic Assembly Lemma)**.** *Suppose $C_{\bullet\bullet}$ is a double complex in mod-R. Suppose C is an upper half plane complex (i.e. $C_{p,q} = 0$ if $q < 0$) and columns are exact. Then* $\mathrm{Tot}^{\pi}(C)$ *is acyclic (exact).*

Proof. We claim that $H_0(\mathrm{Tot}^{\pi}(C_{\bullet\bullet})) = 0$. By symmetry, it is enough to restrict our attention to the $p = 0$ portion of the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,2} & \longleftarrow & C_{0,2} & \longleftarrow & C_{1,2} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,1} & \longleftarrow & C_{0,1} & \longleftarrow & C_{1,1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{-1,0} & \longleftarrow & C_{0,0} & \longleftarrow & C_{1,0} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Suppose

$$(c_0, c_1, c_2, \ldots) \in C_{00} \times C_{-1,1} \times C_{-2,2} \times \cdots$$

is a cycle. Then

$$d(c_0, c_1, c_2, \ldots) = (d^h c_0 + d^v c_1, d^h c_1 + c^v c_2, \ldots) \in C_{-1,0} \times C_{-2,1} \times \cdots .$$

We want to find

$$(b_0, b_1, \ldots) \in C_{10} \times C_{01} \times C_{-12} \times \cdots$$

with

$$d(b_0, b_1, \ldots) = (d^h b_0 + d^v b_1, d^h b_1 + d^v b_2, \ldots) = (c_0, c_1, \ldots).$$

Pick $b_0 = 0$. Then $d^v c_0 = 0$ so $c_0 = d^v b_1$ for some b_1. Then

$$0 = d^h c_0 + d^v c_1 = d^h d^v b_1 + d^v c_1 = -d^v d^h b_1 + d^v c_1 = d^v(c_1 - d^h b_1).$$

Hence there exists $b_2 \in C_{-1,2}$ such that $d^v b_2 = c_1 - d^h b_1$. Then $c_1 = d^v b_2 + d^h b_1$, and proceed by induction to construct b_3, b_4, ... This completes the proof. □

Corollary 9.8. *If $C_{\bullet\bullet}$ is a double complex in the right half plane with exact rows, then* $\mathrm{Tot}^{\pi}(C_{\bullet\bullet})$ *is exact.*

Corollary 9.9. *If $C_{\bullet\bullet}$ is a double complex in the right half plane with exact columns, then* $\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})$ *is exact.*

Proof. Define new complex $\tau_n C_{\bullet\bullet}$ by

$$(\tau_n C)_{p,q} = \begin{cases} C_{pq} & \text{if } q > n, \\ \ker d^v \colon C_{pn} \to C_{pn-1} & \text{if } q = n, \\ 0 & \text{if } q < n. \end{cases}$$

Then we get the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & C_{0,n+2} \arrow[l] \arrow[d] & C_{1,n+2} \arrow[l] \arrow[d] & C_{1,n+2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & C_{0,n+1} \arrow[l] \arrow[d] & C_{1,n+1} \arrow[l] \arrow[d] & C_{1,n+1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & \ker d^v \arrow[l] \arrow[d] & \ker d^v \arrow[l] \arrow[d] & \ker d^v \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence $\mathrm{Tot}^\pi(\tau_n C) = \mathrm{Tot}^\oplus(\tau_n C)$ is exact. This shows that $\mathrm{Tot}^\oplus(C)$ is exact. □

Theorem 9.10. *We have that*

$$L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A)$$

and we call them $\mathrm{Tor}_n^R(A, B)$*.*

We will actually prove a more general statement, following [Wei94, Exer. 2.7.4]. (This was actually a homework exercise, but we include it here for completeness.)

Theorem 9.11. *Suppose* $\mathcal{C}$ *is an abelian category and*

$$T \colon \mathcal{C} \times \cdots \times \mathcal{C} \to \mathcal{D}$$

is an additive functor in p *variables, some of the covariant and some contravariant. Assume moreover that* T *is right-balanced:*

(1) *when any covariant variable is replaced by an injective module,* T *becomes exact in the other variables,*
(2) *when any contravariant variable is replaced by a projective module,* T *becomes exact in the other variables.*

Then for any i, j *there is a natural isomorphism*

$$R^*T(A_1, \dots, \hat{A}_i, \dots, A_p)(A_i) \cong R^*T(A_1, \dots, \hat{A}_j, \dots, A_p)(A_j).$$

Proof. Note that if we fix modules $A_1, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_p$, then the functor

$$T(A_1, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_p) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{D}$$

is right-balanced, and it is enough to show the assertion for this functor in 2 variables. Hence suppose that

$$T\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D}$$

is right-balanced. We mimic the proof of [Wei94, Theorem 2.7.2] to show

$$R^*T(A,-)(B) \cong R^*T(-,B)(A).$$

If the first variable is covariant, choose an injective resolution $\epsilon\colon A \to P_\bullet$ and if it is contravariant, choose a projective resolution $\epsilon\colon P_\bullet \to A$. Similarly, if the second variable is covariant, choose an injective resolution $\eta\colon A \to Q_\bullet$ and if it is contravariant, choose a projective resolution $\eta\colon Q_\bullet \to A$. We then get the double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & T(A,Q_2) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_2) \arrow[u] \arrow[r] & T(P_1,Q_2) \arrow[u] \arrow[r] & T(P_2,Q_2) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & T(A,Q_1) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_1) \arrow[u] \arrow[r] & T(P_1,Q_1) \arrow[u] \arrow[r] & T(P_2,Q_1) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & T(A,Q_0) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_0) \arrow[u] \arrow[r] & T(P_1,Q_0) \arrow[u] \arrow[r] & T(P_2,Q_0) \arrow[u] \arrow[r] & \cdots \\
 & 0 \arrow[u] \arrow[r] & T(P_0,B) \arrow[u, "T(1,\eta)"] \arrow[r] & T(P_1,B) \arrow[u, "T(1,\eta)"] \arrow[r] & T(P_2,B) \arrow[u, "T(1,\eta)"] & \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We will show that the maps

$$T(\epsilon,1)\colon\ \mathrm{Tot}(T(P_\bullet,Q_\bullet)) \to \mathrm{Tot}(T(A,Q_\bullet)) = T(A,Q_\bullet)$$

$$T(1,\eta)\colon\ \mathrm{Tot}(T(P_\bullet,Q_\bullet)) \to \mathrm{Tot}(T(P_\bullet,B)) = T(P_\bullet,B)$$

are quasi-isomorphisms, and hence they induce natural isomorphisms

$$H^*(\mathrm{Tot}(T(P_\bullet,Q_\bullet))) \cong R^*(T(A,-))(B),$$

$$H^*(\mathrm{Tot}(T(P_\bullet,Q_\bullet))) \cong R^*(T(-,B))(A),$$

which gives the result.

We only show $T(\epsilon,1)$ is a quasi-isomorphism using the Acyclic Assembly Lemma 9.7; the fact that $T(1,\eta)$ is a quasi-isomorphism is symmetric. Let $C_{\bullet\bullet}$ be the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & T(A,Q_2) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_2) \arrow[u] \arrow[r] & T(P_1,Q_2) \arrow[u] \arrow[r] & T(P_2,Q_2) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & T(A,Q_1) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_1) \arrow[u] \arrow[r] & T(P_1,Q_1) \arrow[u] \arrow[r] & T(P_2,Q_1) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & T(A,Q_0) \arrow[u] \arrow[r, "T(\epsilon,1)"] & T(P_0,Q_0) \arrow[u] \arrow[r] & T(P_1,Q_0) \arrow[u] \arrow[r] & T(P_2,Q_0) \arrow[u] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and note that $\mathrm{Tot}(C_{\bullet,\bullet})[1]$ is the mapping cone of $\epsilon \otimes 1\colon \mathrm{Tot}(T(P_\bullet, Q_\bullet)) \to T(A, Q_\bullet)$. Hence, to show that $\epsilon \otimes 1$ is a quasi-isomorphism, it is enough to show that the mapping cone $\mathrm{Tot}(C_{\bullet,\bullet})[1]$ is acyclic. Finally, since Q_i are injective if the second variable is covariant and projecitve if the second variable is contravariant, the right-balanced condition shows that $C_{\bullet\bullet}$ has exact rows. Then by Acyclic Assemply Lemma 9.7, we obtain that $\mathrm{Tot}(C_{\bullet,\bullet})[1]$ is acyclic.

This completes the proof. □

10. Tor and Ext

We restrict our attention to $\mathcal{A} = \mathrm{Ab}$.

Example 10.1. Consider the projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

For an abelian group B, apply $- \otimes_{\mathbb{Z}} B$ to $P_\bullet$ to get

$$0 \longrightarrow B \xrightarrow{n\cdot} B \longrightarrow 0$$

Hence

$$\begin{gathered}
\mathrm{Tor}_0(\mathbb{Z}/n, B) = \mathbb{Z}/n \otimes B = H_0(P_\bullet \otimes B) = B/nB, \\
\mathrm{Tor}_1(\mathbb{Z}/n, B) = B[n] = \{b \in B : nb = 0\}, \\
\mathrm{Tor}_k(\mathbb{Z}/n, B) = 0 \text{ for } k \geq 2, \\
\mathrm{Tor}_0(\mathbb{Z}, B) = B, \\
\mathrm{Tor}_k(\mathbb{Z}, B) = 0 \text{ for } k \geq 1,
\end{gathered}$$

In general,

$$\mathrm{Tor}_i(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/(n,m) \text{ for } i = 0, 1.$$

To calculate Ext, we apply $\mathrm{Hom}_{\mathrm{Ab}}(-, B)$ to $P_\bullet$ to get

$$0 \longrightarrow B \xrightarrow{n\cdot} B \longrightarrow 0$$

and then

$$\operatorname{Ext}^0(\mathbb{Z}/n, B) = H^0(P_\bullet) = B[n]$$
$$\operatorname{Ext}^1(\mathbb{Z}/n, B) = H^1(P_\bullet) = B/nB$$

and in particular

$$\operatorname{Ext}^i(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/(n,m) \text{ for } i = 0, 1.$$

If $A = \varinjlim A_\alpha = \operatorname{colim} A_\alpha$, then

$$\operatorname{Tor}_i(A, B) = \operatorname{Tor}_i(\varinjlim A_\alpha, B) = \varinjlim \operatorname{Tor}_i(A_\alpha, B).$$

Proposition 10.2. *Suppose A and B are abelian group. Then*

(1) $\operatorname{Tor}_1(A, B)$ *is a torsion group,*
(2) $\operatorname{Tor}_n(A, B) = 0$ *for* $n \geq 2$.

Proof. If A is finitely generated, then

$$A \cong \mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$$

and the proposition is clear.

Otherwise, $A = \varinjlim A_\alpha$ for $\{A_\alpha\}$ finitely generated subgroups. The limit of torsion groups is torsion, so

$$\operatorname{Tor}_1(A, B) = \varinjlim \operatorname{Tor}_1(A_\alpha, B)$$

is torsion. □

Example 10.3. We have that

$$\operatorname{Tor}_1(\mathbb{Q}/\mathbb{Z}, B) = \varinjlim \operatorname{Tor}_1\left(\underbrace{\mathbb{Z}\left[\frac{1}{n}\right]/\mathbb{Z}}_{\mathbb{Z}/n}, B\right) = \varinjlim B[n],$$

the torsion subgroup of B.

If A is torsion free then $A = \varinjlim \mathbb{Z}^m$, so

$$\operatorname{Tor}_1(A, B) = \varinjlim \operatorname{Tor}_1(\mathbb{Z}^m, B) = 0.$$

Hence A is torsion-free if and only if $\operatorname{Tor}_1(A, -) = 0$ if and only if $A \otimes_{\mathbb{Z}} -$ is exact, or by definition, A is a flat $\mathbb{Z}$-module.

Definition 10.4. A left R-module B is *flat* if $- \otimes_R B$ is exact, and a right R-module A is flat if $A \otimes_R -$ is exact.

In general, a projective module is flat but the converse is not true. For example, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module but it is not projective.

Suppose R is a ring with 1 and

$$S \subseteq \underbrace{Z(R)}_{\text{center}} \subseteq R.$$

Suppose $1 \in S$ and S is closed under multiplication. The *localization of* R with respect to S, $S^{-1}R$, is

$$S^{-1}R = \{(r, s) \mid r \in R,\ s \in S\}/\sim$$

where

$$(r_1, s_1) \sim (r_2, s_2) \text{ if and only if there exists } s_3 \in S \text{ such that } (r_1 s_2 - r_2 s_1)s_3 = 0.$$

We think of (r, s) as $\frac{r}{s}$. If $[(r, s)]$ is an equivalent class, then

$$[(r_1, s_1)] + [(r_2, s_2)] = [(r_1 s_2 + r_2 s_1, s_1 s_2)],$$
$$[(r_1, s_1)] \cdot [(r_2, s_2)] = [(r_1 r_2, s_1 s_2)].$$

We have a map $\varphi\colon R \to S^{-1}R$, setting $\varphi(r) = [(r, 1)]$. We then set

$$S^{-1}M := S^{-1}R \otimes_R M, \text{ the localization of } M \text{ at } S.$$

Theorem 10.5. *The localization $S^{-1}R$ is a flat R-module, i.e. $S^{-1}R \otimes_R -$ is an exact functor from R-mod to $S^{-1}R$-mod.*

Proof. Define category $\mathcal{I}$ with

$$\mathrm{Obj}(\mathcal{I}) = S,$$
$$\text{for } s_1, s_2 \in S, \text{ we set } \mathrm{Hom}_{\mathcal{I}}(s_1, s_2) = \{s \in S \mid s s_1 = s_2\}.$$

This is a filtered category (see definition below)

$$\begin{tikzcd} s_1 \arrow[dr, "s_2"] & \\ & s_1 s_2 \\ s_2 \arrow[ur, "s_1"'] & \end{tikzcd} \qquad \begin{tikzcd} s_1 \arrow[r, shift left, "t_1"] \arrow[r, shift right, "t_2"'] & s_2 \arrow[r, "s_1"] & s_1 s_2 \end{tikzcd}$$

We then have a functor $\mathcal{F}\colon I \to R$-mod given by $\mathcal{F}(s) = R$ for $s \in \mathrm{Obj}(\mathcal{I}) = S$ and if $s_1 \xrightarrow{s} s_2$ is a morphism then

$$\mathcal{F}(s_1) = R \xrightarrow{\mathcal{F}(s) = s\cdot} \mathcal{F}(s_2) = R$$

We claim that $\mathrm{colim}_{s \in I}\, \mathcal{F}(s)$ exists and in fact $\mathrm{colim}_{s\in I}\, \mathcal{F}(s) = S^{-1}R$. Indeed, we define

$$\varphi_s\colon \mathcal{F}(s) = R \to S^{-1}R$$

by $\varphi_s(r) = [(r, s)]$. Then one can easily check that this gives a map from the colimit to $S^{-1}(R)$ which is an isomorphism by the universal property of localization.

Therefore, for $n \geq 1$

$$\mathrm{Tor}_n(S^{-1}R, B) = \mathrm{colim}\, \mathrm{Tor}_n(\mathcal{F}(s), B) = \mathrm{colim}\, \underbrace{\mathrm{Tor}_n(R, B)}_{=0} = 0,$$

and hence $S^{-1}R$ is flat. $\square$

Definition 10.6. A category $\mathcal{C}$ is *filtered* if

(1) for any A, B, there exists C with morphisms $\alpha\colon A \to C$ and $\beta\colon B \to C$,
(2) if $A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B$ then there exists $\gamma\colon B \to C$ such that $\gamma\alpha = \gamma\beta$.

Exercise. The following conditions are equivalent

(1) A is a *flat* right R-module,
(2) $A \otimes_R -$ is an exact functor,
(3) $\mathrm{Tor}_1(A, B) = 0$ for all left R-modules B,
(4) $\mathrm{Tor}_n(A, B) = 0$ for all left R-modules B and $n \geq 1$.

(The first equivalence is the definition.)

Definition 10.7. If B is a left R-module, then $B^* = \mathrm{Hom}_{\mathrm{Ab}}(B, \mathbb{Q}/\mathbb{Z})$ is the right *Pontryagin dual* of B.

If $B \neq 0$, let C be a maximal subgroup. Then $B/C \cong \mathbb{Z}/p$ for p prime, and hence

$$\mathrm{Hom}_{\mathrm{Ab}}(B/C, \mathbb{Q}/\mathbb{Z}) \neq 0,$$

and thus $B^* = \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \neq 0$.

Lemma 10.8. *A morphism $f\colon B \to C$ is injective if and only if $f^*\colon C^* \to B^*$ is surjective, where $f^* = \mathrm{Hom}_{\mathrm{Ab}}(f, \mathbb{Q}/\mathbb{Z})$.*

Proof. Suppose $A \to B$ is a kernel of f, so $0 \to A \to B \to C$ is exact, and hence

$$C^* \to B^* \to A^* \to 0$$

is exact, since $\mathbb{Q}/\mathbb{Z}$ is injective. Hence f is injective if and only if $A = 0$ if and only if $A^* = 0$ if and only if f^* is surjective. $\square$

Proposition 10.9. *The following are equivalent:*

(1) *B is a flat left R-module,*
(2) *B^* is an injective right R-module,*
(3) *$I \otimes_R B \xrightarrow{\cong} IB$ for every right ideal $I \subseteq R$,*
(4) *$\mathrm{Tor}_1(R/I, B) = 0$ for every right ideal $I \subseteq R$.*

Proof. We first check that (3) is equivalent to (4). Apply $- \otimes_R B$ to the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

to get an exact sequence

$$\underbrace{\mathrm{Tor}_1(R, B)}_{=0} \longrightarrow \mathrm{Tor}_1(R/I, B) \longrightarrow I \otimes_R B \longrightarrow \underbrace{R \otimes_R B}_{\cong B} \xrightarrow{\varphi} \underbrace{R/I \otimes_R B}_{B/IB} \longrightarrow 0$$

and hence

$$0 \longrightarrow \mathrm{Tor}_1(R/I, B) \longrightarrow I \otimes_R B \longrightarrow \underbrace{IB}_{\ker \varphi} \longrightarrow 0$$

is exact. This shows (3) is equivalent to (4).

We now show (1) is equivalent to (2). We have

$$\mathrm{Hom}_R(A, B^*) = \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))) \cong \mathrm{Hom}(A \otimes_R B, \mathbb{Q}/\mathbb{Z}) = (A \otimes_R B)^*.$$

If $A' \subseteq A$ is a submodule, then the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(A, B^*) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(A', B^*) \arrow[d, "\cong"] \\
(A \otimes_R B)^* \arrow[r] & (A' \otimes_R B)^*
\end{tikzcd}$$

commutes. Now,

$$\begin{array}{lll}
B^* \text{ is injective} & \text{if and only if} & \mathrm{Hom}_R(A, B^*) \to \mathrm{Hom}_R(A', B^*) \text{ is surjective for all } A' \subseteq A \\
& \text{if and only if} & (A \otimes_R B)^* \to (A' \otimes_R B)^* \text{ is surjective for all } A' \subseteq A \\
& \text{if and only if} & A' \otimes_R B \to A \otimes_R B \text{ is injective for all } A' \subseteq A \\
& \text{if and only if} & - \otimes_R B \text{ is exact} \\
& \text{if and only if} & B \text{ is flat.}
\end{array}$$

We finally show (2) is equivalent to (3). Note that B^* is injective if and only if

$$\overbrace{\mathrm{Hom}_R(R, B^*)}^{B^*} \to \overbrace{\mathrm{Hom}_R(I, B^*)}^{(I \otimes B)^*}$$

is surjective for all right ideals $I \subseteq R$. But this is equivalent to $I \otimes_R B \to B$ is injective for all I, which holds if and only if $I \otimes_R B \cong IB$. □

Definition 10.10. A module M is *finitely presented* if there exists an exact sequence

$$R^m \to R^n \to M \to 0.$$

Note that projectivity is not equivalent to flatness. Indeed, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (it is a localization of $\mathbb{Z}$), but it is not projective.

We will show that for finitely presented modules, flat modules are projective.

For M, A left R-modules, define

$$\sigma \colon A^* \otimes_R M \to \mathrm{Hom}_R(M, A)^*$$

$$\sigma(f \otimes m)(h) = f(h(m))$$

for $f \in A^*$, $m \in M$, $h \in \mathrm{Hom}_R(M, A)$.

Proposition 10.11. *If M is finitely presented, then σ is an isomorphism for all A.*

Proof. Clear if $M = R^n$. Now, if

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

then we have the following commutative diagram

$$\begin{array}{ccccccccc}
A^* \otimes_R R^m & \longrightarrow & A^* \otimes_R R^n & \longrightarrow & A^* \otimes_R M & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \sigma & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_R(R^m, A)^* & \longrightarrow & \mathrm{Hom}_R(R^n, A)^* & \longrightarrow & \mathrm{Hom}_R(M, A)^* & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

which has exact rows, because $\mathrm{Hom}(-, A)^*$ and $A \otimes_R -$ are right-exact covariant functors. By Five Lemma 1.44, we obtain that σ is an isomorphism. □

Theorem 10.12. *A finitely presented flat R-module is projective.*

Proof. Suppose M is finitely presented, flat, and $f\colon B \to C$ is surjective, so $f^*\colon C^* \to B^*$ is injective. Hence the square

$$\begin{array}{ccc}
C^* \otimes_R M & \xrightarrow{f^* \otimes 1} & B^* \otimes M \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}(M, C)^* & \longrightarrow & \mathrm{Hom}(M, B)^*
\end{array}$$

commutes. Since M is flat, $f^* \otimes 1$ is injective, and hence

$$\mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

is surjective. This shows that $\mathrm{Hom}_R(M, -)$ is exact, so M is projective. □

Lemma 10.13 (Flat resolution lemma)**.** *If $\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$ is a flat resolution, then*

$$\mathrm{Tor}_n^R(A, B) = H_n(F_\bullet \otimes B)$$

for a right R-module A and a left R-module B.

If F_n are in fact projective, this is how we defined $\mathrm{Tor}_n^R(A, B)$, and this lemma shows that we can compute $\mathrm{Tor}_n^R(A, B)$ by considering only a flat resolution.

Proof. For $n = 0$, we have an exact sequence

$$F_1 \otimes_R B \longrightarrow F_0 \otimes_R B \longrightarrow A \otimes_R B \longrightarrow 0$$

and hence $H_0(F_\bullet \otimes_R B) = A \otimes_R B = \mathrm{Tor}_0(A, B)$.

We have an exact sequence

$$0 \longrightarrow K \longrightarrow F_0 \longrightarrow A \longrightarrow 0.$$

We then get a long exact sequence

$$\overbrace{\mathrm{Tor}_2(F_0, B)}^{=0} \longrightarrow \mathrm{Tor}_2(A, B) \longrightarrow \mathrm{Tor}_1(K, B) \longrightarrow \overbrace{\mathrm{Tor}_1(F_0, B)}^{=0}$$

$$\longrightarrow \mathrm{Tor}_1(A, B) \longrightarrow K \otimes B \longrightarrow F_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0$$

and hence $\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_{n-1}(K, B)$. We have the exact sequence

$$F_2 \longrightarrow F_1 \longrightarrow K \longrightarrow 0$$

which gives the exact sequence

$$F_2 \otimes B \longrightarrow F_1 \otimes B \longrightarrow K \otimes B \longrightarrow 0$$

and we get that

$$\mathrm{Tor}_1(A, B) = \ker(K \otimes B \to F_0 \otimes B) = \ker\left(\frac{F_1 \otimes B}{d_2(F_2 \otimes B)} \to F_0 \otimes B\right) = H_1(F_\bullet \otimes B).$$

By induction on n, we finally obtain:

$$\mathrm{Tor}_n(A, B) = \mathrm{Tor}_{n-1}(K, B) = H_{n-1}(F_\bullet[1] \otimes B) = H_n(F_\bullet \otimes B),$$

as required. □

Proposition 10.14. *Suppose $R \to T$ is a ring homomorphism and T is flat as a left R-module. Then for all right R-modules A and left T-modules C, we have that*

$$\mathrm{Tor}_n^R(A, C) \cong \mathrm{Tor}_n^T(A \otimes_R T, C).$$

Proof. Let $P_\bullet \to A$ be a projective resolution so that

$$\mathrm{Tor}_n^R(A, C) = H_n(P_\bullet \otimes_R C).$$

Note that $P_n \otimes_R T$ is a projective T-module: $P_n \oplus Q = F$ for some free R-module F, whence

$$F \otimes T = (P_n \oplus Q) \otimes T = (P_n \otimes T) \oplus (Q \otimes T),$$

so $P_n \otimes T$ is a direct summand of the free T-module $F \otimes T$. Hence

$$P_\bullet \otimes_R T \to A \otimes_R T$$

is a projective resolution (since $- \otimes_R T$ is exact). Hence

$$\mathrm{Tor}_n^T(A \otimes_R T, C) = H_n(P_\bullet \otimes_R T \otimes_T C) = H_n(P_\bullet \otimes_R C) = \mathrm{Tor}_n^R(A, C).$$

This completes the proof. □

Corollary 10.15. *Let R be commutative, T be a flat R-algebra with $\varphi\colon R \to T$ and $\varphi(R) \subseteq Z(T)$. Then*

$$T \otimes_R \mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^T(A \otimes_R T, T \otimes_R B).$$

Proof. We have that:

$$\operatorname{Tor}_n^T(A \otimes_R T, T \otimes_R B) = \operatorname{Tor}_n^R(A, T \otimes_R B) = H_n(P_\bullet \otimes_R T \otimes_R B) = H_n(T \otimes_R P_\bullet \otimes_R B)$$
$$= T \otimes_R H_n(P_\bullet \otimes_R B) = T \otimes_R \operatorname{Tor}_n^R(A, B),$$

as required. □

Example 10.16. Suppose $\mathfrak{p} \subseteq R$ is a prime ideal, R is a commutative ring and $R_\mathfrak{p} = S^{-1}R$ for $S = R \setminus \mathfrak{p}$. If M is an R-module, $M_\mathfrak{p} = R_\mathfrak{p} \otimes_R M$, then

$$\operatorname{Tor}_n^R(A, B)_\mathfrak{p} = R_\mathfrak{p} \otimes \operatorname{Tor}_n^R(A, B) = \operatorname{Tor}_n^{R_\mathfrak{p}}(A \otimes_R R_\mathfrak{p}, B \otimes_R R_\mathfrak{p}) = \operatorname{Tor}_n^{R_\mathfrak{p}}(A_\mathfrak{p}, B_\mathfrak{p}).$$

In general,

$$S^{-1} \operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Example 10.17. Suppose A, B are abelian groups. Then there exists I^0 injective such that

$$0 \to B \to I^0 \to I^0/B \to 0$$

is exact, but since I^0 is divisible, $I^1 = I^0/B$ is also divisible, so it is injective. Hence

$$0 \to B \to I^0 \to I^1 \to 0$$

is an injective resolution. This shows that

$$\operatorname{Ext}^n(A, B) = 0 \text{ for } n \geq 2.$$

For $B = \mathbb{Z}$, we get

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

and hence $\operatorname{Ext}^*(A, \mathbb{Z})$ is the homology of

$$0 \to \operatorname{Hom}(A, \mathbb{Q}) \to \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to 0.$$

If A is torsion, then

$$\operatorname{Hom}(A, \mathbb{Q}) = 0.$$

In that case,

$$\operatorname{Ext}^1(A, \mathbb{Z}) = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) = A^*,$$

the Pontryagin dual we defined before.

Proposition 10.18. *Let A be a finitely generated R-module over a commutative Noetherian ring R, B be an R-module, and $S \subseteq R$ be a multiplicative system. Then*

$$S^{-1} \operatorname{Ext}_R^n(A, B) = \operatorname{Ext}_{S^{-1}R}^n(S^{-1}A, S^{-1}B).$$

In particular, for a prime $\mathfrak{p}$, we have

$$\operatorname{Ext}_R^n(A, B)_\mathfrak{p} = \operatorname{Ext}_{R_\mathfrak{p}}^n(A_\mathfrak{p}, B_\mathfrak{p}).$$

Recall that, to check if an R-module M is 0, it is enough to check that for any prime ideal $\mathfrak{p}$, $M_\mathfrak{p} = 0$. So in this case, to check that $\operatorname{Ext}_R^n(A, B) = 0$, it is enough to check that

$$\operatorname{Ext}_{R_\mathfrak{p}}^n(A_\mathfrak{p}, B_\mathfrak{p}) = 0$$

for any prime $\mathfrak{p}$.

We know $\operatorname{Ext}^\bullet(M, N)$ as a measure of failure to <u>Ext</u>end maps. It is a derived functor of Hom in multiple ways. On the one hand, we see the same Ext in lots of different places, but on the other hand this also means objects of Ext are "*slippery*".

We will compare Ext to something more concrete by asking the following question: When does a short exact sequence (of R-modules) split?

Given a short exact sequence, it splits if there is a section

$$\begin{tikzcd} E: & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[d, "1_C"] & 0 \\ & & & & C \arrow[ul, dotted] & \end{tikzcd}$$

In the long exact sequence, we get

$$\begin{array}{ccccc} \mathrm{Hom}_R(C,B) & \longrightarrow & \mathrm{Hom}_R(C,C) & \xrightarrow{\delta} & \mathrm{Ext}^1_R(C,A) \\ ? & \longrightarrow & 1_C & & \\ & & 1_C & \xrightarrow{?} & 0 \end{array}$$

Answer: The short exact sequence E splits if and only if $\delta(1_C) = 0 \in \mathrm{Ext}^1_R(C,A)$.

Definition 10.19. The *obstruction* of E as above is $\theta(E) := \delta(1_C) \in \mathrm{Ext}^1_R(C,A)$.

Remark 10.20. We can also compute the obstruction of E as follows. Take 1_C and lift it to a projective resolution of C, and a map to E

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha"] & P_0 \arrow[r] \arrow[d, "\beta"] & C \arrow[d, "1_C"] & \\ & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Such a lift is unique up to chain homotopy.

We claim that the map $\alpha \in \mathrm{Hom}_R(P_1, A)$ defines the same class as $\theta(E) = \delta(1_C)$. We have the following diagram:

$$\begin{tikzcd} \delta(1_C) = u \in \mathrm{Hom}(P_1,A) \arrow[r] & \mathrm{Hom}(P_1,B) & \\ & v \in \mathrm{Hom}(P_0,B) \arrow[u] \arrow[r] & \mathrm{Hom}(P_0,C)) \\ & & \mathrm{Hom}(C,C) \ni 1_C \arrow[u] \end{tikzcd}$$

We choose $1_C \in \mathrm{Hom}(C,C)$, map it to $\mathrm{Hom}(P_0,C)$, lift it to $v \in \mathrm{Hom}(P_0,B)$, map it to $\mathrm{Hom}(P_1,B)$, and lift it to $u \in \mathrm{Hom}(P_1,A)$.

The maps $u, v, 1_C$ give a commutative diagram as above, with α replaced with u and β replaced with v. The chain map is chain homotopic to the original map, and hence u and v give the same class in $\mathrm{Ext}^1_R(C,A)$.

Definition 10.21. An *extension* in an abelian category $\mathcal{A}$ is a short exact sequence

$$E: \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

(an *extension of* C *by* A).

An *isomorphism of extensions* is a commutative diagram

$$\begin{tikzcd} E: & 0 \arrow[r] & A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ E': & 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

By Five Lemma 1.44, the map $B \to B'$ is an isomorphism.

Definition 10.22. The *trivial extension* of C by A is

$$0 \longrightarrow A \xrightarrow{(1,0)} A \oplus C \longrightarrow C \longrightarrow 0$$

Example 10.23. What are the extensions of $\mathbb{Z}/p$ by $\mathbb{Z}$ (in $\mathbb{Z}$-mod)?

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \,?\, \xrightarrow{j} \mathbb{Z}/p \longrightarrow 0$$

- $? = \mathbb{Z} \oplus \mathbb{Z}/p$, the trivial extension.
- $? = \mathbb{Z}$, $i = \cdot p$, $j = \cdot k$ for any $k \in (\mathbb{Z}/p)^{\times}$.

  If two of these are isomorphic extensions,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "="] & \mathbb{Z} \arrow[r, "\cdot k"] \arrow[d] & \mathbb{Z}/p \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r, "\cdot k'"] & \mathbb{Z}/p \arrow[r] & 0 \end{tikzcd}$$

  then $k = k'$. So there can be nonisomorphic extensions with isomorphic middles.

We will write $\mathrm{ext}^1(C, A)$ for the set of isomorphism classes of extensions of C by A.

If there are enough projectives in $\mathcal{A}$, the obstruction map

$$\theta\colon\ \mathrm{ext}^1(C, A) \to \mathrm{Ext}^1(C, A)$$

is well-defined. (For an isomorphism $E \cong E'$, we get an isomorphism of long exact sequenes.) In fact, more is true.

Theorem 10.24. *The map* $\theta\colon\ \mathrm{ext}^1(C, A) \to \mathrm{Ext}^1(C, A)$ *is a bijection if there are enough projectives in* $\mathcal{A}$.

Proof. Let us construct an inverse (we work with R-modules, but the same construction works in general).

Given $\eta \in \mathrm{Ext}^1(C, A)$, choose a projective resolution $P_\bullet \to C$, and a representative $\phi \in \mathrm{Hom}_R(P_1, A)$:

$$\begin{tikzcd} P_2 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\phi"] & P_0 \arrow[r, "d_0"] & C \arrow[r] & 0 \\ & A & & & \end{tikzcd}$$

Take the pushout and extend it to an isomorphism of its cokernel:

$$\begin{tikzcd} P_2 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\phi"] & P_0 \arrow[r, "d_0"] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Since ϕ represents a cocycle, it factors through $S = P_1/\operatorname{im}(P_2)$. Hence we have another pushout square (we give it the same name by abuse of notation):

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "d_1"] \arrow[d, "\phi"] & P_0 \arrow[r, "d_0"] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

where the map $A \to B$ is injective, because $S \to P_0$ is injective.

We use this short exact sequence in the setting above:

$$\begin{tikzcd} & P_2 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\phi"] & P_0 \arrow[r, "d_0"] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ E: & 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

By the Remark 10.20 earlier, $\theta(E) = [\phi] \in \operatorname{Ext}^1_R(C, A)$.

To conclude that this inverse construction is well-defined, we need to show that the same Ext^1-classes of maps give the same extension.

This follows from the fact that the construction of our extension from ϕ came as a pushout of

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "d_1"] \arrow[d, "\phi"] & P_0 \arrow[r, "d_0"] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

since the maps $S \to A$ are independent of coboundary. □

Remark 10.25.

- We can generalize this construction to higher Ext's (bijective to isomorphism classes of longer exact sequences).

- There is a way to add extensions that is compatible with θ.
- We can *multiply* $\mathrm{Ext}^i(C,A) \otimes \mathrm{Ext}^j(D,C) \to \mathrm{Ext}^{i+j}(D,A)$ that comes from splicing exact sequences.
- We have a notion of Ext^i in any abelian category.

11. Universal coefficients theorem

Recall that a right R-module is flat if and only if $\mathrm{Tor}_1(A,M)=0$ for all M if and only if $\mathrm{Tor}_n(A,M)=0$ for all M and all $n \geq 1$. Moreover, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then

(1) if A, C are flat then B is flat,
(2) if B, C are flat then A is flat.

To see this, we just look at the long exact sequence for Tor.

Set up: let R be a ring, $P_\bullet$ a complex of flat R-modules, M an R-module.

Theorem 11.1 (Künneth). *Assume $B_n(P_\bullet)$ is R-flat for all n (for example, if $R=\mathbb{Z}$ or any PID or a field). There is a natural short exact sequence*

$$0 \longrightarrow H_n(P_\bullet) \otimes_R M \longrightarrow H_n(P_\bullet \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}(P_\bullet), M) \longrightarrow 0$$

Examples 11.2.

(1) If R is a field, $\mathrm{Tor}_1^R(-,-)=0$, we get the obvious isomorphism
$$H_n(P_\bullet) \otimes_R M \cong H_n(P_\bullet \otimes_R M),$$
because $- \otimes_R M$ is exact when R is a field.

(2) Let $R=\mathbb{Z}$, $P_\bullet = \mathbb{Z} \xrightarrow{2} \mathbb{Z}$, $M=\mathbb{Z}/2$. Then
$$H_i(P_\bullet) = \begin{cases} \mathbb{Z}/2 & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases}$$
But
$$P_\bullet \otimes_R M = \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2$$
and hence
$$H_i(P_\bullet \otimes_R M) = \begin{cases} \mathbb{Z}/2 & \text{if } i=0,1 \\ 0 & \text{otherwise} \end{cases}$$
To see this via Künneth Theorem 11.1, we note that
$$0 \longrightarrow H_1(P_\bullet \otimes M) \xrightarrow{\cong} \underbrace{\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}/2)}_{=\mathbb{Z}/2}$$

(3) (Non-example). Let $R=\mathbb{Z}/4$, $M=\mathbb{Z}/2$, and $P_\bullet = \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$. If Künneth was true, we would get

$$0 \longrightarrow \underbrace{H_1(P_\bullet) \otimes_R M}_{=\mathbb{Z}/2 \otimes_{\mathbb{Z}/4} \mathbb{Z}/2 = \mathbb{Z}/2} \longrightarrow \underbrace{H_1(P_\bullet \otimes_R M)}_{H_1(\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2) = \mathbb{Z}/2} \longrightarrow \underbrace{\mathrm{Tor}_1^R(H_0(P_\bullet), M)}_{\mathrm{Tor}_1^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = Z/2} \longrightarrow 0$$

This is impossible for cardinality reasons.

Above, to find $\mathrm{Tor}_1^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$, we take the resolution

$$K_\bullet : \qquad \cdots \longrightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

and note that

$$\mathrm{Tor}_1^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = H_1(K_\bullet \otimes_{\mathbb{Z}/4} \mathbb{Z}/2) = \mathbb{Z}/2.$$

Proof of Künneth Theorem 11.1. Note that $B_\bullet(P) \subseteq Z_\bullet(P) \subseteq P_\bullet$ are complexes where $Z_\bullet(P)$ and $B_\bullet(P)$ have trivial boundary maps.

We claim that $Z_n(P_\bullet)$ is a flat R-module. We have a short exact sequence

$$0 \longrightarrow Z_n(P_\bullet) \longrightarrow P_n \xrightarrow{d} B_{n-1}(P_\bullet) \longrightarrow 0$$

As each $B_{n-1}(P_\bullet)$ is flat, this also shows that $Z_n(P_\bullet)$ is flat. The long exact sequence in homology gives

$$\begin{tikzcd} \cdots \arrow[r] & H_n(B_\bullet(P) \otimes M) \arrow[r, "\alpha_n"] & H_n(Z_\bullet(P) \otimes M) \arrow[r] & H_n(P_\bullet \otimes M) \arrow[dll] \\ & H_{n-1}(B_\bullet(P) \otimes M) \arrow[r, "\alpha_n"] & H_{n-1}(Z_\bullet(P) \otimes M) \arrow[r] & \cdots \end{tikzcd}$$

and hence we have the short exact sequence

$$(*) \qquad 0 \longrightarrow \mathrm{coker}(\alpha_n) \longrightarrow H_n(P_\bullet \otimes_R) \longrightarrow \ker(\alpha_{n-1}) \longrightarrow 0$$

As $Z_\bullet(P)$ has a trivial differential, the same is true for $Z_\bullet(P) \otimes M$ and $B_\bullet(P) \otimes M$. This shows that

$$H_n(Z_\bullet(P) \otimes M) = Z_n(P) \otimes M,$$
$$H_n(B_\bullet(P) \otimes M) = B_n(P) \otimes M.$$

But since $B_n(P)$ and $Z_n(P)$ are flat, and we have the short exact sequence

$$0 \longrightarrow B_n(P) \longrightarrow Z_n(P) \longrightarrow H_n(P) \longrightarrow 0,$$

this is a flat resolution of $H_n(P)$. Therefore

$$H_\bullet(B_n(P) \otimes M \xrightarrow{\alpha_n} Z_n(P) \otimes M) = \mathrm{Tor}_\bullet^R(H_n(P), M),$$

since Tor can be calculated using flat resolutions 10.13. This shows that

$$\mathrm{coker}(\alpha_n) = \mathrm{Tor}_0^R(H_n(P), M) = H_n(P) \otimes M,$$
$$\ker(\alpha_n) = \mathrm{Tor}_1^R(H_n(P), M).$$

Substituting this into the short exact sequence $(*)$ completes the proof. $\square$

Corollary 11.3. *Assume $R = \mathbb{Z}$ (or $B_n(P_\bullet)$ is free). Then for all M, the Künneth sequence splits (non-canonically), i.e. we have isomorphisms*

$$(H_n(P_\bullet) \otimes M) \oplus \operatorname{Tor}_1^R(H_{n-1}(P), M) \cong H_n(P_\bullet \otimes M).$$

Proof. We know that each $d(P_n)$ is free. Hence the short exact sequence

$$0 \longrightarrow Z_n \longrightarrow P_n \longrightarrow d(P_n) \longrightarrow 0$$

splits (non-canonically!), and so $P_n \cong Z_n \oplus d(P_n)$. Taking $- \otimes M$ of both sides, we obtain

$$Z_n \otimes M \subseteq \ker(d_n \otimes 1) \subseteq P_n \otimes M \cong Z_n \otimes M \oplus d(P_n) \otimes M.$$

We hence get that $\ker(d_n \otimes 1) \cong Z_n \otimes M \oplus C$, for some complement C. Taking the quotient by $\operatorname{im}(d_{n+1} \otimes 1) = \operatorname{im}(d_{n+1}) \otimes M$, we get

$$H_n(P_\bullet \otimes M) \cong H_n(P_\bullet) \otimes M \oplus C.$$

Hence the Künneth exact sequence

$$0 \longrightarrow H_n(P_\bullet) \otimes_{\mathbb{Z}} M \longrightarrow H_n(P_\bullet \otimes_{\mathbb{Z}} M) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P_\bullet), M) \longrightarrow 0$$

splits, with $C \cong \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P), M)$. $\square$

Example 11.4. Let X be a topological space and $S_\bullet(X)$ be the singular chain complex. If M is some abelian group,

$$H_n(X; M) = H_n(S_\bullet(X) \oplus M) = H_n(S_\bullet(X)) \otimes M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(S_\bullet(x)), M),$$

and hence

$$H_n(X; M) = H_n(X) \otimes M \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), M).$$

So, to calculate the homology groups with coefficients in M, it is enough to calculate them with coefficients in $\mathbb{Z}$ (but the splitting is **not** functorial, so this does not tell us anything about the maps between the homology groups).

For example, let $X = \mathbb{P}^2(\mathbb{R})$ and $M = \mathbb{Z}/2$. We then have:

$$H_0(X) = \mathbb{Z},\ H_1(X) = \mathbb{Z}/2,\ H_2(X) = 0,$$

and hence

$$H_2(X; \mathbb{Z}/2) = \underbrace{H_2(X) \otimes_Z \mathbb{Z}/2}_{=0} \oplus \operatorname{Tor}_1(H_1(X), \mathbb{Z}/2) = \operatorname{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2.$$

There is an analog of the corollary for cohomology.

Theorem 11.5. *Suppose $P_\bullet$ is a chain complex of left R-modules such that $d(P_n)$ is projective for all n. Then we have*

$$H^n(\operatorname{Hom}_R(P, M)) \cong \operatorname{Hom}_R(H_n(P_\bullet), M) \oplus \operatorname{Ext}_R^1(H_{n-1}(P_\bullet), M).$$

Example 11.6. Again, let $X = \mathbb{P}^2(\mathbb{R})$. By the above result and knowing the homology groups of X, we obtain

$$\begin{aligned} H^0(X) &= \mathbb{Z}, \\ H^1(X) = H^1(X;\mathbb{Z}) &= \underbrace{\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z})}_{=0} \oplus \underbrace{\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z})}_{=0}, \\ H^2(X) &= \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2,\mathbb{Z}) = \mathbb{Z}/2. \end{aligned}$$

Let $P_\bullet$ be a complex of right R-modules and $Q_\bullet$ be a complex of left R-modules. We have the double complex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & P_0 \otimes Q_2 \arrow[l] \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & P_0 \otimes Q_1 \arrow[l] \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & P_0 \otimes Q_0 \arrow[l] \arrow[d] \arrow[r, leftrightarrow] & P_1 \otimes Q_0 \arrow[d] & P_2 \otimes Q_0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and the total complex is

$$(P \otimes_R Q)_n = \bigoplus_{p+q=n} P_p \otimes Q_q.$$

There is an analog of Künneth Theorem for the total complex.

Theorem 11.7. *If P_n, $d(P_n)$ are flat for all n, then*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(P_\bullet) \otimes H_q(Q_\bullet) \longrightarrow H_n(P_\bullet \otimes Q_\bullet) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(P_\bullet), H_q(Q_\bullet)) \longrightarrow 0$$

is exact.

For topological spaces X, Y, (after some work) this gives the result

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \oplus \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(Y)).$$

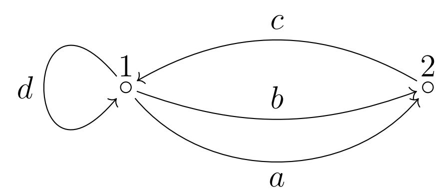
12. Quivers

Definition 12.1. A *quiver* is a finite direction graph, consisting of $Q = (Q_0, Q_1, h, t)$, where

- Q_0 is a finite set of vertices,
- Q_1 is a finite set of arrows,

- $h\colon Q_1 \to Q_0$ is the *head* map; $h(a) \in Q_0$ is the *head* of arrow $a \in Q_1$,
- $t\colon Q_1 \to Q_0$ is the *tail* map; $t(a) \in Q_0$ is the *tail* of arrow $a \in Q_1$.

Example 12.2. The picture



represents a quiver with $Q_0 = \{1, 2\}$, $Q_1 = \{a, b, c, d\}$ and

$$t(a) = t(b) = h(c) = t(d) = h(d) = 1,$$
$$h(a) = h(b) = t(c) = 2.$$

Definition 12.3. A *path* p of length $d \geq 1$ is a sequence

$$p = a_d a_{d-1} \dots a_2 a_1$$

where $t(a_{i+1}) = h(a_i)$ for $i = 1, 2, \dots, d-1$. Then $h(p) = h(a_d)$ is the *head* of path p, and $t(p) = t(a_1)$ is the *tail* of path p. Also, for every $x \in Q_0$, we have a path e_x of length 0 with $h(e_x) = t(e_x) = x$.

Example 12.4. In the example above, $p = bdca$ is a path with $t(p) = t(a) = 1$ and $h(p) = h(b) = 2$.

While the order in which the arrows in a path are written may seem strange at first, note that it is the same as composition of functions. This will be useful later on, when we discuss representations of quivers — the arrows will be *represented by* certain functions and paths indeed become compositions of them.

Definition 12.5. If p, q are paths and $t(p) = h(q)$, say

$$p = a_d a_{d-1} \dots a_1,\ q = b_e b_{e-1} \dots b_1,$$

then

$$pq = a_d a_{d-1} \dots a_1 b_e b_{e-1} \dots b_1$$

is the *composition*. If $t(p) = x$, then $pe_x = p$, and if $h(p) = y$, then $e_y p = p$.

We can associate a category P_Q to a quiver Q:

- objects are elements of Q_0,
- $\mathrm{Hom}_{P_Q}(y, x) = \{\text{paths } p \text{ from } x \text{ to } y\}$, i.e. $h(p) = y$, $t(p) = x$,
- $\mathrm{id}_x = e_x$,
- the composition map $\mathrm{Hom}_{P_Q}(z, y) \times \mathrm{Hom}_{P_Q}(y, x) \to \mathrm{Hom}_{P_Q}(z, x)$ is given by path composition, as defined above: $(p, q) \mapsto pq$.

Throughout the rest of this section, we will make the following distinction: for a field K, we will write K-mod for the category of **finite-dimensional** K-vector spaces, and K-Mod for the category of all K-vector spaces.

Definition 12.6. The category of *representations of* Q over a field K is

$$\mathrm{Rep}_K(Q) = (K\text{-mod})^{P_Q}.$$

Explicitly, the objects in $\mathrm{Rep}_K(Q)$ are determined by a set of finite-dimensional vector spaces $V(x)$ for each $x \in Q_0$, and K-linear maps

$$V(a)\colon V(ta) \to V(ha)$$

for each $a \in Q_1$. Moreover,

$$V(e_x) = \mathrm{id}_{V(x)},$$
$$V(a_d a_{d-1} \dots a_2 a_1) = V(a_d)V(a_{d-1}) \dots V(a_2)V(a_1).$$

If V, W are representations, a morphism $\varphi\colon V \to W$ is a collection of linear maps

$$\varphi(x)\colon V(x) \to W(x)$$

such that

$$\begin{tikzcd} V(t(a)) \arrow[r, "V(a)"] \arrow[d, "\varphi(t(a))"] & V(h(a)) \arrow[d, "\varphi(h(a))"] \\ W(t(a)) \arrow[r, "W(a)"] & W(h(a)) \end{tikzcd}$$

commutes for all arrows a.

Example 12.7. Consider the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r, "a"] & \underset{2}{\circ} \arrow[r, "b"] & \underset{3}{\circ} \end{tikzcd}$$

The paths are e_1, e_2, e_2, a, b, ba. The representations of Q are triples of finite-dimensional K-vector spaces $V(1)$, $V(2)$, $V(3)$ together with maps

$$V(a)\colon V(1) \to V(2),$$
$$V(b)\colon V(2) \to V(3).$$

Definition 12.8. Let Q be a quiver and K be a field. The *path algebra* KQ is defined as

- K-vector space with a basis consisting of all paths in Q,
- if p, q are paths, we define

$$p \cdot q = \begin{cases} pq \text{ (the composition)} & \text{if } t(p) = h(q), \\ 0 & \text{otherwise.} \end{cases}$$

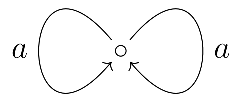
Then KQ is an associative K-algebra with $1 = \sum_{x \in Q_0} e_x$.

Example 12.9. Consider the quiver Q:

$$\begin{tikzcd} \circ\, 1 \arrow[loop left, "a"] \end{tikzcd}$$

The paths are $e_1, a, a^2, a^2, \dots$, and hence $KQ = K[a]$, the polynomial ring in a.

Example 12.10. Consider the quiver Q:



Then $KQ = K\langle a, b\rangle$ is the free associative algebra generated by a and b (non-commutative).

Example 12.11. Consider the quiver

$$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \quad \cdots \quad \underset{n-1}{\circ} \longrightarrow \underset{n}{\circ}$$

Then KQ is isomorphic to the algebra of lower-triangular $n \times n$ matrices.

Theorem 12.12. *The categories KQ-mod (finite-dimensional left KQ-modules) and* $\mathrm{Rep}_K(Q)$ *are equivalent.*

Sketch of the proof. If M is a finite-dimensional KQ-module, then

$$M = \sum_{x \in Q_0} e_x M = \bigoplus_{x \in Q_0} e_x M,$$

and we can define $V(x) = e_x M$. Then

$$e_x e_y = \begin{cases} e_x & x = y, \\ 0 & \text{otherwise,} \end{cases}$$

and for $a \in Q_1$, with $t(a) = x$, $h(a) = y$, the multiplication by a map restricts to

$$V(a) \colon \underbrace{e_x M}_{V(x)} \to \underbrace{e_y M}_{V(y)}.$$

Then we can define $\mathcal{F} \colon KQ\text{-mod} \to \mathrm{Rep}_K(Q)$ by $\mathcal{F}(M) = V$. Conversely, if V is a representation of Q, let

$$M = \bigoplus_{x \in Q_0} V(x),$$

and an arrow $a \in Q_1$ acts on M by

$$\begin{tikzcd} M \arrow[d] & M \\ V(x) \arrow[r, "V(a)"] & V(y) \arrow[u] \end{tikzcd}$$

This defines a map $\mathcal{G} \colon \mathrm{Rep}_K(Q) \to KQ$-mod by letting $\mathcal{G}(V) = M$. Checking the axioms and that $F \circ G$, $G \circ F$ are naturally isomorphic to the identities, the result follows. $\square$

Note that $\dim_K KQ < \infty$ if and only if there are finitely many paths if and only if there are no oriented cycles.

If M is a finite-dimensional KQ-module, then $M = \bigoplus_{x \in Q_0} M(x)$, where $M(x) = e_x M$ and the map

$$a \cdot \colon M \to M$$

restricts to

$$M(a)\colon M(t(a)) \to M(h(a)).$$

Then

$$KQ = \bigoplus_{x \in Q_0} KQe_x = \bigoplus_{x \in Q_0} P_x$$

as left KQ-modules. Then $P_x = KQe_x$ is a projective KQ-module and P_x has a basis consisting of all paths starting at x. Note also that $P_x(y) = e_y P_x = e_y KQ e_x$ is spanned by all paths from x to y.

Example 12.13. For the quiver Q given by

$$\begin{tikzcd} \circ \, 1 \arrow[loop left, "a"] \end{tikzcd}$$

the category $\mathrm{Rep}_K(Q)$ is naturally isomorphic to $K[a]$-mod.

Example 12.14. For the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r] & \underset{2}{\circ} \arrow[r] & \underset{3}{\circ} \end{tikzcd}$$

we have

$$\begin{array}{ll} P_1 : & Ke_1 \longrightarrow Ka \longrightarrow Kba \\ & K \xrightarrow{1} K \xrightarrow{1} K \\ P_2 : & 0 \longrightarrow K \longrightarrow K \\ P_3 : & 0 \longrightarrow 0 \longrightarrow K \end{array}$$

Then note that KQ will be

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

with

$$P_1 = \begin{pmatrix} * \\ * \\ * \end{pmatrix}, P_2 = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix}, P_3 = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}.$$

The map $a\colon x \to y$ corresponds to $P_z(a)\colon P_z(x) \to P_z(y)$ which maps a path p from z to x to the path ap.

Moreover,

$$\mathrm{Hom}_{KQ}(P_x, M) \to M(x)$$
$$(\varphi\colon P_x \to M) \mapsto \varphi(e_x) \in M(X)$$

is an isomorphism, and $\mathrm{Hom}_{KQ}(P_x, -)$ is an exact functor.

Consider $\mathfrak{m} \subseteq KQ$, the (two-sided) ideal generated by all arrow. Then $\mathfrak{m}$ is spanned by all paths of length ≥ 1, and, in general, $\mathfrak{m}^d$ is the ideal spanned by all paths of length $\geq d$.

An ideal $J \subseteq KQ$ is *admissible* if $\mathfrak{m}^d \subseteq J \subseteq \mathfrak{m}^2$ for some d. Then $A = KQ/J$ is a finite-dimensional K-algebra.

Definition 12.15. Two rings A, B are called *Morita equivalent* if A-Mod and B-Mod are equivalent categories.

Theorem 12.16. *A finite-dimensional K-algebra is Morita equivalent to KQ/J where J is an admissible ideal for some quiver Q.*

Denote $e_x + J$ by e_x and $\mathfrak{m} + J/J$ by $\mathfrak{m}$. Then

$$A = \bigoplus_{x \in Q_0} Ae_x = \bigoplus_{x \in Q_0} P_x$$

where $P_x = Ae_x$ is projective. Again,

$$\mathrm{Hom}_A(P_x, M) = M(x) = e_x M$$

and P_x is indecomposable (in fact, the only indecomposable ones).

We then see that A-mod (the category of finite-dimensional left A-modules) has enough projectives.

Note that A^{op}-mod $=$ mod-A is equivalent to A-mod via the map

$$M \mapsto D(M) = M^* = \mathrm{Hom}_K(M, K),$$

$$f \mapsto D(f) = f^*.$$

Then $I_x = (e_x A)^*$, $x \in Q_0$ are the indecomposable injectives.

If Q is a quiver, the simple representations are $P_x/mP_x = S_x$ with

$$S_x(y) = \begin{cases} K & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

and $\mathfrak{m}S_x = 0$.

We then have an exact sequence

$$0 \longrightarrow \bigoplus_{a,\ t(a)=x} P_{h(a)} \longrightarrow P_x \longrightarrow S_x \longrightarrow 0$$

$$P_{h(a)} \ni p \longrightarrow pa$$

which gives a projective resolution of the simple module S_x. In general, if M is any module, a projective resolution is

$$0 \longrightarrow \bigoplus_{a\in Q_1} P(h(a)) \otimes_K M(t(a)) \longrightarrow \bigoplus_{x\in Q_0} P_x \otimes_K M(x) \longrightarrow M \longrightarrow 0$$

$$p \otimes w \longrightarrow pa \otimes w - p \otimes aw$$

$$p \otimes v \longrightarrow pv$$

Then

$$\operatorname{Ext}^n_{KQ}(M, N) = 0 \text{ if } n \geq 2.$$

Example 12.17. Take the quiver

$$\begin{tikzcd} \underset{1}{\circ} \arrow[r] & \underset{2}{\circ} \arrow[r] & \underset{3}{\circ} \end{tikzcd}$$

again and let $A = KQ/(ba)$. Then

$$\begin{aligned} P_1 &: \quad K \longrightarrow K \longrightarrow 0, \\ P_2 &: \quad 0 \longrightarrow K \longrightarrow K, \\ P_3 &: \quad 0 \longrightarrow 0 \longrightarrow K. \end{aligned}$$

We then have that

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0$$

and applying $\operatorname{Hom}_A(-, S_3)$ to $P_\bullet$, we get

$$0 \longrightarrow \underbrace{\operatorname{Hom}_A(P_1, S_3)}_{=0} \longrightarrow \underbrace{\operatorname{Hom}_A(P_2, S_3)}_{=0} \longrightarrow \underbrace{\operatorname{Hom}_A(P_3, S_3)}_{\cong K} \longrightarrow 0.$$

Hence

$$\operatorname{Ext}^2_A(S_1, S_3) = K.$$

Example 12.18. Take the quiver Q

$$\begin{tikzcd} \circ\, 1 \arrow[loop left, "a"] \end{tikzcd}$$

and consider $J = (a^2) \subseteq KQ = K[a]$. Then

$$P_1 = A = KQ/J = K[a]/(a^2).$$

In this case,

$$\cdots \longrightarrow P_1 \xrightarrow{\cdot a} P_1 \xrightarrow{\cdot a} P_1 \longrightarrow S_1 \longrightarrow 0$$

and $\operatorname{Ext}^n(S_1, S_1) = K$ for any $n \geq 0$.

If we look at A-mod for $A = KQ/J$, how can we recover the quiver Q?

- The simple representations are S_x, $x \in Q_0$.
- $\mathrm{Ext}^1_A(S_x, S_y) = K^l$ where l is the number of arrows $x \to y$.

13. Homological dimension

Definition 13.1. Let A be a right R-module.

(1) The *projective (resp. flat) dimension* , $\mathrm{pd}(A) = n$ (resp. $\mathrm{fd}(A)$) is the smallest n such that there is a resolution
$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
such that $P_0, \dots, P_n$ are projective (resp. flat).

(2) The *injective dimension*, $\mathrm{id}(A)$ is the smallest n such that there is an injective resolution
$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0.$$

Lemma 13.2. *The following are equivalent*

(1) $\mathrm{pd}(A) \leq d$,
(2) $\mathrm{Ext}^n_R(A, B) = 0$ *for* $n > d$ *and all right* R*-modules* B,
(3) $\mathrm{Ext}^{d+1}_R(A, B) = 0$ *for all* B,
(4) *if*
$$0 \longrightarrow A_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$
is a resolution of A *with* $P_0, \dots, P_{d-1}$ *projective, then* A_d *is projecitve.*

Proof. We note that trivially, (4) implies (1) implies (2) implies (3). We show (3) implies (4). Suppose (3) is true. Let $A_0 = A$ and define P_k projective and A_{k+1} recursively so that
$$0 \longrightarrow A_{k+1} \longrightarrow P_k \longrightarrow A_k \longrightarrow 0$$
is exact. Then the long exact sequence for Ext gives
$$\underbrace{\mathrm{Ext}^\ell(P_k, B)}_{=0} \longrightarrow \mathrm{Ext}^\ell(A_{k+1}, B) \longrightarrow \mathrm{Ext}^{\ell+1}(A_k, B) \longrightarrow \underbrace{\mathrm{Ext}^{\ell+1}(P_k, B)}_{=0}.$$
Then $\mathrm{Ext}^1(A_d, B) = \cdots = \mathrm{Ext}^{d+1}(A_0, B) = 0$ for all B, and hence $\mathrm{Ext}^1(A_d, B) = 0$, so A_d is projective. □

Dually, we get the following statement.

Lemma 13.3. *The following are equivalent:*

(1) $\mathrm{id}(B) \leq d$,
(2) $\mathrm{Ext}^n_R(A, B) = 0$ *for* $n > d$ *and all* A,
(3) $\mathrm{Ext}^{d+1}_R(A, B) = 0$ *for all* A,
(4) *if*

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{d-1} \longrightarrow A^d \longrightarrow 0$$

is exact and $E^0, \ldots, E^{d-1}$ *are injective then* A^d *is injective.*

Note that

$$\begin{aligned} \sup\{\mathrm{id}(B) \mid B \text{ right } R\text{-module}\} &= \sup\{d \mid \operatorname{Ext}^d(A,B) \neq 0 \text{ for some right } R\text{-modules } A, B\} \\ &= \sup\{\mathrm{pd}(A) \mid A \text{ right } R\text{-module}\}. \end{aligned}$$

Definition 13.4. This is called the *right global dimension of* R, $\mathrm{rgldim}(R)$.

If R is left and right Noetherian, then $\mathrm{lgldim}(R) = \mathrm{rgldim}(R)$.

Recall that for a path algebra KQ, there is a 2-step resolution of any M:

$$0 \to P_1 \to P_0 \to M \to 0,$$

and hence the global dimension KQ is at most 1. Moreover, KQ is semisimple if the global dimension is 0.

We immediately get the following corollary to Baer's criterion for injectivity 6.3.

Corollary 13.5. *We have that*

$$\mathrm{rgldim}(R) = \sup\{\mathrm{pd}(R/I) \mid I \textit{ right } R\textit{-ideal}\}.$$

We also have a similar construction for Tor (and $A \otimes_R B$). The following numbers are the same:

$$\begin{aligned} &\sup\{\mathrm{fd}(A) \mid A \text{ right } R\text{-module}\} \\ &= \sup\{d \mid \operatorname{Tor}_d^R(A,B) \neq 0 \text{ for some } A \in \text{mod-}R, B \in R\text{-mod}\} \\ &= \sup\{\mathrm{fd}(B) \mid B \text{ left } R\text{-module}\} \\ &= \sup\{\mathrm{fd}(R/J) \mid J \text{ right ideal}\} \\ &= \sup\{\mathrm{fd}(R/J) \mid J \text{ left ideal}\} \end{aligned}$$

Definition 13.6. This number is the Tor-*dimension* of R, $\mathrm{tordim}(R)$.

Proposition 13.7. *Assume that R is right Noetherian. Then:*

(1) *for every finitely-generated right R-module A,* $\mathrm{pd}(A) = \mathrm{fd}(A)$*,*
(2) $\mathrm{tordim}(R) = \mathrm{rgldim}(R)$*.*

Proof. We first prove (1). Note that any finitely generated projective module is flat, so $\mathrm{fd}(A) \leq \mathrm{pd}(A)$. If $\mathrm{fd}(A) = d < \infty$, take a resolution of A

$$0 \longrightarrow A_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

with $P_0, \ldots, P_{d-1}$ finitely generated and free. Then A_d is finitely generated (by a lemma analogous to Lemmas 13.2 and 13.3 but for flat dimension), and A_d is flat. Hence A_d is finitely presented and flat, which shows that it is projective. Hence $\mathrm{pd}(A) \leq d$.

Then (2) immediately follows:

$$\begin{aligned} \mathrm{rgldim}(R) &= \sup\{\mathrm{pd}(R/J) \mid J \text{ right ideal}\} \\ &= \sup\{\mathrm{fd}(R/J) \mid J \text{ right ideal}\} \\ &= \mathrm{tordim}(R), \end{aligned}$$

completing the proof. □

Global dimension 0.

Definition 13.8. A ring R is *semi-simple* if every right (equivalently, left) ideal is a direct summand of R.

Theorem 13.9 (Wedderburn's Theorem)**.** *If R is semi-simple, then*

$$R \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i,n_i}(D_i),$$

for division rings D_i.

Theorem 13.10. *The following are equivalent:*

(1) *R is semi-simple,*
(2) *R has right (left) global dimension 0,*
(3) *every R-module is projective,*
(4) *every R-module is injective,*
(5) *all exact sequences split.*

Proof. The proof is clear. □

Example 13.11. Let Q be a quiver and KQ be a path algebra. We have seen that for any KQ-module A, we have a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and hence $\mathrm{lgldim}(KQ) = \mathrm{rgldim}(KQ) \leq 1$.

If $R = KQ/J$ (a finite-dimensional K-algebra) with $\mathfrak{m}^d \subseteq J \subseteq \mathfrak{m}^2$ (so J is admissible). If $J = 0$, $\operatorname{gldim} KQ \leq 1$ and in fact

$$\operatorname{gldim} KQ = \begin{cases} 0 & \text{if } Q \text{ has no arrows,} \\ 1 & \text{if } Q \text{ has arrows.} \end{cases}$$

If $J \neq 0$, then in fact $\operatorname{gldim} KQ/J \geq 2$.

13.1. **Von Neumann regular rings.**

Definition 13.12. A ring R is von Neumann regular if for any $a \in R$, there exists $b \in R$ such that $aba = a$.

Example 13.13. Let k^X be the set of functions $X \to k$ where k is a field and X is a set. For $a\colon X \to k$, define

$$b(x) = \begin{cases} \frac{1}{a(x)} & \text{if } a(x) \neq 0, \\ 0 & \text{if } a(x) = 0, \end{cases}$$

whence $aba = a^2 b = a$.

Example 13.14. The ring $\mathrm{Mat}_{n\times n}(k)$ for a field k. For a matrix A, we have a map $k^n \to \operatorname{im} A$ and we can choose a splitting $B\colon \operatorname{im} A \to k^n$. Extend B to $k^n \to k^n$ to get $ABA = A$.

Suppose R is von Neumann regular. If $a \in R$, then there exists $b \in R$ such that $aba = a$. Then $e = ab$ is an idempotent, $e^2 = ababa = ab = e$. We then have that

$$aR \supseteq abR = eR \supseteq abaR = aR,$$

so $aR = eR$.

Lemma 13.15. *A finitely generated right (or left) ideal is generated by one idempotent.*

Proof. Suppose R is commutative. If e, f are both idempotent,

$$(e + f - ef) = (e, f),$$

since $e(e+f-ef) = e^2+ef-e^2f = e+ef-ef = e$ and similarly for f. The non-commutative case is similar. □

Example 13.16. Define $R \subseteq \mathbb{R}^{\mathbb{R}}$ be the ring

$$R = \{f \mid f \text{ almost constant}\},$$

so for $f \in R$, there exists c such that $f(x) = c$ for all but finitely many x. Then R is von Neumann regular. However,

$$I = \{f \in R \mid f(x) = 0 \text{ for } x \in \mathbb{R} \setminus \mathbb{Z}\}$$

is not finitely generated; indeed, we need elements with $f(a) \neq 0$ for arbitrarily large $a \in \mathbb{Z}$, but for finitely many $f \in I$ will have $f(a) = 0$ for a large enough.

If I is a finitely generated right ideal then for an idempotent e

$$I = eR$$

and hence

$$R = eR \oplus (1 - e)R \cong I \oplus R/I,$$

so R/I is projective, and hence flat.

If I is not finitely generated, then

$$I = \varinjlim I_\alpha \text{ for } I_\alpha \text{ finitely generated ideal}$$

and

$$R/I = \varinjlim R/I_\alpha,$$

For all left R-modules M, we have

$$\mathrm{Tor}_1(R/I, M) = \varinjlim \mathrm{Tor}_1(R/I_\alpha, M) = 0$$

and hence R/I is flat. Therefore, $\mathrm{fd}(M) = 0$, and hence

$$\mathrm{tordim}(R) = 0.$$

13.2. **Global dimension of polynomial rings.** We will show that for a field k,

$$\operatorname{gldim}(k[x_1,\dots,x_n]) = n. \tag{1}$$

Hilbert showed that if M is finitely generated, then it has a free resolution of length n

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

showing that the global dimension is at most n.

Writing $R = k[x_1,\dots,x_n]$, the R-module k has the *Koszul resolution*

$$0 \longrightarrow R^{\binom{n}{n}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{2}} \longrightarrow R^n \longrightarrow R \longrightarrow k \longrightarrow 0$$

of length n. Taking $\operatorname{Hom}(-,k)$ of this sequence of this resolution, we get

$$0 \longrightarrow k^{\binom{n}{n}} \xrightarrow{\ 0\ } \cdots \xrightarrow{\ 0\ } k^{\binom{n}{2}} \xrightarrow{\ 0\ } k^{\binom{n}{1}} \xrightarrow{\ 0\ } k^{\binom{n}{0}} \xrightarrow{\ 0\ } 0$$

showing that

$$\operatorname{Tor}_j^R(k,k) = k^{\binom{n}{j}}.$$

In particular, this will show equation (1).

Proposition 13.17. *If $f\colon R \to S$ is a ring homomorphism and M is an S-module, then*

$$\operatorname{pd}_R(M) \le \operatorname{pd}_S(M) + \operatorname{pd}_R(S).$$

Proof. Let $\operatorname{pd}_S(M) = n$, $\operatorname{pd}_R(S) = d$ and choose a projective S-resolution of M,

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

and let $M_0 = M$ and

$$0 \longrightarrow M_{i+1} \longrightarrow Q_i \longrightarrow M_i \longrightarrow 0.$$

Choose projective R-resolutions of M_i's. Then the Horseshoe Lemma 5.24 gives projective resolutions $P_{\bullet j} \to Q_j$. We then have a double complex (by adjusting the signs of the maps appropriately)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & Q_2 \arrow[l] \arrow[d] & P_{02} \arrow[l] \arrow[d] & P_{12} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & Q_1 \arrow[l] \arrow[d] & P_{01} \arrow[l] \arrow[d] & P_{11} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & Q_0 \arrow[l] \arrow[d] \arrow[r, leftrightarrow] & P_{01} & P_{10} \arrow[l] & \cdots \arrow[l] \\
 & M \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

with exact rows. The total complex gives a map $\mathrm{Tot}(P_{\bullet\bullet}) \to M$ but this projective resolution could be large, even infinite. However, note that $\mathrm{pd}_R(Q_1) \leq \mathrm{pd}_R(S) = d$ because Q_i is a direct summand of a free S-module. We replace $P_{d,i}$ by $P_{d,i}/\operatorname{im} P_{d+1,i}$ to get

$$\mathrm{Tot}(P_{\bullet\bullet}) \to M,$$

a projective R-resolution of M. Then we obtain

$$\mathrm{pd}_R(M) \leq n + d = \mathrm{pd}_S(M) + \mathrm{pd}_R(S),$$

as required. □

Lemma 13.18. *Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of R-modules. Then

$$\mathrm{pd}_R(B) \leq \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(C)\}$$

and if the inequality is strict, then $\mathrm{pd}_R(C) = \mathrm{pd}_R(A) + 1$.

Proof. By the long exact sequence for Ext, we get

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^i(C,M) \arrow[r] & \mathrm{Ext}^i(B,M) \arrow[r] & \mathrm{Ext}^i(A,M) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
 & \mathrm{Ext}^{i+1}(C,M) \arrow[r] & \mathrm{Ext}^{i+1}(B,M) \arrow[r] & \mathrm{Ext}^{i+1}(A,M) \arrow[r] & \cdots
\end{tikzcd}$$

If $i = \mathrm{pd}_R(B)$, then for some R-module M, we obtain $\mathrm{Ext}^i(B,M) \neq 0$, and hence one of the neighboring terms in the long exact sequence above are non-zero, so $\mathrm{Ext}^i(C,M) \neq 0$ or $\mathrm{Ext}^i(A,M) \neq 0$, showing that $\mathrm{pd}_R(C) \geq i$ or $\mathrm{pd}_R(A) \geq i$.

If the inequality is strict, then for any $i > \mathrm{pd}_R(B)$, we get $\mathrm{Ext}^i(B,M) = \mathrm{Ext}^{i+1}(B,M) = 0$, so $\mathrm{Ext}^i(A,M) \cong \mathrm{Ext}^{i+1}(C,M)$ by the long exact sequence above, which shows that $\mathrm{pd}_R(C) = \mathrm{pd}_R(A) + 1$. □

Let R be a ring, $x \in R$ be central, A a left R-module.

Definition 13.19. An element x is a nonzero divisor on A if $xy = 0$ implies that $y = 0$ for all $y \in A$.

Suppose x is a nonzero divisor on R. We have a short exact sequence

$$0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow R/x \longrightarrow 0.$$

We apply $- \otimes_R A$ to get

$$\underbrace{\mathrm{Tor}_1(R,A)}_{=0} \longrightarrow \underbrace{\mathrm{Tor}_1(R/x,A)}_{\{y\in A \ \mid \ xy=0\}} \longrightarrow A \xrightarrow{x\cdot} A \longrightarrow A/xA \longrightarrow 0$$

and x is a nonzero divisor on A if and only if $\mathrm{Tor}_1(R/x,A) = 0$.

Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring with $\mathfrak{m}$ its unique maximal ideal.

Definition 13.20. A *regular sequence* in a finitely-generated R-module A is a sequence $x_1,\dots,x_n \in \mathfrak{m}$ such that x_i is a nonzero divisor on $A/(x_1,\dots,x_{i-1})A$.

The *depth of* A, $\mathrm{depth}(A)$ is the largest n such that there exists a regular sequence of length n on A.

Theorem 13.21 (Auslander–Buchsbaum)**.** *If R is a commutative Noetherian local ring and A is a finitely generated R-module with $\mathrm{pd}(A) < \infty$, then*

$$\mathrm{depth}(R) = \mathrm{depth}(A) + \mathrm{pd}(A).$$

Definition 13.22. The *Krull dimension*, $\dim R$, of R is the maximal n such that there exists a chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n \subset R.$$

If $k = R/\mathfrak{m}$, a field, we get

$$\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geq \dim R.$$

Definition 13.23. The local ring R is *regular* if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim R$.

We also have that

$$\mathrm{depth}(R) \leq \dim R.$$

Definition 13.24. A local ring R is called *Cohen–Maccaulay* if $\mathrm{depth}(R) = \dim R$.

There are various relationships between regular rings and Cohen–Maccaulay rings, even though they are not equivalent.

For a proof of Theorem 13.21, see [Eis95, Chap. 19]. The general idea is to understand what happens when we go from an R-module A to the R/x-module A/xA.

Theorem 13.25 (First Change of Rings Theorem)**.** *Let R be a ring, $x \in R$ be central, nonzero divisor, A is an R/x-module with $\mathrm{pd}_{R/x}(A)$ finite. Then $\mathrm{pd}_R(A) = 1 + \mathrm{pd}_{R/x}(A)$.*

Sketch of proof. If $\mathrm{pd}_{R/x}(A) = 0$, A is a projective R/x-module, and then A is not a projective R-module, because $xA = 0$. Then

$$1 \leq \mathrm{pd}_R(A) \leq \mathrm{pd}_R(A/x) = 1,$$

since $0 \to R \to R \to R/x \to 0$ is a projective resolution of R/x.

The general argument now goes by induction of $\mathrm{pd}_{R/x}(A)$. Assume $\mathrm{pd}_{R/x}(A) \geq 1$. Take P projective R/x-module with an exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow A \longrightarrow 0.$$

Since $1 + \mathrm{pd}_{R/x}(M) = \mathrm{pd}_{R/x}(A)$, we can apply the inductive hypothesis to get

$$\mathrm{pd}_R(M) = 1 + \mathrm{pd}_{R/x}(M).$$

By Lemma 13.18, we get that

$$\mathrm{pd}_R(P) \leq \max\{\mathrm{pd}(M), \mathrm{pd}_R(A)\}$$

and either equality holds or $\mathrm{pd}_R(A) = \mathrm{pd}_R(M) + 1$. In the first case, we get a contradiction. In the second case,

$$\mathrm{pd}_R(A) = \mathrm{pd}_R(M) + 1 = \mathrm{pd}_{R/x}(A) + 1,$$

as required. □

Theorem 13.26 (Second Change of Rings Theorem)**.** *Let $x \in R$ be a central nonzero divisor on R and on A. Then*

$$\mathrm{pd}_R(A) \geq \mathrm{pd}_{R/x}(A/xA).$$

Corollary 13.27. *If A is an R-module and we write $A[x] = R[x] \otimes_R A$, we get that*

$$\mathrm{pd}_{R[x]}(A[x]) = \mathrm{pd}_R(A).$$

Proof. The $\geq$ inequality follows from Second Change of Rings Theorem 13.26. The $\leq$ is immediate, since a projective resolution $P_\bullet \to A$ of A gives a projective resolution

$$P_\bullet[x] \to A[x]$$

of $A[x]$. □

Theorem 13.28. *We have that* $\mathrm{gldim}\, R[x] = \mathrm{gldim}\, R + 1$.

Proof. If M is an $R[x]$-module, then M is an R-module, and we will write $\widetilde{M}$ for M as an R-module. We have the following exact sequence

$$0 \longrightarrow R[x] \otimes_R \widetilde{M} \longrightarrow R[x] \otimes_R \widetilde{M} \longrightarrow \underbrace{R[x] \otimes_{R[x]} M}_{=M} \longrightarrow 0$$

$$p \otimes v \longrightarrow px \otimes v - p \otimes xv$$

By a similar result to Lemma 13.18: if we have a short exact sequence $0 \to A \to B \to C \to 0$, we get

$$\mathrm{pd}(C) \leq \max\{\mathrm{pd}(B), \mathrm{pd}(A) + 1\}.$$

Hence

$$\mathrm{pd}_{R[x]}(M) \leq \mathrm{pd}_{R[x]}(R[x] \otimes_R \widetilde{M}) + 1 \leq \mathrm{pd}_R(\widetilde{M}) + 1 = \mathrm{pd}_R(M) + 1 \leq \mathrm{gldim}\, R + 1.$$

Taking the supremum over all M, we get one inequality. We skip the proof of the other inequality. $\square$

Corollary 13.29. *For a field k, we have that*

$$\mathrm{gldim}\, k[x_1, \ldots, x_n] = n.$$

Let R be a ring and R^* be the group of unites of R. We define the *Jacobson radical* as

$$J(R) = \{r \in R \mid \text{for any } s \in R,\ 1 - rs \in R^*\}$$

and one can prove that

$$J(R) = \{r \in R \mid \text{for any } s \in R,\ 1 - sr \in R^*\}$$

$$J(R) = \bigcap_{\mathfrak{m}} \mathfrak{m}$$

where the intersection can be over left maximal ideals $\mathfrak{m}$ or over right maximal ideals $\mathfrak{m}$.

Example 13.30. If $R = KQ/I$ for an admissible ideal I so that $\mathfrak{m}^d \subseteq I \subseteq \mathfrak{m}^2$, then $J(R) = \mathfrak{m}$.

If $(R, \mathfrak{m})$ is a local commutative ring, clearly $J(R) = \mathfrak{m}$.

Proposition 13.31 (Nakayama Lemma)**.** *Let $\mathfrak{m}$ be the Jacobson radical of R. If B is a finitely generated left R-module and $\mathfrak{m}B = B$ then $B = 0$.*

Proof. Suppose $B \neq 0$, $\{b_1, \ldots, b_n\}$ minimal set of generators over B. Then $b_n \in B = \mathfrak{m}B$, and we can write

$$b_n = \sum_{i=1}^{n} r_i b_i \quad \text{for } r_i \in \mathfrak{m}.$$

Then

$$(1 - r_n)b = \sum_{i=1}^{n-1} r_i b_i \in Rb_1 + \cdots + Rb_{n-1}$$

but $r_n \in \mathfrak{m}$ so $1 - r_n \in R^*$, and so

$$b_n \in Rb_1 + \cdots + Rb_{n-1},$$

and hence $b_1, \ldots, b_{n-1}$ generate B. This contradicts minimality of the set of generators. $\square$

In what follows, assume $(R, \mathfrak{m})$ is a local Noetherian, commutative ring and $k = R/\mathfrak{m}$.

Corollary 13.32. *Let B be a left finitely generated R-module. Elements $b_1, \ldots, b_n \in B$ generate B if and only if the images $b_1, \ldots, b_n$ in $B/\mathfrak{m}B$ span $B/\mathfrak{m}B$ as a k-vector space.*

Note that the finitely-generated assumption is necessary: $\mathbb{Q}_2$ is a $\mathbb{Z}_2$-module with

$$\mathfrak{m}\mathbb{Q}_2 = (2)\mathbb{Q}_2 = \mathbb{Q}_2,$$

so $\mathbb{Q}_2/\mathfrak{m}\mathbb{Q}_2 = 1$ which is spanned by the only element as a k-vector space, but $\mathbb{Q}_2$ over $\mathbb{Z}_2$ is not finitely generated.

Proof. For the 'only if' direction, if $A = Rb_1 + \cdots + Rb_n$ and $B = A + \mathfrak{m}B$. Then $\mathfrak{m} \cdot B/A = B/A$, so $B/A = 0$ and hence $A = B$. □

Corollary 13.33. *Elements $b_1, \ldots, b_n \in B$ are a minimal set of generators if and only if the images of $b_1, \ldots, b_n$ in $B/\mathfrak{m}B$ form a basis of $B/\mathfrak{m}B$ as a k-vector space.*

Proposition 13.34. *If P is a finitely generated projective R-module, then P is free.*

Proof. Let $n = \dim_k(P/\mathfrak{m}P)$. By lifting the generators of $P/\mathfrak{m}P$ as a k-vector space, we get a minimal set of generators for P, giving a short exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

where K is the kernel of the map $R^n \to P$. Since P is projective, this sequence splits, so

$$R^n \cong P \oplus K.$$

Taking $- \otimes_R R/\mathfrak{m}$, we get

$$k^n \cong P/\mathfrak{m}P \oplus K/\mathfrak{m}K \cong k^n \oplus K/\mathfrak{m}K$$

and hence $K = \mathfrak{m}K$, so by Nakayama Lemma 13.31, $K = 0$. This shows $R^n \cong P$. □

If A is a finitely generated R-module. Let $A_0 = A$ and, recursively, having defined A_i, let

$$\beta_i = \dim_k A_i/\mathfrak{m}A_i < \infty$$

and define A_{i+1} as the following kernel

$$0 \longrightarrow A_{i+1} \longrightarrow R^{\beta_i} \longrightarrow A_i \longrightarrow 0.$$

This gives a free resolution of A:

$$\cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow A \longrightarrow 0.$$

Apply $- \otimes_R R/\mathfrak{m}$ to

$$\cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow 0$$

to get

$$\cdots \longrightarrow k^{\beta_2} \xrightarrow{0} k^{\beta_1} \xrightarrow{0} k^{\beta_0} \longrightarrow 0$$

(checking that the maps are indeed 0 is an exercise). This shows that

$$\mathrm{Tor}_i^R(A, R/\mathfrak{m}) = k^{\beta_i}.$$

Definition 13.35. The numbers β_i are called *Betti numbers* of A.

Lemma 13.36. *If* $\operatorname{depth}(R) = 0$ *and* A *is a finitely generated* R*-module, then* $\operatorname{pd}(A) = 0$ *or* $\operatorname{pd}(A) = \infty$*.*

Proof. Suppose $0 < \operatorname{pd}(A) = n < \infty$. Take a resolution

$$0 \longrightarrow B \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

with $F_0, \ldots, F_{n-2}$ free. Then

$$\operatorname{pd}(B) = n - (n-1) = 1.$$

Let $t = \dim_k B/\mathfrak{m}B$ and consider

$$0 \longrightarrow P \longrightarrow R^t \longrightarrow B \longrightarrow 0.$$

Then P is projective (since $\operatorname{pd}(P) = \operatorname{pd}(B) - 1 = 0$), so it finitely generated, and hence free.

Since $\operatorname{depth}(R) = 0$, every element in $\mathfrak{m}$ is a zero divisor. By [Eis95, Cor. 3.2], there exists $s \in \mathfrak{m}$,

$$\{r \in R \mid rs = 0\} = \mathfrak{m}.$$

Now, $P \subseteq \mathfrak{m}R^t$, and hence $sP \subseteq smR^t = 0$, but P is free, so $P = 0$. Since $P = 0$, this shows that $\operatorname{pd}(B) = 0$, a contradiction. □

The Auslander–Buchsbaum Theorem 13.21 follows from similar arguments to this lemma and Change of Rings Theorems 13.25 and 13.26.

If $\operatorname{depth}(R) = 0$, then $\operatorname{pd}(A) = 0$, so A is projective, and hence free, so $\operatorname{depth}(A) = \operatorname{depth}(R) = 0$.

Theorem 13.37. *A ring* R *is regular if and only if* $\operatorname{gldim} R < \infty$*, and in that case*

$$\operatorname{gldim} R = \dim R = \operatorname{pd}_R(R/\mathfrak{m}).$$

Note that $R = k[x]/(x^2)$ has infinite global dimension and it is not regular, and in this case the Krull dimension is 0.

Theorem 13.38. *A regular local ring is Cohen-Macaulay.*

Proof. In general, $\operatorname{depth}(R) \leq \dim R$. If R is regular, let

$$\mathfrak{m} = (x_1, \ldots, x_n),$$

where $n = \dim_R \mathfrak{m}/\mathfrak{m}^2$ and $x_1, \ldots, x_n$ is a regular sequence, so

$$\operatorname{depth}(R) \geq n = \dim R,$$

and hence $\operatorname{depth}(R) = \dim(R)$. □

13.3. **Koszul resolution.** Let R be regular and $\mathrm{pd}_R(R/\mathfrak{m}) = n$. The minimal free R-resolution of $R/\mathfrak{m}$ is the *Koszul resolution.*

Let $x \in \mathfrak{m}$ be a nonzero divisor and consider

$$K(x): \qquad 0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow 0$$

a resolution of $R/(x)$. Suppose x_1, x_2 is a regular sequence, and consider $K(x_1) \otimes_R K(x_2)$:

$$\begin{tikzcd} R & R \arrow[l, "x_1"'] \arrow[d, "-x_2"] \\ R \arrow[from=u, "x_2"'] & R \arrow[l, "x_1"] \end{tikzcd}$$

and the total complex gives a resolution of $R/(x_1, x_2)$:

$$0 \longrightarrow R \xrightarrow{\binom{-x_2}{x_1}} R \oplus R \xrightarrow{(x_1, x_2)} R \longrightarrow 0$$

In general, if $\underline{x} = (x_1, \dots, x_n)$ is a regular sequence, we let $K(\underline{x}) = K(x_1, \dots, x_n)$ be the total complex of $K(x_1) \otimes_R K(x_2) \otimes_R \cdots \otimes_R K(x_n)$. Explicitly, we can write it as

$$0 \longrightarrow R^{\binom{n}{n}} \xrightarrow{\partial} \cdots \xrightarrow{\partial} R^{\binom{n}{2}} \xrightarrow{\partial} R^{\binom{n}{1}} \xrightarrow{\partial} R^{\binom{n}{0}} \longrightarrow 0$$

where we identify

$$\begin{aligned} R^{\binom{n}{1}} &= \bigoplus Re_i, \\ R^{\binom{n}{2}} &= \bigoplus_{i<j} R(e_i \wedge e_j), \\ &\vdots \\ R^{\binom{n}{n}} &= R(e_1 \wedge \cdots \wedge e_n), \end{aligned}$$

and $\partial = \sum x_i \frac{\partial}{\partial e_i}$, so

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \sum_j (-1)^{j-1} x_j e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_k}.$$

Theorem 13.39. *The resolution $K(\underline{x})$ is a free resolution of $R/(x_1, \dots, x_n)$ and*

$$H_q(K(\underline{x})) = \begin{cases} 0 & \text{if } q > 0, \\ R/(x_1, \dots, x_n) & \text{if } q = 0. \end{cases}$$

We first prove the theorem in the $n = 1$ case.

Proposition 13.40. *If $x \in \mathfrak{m}$ is a nonzero divisor, $C_\bullet$ is a chain complex of R-module, then we have an exact sequence*

$$0 \longrightarrow H_0(K(x) \otimes H_q(C_\bullet)) \longrightarrow H_q(K(x) \otimes_R C_\bullet) \longrightarrow H_1(K(x) \otimes_R H_{q-1}(C_\bullet)) \longrightarrow 0.$$

Proof. We have an exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[d] & & & & & & \\
R \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
K(x) \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r, "x"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
R[-1] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 & & & & & &
\end{tikzcd}$$

Taking $- \otimes_R C_\bullet$, we get

$$0 \longrightarrow C_\bullet \longrightarrow K(x) \otimes_R C_\bullet \longrightarrow C_\bullet[-1] \longrightarrow 0$$

and the long exact sequence of this complex gives the desired result. $\square$

Proof of Theorem 13.39. The case $n = 1$ is the Proposition 13.40. We apply the Proposition 13.40 with

$$C_\bullet = K(x_1, \dots, x_{n-1})$$
$$x = x_n$$

to get the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_0(K(x_n) \otimes H_q(K(x_1, \dots, x_{n-1}))) \arrow[r] & H_q(K(x)) \arrow[dll, out=0, in=180] \\
H_1(K(x_n) \otimes_R H_{q-1}(K(x_1, \dots, x_{n-1}))) \arrow[r] & 0. &
\end{tikzcd}$$

For $q \geq 2$, both the kernel and the cokernel in this exact sequence are 0, so $H_q(K(x)) = 0$.

For $q = 1$, the kernel is 0 and the cokernel is

$$H_1(K(x_n) \otimes R/(x_1, \dots, x_{n-1})) = \ker(R/(x_1, \dots, x_{n-1}) \xrightarrow{\cdot x_n} R/(x_1, \dots, x_{n-1})) = 0,$$

and hence $H_1(K(x)) = 0$.

For q_0, we have

$$0 \longrightarrow \underbrace{R/(x_n) \otimes_R R/(x_1, \dots, x_{n-1})}_{\cong R/(x_1, \dots, x_n)} \xrightarrow{\cong} H_0(K(x)) \longrightarrow 0$$

which completes the proof. $\square$

Tensoring the resolution

$$0 \longrightarrow R^{\binom{n}{n}} \xrightarrow{\partial} \cdots \xrightarrow{\partial} R^{\binom{n}{2}} \xrightarrow{\partial} R^{\binom{n}{1}} \xrightarrow{\partial} R^{\binom{n}{0}} \longrightarrow 0$$

with $- \otimes_R R/\mathfrak{m}$, we get

$$0 \longrightarrow k^{\binom{n}{n}} \xrightarrow{0} \cdots \xrightarrow{0} k^{\binom{n}{2}} \xrightarrow{0} k^{\binom{n}{1}} \xrightarrow{0} k^{\binom{n}{0}} \longrightarrow 0$$

and hence

$$\mathrm{Tor}_i(R/(x_1, \dots, x_n), k) = k^{\binom{n}{k}}.$$

If $\mathfrak{m} = (x_1, \dots, x_n)$, this gives

$$\mathrm{Tor}_i(k, k) = k^{\binom{n}{i}}.$$

The resolution

$$0 \longrightarrow R^{\binom{n}{n}} \xrightarrow{\partial} \cdots \xrightarrow{\partial} R^{\binom{n}{2}} \xrightarrow{\partial} R^{\binom{n}{1}} \longrightarrow \mathfrak{m} \longrightarrow 0$$

is called the *Koszul resolution* of $\mathfrak{m}$.

Definition 13.41. Suppose $(R, \mathfrak{m})$ is a local Noetherian ring and A is a finitely generated R-module. We say $x_1, \dots, x_n \in \mathfrak{m}$ is a *maximal A-sequence* if

$$A/(x_1, \dots, x_n)A$$

has no nonzero divisors in $\mathfrak{m}$.

Proposition 13.42. *All maximal A-sequences have same length, and this length is equal to* $\mathrm{depth}(A)$.

Proof. The proof is in [Wei94] and we omit it here. ∎

14. Local cohomology

Let R be a commutative ring, $I \subseteq R$ be an ideal. We define a functor

$$\mathcal{F}_I \colon R\text{–mod} \to R\text{–mod}$$

by

$$\mathcal{F}_I(A) = \{a \in A \mid \text{there exists } d \text{ such that } I^d a = 0\}.$$

and if $f \colon A \to B$ is an R-module homomorphism, the square

$$\begin{array}{ccc} \mathcal{F}_I(A) & \xrightarrow{\mathcal{F}_I(f)} & \mathcal{F}_I(B) \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

commutes. Then $\mathcal{F}_I$ is left exact and

$$H_I^\bullet(A) = R^\bullet \mathcal{F}_I(A)$$

is called the *local cohomology*. If $I^d \subseteq J$ and $J^e \subseteq I$, then

$$\mathcal{F}_I = \mathcal{F}_J$$

so

$$H_I^\bullet = H_J^\bullet.$$

Moreover, we have that

$$\mathcal{F}_I(A) = H_I^0(A) = \{a \in A \mid \text{there exists } d \text{ such that } I^d a = 0\} = \varinjlim \operatorname{Hom}_R(R/I^d, A).$$

In general,

$$H_I^n(A) = \varinjlim \operatorname{Ext}_R^n(R/I^d, A).$$

15. Spectral sequences

This chapter largely follows [Wei94], but another reference for this topic is [McC01].

In this chapter, we work in the category of R-modules.

Definition 15.1. A *spectral sequence* consists of

- *objects* R_{pq}^r for $p, q \in \mathbb{Z}$, $r \geq a$,
- *differentials* $d_{pq}^r \colon E_{pq}^r \to E_{p-r,q+r-1}^r$ satisfying $d^r \circ d^r = 0$,
- $R_{pq}^{r+1} = \frac{\ker d_{pq}^r}{\operatorname{im}(d_{p+r,q-r+1}^r)}$.

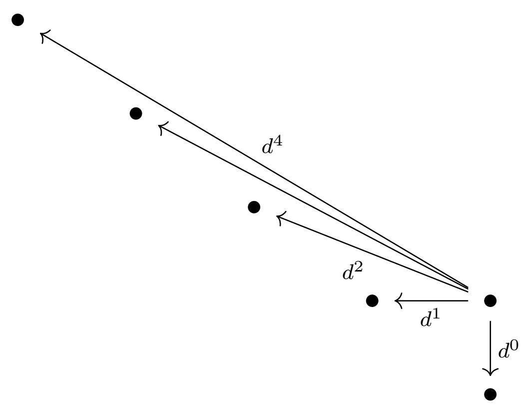
On a diagram, we can represent d^0, d^1, d^2 as follows

$$\begin{tikzcd}
d^0 & \bullet \arrow[d] & E^0_{p,q+1} \arrow[d] & \bullet \arrow[d] \\
 & \bullet \arrow[d] & E^0_{pq} \arrow[d] & E^0_{p+1} \arrow[d] \\
 & \bullet & \bullet & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
d^1 & \bullet & E^1_{p,q+1} \arrow[l] & \bullet \arrow[l] \\
 & \bullet & E^1_{pq} \arrow[l] & E^1_{p+1} \arrow[l] \\
 & \bullet & \bullet \arrow[l] & \bullet \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
d^2 & \bullet & E^2_{p,q+1} & \bullet \\
 & \bullet & E^2_{pq} \arrow[ul] & E^2_{p+1} \arrow[ul] \\
 & \bullet & \bullet \arrow[ul] & \bullet \arrow[ul]
\end{tikzcd}$$

and similarly for d^r for $r \geq 3$. We can think of these pictures as "*pages*" or "*sheets*": for each r, there is a "*page*" with arrows d^r as follows



We let $E^r = \bigoplus_{p,q \in \mathbb{Z}} E^r_{pq}$. Then E^{r+1} is a subquotient of E^r.

Suppose that for $r = a$, we have $B^a = 0$, $Z^a = E^a$.

Let $Z^{r+1} \supseteq B^r$ such that $\ker d^r = \frac{Z^{r+1}}{B^r}$ and $B^{r+1} \supseteq B^r$ such that $\operatorname{im} d^r = \frac{B^{r+1}}{B^r}$, and then $B^{r+1} \subseteq Z^{r+1}$. We then have

$$B^a \subseteq B^{a+1} \subseteq B^{a+2} \subseteq \cdots \subseteq B^\infty \subseteq Z^\infty \subseteq \cdots \subseteq Z^{a-a} \subseteq Z^{a-1} \subseteq Z^a = R^a$$

where

$$B^\infty = \bigcup_i B^i, \quad Z^\infty = \bigcap_i Z^i, \quad E^\infty = Z^\infty / B^\infty.$$

Definition 15.2. A *spectral sequence* is *bounded* if for any n, there are finitely many p such that $E^a_{p,n-p} \neq 0$.

Definition 15.3. A bounded spectral sequence *converges to* $H_\bullet$ if for every n there is a filtration

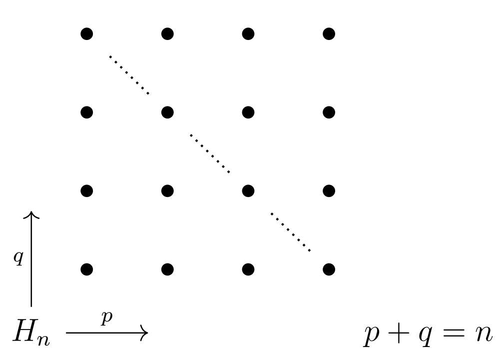
$$0 = F_s H_n \cdots \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots F_t H_n = H_n$$

such that

$$E^\infty_{pq} = F_p H_{p+q} / F_{p-1} H_{p+q}.$$

In that case, we write $E^1_{pq} \Rightarrow H_{p+q}$.

Note that E^1_{pq} converges to the $(p+q)$th homology group, which can be represented as follows:



15.1. **Homology spectral sequences.** We construct a spectral sequence from a filtration of a chain complex. Let $C_\bullet$ be a chain complex with a filtration
$$\cdots \subseteq F_pC_\bullet \subseteq F_{p+1}C_\bullet \subseteq \cdots$$
and assume
$$\bigcup_p F_pC_\bullet = C_\bullet.$$
We construct a spectral sequence with
$$E^0_{pq} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}$$
and
$$E^1_{pq} = H_{p+q}(E^0_{p\bullet}).$$
Note that E^0_{pq} in $E^0_{p\bullet}$ has degree $p+q$, and, to compute the homology, we note that the boundary maps to compute the homology are induced by the boundary maps in $C_\bullet$:
$$E^0_{pq} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} \to \frac{F_pC_{p+q-1}}{F_{p-1}C_{p+q-1}} = E^0_{pq-1}.$$
In general, E^{r+1}_{pq} will be the homology of E^r_{pq}.

In what follows, we drop the "q" from the notation and simply write
$$E^0_p = \frac{F_pC}{F_{p-1}C}$$
and so on. We let
$$\eta_p\colon F_pC \to \frac{F_pC}{F_{p-q}C} = E^0_p$$
be the projection and set
$$A^r_p = \{c \in F_pC \mid d(c) \in F_{p-r}C\},$$
$$Z^r_p = \eta_p(A^t_p) = \frac{A^r_p + F_{p-1}C}{F_{p-1}C} \in E^0_p,$$
$$B^{r+1}_{p-r} = \eta_{p-r}(d(A^r_p)) = \frac{d(A^r_p) + F_{p-r-1}C}{F_{p-r-1}C} \in E^0_{p-r},$$
$$B^r_p = \eta_p(d(A^{r-1}_{p+r-1})) = \frac{d(A^{r-1}_{p+r-1}) + F_{p-1}C}{F_{p-1}(C)} \subseteq Z^r_p.$$
This will simplify calculations, for example:
$$Z^r_p = \frac{A^r_p + F_{p-1}C}{F_{p-1}C} \cong \frac{A^r_p}{A^r_p \cap F_{p-1}C} = \frac{A^r_p}{A^{r-1}_{p-1}}.$$
In what follows, we will use that, for $B \subseteq A$, we have $A \cap (B + C) = B + (A \cap C)$. We set
$$E^r_p = \frac{Z^r_p}{B^r_p} = \frac{A^r_p + F_{p-1}C}{d(A^{r-1}_{p+r-1}) + F_{p-1}C} = \frac{A^r_p}{A^r_p \cap (d(A^{r-1}_{p+r-1}) + F_{p-1}C)} = \frac{A^r_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}}$$
and since
$$E^r_p = \frac{A^r_p}{dA^{r-1}_{p+r-1} + A^{r-1}_{p-1}},$$

$$E^r_{p-r} = \frac{A^r_{p-r}}{dA^{r-1}_{p-1} + A^{r-1}_{p-r-1}},$$

the boundary map d induces

$$d^r_p \colon E^r_p \to E^r_{p-r}.$$

We claim that

$$E^{r+1}_{pq} = \frac{\ker d^r_{pq}}{\operatorname{im} d^r_{p+rq-r+1}}.$$

For that sake, we will compute the kernel of the map d^r_p. If

$$a + dA^{r-1}_{p+r-1} + A^{r-1}_{p-1} \in \ker d^r_p,$$

then without loss of generality, assume that

$$d(a) \in A^{r-1}_{p-r-1}$$

(otherwise, we could choose a different representative a that would satisfy this). Then $a \in A^{r+1}_p$. This shows that

$$\ker d^r_p = \frac{A^{r+1}_p + A^{r-1}_p}{d(A^{r-1}_{p+r-1}} \cong \frac{A^{r+1}_p + F_{p-1}C}{dA^{r-1}_{p+r-1} + F_{p-1}C} \cong \frac{Z^{r+1}_p}{B^r_p}.$$

Then

$$d^r_p \colon E^r_p = \frac{Z^r_p}{B^r_p} \twoheadrightarrow \frac{Z^r_p}{Z^{r+1}_p} \cong \frac{B^{r+1}_{p-r}}{B^r_{p-r}} \hookrightarrow \frac{Z^r_{p-r}}{B^r_{p-r}} = E^r_{p-r}$$

and so

$$\operatorname{im} d^r_p = \frac{B^{r+1}_{p-r}}{B^r_{p-r}}$$

and shifting the index

$$\operatorname{im} d^r_{p+r} = \frac{B^{r+1}_p}{B^r_p} \subseteq \frac{Z^{r+1}_p}{B^r_p} = \ker d^r_p.$$

Hence

$$\frac{\ker d^r_p}{\operatorname{im} d^r_{p+r}} = \frac{Z^{r+1}_p}{B^{r+1}_p} = E^{r+1}_p.$$

This shows that E^{r+1}_p is the homology of E^r_p.

One can also show that $\frac{Z^r_p}{Z^{r+1}_p} \cong \frac{B^{r+1}_{p-r}}{B^r_{p-r}}$ with a similar calculation, but we omit it here.

Assume that a filtration is bounded, i.e. for every n there exist s, t such that

$$0 = F_s C_n \subseteq F_{s+1} C_n \subseteq \cdots \subseteq F_t C_n = C_n.$$

Then E^0_{pq} is bounded and for any n, there are only finitely many p such that $E^0_{p,n-p} \neq 0$.

Theorem 15.4 (Spectral Convergence). *There exists a filtration on $H_n(C_\bullet)$,*

$$\cdots \subseteq F_p H_n(C_\bullet) \subseteq F_{p+1} H_n(C_\bullet) \subseteq \cdots$$

such that

$$F_p H_{p+q}(C_\bullet)/F_{p-1} H_{p+q}(C_\bullet) \cong E^\infty_{pq}.$$

Concisely, we write

$$E^1_{pq} = H_{p+q}(F_p C_\bullet / F_{p-1} C_\bullet) \longrightarrow H_{p+q}(C_\bullet).$$

Here, E^1_{pq} is homology of the associated graded, E^∞_{pq} is the associated graded of homology.

Proof. Recall that we had

$$0 \subseteq B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots \subseteq Z^1_p \subseteq Z^0_p = F_pC,$$

where $B^\infty_p = \bigcup_i B^i_p$, $Z^\infty_p = \bigcap Z^i_p$, and

$$E^r_p = E^r_{p\bullet} = Z^r_p/B^r_p,$$
$$E^\infty_p = Z^\infty_p/B^\infty_p.$$

Suppose $p+q=n$. Then

$$A^r_{pq} = \{c \in F_pC_n \mid d(c) \in F_{p-r}C_n\}.$$

For $r \geq r_0(n,p)$ (when $F_{p-r}C_n = 0$, i.e. $p-r \leq s$), so

$$A^r_{pq} = \ker d \cap F_pC_n = A^\infty_{pq}.$$

Then

$$Z^r_{pq} = \eta_p(A^r_{pq}) = \eta_p(A^\infty_{pq}) = Z^\infty_{pq} = \frac{\ker d \cap F_pC_n + F_{p-1}C_n}{F_{p-1}C_n}.$$

Note that in general, $f\left(\bigcap_i A_i\right) \neq \bigcap_i f(A_i)$, but because of boundedness from below these are finite intersections, so equality does hold.

Moreover,

$$B^{r+1}_{pq} = \eta_p(A^r_{p+r,q-r}) = \frac{d(A^r_{p+r,q-r}) + F_{p-1}C_n}{F_{p-1}C_n},$$

$$B^\infty_{pq} = \frac{d\left(\bigcup A^r_{p+r,q-r} + F_{p-1}(C_n)\right)}{F_{p-1}C_n}.$$

Define:

$$F_pH_n(C) = \frac{\ker d_n \cap f_pC_n}{\operatorname{im} d_{n+1} \cap f_pC_n} = \frac{A^\infty_{pq}}{d\left(\bigcup_r A^r_{p_r,p-r+1}\right)}.$$

Then we have that

$$\frac{F_pH_n(C))}{F_{p-1}H_n(C)} = \frac{A^\infty_{pq}}{d\left(\bigcup_r A^r_{p+q,q-r+1}\right) + A^\infty_{p-1,q-1}}.$$

Applying η_p to the right hand side of the above, we get

$$\frac{\eta_p(A^\infty_{pq})}{\eta_p\left(d\left(\bigcup_r A^r_{p+q,q-r+1}\right) + A^\infty_{p-1,q-1}\right)} \cong \frac{\eta_p(A^\infty_{pq})}{\eta_p\left(\bigcup_r A^r_{p+q,q-r+1}\right)} = \frac{Z^\infty_{pq}}{B^\infty_{pq}}.$$

We claim that η_p actually gives an isomorphism above. Indeed, consider

$$\eta_p \colon A^\infty_{pq} \to \frac{\eta_p(A^\infty_{pq}}{\eta_p\left(d\left(\bigcup_r A^r_{p+q,q-r+1}\right)\right)}.$$

Suppose $a \in A^\infty_{pq}$, so

$$a + f_{p-1}C \in d\left(\bigcup A^r_{p+r,q-r+1}\right) + F_{p-1}C$$

and we can write $a = b + c$ for

$$b \in d\left(\bigcup A^r_{p+r,q-r+1}\right)$$
$$c \in F_{p-1}C \cap A^\infty_{pq} = A^\infty_{p+q-1}$$

which completes the proof. $\square$

Example 15.5. Suppose

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

is an exact sequence of complexes. We will recover the long exact sequence in homology using the Convergence Theorem 15.4. Consider the following filtration on B:

$$0 = F_{-1}B \subseteq \underbrace{F_0B}_{=A} \subseteq F_1B = B.$$

Then

$$E^0_{0q} = \frac{F_0B_q}{F_{-1}B_q} = A_q,\ E^0_{1q} = \frac{F_1B_{q+1}}{F_0B_q} = \frac{B_{q+1}}{A_{q+1}} = C_{q+1},$$

so E^0_{pq} can be represented as

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[d] & A_2 \arrow[d, "d^0"] & C_3 \arrow[d, "d^0"] & 0 \arrow[d] \\
0 \arrow[d] & A_1 \arrow[d, "d^0"] & C_2 \arrow[d, "d^0"] & 0 \arrow[d] \\
0 \arrow[d] & A_0 \arrow[d] & C_1 \arrow[d, "d^0"] & 0 \arrow[d] \\
0 \arrow[d] & 0 \arrow[d] & C_0 \arrow[d] & 0 \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Hence $E^1_{0q} = H_q(A)$, $E^1_{1q} = H_{q+1}(C)$ and in general E^1_{pq} can be represented as

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
0 & H_1(A) \arrow[l] & H_2(C) \arrow[l, "d^1"'] & 0 \arrow[l] \\
0 & H_0(A) \arrow[l] & H_1(C) \arrow[l, "d^1"'] & 0 \arrow[l] \\
0 & 0 \arrow[l] & H_0(C) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Moreover, by definition of E^2_{1q}, we get the following exact sequence

$$0 \longrightarrow E^2_{1q} \longrightarrow H_{q+1}(C) \longrightarrow H_q(A) \longrightarrow E^2_{0q} \longrightarrow 0.$$

Finally, looking at the diagram for E^0_{pq}, we see that the maps $d^r \colon E^r_{pq} \to E^r_{p-r,q+r-1}$ for $r \geq 2$ are all 0, and hence

$$E^2_{pq} = E^\infty_{pq}.$$

By the Convergence Theorem 15.4, there is a filtration on $H_\bullet(B)$ such that

$$F_0 H_q(B) = \frac{F_0 H_q(B)}{F_{-1} H_q(B)} = E^\infty_{0p},$$

$$\frac{H_{q+1}(B)}{F_0 H_{q+1}(B)} = \frac{F_1 H_{q+1}(B)}{F_0 H_{q+1}(B)} = E^\infty_{1q},$$

since $F_1 H_{q+1}(B) = H_{q+1}(B)$ and $F_{-1} H_q(B) = 0$. We then have

$$0 \longrightarrow \underbrace{F_0 H_1(B)}_{=E^\infty_0} \longrightarrow H_q(B) \longrightarrow \underbrace{\frac{H_q(B)}{F_0 H_q(B)}}_{=H^\infty_{q,q-1}} \longrightarrow 0.$$

We then obtain

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(A) \arrow[rr] \arrow[dr, two heads] & & H_q(B) \arrow[rr] \arrow[dr, two heads] & & H_q(C) \arrow[r] & H_{q-1}(A) \arrow[r] & \cdots \\
& & E^\infty_{0q} \arrow[ur, hook] & & E^\infty_{1,q-1} \arrow[ur, hook] & & &
\end{tikzcd}$$

which recovers the long exact sequence of homology.

15.2. **Cohomology spectral sequences.**

One can dualize all the results in the previous section to cohomology.

The objects are E^{pq}_r, $r \geq a$, and the maps are

$$d^{pq}_r \colon E^{pq}_r \to E^{p+r,q-r+1}_r$$

and

$$E^{pq}_{r+1} = \frac{\ker d^{pq}_r}{\operatorname{im} d^{p-r,q+r-1}_r}.$$

Similarly to the Convergence Theorem 15.4, one can prove that if the spectral sequence is bounded, then

$$E^{pq}_r \longrightarrow H^{p+q},$$

i.e. there exists filtration $F^t H^n \subseteq F^{t-1} H^n \subseteq \cdots$ with quotients E^{pq}_∞.

15.3. **Spectral sequences in topology.** We show an example of a spectral sequence in topology and an application of that spectral sequence.

Definition 15.6. A mapping $p\colon E \to B$ has the *homotopy lifting property* (HLP) for space Y if given a homotopy $G\colon Y \times [0,1] \to B$ and a mapping $g\colon Y \times \{0\} \to E$ with $p \circ g(y,0) = G(y,0)$, then there exists a homotopy $\widetilde{G}\colon Y \times [0,1] \to E$ with $\widetilde{G}(y,0) = g$ and $p \circ \widetilde{G} = G$, i.e. the two triangles

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "G"] \arrow[ur, dotted, "\widetilde{G}"] & B \end{tikzcd}$$

commute.

Definition 15.7. We say p is a *(Hurewicz) fibration* if p has HLP for all Y. We say p is a *Serre fibration* if p has HLP for all n-cells.

Proposition 15.8. *Suppose p is a fibration and if B is path-connected then all fibers $p^{-1}(b)$, $b \in B$ are homotopy equivalent.*

In particular, p is surjective. Moreover, $H_\bullet(p^{-1}(b))$ does not depend on b.

Theorem 15.9 (Leray Spectral Sequence)**.** *Suppose $\pi\colon E \to B$ is a fibration with $F = \pi^{-1}(b)$ for some $b \in B$, with B simply connected. Let M be an abelian group. There is a spectral sequence*

$$E^2_{pq} = H_p(B; H_q(F; M) \longrightarrow H_{p+q}(E; M).$$

Corollary 15.10. *Suppose $\pi\colon E \to S^n$ is a fibration and F is a fiber, with $n \geq 2$. Then there exists a long exact sequence*

$$\cdots \longrightarrow H_q(F) \longrightarrow H_q(E) \longrightarrow H_{q-n}(F) \longrightarrow H_{q-1}(F) \longrightarrow H_{q-1}(E) \longrightarrow \cdots$$

Proof. We have

$$E^2_{pq} = H_p(S^n; H_q(F)) = \begin{cases} H_q(F) & \text{if } p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

which gives the following diagram of E^2_{pq}

$$
\begin{array}{ccccccc}
\vdots & \vdots & & \vdots & \vdots & \vdots & \\
H_2(F) & 0 & \cdots & 0 & H_2(F) & 0 & \cdots \\
H_1(F) & 0 & \cdots & 0 & H_1(F) & 0 & \cdots \\
H_0(F) & 0 & \cdots & 0 & H_0(F) & 0 & \cdots \\
\hline
p=0 & p=1 & \cdots & p=n-1 & p=n & p=n+1 &
\end{array}
$$

and if $n > 2$ then $d^2 = 0$ and $E^3_{pq} = E^2_{pq}$, and in general

$$E^2_{pq} = E^3_{pq} = \cdots = E^n_{pq}.$$

For E^n_{pq}, the map d^n is non-trivial:

$$
\begin{tikzcd}
H_{n-1}(F) & 0 & & & & & \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \\
H_2(F) & 0 & \cdots & 0 & H_2(F) & 0 & \cdots \\
H_1(F) & 0 & \cdots & 0 & H_1(F) & 0 & \cdots \\
H_0(F) & 0 & \cdots & 0 & H_0(F) \arrow[uuuullll, "d^n"'] & 0 & \cdots \\
p=0 & p=1 & \cdots & p=n-1 & p=n & p=n+1 &
\end{tikzcd}
$$

This gives the exact sequence

$$(*) \qquad 0 \longrightarrow E^{n+1}_{np} \longrightarrow H_q(F) \xrightarrow{d^n} H_{q+n-1}(F) \longrightarrow E^{n+1}_{0,q+n-1} \longrightarrow 0$$

Moreover, for $r > n$, we have that $d^r = 0$ again. Therefore,

$$E^\infty_{pq} = E^{n+1}_{pq}.$$

This gives the diagram

$$
\begin{array}{ccccc}
E^\infty_{0n} & 0 & \cdots & 0 & E^\infty_{nn} \\
\vdots & \vdots & \cdots & \vdots & \\
E^\infty_{01} & 0 & \cdots & 0 & E^\infty_{n1} \\
E^\infty_{00} & 0 & \cdots & 0 & E^\infty_{n0}
\end{array}
$$

which gives a short exact sequence

$$(**) \qquad 0 \longrightarrow E^\infty_{0q} \longrightarrow H_q(E) \longrightarrow E^\infty_{n,q-n} \longrightarrow 0$$

We then splice the exact sequences $(*)$ and $(**)$ to get the long exact sequence required. $\square$

15.4. **Spectral sequences of double complexes and their applications.** Suppose $C_{\bullet\bullet}$ is a double complex, with nonzero terms only in the first quadrant. Let $\mathrm{Tot}(C)$ be the total complex. We define a bounded filtration on $\mathrm{Tot}(C)$

$$\cdots \subseteq F_p \operatorname{Tot}(C) \subseteq F_{p+1} \operatorname{Tot}(C) \subseteq \cdots$$

where

$$F_k \operatorname{Tot}(C)_n = \bigoplus_{p=0}^{k} C_{p,n-p}.$$

Let d^h be the horizontal maps and d^v be the vertical maps in the double complex, with $d = d^h + d^v$ and $d^h d^v + d^v d^h = 0$. We set

$$E^0_{pq} = \frac{F_p(\operatorname{Tot}(C)_{p+q}}{F_{p-1}(\operatorname{Tot}(C))_{p+q}} = C_{pq}$$

and $d^0 = d^v$. Then

$$E^1_{pq} = H_q(C_{p\bullet}) \longrightarrow H_{p+q}(\operatorname{Tot}(C_{\bullet\bullet}))$$

by Theorem 15.4. We have

$$d^1 \colon E^1_{pq} = H^v_q(C_p) \to E^1_{p-1,q} = H^v_q(C_{p-q,\bullet})$$

is induced by d^h, and so we write $d^1 = d^h$. Then also

$$E^2_{pq} = H^h_p H^v_p(C_{\bullet\bullet}) \longrightarrow H_{p+q}(\operatorname{Tot}(C_{\bullet\bullet})).$$

We could also define $D_{pq} = C_{qp}$ (the *transposition* of the double complex) and apply the above construction to that. This gives the two spectral sequences

$${}^{\mathrm{II}}E^2_{qp} = H^v_q H^h_p(C_{\bullet\bullet}) \longrightarrow H_{p+q}(\operatorname{Tot}(C_{\bullet\bullet}))$$

$${}^{\mathrm{I}}E^2_{pq} = H^h_p H^v_q(C_{\bullet\bullet}) \longrightarrow H_{p+q}(\operatorname{Tot}(C_{\bullet\bullet}))$$

Let A be a right R-module and B be a left R-module. Then

$$\mathcal{F} = A \otimes_R - \qquad \text{is right exact}$$

$$\mathcal{G} = - \otimes_R B \qquad \text{is right exact}$$

We already proved (Theorem 9.10) that

$$L_p\mathcal{F}(B) \cong L_p\mathcal{G}(A) = \mathrm{Tor}_p(A, B)$$

but the proof can be reinterpreted using spectral sequences of double complexes.

Alternative proof of Theorem 9.10. Suppose $P_\bullet \to A$, $Q_\bullet \to B$ are two projective resolutions, and $P \otimes Q$ is the double complex. Applying the above result to this double complex, we obtain

$$^{\mathrm{II}}E^2_{qp} = H^v_q H^h_p(P \otimes Q) \longrightarrow H_{p+q}(\mathrm{Tot}(P \otimes Q)),$$

$$^{\mathrm{I}}E^2_{pq} = H^h_p H^v_q(P \otimes Q) \longrightarrow H_{p+q}(\mathrm{Tot}(P \otimes Q)).$$

We have that

$$H^v_q(P_\bullet \otimes Q_\bullet) = P_\bullet \otimes H_q(Q_\bullet) = \begin{cases} P_\bullet \otimes B & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Then

$$^{\mathrm{I}}E^2_{pq} = \begin{cases} H^h_p(P_\bullet \otimes B) = L_p\mathcal{G}(A) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

Hence the diagram for $^{\mathrm{I}}E^2_{pq}$ is

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
L_0\mathcal{G}(A) & L_1\mathcal{G}(A) & L_2\mathcal{G}(A) & \cdots
\end{array}$$

from which it is clear that $d^n = 0$ for $n \geq 2$. Hence

$$E^2_{pq} = E^3_{pq} = \cdots = E^\infty_{pq}.$$

This shows that

$$H_n(\mathrm{Tot}(P \otimes Q)) = E^\infty_{n0} = L_n\mathcal{G}(A).$$

Similarly, we obtain that

$$^{\mathrm{II}}E^2_{pq} = H_n(\mathrm{Tot}(P \otimes Q)) = L_n\mathcal{F}(B),$$

which proves the theorem. $\square$

One can also prove Künneth's Formula 11.1 using spectral sequences. Suppose P is a complex of flat R-modules, bounded from below. Let M be an R-module. Then there is a convergent spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p(H_q(P), M) \longrightarrow H_{p+q}(P \otimes M),$$

called the *Künneth spectral sequence.*

Alternative proof of Kunneth's Formula 11.1. Let P be a complex of flat R-modules, bounded from below. Assume that $B_n = d(P_{n+1})$ is flat for all n. We show that there is an exact sequence

$$0 \longrightarrow H_q(P) \otimes M \longrightarrow H_q(P \otimes M) \longrightarrow \mathrm{Tor}_1^R(H_{q-1}(P), M) \longrightarrow 0$$

Let $Z_n = \ker(d\colon P_n \to P_{n-1})$. We showed before that Z_n is also flat, and we have a short exact sequence

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(P) \longrightarrow 0$$

which gives a flat resolution of $H_n(P)$, showing that the tor dimension of $H_n(P)$ is at most 1. Then

$$E^2_{pq} = \mathrm{Tor}_p(H_q(P), M) = 0, \qquad \text{for } p \geq 2 \text{ and } p < 0.$$

The diagram for E^2_{pq} is

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & H_2(P) \otimes M & \mathrm{Tor}_1(H_2(P), M) & 0 \\
0 & H_1(P) \otimes M & \mathrm{Tor}_1(H_1(P), M) & 0 \\
0 & H_0(P) \otimes M & \mathrm{Tor}_1(H_0(P), M) & 0
\end{array}$$

which shows that $d^2 = d^3 = \cdots = 0$, and hence $E^\infty_{pq} = E^2_{pq}$. Thus $H_q(P \otimes M)$ has a filtration with quotients E^2_{0q} and $E^2_{1,q-1}$. This gives the short exact sequence

$$0 \longrightarrow H_q(P) \otimes M \longrightarrow H_q(P \otimes M) \longrightarrow \mathrm{Tor}_1^R(H_{q-1}(P), M) \longrightarrow 0$$

as required. □

Another application is the base change for Tor.

Theorem 15.11 (Base change for Tor). *Let $f\colon R \to S$ be a ring homomorphism. If A is a right R-module, B is a left S-module, then*

$$E^2_{pq} = \operatorname{Tor}^S_p(\operatorname{Tor}^R_q(A,S),B) \longrightarrow \operatorname{Tor}^R_{p+q}(A,B).$$

Proof. Let $P_\bullet \to A$ be a projective R-resolution and $Q_\bullet \to B$ be a projective S-resolution. Consider the double complex $P \otimes_R Q$. Then

$$ {}^{\mathrm{I}}E^2_{pq} = H^v_p H^h_q(P \otimes_R Q) = H^v_p(P \otimes_R H^h_q(Q)) = \begin{cases} H_p(P \otimes_R B) = \operatorname{Tor}^R_p(A,B) & \text{if } q = 0,\\ 0 & \text{if } q \neq 0.\end{cases}$$

Note that $P \otimes_R -$ commutes with homology, because P is a projective and hence flat R-module.

Then, by the usual argument, $d^2 = d^3 = \cdots = 0$ and so ${}^{\mathrm{I}}E^2_{pq} = {}^{\mathrm{I}}E^\infty_{pq}$, and hence

$$H_p(\operatorname{Tot}(P \otimes Q)) = E^\infty_{p0} = H_p(P \otimes_R B) = \operatorname{Tor}^R_p(A,B).$$

Moreover,

$${}^{\mathrm{II}}E^2_{pq} = H^v_p H^h_q(P \otimes_R Q) = H^v_p H^h_q((P \otimes_R S) \otimes_S Q) = H^v_p(H^h_q(P \otimes_R S) \otimes_S Q).$$

Here, Q is a projective and hence flat S-module, but not necessarily a flat R-module, so we have to tensor with S first. Hence

$${}^{\mathrm{II}}E^2_{pq} = H^v_p(H^h_q(P \otimes_R S) \otimes_S Q) = H^v_p(\operatorname{Tor}^R_q(A,S) \otimes_S Q) = \operatorname{Tor}^S_p(\operatorname{Tor}^R_q(A,S),B),$$

completing the proof. □

15.5. **Hyperhomology and hyperderived functors.** Let $\mathcal{A}$ be an abelian category with enough projectives and $A_\bullet$ be a chain complex in $\mathcal{A}$.

Definition 15.12. A *Cartan–Eilenberg resolution* (*CE resolution*) is an upper half plane double complex $P_{\bullet\bullet}$ of projectives together with augmentation $\epsilon\colon P_{\bullet 0} \to A_\bullet$

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & P_{-11} & \longleftarrow & P_{01} & \longleftarrow & P_{11} & \longleftarrow & P_{21} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & P_{-10} & \longleftarrow & P_{00} & \longleftarrow & P_{10} & \longleftarrow & P_{20} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & A_{-1} & \longleftarrow & A_{0} & \longleftarrow & A_{1} & \longleftarrow & A_{2} & \longleftarrow & \cdots
\end{array}$$

such that

(1) $P_{p\bullet} \to A_p$ is a resolution and if we define

$$B_p(P_{\bullet q}, d^h) = \operatorname{im}(d^h\colon P_{p+1,q} \to P_{pq})$$
$$Z_p(P_{\bullet q}, d^h) = \ker(d^h\colon P_{pq} \to P_{p-1,q})$$

$$H_p(P_{\bullet q}, d^h) = Z_p(P_{\bullet q}, d^h)/B_p(P_{\bullet q}, d^h)$$

then

$$B_p(P_{\bullet q}, d^h) \to B_p(A_\bullet)$$
$$H_p(P_{\bullet q}, d^h) \to H_p(A_\bullet)$$

are projective resolutions (which implies that $Z_p(P_{\bullet q}, d^h) \to Z_p(A_\bullet)$ is a projective resolution),

(2) if $A_p = 0$ then $P_{p\bullet}$ is the zero complex.

Lemma 15.13. *Every chain complex has a CE-resolution.*

Proof. The proof is omitted but can be found in [Wei94]. It is similar to the proof that any $A \in \mathcal{A}$ has a projective resolution 5.21. □

Definition 15.14. Suppose $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ is a right exact functor. We define the *left hyperderived functor* $\mathbb{L}_p\mathcal{F} \colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{B}$ of $\mathcal{F}$ as follows: if $A_\bullet$ is a chain complex in $\mathcal{A}$, then

$$\mathbb{L}_p\mathcal{F}(A_\bullet) = H_p(\mathrm{Tot}(\mathcal{F}(P_\bullet)))$$

where $P_{\bullet\bullet}$ is a CE-resolution of $A_\bullet$. Dually, we can define *the right hyperderived functor* $\mathbb{R}_p\mathcal{G}$ for a left exact functor $\mathcal{G}$.

There are a lot of details which we will leave out: the fact that this functor is well-defined, what this functor does to morphisms and so on. These are analogous to these properties for left derived functors presented in Chapter 5.

Suppose for simplicity that A is bounded from below. We consider the two spectral sequences for the double complex $\mathcal{F}P_{\bullet\bullet}$ from Section 15.4. We have

$${}^{\mathrm{I}}E^2_{pq} = H^h_p H^v_q(\mathcal{F}(P)) = H^h_p(L_q\mathcal{F}(A_\bullet)),$$

since $P_{p\bullet} \to A_p$ is a projecitve resolution, so

$$H^v_q(\mathcal{F}(P_{p\bullet})) = L_q(\mathcal{F}A_p)$$

by definition. For the other spectral sequence, we note that

$$H_q(\mathcal{F}(P_{\bullet\bullet})) = \mathcal{F}H^v_q(P_{\bullet\bullet})$$

because the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z^h_{pq} \arrow[r] & P_{pq} \arrow[r] & B^h_{p-1q} \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & B_{pq} \arrow[r] & Z_{pq} \arrow[r] & H^h_{pq} \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd}$$

split. We then have

$${}^{\mathrm{II}}E^2_{pq} = H^v_p H^h_q(\mathcal{F}(P_{\bullet\bullet})) = H^v_p(\mathcal{F}H^h_q(P_{\bullet\bullet})) = (L_p\mathcal{F})(H^h_q(A_\bullet))$$

because

$$H^h_q(P_{\bullet\bullet}) \to H_q(A_\bullet)$$

is a projective resolution.

Both of these spectral sequences converge to

$$H_{p+q}(\mathrm{Tot}(\mathcal{F}(P_{\bullet\bullet}))) = \mathbb{L}_{p+q}\mathcal{F}(A_\bullet),$$

i.e.

$$\begin{aligned} {}^{\mathrm{I}}E^2_{pq} &= H^h_p(L_q\mathcal{F}(A_\bullet)) \longrightarrow \mathbb{L}_{p+q}\mathcal{F}(A_\bullet), \\ {}^{\mathrm{II}}E^2_{pq} &= (L_p\mathcal{F})(H^h_q(A_\bullet)) \longrightarrow \mathbb{L}_{p+q}\mathcal{F}(A_\bullet). \end{aligned}$$

Dually, if $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor, $A^\bullet$ is a cochain complex, bounded from below, then

$$\begin{aligned} {}^{\mathrm{I}}E_2^{pq} &= H^p(R^q\mathcal{F}(A^\bullet)) \longrightarrow \mathbb{R}^{p+q}\mathcal{F}(A^\bullet), \\ {}^{\mathrm{II}}E_2^{pq} &= R^p\mathcal{F}(H^q(A^\bullet)) \longrightarrow \mathbb{R}^{p+q}\mathcal{F}(A^\bullet). \end{aligned}$$

Theorem 15.15 (Grothendieck spectral sequence)**.** *Suppose $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories where $\mathcal{A}$, $\mathcal{B}$ have enough injectives, and*

$$\mathcal{G}\colon \mathcal{A} \to \mathcal{B}$$

$$\mathcal{F}\colon \mathcal{B} \to \mathcal{C}$$

be left exact functors where $\mathcal{G}$ sends injectives to $\mathcal{F}$-acyclic objects ($R^p\mathcal{F}(A) = 0$ for $p > 0$):

$$\begin{tikzcd}
I,\ \textit{injective} \arrow[rrrr] & & & & \mathcal{G}I, \mathcal{F}\textit{-acyclic} \arrow[dddll, bend left] \\
 & \mathcal{A} \arrow[rr, "\mathcal{G}"] \arrow[dr, "\mathcal{F}\mathcal{G}"'] & & \mathcal{B} \arrow[dl, "\mathcal{F}"] & \\
 & & \mathcal{C} & & \\
 & & R^p\mathcal{F}(\mathcal{G}(I)) = 0,\ p > 0 & &
\end{tikzcd}$$

We then have that

$$E_2^{pq} = (R^p\mathcal{F})(R^q\mathcal{G})(A) \longrightarrow R^{p+q}(\mathcal{F}\mathcal{G})(A).$$

The idea is that we can compute the derived functors using acyclic objects, instead of projective resolutions. For example, we showed that to compute Tor, it is enough to consider flat resolutions, and, indeed, flat objects are acyclic with respect to tensor products.

Proof. Suppose $A \to I^\bullet$ is an injective resolution. Then $\mathcal{G}(I^\bullet)$ is a cocomplex, and we can apply the above construction to it. We obtain

$${}^{\mathrm{I}}E_2^{pq} = H^p((R^q\mathcal{F})(\mathcal{G}(I^\bullet))) \longrightarrow (\mathbb{R}^{p+q}\mathcal{F})(\mathcal{G}(I^\bullet)).$$

Now, $\mathcal{G}(I^\bullet)$ is $\mathcal{F}$-acyclic by assumption, and hence

$$R^q\mathcal{F}(\mathcal{G}(I^p)) = \begin{cases} \mathcal{F}\mathcal{G}(I^p) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

This shows that

$${}^{\mathrm{I}}E_2^{pq} = \begin{cases} R^p(\mathcal{F}\mathcal{G})(A) & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Hence

$$(\mathbb{R}^{p+q}\mathcal{F})(\mathcal{G}(I^\bullet)) = R^{p+q}(\mathbb{F}\mathcal{G})(A).$$

Using the second spectral sequence, we immediately get that

$${}^{\mathrm{II}}E_2^{pq} = R^p\mathcal{F}(H^q(\mathcal{G}(I^\bullet))) = R^p\mathcal{F}(R^q\mathcal{G}(A)).$$

Altogether, this shows that

$$R^p\mathcal{F}(R^q\mathcal{G}(A)) \longrightarrow R^{p+q}(\mathcal{F}\mathcal{G})(A),$$

as required. □

Example 15.16. Let X, Y be topological spaces and $f\colon X \to Y$ be a continuous map. Then the functor

$$f_*\colon \text{Sheaves}(X) \to \text{Sheaves}(Y),$$

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

for a sheaf $\mathcal{F}$ on X, sends injectives to injectives. Then

$$\Gamma(X;\mathcal{F}) = \mathcal{F}(X)$$

gives a functor

$$\Gamma(X;-)\colon \text{Sheaves}(X) \to \text{Ab}$$

and

$$R^p\Gamma(X,\mathcal{F}) = H^p(X;\mathcal{F})$$

the sheaf cohomology.

Then

$$(R^p\Gamma)(R^qf_*)(\mathcal{F}) = H^p(Y;R^qf_*\mathcal{F}),$$

and since

$$\Gamma f_*\mathcal{F} = f_*\mathcal{F}(Y) = \mathcal{F}(f^{-1}(Y)) = \mathcal{F}(X) = \Gamma(X),$$

we get that

$$R^{p+q}(\Gamma f_*)(\mathcal{F}) = H^{p+q}(X;\mathcal{F}).$$

Then the Grothendieck spectral sequence 15.15 gives

$$H^p(Y;R^qf_*\mathcal{F}) \longrightarrow H^{p+q}(X;\mathcal{F}).$$

In particular, if $R^qf_*\mathcal{F} = 0$ for $q > 0$, then

$$H^p(Y;f_*\mathcal{F}) = H^p(X;\mathcal{F}).$$

16. Triangulated categories

Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ be the category of cochain complexes on $\mathcal{A}$. We recall a few definitions.

Suppose $f\colon A^\bullet \to B^\bullet$. The cone of f is given by

$$\begin{tikzcd}
A^{n+2} & A^{n+1} \arrow[l, "-d_A"'] \arrow[ddl, "-f"] \\
\oplus & \oplus \\
B^{n+1} & B^{n} \arrow[l, "d_B"] \\
\mathrm{cone}(f)^{n+1} & \mathrm{cone}(f)^{n} \arrow[l, "d"']
\end{tikzcd}$$

and the boundary map $d\colon \mathrm{cone}(f)^n \to \mathrm{cone}(f)^{n+1}$ is given by the matrix

$$d = \begin{pmatrix} -d_A & 0 \\ -f & d_B \end{pmatrix}.$$

We then have an exact sequence

$$0 \longrightarrow B^\bullet \longrightarrow \mathrm{cone}(f)^\bullet \xrightarrow{\ \delta\ } A[-1]^\bullet \longrightarrow 0.$$

Similarly, we define the cylinder of f:

$$\begin{tikzcd}
A^{n+1} & A^{n} \arrow[l, "d_A"'] \\
\oplus & \oplus \\
A^{n+2} & A^{n+1} \arrow[l, "-d_A"'] \arrow[uul, "\mathrm{id}_A"'] \arrow[ddl, "-f"] \\
\oplus & \oplus \\
B^{n+1} & B^{n} \arrow[l, "d_B"] \\
\mathrm{cyl}(f)^{n+1} & \mathrm{cyl}(f)^{n} \arrow[l, "d"']
\end{tikzcd}$$

and the boundary map $d\colon \mathrm{cyl}(f)^n \to \mathrm{cyl}(f)^{n+1}$ is given by the matrix

$$d = \begin{pmatrix} d_A & \mathrm{id}_A & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}.$$

We have exact sequences

$$0 \longrightarrow A^\bullet \longrightarrow \mathrm{cyl}(f)^\bullet \longrightarrow \mathrm{cone}(f)^\bullet \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & B^\bullet \arrow[r, "\alpha"] & \mathrm{cyl}(f)^\bullet \arrow[r] \arrow[l, bend left, dotted, "\beta"] & \mathrm{cone}(-\mathrm{id}_A)^\bullet \arrow[r] & 0
\end{tikzcd}$$

with

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ \mathrm{id}_B \end{pmatrix}, \qquad \beta = (\mathrm{id}_A \ 0 \ \mathrm{id}_B)$$

and $\alpha\beta \sim \mathrm{id}_B$, $\beta\alpha = \mathrm{id}_B$, so α and β are homotopy equivalences.

We construct a quotient category $\mathcal{K} = \mathcal{K}(A)$ of $\mathrm{Ch} = \mathrm{Ch}(\mathcal{A})$ by

$$\mathrm{Obj}(\mathcal{K}) = \mathrm{Obj}(\mathrm{Ch})$$

$$\mathrm{Hom}_{\mathcal{K}}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Ch}}(A^\bullet, B^\bullet)/\sim$$

where $\sim$ is the chain homotopy equivalence. It is easy to check that composition in $\mathcal{K}$ is well-defined.

This makes $\mathcal{K}$ into an additive category with an additive functor

$$\mathrm{Ch} \to \mathcal{K}.$$

The cohomology functor $H^n \colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ factors through $\mathcal{K}$ because homotopy equivalent maps induce the same maps on cohomology, and hence the triangle

$$\begin{tikzcd} \mathrm{Ch} \arrow[r, "H^n"] \arrow[d] & \mathcal{A} \\ \mathcal{K} \arrow[ur, "H^n"'] & \end{tikzcd}$$

commutes.

The category $\mathcal{K}$ is universal with this property. Suppose $\mathcal{F} \colon \mathrm{Ch} \to \mathcal{B}$ is a functor such that if $f \colon A^\bullet \to B^\bullet$ is a chain homotopy equivalence, then $F(f)$ is an isomorphism, then $\mathcal{F}$ factors through $\mathcal{K}$.

To show this, we first note that we have maps

$$\begin{tikzcd} B^\bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] & \mathrm{cyl}(\mathrm{id}_B) \arrow[l, "\beta"'] \end{tikzcd}$$

where

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ \mathrm{id} \end{pmatrix}, \qquad \alpha' = \begin{pmatrix} \mathrm{id} \\ 0 \\ 0 \end{pmatrix}.$$

We then have that

$$\mathrm{id} = \mathcal{F}(\mathrm{id}) = \mathcal{F}(\beta\alpha) = \mathcal{F}(\beta)\mathcal{F}(\alpha),$$

so $\mathcal{F}(\alpha)$ and $\mathcal{F}(\beta)$ are inverses, and similarly $F(\alpha')$ and $F(\beta)$. Hence:

$$\mathcal{F}(\alpha') = \mathcal{F}(\alpha)\mathcal{F}(\beta)\mathcal{F}(\alpha') = \mathcal{F}(\alpha).$$

Suppose $f, g \colon B \to C$ and $f \sim g$, so

$$f - g = ds + sd.$$

Then $\gamma = (f\ s\ g)\colon \operatorname{cyl}(B) \to C$ is a chain map, and

$$\gamma\alpha = g, \quad \gamma\alpha' = f,$$

so

$$\mathcal{F}(g) = \mathcal{F}(\gamma\alpha) = \mathcal{F}(\gamma)\mathcal{F}(\alpha),$$
$$\mathcal{F}(f) = \mathcal{F}(\gamma\alpha') = \mathcal{F}(\gamma)\mathcal{F}(\alpha').$$

If $i\colon A^\bullet \to B^\bullet$ is a map, then we have a triangle

$$\begin{tikzcd} & \operatorname{cone}(u) \arrow[dl, "\delta"'] & \\ A^\bullet \arrow[rr, "u"] & & B^\bullet \arrow[ul, "v"'] \end{tikzcd}$$

We will call this a *strict exact triangle*.

Definition 16.1. For $u\colon A^\bullet \to B^\bullet$, $v\colon B^\bullet \to C^\bullet$, $w\colon C^\bullet \to A[-1]^\bullet$, the triangle

$$\begin{tikzcd} & C^\bullet \arrow[dl, "w"'] & \\ A^\bullet \arrow[rr, "u"] & & B^\bullet \arrow[ul, "v"'] \end{tikzcd}$$

is called an *exact triangle* if there exists $\widetilde{u}\colon \widetilde{A^\bullet} \to \widetilde{B^\bullet}$ and an isomorphism in $\mathcal{K}$

$$f\colon A \to \widetilde{A},\ g\colon B \to \widetilde{B},\ h\colon C \to \operatorname{cone}(\widetilde{u})$$

such that the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[-1] \arrow[d, "f[-1]"] \\ \widetilde{A} \arrow[r, "\widetilde{u}"] & \widetilde{B} \arrow[r, "\widetilde{v}"] & \operatorname{cone}(\widetilde{u}) \arrow[r] & \widetilde{A}[-1] \end{tikzcd}$$

commutes.

Example 16.2. The diagram

$$\begin{tikzcd} & 0 \arrow[dl] & \\ A^\bullet \arrow[rr, "\mathrm{id}_A"] & & A^\bullet \arrow[ul] \end{tikzcd}$$

is an exact triangle, because the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "="] & A \arrow[r] \arrow[d, "="] & 0 \arrow[r] \arrow[d] & A[-1] \arrow[d, "="] \\ A \arrow[r, "\mathrm{id}_A"] & A \arrow[r] & \operatorname{cone}(\mathrm{id}_A) \arrow[r] & A[-1] \end{tikzcd}$$

commutes and letting $C = \operatorname{cone}(\mathrm{id}_A)$

$$s = \begin{pmatrix} 0 & -\mathrm{id} \\ 0 & 0 \end{pmatrix},\ d_C = \begin{pmatrix} -d_A & 0 \\ \mathrm{id}_A & d_A \end{pmatrix}$$

we get

$$sd_C + d_C s = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_A \end{pmatrix} = \mathrm{id}_C,$$

so $\mathrm{id}_C \sim 0$.

Example 16.3. Suppose

$$\begin{tikzcd}
& C \arrow[dl, "w"'] & \\
A \arrow[rr, "u"] & & B \arrow[ul, "v"']
\end{tikzcd}$$

is exact. We show that

$$\begin{tikzcd}
& A[-1] \arrow[dl, "-u[-1]"'] & \\
B \arrow[rr, "v"] & & C \arrow[ul, "w"']
\end{tikzcd}$$

is exact. Assume without loss of generality that $C = \operatorname{cone}(u)$ and

$$v = \begin{pmatrix} 0 \\ \mathrm{id}_B \end{pmatrix},\ w = \delta = (\mathrm{id}_A\ 0).$$

Letting $D = \operatorname{cone}(v)$, we get that

$$\begin{tikzcd}
A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[d, "="] & C \arrow[r, "\delta"] \arrow[d, "="] & A[-1] \arrow[r, "-u[-1]"] \arrow[d, bend left, "h"] & B[-1] \arrow[d, "="] \\
& B \arrow[r, "v"] & C \arrow[r] & D \arrow[r] \arrow[u, bend left, "\pi"] & B[-1]
\end{tikzcd}$$

where

$$\pi = (0\ \mathrm{id}_A\ 0),\ h = \begin{pmatrix} -u \\ \mathrm{id}_A \\ 0 \end{pmatrix},$$

and the map $C \to D$ is given by the matrix

$$\begin{pmatrix} 0 & 0 \\ \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_B \end{pmatrix}.$$

We have that

$$\mathrm{id}_D - h\pi = \begin{pmatrix} \mathrm{id} & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} = sd_D + d_D s$$

for the map

$$s = \begin{pmatrix} 0 & 0 & -\mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we obtain that

$$\begin{tikzcd} & B \arrow[dl, "v"'] & \\ C[-1] \arrow[rr, "-w[1]"] & & A \arrow[ul, "u"'] \end{tikzcd}$$

is an exact triangle.

Definition 16.4. An additive category $\mathcal{K}$ is called a *triangulated category* if it has an automorphism $T\colon \mathcal{K} \to \mathcal{K}$ and distinguished triples (u, v, w), called *exact triangles*, where

$$u\colon A \to B,\ v\colon B \to C,\ w\colon C \to TA$$

for some A, B, C, such that the following axioms are satisfied:

(TR 1) Every $u\colon A \to B$ can be embedded in a triangle (u, v, w)

$$\begin{tikzcd} & C \arrow[dl, dotted, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, dotted, "v"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "\mathrm{id}_A"] & & A \arrow[ul] \end{tikzcd}$$

is an exact triangle, and if (u, v, w) is isomorphic to (u', v', w') and (u, v, w) is an exact triangle, then (u', v', w') is an exact triangle.

(TR 2) If (u, v, w) is an exact triangle, then

$$(v, w, -Tu) \text{ and } (-T^{-1}w, u, v)$$

are exact triangles.

(TR 3) If

$$\begin{tikzcd} & C \arrow[dl, "w"'] & & & C' \arrow[dl, "w'"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] & A' \arrow[rr, "u'"] & & B' \arrow[ul, "v'"'] \end{tikzcd}$$

are exact triangles and $f\colon A \to A'$, $g\colon B \to B'$ with $gu = u'f$ are morphisms, then there exists $h\colon C \to C'$ such that

$$(f, g, h)\colon (u, v, w) \to (u', v', w')$$

is a morphism of exact triangles, i.e. the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dotted, "h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA'
\end{tikzcd}$$

commutes.

(TR 4) Suppose A, B, C, A', B', C' are objects in $\mathcal{K}$ and
- (u, j, ∂) is an exact triangle on (A, B, C'),
- (v, x, i) is an exact triangle on (B, C, A'),
- (vu, y, δ) is an exact triangle on (A, C, B'),

then there exists an exact triangle $(f, g, (Tj)i)$ on (C', B', A') such that

$$\partial = \delta f, \ x = gy, \ ig = (Tu)\delta.$$

We can represent this as the diagram

$$\begin{tikzcd}
B' \arrow[drr, "\delta"] \arrow[dddrr, dotted, bend right, "g"'] & & & & C' \arrow[llll, dotted, bend right, "f"'] \arrow[dll, "\partial"'] \\
& & A \arrow[dl, "vu"'] \arrow[dr, "u"] & & \\
& C \arrow[uul, "y"] \arrow[dr, "x"] & & B \arrow[ll, "v"] \arrow[uur, "j"'] & \\
& & A' \arrow[ur, "i"] \arrow[uuurr, bend right, "(Tj)i"'] & &
\end{tikzcd}$$

in which all the triangles commute. (Note that, as described above, only some of these triangles are exact. We distinguish in blue the arrows that go to T applied to the objects.)

Theorem 16.5. *For an abelian category $\mathcal{A}$, the quotient $\mathcal{K}(\mathcal{A})$ is a triangulated category with the automorphism $T(A) = A[-1]$.*

Proof. Axioms (TR1), (TR2), (TR3) have already been verified in the discussion above. We only have to show that (TR4) holds.

Without loss of generality,

$$C' = \operatorname{cone}(u), \ B' = \operatorname{cone}(vu), \ A' = \operatorname{cone}(v)$$

and we can represent the maps as follows

$$j = \begin{pmatrix} 0 \\ \mathrm{id}_B \end{pmatrix} : B \to C', \quad \partial = \begin{pmatrix} \mathrm{id}_A & 0 \end{pmatrix} : C' \to A,$$

$$y = \begin{pmatrix} 0 \\ \mathrm{id}_C \end{pmatrix} : C \to B', \quad \delta = \begin{pmatrix} \mathrm{id}_A & 0 \end{pmatrix} : B' \to A,$$

$$x = \begin{pmatrix} 0 \\ \mathrm{id}_C \end{pmatrix} : C \to A', \quad i = \begin{pmatrix} \mathrm{id}_B & 0 \end{pmatrix} : A' \to B.$$

Taking

$$f = \begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & v \end{pmatrix}, \quad g = \begin{pmatrix} u & 0 \\ 0 & \mathrm{id}_C \end{pmatrix},$$

one can easily verify that

$$\delta f = \begin{pmatrix} \mathrm{id}_A & 0 \end{pmatrix} = \partial, \quad gy = x,$$

$$yv = \begin{pmatrix} 0 \\ v \end{pmatrix} = fj, \quad ig = (Tu)\delta.$$

We have to prove that $(f, g, (Tj)i)$. We have

$$\begin{tikzcd}
C' \arrow[r, "\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & v \end{pmatrix}"] \arrow[d, "="] & B' \arrow[r, "\begin{pmatrix} u & 0 \\ 0 & \mathrm{id}_C \end{pmatrix}"] \arrow[d, "="] & A' \arrow[r, "\begin{pmatrix} 0 & 0 \\ \mathrm{id}_B & 0 \end{pmatrix}"] \arrow[d, "h=\begin{pmatrix} 0 & 0 \\ \mathrm{id}_B & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_C \end{pmatrix}"] & C'[-1] \arrow[d, "="] \\
C' \arrow[r, "\begin{pmatrix} \mathrm{id}_A & 0 \\ 0 & v \end{pmatrix}"'] & B' \arrow[r, "\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_A & 0 \\ 0 & \mathrm{id}_C \end{pmatrix}"'] & D \arrow[r, "\begin{pmatrix} \mathrm{id}_A & 0 & 0 & 0 \\ 0 & \mathrm{id}_B & 0 & 0 \end{pmatrix}"'] & C'[-1]
\end{tikzcd}$$

where $D = \mathrm{cone}(f)$. We construct a chain map

$$\pi = \begin{pmatrix} 0 & \mathrm{id}_b & u & 0 \\ 0 & 0 & 0 & \mathrm{id}_C \end{pmatrix} : \mathrm{cone}(f) \to A',$$

and claim that this gives an inverse map in the quotient category. We have that

$$d_{A'} = \begin{pmatrix} -\mathrm{id}_b & 0 \\ -v & d_C \end{pmatrix}, \quad d_D = \begin{pmatrix} d_A & 0 & 0 & 0 \\ u & -d_B & 0 & 0 \\ -\mathrm{id}_A & 0 & -d_A & 0 \\ 0 & -v & -vu & d_C \end{pmatrix}$$

and $d_D h = h d_{A'}$, $d_{A'}\pi = \pi d_D$, $\pi h = \mathrm{id}_A$,

$$h\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}_B & u & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_C \end{pmatrix}.$$

Setting

$$s = \begin{pmatrix} 0 & 0 & -\mathrm{id}_B & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we get that

$$sd_D + d_D s = \begin{pmatrix} \mathrm{id}_A & 0 & 0 & 0 \\ 0 & 0 & -u & 0 \\ 0 & 0 & \mathrm{id}_A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{id}_D - h\pi.$$

This shows that $\mathrm{id}_D \sim h\pi$, and hence h, π are homotopy equivalences, i.e. isomorphisms in $\mathcal{K}$. Finally, $\pi p = g$, $p = h\pi p = hg$ (in $\mathcal{K}$). $\square$

Similarly, $\mathcal{K}^b = \mathrm{Ch}^b / \sim$, bounded chain complexes, $\mathcal{K}^+$, positive chain complexes, $\mathcal{K}^-$ negative chain complexes, are all triangulated categories.

Definition 16.6. If $H\colon \mathcal{K} \to \mathcal{A}$ is an additive functor where $\mathcal{K}$ is a triangulated category and $\mathcal{A}$ is an abelian category, then H is called a *cohomological functor* if for every exact triangle Δ

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

we have a long exact sequence

$$\cdots \longrightarrow H(T^{i-1}C) \longrightarrow H(T^iA) \xrightarrow{H(T^iu)} H(T^iB) \xrightarrow{H(T^iv)} H(T^iC) \xrightarrow{H(T^iw)} H(T^{i+1}A) \longrightarrow \cdots$$

We will then write $H^i(A) = H(T^iA)$ and $H^i(u) = H(T^iu)$.

Example 16.7. The functor

$$\mathcal{K}(\mathcal{A}) \longrightarrow \mathcal{A}$$
$$A^\bullet \longrightarrow H^0(A^\bullet)$$

is a cohomological functor. Indeed, for an exact triangle

$$\begin{tikzcd} & \mathrm{cone}(u) \arrow[dl] & \\ A \arrow[rr, "u"] & & B \arrow[ul] \end{tikzcd}$$

we have a long exact sequence

$$\cdots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(\mathrm{cone}(u)) \longrightarrow H^{i+1}(A^\bullet) = H^i(A[-1]) \longrightarrow \cdots$$

Proposition 16.8. *If $\mathcal{K}$ is a triangulated category, $X \in \operatorname{Obj}\mathcal{K}$, then $\operatorname{Hom}_{\mathcal{K}}(X,-)$ is a cohomological functor $\mathcal{K} \to \mathcal{A}$.*

Proof. Suppose

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

is an exact triangle. By (TR 1), there is an extra triangle

$$\begin{tikzcd} & 0 \arrow[dl] & \\ A \arrow[rr, "\mathrm{id}_A"] & & A \arrow[ul] \end{tikzcd}$$

and by (TR 3), there exists h such that the squares in

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r, "0"] \arrow[d, "u"] & 0 \arrow[r, "0"] \arrow[d, dotted, "h"] & TA \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & TA \end{tikzcd}$$

commute, so that $vu = h0 = 0$. By (TR 2), we also get that $wv = (Tu)w = 0$.

We have a chain complex

$$\begin{tikzcd} \cdots \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[dll, out=-10, in=170] & \\ & TA \arrow[r] & TB \arrow[r] & TC \arrow[r] & \cdots \end{tikzcd}$$

and this gives a chain complex

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Hom}_{\mathcal{K}}(X,A) \arrow[r] & \operatorname{Hom}_{\mathcal{K}}(X,B) \arrow[r] & \operatorname{Hom}_{\mathcal{K}}(X,C) \arrow[dll, out=-10, in=170] & \\ & \operatorname{Hom}_{\mathcal{K}}(X,TA) \arrow[r] & \operatorname{Hom}_{\mathcal{K}}(X,TB) \arrow[r] & \operatorname{Hom}_{\mathcal{K}}(X,TC) \arrow[r] & \cdots \end{tikzcd}$$

Suppose $b\colon X \to B$ with $vb = 0$. We have

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "T^{-1}h"] & X \arrow[r] \arrow[d, "b"] & 0 \arrow[r] \arrow[d] & TX \arrow[r, "-\mathrm{id}"] \arrow[d, dotted, "h"] & TX \arrow[d, "TB"] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & TA \arrow[r, "-Tu"] & TB \end{tikzcd}$$

then by (TR 3) we get $-(Tu)h = -Tb$, so $u(T{-}1h) = b$. This shows exactness at $\mathrm{Hom}_{\mathcal{K}}(X, B)$. We can first rotate the triangles, and then shift using T to show exactness everywhere. □

Example 16.9. In $\mathcal{K}(\mathrm{Ab})$, we have

$$\begin{tikzcd}
\mathbb{Z}[0] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
& \cdots \arrow[r] & A^{-1} \arrow[r] & A^0 \arrow[r, "d^0"] & A^1 \arrow[r] & A^2 \arrow[r] & \cdots
\end{tikzcd}$$

and so

$$\mathrm{Hom}_{\mathrm{Ch}}(\mathbb{Z}[0], A^\bullet) = \ker(d^0)$$

$$\mathrm{Hom}_{\mathcal{K}}(\mathbb{Z}[0], A^\bullet) = \ker(d^0)/\sim\, = \ker(d^0)/\operatorname{im}(d^{-1}) = H^0(A).$$

This shows that the cohomology functor is representable.

17. Derived categories

Suppose $\mathcal{C}$ is a category and S is a collection of morphisms.

Definition 17.1. A *localization of $\mathcal{C}$ with respect to S* is a category $S^{-1}\mathcal{C}$ together with a functor $q\colon \mathcal{C} \to S^{-1}\mathcal{C}$ such that

(1) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for all $s \in S$,
(2) $(S^{-1}\mathcal{C}, q)$ is universal with property (1), i.e. for every category $\mathcal{D}$ and functor

$$\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$$

such that $\mathcal{F}(s)$ is an isomorphism for all $s \in S$, there exists a unique functor $\widetilde{\mathcal{F}}\colon S^{-1}\mathcal{C} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "q"] \arrow[d, "\mathcal{F}"'] & S^{-1}\mathcal{C} \arrow[dl, dotted, "\widetilde{\mathcal{F}}"] \\
\mathcal{D} &
\end{tikzcd}$$

commutes.

If $\mathcal{C}$ is a small category, then $S^{-1}\mathcal{C}$ exists. Indeed, let $S^{-1}\mathcal{C}$ be the free category on $\mathrm{Obj}\,\mathcal{C}$ generated by all morphisms in $\mathcal{C}$ and all $\widetilde{s}$, $s \in S$, modulo relations from $\mathcal{C}$ and $s\widetilde{s} = \mathrm{id}$, $\widetilde{s}s = \mathrm{id}$ for all $s \in S$. Morphisms in $S^{-1}\mathcal{C}$ are of the form $\widetilde{s_4}s_3s_2\widetilde{s_1}$ and so on.

Example 17.2. If S is the collection of chain homotopy equivalences in $\mathrm{Ch}(\mathcal{A})$, then

$$\mathcal{K} = S^{-1}\,\mathrm{Ch}(\mathcal{A}).$$

Definition 17.3. If Q is the collection of all quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})$, then the *derived category* of $\mathcal{A}$ is

$$D(\mathcal{A}) = Q^{-1}\,\mathrm{Ch}(A).$$

This definition is very abstract so we will try to go via the quotient category $\mathcal{K}$ to get a better understanding.

Let R be the collection of quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$, then $R^{-1}\mathcal{K}(\mathcal{A}) = D(\mathcal{A})$:

$$\begin{tikzcd}
\mathrm{Ch}(\mathcal{A}) \arrow[r] \arrow[d] & S^{-1}\,\mathrm{Ch}(\mathcal{A}) = \mathcal{K}(\mathcal{A}) \arrow[r] \arrow[dl, dotted] & R^{-1}\mathcal{K}(\mathcal{A}) \arrow[dll, dotted] \\
Q^{-1}\,\mathrm{Ch}(A) \arrow[urr, dotted, bend right] & &
\end{tikzcd}$$

where we get the dotted arrows from the universal properties, and they are unique so they are inverses.

Definition 17.4. Let $\mathcal{C}$ be any category. A set of morphisms S is a *multiplicative system* if

(1) S is closed under composition,
(2) $\mathrm{id}_X \in S$ so any $x \in \mathrm{Obj}\,\mathcal{C}$,
(3) *Ore condition*: if $t\colon Z \to Y$ is in S and $g\colon X \to Y$ (in $\mathcal{C}$), then there exists a commuting square

$$\begin{tikzcd}
W \arrow[r, dotted, "f"] \arrow[d, dotted, "s"] & Z \arrow[d, "t"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

for $s \in S$, $f \in \mathrm{Mor}(\mathcal{C})$ and also the dual statement holds,
(4) if $f, g\colon X \to Y$ in $\mathcal{C}$, then: there exists $s \in S$ such that $sf = sg$ if and only if there exists $t \in S$ such that $ft = gt$.

The idea behind this definition is to represent morphisms in $S^{-1}\mathcal{C}$ in the form $fs^{-1} = f\widetilde{s}$ with $s \in S$, $f \in \mathcal{C}$:

$$\begin{tikzcd}
A & B \arrow[l, "s"'] \arrow[r, "f"] & C \\
\arrow[from=1-1, to=1-3, dotted, bend right, "\text{in } S^{-1}\mathcal{C}"']
\end{tikzcd}$$

The Ore condition shows that we can write $s_1^{-1}f_2$ where $s_1 \in S$ as $f_3s_3^{-1}$ where $s_3 \in S$, and we can write the composition in the same form again

$$(f_1s_1^{-1})(f_2s_2^{-1}) = f_1f_3s_3^{-1}s_1^{-1} = (f_1f_3)(s_1s_3)^{-1}.$$

Define a category $\mathcal{D}$ with $\mathrm{Obj}(\mathcal{D}) = \mathrm{Obj}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{D}}(A, B)$ as the set of all diagrams

$$A \xleftarrow{s} C \xrightarrow{f} B$$

with $s \in S$ and $f \in \mathcal{C}$, modulo $\equiv$ where

$$[A \xleftarrow{s_1} C_1 \xrightarrow{f_1} B] \equiv [A \xleftarrow{s_2} C_2 \xrightarrow{f_2} B]$$

if there exist $t_1, t_2 \in S$ with $f_1t_1 = f_2t_2$ and $s_1t_2 = s_2t_2$:

$$\begin{tikzcd}
& C_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\
A & D \arrow[u, "t_1"] \arrow[d, "t_2"'] & B \\
& C_2 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

The composition is defined by considering the following diagram

$$\begin{tikzcd}
D \arrow[r, dotted, "\widetilde{f}_1"] \arrow[d, dotted, "\widetilde{s}_2"] & C_2 \arrow[r, "f_2"] \arrow[d, "s_2"] & A_3 \\
C_1 \arrow[r, "f_1"] \arrow[d, "s_1"] & A_2 \arrow[ur, "h_2 \in \mathcal{D}"'] & \\
A_1 \arrow[ur, "h_1 \in \mathcal{D}"'] & &
\end{tikzcd}$$

and letting

$$A_1 \xleftarrow{\widetilde{s}_2 s_1} D \xrightarrow{f_2 \widetilde{f}_1} A_3$$

to be the composition. One can check that $\equiv$ is an equivalence relation and composition is well-defined.

We have a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, sending $f\colon A \to B$ in $\mathcal{C}$ to

$$\mathcal{F}(f) = [A \xleftarrow{\mathrm{id}_A} A \xrightarrow{f} B] \in \mathrm{Hom}_{\mathcal{D}}(A, B).$$

We claim that if $s \in S$, then $\mathcal{F}(s)$ is an isomorphism in $\mathcal{D}$. Indeed,

$$\mathcal{F}(s) = [A \xleftarrow{\mathrm{id}_A} A \xrightarrow{s} B]$$

has inverse

$$[B \xleftarrow{s} A \xrightarrow{\mathrm{id}_A} A]$$

because the composition

$$[B \xleftarrow{s} A \xrightarrow{s} B]$$

is equivalent to

$$[B \xleftarrow{\mathrm{id}_B} B \xrightarrow{\mathrm{id}_B} B]$$

via the diagram

$$\begin{tikzcd}
& A \arrow[dl, "s"'] \arrow[dr, "a"] & \\
B & A \arrow[u, "\mathrm{id}_A"] \arrow[d, "s"'] & B \\
& B \arrow[ul, "\mathrm{id}_B"] \arrow[ur, "\mathrm{id}_B"'] &
\end{tikzcd}$$

Finally, one can show that $(\mathcal{D}, \mathcal{F})$ has the universal property of localization. This gives a very concrete description of a localization with respect to a multiplicative system.

For an abelian category $\mathcal{A}$, we can hence describe the derived category as follows:

(1) $\mathcal{K}(\mathcal{A}) = U^{-1}\operatorname{Ch}(\mathcal{A})$ where U is a collection of homotopy equivalences (so we replace morphisms by equivalence classes, making the set of morphisms smaller),
(2) $\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{K}(\mathcal{A})$ where S is a collection of quasi-isomorphisms (this S is actually a multiplicative system, and hence morphisms in $\mathcal{D}(\mathcal{A})$ can be described as *fractions* of morphisms in $\mathcal{K}(\mathcal{A})$).

We still have to show that the collection of quasi-isomorphisms is a mulitplicative system. We do this in more generality.

Proposition 17.5. *Suppose $\mathcal{K}$ is a triangulated category, $\mathcal{A}$ is an abelian category, and $H\colon \mathcal{K} \to \mathcal{A}$ is a cohomological functor. Let S be the collection of all s such that $H^n(s)$ is an isomorphism for all n. Then S is a multiplicative system.*

Proof. We check the axioms:

(1), (2) By functoriality of H, S is closed under composition and $\mathrm{id}_x \in S$.
(3) Ore property. Given $s \in S$, $f \in \mathcal{K}$, we want to find $t \in S$, $g \in \mathcal{K}$ such that

$$\begin{tikzcd} W \arrow[r, dotted, "t"] \arrow[d, dotted, "g"'] & Z \arrow[d, "f"] \\ X \arrow[r, "s"] & Y \end{tikzcd}$$

commutes. Embed s in an exact triangle

$$\begin{tikzcd} & C \arrow[dl, "\delta"'] & \\ Z \arrow[rr, "s"] & & Y \arrow[ul, "u"'] \end{tikzcd}$$

and then embed uf in an exact triangle and rotate it to get an exact triangle

$$\begin{tikzcd} & C \arrow[dl, "v"'] & \\ W \arrow[rr, "t"] & & X \arrow[ul, "uf"'] \end{tikzcd}$$

Together, by (TR 3), there exists $g\colon W \to Z$ such that the following diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, dotted, "g"] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r, "v"] \arrow[d, "="] & TW \arrow[r] \arrow[d, "Tg"] & TX \arrow[d, "Tf"] \\ Z \arrow[r, "s"] & Y \arrow[r, "u"] & C \arrow[r] & TZ \arrow[r] & TY \end{tikzcd}$$

commutes. Since $H^n(s)$ is an isomorphism for all n, we get

$$H^n(Z) \xrightarrow{\cong} H^n(Y) \longrightarrow H^n(C) \longrightarrow H^{n+1}(Z) \xrightarrow{\cong} H^{n+1}(Y)$$

so $H^n(C) = 0$, and hence $H^n(t)$ is an isomorphism for all n. The dual property holds by considering the dual category (the dual category of a triangulated category is also triangulated).

(4) If $f, g\colon X \to Y$, we show that $sf = sg$ for some $s \in S$ if and only if $ft = gt$ for some $t \in S$.

We show the '*only if*' implication; the other implication is symmetric. Suppose $s\colon Y \to Y'$ satisfies $sf = sg$ and $s \in S$. Let $h = f - g$ and embed s in an exact triangle

$$\begin{tikzcd} & Y' \arrow[dl, "\delta"'] & \\ Z \arrow[rr, "u"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

From the long exact sequence, as above, we get $H^\bullet(Z) = 0$. Since $\mathrm{Hom}_{\mathcal{K}}(X, -)$ is a cohomological functor, we have the exact sequence

$$\mathrm{Hom}_{\mathcal{K}}(X, Z) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{K}}(X, Y')$$

$$v \longrightarrow uv = h \longrightarrow sh = 0$$

and, since sh, by exactness, there exists $v\colon X \to Z$ such that $uv = h$. Now, v lies in an exact triangle

$$\begin{tikzcd} & Z \arrow[dl, "w"'] & \\ X' \arrow[rr, "t"] & & X \arrow[ul, "v"'] \end{tikzcd}$$

But $vt = 0$, so $0 = uvt = ht = ft - gt$. Hence $ft = gt$. Finally, since $H^\bullet(Z) = 0$, $H^n(t)$ is an isomorphism for all n, and hence $t \in S$.

This shows that S is a multiplicative system. $\square$

By the above discussion, this shows that morphisms in $S^{-1}\mathcal{K}$ are of the form fs^{-1} for $s \in S$, $f \in \mathcal{K}$. A morphism between X and Y is

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and is sometimes referred to as a *roof*.

Proposition 17.6. *The derived category $\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{K}(\mathcal{A})$ is a triangulated category with $T(fs^{-1}) = T(f)T(s)^{-1}$.*

Proof. First note that T is well-defined: if $f_1 s_1^{-1} = f_2 s_2^{-1}$, then $T(f_1)T(s_1)^{-1} = T(f_2)T(s_2)^{-1}$. An exact triangle in $\mathcal{D}(\mathcal{A})$ is a triangle that is isomorphic to an exact triangle in $\mathcal{K}(\mathcal{A})$. We need to check the 4 axioms, but we will only check some of them, the rest can be found in [Wei94].

(TR 1) Suppose $f = us^{-1}$ is a morphism $X \to Y$ in $\mathcal{D}(\mathcal{A})$:

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X & & Y \end{tikzcd}$$

We have that u lies in a exact triangle in $\mathcal{K}(\mathcal{A})$

$$\begin{tikzcd} & U \arrow[dl, "w"'] & \\ Z \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

and we have the diagram

$$\begin{tikzcd} Z \arrow[r, "u"] & Y \arrow[r] & U \arrow[r] & Z[-1] \\ X \arrow[u, "s^{-1}"] \arrow[r, "f"] & Y \arrow[u, "="] \arrow[r, "v"] & U \arrow[u, "="] \arrow[r, "{s[-1]}"] & X[-1] \arrow[u, "{s^{-1}[-1]}"'] \end{tikzcd}$$

Then

$$\begin{tikzcd} & U \arrow[dl, "{s[-1]w}"'] & \\ X \arrow[rr, "f"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

is an exact triangle.

The (TR 2) axioms is clear. The axioms (TR 3) and (TR 4) require a proof, but we omit it here. ☐

Similarly, we can define $\mathcal{D}^b(\mathcal{A})$ from bounded chain complexes and $\mathcal{D}^+(\mathcal{A})$ from positive chain complexes.

Proposition 17.7. *Suppose $I^\bullet$ is a cochain complex of injectives, bounded from below, $Z^\bullet$ is a cochain complex. If $t\colon I^\bullet \to Z^\bullet$ is a quasi-isomorphism in $\mathcal{K}(\mathcal{A})$, then there exists $s\colon Z^\bullet \to I^\bullet$ with $st = \mathrm{id}$ in $\mathcal{K}(\mathcal{A})$.*

Corollary 17.8. *Suppose $I^\bullet$ is a cochain complex of injectives, bounded from below, in $\mathcal{D}(\mathcal{A})$. Then $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X, I) = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(X, I)$.*

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