MATH 631: ALGEBRAIC GEOMETRY I
INTRODUCTION TO ALGEBRAIC VARIETIES

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These are notes from Math 631: Algebraic geometry I taught by Professor Mircea Mustaţă in Fall 2017, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from January 5, 2018. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

References for the first part of the course:

(1) Mumford, *The Red Book of varieties and schemes* [Mum99], Chapter I
(2) Shafarevich, *Basic Algebraic Geometry* [Sha13], Chapter I,
(3) Hartshorne, *Algebraic Geometry* [Har77], Chapter I.

The problem sets, homeworks, and official notes can be found on the course website:

http://www-personal.umich.edu/~mmustata/631_2017.html

Contents

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1. Affine algebraic varieties

1.1. **Algebraic sets and ideals.** The goal is to establish a correspondence

$$\left\{ \begin{array}{c} \text{geometric objects defined} \\ \text{by polynomial equations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{ideals in a} \\ \text{polynomial ring} \end{array} \right\}.$$

Let k be a fixed algebraically closed field of arbitrary characteristic. For example, k could be $\mathbb{C}$, $\overline{\mathbb{Q}}$, or $\overline{\mathbb{F}_p}$ for p prime.

Say n is fixed. The *n-dimensional affine space* $\mathbb{A}^n$ or $\mathbb{A}^n_k$ is (as a set) k^n and the *polynomial ring in n variables over k* is $R = k[x_1, \dots, x_n]$.

How do these two objects correspond to each other? First, note that if $f \in R$ and $u = (u_1, \dots, u_n) \in k^n$, we can evaluate f at u to get $f(u) \in k$. More specifically, for $u = (u_1, \dots, u_n) \in k^n$, we get a homomorphism

$$R = k[x_1, \dots, x_n] \to k, \; x_i \mapsto u_i$$

which is surjective with kernel $(x_1 - u_1, \dots, x_n - u_n)$.

Definition 1.1.1. Given a subset $S \subseteq R$, define

$$V(S) = \{u = (u_1, \dots, u_n) \in \mathbb{A}^n \mid f(u) = 0 \text{ for all } f \in S\}$$

called the *zero locus of S* or the *subset of $\mathbb{A}^n$ defined by S*.

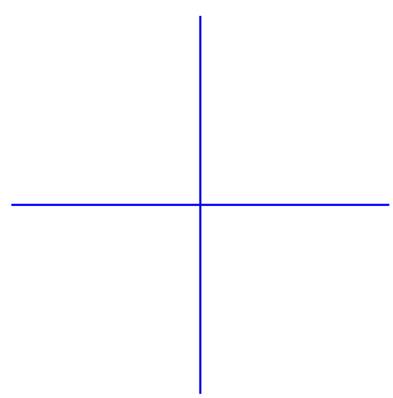
An *algebraic subset* of $\mathbb{A}^n$ is a subset of the form $V(S)$ for some $S \subseteq R$.

Example 1.1.2. A linear subspace of k^n is an algebraic subset. We can take S to a finite collection of linear polynomials:

$$\left\{ \sum_{i=1}^n a_i x_i : a_i \in k, \text{not all } 0 \right\}.$$

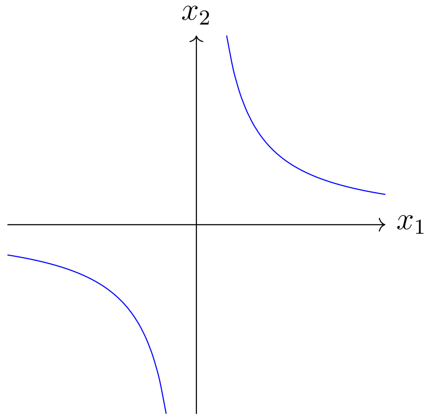
More generally, any translation of a linear subspace is an algebraic subset (called an *affine subspace*).

Example 1.1.3. A union of 2 lines in $\mathbb{A}^2$



$V(x_1 x_2)$

Example 1.1.4. A hyperbola in $\mathbb{A}^2$.



$V(x_1x_2-1)$

Remark 1.1.5. The pictures are drawn in $\mathbb{R}^2$. For obvious reasons, it is impossible to draw pictures in $\mathbb{C}^2$ (or especially $\overline{\mathbb{Q}}^2$ or $\overline{\mathbb{F}_p}^2$), but even the pictures in $\mathbb{R}^2$ can be used to develop a geometric intuition.

Proposition 1.1.6.

(1) $\emptyset = V(1) = V(R)$ *(hence $\emptyset$ is an algebraic subset).*
(2) $\mathbb{A}^n = V(0)$ *(hence $\mathbb{A}^n$ is an algebraic subset).*
(3) *If I is the ideal generated by S, then $V(S) = V(I)$.*
(4) *If $I \subseteq J$ are ideals, then $V(J) \subseteq V(I)$.*
(5) *If $(I_\alpha)_\alpha$ is a family of ideals, then*

$$\bigcap_\alpha V(I_\alpha) = V\left(\bigcup_\alpha I_\alpha\right) = V\left(\sum_\alpha I_\alpha\right).$$

(6) *If I, J are ideals, then*

$$V(I) \cup V(J) = V(I \cdot J) = V(I \cap J).$$

Proof. Properties (1) and (2) are trivial. For (3), recall that

$$I = \{g_1 f_1 + \cdots + g_m f_m \mid m \geq 0, f_i \in S, g_i \in R\}$$

and

$$(g_1 f_1 + \cdots + g_m f_m)(u) = g_1(u) f_1(u) + \cdots + g_m(u) f_m(u).$$

Properties (4) and (5) follow easily from definitions. For (6), note first that

$$V(I) \cup V(J) \subseteq V(I \cap J) \subseteq V(I \cdot J)$$

by property (4). To show $V(I \cdot J) \subseteq V(I) \cup V(J)$, suppose $u \in V(I \cdot J) \setminus (V(I) \cup V(J))$. Then there exist $f \in I$ and $g \in J$ such that $f(u) \neq 0$ and $g(u) \neq 0$. But then $f \cdot g \in I \cdot J$ and $(f \cdot g)(u) = f(u)g(u) \neq 0$, which is a contradiction. $\square$

Remark 1.1.7. By Proposition 1.1.6, the algebraic subsets of $\mathbb{A}^n$ form the closed sets of a topology on $\mathbb{A}^n$, the *Zariski topology*.

Suppose now that $W \subseteq \mathbb{A}^n$ is any subset. Then

$$I(W) = \{f \in R \mid f(u) = 0 \text{ for all } u \in W\} \subseteq R$$

is an ideal in R.

Recall that an ideal $I \subseteq R$ is *radical* if whenever $f^q \in I$ for some $f \in R$, $q \geq 1$, then $f \in I$. A ring R is *reduced* if 0 is a radical ideal, i.e. the ring has no nilpotents.

Since $R = k[x_1, \ldots, x_n]$ is reduced, $I(W)$ is a radical ideal.

Easy properties:

(1) $I(\emptyset) = R$.
(2) $I(\mathbb{A}^n) = 0$ (exercise: show this holds for any infinite field k).
(3) $I(W_1 \cup W_2) = I(W_1) \cap I(W_2)$.
(4) If $W_1 \subseteq W_2$, then $I(W_1) \supseteq I(W_2)$.

We hence have the following maps:

$$\begin{tikzcd} \left\{ \begin{array}{c} \text{subsets} \\ \text{of } \mathbb{A}^n \end{array} \right\} \arrow[r, bend left, "I(-)"] & \left\{ \begin{array}{c} \text{ideals} \\ \text{in } R \end{array} \right\} \arrow[l, bend left, "V(-)"] \end{tikzcd}$$

Proposition 1.1.8. *If $W \subseteq \mathbb{A}^n$ is any subsect, then*

$$V(I(W)) = \overline{W},$$

the closure of W. In particular, if W is an algebraic subset, then $V(I(W)) = W$.

Proof. The containment $W \subseteq V(I(W))$ is clear, and as the right hand side is closed,

$$\overline{W} \subseteq V(I(W)).$$

Let us show that

$$V(I(W)) \subseteq \overline{W} = \bigcap_{Z \supseteq W} Z$$

where the intersection is taken over $Z \supseteq W$ closed. If $Z \supseteq W$ is closed, then $Z = V(J)$ for some J. We have that

$$J \subseteq I(Z) \subseteq I(W)$$

and hence

$$Z = V(J) \supseteq V(I(W)).$$

This completes the proof. $\square$

Recall: given an ideal J in a ring R,

$$\sqrt{J} := \{f \in R \mid \text{there exists } q > 0 \text{ such that } f^q \in J\}.$$

This is a radical ideal, the smallest such that contains J.

Theorem 1.1.9 (Hilbert Nullstellensatz)**.** *For every ideal $J \subseteq R = k[x_1, \ldots, x_n]$, we have*

$$I(V(J)) = \sqrt{J}.$$

Homework. Read Review Sheet #1: *Finite and integral ring homomorphisms.*

Corollary 1.1.10. *We have inverse, order reversing bijections*

$$\left\{ \begin{array}{c} \textit{algebraic subsets} \\ \textit{of } \mathbb{A}^n \end{array} \right\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{radical ideals} \\ \textit{in } k[x_1, \ldots, x_n] \end{array} \right\}$$

Note that if $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$, $\{P\}$ is a closed subset. Clearly

$$I(P) \supseteq (x_1 - a_1, \ldots, x_n - a_n),$$

and they are equal since the right hand side is a maximal ideal. Hence, the points are the minimal nonempty algebraic subsets.

Thus the corollary implies that all maximal ideals are of the form

$$(x_1 - a_1, \ldots, x_n - a_n).$$

We will prove this first, and then the Hilbert Nullstellensatz 1.1.9.

Recall that a ring homomorphism $A \to B$ is *finite* if B is finitely generated as an A-module. The following assertions clearly hold.

(1) The composition of finite homomorphisms is finite,
(2) If f is finite, then $f \otimes 1_C$ is finite:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f\otimes 1_C"] & B \otimes_A C \end{tikzcd}$$

(3) If $f\colon A \to B$ is injective and finite, with A, B domains, then A is a field if and only if B is a field. (Note: it is enough for f to be integral.)

Corollary 1.1.11. *Suppose $A \hookrightarrow B$ is finite with A and B domains. If $K = \mathrm{Frac}(A)$, $L = \mathrm{Frac}(B)$, the induced extension $K \hookrightarrow L$ is finite.*

Proof. Since $A \hookrightarrow B$ is finite and injective,

$$\begin{tikzcd} A \otimes_A K \arrow[r] \arrow[d, "="] & B \otimes_A K \arrow[d, hook] \\ K \arrow[r, hook] & L \end{tikzcd}$$

finite injection of domains

Since K is a field, $B \otimes_A K$ is also a field by property (3) above, and hence $L = B \otimes_A K$ is finite over K. □

Theorem 1.1.12 (Noether Normalization Lemma)**.** *If k is a field (not necessarily algebraically closed), A is an algebra of finite type over k which is a domain, $K = \mathrm{Frac}(A)$, $n = \mathrm{trdeg}(K/k)$, then there exists a k-subalgebra B of A such that*

(1) *B is isomorphic as a k-algebra to $k[x_1, \ldots, x_n]$,*
(2) *the inclusion map $B \hookrightarrow A$ is finite.*

Proof. We will assume k is infinite for simplicity. (For a proof in the general case, see [Mum99]).

We use this as follows: given any nonzero $f \in k[x_1, \ldots, x_r]$, there exists $\lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$ such that $f(\lambda) \neq 0$. (Exercise: prove this by induction on $r \geq 1$.)

Choose generators $y_1, \ldots, y_m$ of A as a k-algebra. Then

$$K = k(y_1, \ldots, y_m),$$

so $n = \operatorname{trdeg}(K/k) \leq m$.

We use induction on $m \geq n$ to show that there exists a linear change of variables

$$y_i = \sum_{j=1}^{m} b_{ij} z_j,$$

where $b_{ij} \in k$ and $\det(b_{ij}) \neq 0$ (hence $A = k[z_1, \ldots, z_n]$) such that

$$k[z_1, \ldots, z_n] \hookrightarrow A$$

is finite.

We claim that this is enough. Take $B = k[z_1, \ldots, z_n]$. Then $\operatorname{Frac}(B) \hookrightarrow \operatorname{Frac}(A) = K$ is finite by Corollary 1.1.11. This implies $n = \operatorname{trdeg}(\operatorname{Frac}(B)/k)$. Hence $z_1, \ldots, z_n$ are algebraically independent over k.

It remains to prove the above statement. If $m = n$, we can use identity as the change of variables because A is a polynomial algebra. Suppose $m > n$. Then $y_1, \ldots, y_m$ are algebraically dependent over k, so there exists a nonzero $f \in k[x_1, \ldots, x_m]$ such that $f(y_1, \ldots, y_m) = 0$. Write

$$f = \underbrace{f_d}_{\neq 0} + \text{lower degree terms } = \sum_{\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}} c_\alpha x_1^{\alpha_1} \ldots x_m^{\alpha_m}.$$

Then

$$\begin{aligned} 0 &= f(y_1, \ldots, y_m) \\ &= f(b_{11}z_1 + \cdots + b_{1m}z_m, \ldots) \\ &= \textstyle\sum_\alpha c_\alpha (b_1 1 z_1 + \cdots + b_{1m}z_m)^\alpha_1 \ldots (b_{m1}z_1 + \cdots + b_{mm}z_m)^{\alpha_m} \\ &= \underbrace{f_d(b_{1m}, \ldots, b_{mm}) z_m^d}_{\in k} + \text{lower order terms in } z_m. \end{aligned}$$

Choose the b_{ij} so that $f_d(b_{1m}, \ldots, b_{mm}) \cdot \det(b_{ij}) \neq 0$. We can do this since k is infinite.

After this change of variables and relabelling z_i as y_i,

$$k[y_1, \ldots, y_{m-1}] \hookrightarrow k[y_1, \ldots, y_m]$$

is **finite**: generated as a module over the left hand side by $1, y_m, \ldots, y_m^{d-1}$.

By induction, after another change of variables in $y_1, \ldots, y_{m-1}$, we may assume

$$k[y_1, \ldots, y_n] \underset{\text{finite}}{\hookrightarrow} k[y_1, \ldots, y_{m-1}] \underset{\text{finite}}{\hookrightarrow} k[y_1, \ldots, y_m]$$

and hence the composition is finite. This completes the proof of the claim and hence the theorem. $\square$

Corollary 1.1.13 (Hilbert Nullstellensatz, Version I)**.** *If k is a field and A is a finitely generated k-algebra, $\mathfrak{m}$ is a maximal ideal in A, then*

$$[A/\mathfrak{m} : k] < \infty.$$

Proof. Let $K = A/\mathfrak{m}$. Since K is a finitely generated k-algebra, by Theorem 1.1.12, there exists a finite inclusion $B \hookrightarrow K$ with $B \cong k[x_1, \ldots, x_n]$. As K is a field, B is also a field by property (3) above. Hence $n = 0$, and K/k is finite. □

Homework. Read Review Sheet #2 about Noetherian rings.

Corollary 1.1.14 (Hilbert Nullestellensatz, Version II)**.** *Suppose $k = \overline{k}$ and let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n] = R$. Then there exist $a_1, \ldots, a_n \in k$ such that*

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n).$$

Proof. By Corollary 1.1.13, $k \hookrightarrow R/\mathfrak{m}$ is finite and since k is algebraically closed, this extension is an isomorphism.

For all i, there exists $a_i \in k$ such that $x_i \equiv a_i \mod \mathfrak{m}$. Then

$$(x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathfrak{m},$$

but the left hand side is a maximal ideal, and hence equality holds. □

Proof of Hilbert Nullstellensatz 1.1.9. Note that $J \subseteq I(V(J))$ by definition, and since the right hand side is a radical ideal:

$$\sqrt{J} \subseteq I(V(J)).$$

We need to prove the reverse inclusion.

Weak version: given an ideal $\mathfrak{a} \subsetneq R$, we have $V(\mathfrak{a}) \neq 0$. This is clear, since any $\mathfrak{a} \neq R$ is contained in a maximal ideal $\mathfrak{m}$, and by Corollary 1.1.14,

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

for some $a_1, \ldots, a_n \in k$. Then

$$(a_1, \ldots, a_n) \in V(\mathfrak{m}) \subseteq V(\mathfrak{a}).$$

Rabinovich's trick: use this property in $k[x_1, \ldots, x_n, y]$. Let $f \in I(V(J))$ and

$$\mathfrak{a} \subseteq R[y] = k[x_1, \ldots, x_n, y]$$

be the ideal generated by J and $1 - f(x)y$, where $x = (x_1, \ldots, x_n)$.

If $\mathfrak{a} \neq R[y]$, then $V(\mathfrak{a}) \neq \emptyset$, so let $(a_1, \ldots, a_n, b) \in V(\mathfrak{a})$. Then

$$g(a_1, \ldots, a_n) = 0 \text{ for all } g \in J \text{ and hence } (a_1, \ldots, a_n) \in V(J),$$

$$1 - f(a_1, \ldots, a_n)b = 0 \text{ and hence } f(a_1, \ldots, a_n) \neq 0.$$

This is a contradiction with $f \in I(V(J))$.

Hence $\mathfrak{a} = R[y]$. Then we can write

$$1 = f_1(x)g_1(x, y) + \cdots + f_r(x)g_r(x, y) + (1 - f(x)y)g(y)$$

where $f_1, \dots, f_r \in J$ and $g_1, \dots, g_r, g \in R[y]$. Consider

$$R[y] \to R_f$$

given by $y \mapsto \frac{1}{f}$ (i.e. we *substitute* $1/f$ for y). Such a map exists by the universal property of polynomial rings. Under the image of this map

$$1 = \sum_{i=1}^{r} f_i(x) g_i\left(x, \frac{1}{f(x)}\right).$$

Clearing denominators (recalling R is a domain), we obtain

$$f^N = \sum_{i=1}^{r} f_i h_i \in J$$

for some $h_i \in R$ and some N. Then $f \in \sqrt{J}$. □

1.2. **The topology on $\mathbb{A}^n$.** Recall that a commutative ring R is Noetherian if one of the following equivalent conditions holds:

(1) every ideal is finitely generated,
(2) there is no infinite strictly increasing chain of ideals,
(3) any nonempty family of ideals has a maximal element.

Theorem 1.2.1 (Hilbert's basis theorem)**.** *If R is a Noetherian ring, then $R[x]$ is a Noetherian ring.*

Corollary 1.2.2. *Any polynomial ring $k[x_1, \dots, x_n]$ over a field k is Noetherian.*

Proof. Note that k is Noetherian and apply Hilbert's basis theorem 1.2.1 repeatedly. □

Suppose again that $k = \overline{k}$, $n \geq 1$, and $R = k[x_1, \dots, x_n]$. Then R is Noetherian, and we get the following consequences:

- Since every algebraic subset $X \subseteq \mathbb{A}^n$ can be written as $V(J)$, J ideal, and there exist $f_1, \dots, f_r \in J$ such that $J = (f_1, \dots, f_r)$, we can in fact write
$$X = V(f_1, \dots, f_r), \text{ the zero locus of finitely many polynomials.}$$
- There is no infinite strictly decreasing chain of closed subset of $\mathbb{A}^n$.

Definition 1.2.3. A topological space X with no infinite strictly decreasing chain of closed subsets is called *Noetherian*.

Example 1.2.4. The real line $\mathbb{R}$ with the usual topology is **not** Noetherian:

$$[0,1] \supsetneq [0,1/2] \supsetneq \cdots \supsetneq [0,1/n] \supsetneq \cdots$$

Most non-algebraic examples are not Noetherian.

Remark 1.2.5. If X is a Noetherian topological space, there is no infinite strictly increasing sequence of open subsets, and hence X is quasi-compact (every open cover has a finite subcover).

Remark 1.2.6. If X is a Noetherian topological space, then every subspace of Y is Noetherian (and hence every subspace of X is quasi-compact). Indeed, if

$$Y_0 \supsetneq Y_1 \supsetneq \cdots$$

is an infinite decreasing sequence of closed subsets of Y, consider

$$\overline{Y_0} \supsetneq \overline{Y_1} \supsetneq \cdots .$$

Since Y_j is closed in Y, $\overline{Y_j} \cap Y = Y_j$. This shows $\overline{Y_{j+1}} \neq \overline{Y_j}$ for all j, contradicting that X is Noetherian.

Definition 1.2.7. A topological space X is *irreducible* if

- $X \neq \emptyset$ and
- whenever $X = X_1 \cup X_2$ with X_1 and X_2 closed, we have $X_1 = X$ or $X_2 = X$.

Otherwise, we say X is *reducible*. We say $Y \subseteq X$ is irreducible or reducible if it is so with the subspace topology.

Remark 1.2.8. We note that X is irreducible if and only if $X \neq \emptyset$ and for every two nonempty open $U, V \subseteq X$, we have $U \cap V \neq 0$. Equivalently, every nonempty open subset of X is dense in X.

Proposition 1.2.9. *If $X \subseteq \mathbb{A}^n$ is a closed subset, then X is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. Note that X is nonempty if and only if $I(X) \neq R$. Suppose X is irreducible. If $f, g \in R$ and $f \cdot g \in I(X)$, then $X = (X \cap V(f)) \cup (X \cap V(g))$ is a decomposition into closed subsets. Since X is irreducible, we may assume $X = X \cap V(f)$, and hence $f \in I(X)$.

Conversely, suppose that $I(X)$ is prime and $X = X_1 \cup X_2$ with X_1, X_2 closed, and $X_1 \neq X$, $X_2 \neq X$. Then

$$I(x) \subsetneq I(X_i), \text{ for } i = 1, 2.$$

Let $f_i \in I(X_i) \setminus I(X)$. Then $f_1 f_2 \in I(X_1) \cap I(X_2) = I(X)$, contradicting that $I(X)$ is prime. $\square$

Examples 1.2.10.

(1) Since $k[x_1, \ldots, x_n]$ is a domain, 0 is a prime ideal, so $\mathbb{A}^n = V(0)$ is irreducible.
(2) Any linear subspace $L \subseteq \mathbb{A}^n$ is irreducible: after a linear change of variables in R, $R = k[y_1, \ldots, y_n]$ such that $I(L) = (y_1, \ldots, y_r)$ and this ideal is prime.
(3) A union of 2 lines in $\mathbb{A}^2$ is reducible.

Remark 1.2.11. If we have a subset X of $\mathbb{A}^n$, it is usually not hard to prove it is an algebraic subset by finding an ideal I such that $X = V(I)$. However, it is sometimes not easy to find $I(X)$, the radical of the ideal I.

Remark 1.2.12.

(1) If $Y \subseteq X$ is a subset, then Y is irreducible if and only if $Y \neq \emptyset$ and if $Y \subseteq F_1 \cup F_2$, $F_1, F_2 \subseteq X$ closed, then $Y \subseteq F_1$ or $Y \subseteq F_2$.
(2) If $Y \subseteq X$, Y irreducible and $Y \subseteq F_1 \cup \cdots \cup F_r$ for F_i closed in X, then there exists an i such that $Y \subseteq F_i$. (To prove this, use induction on r.)

(3) Using the description of irreducibility in (1) and noting that for a closed F we have $Y \subseteq F$ if and only if $\overline{Y} \subseteq F$, we conclude that $Y \subseteq X$ is irreducible if and only if $\overline{Y}$ is irreducible.
(4) If X is irreducible and $U \subseteq X$ is a nonempty open set, then U is irreducible by property (3), because $\overline{U} = X$.

We saw in Proposition 1.2.9 that if $X \subseteq \mathbb{A}^n$ is closed, then X is irreducible if and only if $I(X)$ is prime. The following proposition shows that any such X is a finite union of irreducible closed sets, so $I(X)$ is a finite intersection of prime ideals.

Proposition 1.2.13. *If X is Noetherian topological space and $Y \subseteq X$ is closed and nonempty, then Y is a finite union of irreducible closed sets, i.e. $Y = Y_1 \cup \cdots \cup Y_r$ for some Y_i closed and irreducible. We may assume this decomposition is minimal (so $Y_i \subsetneq Y_j$ for $i \neq j$), and in this case the Y_i are unique up to reordering.*

Definition 1.2.14. In this case, Y_i are the *irreducible components* of Y and

$$Y = Y_1 \cup \cdots \cup Y_r$$

is the *irreducible decomposition* of Y.

Proof of Proposition 1.2.13. If there exists a Y without such a decomposition, we may choose Y to be minimal without a decomposition, since X is Noetherian. In particular, this Y is not irreducible, so there exists a decomposition $Y = A \cup B$ with A and B closed and $A \neq Y$ and $B \neq Y$. Since $A, B \neq \emptyset$, by minimality of Y, both A and B are finite unions of irreducibles. But then $Y = A \cup B$ admits such a decomposition, contradiction.

The fact that we can assume the decomposition is minimal is clear. We show uniqueness. Suppose now

$$Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots \cup Y_s'$$

are minimal decompositions with all Y_i, Y_j' closed and irreducible. For all i, $Y_i \subseteq Y_1' \cup \cdots \cup Y_s'$ and Y_i is irreducible, so there exists $j \leq s$ such that $Y_i \subseteq Y_j'$. Similarly, there exists $l \leq r$ such that $Y_j' \subseteq Y_l$. Then $Y_i \subseteq Y_l$, so $i = l$ by minimality of the decomposition, which implies $Y_i = Y_j'$.

Changing the roles of Y_i and Y_j' in the above argument, we get the result. □

Definition 1.2.15. If X is a topological space and $Y \subseteq X$, Y is *locally closed* if for any $y \in Y$, there exists an open neighborhood U_y of y in X such that $U_y \cap Y \subseteq U_y$ is closed.

Remark 1.2.16. If $Y \subseteq X$ is a subset, then Y is closed in X if and only if for any $y \in X$, there exists an open neighborhood U_y of y in X such that $Y \cap U_y \subseteq U_y$ is closed. Exercise: prove this.

Proposition 1.2.17. *Let $Y \subseteq X$ be a subset. The following are equivalent:*

(1) *Y is locally closed in X,*
(2) *Y is open in $\overline{Y}$,*
(3) *we can write $Y = U \cap F$ with U open and F closed.*

Condition (3) gives a good way to think about locally closed subsets.

Proof. We first prove (1) implies (3). If Y is locally closed, then for all $y \in Y$ choose U_y as in the definition. Consider the open set

$$U = \bigcup_{y \in Y} U_y$$

in X. By Remark 1.2.16, Y is closed in U. Then $Y = F \cap U$ for F closed.

We now show (3) implies (2). If $Y = U \cap F$ for U open and F closed, then $\overline{Y} \subseteq F$, and so $Y = U \cap \overline{Y}$, showing that Y is open in $\overline{Y}$.

Finally, we show (2) implies (1). Suppose $Y = \overline{Y} \cap U$. Then for any $y \in Y$, take $U_y = U$. This works, since $Y \cap U = \overline{Y} \cap U$ is closed in U. □

Principal affine open subsets. Let $X \subseteq \mathbb{A}^n$ be closed. If $f \in k[x_1, \ldots, x_n]$, write

$$D_X(f) = \{u \in X \mid f(u) \neq 0\}.$$

This is clearly an open subset of X, since $D_X(f) = X \setminus V(f)$.

Note that $D_X(f) \cap D_X(g) = D_X(fg)$. In fact, any open subset of X is of the form:

$$X \setminus D_X(J)$$

for some $J \subseteq k[x_1, \ldots, x_n]$. Since J is finitely generated, we can write it as $J = (f_1, \ldots, f_r)$ for some $f_1, \ldots, f_r \in J$, whence

$$X \setminus V(J) = D_X(f_1) \cup \cdots \cup D_X(f_r).$$

Hence the sets $D_X(f)$ give a basis for the topology on X. We call them the *principal affine open subsets*. This name will be justified later.

1.3. **Regular functions and morphisms.**

Definition 1.3.1. An *affine variety* is a closed subset of some $\mathbb{A}^n$. A *quasi-affine variety* is a locally closed subset of some $\mathbb{A}^n$.

These are topological spaces with the induced topology. We will see later (when we talk about sheaves) that these are *ringed spaces*.

We will do this by specifying the "*allowable*" functions.

Definition 1.3.2. Let $Y \subseteq \mathbb{A}^n$ be a locally closed subset. A *regular function* on Y is a map $\varphi \colon Y \to k$ which can locally be given by the quotient of two polynomial functions, i.e. for any $y \in Y$, there exists an open neighborhood U_y of y in Y and $f, g \in k[x_1, \ldots, x_n]$ such that for any $u \in U_y$ we have

$$g(u) \neq 0 \text{ and } \varphi(u) = \frac{f(u)}{g(u)} \in k.$$

We write $\mathcal{O}(Y)$ to be the set of regular functions on Y. (By convention, $\mathcal{O}(\emptyset) = 0$.)

Remark 1.3.3. Just like the notion of a differentiable function in differential topology, we define regular functions by a local property. Examples of functions which are not globally rational will be given later.

Note that

$$\mathcal{O}(Y) = \{\text{regular functions on } Y\} \subseteq \underbrace{\{\text{functions } Y \to k\}}_{k\text{-algebra}}.$$

It is clear that $\mathcal{O}(Y)$ is actually a subalgebra. For example, suppose $\varphi_1, \varphi_2 \in \mathcal{O}(Y)$, so for any $y \in Y$, there exist open neighborhoods U_1, U_2 of y and f_1, f_2, g_1, g_2 such that on U_i, $g_i(u) \neq 0$ and $\varphi_i(u) = \dfrac{f_i(u)}{g_i(u)}$. Then on $U = U_1 \cap U_2 \ni y$, we have that

$$(\varphi_1 + \varphi_2)(u) = \frac{f_1(u)}{g_1(u)} + \frac{f_2(u)}{g_2(u)} = \frac{f_1(u)g_2(u) + f_2(u)g_1(u)}{g_1(u)g_2(u)}$$

and $g_1(u)g_2(u) \neq 0$. By a similar argument, $\varphi_1 \cdot \varphi_2$ is regular.

Suppose X is closed in $\mathbb{A}^n$. We have a map

$$k[x_1, \dots, x_n] \to \mathcal{O}(X),$$
$$f \mapsto (u \mapsto f(u)).$$

The kernel is $I(X)$.

Proposition 1.3.4. *The induced k-algebra homomorphism*

$$k[x_1, \dots, x_n]/I(X) \to \mathcal{O}(X)$$

is an isomorphism.

More generally, suppose $X \subseteq \mathbb{A}^n$ is closed and $U = D_X(g) = \{u \in X \mid g(u) \neq 0\}$ for $g \in k[x_1, \dots, x_n]$. We have a k-algebra homomorphism

$$\Phi \colon k[x_1, \dots, x_n]_g \to \mathcal{O}(U)$$
$$\frac{f}{g^n} \mapsto \left(p \mapsto \frac{f(p)}{g(p)^n}\right).$$

Proposition 1.3.5. *The induced k-algebra homomorphism*

$$k[x_1, \dots, x_n]_g/I(X)_g \to \mathcal{O}(U)$$

is an isomorphism.

Note that Proposition 1.3.4 is the case $g = 1$ of Proposition 1.3.5.

Proof of Proposition 1.3.5. We first show $I(X)_g = \ker \Phi$. Consider

$$\ker \Phi = \left\{ \frac{f}{g^m} \;\middle|\; f(u) = 0 \text{ for any } u \in X \setminus V(g) \right\}.$$

Clearly, $I(X)_g \subseteq \ker(\Phi)$. For the other inclusion, if $\frac{f}{g^m} \in \ker(\Phi)$, $(fg)(u) = 0$ for any $u \in X$, so $fg \in I(X)$, and

$$\frac{f}{g^m} = \frac{fg}{g^{m+1}} \in I(X)_g.$$

Thus Φ induces an injective homomorphism as in the statement.

We need to prove surjectivity. Fix $\varphi \in \mathcal{O}(U)$. Then there exist open $V_1, \ldots, V_r$ such that $U = V_1 \cup \cdots \cup V_r$ and $f_i, g_i \in k[x_1, \ldots, x_n]$ such that for any $u \in V_i$, $g_i(u) \neq 0$ and

$$\varphi(u) = \frac{f_i(u)}{g_i(u)}.$$

Since the principal affine open subsets form a basis, we may assume $V_i = D_x(h_i)$ for some polynomial $h_i \in k[x_1, \ldots, x_n]$, for any i. Since $g_i(u) \neq 0$ for any $u \in D_x(h_i)$,

$$X \cap V(g_i) \subseteq V(h_i).$$

By Hilbert Nullstellensatz 1.1.9, this shows that

$$h_i \in \sqrt{I(X) + (g_i)}.$$

Without changing V_i, we may replace h_i by a power or by an element with the same class modulo $I(X)$. Hence we may assume without loss of generality that $h_i \in (g_i)$, say $h_i = g_i h_i'$. After replacing f_i, g_i by $f_i h_i'$, $g_i h_i'$ respectively, we may assume $g_i = h_i$ for any i:

$$\varphi(u) = \frac{f_i(u)}{g_i(u)} = \frac{f_i(u)h_i'(u)}{g_i(u)h_i'(u)} = \frac{f_i(u)h_i'(u)}{h_i(u)}$$

for $u \in V_i$.

Since $\frac{f_i(u)}{g_i(u)} = \frac{f_j(u)}{g_j(u)}$ for all $u \in D_x(g_i g_j)$,

$$\frac{f_i}{g_i} = \frac{f_j}{g_j} \text{ in } \frac{k[x_1, \ldots, x_n]_{g_i g_j}}{I(X)_{g_i g_j}}$$

by injectivity of the map. Therefore, there exists N such that

$$(g_i g_j)^N (f_i g_j - f_j g_i) \in I(X).$$

Replacing f_i by $f_i g_i^N$ and g_i by g_i^{N+1}, we may assume

$$f_i g_j - f_j g_i \in I(X) \tag{1}$$

for any i, j.

By definition of U, we have that:

$$U = D_X(g) = D_X(g_1) \cup \cdots \cup D_X(g_r).$$

Thus

$$X \setminus V(g) = X \setminus V(g_1, \ldots, g_r)$$

and hence

$$X \cap V(g) = X \cap V(g_1, \ldots, g_r).$$

By Hilbert Nullstellensatz 1.1.9,

$$\sqrt{I(X) + (g)} = \sqrt{I(X) + (g_1, \ldots, g_r)}.$$

Thus, there exists $m \geq 1$ and $a_1, \ldots, a_r \in k[x_1, \ldots, x_n]$ such that

$$g^m - \sum_{i=1}^{r} a_i g_i \in I(X). \tag{2}$$

We claim that φ is the image of $\dfrac{\sum_{i=1}^{r} a_i f_i}{g^m}$. We need to show that for any $u \in D_X(g_j)$, we have

$$\frac{\sum_{i=1}^{r} a_i(u) f_i(u)}{g^m(u)} = \frac{f_j(u)}{g_j(u)}.$$

Hence it is enough to show that

$$\sum_{i=1}^{r} a_i(u) f_i(u) g_j(u) = f_j(u) g(u)^m$$

for any $u \in U$. We have that

$$\begin{aligned} \sum_{i=1}^{r} a_i(u) f_i(u) g_j(u) &= \sum_{i=1}^{r} a_i(u) f_j(u) g_i(u) && \text{by equation (1)} \\ &= f_j(u) g(u)^m && \text{by equation (2).} \end{aligned}$$

This completes the proof. □

Example 1.3.6. Suppose $X \subseteq \mathbb{A}^4$ is defined by $x_1 x_2 = x_3 x_4$ and $L \subseteq X$ is defined by $x_2 = x_3 = 0$. Let

$$\varphi(u_1, u_2, u_3, u_4) = \begin{cases} \frac{u_1}{u_3} & \text{if } u_3 \neq 0 \\ \frac{u_4}{u_2} & \text{if } u_2 \neq 0 \end{cases}$$

Exercise. Show that there are no $f, g \in k[x_1, \ldots, x_4]$ such that

$$\varphi(u) = \frac{f(u)}{g(u)}$$

for all $x \in X \setminus L$.

Note that this does not contradict Proposition 1.3.5, because $X \setminus L$ is not a principal affine open subset.

Let $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ be locally closed subsets. Any map $f\colon X \to Y$ makes the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dotted] \arrow[ddr, dotted, bend right, "f_i"'] & Y \arrow[d, hook] \\ & k^n \arrow[d, "\mathrm{pr}_i"] \\ & k \end{tikzcd}$$

commute, and hence we can write $f = (f_1, \ldots, f_n)$ for $f_i = \mathrm{pr}_i \circ f\colon X \to k$.

Definition 1.3.7. A map $f\colon X \to Y$ is a *morphism* if $f_1, \ldots, f_n \in \mathcal{O}(X)$.

Remark 1.3.8.

(1) A map $f\colon X \to \mathbb{A}^1$ is a morphism if and only if $f \in \mathcal{O}(X)$. So a morphism $f\colon X \to \mathbb{A}^1$ is a regular map.

(2) A map $f\colon X \to Y$ is a morphism if and only if the composition $X \to Y \hookrightarrow \mathbb{A}^n$ is a morphism.

Proposition 1.3.9. *Every morphism $f\colon X \to Y$ is continuous with respect to the Zariski topology.*

Proof. Exercise. (Can also be found in the official notes.) □

Proposition 1.3.10. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then $g \circ f$ is a morphism.*

Proof. Exercise. (Can also be found in the official notes.) □

Using Proposition 1.3.10, we can consider a category with locally closed subsets of affine spaces over k as objects and the morphisms defined this way.

Morevoer, given a morphism $f\colon X \to Y$, we get a morphism of k-algebras

$$f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X),$$
$$\varphi \mapsto \varphi \circ f.$$

This gives a contravariant functor

$$\left\{\begin{array}{c}\text{category of}\\ \text{quasi-affine varieties}\end{array}\right\} \to \{\text{category of } k\text{-algebras}\},$$
$$X \mapsto \mathcal{O}(X),$$
$$f \mapsto f^{\#}.$$

Theorem 1.3.11. *This induces an anti-equivalence of categories*

$$\left\{\begin{array}{c}\text{category of}\\ \text{affine varieties}\end{array}\right\} \to \left\{\begin{array}{c}\text{category of}\\ \text{finitely generated}\\ \text{reduced } k\text{-algebras}\end{array}\right\},$$

Proof. Note that this map is well-defined. If $X \subseteq \mathbb{A}^n$ is closed, Proposition 1.3.4 implies that

$$\mathcal{O}(X) \cong k[x_1, \ldots, x_n]/I(X)$$

is a finitely generated reduced k-algebra.

It is enough to show the following statements:

(1) For any affine varieties X, Y, the map
$$\alpha\colon \operatorname{Hom}_{\mathrm{Var}}(X, Y) \to \operatorname{Hom}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X))$$
is a bijection.

(2) For a finitely generated reduced k-algebra R, there exists an affine variety X such that $R \cong \mathcal{O}(X)$.

We first show (2). If R is a finitely generated reduced k-algebra, there exists a radical ideal J such that

$$R \cong k[x_1, \ldots, x_m]/J.$$

For $X = V(J)$, $J = I(X)$, and hence

$$\mathcal{O}(X) \cong k[x_1, \ldots, x_m]/J \cong R$$

by Proposition 1.3.4.

It remains to show (1). Suppose $X \subseteq \mathbb{A}^m$ with coordinates $x_1, \ldots, x_m$ and $Y \subseteq \mathbb{A}^n$ with coordinates $y_1, \ldots, y_n$ are closed. Given a morphism $f\colon X \to Y$, we recall that $f = (f_1, \ldots, f_n)$, $f_i \in \mathcal{O}(X)$. Then by Proposition 1.3.4

$$\begin{tikzcd}
\mathcal{O}(Y) \arrow[r, "f^\#"] \arrow[d, "\cong"] & \mathcal{O}(X) \arrow[d, "\cong"] \\
\dfrac{k[y_1, \ldots, y_n]}{I(Y)} \arrow[r] & \dfrac{k[x_1, \ldots, x_n]}{I(X)} \\
\overline{y_i} \arrow[r] & f_i
\end{tikzcd}$$

Since f is determined by $f_1, \ldots, f_n$, it is clear that α is injective. For surjectivity, given a morphism of k-algebras

$$\varphi\colon \frac{k[y_1, \ldots, y_n]}{I(Y)} \to \frac{k[x_1, \ldots, x_m]}{I(X)},$$

choose $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ such that

$$\overline{y_i} \mapsto \overline{f_i}.$$

Take $f\colon X \to \mathbb{A}^n$ given by $f(u) = (f_1(u), \ldots, f_n(u))$. By definition, this is a morphism. We claim that $f(X) \subseteq Y$. Indeed, for $g \in I(Y)$, we have that $g(f(u)) = 0$ for all $u \in X$, because

$$g(f_1, \ldots, f_n) \in I(X).$$

This completes the proof of (1) and hence the proves the theorem. □

Definition 1.3.12. A map $f\colon X \to Y$ is an *isomorphism* if it is an isomorphism in the category of quasi-affine schemes, i.e. there exists a morphism $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

We note that f is an isomorphism if and only if f is bijective and f^{-1} is a morphism.

Definition 1.3.13. We say that a quasi-affine variety is *affine* if it is isomorphic to a closed subset in some $\mathbb{A}^n$.

The following is an important example of an affine variety.

Proposition 1.3.14. *Suppose $X \subseteq \mathbb{A}^n$ is closed. For $f \in \mathcal{O}(X)$, $D_X(f) = \{u \in X \mid f(u) \neq 0\}$ is an affine variety.*

(This justifies the name *principal affine open subsets*.)

Proof. Consider $Y \subseteq \mathbb{A}^{n+1}$ defined by

$$I(X) \cdot k[x_1, \ldots, x_n, y] + (1 - f(x)y).$$

Explicitly,

$$Y = \{(x, y) \mid x \in X,\ 1 = f(x)y\}.$$

Let $\varphi\colon D_X(f) \to Y$ be given by $\varphi(u) = \left(u, \frac{1}{f(u)}\right)$. This is clearly a morphism and the map $\psi\colon Y \to D_X(f)$, $\psi(u,t) = u$ is its inverse morphism. $\square$

Remark 1.3.15. Let $X \subseteq \mathbb{A}^n$ be locally closed. Then X is open in $\overline{X}$, which is affine. Hence by Proposition 1.3.14, we have a basis of open subsets in X which are affine varieties.

Proposition 1.3.16. *Suppose that $f\colon X \to Y$ is a morphism of affine varieties and*

$$\varphi = f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X).$$

Let $W \subseteq X$ be closed. Then

$$I_Y\left(\overline{f(W)}\right) = \varphi^{-1}(I_X(W)).$$

Proof. We have

$$\begin{aligned} I_Y\left(\overline{f(W)}\right) &= I_Y(f(W)) \\ &= \{g \in \mathcal{O}(Y) \mid g(f(u)) = 0 \text{ for any } u \in W\} \\ &= \{g \in \mathcal{O}(Y) \mid (\varphi(g))(u) = 0 \text{ for any } u \in W\} \\ &= \varphi^{-1}(I_X(W)). \end{aligned}$$

as required. $\square$

In particular, if $x \in X$ with maximal ideal $\mathfrak{m}_x$ and $y = f(x)$ with maximal ideal $\mathfrak{m}_y$, then $\mathfrak{m}_y = \varphi^{-1}(\mathfrak{m}_x)$.

Another special case is when $W = X$.

Definition 1.3.17. We say that a morphism $f\colon X \to Y$ is *dominant* if $\overline{f(X)} = Y$.

By Proposition 1.3.16, f is dominant if and only if φ is injective. More generally, $I_Y\left(\overline{f(X)}\right) = \ker\varphi$.

Proposition 1.3.18. *Suppose $Z \subseteq Y$ is closed and $Z = V(J)$. Then $f^{-1}(Z) = V(J\cdot\mathcal{O}(X))$, where $J \cdot \mathcal{O}(X)$ is the ideal of $\mathcal{O}(X)$ generated by elements in the image of J under $f^{\#}$.*

Proof. We have that

$$\begin{aligned} u \in f^{-1}(Z) \quad &\text{if and only if} \quad f(u) \in Z \\ &\text{if and only if} \quad \varphi(g)(u) = g(f(u)) = 0 \text{ for any } g \in J \\ &\text{if and only if} \quad u \in V(\varphi(J)) = V(J \cdot \mathcal{O}(X)). \end{aligned}$$

This completes the proof. $\square$

1.4. **Local rings.**

Definition 1.4.1. Let X be a variety and $W \subseteq X$ be closed and irreducible. We define the *local ring of X at W* by

$$\mathcal{O}_{X,W} = \varinjlim_{\substack{U \text{ open}, \\ U \cap W \neq \emptyset}} \mathcal{O}(U)$$

where the order on such U is given by reverse inclusion.

Note that this is a filtering poset, i.e. for any open U_1, U_2 such that $U_i \cap W \neq \emptyset$, we also have $U_1 \cap U_2 \cap W \neq \emptyset$ by irreducibility, so $U_1 \cap U_2$ is a set *bigger* than U_1 and U_2 (under reverse inclusion).

This gives a nice description of $\mathcal{O}_{X,W}$:

$$\mathcal{O}_{X,W} = \left\{ (U, \varphi) \;\middle|\; \begin{array}{c} U \text{ open}, \; U \cap W \neq \emptyset \\ \varphi \in \mathcal{O}(U) \end{array} \right\} \Big/ \sim$$

where $(U_1, \varphi_1) \sim (U_2, \varphi_2)$ if there exists $U \subseteq U_1 \cap U_2$ such that $U \cap W \neq \emptyset$ and $(\varphi_1)|_U = (\varphi_2)|_U$.

Then it is clear that $\mathcal{O}_{X,W}$ is a k-algebra. For example,

$$[(U_1, \varphi_1)] + [(U_2, \varphi_2)] = [(U_1 \cap U_2, \varphi|_{U_1, \cap U_2} + \varphi|_{U_1 \cap U_2})].$$

We will show that if X is affine and $\mathfrak{p} = I_X(W)$ is prime, then

$$\mathcal{O}_{X,W} = \mathcal{O}(X)_{\mathfrak{p}}.$$

In particular, this is indeed a local ring.

The general goal is to describe $\mathcal{O}_{X,W}$ more concretely. Intuitively, $\mathcal{O}_{X,W}$ carries local information about the functions on W. In particular, we have the following lemma.

Lemma 1.4.2. *If $V \subseteq X$ is open, $V \cap W \neq \emptyset$, then there exists a canonical isomorphism $\mathcal{O}_{X,W} \to \mathcal{O}_{V,V \cap W}$.*

Proof. This follows since

$$\left\{ U \subseteq V \;\middle|\; \begin{array}{c} U \text{ open}, \; U \cap W \cap V \neq \emptyset \\ \varphi \in \mathcal{O}(U) \end{array} \right\} \subseteq \left\{ U \subseteq X \;\middle|\; \begin{array}{c} U \text{ open}, \; U \cap W \neq \emptyset \\ \varphi \in \mathcal{O}(U) \end{array} \right\}$$

and the right hand side is final. Explicitly, the map

$$\begin{aligned} \mathcal{O}_{X,W} &\to \mathcal{O}_{V,V \cap W} \\ (U, \varphi) &\mapsto (U \cap V, \varphi_{U \cap V}) \end{aligned}$$

is the isomorphism, with inverse $(U', \varphi') \mapsto (U', \varphi')$. $\square$

Remark 1.4.3. If $(I, \leq)$ is a poset, a subset $J \subseteq I$ if *final* if for any $a \in I$, there exists $b \in J$ such that $a \leq b$.

Proposition 1.4.4. *If X is affine, $W \subseteq X$ is closed and irreducible, with ideal $I_X(W) = \mathfrak{p}$, then*

$$\mathcal{O}_{X,W} \cong \mathcal{O}(X)_{\mathfrak{p}}.$$

In particular, $\mathcal{O}_{X,W}$ is a local ring, with maximal ideal

$$\{(U, \varphi) \mid \varphi|_{W \cap U} = 0\}.$$

Proof. Since principal affine open subsets of X form a basis for the topology:

$$\mathcal{O}_{X,W} = \varinjlim_{\substack{g\in\mathcal{O}(X)\\ D_X(g)\cap W\neq\emptyset}} \underbrace{\mathcal{O}(D_X(g))}_{\mathcal{O}(X)_g}$$

and $D_X(g)\cap W\neq\emptyset$ if and only if $W\not\subseteq V(g)$ if and only if $g\notin\mathfrak{p}$. Then

$$\mathcal{O}_{X,W} = \varinjlim_{\substack{g\in\mathcal{O}(X)\\ g\notin\mathfrak{p}}} \mathcal{O}(X)_g.$$

Exercise: show that the natural map from the right hand size to $\mathcal{O}(X)_\mathfrak{p}$ is an isomorphism.

Finally, $\mathcal{O}(X)_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}\mathcal{O}(X)_\mathfrak{p}$. Since $\mathfrak{p} = I_X(W)$, the corresponding ideal of $\mathcal{O}_{X,W}$ is $\{(U,\varphi)\mid \varphi|_{W\cap U}=0\}$. ☐

Remark 1.4.5. This is the origin of the term *local ring.* The maximal ideal $\mathfrak{p}$ corresponds to a point $p\in X$, and the localization $\mathcal{O}(X)_\mathfrak{p}$ describes the functions in a neighborhood of the corresponding point. (See the first special case below.)

Two special cases.

(1) If $W=\{p\}$ is a point

$$\mathcal{O}_{X,p} = \{\text{germs of regular functions at } p\},$$

the local ring that captures the information about X in a neighborhood at p.

(2) If $W=X$ (and hence X itself is irreducible), we define

$$k(X) := \mathcal{O}_{X,X}.$$

If $U\subseteq X$ is a nonempty open affine subset, $k(X)=k(U)=\operatorname{Frac}(\mathcal{O}(U))$. Then $k(X)$ is a finite type field extension of k, i.e. $k(X)=k(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n\in k(X)$.

The elements of $k(X)$ are called *rational functions on* X.

1.5. **Rational functions and maps.**

Lemma 1.5.1. *If X, Y are quasi-affine varieties, $f,g\colon X\to Y$ are morphisms, then*

$$\{x\in X\mid f(x)=g(x)\}$$

is closed in X.

Proof. Suppose $Y\subseteq\mathbb{A}^n$ is locally closed. Write $f=(f_1,\ldots,f_n)$ and $g=(g_1,\ldots,g_n)$. Then

$$A=\{x\in X\mid (f_i-g_i)(x)=0 \text{ for } 1\le i\le n\}$$

and f_i-g_i are regular functions, and hence are continuous. Thus A is closed as an intersection of closed sets $(f_i-g_i)^{-1}(0)$. ☐

Definition 1.5.2. Let X be a quasi-affine variety. The *ring of rational functions* on X is

$$\{(U,\varphi)\mid U\subseteq X \text{ is open, dense and } \varphi\in\mathcal{O}(U)\}/\sim$$

where $(U_1,\varphi_1)\sim(U_2,\varphi_2)$ if there exists $V\subseteq U_1\cap U_2$ is open, dense, and $(\varphi_1)|_V=(\varphi_2)|_V$.

Note that for open and dense sets U_1, U_2, the set $U_1 \cap U_2$ is also open and dense. Moreover, by Lemma 1.5.1, the condition $(\varphi_1)|_V = (\varphi_2)|_V$ is equivalent to $(\varphi_1)|_{U_1 \cap U_2} = (\varphi_2)|_{U_1 \cap U_2}$.

Hence we can make the following definition.

Definition 1.5.3. A *rational function* from X to Y, written $f\colon X \dashrightarrow Y$, is an equivalence class under $\sim$ of a pair (U, φ), where $U \subseteq X$ is open and dense, and $\varphi\colon U \to Y$, and $(U_1, \varphi_1) \sim (U_2, \varphi_2)$ if $(\varphi_1)|_{U_1 \cap U_2} = (\varphi_2)|_{U_1 \cap U_2}$.

Suppose $f\colon X \dashrightarrow Y$ is a rational map, and (U_1, φ_1), (U_2, φ_2) are both representatives of f. Define

$$\varphi\colon U_1 \cup U_2 \to Y$$

by

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in U_1, \\ \varphi_2(x) & \text{if } x \in U_2. \end{cases}$$

Then φ is a morphism $U_1 \cup U_2 \to Y$ and $(U_1 \cup U_2, \varphi)$ is also a representative of f.

By Noetherianity, we can choose a representative of f, say (V, ψ), such that V is maximal. Then by the above argument, any representative of f is of the form $(V', \psi|_{V'})$ for some dense subset $V' \subseteq V$.

Definition 1.5.4. If (V, ψ) is a representative of f and V is maximal, we say V is the *domain* of the rational function f.

To compose rational functions, we recall Definition 1.3.17.

Definition. A morphism $f\colon X \to Y$ is *dominant* if $\overline{f(X)} = Y$, i.e. for any nonempty open subset $V \subseteq Y$, $f^{-1}(V) \neq \emptyset$.

If $U \subseteq X$ is dense, then $f\colon X \to Y$ is dominant if and only if $f|_U\colon U \to Y$ is dominant. Hence we can make the following definition.

Definition 1.5.5. A rational map $f\colon X \dashrightarrow Y$, represented by (U, φ), f is *dominant* if φ is dominant.

This allows us to define composition.

Definition 1.5.6. Let $f\colon X \dashrightarrow Y$, $g\colon Y \dashrightarrow Z$ be rational maps with X, Y, Z irreducible. If f is dominant, we define the *composition* $g \circ f$ as follows: choose representatives $\varphi\colon U \to Y$, $\psi\colon V \to Z$ and since $U \cap \varphi^{-1}V \neq \emptyset$ (as f is dominant), we let the map

$$U \cap \varphi^{-1}(V) \xrightarrow{\varphi|_{U \cap \varphi^{-1}(V)}} V \xrightarrow{\psi} Z$$

be a representative of $g \circ f$.

Remark 1.5.7.

- The above construction is independent of choices for φ, ψ.
- If g is dominant, $g \circ f$ is also dominant.
- The identity 1_X is dominant for all X.

Therefore, we get a category whose objects are irreducible quasi-affine varieties and whose morphisms are dominant rational maps. We write

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) = \{f\colon X \dashrightarrow Y \text{ dominant rational maps}\},$$

the set of morphisms between X and Y in this category. This leads to an interesting notion of isomorphism.

Definition 1.5.8. A dominant rational map $f\colon X \dashrightarrow Y$ is *birational* if it is an isomorphism in this category, i.e. there exists a dominant rational map $g\colon Y \dashrightarrow X$ such that $f \circ g = 1_Y$, $g \circ f = 1_X$ (as rational maps).

A *birational morphism* is a morphism $f\colon X \to Y$ which is birational.

Two irreducible varieties X and Y are *birational* if there exists a birational map $X \dashrightarrow Y$.

Since the open sets in affine varieties are very large, varieties which are birational will share a lot of the same properties. We will make this more precise later.

Examples 1.5.9.

(1) If X is an irreducible variety, $U \subseteq X$, then the inclusion $i\colon U \hookrightarrow X$ is a birational morphism with inverse $X \dashrightarrow U$ represented by 1_U.

(2) Let $f\colon \mathbb{A}^n \to \mathbb{A}^n$ be given by $f(x_1, \ldots, x_n) = (x_1, x_1x_2, \ldots, x_1x_n)$. For $L = (x_1 = 0)$, $f(L) = \{(0, \ldots, 0)\}$, and hence f induces a map

$$\mathbb{A}^n \setminus L \to \mathbb{A}^n \setminus \{0\}$$

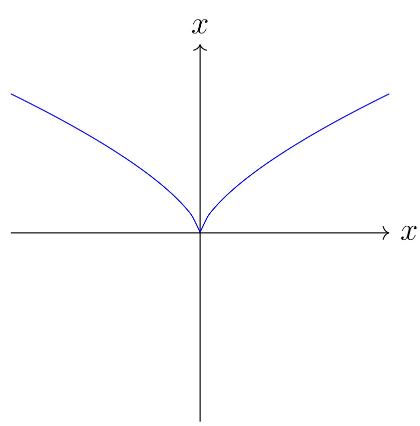
which is an isomorphism with inverse

$$g\colon \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^n \setminus L$$

$$g(y_1, \ldots, y_n) = \left(y_1, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}\right).$$

Hence f is a birational morphism, although it is clearly not an isomorphism.

(3) Let $X \subseteq \mathbb{A}^2$ be defined by $x^2 - y^3 = 0$:



$V(x^2 - y^3) = X$

Let $f\colon \mathbb{A}^1 \to X$, $f(u) = (u^3, u^2)$ be a morphism. Then letting

$$g\colon X \setminus \{(0,0)\} \to \mathbb{A}^1,$$

$$g(x,y) = \frac{x}{y},$$

we see that $f \circ g = 1_X$, $g \circ f = 1_{\mathbb{A}^1}$ and hence f is birational. In fact,

$$\begin{array}{ccc} f^{-1}(X \setminus \{(0,0)\}) & \xrightarrow{=} & \mathbb{A}^1 \setminus \{0\} \\ \big\downarrow \cong & & \\ X \setminus \{(0,0)\} & & \end{array}$$

Even though f is a bijection, it is not an isomorphism, since

$$f^{\#} \colon \frac{k[x,y]}{(x^2 - y^3)} \to k[t]$$
$$x \mapsto t^3$$
$$y \mapsto t^2$$

is not an isomorphism.

Theorem 1.5.10. *The contravariant functor*

$$\left\{ \begin{array}{c} \text{quasi-affine irreducible varieties} \\ \text{and} \\ \text{rational dominant maps} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{finite type field extensions } K/k \\ \text{and} \\ k\text{-algebra homomorphisms} \end{array} \right\}$$

$$X \longrightarrow k(X)$$

$$f \longrightarrow f^{\#}$$

is an anti-equivalence of categories.

Note that if $f \colon X \dashrightarrow Y$ is a dominant rational map, we get a map $f^{\#} \colon k(Y) \to k(X)$ by $f^{\#}(\varphi) = \varphi \circ f$, and this is an k-algebra homomorphism. Hence the functor is well-defined.

Proof. It suffices to show

(1) for any X, Y, the map

$$\mathrm{Hom}_{\mathrm{rat}}(X,Y) \to \mathrm{Hom}_{k\text{-alg}}(k(Y), k(X)),$$

(2) for any field extension K/k of finite type, there exists an irreducible quasi-affine variety such that $K \cong k(X)$.

Part (2) is clear. If K/k is of finite type, $K = k(a_1, \dots, a_n)$ for some $a_1, \dots, a_n \in K$. Then

$$A = k[a_1, \dots, a_n] \cong \frac{k[x_1, \dots, x_n]}{P}$$

for some ideal P, and since $A \subseteq K$ is a domain, P is a prime ideal. Hence set $X = V(P)$, which is irreducible, and $\mathcal{O}(X) \cong A$, so $k(X) \cong K$.

It remains to prove (2). Suppose $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are locally closed and irreducible. Then X is open in $\overline{X}$ and Y is open in $\overline{Y}$. Thus $X \hookrightarrow \overline{X}$, $Y \hookrightarrow \overline{Y}$ are birational morphisms, so

$$\mathrm{Hom}_{\mathrm{rat}}(X, Y) \cong \mathrm{Hom}_{\mathrm{rat}}(\overline{X}, \overline{Y}),$$

and $k(\overline{X}) \cong k(X)$, $k(\overline{Y}) \cong k(Y)$.

Replacing X by $\overline{X}$, Y by $\overline{Y}$, we may assume that X is closed in $\mathbb{A}^m$ and Y is closed in $\mathbb{A}^n$. We note that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{rat}}(X, Y) &= \bigcup_{g \in \mathcal{O}(X)} \underbrace{\mathrm{Hom}_{\mathrm{dom}}(D_X(g), Y)}_{\text{dominant morphisms}} \\
&= \bigcup_{g \in \mathcal{O}(X)} \underbrace{\mathrm{Hom}^{\mathrm{inj}}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X)_g)}_{\text{injective morphisms}}.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathrm{Hom}_{k\text{-alg}}(k(Y), k(X)) &= \mathrm{Hom}^{\mathrm{inj}}_{k\text{-alg}}(\mathcal{O}(Y), k(X)) \\
&= \bigcup_{g \in \mathcal{O}(X)} \mathrm{Hom}^{\mathrm{inj}}_{k\text{-alg}}(\mathcal{O}(Y), \mathcal{O}(X)_g)
\end{aligned}$$

since $\mathcal{O}(Y)$ is a finitely generated k-algebra.

All these isomorphisms are compatible with each other, which completes the proof. □

Corollary 1.5.11. *A dominant rational map $f\colon X \dashrightarrow Y$ between irreducible quasi-affine varieties is birational if and only if $f^{\#}\colon k(Y) \to k(X)$ is an isomorphism.*

Exercise. A rational map $f\colon X \dashrightarrow Y$ for X, Y irreducible is birational if and only if there exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that f induces an isomorphism $U \xrightarrow{\cong} V$.

1.6. **Products of affine and quasi-affine varieties.** We have an identification

$$\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}.$$

We we show that the topology on $\mathbb{A}^{m+n}$ is finer than the product topology.

Proposition 1.6.1. *If $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are closed, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is closed.*

Proof. Suppose $X = V(I)$ for $I \subseteq k[x_1, \ldots, x_m]$ and $Y = V(J)$ for $J \subseteq k[y_1, \ldots, y_n]$. Then

$$X \times Y = V(I \cdot R + J \cdot R)$$

for $R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. This shows that $X \times Y$ is closed in $\mathbb{A}^{m+n}$. □

Corollary 1.6.2. *If $U \subseteq \mathbb{A}^m$ and $V \subseteq \mathbb{A}^n$ are open subsets, then $U \times V \subseteq \mathbb{A}^{m+n}$ is open. (Hence the topology on $\mathbb{A}^{m+n}$ is finer than the product topology.) Similarly, a product of locally closed subsets is locally closed.*

Proof. We have that

$$\mathbb{A}^{m+n} \setminus (U \times V) = (\mathbb{A}^m \setminus U) \times \mathbb{A}^n \cup \mathbb{A}^m \times (\mathbb{A}^n \setminus V)$$

is closed by 1.6.1. The second assertion follows from the ones for open and closed sets. □

Example 1.6.3. The topology on $\mathbb{A}^2$ is strictly finer than the product topology. For example, the diagonal $V(x_1 - x_2)$ is closed in $\mathbb{A}^2$ but not closed in the product topology.

Remark 1.6.4. If $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are locally closed, then $X \times Y \subseteq \mathbb{A}^{m+n}$ is locally closed and the diagram

$$\begin{tikzcd} X \arrow[d, hook] & X \times Y \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{A}^m & \mathbb{A}^{m+n} \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathbb{A}^n \end{tikzcd}$$

commutes.

We claim that $(X \times Y, q_1, q_2)$ is the direct product of X and Y in the category of quasi-affine varieties. Given two morphisms $f\colon Z \to X$, $g\colon Z \to Y$, there is a unique morphism $h\colon Z \to X \times Y$ such that $q_1 \circ h = f$, $q_2 \circ h = g$, namely $h = (f, g)$.

The upshot of this is that if $f\colon X \to X'$ and $g\colon Y \to Y'$ are isomorphisms, then

$$(f, g)\colon X \times Y \to X' \times Y'$$

is an isomorphism.

Proposition 1.6.5. *Let X, Y be affine varieties. The two projections*

$$p\colon X \times Y \to X$$

$$q\colon X \times Y \to Y$$

are open, i.e. the image of an open set under both p and q is open.

Proof. By symmetry, it is enough to prove the assertion for p. Note that if $X = U_1 \cup \cdots \cup U_r$ and $Y = V_1 \cup \cdots \cup V_s$ are open covers, then for an open subset $W \subseteq X \times Y$, we have that

$$p(W) = \bigcup_{i,j} p(W \cap (U_i \times V_j)).$$

It is hence enough to prove the proposition for $U_i \times V_j$.

Since

- we are allowed to replace X, Y by isomorphic varieties,
- X, Y can be covered by open subsets which are affine varieties,

we may assume $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are closed subsets. Also, every open subset $W \subseteq X \times Y$ is a union of principal affine open subsets, so it is enough to show that $p(W)$ is open for $W = D_{X\times Y}(g)$, where $g \in k[x, y]$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$.

We can write

$$g(x, y) = \sum_{i=1}^{r} p_i(x) q_i(y)$$

for polynomials $p_i \in k[x]$, $g_i \in k[y]$. Given W, choose g and p_i, q_i such that r is minimal.

We claim that $\overline{q_1}, \ldots, \overline{q_r} \in \mathcal{O}(Y)$ are linearly independent over k. Otherwise, we can write

$$\sum_{i=1}^{r} \lambda_i q_i = P(x) \in I(Y)$$

with some $\lambda_j \neq 0$. Take

$$g' = g - p_j \lambda_j^{-1} P.$$

Then $D_{X\times Y}(g) = D_{X\times Y}(g')$, since $P \in I(Y)$, and

$$g' = \sum_{i\neq j} (p_i - \lambda_j^{-1}\lambda_i) q_i,$$

contradicting the minimality of r. This proves the claim.

Suppose $u \in p(D_{X\times Y}(g))$. Then there exists $v \in Y$ such that $g(u,v) \neq 0$. Then there exists $K \in \{1, \ldots, r\}$ such that $p_K(u) \neq 0$, i.e. $u \in D_X(p_K)$. It is enough to show that

$$D_X(p_K) \subseteq p(D_{X\times Y}(g)).$$

Suppose $u' \in D_X(p_K)$, so $p_K(u') \neq 0$. Suppose for a contradiction that for any $v' \in Y$ we have that $g(u', v') = 0$. Then

$$\sum_{i=1}^{r} \underbrace{p_i(u')}_{\in k} q_i \in I(Y).$$

Since $p_K(u') \neq 0$, this contradicts the linear independence of $\overline{q_i}$ proved above. □

Corollary 1.6.6. *If X, Y are quasi-affine irreducible varieties, then $X \times Y$ is irreducible.*

Proof. It is enough to show that, given nonempty open subsets $U, V \subseteq X \times Y$, the intersection $U \cap V$ is nonempty. We know that $p(U)$, $p(V)$ are nonempty and open in X by Proposition 1.6.5. Since X is irreducible,

$$p(U) \cap p(V) \neq 0,$$

so let $a \in p(U) \cap p(V)$. Consider

$$U_a = \{b \in Y \mid (a,b) \in U\},$$

$$V_a = \{b \in Y \mid (a,b) \in V\}.$$

Since the map $Y \to X \times Y$ given by $y \mapsto (a, y)$ is a morphism, it is continuous, so U_a is open. Similarly, V_a is open. Hence for $a \in p(U) \cap p(V)$, $U_a, V_a \neq \emptyset$ and hence by irreducibility of Y, we have that $U_a \cap V_a \neq \emptyset$, and hence $(a, b) \in U \cap V$. This shows that $U \cap V$ is nonempty, and hence $X \times Y$ is irreducible. □

Suppose $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are closed subsets, and $I(X) \subseteq k[x]$, $I(Y) \subseteq k[y]$ are their ideals. It is easy to see that

$$X \times Y = V(I(X) \cdot k[x,y] + I(Y) \cdot k[x,y]).$$

The goal is to show that

$$I(X \times Y) = V(I(X) \cdot k[x,y] + I(Y) \cdot k[x,y]),$$

i.e. the ideal $I(X) \cdot k[x,y] + I(Y) \cdot k[x,y]$ is radical. This is equivalent to showing that the quotient ring

$$\frac{k[x,y]}{I(X) \cdot k[x,y] + I(Y) \cdot k[x,y]}$$

is reduced. Note that this ring is isomorphic to

$$\mathcal{O}(X) \otimes_k \mathcal{O}(y).$$

Recall that $k[x,y] = k[x] \otimes_k k[y]$. In general, given

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, dotted] \\ C \arrow[r, dotted] & B \otimes_A C \end{tikzcd}$$

and ideals $I \subseteq B$, $J \subseteq C$, then

$$B/I \otimes_A C/J \cong \frac{B \otimes_A C}{I \cdot (B \otimes_A C) + J \cdot (B \otimes_A C}$$

by right-exactness of $\otimes$. **Exercise.** Prove this.

Proposition 1.6.7. *If X, Y are affine varieties, then*

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$$

is reduced.

This is related to separability of field extensions. Thus, before we prove Proposition 1.6.7, we recall a few properties of separability.

Lemma 1.6.8. *If k is a field, K/k is a finite, separable extension, then $K \otimes_k K'$ is reduced for every field extension K'/k.*

Proof. If K/k is finite and separable, by the Primitive Element Theorem, there exists $u \in K$ such that $K = k(u)$. Morevoer, if $f \in k[x]$ is the minimal polynomial of u, then f has no multiple roots in any field extension of k. Moreover,

$$K \cong \frac{k[x]}{(f)}.$$

Then we have that

$$K \otimes_k K' \cong \frac{K'[x]}{f \cdot K'[x]}.$$

If the irreducible decomposition of f in $K'[x]$ is $g_1, \ldots, g_s$, then the g_i's are relatively prime. Indeed, if g_i and g_j had a common factor of positive degree, then they would have a common linear factor in the algebraic closure of K', i.e. f would have a double zero, contradicting that f has no multiple roots.

Thus, by the Chinese Remainder Theorem,

$$K \otimes_k K' = \prod_{i=1}^{s} \overbrace{\frac{K'[x]}{(g_i)}}^{\text{field}}.$$

Hence $K \otimes_k K'$ is reduced. $\square$

Lemma 1.6.9. *Let k be a perfect field. If K/k is a finite type field extension. Then there exists a transcendence basis $x_1, \ldots, x_n$ of K/k such that*

$$k(x_1, \ldots, x_n) \hookrightarrow K$$

is separable.

Recall that k is a perfect field if every finite extension of k is separable. For example, characteristic 0 fields are separable, and characteristic p fields are separable if $k^p = k$. In particular, algebraically closed fields are perfect.

Proof. We may assume the characteristic of k is $p > 0$. Let $x_1, \ldots, x_m$ be a system of generators of K/k. We may assume that

- $x_1, \ldots, x_n$ are a transcendence basis of K/k,
- $x_{n+1}, \ldots, x_{n+r}$ are not separable over $k(x_1, \ldots, x_n) = K'$,
- $x_{n+r+1}, \ldots, x_m$ are separable over K'

If $r = 0$, there is nothing to prove. Suppose $r > 0$. Then there exists an irreducible polynomial $f \in K'[T]$ such that $f \in K'[T^p]$ and $f(x_{n+1}) = 0$. Choose $0 \neq u \in k[x_1, \ldots, x_n]$ such that

$$g = uf \in k[x_1, \ldots, x_n, T^p].$$

We claim that there exists $i \leq n$ such that $\frac{\partial g}{\partial x_i} \neq 0$. Otherwise,

$$g \in k[x_1^p, \ldots, x_n^p, T^p]$$

and hence $g = h^p$ for a polynomial h, since k is perfect, contradicting the irreducibility of f.

Reordering, we may assume that $\frac{\partial g}{\partial x_n} \neq 0$. Then x_n is separable over $k(x_1, \ldots, x_{n-1}, x_{n+1})$. Hence switching x_n and x_{n+1} will lower r. Repeating this procedure completes the proof of the lemma. $\square$

Proposition 1.6.10. *If k is a perfect field, then for any field extensions K/k and K'/k, the tensor product $K \otimes_k K'$ is reduced.*

Proof. We first show that we may assume that K/k is of finite type. We can write

$$K = \varinjlim_{\substack{k \subseteq K_i \subseteq K \\ K_i/k \text{ finite type}}} K_i.$$

Then

$$K \otimes_k K' \cong \varinjlim (K_i \otimes_k K')$$

and a direct limit of reduced rings is reduced. Hence we may assume K/k is of finite type.

By Lemma 1.6.9, there exists a transcendence basis $x_1, \ldots, x_n$ such that K is separable over $K'' = k(x_1, \ldots, x_n)$. We have that

$$K \otimes_k K' = K \otimes_{K''} K'' \otimes_k K',$$

and $K'' \otimes_k K'$ is a localization of $K'[x_1, \ldots, x_n]$. Therefore,

$$K'' \otimes_k K' \hookrightarrow K'(x_1, \ldots, x_n).$$

Tensoring with K, we obtain

$$K \otimes_k K' \hookrightarrow K \otimes_{K''} K'(x_1, \dots, x_n)$$

and $K \otimes_{K''} K'(x_1, \dots, x_n)$ is reduced by Lemma 1.6.8. □

We can finally prove that $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced for affine varieties X, Y over k. In particular, as we have seen, this implies that the natural map

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$$

is an isomorphism.

Proof of Proposition 1.6.7. We may assume that X and Y are irreducible. Let $X_1, \dots, X_n$ are irreducible components of X and $Y_1, \dots, Y_s$ are the irreducible components of Y. We have a morphism

$$\mathcal{O}(X) \hookrightarrow \prod_{i=1}^{r} \mathcal{O}(X_i),$$

which is injective since $X = \bigcup_i X_i$. Similarly, the map

$$\mathcal{O}(Y) \hookrightarrow \prod_{j=1}^{s} \mathcal{O}(Y_j)$$

is injective. Then

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow \prod_{i,j} \mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j).$$

The product is reduced if all $\mathcal{O}(X_i) \otimes_k \mathcal{O}(Y_j)$ are reduced, and in this case $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ are reduced.

Hence we may assume X and Y are irreducible. Then $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are domains and

$$\mathcal{O}(X) \hookrightarrow k(X), \quad \mathcal{O}(Y) \hookrightarrow k(Y),$$

so

$$\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \hookrightarrow k(X) \otimes_k k(Y).$$

Since k is algebraically closed, it is perfect, so $k(X) \otimes_k k(Y)$ is reduced by Proposition 1.6.10, and hence $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is reduced. □

Definition 1.6.11. A *hypersurface* in $\mathbb{A}^n$ is a closed subset defined by

$$\{u \mid f(u) = 0\}$$

for some $f \in k[x_1, \dots, x_n] \setminus k$.

Recall that if R is a UFD then $R[x]$ is a UFD.[1] By induction, $k[x_1, \dots, x_n]$ is a UFD. In particular, $f \in k[x_1, \dots, x_n] \setminus k$ is irreducible if and only if (f) is prime. This shows that $V(f)$ is irreducible if and only if f is a power of an irreducible polynomial.

Proposition 1.6.12. *If X is an irreducible quasi-affine variety, then X is birational to some irreducible hypersurface in some $\mathbb{A}^N$.*

[1]This is a standard theorem due to Gauss.

Proof. We apply Lemma 1.6.9 to $K = k(X)$ over k. There exists a transcendence basis $x_1, \dots, x_n$ of K/k such that K is finite and separable over $k(x_1, \dots, x_n)$. By the Primitive Element Theorem, there exists $u \in K$ such that $K = k(x_1, \dots, x_n)(u)$. Let $f \in k(x_1, \dots, x_n)[T]$ be the minimal polynomial of u. Then

$$K \cong \frac{k(x_1, \dots, x_n)[T]}{(f)}.$$

After multiplying u be a suitable element of $k(x_1, \dots, x_n)$, we may assume that

$$f \in k[x_1, \dots, x_n, T]$$

and it is irreducible. In this case,

$$k(X) = \operatorname{Frac}\left(\frac{k[x_1, \dots, x_n, T]}{(f)}\right),$$

and hence X is birational to

$$\{u \mid f(u) = 0\},$$

which is a hypersurface. $\square$

1.7. **Affine toric varieties.** We first recall a few definitions.

Definition 1.7.1. A *semigroup* is a set with an operation $+$ which commutative, associative, and has an identity element, 0.

A *morphism of semigroups* is a map $f\colon S \to S'$ satisfying $f(s_1 + s_2) = f(s_1) + f(s_2)$ and $f(0) = 0$.

A *subsemigroup* of a semigroup S is a subset $S' \subseteq S$ such that $0_S \in S'$ and $s_1 + s_2 \in S'$ for $s_1, s_2 \in S'$. (Then S' is also a semigroup.)

Examples 1.7.2.

(1) Every abelian group is a semigroup.
(2) For a field k, $(k, \cdot)$ is a semigroup.
(3) The natural numbers $\mathbb{N} = \mathbb{Z}_{\geq 0}$ are a semigroup $(\mathbb{N}, +)$.
(4) For semigroups S_1, S_2, $S_1 \times S_2$ is a semigroup under pointwise addition.
(5) The set $\{m \in \mathbb{N} \mid m \neq 1\}$ is a semisubgroup of $\mathbb{N}$.

We will require two properties:

- The semigroup S is *finitely generated*: there exist generators $s_1, \dots, s_n \in S$, i.e. for any $u \in S$, there exist $a_1, \dots, a_n \in \mathbb{N}$ such that
$$u = a_1 s_1 + \cdots + a_n s_n.$$
In other words, there is a surjective morphism of semigroups
$$\mathbb{N}^n \twoheadrightarrow S$$
$$(0, \dots, \overset{i}{1}, \dots, 0) \longrightarrow s_i$$
- The semigroup S is *integral*, i.e. S is isomorphic to a subsemigroup of a finitely generated free abelian group.

If S is a finitely generated, integral semigroup, then we can form the semigroup algebra

$$k[S] = \bigoplus_{s \in S} k\chi^u$$

with multiplication given by $\chi^u \cdot \chi^v = \chi^{u+v}$ and identity given by χ^0.

If $S_1 \to S_2$ is a semigroup morphism, we get an induced morphism of semigroup algebras $k[S_1] \to k[S_2]$ given by $\chi^u \mapsto \chi^{\varphi(u)}$.

Examples 1.7.3. We have the following semigroup algebras associated to some semigroups:

(1) $k[\mathbb{N}] \cong k[x]$,
(2) $k[\mathbb{N}^r] \cong k[x_1, \ldots, x_r]$,
(3) $k[\mathbb{Z}^r] \cong k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$,
(4) $S = \{m \in \mathbb{N} \mid m \neq 1\}$, a semisubgroup of $\mathbb{N}$ generated by 2 and 3, then

$$k[S] = k[t^2, t^3] \subseteq k[t].$$

If S is a finitely generated, integral semigroup, then

- there exists a surjective map $\mathbb{N}^r \to S$, and hence a surjective homomorphism

$$k[x_1, \ldots, x_r] = k[\mathbb{N}^r] \to k[S],$$

  and hence $k[S]$ is a finitely generated k-algebra,
- there exists an injective map $S \hookrightarrow \mathbb{Z}^r$, and hence an injective morphism

$$k[S] \hookrightarrow k[\mathbb{Z}^r],$$

  and hence $k[S]$ is a domain.

There exists an affine algebraic variety, unique up to canonical isomorphism, $\mathrm{TV}(S)$ such that $\mathcal{O}(\mathrm{TV}(S)) \cong k[S]$. It is called the *toric variety* associated to S. The points of $TV(S)$ are in bijection with maximal ideals in $k[S]$, i.e. with k-algebra homomorphisms $k[S] \to k$, which are in bijection with semigroup homomorphisms $S \to (k, \cdot)$.

For example, if $\varphi \colon S \to k$ is a semigroup homomorphism, the corresponding point of $\mathrm{TV}(S)$ satisfies

$$\chi^u(\varphi) = \varphi(u)$$

for $u \in S$.

In summary, we have the following procedure

$$S \rightsquigarrow k[S] \rightsquigarrow \mathrm{TV}(S)$$

semigroup	domain of finite type over k	irreducible affine variety

and

$$\text{points on } \mathrm{TV}(S) \leftrightsquigarrow \text{semigroup homomorphisms } S \to (k, \cdot).$$

If $S \to S'$ is a semigroup morphism, we get a map $k[S] \to k[S']$ which gives a map $\mathrm{TV}(S') \to \mathrm{TV}(S)$.

Exercise. Consider S to be the image of a semigroup morphism $\varphi\colon \mathbb{N}^r \to \mathbb{Z}^s$. Then

$$k[S] \cong k[\mathbb{N}^n]/I,$$

where $I = (\chi^a - \chi^b \mid a, b \in \mathbb{N}^r,\ \varphi(a) = \varphi(b))$.

There is some extra structure on $\mathrm{TV}(S)$ coming from the semigroup structure on S. Specifically, we get a morphism

$$\mathrm{TV}(S) \times \mathrm{TV}(S) \to \mathrm{TV}(S)$$

corresponding to the k-algebra homomorphism

$$\begin{aligned} k[S] &\to k[S] \otimes k[S] \\ \chi^u &\mapsto \chi^u \otimes \chi^u. \end{aligned}$$

To describe it at the level of points, let $\varphi, \psi\colon S \to (k, \cdot)$ be a semigroup morphism. Then the map above corresponds to the map $(\varphi, \psi) \mapsto \varphi \cdot \psi$, which is given by $(\varphi \cdot \psi)(u) = \varphi(u)\psi(u)$. This operation is commutative, associative, and has the identity element

$$\begin{aligned} S &\to k \\ u &\mapsto 1. \end{aligned}$$

Examples 1.7.4.

(1) Let $S = \mathbb{N}^r$ so that $TV(S) = \mathbb{A}^r$. The operation on $\mathbb{A}^r$ is coordinate-wise multiplication. In particular, $\mathrm{TV}(S)$ is not a group in general.

(2) Let $S = M$ be a free, finitely generated abelian group, so $M \cong \mathbb{Z}^r$. Then

$$\mathrm{TV}(M) \cong (k^*)^r$$

is the *algebraic torus*. This is an algebraic group. Note that we can recover M from $\mathrm{TV}(M)$ via

$$M \cong \mathrm{Hom}_{\text{alg-gp}}(\mathrm{TV}(M), k^*).$$

Proposition 1.7.5. *If M and M' are free, finitely generated abelian groups, then the canonical map*

$$\mathrm{Hom}_{\mathbb{Z}}(M, M') \to \mathrm{Hom}_{\textit{alg-gp}}(TV(M'), \mathrm{TV}(M))$$

is bijective.

Note that, more generally, if $\varphi\colon S \to S'$ is a semigroup morphism, then we have a commutative square

$$\begin{array}{ccc} k[S] & \longrightarrow & k[S] \otimes k[S] \\ \downarrow & & \downarrow \\ k[S'] & \longrightarrow & k[S'] \otimes k[S'] \end{array}$$

i.e. the induced map $\mathrm{TV}(S') \to \mathrm{TV}(S)$ is compatible with our operations. Hence the map in Proposition 1.7.5 is well-defined.

Proof of Proposition 1.7.5. A morphism of algebraic groups $\mathrm{TV}(M') \to \mathrm{TV}(M)$ corresponds (bijectively) to a k-algebra homomorphism $f\colon k[M] \to k[M']$ such that

$$\begin{tikzcd} k[M] \arrow[r, "f"] \arrow[d] & k[M'] \arrow[d] \\ k[M] \otimes k[M] \arrow[r, "f\otimes f"] & k[M'] \otimes k[M'] \end{tikzcd}$$

commutes. Suppose $f(X^u) = \sum_{u' \in M'} a_{u,u'} X^{u'}$. Then the commutativity condition shows that

$$\begin{aligned} \sum_{u'\in M'} a_{u,u'} X^{u'} \otimes X^{u'} &= \left(\sum_{u'\in M'} a_{u,u'} X^{u'}\right) \otimes \left(\sum_{v'\in M'} a_{u,v'} X^{v'}\right) \\ &= \sum_{u',v'\in M'} a_{u,u'} a_{u,v'} X^{u'} \otimes X^{v'} \end{aligned}$$

This shows that if $u' \neq v'$, then $a_{u,u'} \cdot a_{u,v'} = 0$, so there exists a unique u' such that $a_{u,u'} \neq 0$. (Note that X^u is invertible in $k[M]$, so $f(X^u) \neq 0$.) Using the equality again, we get that $a_{u,u'}^2 = a_{u,u'}$, so $a_{u,u'} = 1$. Therefore, there exists $f\colon M \to M'$ such that $f(X^u) = X^{\varphi(u)}$.

We have hence shown that any morphism $\mathrm{TV}(M') \to \mathrm{TV}(M)$ comes from a morphism $\varphi\colon M \to M'$. It is clear that φ is unique, and if f is a k-algebra homomorphism, then φ is a group homomorphism. $\square$

We go back to the general case where S is a semigroup (not a group).

Exercise. Given an integral, finitely generated semigroup, there exists a semigroup homomrphism $i\colon S \to S^{\mathrm{gp}}$, where S^{gp} is an abelian group, which is universal: given any morphism $g\colon S \to A$ where A is an abelian group, there exists a unique morphism of semigroups $\widetilde{g}\colon S^{\mathrm{gp}} \to A$ such that the diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "i"] & A \\ S^{\mathrm{gp}} \arrow[ur, dotted, "\widetilde{g}"'] & \end{tikzcd}$$

commutes.

In fact, if $S \hookrightarrow M$, where M is a finitely generated free abelian group, we can take S^{gp} to be the subgroup of M generated by S. This clearly shows that:

(1) S^{gp} is a finitely generated free abelian group,
(2) S^{gp} is generated as a group by S,
(3) $i\colon S \to S^{\mathrm{gp}}$ is injective.

Let S be a finitely generated, integral semigroup. Then the map $i\colon \hookrightarrow S^{\mathrm{gp}}$ induces a map

$$\mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S).$$

Proposition 1.7.6. *This map gives an isomorphism of* $\mathrm{TV}(S^{\mathrm{gp}})$ *onto a principal affine open subset of* $\mathrm{TV}(S)$.

Proof. Let $u_1, \ldots, u_n \in S$ be a system of generators. Then S^{gp} is generated as a semigroup by $u_1, \ldots, u_n$ and $-(u_1 + \cdots + u_n)$. Then

$$k[S] \to k[S^{\mathrm{gp}}]$$

is the localization at $X^{u_1+\cdots+u_n}$. $\square$

Note that the map $\mathrm{TV}(S^{\mathrm{gp}}) \to \mathrm{TV}(S)$ is such that the action of $\mathrm{TV}(S^{\mathrm{gp}})$ on itself extends to an algebraic group action of $\mathrm{TV}(S^{\mathrm{gp}})$ on $\mathrm{TV}(S)$ via the following commutative diagram

$$\begin{tikzcd}
\mathrm{TV}(S^{\mathrm{gp}}) \times \mathrm{TV}(S^{\mathrm{gp}}) \arrow[rr] \arrow[dd, hook] \arrow[dr, hook] & & \mathrm{TV}(S^{\mathrm{gp}}) \arrow[dd] \arrow[dr] & \\
& \mathrm{TV}(S^{\mathrm{gp}}) \times \mathrm{TV}(S) \arrow[rr, dotted] \arrow[dl, hook] & & \mathrm{TV}(S) \\
\mathrm{TV}(S) \times \mathrm{TV}(S) \arrow[rr] & & \mathrm{TV}(S) \arrow[ur, "="] &
\end{tikzcd}$$

Definition 1.7.7. An *affine toric variety* is an irreducible affine variety X with an open subset T isomorphic to a torus such that the action of T on itself extends to an algebraic action of T on X.

Remark 1.7.8. Normally, one adds the assumption that X is *normal* in the definition of an affine toric variety. We will discuss this later on.

Hence we saw that if S is a finitely generated integral semigroup, then $\mathrm{TV}(S)$ with $\mathrm{TV}(S^{\mathrm{gp}}) \hookrightarrow \mathrm{TV}(S)$ gives an affine toric variety.

Proposition 1.7.9. *Given any affine toric variety X with a torus $T \subseteq X$, there exists a finitely generated semigroup S such that $X \cong \mathrm{TV}(S)$ as algebraic groups, and the diagram*

$$\begin{tikzcd}
X \arrow[r, "\cong"] & \mathrm{TV}(S) \\
T \arrow[u, hook] \arrow[r, "\cong"] & \mathrm{TV}(S^{gp}) \arrow[u, hook]
\end{tikzcd}$$

commutes.

Proof. We have an induced injective k-algebra homomorphism

$$\mathcal{O}(X) \hookrightarrow \mathcal{O}(T) = k[M],$$

where $T = \mathrm{TV}(M)$. The commutative diagram

$$\begin{tikzcd}
T \times T \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\
T \times X \arrow[r] & X
\end{tikzcd}$$

gives a commutative diagram

$$\begin{tikzcd} k[M] \arrow[r] & k[M]\otimes_k k[M] \\ \mathcal{O}(X) \arrow[u, hook] \arrow[r, dotted] & k[M]\otimes_k \mathcal{O}(X) \arrow[u, hook] \end{tikzcd}$$

The commutativity condition shows that given $f = \sum a_u X^u \in \mathcal{O}(X)$, we have that

$$\sum a_u \chi^u \otimes \chi^u \in k[M]\otimes_k \mathcal{O}(X)$$

if and only if whenever $a_u \neq 0$, we have $\chi^u \in \mathcal{O}(X)$. Hence if $S = \{u \in M \mid \chi^u \in \mathcal{O}(X)\}$, we have $\mathcal{O}(X) = k[S]$. It is clear that S is a semisubgroup of M and hence S is integral. Moreover, as $k[S]$ is finitely generated, so S is finitely generated as a semigroup.

We finally show that $S^{\mathrm{gp}} = M$. We note that $S \subseteq S^{\mathrm{gp}} \subseteq M$. Then the corresponding map

$$T = \mathrm{TV}(M) \to \mathrm{TV}(S^{\mathrm{gp}}) \to X$$

is injective because $T \subseteq X$, and hence the first map $T \to \mathrm{TV}(S^{\mathrm{gp}})$ is injective. By the structure theorem for modules over PIDs, there exists a basis $e_1, \ldots, e_n$ of M such that a basis of S^{gp} is given by $a_1e_1, \ldots, a_ne_n$ for $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$. Thus the map $T \to \mathrm{TV}(S^{\mathrm{gp}})$ can be written as

$$\begin{gathered} \mathrm{TV}(M) \cong (k^*)^n \to (k^*)^r \cong \mathrm{TV}(S^{\mathrm{gp}}) \\ (\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1^{a_1}, \ldots, \lambda_r^{a_r}). \end{gathered}$$

Since this map is injective, we must have $r = n$ and $a_i = 1$ for $1 \leq i \leq n$ if k has characteristic 0. Hence $M = S^{\mathrm{gp}}$.

Exercise. Check that the fact that $T \to X$ is an isomorphism onto an open subset implies that $r = n$ and $a_i = 1$ for all i even in positive characteristic. $\square$

Definition 1.7.10. Suppose X, Y are affine toric varieties with tori $T_X \subseteq X$ and $T_Y \subseteq Y$. A *toric morphism* $f\colon X \to Y$ is a morphism of algebraic varieties that induces a morphism of algebraic groups $g\colon T_X \to T_Y$.

Note that in this case $f(\lambda \cdot u) = g(\lambda)f(u)$, since this holds on $T_X \times T_X \subseteq T_X \times X$ which is open and dense.

Example 1.7.11. If $\varphi\colon S_1 \to S_2$ is a morphism of semigroups, then we have the induced morphism $S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}}$, and hence a commutative diagram

$$\begin{tikzcd} \mathrm{TV}(S_2) \arrow[r, "f"] & \mathrm{TV}(S_1) \\ \mathrm{TV}(S_2^{\mathrm{gp}}) \arrow[u, hook] \arrow[r, "\text{morphism}", "\text{of alg. groups}"'] & TV(S_1^{\mathrm{gp}}) \arrow[u, hook] \end{tikzcd}$$

Hence f is a toric morphism.

Proposition 1.7.12. *Given finitely generated integral semigroups S_1, S_2 the canonical map*

$$\mathrm{Hom}_{\mathrm{sgp}}(S_1, S_2) \to \mathrm{Hom}_{\mathrm{toric}}(\mathrm{TV}(S_2), \mathrm{TV}(S_1))$$

is a bijection.

Proof. By Proposition 1.7.5,

$$\left\{ \begin{array}{c} \text{morphisms of algebraic groups} \\ \mathrm{TV}(S_2^{\mathrm{gp}}) \to \mathrm{TV}(S_1^{\mathrm{gp}}) \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{group homomorphisms} \\ \varphi\colon S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}} \end{array} \right\}.$$

The map φ extends to

$$\begin{tikzcd} \mathrm{TV}(S_2) \arrow[r, dotted] & \mathrm{TV}(S_1) \\ \mathrm{TV}(S_2^{\mathrm{gp}}) \arrow[u, hook] \arrow[r] & TV(S_2^{\mathrm{gp}}) \arrow[u, hook] \end{tikzcd}$$

if and only if $\varphi(S_1) \subseteq S_2$. This completes the proof, since the natural map

$$\mathrm{Hom}_{\mathrm{sgp}}(S_1, S_2) \to \{\varphi\colon S_1^{\mathrm{gp}} \to S_2^{\mathrm{gp}} \mid \varphi(S_1) \subseteq S_2\}$$

is clearly bijective. $\square$

Examples 1.7.13.

(1) If $S = \mathbb{N}^n$, then $(k^*)^n = \mathrm{TV}(\mathbb{Z}^n) \subseteq \mathrm{TV}(S) = \mathbb{A}^n$ and the action is component-wise multiplication.

(2) For $S = \mathbb{N} \setminus \{1\} \subseteq \mathbb{N}$, S is generated by 2 and 3, so

$$k[S] = k[t^2, t^3] \subseteq k[t]$$

and the image of the map

$$\mathrm{TV}(S) \hookrightarrow \mathbb{A}^2$$

is the set X defined by $x^3 - y^2 = 0$. Note that $k^* \hookrightarrow X$ under $\lambda \mapsto (\lambda^2, \lambda^3)$, and via this embedding the action is given by component-wise multiplication.

Recall from the homework that if G is an algebraic group acting algebraically on an affine variety X, there is an induced linear action of G on $\mathcal{O}(X)$ is given by

$$(\lambda f)(u) = f(\lambda^{-1} u).$$

Example 1.7.14. Consider $X = \mathrm{TV}(S) \supseteq V(S^{\mathrm{gp}}) = T$. For $\varphi \in T$, $\varphi\colon S^{\mathrm{gp}} \to k^*$, and $\chi^\omega \in \mathcal{O}(X)$, we have that

$$(\varphi \cdot \chi^\omega)(\psi) = \chi^\omega(\varphi^{-1} \cdot \psi) = \varphi^{-1}(\omega)\psi(\omega)$$

for $\psi\colon S \to k$. Hence the induced action of T on $\mathcal{O}(X)$ is

$$\varphi \cdot \chi^\omega = \varphi(\omega)^{-1}\chi^\omega.$$

Lemma 1.7.15. *A linear subspace $V \subseteq k[S]$ is preserved by the torus action, i.e. $\lambda V \subseteq V$ for all $\lambda \in T$, if and only if V is S-homogeneous, i.e. for all $f = \sum_{u \in S} a_u \chi^u \in V$ we have $\chi^u \in V$ if $a_u \neq 0$.*

Proof. By Example 1.7.14, the "if" implication is clear. Conversely, suppose V is preserved by the T-action and let $f = \sum_u a_u \chi^u \in V$. Given $\varphi \in T$, by Example 1.7.14, we have that

$$\sum_u a_u \varphi(u)^{-1} \chi^u \in V.$$

Repeating this, we see that

$$\sum_u a_u \varphi(u)^{-m} \chi^u \in V \qquad \text{for any } m \geq 0.$$

If $u_1, \dots, u_r \in S^{\mathrm{gp}}$ are pairwise distinct, there exists a map $\varphi\colon S^{\mathrm{gp}} \to k^*$ such that $\varphi(u_i) \neq \varphi(u_j)$ for $i \neq j$. Indeed, given $(a_1^i, \dots, a_n^i) \in \mathbb{Z}^n$, there exist $\lambda_1, \dots, \lambda_n \in (k^*)^n$ such that

$$\prod_{j=1}^n \lambda_j^{a_j^\alpha} \neq \prod_{j=1}^n \lambda_j^{a_j^\beta}$$

for $\alpha \neq \beta$.

If $f = \sum_{i=1}^r a_{u_i} \chi^{u_i}$ and $\varphi\colon S^{\mathrm{gp}} \to k^*$ satisfies $\varphi(u_i) \neq \varphi(u_j)$ for $i \neq j$, we see that $\chi^{u_1}, \dots, \chi^{u_r}$ are linear combinations of

$$\sum_{i=1}^r a_{u_i} \varphi(u_i)^{-m} \chi^{u_i}$$

for $0 \leq m \leq r-1$. ☐

1.8. **Normal varieties.** We first recall a few definitions from Review Sheet #2. Let $A \to B$ be a ring homomorphism, and let

$$B' = \{u \in B \mid u \text{ is integral over } A\},$$

i.e. $u \in B'$ satisfies an equation of the form

$$u^n + a_1 u^{n-1} + \cdots + a_0 = 0, \qquad a_i \in A.$$

Then B' is a subring of B, called the *integral closure of A in B*.

If $A \subseteq B$, then A is *integrally closed* in B if $B' = A$.

Definition 1.8.1. A domain A is *integrally closed* if it integrally closed in $\mathrm{Frac}(A)$. It is *normal* if it is Noetherian and integrally closed.

Definition 1.8.2. An irreducible affine variety X is *normal* if $\mathcal{O}(X)$ is normal, i.e. it is integrally closed in its fraction field $k(X)$.

Example 1.8.3. If a ring R is a UFD, then R is normal. In particular, $\mathbb{A}^n$ and $(k^*)^n$ are normal varieties.

Example 1.8.4. Let S be a finitely generated integral semigroup. Then $\mathrm{TV}(S)$ is *normal* if and only if S is *saturated*, i.e. for any $u \in S^{\mathrm{gp}}$ such that $mu \in S$ for some $m \geq 1$ implies that $u \in S$.

The *only if* implication is immediate, since $(\chi^u)^m \in k[S]$ is integral. The converse is left as an exercise. The solution can be found in the official notes.

2. Dimension of algebraic varieties

2.1. **Krull dimension.**

Remark 2.1.1. Later on, we will define algebraic varieties and all the results about affine algebraic varieties will hold also for algebraic varieties. For simplicity, we present everything for affine varieties for now.

Definition 2.1.2. Let X be a nonempty topological space. The *Krull dimension* of X is

$$\dim X = \sup \left\{ r \geq 0 \;\middle|\; \begin{array}{c} \text{there exists a sequence} \\ Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r \\ \text{of closed and irreducible subsets } Y_i \text{ of } X \end{array} \right\}.$$

By convention, we set $\dim(\emptyset) = -1$.

Definition 2.1.3. Let R be a commutative ring, $R \neq 0$. The *Krull dimension* of R is

$$\dim R = \sup \left\{ r \geq 0 \;\middle|\; \begin{array}{c} \text{there exists a sequence} \\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r \\ \text{of prime ideals } \mathfrak{p}_i \text{ of } R \end{array} \right\}.$$

Example 2.1.4. If X is an affine variety over k, then $\dim X = \dim(\mathcal{O}(X))$, where $\dim X$ is the Krull dimension of X with the Zariski topology.

Remark 2.1.5. For every R, there is a topological space $\operatorname{Spec} R$ such that $\dim \operatorname{Spec}(R) = \dim R$. This will be made precise on the next homework.

Lemma 2.1.6. *If X is a topological space and Y is a subspace of X, then* $\dim Y \leq \dim X$.

Proof. If $Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r$ are irreducible and closed in Y, then $\overline{Y_0} \supseteq \overline{Y_1} \supseteq \cdots \supseteq \overline{Y_r}$ is a sequence of irreducible closed sets. Moreover, since Y_i is closed in Y, $Y_i = \overline{Y_i} \cap Y$ for all i, and hence the inclusion are strict. This shows that $\dim X \geq r$. □

Lemma 2.1.7. *Suppose X is a topological space, $Y_1, \ldots, Y_r$ are closed subsets of X, and $Y = Y_1 \cup \cdots \cup Y_r$. Then*

$$\dim Y = \max_i \dim Y_i.$$

This applies, for example, if X is Noetherian space, Y is closed, and Y_i are the irreducible components of Y.

Proof. The inequality $\geq$ follows by Lemma 2.1.6. To prove the other inequality, suppose

$$Z_0 \supsetneq \cdots \supsetneq Z_m$$

are closed irreducible subsets of Y. Then

$$Z_0 \subseteq Y = Y_1 \cup \cdots \cup Y_r$$

and since Z_0 is irreducible, there is an i such that $Z_0 \subseteq Y_i$. Then $Z_j \subseteq Y_i$ for all j and hence $\dim(Y_i) \geq m$. This implies the other inequality. □

Lemma 2.1.8. *Suppose X is a topological space, $X = U_1 \cup \cdots \cup U_r$ for open subsets $U_i \subseteq X$. Then* $\dim X = \max_i \dim U_i$.

Proof. The inequality $\geq$ follows by Lemma 2.1.6. To prove the other inequality, suppose

$$Z_0 \supsetneq \cdots \supsetneq Z_m$$

are closed irreducible subsets of X. Since $Z_m \neq \emptyset$, there is an i such that $Z_m \cap U_i \neq \emptyset$, and hence

$$Z_0 \cap U_i \supseteq \cdots \supseteq Z_m \cap U_i$$

are closed in U_i. Since the Z_j's are irreducible and $U_i \cap Z_j \neq \emptyset$, $U_i \cap Z_j$ is irreducible and dense in Z_j. In particular, $Z_j \cap U_i \supsetneq Z_{j+1} \cap U_j$ for all j. Hence $\dim(U_i) \geq m$. □

Definition 2.1.9. Let X be a topological space and $Y \subseteq X$ be an irreducible closed subset. Then *codimension of Y in X* is

$$\operatorname{codim}_X(Y) = \sup \left\{ r \geq 0 \;\middle|\; \begin{array}{c} \text{there exists a sequence} \\ Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_r = Y \\ \text{for closed and irreducible subsets } Y_i \text{ of } X \end{array} \right\}.$$

Definition 2.1.10. Let R be a commutative ring and $\mathfrak{p} \subseteq R$ be a prime ideal. Then *height of* $\mathfrak{p}$ or *codimension of* $\mathfrak{p}$ is

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{codim}(\mathfrak{p}) = \sup \left\{ r \geq 0 \;\middle|\; \begin{array}{c} \text{there exists a sequence} \\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p} \\ \text{for prime ideals } \mathfrak{p}_i \text{ of } R \end{array} \right\}$$

Remark 2.1.11. If X is an affine variety and $Y \subseteq X$ is an irreducible closed subset with ideal $\mathfrak{p} = I_X(Y) \subseteq \mathcal{O}(X)$, then $\operatorname{codim}(\mathfrak{p}) = \operatorname{codim}_X(Y)$.

Remark 2.1.12. Arguing as in Lemmas 2.1.6, 2.1.7, and 2.1.8, if Y is a closed irreducible subset of X and $U \subseteq X$ is an open subset such that $U \cap V \neq \emptyset$, then

$$\operatorname{codim}_X(Y) = \operatorname{codim}_U(Y \cap U).$$

2.2. **Finite morphisms between affine varieties.**

Definition 2.2.1. A morphism of affine varieties $f\colon X \to Y$ is *finite* if $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is *finite*, i.e. it makes $\mathcal{O}(X)$ into a finitely generated $\mathcal{O}(Y)$-module.

Example 2.2.2. Let Y be any affine variety and let $a_1, \ldots, a_n \in \mathcal{O}(Y)$. Consider

$$X = \{(u,t) \mid t^n + a_1(u)t^{n-1} + \cdots + a_n(u) = 0\} \subseteq Y \times \mathbb{A}^1.$$

This is a closed subset in $Y \times \mathbb{A}^1$. Clearly,

$$\mathcal{O}(X) \cong \left(\frac{\mathcal{O}(y)[t]}{(t^n + a_1(u)t + \cdots + a_n(u))} \right)_{\text{red}}.$$

Then $\mathcal{O}(X)$ is generated as an $\mathcal{O}(Y)$-module by $\bar{1}, \bar{t}, \ldots, \bar{t}^{n-1}$. Hence the composition

$$X \hookrightarrow Y \times \mathbb{A}^1 \to Y$$

is a finite morphism.

Example 2.2.3. Let X be an affine variety and $i\colon Y \hookrightarrow X$ be an inclusion of a closed subset of X. Then i is finite, since $\mathcal{O}(X) \to \mathcal{O}(Y)$ is surjective, and hence finite.

Example 2.2.4. Let $X \subseteq \mathbb{A}^N$ be a closed subset. Recall that in the proof of Noether Normalization Lemma 1.1.12, we show that if $x_1, \ldots, x_N$ are the coordinates on $\mathbb{A}^N$, then there exists a change of coordinates

$$y_i = \sum a_{ij} x_j \qquad a_{ij} \in k,\ \det(a_{ij}) \neq 0$$

such that

$$k[y_1, \ldots, y_r] \hookrightarrow \mathcal{O}(X)$$

is finite, where $r = \operatorname{trdeg}_k(k(X))$.

In other words, we have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[drr, bend right, "\text{finite and dominant}"'] & \mathbb{A}^N \arrow[r, "\cong", "\text{linear}"'] & \mathbb{A}^N \arrow[d] & (u_1, \ldots, u_N) \arrow[d] \\
 & & \mathbb{A}^r & (u_1, \ldots, u_r)
\end{tikzcd}$$

Example 2.2.5. A map $f\colon X \to Y = \{\text{point}\}$ is finite if and only if X is a finite set. For the "if" implication, note that $\mathcal{O}(X) = k^{\#X}$ is finitely generated over k. Conversely, if $X \to Y$ is finite, then there exist irreducible components X_i of X such that the induced map $X_i \to Y$ is finite (as a composition of finite maps). We may hence assume that X is irreducible.

The induced map $k \hookrightarrow \mathcal{O}(X)$ is finite and $\mathcal{O}(X)$ is a domain, and hence $\mathcal{O}(X)$ is a field which is a finite extension of $k = \bar{k}$. Thus $\mathcal{O}(X) = k$, which means that X is a point.

Remark 2.2.6. Note that a composition of finite morphisms between affine varieties is finite.

Remark 2.2.7. If $f\colon X \to Y$ is a finite morphism between affine varieties X and Y, and $Z \subseteq X$, $W \subseteq Y$ satisfy $f(Z) \subseteq W$, then the induced morphism $g\colon Z \to W$ is finite. This is clear by the diagram

$$\begin{tikzcd}
X \arrow[r] & Y & & \mathcal{O}(Y) \arrow[r, "f^{\#}"] \arrow[d, two heads] & \mathcal{O}(X) \arrow[d, two heads] \\
Z \arrow[u, hook] \arrow[r] & W \arrow[u, hook] \arrow[r, phantom, "\rightsquigarrow"] & & \mathcal{O}(W) \arrow[r, "g^{\#}"] & \mathcal{O}(Z)
\end{tikzcd}$$

Since $f^{\#}$ is finite, $g^{\#}$ is also finite.

In particular, for any $y \in Y$, the map $f^{-1}(y) \to \{y\}$ is finite, so the set $f^{-1}(y)$ is finite.

Proposition 2.2.8. *Let $\varphi\colon R \to S$ be a finite (or just integral) ring homomorphism.*

(1) *If $\mathfrak{q} \subseteq S$ is a prime ideal and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ then $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is maximal.*

(2) *If* $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$ *are prime ideals in* S *then* $\varphi^{-1}(\mathfrak{q}_1) \neq \varphi^{-1}(\mathfrak{q}_2)$.
(3) *If* φ *is also injective, then for any prime* $\mathfrak{p} \subseteq R$ *there exists a prime* $\mathfrak{q} \subseteq S$ *such that* $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.
(4) *If* $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ *are primes in* R *and* $\mathfrak{q}_1 \subseteq S$ *is a prime in* S *such that* $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$*, then there exists a prime* $\mathfrak{q}_2 \supseteq \mathfrak{q}_1$ *such that* $\varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$.

Proof. For (1), note that we have a finite injective homomorphism

$$\underbrace{R/\mathfrak{p}}_{\text{domain}} \hookrightarrow \underbrace{S/\mathfrak{q}}_{\text{domain}},$$

and hence $S/\mathfrak{q}$ is a field if and only if $R/\mathfrak{p}$ is a field. (See Review Sheet #1 for the proof of the last assertion.)

For (2), suppose $\varphi^{-1}(\mathfrak{q}_1) = \varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}$. Recall that if a map $A \to B$ is finite, then the map $C \to B \otimes_A C$ is also finite:

$$\begin{array}{ccccc} A & \longrightarrow & B & \qquad & \text{finite} \\ \downarrow & & \downarrow & & \\ C & \longrightarrow & B \otimes_A C & & \text{finite} \end{array}$$

Therefore, the map

$$R_{\mathfrak{p}} \to S_{\mathfrak{p}} = S \otimes_R R_{\mathfrak{p}}$$

is finite. By part (1), both $\mathfrak{q}_1 S_{\mathfrak{p}}$ and $\mathfrak{q}_2 S_{\mathfrak{p}}$ are maximal ideals contained one in the other. Then they must be equal, which is a contradiction.

Recall that there is a bijection

$$\left\{ \begin{array}{c} \text{primes } \mathfrak{q} \text{ in } S \\ \text{with } \varphi^{-1}(\mathfrak{q}) = \mathfrak{p} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{primes in} \\ S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \cong S \otimes R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \end{array} \right\}$$

$$\mathfrak{q} \longrightarrow \mathfrak{q}S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$$

The reason is the primes in $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ are all of the form $\mathfrak{q}S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ where $\mathfrak{q}$ is a prime in S such that $\mathfrak{p}S_{\mathfrak{p}} \subseteq \mathfrak{q} \subseteq S_{\mathfrak{p}}$ and $\varphi(R \setminus \mathfrak{p}) \cap \mathfrak{q} = \emptyset$.

We will use this and Nakayama Lemma (which we recall in Remark 1.2.16) to prove (3). If φ is injective, then $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is also injective. Since $S_{\mathfrak{p}} \neq 0$ is a finitely generated $R_{\mathfrak{p}}$ module and $R_{\mathfrak{p}}$ is local, by Nakayama lemma $S_{\mathfrak{p}} \neq \mathfrak{p}S_{\mathfrak{p}}$, i.e. $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \neq 0$. Therefore, this ring has a prime ideal, which must be of the form

$$\mathfrak{q}S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$$

with a prime $\mathfrak{q} \subseteq S$ such that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. This implies (3).

For (4), consider the map

$$\overline{\varphi}\colon R/\mathfrak{p}_1 \to S/\mathfrak{q}_1$$

which is injective and finite. Using (3) for $\mathfrak{p}_2/\mathfrak{p}_1$, we get a prime ideal in $S/\mathfrak{q}_1$ equal to $\mathfrak{q}_2/\mathfrak{q}_1$, whose inverse image under $\overline{\varphi}$ is $\mathfrak{p}_2/\mathfrak{p}_1$. This implies (4). $\square$

Remark 2.2.9 (Nakayama Lemma)**.** If $(A, \mathfrak{m})$ is a local ring and M is a finitely generated A-module such that $M = \mathfrak{m}M$, then $M = 0$.

Corollary 2.2.10. *Let $f\colon X \to Y$ be a finite morphism between affine varieties and $\varphi = f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$.*

1. *The map f is closed, i.e. $f(Z)$ is closed for $Z \subseteq X$ closed. In particular, f is surjective if and only if φ is injective.*
2. *If $Z_1 \subsetneq Z_2$ are irreducible and closed in X, then $f(Z_1) \neq f(Z_2)$.*
3. *If f is surjective, then for any irreducible and closed subset $W \subseteq Y$, there exists an irreducible and closed subset $Z \subseteq X$ such that $f(Z) = W$.*
4. *If $W_1 \subseteq W_2$ is irreducible and closed in Y and $Z_2 \subseteq X$ is irreducible, closed, and $f(Z_2) = W_2$, then there exists an irreducible and closed subset $Z_1 \subseteq X$ such that $Z_1 \subseteq Z_2$ and $f(Z_1) = W_1$.*

Proof. Let $Z \subseteq X$ be the closed subset corresponding to $I = I_X(Z) \subseteq \mathcal{O}(X)$. We saw that

$$\overline{f(Z)} = V(\varphi^{-1}(I)).$$

We will show that $V(\varphi^{-1}(I)) \subseteq f(Z)$. Consider the finite injective homomorphism

$$\mathcal{O}(Y)/\varphi^{-1}(I) \hookrightarrow \mathcal{O}(X)/I.$$

Given a point $y \in V(\varphi^{-1}(I))$ corresponding to a maximal ideal in $\mathfrak{n}$ in $\mathcal{O}(Y)/\varphi^{-1}(I)$, by Proposition 2.2.8 (3), there exists a prime ideal $\mathfrak{m} \subseteq \mathcal{O}(X)$ such that $I \subseteq \mathfrak{m}$. Since $\varphi^{-1}(\mathfrak{m}) = \mathfrak{n}$, $\mathfrak{m}$ is maximal by Proposition 2.2.8 (1). Hence $y \in f(Z)$.

Finally, note that φ is injective if and only if $\overline{f(X)} = Y$ if and only if f is surjective, since f is closed.

We have hence proved (1), and assertions (2), (3), (4) follow from the corresponding results in Proposition 2.2.8 via the correspondence between prime ideals and irreducible closed sets. $\square$

Corollary 2.2.11. *If $f\colon X \to Y$ is a surjective finite morphism of affine varieties, then $\dim X = \dim Y$. Also, if Z is an irreducible closed subset of X, then $\operatorname{codim}_X(Z) = \operatorname{codim}_Y(f(Z))$.*

Proof. If $Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_r$ are irreducible and closed in X, then

$$f(Z_0) \supseteq f(Z_1) \supseteq \cdots \supseteq f(Z_r)$$

in Y are closed by Corollary 2.2.10 (1), irreducible, and the inclusions are strict by Corollary 2.2.10 (2). Therefore, $\dim Y \geq \dim X$.

Suppose $W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_s$ are irreducible and closed in Y. Since f is surjective, by Corollary 2.2.10, there exists a closed irreducible subset $Z_0 \subseteq X$ such that $f(Z_0) = W_0$. By Corollary 2.2.10 (4), there exists a sequence

$$Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_s$$

of irreducible and closed subset such that $f(Z_i) = W_i$ for all i. Since the inclusions have to be strict, $\dim X \geq \dim Y$.

Exercise. Prove the assertion above codimension. $\square$

2.3. **The Principal Ideal Theorem (Krull).** The goal of this section will be to prove the following theorem.

Theorem 2.3.1 (Principal Ideal Theorem, Krull)**.** *Let X be a quasi-affine variety, $f \in \mathcal{O}(X)$, and let Y be an irreducible component of $V(f) = \{u \in X \mid f(u) = 0\}$. Then* $\operatorname{codim}_X(Y) \leq 1$.

Remark 2.3.2. The corresponding statement holds in arbitrary Noetherian rings.

Remark 2.3.3. We show that this holds if X is affine and $\mathcal{O}(X)$ is a UFD.

If $f = 0$, $Y = X$, and hence $\operatorname{codim}_X(Y) = 0$.

If $f \neq 0$, we can write $f = u f_1^{m_1} \dots f_r^{m_r}$ for a unit u and irreducibles f_i. Then $Y = V(f_i)$ for some i. If $\operatorname{codim}_X(Y) \geq 2$, there is a prime ideal $\mathfrak{p}$ such that $0 \subsetneq \mathfrak{p} \subsetneq (f_i)$. Choose $a \in \mathfrak{p} \setminus \{0\}$. Let m be the exponent of f_i in the prime factorization of a, i.e. $a = f_i^m b \in \mathfrak{p}$ for b coprime f_i. Since $f_i \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, this shows that $b \in \mathfrak{p}$. However, $\mathfrak{p} \subset (f_i)$, so $f_i | b$, contradicting the definition of m.

Homework. Read Review Sheet #3 on norms of elements with respect to finite field extensions.

Before we proceed with the proof of Krull's Theorem 2.3.1, we recall a few notions from Review Sheet #3. Suppose L/K is a finite field extension. For $u \in L$, let $\varphi_u \colon L \to L$ be the K-linear map given by $\varphi_u(v) = uv$. We define the norm of u as

$$N_{L/K}(u) = \det(\varphi_u) \in K.$$

Then $N_{L/K}$ has the following properties

- $N_{L/K}(0) = 0$ and $N_{L/K}(u) \neq 0$ if $u \neq 0$,
- $N(u_1 u_2) = N(u_1) N(u_2)$,
- if $u \in K$, then $N(u) = u^{[L:K]}$,
- if f is a minimal polynomial of u over K, then $N(u) = \pm f(0)^{[L:K(u)]}$.

Proposition 2.3.4. *Let $A \hookrightarrow B$ be an injective integral homomorphism of integral domains such that $K = \operatorname{Frac}(A) \hookrightarrow L = \operatorname{Frac}(B)$ is finite. If A is integrally closed, then for any $u \in B$, $N_{L/K}(u) \in A$. Moreover, if J is an ideal in B and $u \in J$, then $N_{L/K}(u) \in J \cap A$.*

Proof. Let $u \in B$ and $f \in K[x]$ be the monic minimal of u over K. By hypothesis, there exists a monic polynomial $g \in A[x]$ such that $g(u) = 0$. Then $f | g$ in $K[x]$. For every root a of f in some algebraic closure $\overline{L}$ of L we have that $g(a) = 0$, so a is also integral over A. Writing $f = x^n + \alpha_1 x^{n-1} + \dots + \alpha_n$, we know that each coefficient α_i is a (symmetric) polynomial function of the roots of f, so it is integral over A. Since A is integrally closed in K, $\alpha_i \in A$ for all i. This implies that $f(0) = \alpha_n \in A$. Hence

$$N_{L/K}(u) = \pm(\text{power of } f(0)) \in A.$$

Finally, if $u \in J$, then

$$u^n + \alpha_1 u^{n-1} + \dots + \alpha_n = 0,$$

and hence

$$\alpha_n = -u(\underbrace{u^{n-1} + \dots + \alpha_{n-1}}_{\in B}) \in J,$$

since $\alpha_i \in A \subseteq B$ and $u \in B$. This shows that

$$N_{L/K}(u) = \pm(\text{power of } \alpha_n) \in J,$$

completing the proof. $\square$

Before proving Theorem 2.3.1, we restate it here for convenience and make a few remarks.

Theorem (Krull's Theorem 2.3.1)**.** *Let X be a quasi-affine variety and $f \in \mathcal{O}(X)$. Let Y be an irreducible component of $V(f) = \{u \in X \mid f(u) = 0\}$. Then $\operatorname{codim}_X(Y) \leq 1$.*

Remark 2.3.5.

(1) If $X_1, \ldots, X_r$ are the irreducible components of X and $f|_{X_i} \neq 0$ for all i, then there exists i such that $Y \subsetneq X_i$, so $\operatorname{codim}_X Y \geq 1$. Hence by Theorem 2.3.1, $\operatorname{codim}_X Y = 1$.

(2) If U is an open subset of X and $U \cap Y \neq \emptyset$, then $\operatorname{codim}_X Y = \operatorname{codim}_U(Y \cap U)$. Moreover, $Y \cap U$ is an irreducible component of $V(f|_U)$. Hence we may replace X by U and Y by $U \cap Y$ in Krull's Theorem 2.3.1.

(3) To prove Krull's Theorem 2.3.1, we may assume X is affine, irreducible and $Y = V(f)$. Indeed, if there is a chain $Y \subsetneq Y_1 \subsetneq Y_2$ of irreducible closed subsets in X, then Y is an irreducible component of $V(f) \cap Y_2 = V(f|_{Y_2})$. Hence we may replace X by Y_2 to assume X is irreducible. Moreover, choose $U \subseteq X$ open, affine such that $U \cap Y \neq \emptyset$ and $U \subsetneq X \setminus \bigcup(\text{other components of } V(f))$. By (2), we may replace X by U to assume that X is affine and irreducible and $Y = V(f)$.

Proof of Theorem 2.3.1. By Remark 2.3.5 (3), we may assume that X is irreducible, affine, and $Y = V(f)$. Since $\mathcal{O}(X)$ is a domain of finite type over k, by Noether Normalization Lemma 1.1.12, there exists a finite injective homomorphism $B \hookrightarrow \mathcal{O}(X)$ with $B \cong k[x_1, \ldots, x_r]$ where r is the transcendence degree of $k(X)$ over k.

Let $\mathfrak{p} = I_X(Y) \subseteq \mathcal{O}(X)$, which is a prime ideal, and let $\mathfrak{q} = \mathfrak{p} \cap B$, which is a prime ideal in B. Morevoer, let $K = \operatorname{Frac}(B)$, $L = k(X) = \operatorname{Frac}(\mathcal{O}(X))$, and $u = N_{L/K}(f) \neq 0$. Since $f \in \mathfrak{p}$, by Proposition 2.3.4, $u \in \mathfrak{q}$.

We will show that $\mathfrak{q} = \sqrt{(u)}$. The '$\supseteq$' inclusion is clear since $u \in \mathfrak{q}$ and $\mathfrak{q}$ is prime. For the other inclusion, '$\subseteq$', suppose $g \in \mathfrak{q}$. Then $g \in \mathfrak{p} = \sqrt{(f)}$, so there exists $s \geq 1$ such that $g^s = fh$ for some $h \in \mathcal{O}(X)$. Applying $N_{L/K}$ to this, we obtain

$$\begin{aligned} u^{s \cdot [L:K]} &= N_{L/K}(g)^s \\ &= N_{L/k}(f) \cdot N_{L/K}(g) \in (u), \end{aligned}$$

since $N_{L/K}(f) = u$ and $N_{L/K}(h) \in B$ by Proposition 2.3.4, and hence $g \in \sqrt{(u)}$.

Consider the finite surjective morphism $\varphi\colon X \to \mathbb{A}^r$ such that $Y \subseteq X$ goes to $\varphi(Y) = V(\mathfrak{q})$. By Corollary 2.2.11,

$$\operatorname{codim}_X Y = \operatorname{codim}_{\mathbb{A}^r} \varphi(Y).$$

Since B is a UFD, we can apply Remark 2.3.3 to conclude that $\operatorname{codim}_{\mathbb{A}^r}(\varphi(Y)) \leq 1$, because $\varphi(Y) = V(u)$ by the claim. This completes the proof. $\square$

Corollary 2.3.6. *Let X be a quasi-affine variety and $f_1, \ldots, f_r \in \mathcal{O}(X)$. If Y is an irreducible component of*

$$V(f_1, \ldots, f_r) = \{x \mid f_i(x) = 0 \text{ for all } i\},$$

then $\operatorname{codim}_X(Y) \leq r$.

Proof. We argue by induction on r. The base case $r = 1$ is Krull's Theorem 2.3.1. As we did in the proof of Krull's Theorem 2.3.1, we may assume that X is affine (see specifically Remark 2.3.5 (3)). Consider a sequence of irreducible closed subsets

$$Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_m.$$

By hypothesis, there exists i such that $Y_i \not\subseteq V(f_i)$, and without loss of generality, assume $i = 1$. By Noetherianity, we may assume that there is no irreducible closed subset Z with $Y \subsetneq Z \subsetneq Y_1$. Then Y is an irreducible component of $Y_1 \cap V(f_1)$.

After replacing X by some U open such that $U \cap Y \neq \emptyset$ and

$$U \subseteq X \setminus \bigcup(\text{other compoments of } Y_1 \cap V(f_1)),$$

we may assume $Y = Y_1 \cap V(f_1)$. (This is the same trick as in Remark 2.3.5 (3).) Then

$$I_X(Y) = \sqrt{I_X(Y_1) + (f_1)},$$

so for any i we can write

$$f_i^{r_i} - g_i \in (f_1)$$

for some $r_i \geq 1$ and $g_i \in I_X(Y_1)$. Note that

$$V(f_1, \ldots, f_r) = V(f_1, g_2, \ldots, g_r).$$

It is enough to show that that Y_1 is an irreducible component of $V(g_2, \ldots, g_r)$. Indeed, if this holds, then by the inductive hypothesis $m - 1 \leq r - 1$, so $m \leq r$.

We have $Y_1 \subseteq V(g_2, \ldots, g_r)$. Suppose $Y_1 \subsetneq Z \subsetneq V(g_2, \ldots, g_r)$ and Z is irreducible and closed. Since Y is an irreducible component of $V(f_1, g_2, \ldots, g_r)$, it is an irreducible component of

$$Z \cap V(f_1) = V((f_1)|_Z).$$

But then Krull's Theorem 2.3.1 implies that $\operatorname{codim}_Z(Y) \leq 1$, which is a contradiction. $\square$

Corollary 2.3.7. *For any* $n \geq 0$, *we have* $\dim(\mathbb{A}^n) = n$.

Proof. For "$\geq$", note that

$$0 = V(x_1, \ldots, x_n) \subsetneq V(x_1, \ldots, x_{n-1}) \subsetneq \cdots \subsetneq X$$

is a sequence of closed irreducible subsets of length n.

For "$\leq$", it is enough to show that $\operatorname{codim}_X\{p\} \leq n$ for any point $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$. But

$$\{p\} = V(x_1 - a_1, \ldots, x_n - a_n),$$

so its codimension is at most n by Corollary 2.3.6. $\square$

Corollary 2.3.8. *If X is an irreducible quasi-affine variety, then* $\dim X = \operatorname{trdeg}_k(k(X))$.

Proof. Let $X = U_1 \cup \cdots \cup U_r$ for affine open subsets U_i. Then

$$\dim(X) = \max_i \dim U_i.$$

Moreover, $k(U_i) = k(X)$, so we may assume X is affine.

By Noether Normalization Lemma 1.1.12, there exists a finite surjective morphism $X \to \mathbb{A}^r$ where $r = \operatorname{trdeg}_k(k(X))$, so $\dim X = \dim \mathbb{A}^r = r$. □

We get the following consequences of Corollary 2.3.8:

- for any quasi-affine variety X, $\dim X < \infty$,
- for any irreducible X and nonempty open subset $U \subseteq X$, $\dim U = \dim X$.

Corollary 2.3.9. *Let X be a quasi-affine variety.*

(1) *If $Y = Y_0 \subsetneq Y_1 \cdots \subsetneq Y_r = Z$ is a saturated chain of irreducible closed subsets (i.e. there are no other such sets between Y_{i-1} and Y_i for any i), then*
$$r = \dim Z - \dim Y.$$

(2) *If X has* pure dimension *(i.e. all irreducible components of X have the same dimension) and $Y \subseteq X$ is irreducible and closed, then*
$$\dim Y + \operatorname{codim}_X Y = \dim X.$$

Proof. The key statement is the following. Suppose $Y \subsetneq Z$ is irrreducible and closed, and there is no irreducible and closed between Y and Z. Then
$$\dim Z = \dim Y + 1.$$

We may assume that $X = Z$. By Corollary 2.3.8, we may replace X and Y by U and $U \cap Y$ if $U \subseteq X$ is open with $U \cap Y \neq \emptyset$. We may hence assume that X is affine. Choose $f \in I_X(Y) \setminus \{0\}$. Then $Y \subseteq V(f) \subsetneq X$, so Y is an irreducible component of $V(f)$, since there are no irreducible closed subsets between Y and X.

Replace X by
$$X \setminus \bigcup (\text{irreducible components of } V(f) \text{ different from } Y).$$
Since this subset is open, we may assume that $Y = V(f)$. Similarly to the proof of Principal Ideal Theorem 2.3.1, we apply Noether Normalization Lemma 1.1.12 to get a commutative square

$$\begin{tikzcd} V(f) = Y \arrow[r, hook] \arrow[d] & X \arrow[d, "p"] \\ V(u) = p(Y) \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

where p is finite and surjective. We have that $I_{\mathbb{A}^n}(p(Y))$ is a principal ideal (g). We claim that $\dim(p(Y)) = n - 1$. We have that
$$\mathcal{O}(p(Y)) = \frac{k[x_1, \ldots, x_n]}{(g)}.$$
After a linear change of coordinates (as in the proof of Noether Normalization Lemma 1.1.12), we may assume that g is a monic polynomial in x_n with coefficients in $k[x_1, \ldots, x_{n-1}]$. The the map
$$k[x_1, \ldots, x_{n-1}] \to \mathcal{O}(p(Y))$$

is finite and injective, and hence

$$\dim(\mathcal{O}(p(Y))) = \dim k[x_1, \dots, x_{n-1}] = n - 1,$$

where the last equality follows from Corollary 2.3.7.

This shows that $\dim Z = \dim Y + 1$, as we claimed at the beginning of the proof.

In the setting of part (1), $Y = Y_0 \subsetneq Y_1 \cdots \subsetneq Y_r = Z$ is saturated, so $\dim Y_i = \dim Y_{i-1} + 1$ for all $1 \le i \le r$, and hence

$$\dim Z = \dim Y + r.$$

This shows (1), and, in particular, r is independent of the choice of chain. Hence $r = \operatorname{codim}_Z(Y)$, proving part (2) when X is irreducible.

In general,

$$\operatorname{codim}_X Y = \max\{\operatorname{codim}_{X_i} Y \mid Y \subseteq X_i\},$$

where $X_1, \dots, X_m$ are the irreducible components of X. By the above assertion,

$$\operatorname{codim}_X Y = \max\{\dim X_i - \dim Y \mid Y \subseteq X_i\},$$

and if X has pure dimension, this shows that

$$\operatorname{codim}_X Y = \dim X - \dim Y,$$

completing the proof. □

We now prove a partial converse to Corollary 2.3.6.

Proposition 2.3.10. *If X is an affine variety and $Y \subseteq X$ is irreducible, closed and has codimension r, then there exist $f_1, \dots, f_r \in \mathcal{O}(X)$ such that Y is an irreducible component of $V(f_1, \dots, f_r)$.*

Proof. When $r = 0$, this is clear because $V(0) = X$.

Suppose $r \ge 1$. We want to show that there is an f_1 such that $Y \subseteq V(f_1)$ and if $X_1, \dots, X_n$ are the irreducible components of X, then $X_i \subsetneq V(f_1)$ for all i. In other words, we want

$$f_1 \in I_X(Y) \setminus \bigcup_{i=1}^{m} I_X(X_i).$$

Since $Y \neq X_i$ for any i, $I_X(Y) \subsetneq I_X(X_i)$, and by the Prime Avoidance Lemma 2.3.11, such an f_1 exists. Then $Y \subseteq V(f_1)$ and

$$\operatorname{codim}_{V(f_1)}(Y) \le r - 1.$$

Repeat this procedure to find $f_2, \dots, f_r$ such that

$$Y \subseteq V(f_1, \dots, f_r)$$

has codimension 0. □

Remark 2.3.11 (Prime Avoidance Lemma)**.** If $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are prime ideals in a ring R and $I \subseteq R$ is an ideal such that $I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$, then $I \subseteq \mathfrak{p}_i$ for some i.

The proof of this is left as an exercise, and will appear on a review sheet.

In general, we cannot choose $f_1, \ldots, f_r \in \mathcal{O}(X)$ such that $Y = V(f_1, \ldots, f_r)$ (even if one is willing to pass to small neighborhoods of a given point).

Example 2.3.12. Let $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}^4$, $\dim X = 3$, and $Y = V(x_1, x_3) \subseteq X$. Then $Y \cong \mathbb{A}^2$, so $\dim Y = 2$. However, there is no f such that $(x_1, x_3) = \sqrt{(f, x_1x_2 - x_3x_4)}$ (even in $\mathcal{O}_{X,0}$).

Challenge. Can you prove this?

2.4. **Dimension of fibers of morphisms.** The setting is as follows. Suppose $f\colon X \to Y$ is a dominant morphism of irreducible quasi-affine varieties. Then $k(Y) \hookrightarrow k(X)$, and let $d = \mathrm{trdeg}_{k(Y)} k(X)$. Then

$$d = \mathrm{trdeg}_k(k(X)) - \mathrm{trdeg}_k(k(Y)) = \dim X - \dim Y$$

by Corollary 2.3.8.

Note that if $y \in Y$, then $f^{-1}(y)$ is a closed subset of X. More generally, if $W \subseteq Y$ is closed, then $f^{-1}(W) \subseteq X$ is closed.

We will show that

- for any $y \in f(X)$, every irreducible component of $f^{-1}(y)$ has dimension at least d,
- there exists $V \subseteq Y$ open such that for any $y \in V$, every irreducible component of $f^{-1}(y)$ is nonempty and has dimension d.

Theorem 2.4.1. *If Z is an irreducible closed subset of Y and W is an irreducible component of $f^{-1}(Z)$ which dominates Z, then $\dim W \geq \dim Z + d$ (and, equivalently, $\mathrm{codim}_X(W) \leq \mathrm{codim}_Y(Z)$). In particular, if $y \in f(X)$, then every irreducible component of $f^{-1}(y)$ has dimension at least d.*

Proof. If we replace f by $f^{-1}(U) \to U$ where U is an open subset such that $U \cap Z \neq \emptyset$, since W dominates Z, $W \cap f^{-1}(U) \neq 0$. Thus the codimensions do not change, and by taking U to be affine, we may assume that Y is affine. If $\mathrm{codim}_Y Z = r$, by Proposition 2.3.10, there exist $f_1, \ldots, f_r \in \mathcal{O}(Y)$ such that Z is an irreducible component of $V(f_1, \ldots, f_r)$. Then W is an irreducible component of $f^{-1}(V(f_1, \ldots, f_r)) = V(f_1 \circ f, \ldots, f_r \circ f)$.

Indeed, if $W \subsetneq W' \subseteq f^{-1}(V(f_1, \ldots, f_r))$, then

$$Z = \overline{f(W)} \subseteq \overline{f(W')} \subseteq V(f_1, \ldots, f_r),$$

so $\overline{f(W')} = Z$, and hence $W' \subseteq f^{-1}(Z)$. Since W is an irreducible component of $f^{-1}(Z)$, we have $W = W'$, which is a contradiction.

By Corollary 2.3.6, $\mathrm{codim}_X W \leq r$. □

Theorem 2.4.2. *Given f as before, there exists $V \subseteq Y$ open, nonempty such that*

1. *$V \subseteq f(X)$,*
2. *for any $Z \subseteq Y$ irreducible, closed such that $Z \cap V \neq \emptyset$, and for every W which is an irreducible component of $f^{-1}(Z)$ which dominates Z, we have that*

$$\dim W = \dim Z + d.$$

In particular, for any $y \in V$, every irreducible component of $f^{-1}(y)$ has dimension d.

Proof. It is clear that we may replace f by $f^{-1}(U) \to U$ whenever U is open in Y. In particular, we may assume Y is affine. We claim that we may also assume that X is affine. Indeed, let $U_1, \dots, U_m \subseteq X$ be affine open subsets such that $X = \bigcup_i U_i$. If $V_i \subseteq Y$ satisfies (1) and (2) for $U_i \hookrightarrow X \xrightarrow{f} Y$, then $V = V_1 \cap \cdots \cap V_m$ satisfies (1) and (2) for f.

Thus suppose X, Y are affine, and consider

$$f^{\#} \colon \mathcal{O}(Y) \to \mathcal{O}(X),$$

which is injective, since f is dominant. (Note that $\mathcal{O}(Y)$ and $\mathcal{O}(X)$ are both domains since X and Y are irreducible.) Consider the induced map

$$k(Y) \hookrightarrow \underbrace{k(Y) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)}_{R}.$$

Then R is a $k(Y)$-algebra of finite type and $\mathrm{Frac}(R) = k(X)$.

By Noether Normalization Lemma 1.1.12, there exist $y_1, \dots, y_d \in R$ such that

(1) they are algebraically independent over $k(Y)$,
(2) $k(Y)[y_1, \dots, y_d] \hookrightarrow R$ is finite.

After possibly multiplying the y_i by an element in $\mathcal{O}(Y)$ to clear denominators, we may assyme that $y_1, \dots, y_d \in \mathcal{O}(X)$.

We claim that there exists $s \in \mathcal{O}(Y) \setminus \{0\}$ such that $\mathcal{O}(Y)_s[y_1, \dots, y_d] \hookrightarrow \mathcal{O}(X)_s$ is finite.[2] Choose generators $x_1, \dots, x_N$ of $\mathcal{O}(X)$ as a k-algebra. By (2) above, we have

$$x_i^{m_i} + a_{i,1} x_i^{m_i - 1} + \cdots + a_{i,m_i} = 0$$

for some $m_i \in \mathbb{Z}_{>0}$ and $a_{ij} \in k(Y)[y_1, \dots, y_d]$. Choose $s \in \mathcal{O}(Y)$ such that $s a_{ij} \in \mathcal{O}(Y)[y_1, \dots, y_d]$ (i.e. clear denominators). Then all x_i's are integral over $\mathcal{O}(Y)_s[y_1, \dots, y_d]$, and they generate $\mathcal{O}(X)_s$ over $\mathcal{O}(Y)_s$ as an algebra, which proves the claim.

Replace f by the restricted map $D_X(f^{\#}(s)) = f^{-1}(D_Y(s)) \to D_Y(s)$. We may assume we can factor f as

$$X \xrightarrow{g} Y \times \mathbb{A}^d \xrightarrow{\mathrm{pr}_1} Y,$$

where g is finite and surjective. Let W be an irreducible component of $f^{-1}(Z)$. Then $g(W) \subseteq Z \times \mathbb{A}^d$. Since g is finite and surjective,

$$\dim W = \dim(g(W)) \leq \dim(Z \times \mathbb{A}^d) = \dim Z + d,$$

where the last equality follows from a homework problem.[3]

The opposite inequality holds by Theorem 2.4.1. ∎

[2] In other words, instead of inverting all elements of $\mathcal{O}(Y) \neq \{0\}$, we can just invert one element and still get a finite morphism between the localizations induced by f.

[3] Specifically, for quasi-affine varieties X and Y, $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Example 2.4.3. Let $f\colon \mathbb{A}^3 \to \mathbb{A}^3$ be given by

$$f(x,y,z) = (x^a y^b z, x^c y, x)$$

for some $a,b,c \in \mathbb{Z}_{>0}$. If $V \subseteq \mathbb{A}^3$ is $V = D_{\mathbb{A}^3}(xy) = (xy \neq 0)$, then $f^{-1}(V) \to V$ is an isomorphism with inverse $(u,v,w) \mapsto (w, vw^{-c}, uv^{-b}, w^{-a+bc})$. Hence f is birational, and on V the fibers are just single points.

What about the other fibers? We have that

$$f^{-1}(x_0,y_0,0) = \begin{cases} \emptyset & \text{if } x_0 \neq 0 \text{ or } y_0 \neq 0, \\ \cong \mathbb{A}^2 & \text{if } x_0 = y_0 = 0. \end{cases}$$

Similarly, for $z_0 \neq 0$,

$$f^{-1}(x_0,0,z_0) = \begin{cases} \emptyset & \text{if } x_0 \neq 0, \\ V(x - z_0, y) \cong \mathbb{A}^1 & \text{if } x_0 = 0. \end{cases}$$

3. General algebraic varieties

General algebraic varieties are objects obtained by gluing finitely many affine algebraic varieties together with a Hausdorff-type condition. We can glue affine varieties using atlases (for example, in differential geometry) or using ringed spaces (which is the more modern point of view). We will take the latter approach.

3.1. **Presheaves and sheaves.** The first goal will be to define sheaves.

Let X be a topological space. We can think of X as a category, Cat_X whose objects are the open subsets of X and $\mathrm{Hom}(U,V)$ has 1 element if $U \subseteq V$ and is empty otherwise.

Let $\mathcal{C}$ be a category and X be a topological space. The main examples will come from $\mathcal{C} = R$-mod or $\mathcal{C} = R$-alg, where R is a fixed commutative ring, but $\mathcal{C}$ could also be the category of sets or the category of rings.

Definition 3.1.1. A *presheaf on X* of objects in $\mathcal{C}$ is a contravariant functor

$$\mathcal{F}\colon\ \mathrm{Cat}_X \to \mathcal{C}.$$

In other words, for any open set $U \subseteq X$, we have an object $\mathcal{F}(U)$ in $\mathcal{C}$, and for any inclusion $U \subseteq V$ we have a morphism $\mathcal{F}(V) \to \mathcal{F}(U)$ in $\mathcal{C}$, denoted by $s \mapsto s|_U$, which satisfies:

(1) if $U = V$, $s|_U = s$,
(2) if $U \subseteq V \subseteq W$, $s \in \mathcal{F}(W)$, then $(s|_V)|_U = s|_U$.

The object $\mathcal{F}(U)$ in $\mathcal{C}$ is also written $\Gamma(U,\mathcal{F})$ and its elements are called *sections of $\mathcal{F}$ on U*. The morphisms $\mathcal{F}(V) \to \mathcal{F}(U)$ are called *restrictions of sections.*

Suppose that $\mathcal{C}$ is a subcategory of the category of sets and a morphism in $\mathcal{C}$ is an isomorphism if and only if it is a bijection.

Definition 3.1.2. A presheaf $\mathcal{F}$ is a *sheaf* if given any family $(U_i)_{i\in I}$ of open subsets of X and $U = \bigcup_{i\in I} U_i$ together with $s_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $(s_i)|_{U_i\cap U_j} = (s_j)|_{U_i \cap U_j}$ for all $i,j \in I$, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

In other words, sheaves are presheaves such that sections can be described locally.

Note that if we take $I = \emptyset$, the condition in the definition is that $\mathcal{F}(\emptyset)$ contains precisely one element.

Remark 3.1.3. We can write the above definition categorically, without requiring that $\mathcal{C}$ is a subcategory of sets. However, all the categories we consider are subcategories of sets, so we simplify the definition to this case.

Remark 3.1.4. If $\mathcal{F}$ is a presheaf of objects in $\mathcal{C}$ which is a subcategory of the category of abelian groups, the condition for $\mathcal{F}$ to be a sheaf can be reformulated as follows: for any union $U = \bigcup_i U_i$ of open sets U_i, the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

$$(s_i)_{i\in I} \longrightarrow ((s_i)_{U_i\cap U_j} - (s_j)|_{U_i\cap U_j})$$

is exact.

Examples 3.1.5.

(1) Let X be a topological space, and let $\mathcal{C} = \mathbb{R}$-alg, and define for $U \subseteq X$ open

$$\mathcal{C}_{X,\mathbb{R}}(U) = \{\text{continuous functions } f\colon U \to \mathbb{R}\}$$

with restriction maps given by restrictions of functions. This is the *sheaf of real continuous functions on* X.

(2) Let X be a $\mathcal{C}^\infty$-manifold. For $U \subseteq X$ open, define

$$\mathcal{C}^\infty_{X,\mathbb{R}}(U) = \{f\colon U \to \mathbb{R} \mid f \text{ is } \mathcal{C}^\infty\}$$

with restriction maps given by restrictions of functions. Then $\mathcal{C}^\infty_{x,\mathbb{R}}$ is a sheaf of $\mathbb{R}$-algebras.

(3) Let X be a quasi-affine variety. For $U \subseteq X$ open, let

$$\mathcal{O}_X(U) = \{f\colon U \to k \mid f \text{ regular function}\}$$

with restriction of functions. Then $\mathcal{O}_X$ is a sheaf of k-algebras.

(4) Let $f\colon X \to Y$ be a continuous function, and define a sheaf of sets on Y by setting for a set $U \subseteq Y$,

$$\mathcal{F}(U) = \{s\colon U \to X \text{ continuous} \mid f(s(y)) = y \text{ for all } y \in U\}$$

with restriction of functions. In this example, the *section of* $\mathcal{F}$ are actually sections of a function, justifying the name.

Definition 3.1.6. If $\mathcal{F}$, $\mathcal{G}$ are presheaves on X of objects in $\mathcal{C}$, define $\varphi\colon \mathcal{F} \to \mathcal{G}$ by defining for any $U \subseteq X$ open a morphism in $\mathcal{C}$

$$\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$$

which is compatible with the restriction maps: if $V \subseteq U$ are open sets, then

$$\begin{array}{ccc}\mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V)\end{array}$$

commutes, i.e. $\varphi_V(s|_V) = \varphi_U(s)|_V$.

The same definition applies for sheaves.

Since the morphisms can be composed, we get a category of presheaves on X of objects in $\mathcal{C}$, and the full subcategory of sheaves on X of objects in $\mathcal{C}$.[4]

Example 3.1.7. Let X and $\mathcal{C}$ be as before, $W \subseteq X$ be open, and $\mathcal{F}$ be a presheaf on X. Then $\mathcal{F}|_W$ is a presheaf on W given by $(\mathcal{F}|_W)(U) = \mathcal{F}(U)$ for all $U \subseteq W$ open with the same restriction maps.

Clearly:

- if $\mathcal{F}$ is a sheaf, then $\mathcal{F}|_W$ is a sheaf,
- if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism, we get
$$\varphi|_W \colon \mathcal{F}|_W \to \mathcal{G}|_W.$$

Hence *restriction to* W is a functor from presheaves on X to presheaves on W.

Example 3.1.8. Let X be a quasiaffine, $W \subseteq X$ be open. Then $\mathcal{O}_W = (\mathcal{O}_X)|_W$ is the sheaf of function on W.

Assume $\mathcal{C}$ has direct limits.

Definition 3.1.9. If $\mathcal{F}$ is a presheaf on X of objects in $\mathcal{C}$ and $x \in X$, then the *stalk of* $\mathcal{F}$ *at* x is
$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U),$$
where U varies over the open neighborhoods of x, ordered by reverse inclusion.

More generally, if $W \subseteq X$ is closed and irreducible, $\mathcal{F}_W = \varinjlim_{U \cap W \neq \emptyset} \mathcal{F}(U)$.

Remark 3.1.10. This is a generalization of the definition of $\mathcal{O}_{X,W}$ for a closed irreducible subset $W \subseteq X$ (cf. Definition 1.4.1).

Note that both sets are filtered, so the colimit is indeed a direct limit.

If $\mathcal{C}$ is R-mod or R-alg, we know how to describe $\mathcal{F}_W$ (similarly to $\mathcal{O}_{X,W}$).

This gives a functor

$$\left\{\begin{array}{c}\text{presheaves on } X \\ \text{of objects in } \mathcal{C}\end{array}\right\} \longrightarrow \mathcal{C}$$

$$\mathcal{F} \longrightarrow \mathcal{F}_W$$

[4] A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *full* if $\mathrm{obj}\mathcal{D} \subseteq \mathrm{obj}\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{D}}(D, D') = \mathrm{Hom}_{\mathcal{C}}(D, D')$.

since $\varphi\colon \mathcal{F} \to \mathcal{G}$ induces $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ and taking $\varinjlim$ gives a map $\mathcal{F}_W \to \mathcal{G}_W$.

Notation. If $s \in \mathcal{F}(U)$ and $x \in X$, we write s_x for the image of s in $\mathcal{F}_x$.

Note that if $\mathcal{F}$ is a sheaf and $s_1, s_2 \in \mathcal{F}(U)$ satisfy $(s_1)_x = (s_2)_x$ for all $x \in U$, then $s_1 = s_2$.

Definition 3.1.11. Let $f\colon X \to Y$ be a continuous map and $\mathcal{F}$ be a presheaf on X. We get a presheaf on Y, denoted $f_*\mathcal{F}$ and called the *pushforward of $\mathcal{F}$ by f*, given by

$$\Gamma(V, f_*\mathcal{F}) = \Gamma(f^{-1}(V), \mathcal{F})$$

for $V \subseteq Y$ open, and the restriction maps induced by those of $\mathcal{F}$.

If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is a sheaf.

This gives a functor: if $\varphi\colon \mathcal{F} \to \mathcal{G}$, then $f_*\mathcal{F} \to f_*\mathcal{G}$ is induced by the following commutative diagram

$$\begin{tikzcd}
\Gamma(V, f_*\mathcal{F}) \arrow[r, dotted] \arrow[d, "="] & \Gamma(V, f_*\mathcal{G}) \arrow[d, "="] \\
\Gamma(f^{-1}(V), \mathcal{F}) \arrow[r, "\varphi|_{f^{-1}(V)}"] & \Gamma(f^{-1}(V), \mathcal{G})
\end{tikzcd}$$

Definition 3.1.12. If $\mathcal{F}$ is a presheaf on X, a *subpresheaf* is a presheaf $\mathcal{G}$ such that $\mathcal{G}(U)$ is a subset of $\mathcal{F}(U)$ and the restriction maps for $\mathcal{G}$ are induced by the restriction maps for $\mathcal{F}$.

If moreover $\mathcal{F}$, $\mathcal{G}$ are both sheaves, $\mathcal{G}$ is called a *subsheaf* of $\mathcal{F}$.

If $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, the inclusion map gives a morphism of presheaves $\mathcal{G} \to \mathcal{F}$.

3.2. **Prevarieties.** Let $k = \bar{k}$ be fixed. Let X be a topological space. Define a sheaf on X by setting for U open

$$\mathrm{Fun}_X(U) = \{f\colon U \to k\}$$

which is a k-algebra with respect to point-wise operations, and the restriction maps are restrictions of functions.

If $f\colon X \to Y$ is a continuous map, we have a morphism of sheaves on Y given by

$$(*) \qquad \mathrm{Fun}_Y \to f_*\,\mathrm{Fun}_X$$

where for $U \subseteq Y$ we set

$$\mathrm{Fun}_Y(U) \to \mathrm{Fun}_X(f^{-1}(U))$$
$$\varphi \mapsto \varphi \circ f$$

We define the category Top_k as follows:

- objects are $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a subsheaf of k-algebras of Fun_X,
- morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are continuous functions $f\colon X \to Y$ such that for any $U \subseteq Y$ open and any $\varphi \in \mathcal{O}_Y(U)$, we have $\varphi \circ f \in \mathcal{O}_X(f^{-1}(U))$, i.e. $(*)$ induces a morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remarks 3.2.1.

(1) Let $(X, \mathcal{O}_X) \in \mathrm{Top}_k$. If $U \subseteq X$ is open, then we define $\mathcal{O}_U = (\mathcal{O}_X)_U \subseteq \mathrm{Fun}_U$, and then $(U, \mathcal{O}_U) \in \mathrm{Top}_k$. The inclusion map $i\colon U \to X$ defines a morphism in Top_k.
(2) If $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y) \in \mathrm{Top}_k$, $X = \bigcup_{i\in I} U_i$ is an open cover, $\alpha_i\colon U_i \hookrightarrow X$ are the inclusions, and $f\colon X \to Y$ is any map, then
$$f \text{ is a morphism if and only if } f \circ \alpha_i \text{ is a morphism for all } i.$$
This is because of the sheaf condition for $\mathcal{O}_X$.
(3) Isomorphisms in Top_k: $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if f is a homeomorphism and for any open $U \subseteq Y$
$$\varphi\colon U \to k \text{ is in } \mathcal{O}_Y(U) \text{ if and only if } \varphi \circ f \in \mathcal{O}_X(f^{-1}(U)).$$

Terminology: $\mathcal{O}_X$ is the *structure sheaf* of $(X, \mathcal{O}_X)$. We sometimes abuse notation and write X to mean $(X, \mathcal{O}_X)$, leaving the structure sheaf implicit.

Example 3.2.2. Let $X \subseteq \mathbb{A}^n$ be a locally closed subset. We define a sheaf $\mathcal{O}_X$ by setting
$$U \mapsto \mathcal{O}_X(U) = \{f\colon U \to k \mid f \text{ regular function}\}.$$
Then $(X, \mathcal{O}_X)$ is an object in Top_k.

If $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are locally closed, then $f\colon X \to Y$ is a morphism in the new sense if and only if it is a morphism in the old sense.

The 'if' implication follows since for f continuous and $\varphi\colon U \to k$ regular on $U \subseteq Y$ open, we have $\varphi \circ f\colon f^{-1}(U) \to k$ is regular.

For the 'only if' implication, apply the definition for $Y \hookrightarrow \mathbb{A}^n \xrightarrow{\mathrm{pr}_i} k$ to conclude that if $f = (f_1, \ldots, f_n)$, then each f_i is a regular function.

Definition 3.2.3. An element $(X, \mathcal{O}_X) \in \mathrm{Top}_k$ is an *affine variety* if it isomorphic in Top_k to $(Y, \mathcal{O}_Y)$ where Y is a closed subset of some $\mathbb{A}^n$.

Definition 3.2.4. A *prevariety* is an object $(X, \mathcal{O}_X)$ in Top_k such that there is a finite open cover $X = U_1 \cup \cdots \cup U_n$ such that each $(U_i, \mathcal{O}_{U_i})$ is an affine variety. *Morphisms of prevarieties* are just morphisms in Top_k.

Definition 3.2.5. A *quasi-affine variety* $(X, \mathcal{O}_X)$ is an object in Top_k isomorphic to $(Y, \mathcal{O}_Y)$ where $Y \subseteq \mathbb{A}^n$ is locally closed.

Note that any quasi-affine variety is a prevariety.

Properties.

(1) If $(X, \mathcal{O}_X)$ prevariety, then X is a Noetherian topological space.

Proof. If $X = U_1 \cup \cdots \cup U_r$ where $(U_i, \mathcal{O}_{U_i})$ is affine, so U_i is Noetherian for all i, then any chain
$$F_1 \supseteq F_2 \supseteq \cdots$$
of closed subsets in X gives a chain
$$F_1 \cap U_i \supseteq F_2 \cap U_i \supseteq \cdots$$

of closed subsets in U_i, so there exists n_i such that $F_j \cap U_i = F_{j+1} \cap U_i$ for $j > n_i$, whence $F_j = F_{j+1}$ for $j > \max\{n_1, \ldots, n_r\}$. $\square$

(2) If $(X, \mathcal{O}_X)$ is a prevariety, then $\mathcal{O}_X$ is a subsheaf of the sheaf $\mathcal{C}_X$ of continuous functions to k, i.e. $\mathcal{C}_X(U) = \{f : U \to k \mid f \text{ continuous}\}$.
   This is clear since $\mathcal{O}_X$ is a sheaf and the property holds for affines.
(3) If $(X, \mathcal{O}_X)$ is a prevariety, $\varphi \in \mathcal{O}_X(X)$, then $U = \{x \mid \varphi(x) \neq 0\}$ is open by (2), and we have $\frac{1}{\varphi} \in \mathcal{O}_X(U)$.
   This is clear since $\mathcal{O}_X$ is a sheaf and the property holds for affines by definition of regular functions.
(4) All statements about dimension theory of quasi-affine varieties extends to all prevarieties.
   This is because in all the proofs we reduced to the affine case, and the reduction step is the same in the case of prevarieties.
(5) If $(X, \mathcal{O}_X)$ is a prevariety, then the open subsets of X that are affine form a basis for the topology of X.
   This is clear since $\mathcal{O}_X$ is a sheaf and the property holds for affines.

3.3. **Subvarieties.** Let $(X, \mathcal{O}_X)$ be an object in Top_k and Z be a locally closed subset of X. Define $\mathcal{O}_Z$ on Z as follows

$$\mathcal{O}_Z(U) = \left\{ f : U \to k \;\middle|\; \begin{array}{c} \text{for any } x \in U \text{ there exists } V \ni x \text{ open in } X \text{ and } g \in \mathcal{O}_X(V) \\ \text{such that } V \cap Z \subseteq U \text{ and } g|_{V \cap Z} = f|_{V \cap Z} \end{array} \right\}.$$

Intuitively, elements of $\mathcal{O}_Z$ are functions on Z that extend locally to sections of $\mathcal{O}_X$.

It is clear that

(1) $\mathcal{O}_Z$ is a subsheaf of Fun_Z,
(2) If Z, X are as above and W is locally closed in Z, the sheaf $\mathcal{O}_W$ is the same whether we consider W as a subset of X or of Z.

Note that if Z is open in X, we recover the previous definition, $\mathcal{O}_Z = (\mathcal{O}_X)|_Z$.

Example 3.3.1. If Z is a locally closed subset in $\mathbb{A}^n$, then the sheaf of regular functions on Z is obtained in this way from $\mathcal{O}_{\mathbb{A}^n}$.

Proposition 3.3.2. *If $(X, \mathcal{O}_X)$ is a prevariety and $Z \subseteq X$ is locally closed, then $(Z, \mathcal{O}_Z)$ is a prevariety.*

Proof. There is an affine open cover $X = U_1 \cup \cdots \cup U_n$. It is enough to show that

$$(U_i \cap Z, \mathcal{O}_Z|_{U_i \cap Z})$$

is a prevariety.

We have that $U_i \cap Z$ is a locally closed subset of U_i, which is affine, and $\mathcal{O}_Z|_{U_i \cap Z}$ is a sheaf induced by $\mathcal{O}_X$ on a locally closed subset $U_i \cap Z$, which is the same the the sheaf induced by $\mathcal{O}_{U_i}$ on $U_i \cap Z$. Therefore, $(U_i \cap Z, \mathcal{O}_{U_i \cap Z})$ is isomorphic to a quasi-affine variety, and hence it is a prevariety. $\square$

Remark 3.3.3. If $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are as in Proposition 3.3.2, then the inclusion map $Z \hookrightarrow X$ gives a morphism of prevarieties.

Definition 3.3.4. A *locally closed* (*open*, *closed*) subvariety of the prevariety $(X, \mathcal{O}_X)$ is a prevariety $(Z, \mathcal{O}_Z)$ where $Z \subseteq X$ is locally closed (open, closed) and $\mathcal{O}_Z$ is defined as before from $\mathcal{O}_X$.

Definition 3.3.5. A *locally closed immersion* (or *embedding*) is a morphism of prevarieties $f\colon X \to Y$ which factors as

$$X \xrightarrow{g} Z \xrightarrow{i} Y$$

where g is an isomorphism and i is the inclusion map of a locally closed subvariety. If Z is open (closed), we say that f is an *open* (*closed*) *immersion*.

Proposition 3.3.6. *Let $f\colon X \to Y$ be a locally closed immersion of prevarieties. Given $g\colon Z \to Y$, there exists $h\colon Z \to X$ such that $f \circ h = g$, i.e. the triangle*

$$\begin{tikzcd} & Z \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes, if and only if $g(Z) \subseteq f(X)$.

Proof. If there is such an h, then $g(Z) = f(h(Z)) \subseteq f(X)$. Conversely, suppose $g(Z) \subseteq f(X)$. Since f is injective, there is a unique function (set-theoretically) $h\colon Z \to X$ such that $f \circ h = g$. We need to show that h is a morphism. By definition of a locally closed immersion, there exists a commuting triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & W \arrow[ul, "f'", "\cong"'] \arrow[ur, hook, "i"'] & \end{tikzcd}$$

where i is an inclusion of a locally closed subvariety and f' is an isomorphism. Altogether, the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & W \arrow[ul, "f'", "\cong"'] \arrow[ur, hook, "i"'] & \\ & Z \arrow[uul, bend left, "h"] \arrow[uur, bend right, "g"'] & \end{tikzcd}$$

commutes and it is enough to show that $(f')^{-1} \circ h$ is a morphism. After replacing $f\colon X \to Y$ by $W \hookrightarrow Y$, we may assume that f is the inclusion of a subvariety.

We now have the commuting triangle

$$\begin{tikzcd} X \arrow[rr, hook, "f"] & & Y \\ & Z \arrow[ul, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

Choose $Y = U_1 \cup \cdots \cup U_r$ where U_i is open and affine, and write

$$X = \bigcup_{i=1}^{r} f^{-1}(U_i).$$

We can then restrict the above triangles to triangles

$$\begin{tikzcd} \text{(quasi-affine) } f^{-1}(U_i) \arrow[rr, hook, "f"] & & U_i \text{ (affine)} \\ & g^{-1}(U_i) \arrow[ul, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

By covering $g^{-1}(U_i)$ by open affine subsets, we see that $g^{-1}(U_i) \to f^{-1}(U_i)$ is a morphism for all i. Therefore, h is a morphism. □

Proposition 3.3.7. *Let $f\colon X \to Y$ be a morphism of prevarieties. Then the following are equivalent:*

(1) *f is a closed immersion,*
(2) *for any $U \subseteq Y$ open affine, $f^{-1}(U)$ is an affine variety and the induced homomorphism $\mathcal{O}(U) \to \mathcal{O}(f^{-1}(U))$ is surjective,*
(3) *there is an open cover $Y = \bigcup_{i=1}^{r} U_i$ such that $f^{-1}(U_i)$ is affine and $\mathcal{O}(U_i) \to \mathcal{O}(f^{-1}(U_i))$ is surjective.*

Proof. We show that (1) implies (2). We have the commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y & \text{closed immersion} \\ f^{-1}(U) \arrow[u, hook] \arrow[r] & U \arrow[u, hook] & \text{closed immersion} \end{tikzcd}$$

As U is affine and open, $f^{-1}(U)$ is affine, and $\mathcal{O}(U) \to \mathcal{O}(f^{-1}(U))$ is surjective by the description of regular functions on closed subsets of $\mathbb{A}^n$. This shows that (1) implies (2), and it is clear that (2) implies (3).

We show that (3) implies (1). Each morphism $f^{-1}(U_i) \to U_i$ is a closed immersion. In particular, it is injective and a homeomorphism onto a closed subset of U_i. Therefore, f is injective and a homeomorphism onto a closed subset Z of Y. Consider on Z the sheaf $\mathcal{O}_Z$ defined from $\mathcal{O}_X$. By Proposition 3.3.6, f factors as $X \xrightarrow{h} Z \xhookrightarrow{i} Y$. Then h is bijection and we need to show h^{-1} is a morphism. This follows since

$$f^{-1}(U_i) = h^{-1}(U_i \cap Z) \to U_i$$

is an isomorphism, since $f^{-1}(U_i) \to U_i$ is a closed immersion. $\square$

3.4. **Fibered products of prevarieties.**

Proposition 3.4.1. *If X and Y are prevarieties, then there is a product on X, Y in the category of prevarieties such that*

- *the underlying set is $X \times Y$,*
- *the topology is finer than the product topology,*
- *the "projection maps" $X \times Y \to X$ and $X \times Y \to Y$ are the projections.*

Sketch of proof. Cover both X and Y by affine open subsets

$$X = \bigcup_{i=1}^{r} U_i, \; Y = \bigcup_{j=1}^{s} Y_j.$$

Then

$$X \times Y = \bigcup_{i,j} (U_i \times V_j).$$

We know that $U_i \times V_j$ with the projections onto U_i and V_j is the direct product of U_i and V_j in the category of quasi-affine varieties, where the topology and the structure sheaf on $U_i \times V_j$ come from embeddings $U_i \subseteq \mathbb{A}^{n_i}$ and $V_j \subseteq \mathbb{A}^{m_j}$.

The key point is that on $(U_{i_1} \times V_{j_1}) \cap (U_{i_2} \times V_{j_2})$, the restrictions of the topology and the struction sheaf coming from both $U_{i_1} \times V_{j_1}$ and $U_{i_2} \times V_{j_2}$ coincide. This is because both of them satisfy the universal property defining the product $(U_{i_1} \cap U_{i_2}) \times (V_{j_1} \cap V_{j_2})$. We can then define a topology and a structure sheaf on $X \times Y$ by

- $W \subseteq X \times Y$ open if and only if $W \cap (U_i \times V_j)$ is open for all i, j,
- $\varphi \colon W \to k$ is in $\mathcal{O}_{X \times Y}(W)$ if
$$\varphi|_{W \cap (U_i \times V_j)} \in \mathcal{O}_{U_i \times V_j}(W \cap (U_i \times V_j))$$
for all i, j.

This defines $X \times Y$ as an object in Top_k. It is a prevariety since it is covered by the affine open subsets $U_i \times V_j$.

Exercise. Check that $X \leftarrow X \times Y \to Y$ satisfies the universal property. $\square$

Remark 3.4.2. If X and Y are irreducible is $X \times Y$ is irreducible. To prove this, you could use the following fact proved on Homework 1.

Suppose X is topological space and $X = U_1 \cup \cdots \cup U_r$ is an open cover and $U_i \neq 0$. Then X is irreducible if and only if U_i is irreducible for all i and $U_i \cap U_j \neq \emptyset$ for all i, j.

This allows to reduce to the case where X and Y are affine, where we know that this result holds.

Definition 3.4.3. Let $f \colon X \to Y$ be a morphism of prevarieties. Define

$$j_f \colon X \to X \times Y, \; x \mapsto (x, f(x))$$

as the morphism defined by the universal property of products. Then j_f is the *graph morphism of f* and $\Gamma_f = \mathrm{im}(j_f)$ is the *graph of f*.

For example, if $f = 1_X$, then $\Gamma_f = \Delta_X \subseteq X \times X$ is the diagonal.

Proposition 3.4.4. *Let $f\colon X \to Y$ be a morphism of prevarieties. Then $j_f\colon X \to X \times Y$ is a locally closed immersion.*

Proof. For any $x \in X$, choose affine open neighborhoods $V_x \subseteq Y$ of $f(x)$ and $U_x \subseteq f^{-1}(V_x)$ of x. Then

$$W = \bigcup_{x \in X} U_x \times V_x$$

is open in $X \times Y$. Since $X \times Y$ is Noetherian, we may take only finitely many x.

Note that j_f factors through W

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "g"'] & & X \times Y \\ & W \arrow[ur, hook, "\text{open subvariety}"'] & \end{tikzcd}$$

and g is a morphism. It is enough to show that g is a closed immersion. We have that

$$g^{-1}(U_x \times V_x) = U_x$$

is affine. By Proposition 3.3.7, we only need to show that $\mathcal{O}(U_x \times V_x) \to \mathcal{O}(U_x)$ is surjective. We may hence assume that X and Y are affine and show that $j_f^{\#}\colon \mathcal{O}(X \times Y) \to \mathcal{O}(X)$ is surjective. If $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$, $f = (f_1, \ldots, f_n)$, we have that $j_f^{\#}$ is the map

$$\frac{k[x_1, \ldots, x_m, y_1, \ldots, y_n]}{I(X \times Y)} \to \frac{k[x_1, \ldots, x_m]}{I(X)}$$
$$x_i \mapsto x_i$$
$$y_j \mapsto f_j$$

The surjectivity is hence immediate. $\square$

Corollary 3.4.5. *The category of prevarieties has fibered products. More explicitly, if $f\colon X \to Z$ and $g\colon Y \to Z$, then*

$$W = \{(x, y) \mid f(x) = g(x)\} \subseteq X \times Y$$

is a locally closed subset and, with the maps inducted by projections. It is the fibered product $X \times_Z Y$:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

(satisfying the obvious universal property).

Proof. The map

$$X \times Y \xrightarrow{h} Z \times Z \supseteq \Delta_Z$$

$$(x, y) \longrightarrow (f(x), g(y))$$

is clearly a morphism, and $W = h^{-1}(\Delta_Z)$ is locally closed in $X \times Y$ by Proposition 3.4.4.

Consider the subvariety structure on W. It is enough to show that for any maps $\alpha\colon T \to X$, $\beta\colon T \to Y$ such that $f \circ \alpha = g \circ \beta$, there is a unique map $\gamma\colon T \to W$ such that $p \circ \gamma = \alpha$, $q \circ \gamma = \beta$, i.e. the following diagram commutes

$$\begin{tikzcd}
T \arrow[drr, bend left, "\beta"] \arrow[dr, dotted, "\gamma"] \arrow[ddr, bend right, "\alpha"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

The uniqueness is clear since the map has to be given by $\gamma(t) = (\alpha(t), \beta(t)) \in W$.

We just need to show that this γ is a morphism. We have that

$$T \xrightarrow{\gamma} W \hookrightarrow_{\text{incl.}} X \times Y$$

is a morphism by the universal property of the product. Therefore, γ is a morphism by the universal property of a subvariety. $\square$

Example 3.4.6. Let $f\colon X \to Y$ be a morphism of prevarieties. Let $W \subseteq Y$ be a locally closed subvariety. Then we have the commuting square

$$\begin{tikzcd}
f^{-1}(W) \arrow[r] \arrow[d, "f"'] & X \arrow[d, "f"] \\
W \arrow[r, hook, "i"] & Y
\end{tikzcd}$$

and $X \times_Y W \cong f^{-1}(W)$.

3.5. **Separated prevarieties.** When we define manifolds, we require the underlying topological space to be Haussdorff. We want to do similar for prevarieties, but algebraic prevarieties are almost never Hausdorff.

The key obvservation is that if X is a topological space then X is Hausdorff if and only if $\Delta_X \subseteq X \times X$ is closed under the product topology. This motivates the following definition.

Definition 3.5.1. A prevariety X is *separated* if $\Delta_X \subseteq X \times X$ is a closed subset. A separated prevariety is a *variety*.

Remarks 3.5.2.

(1) We know that $\Delta\colon X \to X \times X$ given by $x \mapsto (x, x)$ is a locally closed immersion by Proposition 3.4.4. Therefore, X is separated if and only if Δ is a closed immersion.
(2) If X is separated and $f, g\colon Y \to X$ are moprhisms, then

$$\{y \in Y \mid f(y) = g(y)\}$$

is closed in Y. (In particular, our discussion about the domain of a rational map extends to varieties.)

To see this, note that we have a morphism

$$(f, g)\colon Y \to X \times X,\ y \mapsto (f(y), g(y))$$

and the set we want $(f, g)^{-1}(\Delta_X)$, which is closed in Y since X is separated.

Properties.

(1) If X is a variety and Y is a locally closed subvariety of X, then Y is a variety. In particular, every quasi-affine variety is a variety.

Proof. We have the diagram

$$\begin{tikzcd} Y \times Y \arrow[r, hook, "\text{locally closed}"] & X \times X \\ \Delta_Y = \Delta_X \cap Y \times Y \arrow[u, hook] \arrow[r, hook] & \Delta_X \arrow[u, hook, "\text{closed}"'] \end{tikzcd}$$

and hence Δ_Y is closed in $Y \times Y$.

Finally, note that $\mathbb{A}^n$ is separated because $\Delta_{\mathbb{A}^n} = V(x_1 - y_1, \ldots, x_n - y_n)$ is closed in $\mathbb{A}^n \times \mathbb{A}^n$. □

(2) If $f\colon X \to Y$ is a morphism with Y separated, then $j_f\colon X \times X \to Y$ is a closed immersion.

Proof. By Proposition 3.4.4, we only need to show that $\Gamma_f \subseteq X \times Y$ is closed. We have a morphism

$$h\colon X \times Y \to Y \times Y$$
$$(x, y) \mapsto (f(x), y)$$

and $\Gamma_f = h^{-1}(\Delta_Y)$ is closed in $X \times Y$ since Y is separated. □

(3) If X and Y are varieties then $X \times Y$ is a variety. More generally, if $f\colon X \to Z$ and $g\colon Y \to Z$ are morphisms of varieties, then

$$W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

is a subvariety of $X \times Y$. In particular, $X \times_Z Y$ is a variety by (1).

Proof. The first assertion follows from noting noting that $\Delta_{X\times Y} = p_{13}^{-1}(\Delta_X) \cap p_{24}^{-1}(\Delta_Y)$ where

$$\begin{tikzcd} & (X \times Y) \times (X \times Y) \arrow[dl, "p_{13}"'] \arrow[dr, "p_{24}"] & \\ X \times X & & Y \times Y \end{tikzcd}$$

□

Thus this set is closed in $X \times Y$, so $X \times Y$ is separated. For the second assertion, consider the map

$$h \colon X \times Y \to Z \times Z$$
$$(x, y) \mapsto (f(x), g(y))$$

and $W = h^{-1}(\Delta_Z)$ is closed in $X \times Y$.

(4) If X is a variety and $U, V \subseteq X$ are affine open subsets, then $U \cap V$ is affine.

Proof. We have that $\Delta \colon X \to X \times X$ is a closed immersion, and $U \times V$ is affine, so $\Delta^{-1}(U \times V) = U \cap V$ is affine. $\square$

Proposition 3.5.3. *If X is a prevariety and $X = U_1 \cup \cdots \cup U_r$ is an affine open cover, then X is separated if and only if $U_i \cap U_j$ is affine for any i, j and the homomorphism*

$$\begin{tikzcd} \mathcal{O}(U_i \times U_j) \arrow[r] \arrow[d, "="] & \mathcal{O}(U_i \cap U_j) \\ \mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \arrow[ur] & \end{tikzcd}$$

is surjective.

Proof. Apply Proposition 3.3.7, i.e. the crierion for

$$\Delta \colon X \to X \times X$$

to be a closed immersion in terms of the affine open cover $X \times X = \bigcup_{i,j} U_i \times U_j$. $\square$

For gluing varieties, see problem set 5.

Example 3.5.4. Suppose $X_1 = X_2 = \mathbb{A}^1$, and define $U \subseteq X_1$, $V \subseteq X_2$ by

$$U = V = \mathbb{A}^1 \setminus \{0\}.$$

- Glue X_1, X_2 via the isomorphism $\mathrm{id} \colon U \to V$. We get a prevariety X with 2 open subsets $W_1, W_2 \subseteq X$ such that $W_i \cong \mathbb{A}^1$. Then X is separated if and only if $W_1 \cap W_2$ is affine and the induced map $\mathcal{O}(W_1 \cap W_2) = \mathcal{O}(W_1) \otimes_k \mathcal{O}(W_2) \to \mathcal{O}(W_1 \cap W_2)$ is surjective. While $W_1 \cap W_2$ is affine, the induced map is
$$k[x_1] \otimes_k [x_2] \to k[t, t^{-1}]$$
$$x_1 \mapsto t$$
$$x_2 \mapsto t$$
is clearly not surjective. Hence X is not separated.
- On the other hand, let Y be obtained from gluing X_1, X_2 along $U \to V$ given by $t \mapsto t^{-1}$. We get two open affine subsets $W_1, W_2 \subseteq Y$ such that $W_1 \cap W_2 \cong \mathbb{A}^1 \setminus \{0\}$ and the induced map
$$\mathcal{O}(W_1) \otimes_k \mathcal{O}(W_2) \to \mathcal{O}(W_1 \cap W_2)$$
$$x_1 \mapsto t$$
$$x_2 \mapsto t^{-1}$$

is clearly surjective. Hence Y is separeted. In fact, one can see that Y is $\mathbb{P}^1$, which we introduce in the next chapter.

4. Projective varieties

As always, $k = \overline{k}$ is a fixed field throughout.

4.1. **The projective space.** For $n \geq 0$, we define

$$\mathbb{P}^n = \{\text{lines in } k^{n+1}\},$$

where a line is a 1-dimensional subspace. A line in k^{n+1} is given by some $(a_0, \dots, a_n) \in k^{n+1} \setminus \{0\}$, and two tuples $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ give the same line if there is a $\lambda \in k^*$ such that $b_i = \lambda a_i$ for all i. This shows that, as a set,

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\})/k^*$$

where k^* acts by $\lambda(a_0, \dots, a_n) = (\lambda a_0, \dots, \lambda a_n)$. Denote the quotient map by

$$\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n,$$

and write $\pi(a_0, \dots, a_n) = [a_0, \dots, a_n]$.

Note that $S = k[x_0, \dots, x_n]$ is an $\mathbb{N}$-graded k-algebra, where the grading is given by degree.

Definition 4.1.1. A commutative unital ring R is *graded* if

$$R = \bigoplus_{m \in \mathbb{Z}} R_m$$

as abelian groups such that $R_p \cdot R_q \subseteq R_{p+q}$. It is moreover $\mathbb{N}$-*graded* if $R_m = 0$ for all $m < 0$.

An element $u \in R_m \setminus \{0\}$ is *homogeneous of degree m* and 0 is homogeneous of any degree. Accordingly, if $f \in R$ and $f = \sum_{i \geq 0} f_i$ for $f_i \in R_i$, then the f_i are the *homogeneous components* of f.

Note that $R_0 \subseteq R$ is a subring and R_m is an R_0-module for all m. Hence R is actually an R_0-algebra.

Definition 4.1.2. A graded ring R is an A-*algebra* if R_0 is an A-algebra (so R is an A-algebra).

Definition 4.1.3. If R, S are graded rings, a *graded homomorphism* $\varphi \colon R \to S$ is a homomorphism such that $\varphi(R_m) \subseteq S_m$ for all m.

Example 4.1.4. The main example, as we saw above, is $S = k[x_0, \dots, x_n]$ where

$$S_m = \{\text{homogeneous polynomials of degree } m\}.$$

Remark 4.1.5. A closed subset $Y \subseteq \mathbb{A}^{n+1}$ is k^*-invariant if and only if $I_{\mathbb{A}^{n+1}}(Y)$ is a homogeneous ideal.

Exercise. If $I \subseteq R$ is an ideal of a graded ring R, the following are equivalent:

(1) I is generated by homogeneous elements,

(2) if $f \in I$, all the homogeneous components of f are in I,
(3) the decomposition of R induces a decomposition $I = \bigoplus_m (I \cap R_m)$.

We can use this to prove the assertion in Remark 4.1.5. If $f \in S_m$, then

$$f(\lambda(u_0, \ldots, u_n)) = \lambda^m f(u_0, \ldots, u_n) \quad \text{for all } \lambda \in k^*.$$

This implies that I is generated by homogeneous elements, and hence $V(I) \subseteq \mathbb{A}^{n+1}$ is k^*-invariant. Hence $I_{\mathbb{A}^n}(Y)$ homogeneous implies Y is k^*-invariant.

Suppose $Y \subseteq \mathbb{A}^{n+1}$ is k^*-invariant. Let $f \in I = I_{\mathbb{A}^{n+1}}(Y)$ and write

$$f = \sum_{i \geq 0} f_i$$

with f_i homogeneous of degree i. If $u \in Y$, then $\lambda u \in Y$, so

$$0 = f(\lambda u) = \sum_{i \geq 0} \lambda^i f_i(u).$$

This holds for infinitely many λ, and hence $f_i(u) = 0$ for all i. Thus I is homogeneous.

Goal. To establish a correspondence between homogeneous radical ideals in S and the closed subsets in $\mathbb{P}^n$ with respect to a suitable topology.

By the property

$$f(\lambda(u_0, \ldots, u_n)) = \lambda^m f(u_0, \ldots, u_n) \quad \text{for all } \lambda \in k^*$$

for $f \in S_m$, it makes sense to say that a homogeneous polynomial vanishes at a point in $\mathbb{P}^n$.[5]

Given any f, we say that f *vanishes at* $p \in \mathbb{P}^n$ if all the homogeneous components of f vanish at p.

Definition 4.1.6. For a homogeneous ideal $I \subseteq S$, define the *zero locus of I in $\mathbb{P}^n$* as

$$V(I) = \{p \in \mathbb{P}^n \mid \text{all } f \in I \text{ vanish at } p\}.$$

Note that $\pi^{-1}(V_{\mathbb{P}^n}(I)) = V_{\mathbb{A}^{n+1}}(I) \setminus \{0\}$.

The same properties hold for V in the projective case as did in the affine case:

(1) $V(S) = \emptyset$,
(2) $V(0) = \mathbb{P}^n$,
(3) $V\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha V(I_\alpha)$,
(4) $V(I \cap J) = V(I \cdots J) = V(I) \cup V(J)$.

Therefore, $V(I)$ for homogeneous ideals $I \subseteq S$ form the closed sets of a topology on $\mathbb{P}^n$, which we call the *Zariski topology* again.

In the opposite direction, if $X \subseteq \mathbb{P}^n$ is a subset, then we may define

$$I(X) = \{f \in S \mid f \text{ vanishes at all } p \in X\},$$

[5]Note that it does not make sense to evaluate a polynomial at a point of $\mathbb{P}^n$.

which is a homogeneous, radical ideal.

Consider the two compositions.

- If $X \subseteq \mathbb{P}^n$, then $V(I(X)) = \overline{X}$ in $\mathbb{P}^n$ (and the proof is the same as in the affine case).
- If J is a homogeneous radical ideal in S such that $J \neq (x_0, \ldots, x_n)$, then
$$I(V(J)) = J,$$
which is known as the *graded Nullstellensatz*.

Proof. The '$\supseteq$' inclusion is clear. For the other inclusion, suppose $f \in I(V(J))$. We will show that $f \in J$. We may assume that f homogeneous of degree d (by decomposing it into homogeneous components and proving the assertion for each of them separately). Let Z be the zero locus of J in $\mathbb{A}^{n+1}$. By hypothesis, f vanishes on $\pi^{-1}(V(J))) = Z \setminus \{0\}$. If $d > 0$, then $f(0) = 0$, so f vanishes on Z, and by the Nullstellensatz 1.1.9, $f \in J$.

Suppose $d = 0$ and assume that $f \neq 0$. This implies that $V(J) = \emptyset$ as a subset of $\mathbb{P}^n$. Therefore, $Z \subseteq \{0\}$, so by the Nullstellensatz 1.1.9,
$$(x_0, \ldots, x_n) \subseteq I_{\mathbb{A}^{n+1}}(Z) = J.$$
This is a maximal ideal different from J by hypothesis, so $J = S$ and hence $f \in J$. $\square$

Note that $I(V(x_0, \ldots, x_n)) = I(\emptyset) = S \neq (x_0, \ldots, x_n)$. For this reason, $(x_0, \ldots, x_n)$ is sometimes called the *irrelevant ideal*.

Corollary 4.1.7. *We have inverse, order-reversing bijections*

$$\left\{ \begin{array}{c} \textit{closed subsets} \\ \textit{of } \mathbb{P}^n \end{array} \right\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{radical, homogeneous ideals} \\ \textit{in } k[x_0, x_1, \ldots, x_n] \\ \textit{different from } (x_0, \ldots, x_n) \end{array} \right\}$$

Proof. This follows from the above discussion, since for any $X \subseteq \mathbb{P}^n$ closed, $I(X) \neq (x_0, \ldots, x_n)$. Otherwise, $X = V(I(X)) = \emptyset$, so $I(X) = S$. $\square$

Exercise. If $P \subseteq S$ is a homogeneous ideal, then P is prime if and only if for any homogeneous elements f, g of S, if $fg \in P$ then $f \in P$ or $g \in P$.

Using this, if $I \subset S$ is radical, homogeneous, and different from $(x_0, \ldots, x_n)$, then $V(I)$ is irreducible if and only if I is prime.

Notation. If $X \subseteq \mathbb{P}^n$ is closed with corresponding ideal I_X, we let $S_X = S/I_X$ called the *homogeneous coordinate ring for* X. This is an $\mathbb{N}$-graded k-algebra since I_X is homogeneous.

In particular, S is the *homogeneous coordinate ring* of $\mathbb{P}^n$, and $S_+ = (x_0, \ldots, x_n)$ is the *irrelevant ideal*.

Definition 4.1.8. If $X \subseteq \mathbb{P}^n$ is closed and $f \in S_X$ is homogeneous, $\deg(f) > 0$,
$$D_X^+(f) = X \setminus V(\tilde{f})$$
where $\tilde{f} \in S$ lies over f.

This is an open subset of X. Since every radical homogeneous ideal $J \neq S_+, S$ is generated by finitely many homogeneous polynomials of positive degree, the open subsets $D_{\mathbb{P}^n}(f)$ give a basis for the topology on $\mathbb{P}^n$.

Recall that we have a map $\pi\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ sending u to $k \cdot u$.

Definition 4.1.9. If $X \subseteq \mathbb{P}^n$ is closed, the *affine cone* $C(X)$ over X is

$$C(X) = \bigcup_{\ell \in X} \ell,$$

where $\ell \subseteq \mathbb{A}^{n+1}$ is a line.

Note that for nonempty X, $C(X) = \pi^{-1}(X) \cup \{0\}$. Moreover, if $I = I_X$, then $C(X)$ is the zero-locus of I in $\mathbb{A}^{n+1}$, so $C(X)$ is also closed.

Next, we will want to characterize the irreducible components of closed subsets of $\mathbb{P}^n$.

Exercise. If G is an irreducible linear algebraic group acting algebraically on the variety X then every irreducible component Z of X is preserved by the G-action, i.e. $gZ = Z$ for all $g \in G$.

Suppose now that $X \subseteq \mathbb{P}^n$ is a closed subset. Then $C(X) \subseteq \mathbb{A}^{n+1}$ is closed and we have a k^*-action on $C(X)$. By the exercise, the irreducible components of $C(X)$ are preserved by the k^*-action. By Remark 4.1.5, this implies that the ideal defining each irreducible component of $C(X)$ is homogeneous.

Hence the minimal primes in S containing I_X are homogeneous. They correspond to the irreducible components $X_1, \ldots, X_r$ of X. The corresponding irreducible decomposition of $C(X)$ is

$$C(X) = C(X_1) \cup \cdots \cup C(X_r).$$

The next goal is to put a sheaf of functions on $\mathbb{P}^n$, which will make it a variety.

The key observationis that if $F, G \in S$ are homogeneous polynomials of degree d we get a function

$$\begin{aligned} D^+_{\mathbb{P}^n}(G) &\to k \\ [u_0, \ldots, u_n] &\mapsto \frac{F(u_0, \ldots, u_n)}{G(u_0, \ldots, u_n)} \end{aligned}$$

which is well-defined.

Definition 4.1.10. Given a locally closed subset $W \subseteq \mathbb{P}^n$, a *regular function on* W is $f\colon W \to k$ such that for any $p \in W$, there is an open neighborhood $U_p \subseteq W$ of p and homogeneous polynomials $F, G \in S$ of the same degree such that for any $q \in U_p$ we have

$$G(q) \neq 0 \text{ and } f(q) = \frac{F(q)}{G(q)}.$$

We then write

$$\mathcal{O}(W) = \{\text{regular functions on } W\} \subseteq \mathrm{Fun}(W).$$

We note that:

- $\mathcal{O}(W)$ is a k-subalgebra of $\operatorname{Fun}(W)$.
- if $U \subseteq W$ is open, restriction of functions gives $\mathcal{O}(W) \to \mathcal{O}(U)$.
- the presheaf $\mathcal{O}_W \subseteq \operatorname{Fun}_W$ we obtain is in fact a sheaf (by the *local definition* of regular function).

Remark 4.1.11. The sheaf $\mathcal{O}_W$ is the sheaf induced on the locally closed subset W by the sheaf $\mathcal{O}_{\mathbb{P}^n}$.

We will next work towards showing that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is a variety.

Let $(x_i \neq 0) = U_i \subseteq \mathbb{P}^n$ for $0 \leq i \leq n$, an open subset of $\mathbb{P}^n$, and define

$$\begin{aligned} \varphi_i \colon U_i &\to \mathbb{A}^n \\ [u_0, \ldots, u_n] &\mapsto \left(\frac{u_0}{u_i}, \ldots, \frac{u_{i-1}}{u_i}, \frac{u_{i+1}}{u_i}, \ldots, \frac{u_n}{u_i} \right). \end{aligned}$$

This map is bijective with the inverse

$$\begin{aligned} \psi_i \colon \mathbb{A}^n &\to U_i \\ (v_1, \ldots, v_n) &\mapsto [v_1, \ldots, v_i, 1, v_{i+1}, \ldots, v_n]. \end{aligned}$$

Proposition 4.1.12. *The maps φ_i and ψ_i are inverse isomorphisms in* Top_k.

Proof. For simplicity, take $i = 0$. Suppose that $U = D_{\mathbb{A}^n}(f)$ for some $f \in k[x_1, \ldots, x_n]$. Then

$$\varphi_0^{-1}(U) = \left\{ [u_0, \ldots, u_n] \;\middle|\; f\left(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0} \right) \neq 0 \right\}.$$

If $\deg(f) = d$, we can write

$$f\left(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0} \right) = \frac{g(u_0, \ldots, u_n)}{u_0^d},$$

where g is homogeneous of degree d. Then

$$\varphi_0^{-1}(U) = D^+_{\mathbb{P}^n}(x_0 g(x_0, \ldots, x_n))$$

is open, so φ_0 is continuous.

Suppose $V = D^+_{\mathbb{P}^n}(g)$. Then

$$\psi_0^{-1}(V) = \{(v_1, \ldots, v_n) \mid g(1, v_1, \ldots, v_n) \neq 0\} = D_{\mathbb{A}^n}(h),$$

where $h(x_1, \ldots, x_n) = g(1, x_1, \ldots, x_n)$, which is open. Hence ψ_0 is continuous, which shows that φ_0 is a homeomorphism.

It is enough to show that $\alpha \colon U \to k$ on an open set $U \subseteq \mathbb{A}^n$ is regular if and only if $\alpha \circ \varphi_0$ is regular.

If α is regular, for any $p \in U$ we can write

$$\alpha(u_1, \ldots, u_n) = \frac{f(u_1, \ldots, u_n)}{g(u_1, \ldots, u_n)}$$

in some neighborhood of p for some $f, g \in k[x_1, \ldots, x_n]$. Then

$$(\alpha \circ \varphi_0)[u_0, \ldots, u_n] = \frac{f\left(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0}\right)}{g\left(\frac{u_1}{u_0}, \ldots, \frac{u_n}{u_0}\right)} = \frac{u_0^{\deg g} f_1(u_0, \ldots, u_n)}{u_0^{\deg f} g_1(u_0, \ldots, u_n)}$$

for some f_1, g_1 homogeneous with $\deg f_1 = \deg f$, $\deg g_1 = \deg g$. Hence $\alpha \circ \varphi_0$ is regular.

Exercise. Prove the other implication: if $\alpha \circ \varphi_0$ is regular, then α is regular. □

Therefore, $\mathbb{P}^n$ is a prevariety. We show that it is actually a variety.

Proposition 4.1.13. *The prevariety $\mathbb{P}^n$ is separated.*

Proof. We have that $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$ where U_i is affine and open. We recall that by Proposition 3.5.3, we have that $\mathbb{P}^n$ is separated if and only if $U_i \cap U_j$ is affine for all i, j, and the induced map

$$\mathcal{O}(U_i) \otimes_k \mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j) \qquad (*)$$

is surjective. Recall that we have an isomorphism

$$\varphi_i \colon U_i \to \mathbb{A}^n.$$

Suppose $i < j$. Then this induces

$$U_i \cap U_j \xrightarrow{\cong} (x_j \neq 0) \subseteq \mathbb{A}^n,$$

showing that $U_i \cap U_j$ is affine.

Let use write $x_1, \ldots, x_n$ for coordinates on $\varphi_i(U_i)$ and $y_1, \ldots, y_n$ for the coordinates on $\varphi_j(U_j)$.

Exercise. Show that $(*)$ corresponds to a map

$$k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_n] \to k[x_1, \ldots, x_n, x_j^{-1}]$$

such that

$$x_\ell \mapsto x_\ell$$
$$y_{i+1} \mapsto x_j^{-1}.$$

This map is clearly surjective, which completes the proof. □

Altogether, this proves that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is an algebraic variety, and hence for any locally closed subset W of $\mathbb{P}^n$, $(W, \mathcal{O}_W)$ is an algebraic variety.

4.2. **Projective varieties.**

Definition 4.2.1. A variety is *projective* (*quasi-projective*) if it is isomorphic to a closed (locally closed) subvariety of some $\mathbb{P}^n$.

Remark 4.2.2. Since there is an open immersion $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$, quasi-affine varieties are quasi-projective.

Remark 4.2.3. The map $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a morphism. Indeed, it is enough to show that for any i, $0 \leq i \leq n$, if $U_i = D^+_{\mathbb{P}^n}(x_i)$, then $\pi^{-1}(U_i) \to U_i$ is a morphism. This follows from the commutativity of the diagram:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r] \arrow[d, "\cong"] & U_i \arrow[d, "\cong"] \\ \mathbb{A}^{n+1} \setminus V(x_i) \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

$$(x_0, \ldots, x_n) \longrightarrow \left(\tfrac{x_0}{x_i}, \ldots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \ldots, \tfrac{x_n}{x_i}\right)$$

since the bottom map is clearly a morphism.

We will use the following homework problem in the proof of the next proposition.

Remark 4.2.4 (Homework 6, Problem 5)**.** Let X be a prevariety and let $f_1, \ldots, f_r \in \Gamma(X, \mathcal{O}_X)$ be such that the ideal they generate is $\Gamma(X, \mathcal{O}_X)$. If $D_X(f_i)$ is an affine variety for all i, then X is an affine variety.

Proposition 4.2.5. *If $X \subseteq \mathbb{P}^n$ is a closed subvariety and $f \in S_X$ is homogeneous with $\deg f > 0$, then $D_X^+(f) = X \setminus (f = 0)$ is an affine variety.*

Proof. If $\widetilde{f} \in S$ maps to $f \in S_X$, we have that

$$D_X^+(f) = D_{\mathbb{P}^n}^+(\tilde{f}) \cap X.$$

Since X is closed in $\mathbb{P}^n$, it is enough to show the assertion for $\mathbb{P}^n$.

Consider the following regular functions on $D_{\mathbb{P}^n}^+(f)$: $\varphi_0, \ldots, \varphi_n$ such that

$$\varphi_i([u_0, \ldots, u_n]) = \frac{u_i^{\deg f}}{f(u)}.$$

We claim that the functions $\varphi_0, \ldots, \varphi_n$ generate the unit ideal in $\mathcal{O}(D_{\mathbb{P}^n}^+(f))$. Since $f \in (x_0, \ldots, x_n)$, there exists m such that $f^m \in (x_0^d, \ldots, x_n^d)$, where $d = \deg f$. We can hence write

$$f^m = \sum_{i=0}^n h_i x_i^d.$$

Define $\alpha_i \in D_{\mathbb{P}^n}^+(f)$ by

$$\alpha_i([u_0, \ldots, u_n]) = \frac{h_i(u)}{f(u)^{m-1}}.$$

These are regular functions and

$$\sum_{i=0}^n \alpha_i \varphi_i = \sum_{i=0}^n \frac{h_i}{f^{m-1}} \frac{x_i^d}{f} = \frac{\sum_{i=0}^n h_i x_i^d}{f^m} = 1.$$

This shows that $\varphi_0, \ldots, \varphi_n$ generate the unit ideal in $\mathcal{O}(D_{\mathbb{P}^n}^+(f))$.

By Remark 4.2.4, to show that $D_{\mathbb{P}^n}^+(f)$ is affine, we just need to note that $D_{\mathbb{P}^n}(f) \cap D_{\mathbb{P}^n}(x_i) = D_{\mathbb{P}^n}^+(x_i f)$. Indeed, the isomorphism

$$D_{\mathbb{P}^n}^+(x_i) \cong \mathbb{A}^n$$

with coordinates $y_1, \ldots, y_n$ on $\mathbb{A}^n$ induces

$$D_{\mathbb{P}^n}(x_i f) \cong \{f(y_1, \ldots, y_i, 1, y_{i+1}, \ldots, y_n) \neq 0\} \subseteq \mathbb{A}^n,$$

which is a principal affine open subset and we know it is affine. $\square$

Localization in graded rings. If $S = \bigoplus S_m$ is a graded ring and $T \subseteq S$ is a multiplicative system such that every element in T is homogeneous, then $T^{-1}S$ has a natural grading

$$(T^{-1}S)_m = \left\{ \frac{a}{t} \;\middle|\; t \in T, a \in S_{m+\deg t} \right\}.$$

Example 4.2.6. If $f \in S$ is homogeneous, S_f is graded and its degree 0 part is written as $S_{(f)}$.

Example 4.2.7. If $\mathfrak{p} \subseteq S$ is a homogeneous prime ideal, take $T = \{u \in S\backslash\mathfrak{p} \mid u \text{ homogenenous}\}$. Then $T^{-1}S$ is graded and its degree 0 part is written as $S_{(\mathfrak{p})}$.

Fix a closed subvariety $X \subset \mathbb{P}^n$, let S_X be the homogeneous coordinate ring of X and $f \in S_X$ be homogeneous of positive degree. Define

$$\begin{aligned} \Phi \colon (S_X)_{(f)} &\to \mathcal{O}(D_X(f)) \\ \frac{g}{f^m} &\mapsto \left(u = [u_0, \ldots, u_n] \mapsto \frac{g(u)}{f(u)^m} \right) \end{aligned}$$

for g homogeneous of degree $\deg g = m \cdot \deg f$.

Proposition 4.2.8. *The map Φ is an isomorphism.*

Proof. The proof is very similar to the proof of Proposition 1.3.5, and hence it is left as an exercise. $\square$

4.2.1. *Dimension of projective varieties.*

Proposition 4.2.9. *If $\emptyset \neq X \subseteq \mathbb{P}^n$ is a closed subvariety with homogeneous coordinate ring S_X. Then* $\dim X = \dim(S_X) - 1$.

Proof. Suppose first that X is irreducible. Then the diagram

$$\begin{tikzcd} \mathbb{A}^{n+1} \setminus \{0\} \arrow[d, "\pi"] & C(X) \setminus \{0\} \arrow[l, hook'] \arrow[d, "f"] \\ \mathbb{P}^n & X \arrow[l, hook'] \end{tikzcd}$$

commutes. Since $\mathcal{O}(C(X)) = S_X$, we have that

$$\dim(C(X) \setminus \{0\}) = \dim C(X) = \dim(S_X).$$

Since X is irreducible, $C(X)$ is irreducible, and hence $C(X) \setminus \{0\}$ is also irreducible.

The map f is surjective and has 1-dimensional fibers (isomorphic to $\mathbb{A}^1 \setminus \{0\}$). By Theorem 2.4.2, we obtain that

$$\dim(C(X) \setminus \{0\}) = \dim(X) + 1.$$

This completes the proof in the irreducible case.

Generally, if X has irreducible components $X_1, \dots, X_r$, then $C(X)$ has irreducible components $C(X_1), \dots, C(X_r)$. By the irreducible case, $\dim X_i = \dim C(X_i) - 1$, and hence

$$\dim X = \max_i \dim X_i = \max_i \dim C(X_i) - 1 = \dim C(X) - 1$$

by Lemma 2.1.7. □

We will use the following homework problem to prove the next corollary.

Remark 4.2.10 (Homework 5, Problem 3)**.** Suppose X and Y are irreducible closed subsets of $\mathbb{A}^n$. Then any irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.

Corollary 4.2.11. *If $X, Y \subseteq \mathbb{P}^n$ are nonempty closed subsets such that $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$ and every irreducible component of $X \cap Y$ has dimension at least*

$$\dim X + \dim Y - n.$$

Proof. Clearly, $(C(X) \cap C(Y)) \setminus \{0\} = C(X \cap Y) \setminus \{0\}$. Note that

$$0 \in C(X) \cap C(Y),$$

so this set is nonempty. By Remark 4.2.10, every irreducible component of $C(X) \cap C(Y)$ has dimension at least

$$\dim C(X) + \dim C(Y) - (n+1).$$

Then Corollary 4.2.9 implies that any irreducible component of $C(X) \cap C(Y)$ has dimension at least

$$\dim C(X) + \dim C(Y) - (n+1) = \dim X + 1 + \dim Y + 1 - n - 1 \geq 1,$$

Hence $\{0\} \neq C(X) \cap C(Y)$, which shows that $X \cap Y \neq \emptyset$.

We thus showed that $C(X \cap Y) = C(X) \cap C(Y)$. Then any irreducible component Z of $X \cap Y$ gives an irreducible component $C(Z)$ of $C(X) \cap C(Y)$ and

$$\dim Z = \dim C(Z) - 1 \geq \dim X + \dim Y - n + 1 - 1 = \dim X + \dim Y - n$$

by Corollary 4.2.9. □

Grothendieck's philosophy was to work in a relative setting: instead of studying varieties, study varieties over a given variety.

Definition 4.2.12. Given a variety Y, a *variety over* Y is a morphism $f \colon X \to Y$ and a morphism of varieties X_1, X_2 over Y is a commuting triangle

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

This defines a category of varieties over Y. If $Y = \{\mathrm{pt}\}$, this is just the category of all varieties.

Today, we study projective varieties over a fixed affine variety Y. The setting is as follows. Let S be a reduced, $\mathbb{N}$-graded, finitely generated k-algebra. Note that S_0 is a finitely-generated k-algebra and S_m is a finitely-generated S_0-module. We write

$$S_+ = \bigoplus_{m>0} S_m.$$

Exercise. Homogeneous elements $u_1, \ldots, u_n \in S_+$ generate S as an S_0-algebra if and only if they generate S_+ as an ideal.

We always assume S is generated by S_1 as an S_0-algebra, i.e. there exists a surjective graded homomorphism of S_0-algebras

$$S_0[x_0, \ldots, x_n] \twoheadrightarrow S$$

with the standard grading on $S_0[x_0, \ldots, x_n]$.

Since S_0 is a finitely-generated, reduced k-algebra, we can define

$$W_0 = \mathrm{MaxSpec}(S_0),$$

which is an affine variety associated to S_0. Similarly, to S we can associate

$$W = \mathrm{MaxSpec}(S)$$

and the inclusion $S_0 \hookrightarrow S$ given a map $W \to W_0$.

Remark 4.2.13. On Problem Set 6, we showed that given an commutative ring R, there is a natural affine variety $\mathrm{MaxSpec}(R)$ whose points are the maximal ideals in R.

More generally, one would consider $\mathrm{Spec}(R)$, the prime ideals of R, but we restrict our attention to this variety.

We have a k^*-action on W with the morphism $k^* \times W \to W$ corresponding to

$$S \to S[t, t^{-1}]$$
$$f = \sum_i f_i \mapsto \sum_i f_i t^i$$

with $f_i \in S_i$ the homogeneous components.

Choose graded surjective map $S_0[x_0, \ldots, x_n] \twoheadrightarrow S$, which is a morphism of S_0-algebras. This gives a map

$$j \colon W \hookrightarrow \mathrm{MaxSpec}(S_0[x_0, \ldots, x_n]) \cong W_0 \times \mathbb{A}^{n+1},$$

since $S_0[x_0, \ldots, x_n] \cong S_0 \otimes_k k[x_0, \ldots, x_n]$.

We also have the morphism

$$\beta \colon k^* \times W_0 \times \mathbb{A}^{n+1} \to W_0 \times \mathbb{A}^{n+1}$$
$$(\lambda, w_0, x_0, \ldots, x_n) \mapsto (w_0, \lambda x_0, \ldots, \lambda x_n).$$

Then β gives an algebraic group action of k^*. Moreover,

$$j(\alpha(\lambda, w)) = \beta(\lambda, j(w)),$$

which shows that α is a group action as well.

Lemma 4.2.14. *The orbits of the k^*-action on W are either points or 1-dimensional. Moreover:*

(1) *$x \in W$ is fixed by the k^*-action if and only if $x \in V(S_+)$,*
(2) *if O is a 1-dimensional orbit, O is closed in $Q \setminus V(S_+)$, $\overline{O} \cong \mathbb{A}^1$ and $\overline{O} \cap V(S_+)$ is 1 point.*

Proof. Consider $W \hookrightarrow W_0 \times \mathbb{A}^{n+1}$ and note that $V(S_+) = W_0 \times \{0\}$. □

Recall that a subset $Y \subseteq W$ is preserved by the k^*-action if and only if $I_W(Y)$ is homogeneous.

Definition 4.2.15. Given S as before, define the set

$$\operatorname{MaxProj}(S) = \left\{ \begin{array}{c} \text{1-dimensional orbit} \\ \text{closures in } W \end{array} \right\} = \left\{ \begin{array}{c} \text{homogenenous prime ideals } \mathfrak{p} \subseteq S \\ \text{such that } \dim S/\mathfrak{p} = 1 \text{ such that } S_+ \not\subseteq \mathfrak{p} \end{array} \right\}.$$

The topology on this set is defined by declaring that the closets sets are 1-dimensional orbit closure contained in some closed subset of W, preserved by the torus action. These are subsets of the form

$$V(I) = \{\mathfrak{q} \in \operatorname{MaxProj}(S) \mid \mathfrak{q} \supseteq I\}.$$

One can easily check that this defines a topology on $\operatorname{MaxProj}(S)$.

Remark 4.2.16. If $S = k[x_1, \dots, x_n]$, then $\operatorname{MaxProj}(S) = \mathbb{P}^n$.

Remark 4.2.17. We use the notation $\operatorname{MaxProj}(S)$ in analogy to the notation $\operatorname{MaxSpec}(S)$ above. Similarly as before, $\operatorname{Proj}(S)$ would be a similar construction without the restriction to maximal ideals in S.

Note that we have a map

$$\begin{aligned} \operatorname{MaxProj}(S) &\to W_0 \\ \mathfrak{q} &\mapsto \mathfrak{q} \cap S_0 \end{aligned}$$

and it is continuous.

Remark 4.2.18 (Comments about the topology on $X = \operatorname{MaxProj}(S)$)**.** Since every homogeneous ideal $I \subseteq S$ is generated by finitely many homogeneous elements, every open subset in X is a finite union of open subsets of the form

$$D_X^+(f) = \{\mathfrak{q} \in X \mid f \notin \mathfrak{q}\},$$

where f is a homogeneous element of S.

Moreover, we may only take f with positive degree. Suppose that $t_0, \dots, t_n \in S_1$ generate S as an S_0-algebra. Then $S_+ = (t_0, \dots, t_n)$ and hence

$$D_X^+(f) = \bigcup_{i=0}^{n} D_X^+(ft_i).$$

In particular, as in the care of a projective space, we will use the decomposition

$$X = \bigcup_{i=0}^{n} D_X^+(t_i).$$

Exercise. Prove the following version of Nullstellensatz. Suppose $I \subseteq S$ is a homogeneous ideal and $f \in S$ is homogeneous such that $f \in \mathfrak{q}$ for any $\mathfrak{q} \in \operatorname{MaxProj}(S)$ with $\mathfrak{q} \supseteq I$. Then

$$f \cdot S_+ \subseteq I.$$

Moreover, if $\deg(f) > 0$, then $f \in I$.

Recall that if $f \in S$ is a homogeneous element then $S_{(f)}$ is the degree 0 part of S_f, the localization of S at f. Similarly, if $\mathfrak{q} \subseteq S$ is a homogeneous prime ideal, then $S_{(\mathfrak{q})}$ is the degree 0 part of $T^{-1}S$ where $T = \{\text{homogenenous elements in } S \setminus \mathfrak{q}\}$.

Proposition 4.2.19. *Suppose $t \in S_1$.*

(1) *There is an isomorphism of graded rings $S_t \cong S_{(t)}[x, x^{-1}]$.*
(2) *If I is a homogeneous ideal in S_t, then*

$$I = \bigoplus_{m \in \mathbb{Z}} (I \cap S_{(t)}) t^m.$$

(3) *The set $D_X^+(t)$ is homeomorphic to* $\operatorname{MaxSpec}(S_{(t)})$.
(4) *If $\mathfrak{q} \in \operatorname{MaxProj}(S)$, then $S_{(\mathfrak{q})}$ is a local ring with maximal ideal*

$$\left\{ \frac{a}{s} \in S_{(\mathfrak{q})} \;\middle|\; a \in \mathfrak{q} \right\}$$

and residue field k.

Proof. For (1), it is clear that the map

$$\begin{aligned} S_{(t)}[x, x^{-1}] &\to S_t \\ x &\mapsto t \end{aligned}$$

is an isomorphism. Then (2) follows from this isomorphism.

For (3), recall that

$$D_X^+(t) = \{\mathfrak{q} \in \operatorname{MaxProj}(S) \mid t \notin \mathfrak{q}\}.$$

We have a correspondence

$$\left\{ \begin{array}{c} \text{homogeneous prime ideals} \\ \mathfrak{q} \text{ in } S \text{ such that } t \notin S \end{array} \right\} \cong \{\text{homogeneous prime ideals in } S_t\} \cong \{\text{prime ideals in } S_{(t)}\}.$$

with the property that

$$S_t/\mathfrak{q}S_t \cong S_{(t)}/\mathfrak{p}[x, x^{-1}],$$

so $\dim S/\mathfrak{q} = \dim S_t/\mathfrak{q}S_t = \dim S_{(t)}/\mathfrak{p} + 1$. Hence $\dim S/\mathfrak{q} = 1$ if and only if $\mathfrak{p}$ is a maximal ideal. This gives a homeomorphism

$$D_X^+(t) \cong \operatorname{MaxSpec}(S_{(t)}).$$

Finally, for (4), given any $\mathfrak{q} \in \operatorname{MaxProj}(S)$, choose $t \in S_1$ such that $t \notin \mathfrak{q}$. Via the previous correspondence, let $\mathfrak{p}$ correspond to $\mathfrak{q}$. Then

$$S_{(\mathfrak{q})} \supseteq \left\{ \frac{f}{g} \;\middle|\; f \in \mathfrak{q} \right\}$$

is the unique maximal ideal, i.e. this is a local ring. Via the isomorphism

$$S_t/\mathfrak{q}S_t \cong S_{(t)}/\mathfrak{p}[x, x^{-1}]$$

we obtain

$$S_{(\mathfrak{q})}/\text{maximal ideal} \cong S_{(t)}/\mathfrak{p} = k.$$

This completes the proof. $\square$

Define a sheaf of functions on $X = \operatorname{MaxProj}(S)$ as follows: $\varphi\colon U \to k$ is in $\mathcal{O}_X(U)$ if and only if for any $x \in U$, there is an open neighborhood $U_x \subseteq U$ of x and homogeneous elements $f, g \in S$ of the same degree such that $\mathfrak{q} \in U_x$, $g \notin \mathfrak{q}$ and

$$\varphi(\mathfrak{q}) = \text{image of } \frac{f}{g} \in S_{(\mathfrak{q})} \text{ in the residue field.}$$

This gives an object $(X, \mathcal{O}_X) \in \mathrm{Top}_k$.

Properties.

(1) The map

$$\begin{aligned} \operatorname{MaxProj}(S) &\to \operatorname{MaxProj}(S_0) \\ q &\mapsto \mathfrak{q} \cap S_0 \end{aligned}$$

is a morphism in Top_k.

(2) Given a surjective, graded map $T \twoheadrightarrow S$, we get a map

$$\begin{aligned} \operatorname{MaxProj}(S) &\to \operatorname{MaxProj}(T) \\ \mathfrak{q} &\mapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

which is an isomorphism onto a closed subset with the induced sheaf.

(3) For a commutative ring A, we have that

$$\operatorname{MaxProj}(A[x_0, \ldots, x_n]) \cong \operatorname{MaxSpec}(A) \times \mathbb{P}^n$$

such that

$$D_X^+(x_i) \cong \operatorname{MaxSpec}(A[x_0/x_i, \ldots, x_n/x_i])$$

corresponds to

$$\operatorname{MaxSpec}(A) \times D_{\mathbb{P}^n}^+(x_i) \cong \operatorname{MaxSpec}(A) \times \operatorname{MaxSpec}(k[x_0/x_i, \ldots, x_n/x_i]).$$

The above isomorphism glues the isomorphisms

$$\operatorname{MaxSpec}(A[x_0/x_i, \ldots, x_n/x_i]) \cong \operatorname{MaxSpec}(A) \times \operatorname{MaxSpec}(k[x_0/x_i, \ldots, x_n/x_i])$$

on affine charts.

(4) If $f \in S$ is homogeneous of positive degree, $D_X^+(f)$ is affine.

(5) Moreover, the homomorphism $S_{(f)} \to \mathcal{O}(D_X^+(f))$ given by

$$\frac{g}{f^m} \mapsto \left(\mathfrak{q} \mapsto \text{image of } \frac{g}{f^m} \text{in the residue field of } S_{\mathfrak{q}}\right)$$

is an isomorphism.

More details, including proofs of the above facts, are provided in the official notes, but will be skipped here.

5. Classes of morphisms

This chapter will present certain distinguished classes of morphisms.

5.1. **Proper morphisms and complete varieties.** Recall that, topologically, a proper continuous map is one that pulls back compact sets to compact sets. However, as we have seen before, varieties behave differently from regular topological spaces (for example, they are almost never Hausdorff, which is why we introduced the seperatedness condition), so we will need to define this notion differently in this case.

We first want an algebraic analogue for the notion of compactness. To get it, we will argue as we did when we defined sepearatedness as an analogue of the Hausdorff property.

The key point is that given a Hausdorff topological space X, X is compact if and only if for any Hausdorff topological space Y, the projection $X \times Y \to Y$ is closed. There is also a similar characterization of proper continuous maps.

Definition 5.1.1. A variety X is *complete* if for any variety Y the projection $p\colon X \times Y \to Y$ is closed.

More generally, a morphism $f\colon X \to Y$ of algebraic varieties is *proper* if for any morphism $g\colon Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is closed:

$$\begin{array}{ccc} X & \longrightarrow & Y & \qquad \text{proper} \\ \uparrow & & \uparrow & \\ X \times_Y Z & \longrightarrow & Z & \qquad \text{closed.} \end{array}$$

Hence X is complete if and only if $X \to \{*\}$ is proper.

We will eventually prove that $\mathbb{P}^n$ is a complete variety, but we begin with a few elementary properties.

Proposition 5.1.2.

(1) *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both proper, then $g \circ f$ is proper.*
(2) *Given a Cartesian diagram*

$$\begin{tikzcd} X \times_Z Z \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r] & Y, \end{tikzcd}$$

if f is proper, then g is proper. In particular, if f is proper then $f^{-1}(y)$ is complete for all $y \in Y$.
(3) *Closed immersions are proper.*
(4) *If X is complete, then every morphism $f\colon X \to Y$ is proper.*
(5) *If $Y = \bigcup_{i \in I} V_i$ is an open cover and $f\colon X \to Y$ is such that every $f^{-1}(V_i) \to V_i$ is proper, then f is proper.*

Proof. For (1), given $W \to Z$, consider

$$\begin{tikzcd} X \times_Y (Y \times_Z W) \arrow[r, "f'"] \arrow[d] & Y \times_Z W \arrow[r, "g'"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Since the two squares are Cartesian, the rectangle is Cartesian. By hypothesis, f' and g' is closed, so $g' \circ f'$ is closed, so $g \circ f$ is proper.

For (2), given $T \to Z$, consider the diagram

$$\begin{tikzcd} (X \times_Y Z) \times_Z T \arrow[r] \arrow[d, "h"] & X \times_Y Z \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ T \arrow[r] & Z \arrow[r] & Y \end{tikzcd}$$

Since the two squares are Cartesian, so is the rectangle. Hence if f is proper, then h is proper. Since this holds for all $T \to Z$, we conclude that g is proper.

For (3), consider a closed immersion $i \colon X \hookrightarrow Y$ and any morphism $g \colon Z \to Y$. We then have the Cartesian square

$$\begin{tikzcd} \overbrace{X \times_Y Z}^{\cong g^{-1}(i(X))} \arrow[r, "j"] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Since i is a closed immersion, j is a closed immersion, and hence j is closed. This shows i is proper.

For (4), factor f as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \times Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

for $i(x) = (x, f(x))$ and $p(x, y) = y$. Then p is proper since X is complete, by property (2). Since i is a closed immersion (since X, Y are separated), it is proper by (3). Hence $f = p \circ i$ is proper by property (1).

Part (5) is left as an exercise. $\square$

Theorem 5.1.3. *The projective space $\mathbb{P}^n$ is complete.*

Proof. We need to show that for every variety Z, the projection $Z \times \mathbb{P}^n \to Z$ is closed. If $Z = \bigcup_{i=1}^{r} U_i$ is an affine open cover, it is enough to show that each $U_i \times \mathbb{P}^n \to U_i$ is closed. We may hence assume that Z is affine and $Z = \mathrm{MaxSpec}(A)$. The map is then given by

$$Z \times \mathbb{A}^n = \mathrm{MaxProj}(A[x_0, \ldots, x_n]) \to \mathrm{MaxSpec}(A).$$

Choose $V \subseteq Z \times \mathbb{P}^n$ closed, so $V = V(I)$ for some $I \subseteq A[x_0, \ldots, x_n]$ homogeneous. Note that if

$$I' = \{h \in A[x_0, \ldots, x_n] \mid h \cdot (x_0, \ldots, x_n) \subseteq \sqrt{I}\},$$

then clearly $I \subseteq I'$, and in fact $V(I) = V(I')$, since the prime ideals in MaxProj do not contain $(x_0, \ldots, x_n)$.

We will show that $Z \setminus p(V(I))$ is open. Let $\mathfrak{m} \in \mathrm{MaxSpec}(A) \setminus p(V(I))$. Let $U_i = D^+_{Z\times\mathbb{P}^n}(x_i)$, which is an affine variety with ring

$$A[x_0, \ldots, x_n]_{(x_i)} = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

For any i, $\mathfrak{m} \notin p(U_i \cap V(I))$. Note that $V(I) \cap U_i$ is defined by

$$I_{(x_i)} := \left\{ \frac{f}{x_i^m} \;\middle|\; f \in I, f \text{ homogeneous, degree } m \right\}$$

Note that $\mathfrak{m} \notin p(U_i \cap V(I))$ is equivalent to

$$\mathfrak{m} \cdot A[x_0, \ldots, x_n]_{(x_i)} + I_{(x_i)} = A[x_0, \ldots, x_n]_{(x_i)}.$$

By writing $1 \in$ LHS and getting rid of the denominators,

$$x_i^{n_i} \in \mathfrak{m} \cdot A[x_0, \ldots, x_n]_{n_i} + I_{n_i}.$$

If $N \gg 0$, localizing at $\mathfrak{m}$, we obtain

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq \mathfrak{m} \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N + I_N \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N.$$

By Nakayama's Lemma (cf. Remark 2.2.9), we see that

$$A_{\mathfrak{m}}[x_0, \ldots, x_n]_N \subseteq I \cdot A_{\mathfrak{m}}[x_0, \ldots, x_n]_N.$$

Then for some $h \in A \setminus \mathfrak{m}$ we obtain

$$h \cdot (x_0, \ldots, x_n)^N \subseteq I,$$

and hence $h \in I'$.

We claim that $D_Z(h) \subseteq Z \setminus p(V(I'))$. Note that $\mathfrak{m} \in D_Z(h)$, so proving this claim will complete the proof of the Theorem.

To prove the claim, take $\mathfrak{q} \in V(I) = V(I')$, $\mathfrak{q} \subseteq A[x_0, \ldots, x_n]$. Then $p(\mathfrak{q}) = \mathfrak{q} \cap A$. Since $h \in I' \subseteq \mathfrak{q}$, $h \in \mathfrak{q}$. Hence

$$p(V(I)) \cap D(h) = \emptyset,$$

completing the proof. $\square$

Corollary 5.1.4. *If X is a projective variety then X is complete. Therefore, any morphism $X \to Y$ is proper; in particular, it is closed.*

Proof. There is a closed immersion
$$X \hookrightarrow \mathbb{P}^n$$
which is proper by Proposition 5.1.2. Moreover, the map
$$\mathbb{P}^n \to \{\mathrm{pt}\}$$
is proper by Theorem 5.1.3. Hence the composition
$$X \hookrightarrow \mathbb{P}^n \to \{\mathrm{pt}\}$$
is proper by Proposition 5.1.2.

Finally, the other two assertions follow from Proposition 5.1.2. □

Corollary 5.1.5. *If S is a reduced, $\mathbb{N}$-graded k-algebra of finite typed, generated over S_0 by S_1, then*
$$\mathrm{MaxProj}(S) \to \mathrm{MaxSpec}(S_0)$$
is proper.

Proof. Choose a surjective graded morphism
$$S_0[x_0, \ldots, x_n] \twoheadrightarrow S.$$
This gives a closed immersion
$$\mathrm{MaxProj}(S) \hookrightarrow \mathrm{MaxProj}(S_0[x_0, \ldots, x_n]) = \mathrm{MaxSpec}(S_0) \times \mathbb{P}^n,$$
and we obtain the diagram
$$\begin{tikzcd}
\mathrm{MaxProj}(S) \arrow[r, hook, "i"] \arrow[ddr, bend right] & \mathrm{MaxProj}(S_0[x_0, \ldots, x_n]) \arrow[d, "="] & \\
& \mathrm{MaxSpec}(S_0) \times \mathbb{P}^n \arrow[r] \arrow[d, "p"] & \mathbb{P}^n \arrow[d, "g"] \\
& \mathrm{MaxSpec}(S_0) \arrow[r] & \{\mathrm{pr}\}
\end{tikzcd}$$
Since g is proper by Theorem 5.1.3, Proposition 5.1.2 shows that since p and i are proper, the map $p \circ i$ is proper. □

Theorem 5.1.6 (Nagate, Deligne)**.**

- *Any variety X admits an open immersion $i\colon X \hookrightarrow Y$, with Y complete.*
- *More generally, any morphism of varieties $f\colon X \to S$ factors as*
$$\begin{tikzcd}
X \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$
for an open immersion i and a proper morphism g.

The proof of this theorem is difficult and is omitted here.

By Corollary 5.1.4, we know that projective varieties are complete. A natural question to ask is for the converse relation. Chow's Lemma allows to reduce the proof of statements about complete varieties to projective ones.

Theorem 5.1.7 (Chow's Lemma)**.** *If X is a complete variety, there exists $f\colon Y \to X$ such that*

1. *Y is projective,*
2. *f induces an isomorphism between open dense subsets of X and Y.*

In the relative setting, if $g\colon X \to S$ is a proper morphism, there exists $f\colon Y \to X$ such that

1. *The composition $g \circ f$ factors as*

$$\begin{tikzcd} Y \arrow[rr, hook, "\text{closed}", "\text{immersion}"'] \arrow[dr, "g\circ f"'] & & S \times \mathbb{P}^N \arrow[dl, "\mathrm{pr}"] \\ & S & \end{tikzcd}$$

2. *f induces an isomorphism between open dense subsets of X and Y.*

Idea of the proof (in the absolute case).

- Reduce to the case when X is irreducible.
- Cover X by affine open subsets $U_1, \ldots, U_n$ and let $U^* = U_1 \cap \cdots \cap U_n$. Embed U_i in a projective space and take the closed $\overline{U_i}$ to obtain a projective variety $\overline{U_i}$. Then consider 2 locally cloed immersions:

$$\begin{tikzcd} U^* \arrow[r, "\alpha"] & \overline{U_1} \times \cdots \times \overline{U_n} \\ U^* \arrow[u, "="] \arrow[r, "\beta"] & X \times \overline{U_1} \times \cdots \times \overline{U_n} \arrow[u] \end{tikzcd}$$

  Using the Segre embedding (see Homework 8), we know that

$$\overline{U_1} \times \cdots \times \overline{U_n}$$

  is projective as a product of projective varieties. Hence

$$W = \overline{\mathrm{im}(\alpha)}$$

  is projective and letting $Y = \overline{\mathrm{im}(\beta))}$, we have maps

$$h\colon Y \to W$$

$$f\colon Y \to X$$

  The key point is the Y is an isomorphism, so Y is a projective variety.

The proof of this is technical, so it will not be presented here, but one can find it in the official notes. $\square$

5.2. **Finite morphisms.** We now discuss a special case of proper morphisms: finite morphisms.

Definition 5.2.1. A morphism $f\colon X \to Y$ is *affine* if for any affine open subset $V \subseteq Y$, $f^{-1}(V)$ is an affine variety. Moreover, it is *finite* if it is affine and

$$\mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$$

is a finite morphism of k-algebras.

Proposition 5.2.2. *Given $f\colon X \to Y$ and an affine open cover $Y = V_1 \cup \cdots \cup V_r$ such that $f^{-1}(V_i)$ is affine for all i (and $\mathcal{O}(V_i) \to \mathcal{O}(f^{-1}(V_i))$ is finite), then f is affine (respectively, finite).*

Lemma 5.2.3. *If X is any prevariety and $U, V \subseteq X$ are affine open subsets, we can cover $U \cap V$ by principal affine open subsets with respect to both U and V.*

Proof. Given $p \in U \cap V$, choose $p \in W = D_U(f) \subseteq U \cap V$ for some $f \in \mathcal{O}(U)$. Choose $p \in W_1 \subseteq W$, $W_1 = D_V(g)$ for some $g \in \mathcal{O}(V)$. We claim that W_1 is a principal affine open subset of U. Note that

$$g|_W = \frac{h}{f^m}$$

for some $h \in \mathcal{O}(U)$. Then

$$W_1 = D_U(fh),$$

so it is a principal affine open subset. □

Lemma 5.2.4. *Let X be an affine variety, $f_1, \ldots, f_r \in \mathcal{O}(X) = A$, and M be an A-module such that M_{f_i} is a finitely-generated A_{f_i}-module. If $(f_1, \ldots, f_r) = A$ (or equivalently, $X = \bigcap_{i=1}^{r} D_X(f_i)$), then M is finitely-generated.*

Proof. Choose finitely many generators for M_{f_i} over A_{f_i}: $\frac{m_{i,j}}{1}$. Take $N \subseteq M$ generated by all $m_{i,j}$.

Then N is finitely-generated and $N_{f_i} = M_{f_i}$ for all i. Then

$$(M/N))_{f_i} = 0$$

for all i. This implies $M/N = 0$: if $u \in M/N$, for any i, there is an n_i such that $f_i^{n_i} u = 0$, and since $(f_1, \ldots, f_r) = A$, also $(f_1^{n_1}, \ldots, f_r^{n_r}) = A$, so $u = 0$. □

Proof of Proposition 5.2.2. Let $f\colon X \to Y$, $Y = \bigcup_{i=1}^{r} V_i$, $V \subseteq Y$ affine.

If V is a principal affine open subset of one of the V_i, $V = D_{V_i}(\varphi)$, then

$$f^{-1}(V) = D_{f^{-1}(V_i)}(\varphi \circ f)$$

is affine, and

$$\underbrace{\mathcal{O}(V)}_{=\mathcal{O}(V)_\varphi} \to \underbrace{\mathcal{O}(f^{-1}(V))}_{\mathcal{O}(f^{-1}(V))_{\varphi \circ f}}$$

is finite.

Let V be an arbitrary affine open subset. Then

$$V = \bigcup_i (V \cap V_i).$$

By Lemma 5.2.3, we can cover each $V \cap V_i$ by principal affine open subsets with respect to both V and V_i. By the previous argument, the result holds for each of these sets.

Hence we have

$$V = \bigcup_{j=1}^{s} D_V(\varphi_j) \qquad (*)$$

for $\varphi_j \in \mathcal{O}(V)$ such that

$$f^{-1}(D_V(\varphi_j))$$

is affine (and $\mathcal{O}(D_V(\varphi_j)) \to \mathcal{O}(f^{-1}(D_V(\varphi_j)))$ is finite). Note that $(*)$ implies that

$$(\varphi_1 \ldots, \varphi_s) = \mathcal{O}(V)$$

and

$$(\varphi_1 \circ f, \ldots, \varphi_s \circ f) = \mathcal{O}(f^{-1}(V)).$$

Moreover

$$f^{-1}(D_V(\varphi_j)) = D_{f^{-1}(V)}(\varphi_j \circ f).$$

By Remark 4.2.4, this shows that $f^{-1}(V)$ is affine.

Consider $\mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$. We have that

$$\mathcal{O}(V)_{\varphi_i} \to \mathcal{O}(f^{-1}(V))_{\varphi_i}$$

is finite for all i. Lemma 5.2.4 for $\mathcal{O}(f^{-1}(V))$ implies that it is finitely generated over $\mathcal{O}(V)$, completing the proof. $\square$

As a consequence of Proposition 5.2.2, if $f\colon X \to Y$ is a morphism of affine varieties, the new definition agrees with the old one.

All properties of finite morphisms that we discussed for morphisms between affine varieties extend to arbitrary finite morphisms by taking affine open cover of the target. For example:

(1) Finite morphisms have finite fibers.
(2) Finite morphisms are closed.
(3) Finite morphisms are closed under composition and base change: if f is finite and we have a Cartesian square

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

then g is also finite. Indeed, if X, Y, Z are affine and $\mathcal{O}(Y) \to \mathcal{O}(X)$ is finite, then we have the triangle

$$\begin{tikzcd} \mathcal{O}(Y) \otimes_{\mathcal{O}(Y)} \otimes \mathcal{O}(Z) = \mathcal{O}(X) \arrow[r] \arrow[dr, "\text{finite}"'] & \mathcal{O}(X \times_Y Z) \\ & \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \arrow[u, two heads, "\text{quotient by nilradical}"'] \end{tikzcd}$$

Another way to see this is to consider the diagram

$$\begin{tikzcd}
\mathcal{O}(Z) \arrow[r] \arrow[dr, "\text{finite}"'] \arrow[ddr, bend right] & \mathcal{O}(X \times_Y Z) \\
& \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z) \arrow[u, two heads] \\
& \mathcal{O}(X) \otimes_k \mathcal{O}(Z) \arrow[u] \arrow[uu, bend right, two heads]
\end{tikzcd}$$

(4) Closed immersions are finite.

Remark 5.2.5. How to compute $\mathcal{O}(X \times_Y Z)$ for X, Y, Z affine where $Y \subseteq \mathbb{A}^n$ and

$$\begin{tikzcd}
& X \arrow[d, "f=(f_1,\ldots,f_n)"] \\
Z \arrow[r, "h=(h_1,\ldots,h_n)"'] & Y
\end{tikzcd}$$

We have

$$\{(x,z) \mid f(x) = h(z)\} = X \times_Y Z \hookrightarrow X \times Z$$

and hence

$$\begin{aligned}
\mathcal{O}(X \times_Y Z) &= \frac{\mathcal{O}(X \times Z)}{\operatorname{rad}(f_i - h_i)} \\
&= \frac{\mathcal{O}(X) \otimes_k \mathcal{O}(Z)}{\operatorname{rad}(f_i \otimes 1 - 1 \otimes h_i)} \\
&= \frac{\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(Z)}{\text{nilrad}}.
\end{aligned}$$

Another way to see this would be to note that the fibered coproduct in the category of reduced k-algebras is the tensor product.

Remark 5.2.6. Finite morphisms are proper:

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f \text{ finite}"] \\
Z \arrow[r] & Y
\end{tikzcd}$$

and since f is finite, g is finite, and hence closed.

Theorem 5.2.7. *A proper morphism with finite fibers is finite.*

We omit the proof here.

If $f\colon X \to Y$ is finite, then for any Z closed in X, $\dim Z = \dim f(Z)$. Moreover, if Z is irreducible, then $\operatorname{codim}_X Z = \operatorname{codim}_Y f(Z)$. Both of these results follow from the affine case.

5.3. **Flat morphisms.** Next, we consider flat morphisms $f\colon X \to Y$, i.e. morphisms such that for $x \in X$, the k-algebra homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. We start with a review of the underlying commutative algebra.

Recall that if A is a commutative ring and M is an A-module, then

$$- \otimes_A M \text{ is right exact.}$$

We say that M is *flat* if this functor is **exact**.

If $A \to B$ is a ring homomorphism and ${}_A B$ is flat, we say that B is a *flat* A-algebra or simply φ is *flat*.

Examples 5.3.1.

(1) The module $M = A$ is flat over A since $- \otimes_A A = \mathrm{id}$.
(2) A direct sum of flat modules is flat. In particular, free modules are flat (for example, any vector space over a field is flat).
(3) For a filtered set I,
$$\varinjlim_{i \in I} (\text{flat modules}) \text{ is flat.}$$
(4) If $S \subseteq A$ is a multiplicative system, $A \to S^{-1}A$ is flat, since $N \otimes_A S^{-1}A \cong S^{-1}N$.
(5) If M is flat over A and $a \in A$ is non-zero-divisor, the map $A \to A$ given by multiplication by a is injective. Applying $- \otimes_A M$, we see that
$$M \xrightarrow{\cdot a} M \text{ is injective.}$$
In particular, if A is a domain, then M is torsion free.
If A is a PID, the converse also holds: if M is torsion-free, M is a filtered direct limit of finitely-generated submodules (which are free so they are flat), and hence M is flat.

Proposition 5.3.2.

(1) *If ${}_A M$ is flat and $\varphi\colon A \to B$ is a ring homomorphism, then $M \otimes_A B$ is flat over B.*
(2) *If $\varphi\colon A \to B$ is flat and ${}_B M$ is flat, then ${}_A M$ is flat.*
(3) *If $\mathfrak{p}$ prime in A and M is an $A_{\mathfrak{p}}$-module, then ${}_{A_{\mathfrak{p}}} M$ is flat if and only if ${}_A M$ is flat.*
(4) *If $A \to B$ is a ring homomorphism and ${}_B M$, then M is flat over A if and only if for any prime (maximal) ideal $\mathfrak{p} \subseteq B$, $M_{\mathfrak{p}}$ is flat over A.*

Proof. For (1), if N is a B-module, then

$$(M \otimes_A B) \otimes_B N \cong M \otimes_A N.$$

For (2), if N is an A-module, then

$$N \otimes_A M \cong (N \otimes_A B) \otimes_B M,$$

and both $- \otimes_A B$ and $- \otimes_B M$ are exact, so their composition is exact.

The 'only if' implication in (3) follows by (2) since $A \to A_{\mathfrak{p}}$ is flat. Conversely, use the following: if N is an $A_{\mathfrak{p}}$-module, then

$$N \otimes_{A_{\mathfrak{p}}} M \cong N \otimes_A M,$$

which follows from

$$N \otimes_{A_\mathfrak{p}} M \cong \underbrace{(N \otimes_A A_\mathfrak{p})}_{\cong N} \otimes_{A_\mathfrak{p}} M \cong N \otimes_A \underbrace{(A_\mathfrak{p} \otimes_{A_\mathfrak{p}} M)}_{\cong M} \cong N \otimes_A M,$$

since N, M are $A_\mathfrak{p}$-modules.

For (4), if M is a flat A-module and N is an A-module, then

$$N \otimes_A M_\mathfrak{p} \cong (N \otimes_A M) \otimes_A A_\mathfrak{p},$$

so $- \otimes_A M_\mathfrak{p}$ is exact as a composition of exact functors. Conversely, if $M_\mathfrak{p}$ is flat over A for all $\mathfrak{p}$: suppose $N' \hookrightarrow N$ is an injective homomorphism of A-modules, then

$$(N' \otimes_A M)_\mathfrak{p} \hookrightarrow (N \otimes_A M)_\mathfrak{p}$$

is injective for every $\mathfrak{p}$. Therefore, $N' \otimes_A M \to N \otimes_A M$ is injective. □

Remark 5.3.3. To conclude (4), we use the factor that if M is an R-module such that $M_\mathfrak{p} = 0$ for all prime (maximal) ideals $\mathfrak{p}$, then $M = 0$. Applying this to $\ker(f)$ shows that f is injective if and only if $f_\mathfrak{p}$ is injective for all $\mathfrak{p}$.

Remark 5.3.4. Suppose $\varphi\colon A \to B$ is a flat ring homomorphism, $\mathfrak{p} \subseteq A$ is prime, and let $\mathfrak{q} \supseteq \mathfrak{p} \cdot B$ be the minimal prime in B containing $\mathfrak{p}B$. Then $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

To see this, note that by Proposition 5.3.2,

$$A/\mathfrak{p} \xrightarrow{\overline{\varphi}} B/\mathfrak{p}B$$

is flat, so if $\overline{a} \in A/\mathfrak{p} \setminus \{0\}$, $\overline{\varphi}(\overline{a})$ is a non-zero-divisor. But

$$\mathfrak{q}/\mathfrak{p}B \subseteq \{\text{zero divisors of } B/\mathfrak{p}B\},$$

since it is a minimal prime ideal (see Proposition 2.1 in Review Sheet 5). Hence

$$\varphi^{-1}(\mathfrak{q}/\mathfrak{p}B) = \{0\}.$$

Proposition 5.3.5. *Let $f\colon X \to Y$ be a morphism of algebraic varieties. Then the following are equivalent:*

(1) *Given any $U \subseteq X$, $V \subseteq Y$ is an affine open cover such that $f(U) \subseteq V$,*

$$\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$$

is flat.

(2) *There exist affine open covers $X = \bigcup_i U_i$, $Y = \bigcup_i V_i$ such that $f(U_i) \subseteq V_i$ and*

$$\mathcal{O}_Y(V_i) \to \mathcal{O}_X(U_i)$$

is flat.

(3) *For all $x \in X$, the homomorphism*

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is flat.

Definition 5.3.6. A morphism $f\colon X \to Y$ is *flat* if it satisfies any (and all) of the equivalent conditions in Proposition 5.3.5.

Proof of 5.3.5. Suppose $U \subseteq X$, $V \subseteq Y$ affine open such that $f(U) \subseteq V$ and consider

$$A = \mathcal{O}(V) \xrightarrow{\varphi} \mathcal{O}(U) = B.$$

Condition (3) for those $x \in U$ shows that for all maximal ideals $\mathfrak{q}$ in B, if $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is flat. By Proposition 5.3.5 (2), this is equivalent to $A \to B_{\mathfrak{q}}$ being flat for all $\mathfrak{q}$, which is equivalent to $A \to B$ being flat by Proposition 5.3.5 (4). This proves that (2) implies (3) and (3) implies (1), and it is clear that (1) implies (2). ☐

Examples 5.3.7.

(1) Any open immersion is flat.
(2) If X and Y are algebraic varieties, then the projections $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ are flat. To see this, reduce to the case when X and Y are affine. For p, since $\mathcal{O}(Y)$ is flat over k, by Proposition 5.3.2 (1), we see that $\mathcal{O}(X) \to \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is flat.
(3) If $f\colon X \to Y$ is flat and $W \subseteq Y$ is an irreducible closed set with $f^{-1}(W) \neq \emptyset$, $V \subseteq X$ is any irreducible component of $f^{-1}(W)$, then $\overline{f(V)} = W$.

    By choosing affine open subsets in X and Y meeting V and W, we may assume that X and Y are affine. Apply Remark 5.3.4 to $f^{\#}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ and $\mathfrak{p} \subseteq \mathcal{O}(Y)$ corresponding to W, $\mathfrak{q} \subseteq \mathcal{O}(X)$ corresponding to V, to see that $(f^{\#})^{-1}(\mathfrak{q}) = \mathfrak{p}$.
(4) If $f\colon X \to \mathbb{A}^1$ is any morphism, then f if flat if and only if no irreducible component of X is contracted by f.

    The 'only if' implication follows from (3). For the converse implication, reduce to the case when X is affine. Since $k[x]$ is a PID, f is flat if and only if $\mathcal{O}(X)$ is torsion free over $k[x] = \mathcal{O}(\mathbb{A}^1)$. Suppose $u \in k[x] \setminus \{0\}$ such that $f^{\#}(u) \cdot v = 0$ for some $u \in \mathcal{O}(X) \setminus \{0\}$. If X' is an irreducible component of X such that $v|_{X'} \neq 0$, then $f(X') \subseteq \{u = 0\}$, contradiction.

Theorem 5.3.8. *A flat morphism $f\colon X \to Y$ of algebraic varieties is open.*

This generalizes the result that projections are open (cf. Proposition 1.6.5).

We will use the following lemma in the proof.

Lemma 5.3.9. *If Y is a Noetherian topological space, then $W \subseteq Y$ is open if and only if for all closed irreducible subset $Z \subseteq Y$ such that $W \cap Z \neq \emptyset$, W contains an open nonempty subset of Z.*

Proof. The 'only if' implication is clear. For the 'if' implication, argue by Noetherian induction. We may assume that the property holds for all proper closed subsets of Y and show it for Y. We may assume $W \neq \emptyset$. Consider an irreducible decomposition

$$Y = Y_1 \cup \cdots \cup Y_r.$$

Then $W \cap Y_i \neq \emptyset$ for some i. By the assumption, there exists $\emptyset \neq U \subseteq Y_i$ open such that $U \subseteq W$. Replace U by $U \setminus \bigcup_{i \neq j} Y_j$, which is still open in Y.

Consider $W \setminus U \subseteq Y \setminus U$. Note that $Y \setminus U$ is a proper closed subset of Y. If Z is an irreducible closed subset of $Y \setminus U$ such that $(W \setminus U) \cap Z \neq \emptyset$, W contains an open subset of Z, so $W \setminus U$ contains an open subset of Z. By the inductive hypothesis, $W \setminus U$ is open in $Y \setminus U$.

Then $Y \setminus W = (Y \setminus U) \setminus (W \setminus U)$ is closed in $Y \setminus U$, and hence in Y. Hence W is open in Y, completing the proof. $\square$

Proof of Theorem 5.3.8. If $U \subseteq X$ is open, $f|_U : U \to Y$ is flat. Hence it is enough to show that $f(X)$ is open.

Let us show that $f(X)$ satisfies the condition in Lemma 5.3.9. Given $Z \subseteq Y$ closed irreducible such that $f(X) \cap Z \neq \emptyset$, choose V to be an irreducible component of $f^{-1}(Z)$. As we have seen in Example 5.3.7 (3), $\overline{f(V)} = Z$, so $f(V)$ contains an open subset of Z. Then Lemma 5.3.9 implies that $f(X)$ is open. $\square$

Proposition 5.3.10 (Going down for flat homomorphisms)**.** *Let $\varphi\colon A \to B$ be a flat homomorphism. If $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ are prime ideals in A and $\mathfrak{q}_1$ is a prime ideal in B such that $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$, then there is a prime ideal $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ in B such that $\varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$:*

$$\begin{array}{ccc} A & \to & B \\ \mathfrak{p}_1 & & \mathfrak{q}_1 \\ \mathfrak{p}_2 & & \mathfrak{q}_2. \end{array}$$

Proof. We saw that since φ is flat, the induced homomorphism $A_{\mathfrak{p}_1} \to B_{\mathfrak{q}_1}$ is flat. We may hence assume that $(A, \mathfrak{p}_1)$ and $(B, \mathfrak{q}_2)$ are local and φ is a local homomorphism.

Primes in B lying over $\mathfrak{p}_2$ are in bijection with the primes with the primes in

$$B_{\mathfrak{p}_2}/\mathfrak{p}_2 B_{\mathfrak{p}_2} \cong B \otimes_A A_{\mathfrak{p}_2}/\mathfrak{p}_2 A_{\mathfrak{p}_2}.$$

We only need to show that this ring is nonzero.

In fact, for every nonzero module ${}_AM$, the B-module $M \otimes_A B$ is nonzero. Suppose $M \neq 0$ and choose $u \in M$, $u \neq 0$. Since $\mathrm{Ann}_A(u) \neq A$, $\mathrm{Ann}_A(u) \subseteq \mathfrak{p}_1$, a prime ideal. Considering the map $A \to M$ given by $a \mapsto au$, we get an injective map

$$A/\mathrm{Ann}_A(u) \hookrightarrow M,$$

and since B is flat over A, tensoring with B over A gives an injective map

$$A/\mathrm{Ann}_A(u) \otimes_A B \hookrightarrow M \otimes_A B.$$

But since $\mathrm{Ann}_A(u) \cdot B \subseteq \mathfrak{q}_1$, a prime ideal, the quotient $B/\mathrm{Ann}_A(u) \cdot B$ is nonzero. Hence $M \otimes_A B \neq 0$. $\square$

Proposition 5.3.11. *Suppose $\varphi\colon A \to B$ is a ring homomorphism which satisfies the going down property. Let $\mathfrak{q} \subseteq B$ be prime and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then*

$$\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} \leq \dim B_{\mathfrak{q}} - \dim A_{\mathfrak{p}}.$$

(If the dimensions are infinite, this still holds in the form $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} + \dim A_{\mathfrak{p}} \leq \dim B_{\mathfrak{q}}$*.)*

Proof. Let $r = \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ and $s = \dim A_{\mathfrak{p}}$. Then there is a chain of prime ideals in A:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s = \mathfrak{p}$$

and a chain of prime ideals in B:

$$\mathfrak{p}B \subseteq \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_r.$$

Since $\mathfrak{p}B \subseteq \mathfrak{q}_0$, we have that

$$\cdots \subsetneq \mathfrak{p}_s \subsetneq \mathfrak{p} \subseteq \varphi^{-1}(\mathfrak{q}_0).$$

Applying the going down property several primes, we obtain primes in B such that

$$\mathfrak{p}_0' \subsetneq \cdots \subsetneq \mathfrak{p}_s' \subseteq \mathfrak{q}_0$$

such that $\varphi^{-1}(\mathfrak{p}_i') = \mathfrak{p}_i$. The chain

$$\mathfrak{p}_0' \subsetneq \cdots \subsetneq \mathfrak{p}_s' \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_r = \mathfrak{q}$$

shows that $\dim B_{\mathfrak{q}} \geq r + s$. □

Theorem 5.3.12. *If $f\colon X \to Y$ is a flat morphism and $W \subseteq Y$ is an irreducible closed subset such that $W \cap f(X) \neq \emptyset$ and V is an irreducible component of $f^{-1}(W)$, then*

$$\operatorname{codim}_X(V) = \operatorname{codim}_Y(W).$$

For example, if X has pure dimension m and Y has pure dimension n, and if V and W are as in the theorem, then $\dim V = dim W + m - n$. In particular, for any $y \in f(X)$, $f^{-1}(y)$ has pure dimension $m - n$.

Proof of Theorem 2.4.2. Since f is flat, $\overline{f(V)} = W$. The map $\mathcal{O}_{Y,W} \to \mathcal{O}_{X,V}$ is flat, and hence it satisfies the going down property by Proposition 5.3.10. Then

$$\dim \mathcal{O}_{X,V}/\mathfrak{m}_{Y,W}\mathcal{O}_{X,V} = 0,$$

since V is an irreducible component of $f^{-1}(W)$. Hence Proposition 5.3.11 implies that

$$\operatorname{codim}_X(V) = \dim \mathcal{O}_{X,V} \geq \dim \mathcal{O}_{Y,W} = \operatorname{codim}_Y(W).$$

For the reverse inclusion, choose an irreducible closed subset X' of X such that $\operatorname{codim}_X(V) = \operatorname{codim}_{X'}(V)$. Choose Y' to be an irreducible component of Y containing $\overline{f(X')}$. Then by Theorem 2.4.1:

$$\operatorname{codim}_X(V) = \operatorname{codim}_{X'}(V) \leq \operatorname{codim}_{Y'}(W) \leq \operatorname{codim}_Y(W),$$

completing the proof. □

6. Smooth varieties

To introduce smooth varieties, we first have to discuss the tangent space of a variety at a point.

Let X be an algebraic variety. For $x \in X$, we have the local ring $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}_{X,x}$. Zariski's defintion of a tangent space is

$$T_xX = \left(\frac{\mathfrak{m}_{X,x}}{\mathfrak{m}_{X,x}^2}\right)^*,$$

a vector space over $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong k$. We will discuss the geometric description and also show that $\dim T_xX \geq \dim_x X = \dim(\mathcal{O}_{X,x})$. Then X is smooth at x if this is an equality.

6.1. **The tangent space.** If $(R, \mathfrak{m})$ is a Noetherian local ring, then $\mathfrak{m}/\mathfrak{m}^2$ is a finitely-generated vector space over $k = R/\mathfrak{m}$.

By Nakayama's Lemma, $\dim_k \mathfrak{m}/\mathfrak{m}^2$ is the minimal number of generators of $\mathfrak{m}$.

We first consider the case when X is an algebraic variety, $p \in X$, $R = \mathcal{O}_{X,p}$, $\mathfrak{m}_p$ is the maximal ideal, and $R/\mathfrak{m}_p = k$.

Definition 6.1.1 (Zariski)**.** The *tangent space* to X at p is

$$T_pX = (\mathfrak{m}_p/\mathfrak{m}_p^2)^* = \operatorname{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k).$$

If we replace X by an open neighborhood of p, T_pX does not change. Hence to describe this geometrically, we may assume $X \subseteq \mathbb{A}^n$ is a closed subset with ideal I_X.

Proposition 6.1.2. *There is an isomorphism*

$$T_pX = \left\{ (u_1, \dots, u_n) \in k^n \;\middle|\; \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) u_i = 0 \text{ for all } f \in I_X \right\}.$$

Moreover, it is enough to only take f amongst a set of generators of I_X.

Proof. If $p = (a_1, \dots, a_n)$, the ideal of p in $\mathbb{A}^n$ is $(x_1 - a_1, \dots, x_n - a_n)$. We have that

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 = k[x_1, \dots, x_n]/(f_1, \dots, f_r) + (x_1 - a_1, \dots, x_n - a_n)^2,$$

where $I_X = (f_1, \dots, f_r)$. Given any $f \in I_X$, we have

$$f \equiv \underbrace{f(p)}_{=0} + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(x_i - a_i) \mod (x_1 - a_1, \dots, x_n - a_n)^2.$$

Then $m_\mathfrak{p}/m_\mathfrak{p}^2$ is the vector space generated by $e_1, \dots, e_n$, where $e_i = \overline{x_i - a_i}$, with the relations

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) e_i = 0$$

for $f \in I_X$. By taking the dual, we obtain the isomorphism.

Note that if $g = \sum_{j=1}^r g_j f_j$ with $g_j \in k[x_1, \dots, x_n]$, then

$$\sum_{i=1}^r \frac{\partial f}{\partial x_i}(p) u_i = \sum_{j=1}^r g_j(p) \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p) u_i$$

by the product rule and the fact that $f_j(p) = 0$ for all j. This proves the last assertion. □

Terminology. If $X \subseteq \mathbb{A}^n$ is closed, the linear subspace in the proposition is the *embedded tangent space of X at p.*

Functoriality. If $f \colon X \to Y$ is a morphism and $p \in X$, we obtain the morphism

$$\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

mapping the maximal ideal inside the maximal ideal, and hence it gives

$$\mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2.$$

Applying $(-)^*$, we obtain the linear map

$$df_p \colon T_pX \to T_{f(p)}Y.$$

If $g\colon Y \to Z$ is a morphism, then

$$dg_{f(p)} \circ df_p = d(g \circ f)_p.$$

Example 6.1.3. If $i\colon X \hookrightarrow Y$ is a closed immersion, then di_p is injective. If $i\colon X \hookrightarrow Y$ is an open immersion, then di_p is an isomorphism.

Proposition 6.1.4. *Suppose $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ are closed subvarieties and*

$$f = (f_1, \ldots, f_n)\colon X \to Y$$

for $f_i \in k[x_1, \ldots, x_m]$, $1 \leq i \leq n$. Then, under the identification in Proposition 6.1.2, the diagram

$$\begin{tikzcd} T_pX \arrow[r, "df_p"] \arrow[d, hook] & T_{f(p)}Y \arrow[d, hook] \\ k^m \arrow[r, dotted, "\left(\frac{\partial f_i}{\partial x_j}(p)\right)"'] & k^n \end{tikzcd}$$

commutes.

Proof. We have the map

$$\begin{aligned} \mathcal{O}_{Y,f(p)} &\to \mathcal{O}_{X,p} \\ \mathfrak{m}_{f(p)}/\mathfrak{m}_{f(p)}^2 &\to \mathfrak{m}_p/\mathfrak{m}_p^2 \\ y_j &\mapsto f_j(x) \\ y_j - f_j(p) &\mapsto f_j(x) - f_j(p) = \sum_{i=1}^m \frac{\partial f_j}{\partial x_i}(p)(x_i - a_i) \mod (x_1 - a_1, \ldots, x_n - a_n)^2 \end{aligned}$$

where $y_j - f_j(p)$ generate $\mathfrak{m}_{f(p)}$ and $x_i - a_i$ generated $\mathfrak{m}_p$, writing $p = (a_1, \ldots, a_m)$. After applying $(-)^*$, the result follows. $\square$

We present a variant of this for projective varieties. Let $[a_0, \ldots, a_n] = p \in X \subseteq \mathbb{P}^n$ be a closed subvariety and I_X the radical ideal corresponding to X. We define

$$\mathbb{T}_pX = \left\{ [u_0, \ldots, u_n] \in \mathbb{P}^n \;\middle|\; \sum_{i=0}^n \frac{\partial f}{\partial x_i}(a_0, \ldots, a_n)u_i = 0 \text{ for all homogeneous } f \in I_X \right\},$$

which is called the *projective tangent space.*

Note that:

(1) if we rescale the coordinates a_i by $\lambda \in k^*$, the equations get rescaled by a suitable power of λ,

(2) $p \in \mathbb{T}_p X$ by Euler's identity: if f is homogeneous of degree d,

$$\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f.$$

Exercise. Suppose $p \in U_i = (x_i \neq 0) \cong \mathbb{A}^n$. Via this isomorphism, $\mathbb{T}_p X \cap U_i$ is the image of the embedded tangent space to $X \cap U_i$ at p via the translation that maps $0 \mapsto p$.

Note that if $f \in I_X$ and $i = 0$, $g = f(1, x_1, \dots, x_n)$, then

$$\frac{\partial f}{\partial x_i}(1, a_1, \dots, a_n) = \frac{\partial g}{\partial x_i}(a_1, \dots, a_n),$$

$$\frac{\partial f}{\partial x_0}(1, a_1, \dots, a_n) = -\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(a_1, \dots, a_n) a_i$$

by Euler's identity.

Then the equation defining $\mathbb{T}_p X$ becomes

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(a_1, \dots, a_n) u_i - \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(a_1, \dots, a_n) a_i = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(a_1, \dots, a_n)(u_i - a_i).$$

Proposition 6.1.5. *Let $(R, \mathfrak{m})$ be the localization of a k-algebra of finite type at a prime ideal. Then*

$$\dim_K \mathfrak{m}/\mathfrak{m}^2 \geq \dim(R),$$

where $K = R/\mathfrak{m}$.

Proof. Write $R = A_{\mathfrak{p}}$, $\mathfrak{m} = \mathfrak{p} A_{\mathfrak{p}}$, where A is a k-algebra of finite type. Let $I \subseteq A$ be the nilradical. Then consider the quotient

$$R \twoheadrightarrow \overline{R} = \frac{R}{IR}$$

and note that

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \mathfrak{m}/\mathfrak{m}^2 + IR.$$

Then

$$\dim_K \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \leq \dim_K \mathfrak{m}/\mathfrak{m}^2$$

$$\dim R = \dim \overline{R}.$$

It is hence enough to prove the assertion when A is reduced.

Let X be an affine variety such that $\mathcal{O}(x) = A$ and $r = \dim_K \mathfrak{m}/\mathfrak{m}^2$. By Nakayma's Lemma, $\mathfrak{p} A_{\mathfrak{p}}$ is generated by r elements $\frac{b_1}{f_1}, \dots, \frac{b_r}{f_r}$, so there exists $f = f_1 \dots f_r \notin \mathfrak{p}$ such that $\mathfrak{p} A_f \subseteq A_f$ is generated by r elements. Replace A by A_f to assume that $\mathfrak{p} = (a_1, \dots, a_r)$. Then

$$\dim(A_{\mathfrak{p}}) = \operatorname{codim} V(\mathfrak{p}) \leq r,$$

completing the proof. □

Definition 6.1.6. A ring R, which is a localization of a k-algebra at a prime ideal, is *regular* if $\dim(R) = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

6.2. **Smooth points and varieties.**

Definition 6.2.1. If X is an algebraic variety, $x \in X$ is a *smooth point* (or a *nonsingular point*) if $\mathcal{O}_{X,x}$ is regular. Moreover, X is *smooth* (or *nonsingular*) if all points of X are smooth points. More generally, if $V \subseteq X$ is an irreducible closed subset, we say X is *smooth at V* if $\mathcal{O}_{X,V}$ is regular.

We will see that X is smooth at V if and only if there exists $x \in V$ which is a smooth point.

Examples 6.2.2.

(1) Both $\mathbb{A}^n$ and $\mathbb{P}^n$ are smooth varieties.
(2) If $H \subseteq \mathbb{A}^n$ is a hyperplane defined by a radical ideal (F), then for any $p \in H$, $\dim \mathcal{O}_{H,p} = n-1$. Hence $p \in H$ is a smooth point if and only if $\dim T_pH = n-1$. This is equivalent to $\frac{\partial F}{\partial x_i}(p) \neq 0$ for some i.

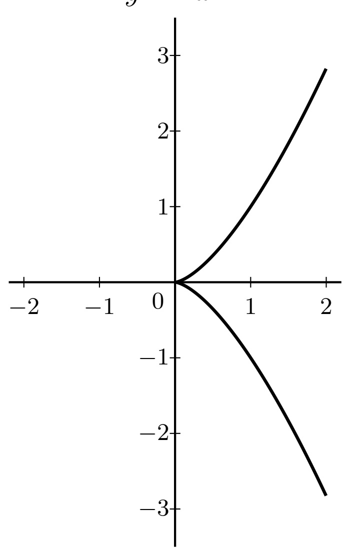
Hence the set of *singular points*[6] of H is defined by the ideal

$$\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right),$$

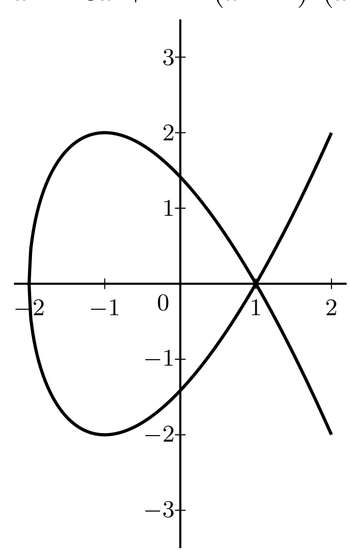
so it is closed.

For example, both $x^2 - y^3 = 0$ has a singular point at $(0,0)$ (this is called a *cuspidal cubic*) and $y^2 - x^3 + 3x - 2 = 0$ has singular points at $(1,0)$ (this is called a *nodal cubic*).

$y^2 = x^3$



$y^2 = x^3 - 3x + 2 = (x-1)^2(x+2)$



Theorem 6.2.3. *If X is an algebraic variety, then*

$$X_{\mathrm{sm}} = \{x \in X \mid x \text{ is a smooth point of } X\}$$

is a nonempty open subset of X.

Assume for now the following proposition. It will only be proved later.

[6]Of course, a point is *singular* if it is not nonsingular.

Proposition 6.2.4. *If $x \in X$ is a smooth point, $\mathcal{O}_{X,x}$ is a domain, i.e. x lies on a unique irreducible component of X.*

This result says that any intersection point of irreducible components is a singular point. Indeed, in the example $y^2 = x^3$ above, $(0,0)$ was in the intersection of two irreducible components of this cubic. However, the converse of this proposition is not true; and indeed $y^2 = x^3 - 3x + 2$ above has a singular point $(1,0)$ which does not lie on the intersection of irreducible components.

Proof of Theorem 6.2.3. Let $X_1, \ldots, X_r$ be the irreducible components of X. Then Proposition 6.2.4, for all $i \neq j$, $X_i \cap X_j \subseteq X \setminus X_{\mathrm{sm}}$, i.e.

$$X \setminus X_{\mathrm{sm}} = \left(\bigcup_{i \neq j} (X_i \cap X_j)\right) \cup \bigcup_{i=1}^{r} (X_i \setminus (X_i)_{\mathrm{sm}}).$$

Hence it is enough to prove the theorem when X is irreducible.

Let $r = \dim X$. By covering X by affine open subsets, we may assume that X is affine. Suppose that $X \subseteq \mathbb{A}^n$ is defined by $I_X = (f_1, \ldots, f_q)$.

For any $p \in X$,

$$T_pX = \left\{ u = (u_1, \ldots, u_n) \in k^n \;\middle|\; \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(p) u_j = 0 \text{ for all } i \right\}$$

by Proposition 6.1.2. Hence

$$\dim T_pX \leq r \text{ if and only if } \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right) \geq n - r.$$

This is an open condition, and hence X_{sm} is open.

To prove $X_{\mathrm{sm}} \neq \emptyset$, we may replace X by any irreducible variety birational to X. By Proposition 1.6.12, we may assume that $X \subseteq \mathbb{A}^{r+1}$ is an irreducible hypersurface. Let F be a generator for I_X, the radical ideal corresponding to X. We saw that $p \in X$ is singular if and only if $\frac{\partial F}{\partial x_i}(p) = 0$ for all i.

Suppose $X_{\mathrm{sm}} = \emptyset$. Then for any $p \in X$, we have that $\frac{\partial F}{\partial x_i}(p) = 0$ for all i, and hence $F \big| \frac{\partial F}{\partial x_i}$ for i. If $d_i = \deg_{x_i} F$, then $\deg_{x_i} \frac{\partial F}{\partial x_i} < d_i$, so $F \big| \frac{\partial F}{\partial x_i}$ if and only if $\frac{\partial F}{\partial x_i} = 0$. Therefore, $\frac{\partial F}{\partial x_i} = 0$ for all i. Hence $\operatorname{char}(k) = p > 0$ and $f \in k[x_1^p, \ldots, x_n^p]$. Since k is a perfect field, $F = G^p$ for some $G \in k[x_1, \ldots, x_n]$, which contradicts the fact that (F) is radical. $\square$

Example 6.2.5. If G is an algebraic group acting algebraically and transitively on a variety X, then X is smooth. Indeed, since there is a smooth point, using the transitivity of the group action, we see that any point is smooth.

In particular, any algebraic group G is smooth, since it acts transitively on itself.

6.3. **Blow ups (of affine varieties).** To prove Proposition 6.2.4, we first need to introduce blow ups. They will also be useful in other settings, so it is an important topic on its own.

The set up is as follows: let X be an affine variety, $A = \mathcal{O}(X)$ and $I \subseteq A$ is an ideal.

Definition 6.3.1. The *Rees algebra* of I is

$$R(A,I)=\bigoplus_{m\geq 0} I^m t^m \subseteq A[t].$$

It is a graded subring of $A[t]$. Note that:

(1) since A is reduced, $R(A,I)$ is reduced,
(2) if A is a domain (i.e. X is irreducible), then $R(A,I)$ is a domain.

Note also that $R(A,I)$ is a finitely generated over k. In fact:

(1) $R(A,I)_0=A$ which is finitely-generated over k,
(2) $R(A,I)$ is generated over A by finitely many elements in $R(A,I)_1$; for example, if $I=(f_1,\ldots,f_r)$, then $f_1t,\ldots,f_rt$ generate $R(A,I)$ over A.

We are now in the setting similar to the one in section 5.1, and we will use the results and notation from that section without further reference. We first make the following definition.

Definition 6.3.2. The *blow-up of X along I* is

$$\widetilde{X}=\mathrm{MaxProj}(R(A,I))$$

together with the natural map

$$\pi\colon \widetilde{X}\to X.$$

Proposition 6.3.3.

(1) *If $Z=V(I)$, then π is an isomorphism over $X\setminus Z$.*
(2) *The set $\pi^{-1}(Z)\subseteq\widetilde{X}$ is locally defined by 1 element which is a non-zero-divisor.*
(3) *If X is irreducible and $I\neq 0$, then π is birational.*
(4) *More generally, if Z does not contain any irreducible component of X, we have a bijection between the irreducible components of $\widetilde{X}$ and irreducible components of X by applying π; each component of $\widetilde{X}$ is birational to the corresponding component of X.*

Proof. For (1), it is enough to show that if $D_X(a)\subseteq X\setminus Z$, then $\pi^{-1}(D_X(a))\to D_X(a)$ is an isomorphism. Note that

$$\pi^{-1}(D_X(a))=\mathrm{MaxProj}(R(A,I)_a).$$

Note also that the square

$$\begin{array}{ccc} A & \longrightarrow & R(A,I) \\ \big\downarrow & & \big\downarrow \\ A_a & \longrightarrow & \underbrace{R(A,I)_a}_{=R(A_a,IA_a)} \end{array}$$

commutes. Since $D_X(a) \subseteq X \setminus Z$, we have that $V(I) = Z \subseteq V(a)$, so $a \in \sqrt{I}$, and hence $IA_a = A_a$. Therefore:

$$\pi^{-1}(D_X(a)) = \operatorname{MaxProj}(A_a[t]) \cong D_X(a) \times \mathbb{P}^0 = D_X(a).$$

This proves (1).

For (2), note that

$$\pi^{-1}(Z) = V(I \cdot R(A, I)).$$

Say $I = (f_1, \ldots, f_r)$. Then $R(A, I)$ is generated over A by $f_1 t, \ldots, f_r t$ and

$$\widetilde{X} = D^+_{\widetilde{X}}(f_1 t) \cup \cdots \cup D^+_{\widetilde{X}}(f_r t).$$

Consider $D^+_{\widetilde{X}}(f_i t) = U_i = \operatorname{MaxSpec}(R(A, I)_{(f_i t)})$. Then

$$\pi^{-1}(Z) \cap U_i \subseteq U_i$$

is defined by $I \cdot R(A, I)_{(f_i t)}$, which is generated by $\dfrac{f_j}{1}$ for all j. However,

$$\frac{f_j}{1} = \frac{f_i}{1} \cdot \frac{f_j t}{f_i t},$$

so $I \cdot R(A, I)_{(f_i t)}$ is generated by $\dfrac{f_i}{1}$. Finally, we note that $\dfrac{f_i}{1}$ is a non-zero-divisor in $R(A, I)_{(f_i t)}$: if $\dfrac{f_i}{1} \dfrac{h}{(f_i t)^m)} = 0$, then $f_i h (f_i t)^p = 0$ for some p, and hence $h \cdot (f_i t)^{p+1} = 0$, so $\dfrac{h}{(f_i t)^m} = 0$.

Since (3) follows from (4), it remains to prove (4). By (1), π is an isomorphism over $X \setminus Z$, so if $X_1, \ldots, X_r$ are irreducible components of X, then each

$$\overline{\pi^{-1}(X_i \setminus Z)}$$

is an irreducible component of $\widetilde{X}$ and

$$\widetilde{X} = \bigcup_{i=1}^{r} \overline{\pi^{-1}(X_i \setminus Z)} \cup \pi^{-1}(Z).$$

By (2), $\pi^{-1}(Z)$ cannot contain any irreducible component of $\widetilde{X}$. Therefore, the irreducible components of $\widetilde{X}$ are the $\overline{\pi^{-1}(X_i \setminus Z)}$. $\square$

Special case. Suppose $I = \mathfrak{m}$ is a maximal ideal defined by a point $a \in X$. Suppose x is not isolated. Then

$$\pi^{-1}(Z) = \operatorname{MaxProj}\left(\frac{R(A, \mathfrak{m})}{\mathfrak{m} R(A, \mathfrak{m})} / \text{nil-rad}\right)$$

and

$$\mathfrak{m} \cdot R(A, \mathfrak{m}) = \bigoplus_{i \geq 0} \mathfrak{m}^{i+1} t^i.$$

We define

$$\frac{R(A, \mathfrak{m})}{\mathfrak{m} R(A, \mathfrak{m})} = \operatorname{gr}_{\mathfrak{m}}(A) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}.$$

Note that in this case the irreducible components of $\widetilde{X}$ meeting $\pi^{-1}(Z)$ correspond to the irreducible components of X containing x. By property (2) and the Principal Ideal Theorem 2.3.1, we see that

$$\dim \pi^{-1}(Z) = \dim(\mathcal{O}_{X,x}) - 1.$$

We also know that

$$\dim \pi^{-1}(Z) = \dim(\mathrm{gr}_{\mathfrak{m}}(A)) - 1.$$

Therefore,

$$\dim \mathcal{O}_{X,x} = \dim(\mathrm{gr}_{\mathfrak{m}}(A))$$

This is also true if x is an isolated point of X, since both of the dimensions are 0.

Remark 6.3.4. For $X = \mathbb{A}^n$, we defined the blow up at 0 as

$$\begin{tikzcd} \{(p,\ell) \mid p \in \ell\} = X' \arrow[r, hook] \arrow[dr] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

and if $x_1, \ldots, x_n$ are coordinates on $\mathbb{A}^n$ and $y_1, \ldots, y_n$ are coordinates on $\mathbb{P}^{n-1}$, then

$$X' = \{x_i y_j - x_j y_j = 0 \text{ for all } i, j\}.$$

We showed in the problem session that X' is irreducible and $X' \to \mathbb{A}^n$ is birational.

We relate this to the blow up $\widetilde{X}$ at the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. We have

$$\widetilde{X} = \mathrm{MaxProj}\left(\bigoplus_{i \geq 0} \mathfrak{m}^i t^i\right)$$

and we have the map

$$\begin{aligned} k[x_1, \ldots, x_n][y_1, \ldots, y_n] &\twoheadrightarrow R(A, \mathfrak{m}) \\ y_i &\mapsto x_i t. \end{aligned}$$

Since all $x_i y_j - x_j y_i$ are contained in the kernel, we have

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[dr] & X' \arrow[d] & \text{closed immersion} \\ & X = \mathbb{A}^n \end{tikzcd}$$

Since both vertical maps are birational, we have that $\widetilde{X} = X'$.

Definition 6.3.5. Let $\pi \colon \widetilde{X} \to X$ be a blow up and $Z = V(I)$. If Y is a closed subvariety of X such that no irreducible component of Y lies inside Z, then

$$\widetilde{Y} = \overline{\pi^{-1}(Y \setminus Z)}$$

is called the *strict transform* (*proper transform*) of Y in $\widetilde{X}$.

We claim that $\widetilde{Y}$ is the blow-up of Y along $J = I \cdot \mathcal{O}(Y)$.

If $B = \mathcal{O}(Y)$, we have the natural map

$$R(A, I) \twoheadrightarrow R(B, J).$$

Then

$$\begin{tikzcd} \widetilde{Y} = \mathrm{MaxProj}(R(B,J)) \arrow[r, hook, "j"] \arrow[d, "\pi_Y"] & \mathrm{MaxProj}(R(A,I)) = \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

We have that $Z \cap Y \hookrightarrow Y$ and Z contains no irreducible component of Y, and hence $\mathrm{im}(j)$ is the strict transform of Y.

We now discuss how to compute strict transforms. For $X = \mathbb{A}^n \ni p$, consider

$$\begin{tikzcd} \mathrm{Bl}_p X = \widetilde{X} \arrow[d, "\pi"] & E = \pi^{-1}(p) \arrow[l, hook'] \arrow[d] \\ X & \{p\}. \arrow[l, hook'] \end{tikzcd}$$

Let $p \in Y$, where Y is a hyperplane with $I(Y) = (f)$ for $f \neq 0$. Then define

$$\mathrm{mult}_p(Y) = \text{largest power of } \mathfrak{m}_p \text{ containing } f.$$

Apply a translation to assume $p = 0$. Then

$$f = f_m(x_1, \dots, x_n) + \cdots + f_d(x_1, \dots, x_n)$$

where f_i is homogeneous of degree i, $f_m \neq 0$ and $d \geq m = \mathrm{mult}_0(\mathrm{Y}) \geq 1$.

What is the strict transform of Y in this case? We want to compute $\pi^{-1}(Y)$. Recall that

$$\widetilde{X} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

is given by $x_i y_j = x_j y_i$ for all i, j. Then

$$U_i = D^+_{\widetilde{X}}(y_i) = (y_i \neq 0).$$

On this set, $x_i = \frac{y_j}{y_i} x_j$ for $j \neq i$, and

$$U_i \cong \mathbb{A}^n \text{ with coordinates } u_1, \dots, u_n$$

and we have the map $\pi \colon U_i \to \mathbb{A}^n$. Then

$$\pi^{\#}(x_i) = u_i,$$

$$\pi^{\#}(x_j) = u_i u_j \text{ for } j \neq i.$$

Recall that $Y = (f = 0)$. Then

$$\pi^{-1}(Y) \cap U_i = \{f(u_1 u_i, \dots, u_i, \dots, u_n u_i) = 0\}.$$

Note that

$$f(u_1 u_i, \dots, u_i, \dots, u_n u_i) = u_i^m (f_m(u_1, \dots, 1, \dots, u_n) + u_i f_{m+1}(\dots) + \cdots).$$

Moreover,

$$\pi^{-1}(\{0\}) = \{0\} \times \mathbb{P}^{n-1}$$

and

$$\pi^{-1}(\{0\}) \cap U_i = (u_i = 0).$$

Since

$$f_m(u_1, \ldots, 1, \ldots, u_n) + u_i f_{m+1}(\ldots) + \cdots$$

does not vanish on $\pi^{-1}(\{0\})$, its zero locus is $\widetilde{Y} \cap U_i$ (in fact, it generates the ideal of $\widetilde{Y} \cap U_i$).

Special case: $n = 2$. We have that

$$f(x_1, x_2) = f_m(x_1, x_2) + \text{higher degree terms.}$$

Note that

$$f_m(x_1, x_2) = \ell_1 \cdot \cdots \cdot \ell_m$$

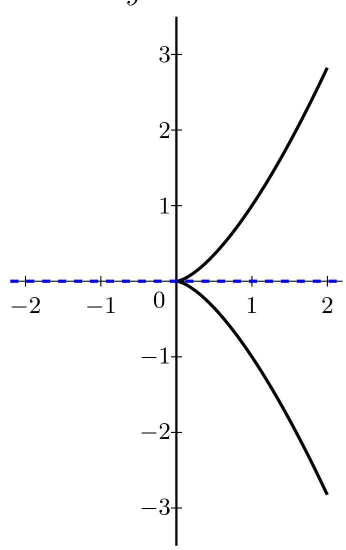
where ℓ_i are linear forms (since $k = \bar{k}$). Indeed:

$$f_m(x_1, x_2) = x_1^m f_m(1 - x_2/x_1) = x_1^m \cdot \prod_{i=1}^{m} \left(\frac{x_2}{x_1} - \lambda_i \right)$$

The forms $\ell_1, \ldots, \ell_m$ define the *tangents* of Y at 0.

Examples 6.3.6. For the cuspidal cubic

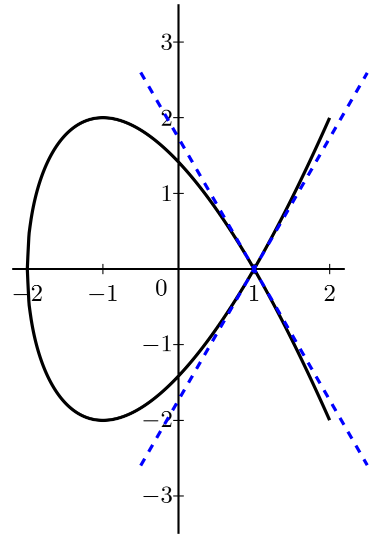
$$y^2 = x^3$$



the unique tangent is $y = 0$.

For the nodal cubic

$$y^2 = x^3 - 3x + 2 = (x-1)^2(x+2)$$



there are two tangents given by $y = \sqrt{3}x + 1$ and $y = -\sqrt{3}x + 1$.[7]

Exercise. Show the the points of $\widetilde{Y} \cap E$, where $E = \mathbb{P}^1$ is parametrized by lines through 0, correspond to presicely to the tangent lines of Y at 0. Show that in these 2 examples (the cusp and the node), $\widetilde{Y}$ is smooth.

6.4. **Back to smooth points.** We still need to prove Proposition 6.2.4. We restate it here for convenience.

Proposition (Proposition 6.2.4)**.** *If $x \in X$ is a smooth point, then $\mathcal{O}_{X,x}$ is a domain.*

We begin with some preparation. Let $x \in X$ is a smooth point, choose $U \ni x$ to be affine open and let

$$A = \mathcal{O}(U) \supseteq \mathfrak{m} \text{ ideal corresponding to } x.$$

If $n = \dim \mathcal{O}_{X,x}$,

$$\text{minimal number of generators for } \mathfrak{m}A_{\mathfrak{m}} = \dim_k \mathfrak{m}/\mathfrak{m}^2 = n.$$

After replacing A by A_f for some $f \notin \mathfrak{m}$, we may assume that $\mathfrak{m} = (f_1, \ldots, f_n)$. Consider

$$\mathrm{gr}_{\mathfrak{m}}(A) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} = k \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$$

which is generated over k by $\overline{f_1}, \ldots, \overline{f_n} \in \mathfrak{m}/\mathfrak{m}^2$.

Hence the map

$$\begin{aligned} \varphi\colon k[x_1, \ldots, x_n] &\to \mathrm{gr}_{\mathfrak{m}}(A) \\ x_i &\mapsto \overline{f_i} \in \mathfrak{m}/\mathfrak{m}^2 \end{aligned}$$

[7]To agree with the above discussion, we should shift the point $(1,0)$ to $(0,0)$ and, indeed, the example considered in class was actually $y^2 = x^2(x+1)$. However, since I already used these graphs for Examples 6.2.2 (because they looked nice), I will stick to the same ones here.

is surjective. We saw in the previous section that

$$n = \dim(\mathcal{O}_{X,x}) = \dim(\mathrm{gr}_{\mathfrak{m}}(A)).$$

Since $k[x_1, \ldots, x_n]$ is a domain of dimension n, $\ker \varphi = 0$.

The conclusion is that if $x \in X$ is a smooth point and X is affine, $\mathcal{O}(X) = A$, $\mathfrak{m}$ is the corresponding maximal ideal, then

$$\mathrm{gr}_{\mathfrak{m}A_{\mathfrak{m}}}(A_{\mathfrak{m}}) = \mathrm{gr}_{\mathfrak{m}}(A) \cong k[x_1, \ldots, x_n].$$

In particular, the left hand side $\mathrm{gr}_{\mathfrak{m}}(A)$ is a domain.

Proof of Proposition 6.2.4. Let $R = \mathcal{O}_{X,x} \supseteq \mathfrak{m}$ be a maximal ideal. Then $\mathrm{gr}_{\mathfrak{m}}(R)$ is a domain by the above discussion.

If R is not a domain, there exist $a, b \in R \setminus \{0\}$ such that $ab = 0$. Since $a, b \neq 0$, by Krull's Intersection Theorem 6.4.1, there exist i, j such that

$$a \in \mathfrak{m}^i \setminus \mathfrak{m}^{i+1},$$

$$b \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}.$$

Then

$$0 \neq \overline{a} \in \mathrm{gr}_{\mathfrak{m}}(R)_i$$

$$0 \neq \overline{b} \in \mathrm{gr}_{\mathfrak{m}}(R)_j$$

so $\overline{a}\overline{b} \neq 0$. But then $0 = ab \notin \mathfrak{m}^{i+j+1}$, which is a contradiction. □

In the proof, we have used the following theorem.

Theorem 6.4.1 (Krull's Intersection Theorem)**.** *If $(R, \mathfrak{m})$ is a local ring, then $\bigcap \mathfrak{m}^i = 0$.*

Proof. This is Theorem 3.1 in Review Sheet #4 and it is proved there. □

Remark 6.4.2. We can use the same method to show that if A is a k-algebra of finite type and $\mathfrak{m} \subseteq A$ is a maximal ideal such that $A_{\mathfrak{m}}$ is regular, then $A_{\mathfrak{m}}$ is a domain.

Indeed, let $I \subseteq A_{\mathfrak{m}}$ be the nilradical and $\overline{A_{\mathfrak{m}}} = A_{\mathfrak{m}}/I$ with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/I$. As before, we have

$$\begin{tikzcd}
k[x_1, \ldots, x_n] \arrow[r, two heads] \arrow[dr, two heads] & \bigoplus\limits_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \arrow[d, two heads] \\
& \underbrace{\bigoplus\limits_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}}_{\dim n}
\end{tikzcd}$$

The composition is an isomorphism for dimension reasons. Hence the first map is injective, and therefore an isomorphism, so $\mathrm{gr}_{\mathfrak{m}}(A_{\mathfrak{m}})$ is a domain. Then we can use the same argument to conclude that $A_{\mathfrak{m}}$ is a domain.

Remark 6.4.3. If $\mathfrak{m} \subseteq A$ is a maximal ideal, then

$$\mathrm{gr}_{\mathfrak{m}}(A) \cong \mathrm{gr}_{\mathfrak{m}A_{\mathfrak{m}}}(A_{\mathfrak{m}}).$$

Indeed,

$$\mathrm{gr}_{\mathfrak{m}A_{\mathfrak{m}}}(A_{\mathfrak{m}}) \cong \bigoplus_{i \geq 0} \underbrace{\frac{\mathfrak{m}^i A_{\mathfrak{m}}}{\mathfrak{m}^{i+1} A_{\mathfrak{m}}}}_{(\mathfrak{m}^i/\mathfrak{m}^{i+1})\mathfrak{m}},$$

since $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is an $A/\mathfrak{m}$-module.

Proposition 6.4.4. *Let X be an algebraic variety, $Y \hookrightarrow X$ be a closed subvariety, and $x \in Y_{\mathrm{sm}}$. Suppose there is an affine open neighborhood U of x such that*

(1) *$I_U(Y \cap U) = (f_1, \ldots, f_r)$ for $f_i \in \mathcal{O}(U)$,*
(2) *$Y \cap U$ is irreducible and*

$$\mathrm{codim}_U(Y \cap U) = r.$$

Then $x \in X_{\mathrm{sm}}$.

Example 6.4.5. If $Y \subseteq X$ and $x \in Y_{\mathrm{sm}}$ is such that for some open affine neighborhood U of x in X, $I_U(Y \cap U) = (f)$ for a function $f \in \mathcal{O}(U)$ which is a non-zero-divisor, then $x \in X_{\mathrm{sm}}$.

Proof of Proposition 6.4.4. We have

$$R = \mathcal{O}_{X,x} \twoheadrightarrow \overline{R} = \mathcal{O}_{Y,x} = R/(f_{1,x}, \ldots, f_{r,x}).$$

Then

$$\overline{\mathfrak{m}} = \mathfrak{m}/(f_{1,x}, \ldots, f_{r,x})$$

so

$$\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \mathfrak{m}/\mathfrak{m}^2 + (f_{1,x}, \ldots, f_{r,x}).$$

Hence

$$\dim(\mathcal{O}_{Y,x}) = \dim_k T_x Y \geq \dim_k T_x X - r.$$

Since

$$\dim(\mathcal{O}_{X,x}) \geq \mathrm{codim}_U(Y \cap U) + \dim \mathcal{O}_{Y,x}$$

we have that

$$\dim(\mathcal{O}_{X,x}) \geq r + \dim_k T_x X - r = \dim_k T_x X.$$

This shows that $x \in X_{\mathrm{sm}}$. $\square$

Corollary 6.4.6. *If X is a variety and $V \subset X$ is an irreducible and closed subset such that X is smooth at V (i.e. $\mathcal{O}_{X,V}$ is regular), then there exists $x \in V$ such that $x \in X_{\mathrm{sm}}$.*

Note that the converse also holds: this is a special case of a theorem by Auslander-Buchsbam and Serre which says that if $(R, \mathfrak{m})$ is regular, then $R_{\mathfrak{p}}$ is regular for any prime ideal $\mathfrak{p}$. A direct proof will be given later, when we discuss the sheaf of differentials.

Proof. Let $\dim \mathcal{O}_{X,V} = r$. By the hypothesis, the maximal ideal in $\mathcal{O}_{X,V}$ is generated by r elements. After replacing X by a suitable U which is affine, open and $U \cap V \neq \emptyset$, we may assume that X is affine and $I_X(V) = (f_1, \ldots, f_r)$. By Proposition 6.4.4, if $x \in Y_{\mathrm{sm}}$, $x \in X_{\mathrm{sm}}$. Since by Theorem 6.2.3, $Y_{\mathrm{sm}} \neq \emptyset$, this completes the proof. $\square$

Example 6.4.7. Let $H \subseteq \mathbb{P}^n$ be a hyperplane and let $X \subseteq H$ be a closed subvariety. Let $p \in \mathbb{P}^n \setminus H$. The *projective cone over* X *with vertex* p is

$$C_p(X) = \bigcup_{x \in X} \{\text{line joining } p \text{ and } x\}.$$

We claim that this is closed in $\mathbb{P}^n$.

Choose coordinates on $\mathbb{P}^n$ such that $H = (x_n = 0)$ and $p = [0, \ldots, 0, 1]$. Then

$$X \hookrightarrow H \cong \mathbb{P}^{n-1}$$

where $[u_0, \ldots, u_{n-1}, 0] \in H$ corresponds to $[u_0, \ldots, u_{n-1}] \in \mathbb{P}^{n-1}$. The line joining p and $x = [a_0, \ldots, a_{n-1}] \in X$ is

$$\overline{px} = [\lambda a_0, \ldots, \lambda a_{n-1}, \mu].$$

Then if $I_X \subseteq k[x_0, \ldots, x_{n-1}]$ is the ideal of X, then $I_X \cdot k[x_0, \ldots, x_n]$ is the ideal of $C_p(X)$.

In particular, note that $C_pX \cap (x_n \neq 0) \subseteq \mathbb{A}^n$ is the affine cone over X.

Finally, we claim that $p \in C_p(X)$ is smooth if and only if X is a linear space.

Exercise. Prove this claim.

Example 6.4.8. Suppose $\mathrm{char}(k) \neq 0$. Let $Q \subset \mathbb{P}^n$ be a *quadric hypersurface*, i.e. $I_Q = (F)$ where F is homogeneous of degree 2. If $x_0, \ldots, x_n$ are the coordinates on $\mathbb{P}^n$, we can write

$$F = \sum_{i,j=0}^{n} a_{i,j} x_i x_j \qquad \text{with } a_{ij} = a_{ji}$$

The rank of Q is $\mathrm{rank}(a_{ij})_{i,j}$. This is independent of the choice of coordinates.

Since $k = \overline{k}$, $\mathrm{char}(k) \neq 2$, after a suitable linear change of variables, we can write

$$F = \sum_{i=0}^{r} x_i^2$$

where $\mathrm{rank}(F) = r + 1$.

Note that this is smooth if and only if $r = n$. Moreover $r \geq 1$ since (F) is radical, and $r \geq 2$ if and only if Q is irreducible.

For example, if $n = 2$, we either have a smooth conic or two intersecting lines.

If $n = 3$, we either have a smooth quadric or a projective cone over smooth conic in $\mathbb{P}^2$ or a union of 2 hypersurfaces.

Exercise. If $X \subseteq \mathbb{P}^3$ is a smooth quadric, the variety of lines on X has two connected components, each isomorphic to $\mathbb{P}^1$.

Note that every smooth quadric in $\mathbb{P}^3$ is isomorphic to the one given by

$$x_0 x_3 = x_1 x_2$$

which is the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. The two families are given by the two copies of $\mathbb{P}^1$.

6.5. **Bertini's Theorem.** Let $(\mathbb{P}^n)^*$ be the parameter space for hyperplaces in $\mathbb{P}^n$.

Terminology: if Z is an irreducible variety, we say that a property holds for a *general point* in Z if there exists an open subset $0 \neq U \subseteq Z$ such that the property holds for all points in U.

Note that if a few properties hold for a general point, then they all hold for a general point simultaneously.

Definition 6.5.1. Given a smooth variety X and two smooth closed subvarieties Y and Z of X, recall that for every $p \in YZ$, we may consider T_pY and T_pY as linear subspaces of T_pX. We say that Y and Z *intersect transversely* if for every $p \in Y \cap Z$, we have

$$\operatorname{codim}_{T_p(X)}(T_pY \cap T_pZ) = \operatorname{codim}_X^p(Y) + \operatorname{codim}_X^p(Z)$$

for $\operatorname{codim}_X^p(Y) = \operatorname{codim}_{X'}(Y')$ where p lies on a unique irreducible components X' and Y' of X and Y.

Note that if $X \subseteq \mathbb{P}^n$ and $H \in (\mathbb{P}^n)^*$, then X and H intersect transversely if and only if for all $p \in X \cap H$, $T_p(X) \subseteq T_p\mathbb{P}^n$ is not contained in T_pH, or in other words for all $p \in X \cap H$, $\mathbb{T}_pX \nsubseteq H$.

Theorem 6.5.2 (Bertini)**.** *If $X \subseteq \mathbb{P}^n$ is a smooth variety, then for a general hypersurface $H \in (\mathbb{P}^n)^*$, H and X intersect transversely. In particular, $X \cap H$ is smooth of dimension* $\dim X - 1$.

Proof. We may assume that X is irreducible. Consider

$$\begin{tikzcd}[column sep=small] & Z = \{(p,H) \in X \times (\mathbb{P}^n)^* \mid \mathbb{T}_pX \subseteq H\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & (\mathbb{P}^n)^* \end{tikzcd}$$

We first claim that Z is a closed subset of $X \times (\mathbb{P}^n)^*$. Let $f_1, \dots, f_r \in I_X$ be generators. If $p \in X$,

$$\mathbb{T}_pX := \left\{ [u_0, \dots, u_n] \;\middle|\; \sum_{j=0}^n \frac{\partial f_i}{\partial x_j}(p) u_j = 0 \text{ for all } i \right\}.$$

If $H = \left(\sum_{j=0}^n a_j u_j = 0 \right)$, $\mathbb{T}_pX \subseteq H$ is equivalent to

$$\operatorname{rank} \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ \frac{\partial f_1}{\partial x_0}(p) & \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_r}{\partial x_0}(p) & \frac{\partial f_r}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix} \leq \left(\frac{\partial f_i}{\partial x_j}(p) \right) = n - \dim X.$$

We get a condition of the form

$$\sum_{j=0}^n a_j g_j(p) = 0$$

where all g_j are homogeneous of the same degree. It is then easy to see that Z is closed in $X \times (\mathbb{P}^n)^*$ by covering X and $(\mathbb{P}^n)^*$ by affine open subsets. In particular, Z is a projective variety.

We now have that

$$\pi_1^{-1}(p) \cong \mathbb{P}^{n-1-\dim X},$$

since $\dim \mathbb{T}_p X = \dim X$. Indeed, suppose

$$\mathbb{T}_p X = \{x_0 = \cdots = x_m = 0\} \text{ where } n - m - 1 = \dim X.$$

Then

$$\mathbb{T}_p X \subseteq \left(\sum a_i x_i = 0\right) \text{ if and only if } a_j = 0 \text{ for } j > m.$$

This is a linear subspace of $(\mathbb{P}_n)^*$ of dimension $m = n - 1 - \dim X$.

By Theorem 2.4.1, we have that

$$\dim Z \leq \dim X + n - 1 - \dim X = n - 1.$$

This shows that $\pi_2(Z)$ is a proper closed subset of $(\mathbb{P}^n)^*$. Hence every hyperplane H such that $[H] \in (\mathbb{P}^n)^* \setminus \pi_2(Z)$ has the property that H and X intersect transversely. Hence $H \cap X$ is smooth of dimension equal to $\dim X - 1$ by Problem 3 on Problem Set 10. $\square$

There are other versions of Bertini's Theorem 6.5.2:

(1) If $X \subseteq \mathbb{P}^n$ is irreducible, $\dim X \geq 2$, then for a general $[H] \in (\mathbb{P}^n)^*$, $X \cap H$ is irreducible.

Proof. Consider $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^{N_d}$, using the dth Veronese embedding. If $H' \subseteq \mathbb{P}^{N_d}$ is the hyperplane corresponding to H, then $\nu_d(X) \cap H' \cong X \cap H$. We can hence apply Bertini's Theorem 6.5.2 for $\nu_d(X)$ to get the result. $\square$

(2) (Due to Kleiman) If $\operatorname{char}(k) = 0$, we get a version in which instead of $X \subseteq \mathbb{P}^n$ we just have a morphism $f \colon X \to \mathbb{P}^n$ and take $f^{-1}(H)$.

Special case. If $X = \mathbb{P}^n$, this implies that a general hypersurface $H \subseteq \mathbb{P}^n$ of degree d is smooth.

Proposition 6.5.3. *If the characteristic of k is bigger than d, then*

$$\left\{ \begin{array}{c} \textit{singular hypersurfaces} \\ \textit{of degree } d \end{array} \right\} \subseteq \left\{ \begin{array}{c} \textit{hypersurfaces of} \\ \textit{degree } d \textit{ in } \mathbb{P}^n \end{array} \right\} = \mathcal{H}_d$$

is an irreducible closed subset of codimension 1.

Proof. Consider

$$\begin{tikzcd}[column sep=small] & \mathcal{Z} = \{(p,H) \mid p \in \mathbb{P}^n, H \in \mathcal{H}_d, P \in H_{\mathrm{sing}}\} \subseteq \mathbb{P}^n \times \mathbb{P}^{N_d} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^n & & \mathcal{H}_d \end{tikzcd}$$

Exericse. Check that $\mathcal{Z}$ is closed in $\mathbb{P}^n \times \mathcal{H}_d$.

We want to find $\alpha^{-1}(p)$. We may assume $p = [1, 0, \ldots, 0]$. Then

$$\alpha^{-1}(p) = \left\{ [F] \in \mathbb{P}^{N_d} \;\middle|\; F(p) = 0,\ \frac{\partial F}{\partial x_i}(p) = 0 \right\}$$

and note that $F(p) = 0$ means that the coefficient of x_0^d is 0 and $\frac{\partial F}{\partial x_i}(p) = 0$ means that the coefficients of $x_0^{d-1}x_i$ are 0.

Let

$$\overline{\mathcal{Z}} = \left\{ (p, [F]) \mid F(p) = 0, \frac{\partial F}{\partial x_i}(p) = 0 \text{ for all } i \right\} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{N_d}.$$

Then fiber over $p \in \mathbb{P}^n$ is a linear subspace of codimension $n+1$ in $\mathbb{P}^{N_d}$.

The map $\overline{\mathcal{Z}} \to \mathbb{P}^n$ is proper and the fibers are irreducible of the same dimension. Therefore, $\overline{\mathcal{Z}}$ is irreducible of dimension $N_d - 1$. Since $\mathcal{Z}$ is open in $\overline{\mathcal{Z}}$, it is irreducible of the same dimension. Therefore, $\beta(\mathcal{Z}) \subseteq \mathcal{H}_d$ is closed and irreducible.

To complete the proof, it is enough to find a 0-dimensional fiber. For example, consider $x_0^d + \cdots + x_{n-1}^d = 0$. Then the singular locus is $[0, \ldots, 0, 1]$.

This shows that β has a 0-dimensional fibers. By Theorem 2.4.1, for any $y \in \beta(\mathcal{Z})$, $\dim \beta^{-1}(y) \geq \dim \mathcal{Z} - \dim \beta(\mathcal{Z})$. If β is not dominant, then every fiber of β has dimension at least 1. $\square$

6.6. **Smooth morphisms between smooth varieties.** Smooth morphisms are the analogue of submersions for differentiable manifolds. In particular, recall that preimages of submersions are submanifolds.

Definition 6.6.1. Let $f : X \to Y$ be a morphism between smooth varieties X and Y. Then f is *smooth* at $x \in X$ if

$$df_x : T_x X \to T_{f(x)} Y$$

is surjective, and f is *smooth* if it is smooth at every $x \in X$.

In general (for varieties which are not smooth), a morphism is smooth if it is flat and the fibers are smooth, but since we did not talk about smoothness of fibers, we postpone this until later.

Example 6.6.2. Consider the map

$$\begin{aligned} f : \mathbb{A}^1 &\to \mathbb{A}^1 \\ t &\mapsto t^n \end{aligned}$$

for $\mathrm{char}(k)$ not dividing n. Then df_t is multiplication by nt^{n-1}:

$$\begin{tikzcd} T_t\mathbb{A}^1 \arrow[r, "df_t"] & T_{f(t)}\mathbb{A}^1 \\ k \arrow[u, "\cong"] \arrow[r, "nt^{n-1}"] & k \arrow[u, "\cong"'] \end{tikzcd}$$

Hence f is smooth at t if and only if $t \neq 0$.

We will next see that for a smooth maps f, every fiber $f^{-1}(y)$ is a smooth variety with $\dim_x(f^{-1}(y)) = \dim_x X - \dim_{f(x)} Y$ for $f(x) = y$.

We have an analog of Sard's Theorem in this case, but we omit the proof here.

Theorem 6.6.3. *If $\operatorname{char}(k) = 0$, then for any dominant $f\colon X \to Y$ with X and Y smooth, there exists an open subset $V \subseteq Y$ such that $f^{-1}(V) \to V$ is smooth. In particular, for all $y \in V$, the fiber $f^{-1}(y)$ is smooth.*

Note that the assumption that $\operatorname{char}(k) = 0$ is essential. For example, the map $\mathbb{A}^1 \to \mathbb{A}^1$, $t \mapsto t^p$ in characteristic $p > 0$ has $df_t = 0$ for all t, so it is not surjective.

Proposition 6.6.4. *If $f\colon X \to Y$ is a smooth morphism between smooth irreducible varieties, then*

- *f is dominant,*
- *for any $y \in f(X)$, the fiber $f^{-1}(y)$ is smooth and has dimension $\dim X - \dim Y$ and*

$$T_x(f^{-1}(y)) = \ker(df_x).$$

Proof. Fix $x \in X$ and let $y = f(x)$. We have

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[drr] & X \arrow[r, "f"] & Y \\ & & \{y\} \arrow[u, hook] \end{tikzcd}$$

and hence $T_x(f^{-1}(y)) \subseteq \ker(df_x)$. Therefore,

$$\dim T_x(f^{-1}(y)) \leq \dim T_x X - \dim T_y Y = \dim X - \dim Y.$$

On the other hand, we know that every irreducible component of $f^{-1}(y)$ has dimension at least $\dim X - \dim Y$ by Theorem 2.4.1 (with strict inequality if f is not dominant). Therefore,

$$\dim \mathcal{O}_{f^{-1}(y)} \geq \dim X - \dim Y \geq \dim T_x(f^{-1}(y)).$$

Since we always have the other inequality 6.1.5, we conclude that $f^{-1}(y)$ is smooth, $T_x(f^{-1}(y)) = \ker(df_x)$, all irreducible components of $f^{-1}(y)$ have dimension equal to $\dim X - \dim Y$, and f is dominant. $\square$

Example 6.6.5. Note that fibers are not always irreducible. Consider the smooth map

$$f\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^{1 \setminus \{0\}}$$

given by $x \mapsto x^2$.

Definition 6.6.6. A morphism $f\colon X \to Y$ of smooth varieties is *étale* at x if f is smooth at x and $\dim_x X = \dim_{f(x)} Y$.

A map is finite and étale if and only if it is a covering space.

6.7. **Resolution of singularities.** We would like to be able to deal with singular points using a variety which is smooth but without losing much information about them. This motivates the following definition.

Definition 6.7.1. Let X be an irreducible algebraic variety. A *resolution of singularities of* X is a proper, birational morphism $f\colon Y \to X$ with Y smooth.

One might require more properties. For example, that f is an isomorphism over X_{sm}, Y is projective if X is projective, etc.

Theorem 6.7.2 (Hironaka)**.** *If* $\mathrm{char}(k) = 0$*, then every irreducible algebraic variety has a resolution of singularities.*

Remark 6.7.3. It is conjectured (and hoped) that this result also holds in characteristic $p > 0$. Hironaka recently posted a paper on his website claiming to have proved it, but it has not been checked as of yet.

The proof of this theorem is difficult and technical, so we omit it here.

One can in fact be more precise with the statement. Assume for simplicity that there is a closed immersion $X \hookrightarrow W$ such that W is smooth. For example, this is always true if X is quasiprojective. Then Hironaka's Theorem 6.7.2 can be stated more precisely by saying that there exists a sequence of blow-ups W_i with smooth centers Z_i

$$\begin{array}{ccc}
Z_r & \hookrightarrow & W_r \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
Z_1 & \hookrightarrow & W_1 \\
\downarrow & & \downarrow \\
Z_0 & \hookrightarrow & W = W_0
\end{array}$$

such that at every step the strict transform of X is not contained in Z_i and the strict transform of X on W_r is smooth.

7. Quasicoherent and coherent sheaves

7.1. **Operations with sheaves.** Fix a commutative ring R and a topological space X. All sheaves and presheaves are of R-modules. (For example, one can think of $R = \mathbb{Z}$ or $R = k$, a field.)

7.1.1. *The sheaf associated to a presheaf.* Recall that we have a functor

$$\mathrm{Sh}_X^R \hookrightarrow \mathrm{Psh}_X^R\,.$$

We construct a left adjoint. For any presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ together with a morphism $\varphi\colon \mathcal{F} \to \mathcal{F}^+$ which satisfies the universal property: for any sheaf $\mathcal{G}$ with a morphism $\psi\colon \mathcal{F} \to \mathcal{G}$ of presheaves, there is a unique morphism of sheaves $\alpha\colon \mathcal{F}^+ \to \mathcal{G}$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \mathcal{F}^+ \arrow[d, dotted, "\alpha"] \\ & \mathcal{G} \end{tikzcd}$$

commute. In other words, φ induces a bijection

$$\mathrm{Hom}_{\mathrm{Sh}}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Psh}}(\mathcal{F}, \mathcal{G}).$$

We call $\mathcal{F}^+$ the *sheafification* of $\mathcal{F}$.

Construction. Define $\mathcal{F}^+(U)$ as the set of maps

$$s\colon U \to \coprod_{x \in U} \mathcal{F}_x$$

such that

(1) for any $x \in X$, $s(x) \in \mathcal{F}_x$,
(2) for any $x \in U$, there exists an open neighborhood $U_x \subseteq U$ of x and $t \in \mathcal{F}(U_x)$ such that $s(y) = t_y$ for all $y \in U_x$.

We check that this $\mathcal{F}^+$ works.

(1) As $\mathcal{F}$ is a presheaf of R-modules, each $\mathcal{F}_x$ is an R-module, so each $\mathcal{F}^+(U)$ is an R-module.
(2) Restriction of functions gives for $V \subseteq U$ the map

$$\mathcal{F}^+(U) \to \mathcal{F}^+(V)$$

which shows that $\mathcal{F}^+$ is a presheaf of R-modules.
(3) It is clear that $\mathcal{F}^+$ is a sheaf, because we deal with functions characterized by a local condition.
(4) Define

$$\begin{aligned} \varphi\colon \mathcal{F} &\to \mathcal{F}^+ \\ \mathcal{F}(U) &\to \mathcal{F}^+(U) \\ s &\mapsto \left(U \ni x \mapsto s_x \in \coprod_{x \in U} \mathcal{F}_x \right). \end{aligned}$$

(5) We check the universal property. Suppose $\psi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves for a sheaf $\mathcal{G}$. Take $s \in \mathcal{F}^+(U)$, $s\colon U \to \coprod_{x \in U} \mathcal{F}_x$. By hypothesis, we can cover U by some U_i open with $t_i \in \mathcal{F}(U_i)$ such that $s(y) = t_{i,y}$ for all $y \in U_i$. Then

$$\psi(t_i) \in \mathcal{G}(U_i)$$

$$\psi(t_i)_y = \psi(t_j)_y \text{ for all } y \in U_i \cap U_j.$$

Since $\mathcal{G}$ is a sheaf and

$$\psi(t_i)|_{U_i \cap U_j} = \psi(t_j)|_{U_i \cap U_j} \text{ for all } i, j,$$

there is a unique $t \in \mathcal{G}(U)$ such that $t|_{U_i} = \psi(t_i)$ for all i.

Define $\alpha(s) = t \in \mathcal{G}(U)$. It is then easy to check that this gives a morphism of sheaves

$$\mathcal{F}^+ \to \mathcal{G}$$

which is the unique one making the diagram comumute.

Properties.

(1) If $\mathcal{F}$ is a sheaf, then $\varphi\colon \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.
(2) This is functorial: if $u\colon \mathcal{F} \to \mathcal{G}$ is a morphism, there is a unique morphism $u^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ making the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "u"] \arrow[d] & \mathcal{G} \arrow[d] \\ \mathcal{F}^+ \arrow[r, "u^+"] & \mathcal{G}^+ \end{tikzcd}$$

commute.
This gives a functor $\mathrm{Psh}_X^R \to \mathrm{Sh}_X^R$.
(3) For any open subset $U \subseteq X$, we an isomorphism

$$(\mathcal{F}^+)_U \cong (\mathcal{F}|_U)^+$$

of sheaves.
(4) The morphism $\mathcal{F} \to \mathcal{F}^+$ induces for any $x \in X$ an isomorphism $\mathcal{F}_x \to \mathcal{F}_x^+$. Indeed, the inverse map is defined as follows: for $x \in U$, $s \in \mathcal{F}^+(U)$, $s\colon U \to \coprod\limits_{y\in U} \mathcal{F}_y$, and the assignment $(U, s) \mapsto s(x) \in \mathcal{F}_x$ gives the inverse of φ_x.

Example 7.1.1. If M is an R-module, we have the constant presheaf M:

$$U \mapsto M$$

for any U. The corresponding sheaf is $\underline{M}$.

If X has the property that for every open set in X is a disjoint union of connected open subsets, then

$$\underline{M}(U) = \{s\colon U \to M \mid s \text{ is constant on connected components of } U\}.$$

7.1.2. *The category* Sh_X^R *is an abelian category.* Note that

- $\mathrm{Hom}_{\mathrm{Sh}_X^R}(\mathcal{F}, \mathcal{G})$ is an R-module,
- composition of morphisms is bilinear,
- given $\mathcal{F}_1, \ldots, \mathcal{F}_n$, define

$$(\mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_n)(U) = \mathcal{F}_1(U) \oplus \cdots \oplus \mathcal{F}_r(U)$$

and it is easy to check this gives a sheaf of R-modules,
- for any $x \in X$, we have

$$\left(\bigoplus_{i=1}^r \mathcal{F}_i\right)_x \cong \bigoplus_{i=1}^r (\mathcal{F}_i)_x,$$

- we have morphisms

$$\mathcal{F}_i \to \bigoplus_{j=1}^{r} \mathcal{F}_j$$

which make $\bigoplus \mathcal{F}_j$ the coproduct of the $\mathcal{F}_j$, and also

$$\bigoplus_{j=1}^{r} \mathcal{F}_j \to \mathcal{F}_i$$

which make the $\bigoplus \mathcal{F}_j$ the product of the $\mathcal{F}_j$.

This makes Sh_X^R into an additive category. To shows that Sh_X^R is abelian, we need to discuss kernels and cokernels.

Kernels. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we may define

$$(\ker(\varphi))(U) = \ker(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)).$$

It is easy to check this is a subsheaf of $\mathcal{F}$. It is the (categorical) kernel of φ: for every morphism $u\colon \mathcal{F}' \to \mathcal{F}$ such that $\varphi \circ u = 0$, there is a unique $v\colon \mathcal{F}' \to \ker(\varphi)$ such that

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "u"] \arrow[dr, dotted, "v"'] & \mathcal{F} \\ & \ker(\varphi) \arrow[u, hook] \end{tikzcd}$$

commutes.

Since filtered direct limits are exact, we see that

$$(\ker(\varphi))_x \cong \ker(\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x).$$

Cokernels. For a map $\varphi\colon \mathcal{F} \to \mathcal{G}$ between sheaves, we define the presheaf

$$\widetilde{\mathrm{coker}(\varphi)}(U) = \mathrm{coker}(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)) = \frac{\mathcal{G}(U)}{\mathrm{im}(\varphi_U)}.$$

Then define

$$\mathrm{coker}(\varphi) = \widetilde{\mathrm{coker}(\varphi)}^{+}.$$

- This is the categorical cokernel of φ: this is clear from the following diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] & \mathcal{G} \arrow[r] \arrow[d] & \widetilde{\mathrm{coker}} \arrow[r] \arrow[dl, dotted, "v_1"'] & \mathrm{coker} = \widetilde{\mathrm{coker}}^{+} \arrow[dll, dotted, "v_2"] \\ & \mathcal{G}' & & \end{tikzcd}$$

where we get v_1 by the universal property of cokernels of R-modules, and v_2 by the universal property of sheafification. Moreover,

$$(\mathrm{coker}(\varphi))_x \cong \mathrm{coker}(\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x)$$

using exactness of filtered direct limits and the fact that $\mathcal{M}_x \cong \mathcal{M}_x^+$ for all presheaves $\mathcal{M}$ and all x.

- If $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, then we define the quotient by
$$\mathcal{F}/\mathcal{F}' = \operatorname{coker}(\mathcal{F}' \hookrightarrow \mathcal{F}).$$
Note that for all $x \in X$, we then have an exact sequence
$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x/\mathcal{F}'_x \longrightarrow 0.$$
- If $\varphi\colon \mathcal{F} \to \mathcal{G}$, define $\operatorname{im}(\varphi) := \ker(\mathcal{G} \to \operatorname{coker}(\varphi))$. In other words,
$$\operatorname{im}(\varphi) = (U \mapsto \operatorname{im}(\varphi_U))^+.$$
Exercise. Show that for any open subset $U \subseteq X$, we have
$$\Gamma(U, \operatorname{im}(\varphi)) = \left\{ s \in \Gamma(U, \mathcal{G}) \;\middle|\; \begin{array}{c} \text{for all } x \in U \text{ there exists an open} \\ \text{neighborhood } U_x \subseteq X \text{ of } x \\ \text{such that } s|_{U_x} \in \operatorname{im}(\mathcal{F}(U_x) \to \mathcal{G}(U_x)) \end{array} \right\}.$$
Note that if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then we have a canonical morphism
$$\mathcal{F}/\ker(\varphi) \to \operatorname{im}(\varphi)$$
induced by the universal properties. This is an isomorphism: for any $x \in X$, the induced morphism is an isomorphism by the following diagram
$$\begin{tikzcd} (\mathcal{F}/\ker(\varphi))_x \arrow[r] \arrow[d, "\cong"] & \operatorname{im}(\varphi)_x \arrow[d, "\cong"] \\ \mathcal{F}_x/\ker(\varphi_x) \arrow[r, "\cong"] & \operatorname{im}(\varphi_x) \end{tikzcd}$$

Altogether, this makes Sh_X^R into an *abelian category*.

Definition 7.1.2. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is *injective* if any of the following equivalent conditions hold:

(1) $\ker(\varphi) = 0$,
(2) $\varphi(U)$ is injective for all U,
(3) φ_x is injective for all x.

If φ is injective, it gives an isomorphism of $\mathcal{F}$ with a subsheaf of $\mathcal{G}$.

Definition 7.1.3. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ is *surjective* if any of the following equivalent conditions hold:

(1) $\operatorname{coker}(\varphi) = 0$,
(2) $\operatorname{im}(\varphi) = \mathcal{G}$,
(3) φ_x is surjective for all x,
(4) φ induces and isomorphism
$$\mathcal{F}/\ker(\mathcal{G}) \to \mathcal{G}.$$

Definition 7.1.4. A sequence
$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$
is *exact* if $\ker(\psi) = \operatorname{im}(\varphi)$ or, equivalently, if $\psi \circ \varphi = 0$ and $\ker(\psi)/\operatorname{im}(\varphi) = 0$.

Such a sequence is exact if and only if

$$\mathcal{F}'_x \xrightarrow{\varphi} \mathcal{F}_x \xrightarrow{\psi} \mathcal{F}''_x$$

is exact for all $x \in X$.

Key point. If $U \subseteq X$ is open, then the functor

$$\Gamma(U, -)\colon \operatorname{Sh}_X^R \to R\text{-mod}$$

is left exact: if

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \xrightarrow{\varphi} \Gamma(U, \mathcal{F}) \xrightarrow{\psi} \Gamma(U, \mathcal{F}'')$$

is exact, but the right map might not be surjective.

Given $\mathcal{F}, \mathcal{G} \in \operatorname{Sh}_X^R$, $\operatorname{Hom}_{\operatorname{Sh}_X^R}(\mathcal{F}, \mathcal{G})$ is an R-module. There exists a sheaf of R-modules:

$$\mathcal{H}om_R(\mathcal{F}, \mathcal{G})$$

given by

$$U \mapsto \operatorname{Hom}_{\operatorname{Sh}_U^R}(\mathcal{F}|_U, \mathcal{G}|_U).$$

If $V \subseteq U$, we get the restriction map

$$\operatorname{Hom}(\mathcal{F}|_U, \mathcal{G}|_U) \to \operatorname{Hom}(\mathcal{F}|_V, \mathcal{G}|_V)$$
$$\varphi \mapsto \varphi|_V.$$

This makes $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ into a presheaf. In fact, it is a sheaf, because morphisms of sheaves can be glued uniquely (see problem session 5).

Note that

$$\Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) = \operatorname{Hom}(\mathcal{F}, \mathcal{G}).$$

Finally, we define functors related to continuous maps between topological spaces. Fix a continuous map $f\colon X \to Y$. Define

$$f_*\colon \operatorname{Sh}_X^R \to \operatorname{Sh}_Y^R$$

by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, the *push-forward functor*.

Note that f_* is left exact: if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact on X, then

$$0 \longrightarrow f_*\mathcal{F}' \xrightarrow{\varphi} f_*\mathcal{F} \xrightarrow{\psi} f_*\mathcal{F}''$$

is exact on Y, since for all $V \subseteq Y$, the sequence

$$0 \longrightarrow \mathcal{F}'(f^{-1}(V)) \xrightarrow{\varphi} \mathcal{F}(f^{-1}(V)) \xrightarrow{\psi} \mathcal{F}''(f^{-1}(V))$$

is exact.

We construct a left adjoint of f_*, the *pull-back* f^{-1}. Given a sheaf $\mathcal{G}$ on Y, define a sheaf $\mathcal{F}$ on X by:

$$X \supseteq U \mapsto \varinjlim_{\substack{V \supseteq f(U) \\ \text{open}}} \mathcal{G}(V),$$

where the open sets $V \supseteq f(U)$ are ordered by reverse inclusion. If $U' \subseteq U$, then for every V such that $f(U) \subseteq V$, we have $f(U') \subseteq V$, and we get a restriction map

$$\mathcal{F}(U) \to \mathcal{F}(U'),$$

which makes $\mathcal{F}$ a presheaf. Then define $f^{-1}\mathcal{G} = \mathcal{F}^+$.

- This is functorial: if $\varphi\colon \mathcal{G} \to \mathcal{G}'$ is a morphism, we get an induced map $\mathcal{F} \to \mathcal{F}'$ which gives a unique map
$$f^{-1}\mathcal{G} = \mathcal{F}^+ \to (\mathcal{F}')^+ = f^{-1}\mathcal{G}.$$
This gives a functor $f^{-1}\colon \mathrm{Sh}_Y^R \to \mathrm{Sh}_X^R$.
- Note that for $x \in X$:
$$(f^{-1}\mathcal{G})_x \cong \mathcal{F}_x = \varinjlim_{U \ni x} \varinjlim_{V \supseteq f(U)} \mathcal{G}(U) = \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}.$$
This implies that $f^{-1}\colon \mathrm{Sh}_Y^R \to \mathrm{Sh}_X^R$ is an exact functor, because it is exact at the level of stalks.

Example 7.1.5. If $\mathcal{F}$ is a sheaf on X, $x \in X$, the inclusion $i_x\colon \{x\} \hookrightarrow X$ gives

$$i_x^{-1}(\mathcal{F}) = \mathcal{F}_x.$$

Proposition 7.1.6. *The pair of functors (f^{-1}, f_*) is an adjoint pair, i.e. we have an natural isomorphism*

$$\mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_*\mathcal{F})$$

for all $\mathcal{F} \in \mathrm{Sh}_X^R$, $\mathcal{G} \in \mathrm{Sh}_Y^R$.

Proof. By the universal property of sheafification, giving a morphism $f^{-1}\mathcal{G} \to \mathcal{F}$ is equivalent to giving a morphism

$$\varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \to \mathcal{F}(U)$$

for any open subset $U \subseteq X$, compatible with restrictions. By the universal property of direct limits, this is equivalent to giving a morphism

$$\mathcal{G}(V) \to \mathcal{F}(U)$$

for any open subsets $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, compatible with restrictions. Finally, $f(U) \subseteq V$ if and only if $U \subseteq f^{-1}(V)$, and hence, by compatibility with restrictions, this is equivalent to giving for all $V \subseteq Y$ a morphism

$$\mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$$

which is compatible with restrictions. This is the same as giving a morphism $\mathcal{G} \to f_*\mathcal{F}$. $\square$

7.2. **$\mathcal{O}_X$-modules.** Fix a ringed space $(X, \mathcal{O}_X)$, i.e. X is topological space and $\mathcal{O}_X$ is a sheaf of rings on X. For example, we can take X to be an algebraic variety and $\mathcal{O}_X$ to be the sheaf of regular functions on X.

Definition 7.2.1. A *presheaf (sheaf) of $\mathcal{O}_X$-modules* is a presheaf (sheaf) $\mathcal{M}$ of sbelian groups such that for any $U \subseteq X$ open, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-modules, and these structures are compatible with restriction maps. Explicitely, if $V \subseteq U$, $a \in \mathcal{O}_X(U)$, $s \in \mathcal{M}(U)$, we have that

$$(a \cdot s)|_V = a|_V \cdot s|_V.$$

Definition 7.2.2. A *morphism of preseaves of $\mathcal{O}_X$-modules* $\mathcal{M} \to \mathcal{N}$ is a morphism of presheaves of abelian groups such that for any $U \subseteq X$ open, $\mathcal{M}(U) \to \mathcal{N}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

Example 7.2.3.

(1) The sheaf $\mathcal{O}_X$ is naturally and $\mathcal{O}_X$-module.
(2) Giving a sheaf of $\mathcal{O}_X$-modules is equivalent to giving a sheaf of abelian groups $\mathcal{M}$ together with a morphism

$$\mathcal{O}_X \to \mathcal{H}om_{\mathbb{Z}}(\mathcal{M}, \mathcal{M}).$$

This implies that if $\mathcal{O} = \underline{R}$ for some ring R, then

$$\{\text{sheaves of } \mathcal{O}_X\text{-modules}\} \cong \{\text{sheaves of } R\text{-modules}\}.$$

Remark 7.2.4. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $\mathcal{M}$ is a sheaf of $\mathcal{O}_X(X)$-modules. Indeed, $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module, and we have a map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$, which makes $\mathcal{M}(U)$ into an $\mathcal{O}_X(X)$-module by restriction of scalars. We thus get a functor

$$\mathcal{O}_X\text{-mod} \to \mathrm{Sh}_X^{\mathcal{O}_X(X)},$$

where $\mathcal{O}_X$-mod is the category of sheaves of $\mathcal{O}_X$-modules.

Remark 7.2.5. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, $\mathcal{F}_x$ is naturally and $\mathcal{O}_{X,x}$-module.

Remark 7.2.6. If $\mathcal{F}$ is a presheaf of $\mathcal{O}_X$-modules, $\mathcal{F}^+$ is naturally a sheaf of $\mathcal{O}_X$-modules such that $\mathcal{F} \to \mathcal{F}^+$ is a morphism of presheaves of $\mathcal{O}_X$-modules. Indeed, define

$$\mathcal{F}^+(U) = \left\{ s \colon U \to \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \text{old conditions and an extra condition} \right\}$$

where the extra condition is: if $a \in \mathcal{O}_X(U)$, $s \in \mathcal{F}^+(U)$, then

$$(as)_{(x)} = \underbrace{a_x}_{\in \mathcal{O}_{X,x}} s(x).$$

Then $\mathcal{F}^+$ will satisfy the same universal proprety with respect to the morphism $\mathcal{F}$, so it is a sheaf of $\mathcal{O}_X$-modules.

Examples 7.2.7.

(1) If $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are sheaves of $\mathcal{O}_X$-modules, then $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ has a natural structure of $\mathcal{O}_X$-modules, which makes it both a product and a coproduct in the category of $\mathcal{O}_X$-modules.

(2) If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then $\ker(\varphi)$, $\operatorname{coker}(\varphi)$, $\operatorname{im}(\varphi)$ carry natural $\mathcal{O}_X$-module structures, where $\ker(\varphi)$ is the kernel of φ in $\mathcal{O}_X$-mod etc.

Altogether, this makes $\mathcal{O}_X$-mod an abelian category.

For an infinite family $(\mathcal{M}_i)_{i\in I}$, we consider the product and direct sum. The product

$$U \mapsto \prod \mathcal{M}_i(U)$$

is an $\mathcal{O}_X$-module, but its stalks are tricky to describe. The direct sum

$$U \mapsto \bigoplus \mathcal{M}_i(U)$$

is only a presheaf, but

$$(U \mapsto \bigoplus \mathcal{M}_i(U))^+$$

is an $\mathcal{O}_X$-module, and its stalks are easy to describe:

$$\Big(\bigoplus \mathcal{M}_i\Big)_x = \bigoplus (\mathcal{M}_i)_x.$$

Multilinear constructions. All of $\otimes$, Sym^p, $\bigwedge^p$ have analogues for $\mathcal{O}_X$-modules. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{O}_X$-modules, then we may set

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

and for $V \subseteq U$ we get a natural map

$$\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U) \to \mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{N}(V).$$

This defines a presheaf and its sheafification is

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}.$$

It is useful to note that

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x \cong \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x,$$

since $\otimes$ commutes with $\varinjlim$ and taking the sheafification does not change stalks.

Similarly, we define

$$\mathrm{Sym}^p(\mathcal{M}) = \Big(U \mapsto \mathrm{Sym}^p_{\mathcal{O}_X(U)} \mathcal{M}(U)\Big)^+,$$

$$\bigwedge^p \mathcal{M} = \left(U \mapsto \bigwedge^p_{\mathcal{O}_X(U)} \mathcal{M}(U)\right)^+,$$

both of which behave nicely with stalks.

Example 7.2.8. An $\mathcal{O}_X$-module $\mathcal{M}$ is locally free (of finite rank) if for all $x \in X$, there is an open set $U \ni x$ such that $\mathcal{M}|_U \cong \mathcal{O}_U^{\oplus n}$ for some n. If n is independent of x, we say that $\mathcal{M}$ is *locally free* of rank n.

Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}(X)$ and hence

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}.$$

Behavior with respect to morphisms of ringed spaces. A morphism of ringed spaces is

$$(X, \mathcal{O}_X) \xrightarrow{(f,f^{\#})} (Y, \mathcal{O}_Y)$$

where $f\colon X \to Y$ is a continuous map and $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

The main example to have in mind is a morphism of algebraic varieties $f\colon X \to Y$. In this case, $f^{\#}$ is determined by f, for $\varphi \in \mathcal{O}_Y(V)$, we get $\varphi \circ f \in \mathcal{O}_X(f^{-1}(V))$ and this gives the morphism

$$\mathcal{O}_Y \to f_*\mathcal{O}_X.$$

Ringed spaces form a category with composition given by

$$\begin{tikzcd} (X, \mathcal{O}_X) \arrow[r, "(f,f^{\#})"] \arrow[rr, bend right, "((g\circ f),(f^{\#}\circ g^{\#}))"'] & (Y, \mathcal{O}_Y) \arrow[r, "(g,g^{\#})"] & (Z, \mathcal{O}_Z) \end{tikzcd}$$

since the map $\mathcal{O}_Z \to (g \circ f)_*\mathcal{O}_X$ is given by the composition

$$\begin{tikzcd} \mathcal{O}_Z(W) \arrow[rr, "(g\circ f)^{\#}"] \arrow[dr, "g^{\#}"'] & & \mathcal{O}_X(f^{-1}(g^{-1}(W))) \\ & \mathcal{O}_Y(f^{-1}(W)) \arrow[ur, "f^{\#}"'] & \end{tikzcd}$$

Suppose $\mathcal{M}$ is an $\mathcal{O}_X$-module. Then $f_*\mathcal{M}$ is an $f_*\mathcal{O}_X$-module, so it is an $\mathcal{O}_Y$-module via restriction of scalars for $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

We want a left adjoint to f_*: if $\mathcal{N}$ is an $\mathcal{O}_Y$-module, $f^{-1}\mathcal{N}$ is an $f^{-1}\mathcal{O}_Y$-module, where $f^{-1}\mathcal{O}_Y$ is a sheaf of rings on X, and $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ corresponds to $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by adjointness.

Define

$$f^*\mathcal{N} = f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

which has a natural structure of an $\mathcal{O}_X$-module.

We have that

$$(f^*\mathcal{N})_x \cong (f^{-1}\mathcal{N})_x \otimes_{(f^{-1}\mathcal{O}_Y)_x} \mathcal{O}_{X,x} \cong \mathcal{N}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x},$$

where $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is obtained by taking the stalk of $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. Explicitely,

$$(V \ni f(x), \varphi \in \mathcal{O}_Y(V)) \mapsto (f^{-1}(V), f^{\#}(\varphi)).$$

As a consequence, we see that f^* is right exact.

Example 7.2.9. If $f\colon X \to Y$ is a morphism of algebraic varieties and f is flat, then f^* is exact, since $\mathcal{O}_{X,x}$ is a flat module over $\mathcal{O}_{Y,f(x)}$ for any $x \in X$.

Example 7.2.10. Suppose $U \subseteq X$ is open. Then $\mathcal{O}_U = \mathcal{O}_X|U$, $i\colon U \hookrightarrow X$ is a morphism of ringed spaces and we have

$$\begin{gathered}\mathcal{O}_X \to i_*\mathcal{O}_U\\ \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)\\ s \mapsto s|_{U\cap V}.\end{gathered}$$

Note that $i^{-1}\mathcal{M} \cong \mathcal{M}|_U$ and $i^{-1}\mathcal{O}_X \cong \mathcal{O}_U$. Therefore,

$$i^*\mathcal{M} \cong \mathcal{M}|_U.$$

Exercise. Show that f^* is compatible with $\otimes$, Sym^p, $\bigwedge^p$. For example, for $\otimes$, if $f\colon X \to Y$ and $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}_Y$-modules, then there is a canonical isomorphism

$$f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}) \cong f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}.$$

Hint. Use universal property or construct a morphism and show it induces and isomorphism on stalks.

Remark 7.2.11. If

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z)$$

are morphisms of ringed spaces, then

$$(g_* \circ f_*)(\mathcal{M}) = g_*(f_*(\mathcal{M}))$$

by definition, and

$$f^*(g^*(\mathcal{N})) \cong (g \circ f)^*(\mathcal{N}),$$

since both sides give a left adjoint to $g_* \circ f_*$.

Remark 7.2.12. If $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \text{ is an } \mathcal{O}_X(X)\text{-module},$$

so

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \text{ is an } \mathcal{O}_X\text{-module}.$$

7.3. **Quasicoherent sheaves on affine varieties.** The goal is to define a functor

$$\mathcal{O}_X(X)\text{-mod} \to \mathcal{O}_X\text{-mod}$$

for an affine variety X.

We first need a few preliminary notions. Suppose $(X, \mathcal{O}_X)$ is a ringed space and $P \subseteq \{\text{open subsets of } X\}$ such that

(1) P is a basis for the topology on X: every open subset $U \subseteq X$ is a union of elements of P,
(2) if $U, V \in P$, then $U \cap V \in P$.

The main example are principal affine open subsets of an affine variety.

Definition 7.3.1. A *P-sheaf of $\mathcal{O}_X$-modules* is

$$P \ni U \mapsto \mathcal{M}(U) \text{ an } \mathcal{O}_X(U)\text{-module},$$

with restruction maps $\mathcal{M}(U) \to \mathcal{M}(V)$ for $V \subseteq U$ satisfying the obvious compatibility conditions and the *sheaf axiom for covers of subsets in P by subsets in P*: for $U = \bigcup_{i\in I} U_i$ for $U, U_i \in P$, we get the sheaf condition.

Define similarly a *morphism of P-sheaves.*

Proposition 7.3.2. *The forgetful functor*

$$\left\{ \begin{array}{c} \textit{sheaves of} \\ \mathcal{O}_X\textit{-modules} \end{array} \right\} \to \left\{ \begin{array}{c} P\textit{-sheaves of} \\ \mathcal{O}_X\textit{-modules} \end{array} \right\}$$

is an equivalent of categories.

Sketch of the proof. Define the inverse functor by sending a P-sheaf α to the sheaf F_* given by

$$\mathcal{F}_\alpha(U) = \ker\left(\prod_{\substack{V\subseteq U \\ V\in P}} \alpha(V) \to \prod_{\substack{V_1,V_2\subseteq U \\ V_1,V_2\in P}} \alpha(V_1\cap V_2) \right)$$

where the map is given by

$$(s_v)_v \mapsto (s_{v_1}|_{V_1\cap V_2} - s_{v_2}|_{V_1\cap V_2}).$$

Then check that this gives an inverse to the forgetful functor. $\square$

For an affine variety X, we mentioned that

$$P = \{\text{principal affine open subsets of } X\}$$

satisfies propeties (1) and (2). Let $A = \mathcal{O}_X(X)$ and M be an A-module. We want to define an $\mathcal{O}_X$-module $\widetilde{M}$ on X such that

$$\Gamma(D_X(f), \widetilde{M}) = M_f = M \otimes_A A_f.$$

We need to check that this is well-defined. If $D_X(g) \subseteq D_X(f)$, then $V(f) \subseteq V(g)$ so $g \in \sqrt{(f)}$, and by the universal property of localization, we have a canonical morphism $A_f \to A_g$ making the triangle

$$\begin{tikzcd} A_f \arrow[rr] & & A_g \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commute. Therefore:

(1) M_f only depends on $D_X(f)$ (up to canonical isomorphism),
(2) if $D_X(g) \subseteq D_X(f)$, we have a restriction map $M_f \to M_g$ and these are compatible.

By Proposition 7.3.2, to get a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$, we only need to show the following lemma.

Lemma 7.3.3. *If*

$$D_X(f) = \bigcup_{i \in I} D_X(f_i),$$

then the sequence $0 \longrightarrow M_f \xrightarrow{\alpha} \prod_{i \in I} M_{f_i} \xrightarrow{\beta} \prod_{i_1, i_2 \in I} M_{f_1, f_2}$ *is exact.*

Proof. We may replace X by $D_X(f)$ and M by M_f to assume $f = 1$.

We first show α is injective. Suppose $u \in M$ such that $\frac{u}{1} = 0$ in M_{f_i} for all i. Note that

$$X = \bigcup_{i \in I} D_X(f_i)$$

is equivalent to $(f_i \mid i \in I) = A$. Moreover, $\frac{u}{1} = 0$ in M_{f_i} is equivalent to $f_i^{n_i} u = 0$ for some $n_i > 0$. This shows that $\operatorname{Ann}(u) \supseteq (f_i^{n_i} \mid i \in I) = A$, so $u = 0$. Therefore, α is injective.

It remains to prove that $\operatorname{im} \alpha = \ker \beta$. Since $\operatorname{im} \alpha \subseteq \ker \beta$ trivially, we just need to show that $\ker \beta \subseteq \operatorname{im} \alpha$. Suppose we have $\frac{a_i}{f_i^{m_i}} = u_i \in M_{f_i}$ such that the images of u_i and u_j in $M_{f_i f_j}$ are the same. After replacing each f_i by a suitable power, we may assume that $m_i = 1$ for all i.

Choose $J \subseteq I$ finite such that $(f_j \mid j \in J) = A$. If there exists $u \in M$ such that the image of u in M_j is u_j for all $j \in J$, the same holds for all $i \in I$. Indeed, given $i \in I$, consider $D_X(f_i) = \bigcup_{j \in J} D_X(f_i f_j)$, so this is true by injectivity of α. This shows that we may assume I is finite.

Recall that $u_i = \frac{a_i}{f_i} \in M_{f_i}$ and $\frac{a_i}{f_i} = \frac{a_j}{f_j}$ in $M_{f_i f_j}$, we have that

$$(f_i f_j)^{q_{ij}} (f_j a_i f_i a_j) = 0$$

in M for all i, j and some q_{ij}.

After replacing each f_i by a large power of f_i, we may assume that $f_j a_i - f_i a_j = 0$ for all i, j. Since $(f_i \mid i \in I) = A$, we may write

$$1 = \sum_{i \in I} f_i b_i.$$

We claim that we can take

$$u = \sum_{i \in I} b_i a_i \in M.$$

We need to show that

$$u = \frac{a_j}{f_j} \text{ in } M_{f_j}.$$

We have that

$$\begin{aligned} f_j u &= \sum_{i \in I} f_j b_i a_i \\ &= \sum_{i \in I} f_i b_i a_j \\ &= a_j. \end{aligned}$$

This completes the proof. □

Altogether, we obtain the following corollary.

Corollary 7.3.4. *For every module M over A, we get a sheaf $\widetilde{M}$ on X such that*
$$\Gamma(D_X(f), \widetilde{M}) \cong M_f,$$
compatible with restriction.

Moreover, if $\varphi\colon M \to N$ is a morphism of A-modules, then for any f, we get a morphism $M_f \to N_f$, compatible with restriction, which gives a morphism of sheaves $\widetilde{M} \to \widetilde{N}$ by Proposition 7.3.2.

Therefore, we have a functor
$$A\text{-mod} \to \mathcal{O}_X\text{-mod}.$$

Definition 7.3.5. An $\mathcal{O}_X$-module $\mathcal{M}$ on X is *quasicoherent* if it is isomorphic to $\widetilde{M}$ for some M. It is *coherent* if, in addition, M is a finitely-generated A-module.

We finally note that for an irreducible closed subset $V \subseteq X$ corresponding to the prime ideal $\mathfrak{p} \subseteq A$, we have
$$\widetilde{M}_V = \varinjlim_{D_X(f)\cap V \neq \emptyset} M_f = \varinjlim_{f \notin \mathfrak{p}} M_f \cong M_{\mathfrak{p}}.$$

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