MATH 632: ALGEBRAIC GEOMETRY II
COHOMOLOGY ON ALGEBRAIC VARIETIES

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These are notes from Math 632: Algebraic geometry II taught by Professor Mircea Mustaţă in Winter 2018, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from May 24, 2018. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The problem sets, homeworks, and official notes can be found on the course website:

http://www-personal.umich.edu/~mmustata/632-2018.html

This course is a continuation of Math 631: Algebraic Geometry I. We will assume the material of that course and use the results without specific references. For notes from the classes (similar to these), see:

http://www-personal.umich.edu/~ahorawa/math_631.pdf

and for the official lecture notes, see:

http://www-personal.umich.edu/~mmustata/ag-1213-2017.pdf

The focus of the previous part of the course was on algebraic varieties and it will continue this course. Algebraic varieties are closer to geometric intuition than schemes and understanding them well should make learning schemes later easy. The focus will be placed on sheaves, technical tools such as cohomology, and their applications.

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1. Sheaves

1.1. **Quasicoherent and coherent sheaves on algebraic varieties.** The object we will consider is a ringed space $(X, \mathcal{O}_X)$ where X is an algebraic variety and $\mathcal{O}_X$ is the sheaf of regular functions on X.

Recall that $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ such that for any open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and if $U \subseteq V$ is an open subset, then

$$\mathcal{F}(V) \to \mathcal{F}(U)$$

is a morphism of $\mathcal{O}_X(V)$-modules, where the $\mathcal{O}_X(V)$-module structure on $\mathcal{F}(U)$ is given by the restriction map $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Fact 1.1.1. *If X has a basis of open subsets $\mathcal{U}$, closed under finite intersection, giving an $\mathcal{O}_X$-module on X is equivalent to giving $\mathcal{O}_X(U)$-modules $\mathcal{F}(U)$ for any $U \in \mathcal{U}$ with restriction maps between these which satisfy the sheaf axiom.*

Example 1.1.2. If X is an affine variety and $A = \mathcal{O}(X)$, M is an A-module, then we obtain an $\mathcal{O}_X$-module $\widetilde{M}$ such that for any $f \in A$, $\Gamma(D_X(f), \widetilde{M}) = M_f$ with the restriction maps induced by localization. (The sheaf axiom was checked in Math 631.)

We get a functor

$$\{A\text{-modules}\} \to \{\mathcal{O}_X\text{-modules}\}$$
$$M \mapsto \widetilde{M}.$$

Definition 1.1.3. Suppose X is affine. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* (*coherent*) if $\mathcal{F} \cong \widetilde{M}$ for some (finitely-generated) A-module M.

Example 1.1.4. The sheaf of regular functions $\mathcal{O}_X$ on X is a coherent sheaf with $\mathcal{O}_X \cong \widetilde{A}$.

If $V \subseteq X$ is irreducible and closed with $\mathfrak{p} = I_X(V)$,

$$\widetilde{M}_V = \varinjlim_{\substack{U \text{ open} \\ U \cap V \neq \emptyset}} \Gamma(U, \widetilde{M}) = \varinjlim_{f \notin \mathfrak{p})} M_f = M_{\mathfrak{p}}.$$

Remarks 1.1.5.

(1) Given any $\mathcal{O}_X$-module $\mathcal{M}$, we have a canonical morphism of $\mathcal{O}_X$-modules:

$$\Phi_M \colon \widetilde{\Gamma(X, \mathcal{M})} \to \mathcal{M}$$

given on $D_X(f)$ by the unique morphism of A_f-modules

$$\Gamma(X, \mathcal{M})_f \to \Gamma(D_X(f), \mathcal{M})$$

induced by the restriction map.

Then the following are equivalent:

- $\mathcal{M}$ is quasicoherent,
- $\Phi_{\mathcal{M}}$ is an isomorphism,
- for any $f \in A$, the canonical map

$$\Gamma(X, \mathcal{M})_f \to \Gamma(D_X(f), \mathcal{M})$$

is an isomorphism.

(2) If $\mathcal{M}$ is quasicoherent (coherent) on X, then $\mathcal{M}|_{D_X(f)}$ is quasicoherent (coherent) on $D_X(f)$ for any $f \in A$.

The following proposition shows that for affine varieties all the operations on modules have natural analogues in $\mathcal{O}_X$-modules.

Proposition 1.1.6.

(1) *If $M_1, \ldots, M_n$ are A-modules, then*

$$\bigoplus_{i=1}^{n} \widetilde{M_i} \cong \widetilde{\bigoplus_{i=1}^{n} M_i}.$$

(2) *The functor $M \mapsto \widetilde{M}$ is exact.*

(3) *Given a morphism $\varphi \colon \widetilde{M} \to \widetilde{N}$ of $\mathcal{O}_X$-modules and $u = \varphi_X \colon M \to N$ induced by φ on global sections, we have that $\varphi = \tilde{u}$.*

(4) *If $\varphi \colon \widetilde{M} \to \widetilde{N}$ is a morphism of quasicoherent (coherent) sheaves, then* $\ker(\varphi)$, $\operatorname{coker}(\varphi)$, $\operatorname{im}(\varphi)$ *are quasicoherent (coherent).*

(5) *The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between A-modules and quasicoherent sheaves on X with the inverse given by $\Gamma(X, -)$.*

(6) *If M, N are A-modules, then $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.*

(7) *If M is an A-module, then*

$$\widetilde{S^p(M)} \cong S^p(\widetilde{M}),$$

$$\widetilde{\bigwedge^p M} \cong \bigwedge^p (\widetilde{M}).$$

(8) *If M, N are A-modules, then*

$$\widetilde{\operatorname{Hom}_A(M, N)} \cong \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

if M is a finitely-generated A-module.

(9) *Let $f \colon X \to Y$ be a morphism of affine varieties and*

$$\varphi = f^{\#} \colon A = \mathcal{O}(Y) \to B = \mathcal{O}(X).$$

Then

$$f_*(\widetilde{{}_B M}) \cong \widetilde{{}_A M} \quad \text{if } M \text{ is a } B\text{-module}$$

and

$$f^*(\widetilde{N}) \cong \widetilde{B \otimes_A N} \quad \text{if } N \text{ is an } A\text{-module.}$$

Proof. For (1), we have that

$$\Gamma\left(X, \bigoplus_{i=1}^{n} \widetilde{M_i}\right) = \bigoplus_{i=1}^{n} \Gamma(X, \widetilde{M_i}) = \bigoplus_{i=1}^{n} M_i.$$

It is hence enough to show that $\bigoplus_{i=1}^{n} M_i$ is quasicoherent. We have that

$$\begin{array}{ccc}
\Gamma\left(X, \bigoplus_{i=1}^{n} \widetilde{M_i}\right)_f & \xrightarrow{\cong} & \Gamma\left(D_X(f), \bigoplus_{i=1}^{n} \widetilde{M_i}\right) \\
\Big\downarrow{=} & & \Big\downarrow{=} \\
\left(\bigoplus_{i=1}^{n} M_i\right)_f & \xrightarrow{\cong} & \bigoplus_{i=1}^{n} (M_i)_f = \bigoplus_{i=1}^{n} \Gamma(D_X(f), \widetilde{M_i})
\end{array}$$

where the bottom arrow is an isomorphism since localization commutes with direct sums.

To prove (2), it is enough to show that if $M' \to M \to M''$ is an exact sequence, then

$$\underbrace{\widetilde{M'}_x}_{=M'_{\mathfrak{m}_x}} \to \underbrace{\widetilde{M}_x}_{=M_{\mathfrak{m}_x}} \to \underbrace{\widetilde{M''}_x}_{M''_{\mathfrak{m}_x}}$$

is exact for all $x \in X$, where $\mathfrak{m}_x$ is the maximal ideal corresponding to x. This is clear since localization is an exact functor.

In (3), it is enough to check that the two maps agree on $D_X(f)$. By definition, we have a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{u=\varphi_X} & N \\
\Big\downarrow & & \Big\downarrow \\
M_f & \xrightarrow{\varphi_{D_X(f)}} & N_f
\end{array}$$

and $\varphi_{D_X(f)}$ is a morphism of A_f-modules. Then $\varphi_{D_X(f)}$ is induced by u by passing to the localization, which completes the proof.

For (4), we know by (3) that $\varphi = \tilde{u}$ for some $u\colon M \to N$, so (2) shows that

$$\ker(\varphi) = \widetilde{\ker(u)}.$$

Similarly for $\mathrm{coker}(\varphi)$ and $\mathrm{im}(\varphi)$. The assertion about coherence follows from a more general fact: if $\mathcal{M}$ is coherent, then any quasicoherent subsheaf or quotient sheaf of $\mathcal{M}$ is coherent (to show this, use the fact that A is Noetherian).

In (5), we already know that we have a functorial isomorphism $\Gamma(X, \widetilde{M}) \cong M$. Then

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(\widetilde{M}, \widetilde{N})$$

is injective, and we saw in (3) that it is surjective, so it is bijective. The result then follows.

We show (6). By definition of $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$, we have a canonical map

$$M \otimes N \to \Gamma(X, \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}).$$

Therefore, we get maps

$$\widetilde{M \otimes N} \longrightarrow \Gamma(X, \widetilde{\widetilde{M} \otimes_{\mathcal{O}_X}} \widetilde{N}) \xrightarrow{\Phi_{\widetilde{M} \otimes \widetilde{N}}} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$$

where $\Phi_{\widetilde{M}\otimes\widetilde{N}}$ is given by Remark 1.1.5 (1). It is now enough to show that it is an isomorphism by checking it on stalks: if $x \in X$ corresponds to $\mathfrak{m} \subseteq A$, then

$$(\widetilde{M \otimes_A N})_x = (M \otimes_A N)_{\mathfrak{m}} = (M \otimes_A N) \otimes_A A_{\mathfrak{m}},$$

$$(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})_x = \widetilde{M}_x \otimes_{\mathcal{O}_{X,x}} \widetilde{N}_x = M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}},$$

and we have that

$$M \otimes_A N \otimes_A A_{\mathfrak{m}} \cong M \otimes_A N_{\mathfrak{m}} \cong M \otimes_A A_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N \cong M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}}.$$

Part (7) follows by a similar argument. Part (8) is given as a homework problem.

In part (9), note that

$$f^*(\widetilde{N}) = f^{-1}(\widetilde{N}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

and proving the assertion about f^* is hence similar to $\otimes$ and left as an exercise. For f_*, if $a \in A$, we have

$$\Gamma(D_Y(a), f_*(\widetilde{M})) = \Gamma(\underbrace{f^{-1}(D_Y(a))}_{D_X(\varphi(a))}, \widetilde{M}) = M_{\varphi(a)} = M_a.$$

This shows the isomorphism for f_*. □

So far, we have assumed that X is finite. The next goal is to *globalize* the definitions and results to the general case. Thus, let now X be any algebraic variety.

Proposition 1.1.7. *If $\mathcal{F}$ is an $\mathcal{O}_X$-module, the following are equivalent:*

(1) *for any affine open $U \subseteq X$, $\mathcal{F}|_U$ is quasicoherent,*
(2) *for any affine open $U \subseteq X$ and every $f \in \mathcal{O}_X(U)$, the canonical map*

$$\mathcal{F}(U)_f \to \mathcal{F}(D_U(f))$$

is an isomorphism,
(3) *there is an affine open cover $X = U_1 \cup \cdots \cup U_n$ such that $\mathcal{F}|_{U_i}$ is quasicoherent for all i.*

Moreover (1) is equivalent to (3) if we replace "quasicoherent" by "coherent".

Proof. Note that (1) and (2) are clearly equivalent, and (1) trivially implies (3). We only need to show that (3) implies (1). Choose an affine open subset $U \subseteq X$. We know that the restriction of $\mathcal{F}$ to any principal affine open subset of some U_i is quasicoherent. We showed in Math 631 that we can cover U by open subsets that are principal with respect to both U and some U_i.

It is hence enough to show that if X is an affine variety, $\mathcal{F}$ is an $\mathcal{O}_X$-module, and

$$X = \bigcup_{i=1}^{r} D_X(f_i),$$

where each $\mathcal{F}|_{D_X(f_i)}$ is quasicoherent, then $\mathcal{F}$ is quasicoherent. Consider $a \in A = \mathcal{O}(X)$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}(X)_a & \longrightarrow & \prod_i \mathcal{F}(D_X(f_i))_a & \longrightarrow & \prod_{i,j} \mathcal{F}(D_X(f_if_j))_a \\
 & & \big\downarrow & & \big\downarrow \cong & & \big\downarrow \cong \\
0 & \longrightarrow & \mathcal{F}(D_X(a)) & \longrightarrow & \prod_i \mathcal{F}(D_X(af_i)) & \longrightarrow & \prod_{i,j} \mathcal{F}(D_X(af_if_j))
\end{array}$$

Since the rows are exact and the second and third vertical maps are isomorphism, the first map is an isomorphism (we can add another two zeros on the left and apply the Five Lemma).

Finally, in the coherent case, it is enough to show that if $X = D_X(f_1) \cup \cdots \cup D_X(f_m)$ is affine, M is an A-module for $A = \mathcal{O}(X)$, and M_{f_i} is a finitely-generated A_{f_i}-module for all i, then ${}_AM$ is finitely-generated. This was already proved in Math 631, so we leave it as an exercise here. □

Definition 1.1.8. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasicoherent* (*coherent*) if it satisfies the equivalent properties in Proposition 1.1.7 (for coherent: replace "quasicoherent" by "coherent" in (1) and (3)).

The categories $\mathrm{Qcoh}(X)$, $\mathrm{Coh}(X)$ are closed under:

- finite direct sums,
- ker, coker, im (so they are abelian categories),
- tensor products, symmetric powers, exterior powers,
- if $\mathcal{F}, \mathcal{G}$ are quasicoherent, $\mathcal{F}$ is coherent, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasicoherent, and if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is also coherent (this is a homework problem).

Proposition 1.1.9. *For a morphism of algebraic varieties $f\colon X \to Y$:*

(1) *if $\mathcal{F}$ is quasicoherent (coherent) on Y, then $f^*(\mathcal{F})$ is quasicoherent (coherent) on X,*
(2) *if $\mathcal{G}$ is quasicoherent on X, then $f_*\mathcal{G}$ is quasicoherent on Y (this is not true for general coherent sheaves, but it is true when f is a finite morphism[1]).*

Proof. For (1), choose for any $x \in X$, affine open neighborhood V of $f(x)$, affine open neighborhood $U \subseteq f^{-1}(V)$ of x. We then have

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\big\uparrow & & \big\uparrow \\
U & \xrightarrow{g} & V
\end{array}$$

and hence

$$f^*(\mathcal{F})|_U \cong g^*(\mathcal{F}|_V).$$

Thus, we are done by the affine case.

Part (2) was discussed during the problem session and we include it here for completeness. □

[1] In fact, this even holds when f is proper. This will be proved later in the class.

1.2. **Locally free sheaves.** Let X be an algebraic variety. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if there exists an open cover $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{\mathcal{U}_i}^{\oplus r_i}$. If all r_i are equal to r, then $\mathcal{F}$ is *locally free of rank r*.

Note that:

- every such $\mathcal{F}$ is coherent,
- if X is connected, every locally free sheaf has well-defined rank.

Proposition 1.2.1. *Suppose X is affine and $\mathcal{O}(X) = A$. If ${}_AM$ if finitely-generated, then the following conditions are equivalent:*

(1) *$\widetilde{M}$ is locally free,*
(2) *for any prime ideal $\mathfrak{p}$, $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$,*
(3) *for any maximal ideal $\mathfrak{m}$, $M_\mathfrak{m}$ is free over $A_\mathfrak{m}$,*
(4) *M is projective.*

Recall that M is projective if $\operatorname{Hom}(M, -)$ is exact.

Remark 1.2.2. If ${}_AM$ is finitely-generated. Then there exists a free finitely generated module F with a surjective map $F \twoheadrightarrow M$. Then M is projective if and only if there is a splitting $M \to F$, which is equivalent to saying that M is a direct summand of a free module.

Proof. We first show that ${}_AM$ is projective if and only if $M_\mathfrak{p}$ is projective over $A_\mathfrak{p}$ for all prime (maximal) ideals $\mathfrak{p}$. Indeed, choose a free finitely generated module F with $F \twoheadrightarrow M$. Then M is projective if and only if $\operatorname{Hom}(M, F) \to \operatorname{Hom}(M, M)$ is surjective. Since Hom commutes with localization and a morphism is surjective if and only if it is surjective after localizing at every prime (maximal) ideal.

We now show that if ${}_RM$ is finitely generated and projective and $(R, \mathfrak{m})$ is local, then ${}_RM$ is free. Choose a minimal set $u_1, \ldots, u_n$ of generators of M and consider the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & R^n & \longrightarrow\!\!\!\!\!\rightarrow & M & \longrightarrow & 0 \\ & & & & e_i & \longrightarrow & u_i & & \end{array}$$

This exact sequence is split since M is projective. Since the sequence is split, tensoring with $R/\mathfrak{m}$ gives an exact sequence (since tensor products commute with direct sums)

$$0 \longrightarrow N/\mathfrak{m}N \longrightarrow (R/\mathfrak{m})^n \xrightarrow{\cong} M/\mathfrak{m}M \longrightarrow 0$$

Hence $N/\mathfrak{m}N = 0$. By Nakayama's Lemma, this shows that $N = 0$, so $M \cong R^n$.

Altogether, we have shown that (2), (3), and (4) are all equivalent. To see that (1) implies (3), note that if $\widetilde{M}$ is lcoally free, then for any $x \in X$ corresponding to the maximal ideal $\mathfrak{m}$:

$$M_\mathfrak{m} = \widetilde{M}_x \cong \mathcal{O}_{X,x}^{\oplus r} = A_\mathfrak{m}^{\oplus r}$$

for some r.

Let us now show that (3) implies (1). Fix $x \in X$ corresponding to $\mathfrak{m} \subset A$. We know that $M_\mathfrak{m}$ is free of rank r, so we may choose a basis $\frac{u_1}{1}, \ldots, \frac{u_r}{1}$. Then the map

$$\begin{aligned} \varphi \colon A^{\oplus r} &\to M \\ e_i &\mapsto u_i \end{aligned}$$

becomes an isomorphism after localizing at $\mathfrak{m}$, i.e. $(\ker \varphi)_\mathfrak{m} = 0 = (\operatorname{coker} \varphi)_\mathfrak{m}$. Since $\ker \varphi$ and $\operatorname{coker} \varphi$ are finitely generated over A, there exists f such that $(\ker \varphi)_f = 0 = (\operatorname{coker} \varphi)_f$. Therefore, $\varphi \otimes_A A_f$ is an isomorphism, and hence

$$\widetilde{M}_{D_X(f)} \cong \mathcal{O}_{D_X(f)}^{\oplus r}.$$

Since $D_X(f)$ is a neighborhood of x, this completes the proof. □

Definition 1.2.3. Given a coherent sheaf $\mathcal{F}$, the *fiber of $\mathcal{F}$ at $x \in X$* is

$$\mathcal{F}_{(x)} := \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal.

Note that

(1) for $i \colon \{x\} \hookrightarrow X$ for $x \in X$, we have that

$$\mathcal{F}_{(x)} \cong i^* \mathcal{F},$$

since for a maximal ideal $\mathfrak{m} \subseteq A$, we have that

$$M_\mathfrak{m} / \mathfrak{m} M_\mathfrak{m} \cong M / \mathfrak{m} M,$$

(2) by Nakayama's Lemma, we have that $\dim_k \mathcal{F}_{(x)}$ is the minimal number of generators of $\mathcal{F}_x$.

Proposition 1.2.4. *A coherent sheaf $\mathcal{F}$ is locally free of rank r if and only if $\dim_k \mathcal{F}_{(x)} = r$ for all $x \in X$.*

Proof. The 'only if' implication is clear: if $\mathcal{F}$ is locally free of rank r, then $\mathcal{F}_x \cong \mathcal{O}_{X,x}^{\oplus r}$, and hence

$$\mathcal{F}_{(x)} \cong k^{\oplus r}.$$

By choosing an affine open neighborhood around each point, we may assume that X is affine, $A = \mathcal{O}(X)$, and $\mathcal{F} = \widetilde{M}$ for a finitely-generated module M. Fix $x \in X$ corresponding to the maximal ideal $\mathfrak{m} \subseteq A$. Then $M_\mathfrak{m}$ has the minimal number of generators equal to r. Choose a morphism

$$\varphi \colon A^{\oplus r} \to M$$

which becomes surjective after localizing at $\mathfrak{m}$. Replacing A by A_f for some $f \notin \mathfrak{m}$ (i.e. replacing the affine neighborhood of x by a smaller one), we may assume that φ is surjective. Consider the short exact sequence

$$0 \longrightarrow N \hookrightarrow A^{\oplus r} \xrightarrow{\varphi} M \longrightarrow 0.$$

For every maximal ideal $\mathfrak{n}$, the minimal number of generators $M_\mathfrak{n}$ is still $\mathfrak{n}$. Then, localizing the exact sequence at $\mathfrak{n}$, we get an exact sequence

$$0 \longrightarrow N_{\mathfrak{n}} \longrightarrow A_{\mathfrak{n}}^{\oplus r} \longrightarrow M_{\mathfrak{n}} \longrightarrow 0$$

and tensoring with $A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}}$, we obtain

$$N_{\mathfrak{n}}/\mathfrak{n}N_{\mathfrak{n}} \longrightarrow (A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}})^{\oplus r} \xrightarrow{\cong} M_{\mathfrak{n}}/\mathfrak{n}M_{\mathfrak{n}} \longrightarrow 0.$$

Therefore, $N_{\mathfrak{n}} \subseteq \mathfrak{n} \cdot A_{\mathfrak{n}}^{\oplus r}$, so $N \subseteq \mathfrak{n} \cdot A^{\oplus r}$. Hence

$$N \subseteq \left(\bigcap_{\mathfrak{n} \in \operatorname{MaxSpec}(A)} \mathfrak{n} \right)^{\oplus r} = 0,$$

showing that $M \cong A^{\oplus r}$ is free. □

Proposition 1.2.5. *Given a short exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

with $\mathcal{F}''$ locally free, for every $\mathcal{O}_X$-module $\mathcal{G}$ the sequence

$$0 \longrightarrow \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow \mathcal{F}'' \otimes_{\mathcal{O}_X} \mathcal{G} \longrightarrow 0$$

is exact. In particular, for any $x \in X$, we have an exact sequence

$$0 \longrightarrow \mathcal{F}'_{(x)} \longrightarrow \mathcal{F}_{(x)} \longrightarrow \mathcal{F}''_{(x)} \longrightarrow 0$$

Proof. For the first assertion, take stalks and use the fact that

$$0 \longrightarrow \mathcal{F}'_x \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}''_x \longrightarrow 0$$

is split exact (since $\mathcal{F}''_x$ is free), and hence tensoring preserves exactness.

The second assertion follows by taking $\mathcal{G} = k(x)$ where

$$k(x)(U) = \begin{cases} U & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $i\colon \{x\} \hookrightarrow X$, $k(x) = i_*\mathcal{O}_{\{x\}}$. □

Corollary 1.2.6. *If* $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ *is exact with $\mathcal{F}''$ locally free, then $\mathcal{F}'$ is locally free if and only if $\mathcal{F}$ is locally free. If two of the above have well-defined rank, then so does the third, and*

$$\operatorname{rank}(\mathcal{F}) = \operatorname{rank}(\mathcal{F}') + \operatorname{rank}(\mathcal{F}'').$$

Proof. Work on the connected components of X and apply Propositions 1.2.4 and 1.2.5. □

The following operations preserve locally free sheaves:

(1) if $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are locally free (with $\operatorname{rank}(\mathcal{M}_i) = r_i$), then $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$ is locally free (with equal to $r_1 + \cdots + r_n$),
(2) if $\mathcal{E}$, $\mathcal{F}$ are locally free, then $\mathcal{E} \otimes \mathcal{F}$ is locally free,
(3) if $\mathcal{E}$ is locally free, then $S^p(\mathcal{E})$ and $\bigwedge^p(\mathcal{E})$ are locally free.

Definition 1.2.7. If $\mathcal{E}$ is a coherent sheaf, then the coherent sheaf

$$\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$$

is the *dual of* $\mathcal{E}$.

The assignment $\mathcal{E} \to \mathcal{E}^\vee$ is a contravariant functor.

Proposition 1.2.8.

(1) *For every coherent sheaves* $\mathcal{E}, \mathcal{F}$, *there exists a morphism*

$$\mathcal{E}^\vee \otimes \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$$

and it is an isomorphism if one of $\mathcal{E}$, $\mathcal{F}$ *is locally free.*
(2) *If* $\mathcal{E}$ *and* $\mathcal{F}$ *are locally free coherent sheaves, then* $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ *is locally free.*
(3) *For every coherent sheaf* $\mathcal{E}$, *there is a canonical morphism* $\mathcal{E} \to (\mathcal{E}^v)^v$ *which is an isomorphism if* $\mathcal{E}$ *is locally free.*
(4) *For every coherent sheaf* $\mathcal{E}$, *there is a canonical morphism*

$$\mathcal{O}_X \to \mathcal{H}om(\mathcal{E}, \mathcal{E})$$

which is an isomorphism if $\mathcal{E}$ *is locally free of rank* 1.

Proof. The proof of this proposition is left as an exercise. □

1.3. **Vectorbundles.** We will use the terminology:

- *vector bundle* for a locally free sheaf,
- *line bundle* for a locally free sheaf of rank 1,
- *trivial vector bundle of rank* r for a sheaf isomorphic to $\mathcal{O}_X^{\oplus r}$.

Definition 1.3.1. The *Picard group of* X, denoted $\operatorname{Pic}(X)$ is the set of isomorphism classes of line bundles on X with the operation

$$(\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$$

(which is clearly associative and commutate) with identity given by $\mathcal{O}_X$ and the inverse given by $\mathcal{L}^\vee$.

For now, we do not have the necessary tools to compute any nontrivial Picard groups, but we will get to this later, when we discuss divisors.

Remark 1.3.2. If $f \colon X \to Y$ is a morphism and $\mathcal{E}$ is locally free on Y, then $f^*\mathcal{E}$ is locally free on X.

In particular, we get a morphism of abelian groups

$$f^*\colon \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X)$$

$$\mathcal{L} \longrightarrow f^*\mathcal{L}$$

Our next goal will be to provide some geometric intuition for vector bundles. We first discuss the description of locally free sheaves in terms of transition maps.

Let X be an algebraic variety and $\mathcal{E}$ a vector bundle of rank r. Then there is an open cover

$$X = \bigcup_{i \in I} U_i$$

such that we have *trivialization maps*

$$\varphi_i\colon \mathcal{E}|_{U_i} \overset{\cong}{\to} \mathcal{O}_{U_i}^{\oplus r}.$$

Given i, j, consider the transition maps

$$\varphi_{i,j} = \varphi_i \circ \varphi_j^{-1}\colon \varphi_i \circ \varphi_j^{-1}\colon \mathcal{O}_{U_i\cap U_j}^{\oplus r} \overset{\cong}{\to} \mathcal{O}_{U_i\cap U_j}^{\oplus r}$$

satisfying the *cocycle condition*:

$$\varphi_{i,i} = \mathrm{id} \text{ for all } i,$$
$$\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k} \text{ on } U_i \cap U_j \cap U_k \text{ for all } i, j, k.$$

Note that an isomorphism

$$\mathcal{O}_{U_i\cap U_j}^{\oplus r} \to \mathcal{O}_{U_i\cap U_j}^{\oplus r}$$

is given by an invertible matrix $M \in M_r(\mathcal{O}_X(U_i \cap U_j))$.

Exercise. Given an open cover

$$X = \bigcup_{i \in I} U_i$$

and invertible matrices $(a_{i,j})_{i,j\in I}$, $a_{i,j} \in M_r(\mathcal{O}_X(U_i \cap U_j))$ satisfying the cocycle condition, there is a locally free sheaf $\mathcal{E}$, unique up to canonical isomorphism, that has this as the family of transition maps.

One can say precisely when the vector bundles corresponding to two covers and transition maps are isomorphic.

Example 1.3.3. Suppose $r = 1$. Consider the sheaf $\mathcal{O}_X^*$ of abelian groups given by

$$\mathcal{O}_X^*(U) = \mathcal{O}_X(U)^* = \{\text{invertible elements of } \mathcal{O}_X(U)\}.$$

A line bundle $\mathcal{L}$ on X is described by an open cover $\mathcal{U} = (U_i)_{i\in I}$ and elements $(a_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j))_{i,j\in I}$ that satisfy the cocycle condition.

Suppose $\mathcal{L}$ corresponds to the family $(a_{i,j})$. Then $\mathcal{L}^{\otimes m}$ corresponds to $(a_{i,j}^m)$ and $\mathcal{L}^\vee$ corresponds to $(a_{i,j}^{-1})$.

If $\mathcal{L}$ is given with respect to $\mathcal{U} = (U_i)_{i\in I}$ by transition maps $(a_{i,j} \in \mathcal{O}_X^*(U_i \cap U_j))$, then

$$\Gamma(X, \mathcal{L}) \cong \left\{ (f_i)_{i\in I} \in \prod_{i\in I} \mathcal{O}_X(U_i) \;\middle|\; f_i = a_{i,j} f_j \text{ on } U_i \cap U_j \text{ for all } i, j \in I \right\}.$$

Proving this is left as an exercise.

1.4. **Geometric constructions via sheaves.** The language of sheaves is used to deal with global objects which are "locally nice" (for example, manifolds, affine varieties etc). In this section, we will discuss how one can use sheaves for certain geometric constructions.

Definition 1.4.1. Let X be an algebraic variety. An $\mathcal{O}_X$*-algebra* is a sheaf of commutative rings $\mathcal{A}$ on X with a morphism of sheaves of rings $\mathcal{O}_X \to \mathcal{A}$. A morphism of $\mathcal{O}_X$-algebras is defined in the obvious way.

Note that every $\mathcal{A}$-module becomes an $\mathcal{O}_X$-module. In particular, $\mathcal{A}$ is an $\mathcal{O}_X$-module.

We will be interested in cases when:

- $\mathcal{A}$ is quasicoherent,
- $\mathcal{A}$ is *reduced*: for any open affine $U \subseteq X$, $\mathcal{A}(U)$ is reduced,
- $\mathcal{A}$ is of *finite type over* $\mathcal{O}_X$: for any open affine $U \subseteq X$, $\mathcal{A}(U)$ is a finitely generated $\mathcal{O}_X(U)$-algebra (or equivalently, $\mathcal{A}(U)$ is a finitely-generated k-algebra).

Example 1.4.2. Let $\mathcal{F}$ be a quasicoherent sheaf. Then set

$$S^\bullet(\mathcal{F}) := \bigoplus_{m \geq 0} S^m(\mathcal{F}).$$

Given an open subset $X \supseteq U$:

$$S^i(\mathcal{F}(U)) otimes_{\mathcal{O}_X(U)} S^j(\mathcal{F}(U)) \to S^{i+1}(\mathcal{F}(U))$$

and passing to the associated sheaves, we get a map

$$S^i(\mathcal{F}) \otimes S^j(\mathcal{F}) \to S^{i+j}(\mathcal{F})$$

making $S^\bullet(\mathcal{F})$ and $\mathcal{O}_X$-algebra.

Note that:

(1) $S^\bullet(\mathcal{F})$ is quasicoherent,
(2) if $\mathcal{F}$ is coherent, then $S^\bullet(\mathcal{F})$ is a finite type $\mathcal{O}_X$-algebra.

The $\mathcal{O}_X$-algebra $S^\bullet(\mathcal{F})$ has the following universal property: if A is any $\mathcal{O}_X$-algebra:

$$\mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(S^\bullet(\mathcal{F}), \mathcal{A}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{F}, \mathcal{A}).$$

Suppose $f \colon Y \to X$ is any morphism. Then $\mathcal{O}_X \to f_*\mathcal{O}_Y$ makes $f_*\mathcal{O}_Y$ an $\mathcal{O}_X$-algebra. It is:

- quasicoherent,
- reduced,
- a finitely-generated $\mathcal{O}_X$-algebra if f is affine: for an affine open subset $U \subseteq X$, $f^{-1}(U)$ is affine and
$$\Gamma(U, f_*\mathcal{O}_Y) \cong \Gamma(f^{-1}(U), \mathcal{O}_Y)$$
is a finitely-generated k-algebra.

Suppose $g \colon Z \to X$ is another variety over X and we have a morphism $h \colon Z \to Y$ over X, i.e. the diagram

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

commutes. We then have the map

$$\mathcal{O}_Y \to h_*\mathcal{O}_Z,$$

and hence a morphism of $\mathcal{O}_X$-algebras:

$$f_*\mathcal{O}_Y \to f_*h_*\mathcal{O}_Z = g_*\mathcal{O}_Z.$$

Altogether, this shows that there is a contravariant functor:

$$\Phi\colon \left\{ \begin{array}{c} \text{varieties over } X \\ \text{affine over } X \end{array} \right\} \to \left\{ \begin{array}{c} \text{quasicoherent, reduced} \\ \text{finitely generated } \mathcal{O}_X\text{-algebras} \end{array} \right\}$$
$$(f\colon Y \to X) \mapsto f_*\mathcal{O}_Y$$

The goal is to construct an inverse functor. Let $\mathcal{A}$ be a quasicoherent, finitely generated, reduced $\mathcal{O}_X$-algebra. Given an affine open subset $U \subseteq X$, consider $\mathcal{A}(U)$. We then have affine varieties over U:

$$Y_U = \operatorname{MaxSpec}(\mathcal{A}(U)) \xrightarrow{\pi_U} U$$

with the map induced by $\mathcal{O}_X(U) \to \mathcal{A}(U)$. We claim that these can be glued together into a variety. In particular, we will show that if $V \subseteq U$ is an affine open subset, the commutative diagram

$$\begin{tikzcd} Y_V \arrow[r] \arrow[d, "\pi_V"] & Y_U \arrow[d, "\pi_U"] \\ V \arrow[r, hook, "j"] & U \end{tikzcd}$$

is Cartesian (where the top map is induced by $\mathcal{A}(U) \to \mathcal{A}(V)$), i.e.

$$\alpha\colon Y_V \to \pi_V^{-1}(V)$$

is an isomorphism.

This is clear if V is a principal affine open subset in U, since for $V = D_U(\varphi)$, $\mathcal{A}(V) = \mathcal{A}(U)_\varphi$. In the general case, write $V = V_1 \cup \cdots \cup V_r$ for principal affine open subsets $V_i \subseteq U$. Then

$$Y_{V_i} = \alpha^{-1}(\pi_U^{-1}(V_i)) \to \pi_U^{-1}(V_i)$$

is an isomorphism, so α is an isomorphism.

Given any two affine open subsets $U, V \subseteq X$, we get an isomorphism

$$\pi_U^{-1}(U \cap V) \cong Y_{U\cap V} \cong \pi_V^{-1}(U \cap V).$$

Therefore, we can glue $Y_U = \operatorname{MaxSpec}(\mathcal{A}(U))$ along these isomorphisms to get

$$\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}).$$

Gluing the π_U, we get a map

$$\pi_X\colon \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) \to X$$

such that for any open affine subset $U \subseteq X$, we have

$$\pi_X^{-1}(U) \cong \mathrm{MaxSpec}(\mathcal{A}(U)).$$

Then the map π_X is affine and there is a canonical isomorphism $(\pi_X)_*\mathcal{O}_Y \cong \mathcal{A}$. Moreover, this is functorial (which can be proved in similar fashion).

Exercise. The functor $A \mapsto \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})$ gives an inverse to the functor

$$\Phi\colon (Y \xrightarrow{\pi} X) \mapsto \pi_*\mathcal{O}_Y.$$

We still need to check separatedness of the variety $\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})$. We first make a definition.

Definition 1.4.3. If $f\colon X \to Y$ is a morphism of prevarieties, f is *separated* if for

$$\begin{tikzcd} X \arrow[r, hook, "\Delta"] \arrow[dr, hook] & X \times X \\ & X \times_Y X \arrow[u, hook] \end{tikzcd}$$

we have that $\Delta(X)$ is closed in $X \times_Y X$.

Note that if Y is a point, this just means that X is separated.

Remark 1.4.4. If X is separated then any $f\colon X \to Y$ is separated.

Remark 1.4.5. If there is an open cover $Y = \bigcap_i V_i$ such that each $f^{-1}(V_i) \to V_i$ is separated, then f is separated. Indeed,

$$\Delta(X) \hookrightarrow \bigcup_{i \in I} f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$$

and the intersection of $\Delta(X)$ with $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$ is $\Delta(f^{-1}(V_i))$, which is closed in $f^{-1}(V_i) \times_{V_i} f^{-1}(V_i)$.

In particular, if each $f^{-1}(V_i)$ is separated, then f is separated.

In our case, this implies that

$$\pi\colon \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) \xrightarrow{\pi_X} X$$

is separated, since for each affine subset $U \subseteq X$, we have that $\pi_X^{-1}(U)$ is affine, hence separeted.

Exercise. A composition of separated morphisms is separated.

As a consequence, $\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})$ is a variety, since the map

$$\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) \to X \to \{*\}$$

is separated.

Exercise. Let $\pi\colon \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) = Y \to X$. Given a map of varieties over X:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

we get a map $\mathcal{A} = \pi_*\mathcal{O}_Y \to g_*\mathcal{O}_Z$. Show that the resulting morphism

$$\mathrm{Hom}_{\mathrm{Var}_X}(Z, \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, g_*\mathcal{O}_Z)$$

is a bijection. Note that by the adjointness property:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, g_*\mathcal{O}_Z) = \mathrm{Hom}_{\mathcal{O}_Z}(g^*\mathcal{A}, \mathcal{O}_Z).$$

Remark 1.4.6. The map $\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) \to X$ is finite if and only if $\mathcal{A}$ is a coherent $\mathcal{O}_X$-module.

1.5. **Geometric vector bundles.** Let X be a variety.

Definition 1.5.1. A *geometric vector bundle on* X is a morphism

$$\pi\colon E \to X$$

such that for any $x \in X$, the fiber

$$E_{(x)} = \pi^{-1}(x)$$

has a k-vector space structure, which is *locally trivial*, i.e. there is an open cover $X = \bigcup\limits_i U_i$ such that there is an isomorphism

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\alpha"] \arrow[dr] & & U_i \times k^{r_i} \arrow[dl] \\ & U_i & \end{tikzcd}$$

which is linear on each fiber.

If all r_i are equal to r, we say that E has *rank* r.

Definition 1.5.2. The category $\mathrm{Vect}(X)$ of geometric vector bundles over X is a category whose

- objects are geometric vector bundles,
- morphisms are morphisms of varieteis over X which are linear on the fibers.

Fix a geometric vector bundle $\pi\colon E \to X$. Define a sheaf $\mathcal{E}$ of sets on X given by

$$\mathcal{E}(U) = \{\text{morphisms } s\colon U \to E \text{ such that } \pi \circ s = \mathrm{id}_U\}.$$

Defining:

$$(s_1 + s_2)(x) = s_1(x) + s_2(x),$$
$$fs(x) = f(x)s(x) \text{ for } f \in \mathcal{O}_X,$$

(both of these defined in $\pi^{-1}(x)$ which is a k-vector space) makes $\mathcal{E}$ into an $\mathcal{O}_X$-module. Indeed, suppose $X = \bigcup\limits_i U_i$, $\pi^{-1}(U_i) \cong U_i \times k^{r_i}$. Then

$$\mathcal{E}(U \cap U_i) \cong \mathcal{O}_X(U \cap U_i)^{\oplus r_i}$$

is an isomorphism $\mathcal{O}_X(U_i \cap U_i)$-modules. This shows that

- the operations are well-defined,
- $\mathcal{E}$ is an $\mathcal{O}_X$-module,
- $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$.

Therefore, $\mathcal{E}$ is a locally free sheaf.

This assignment is actually functorial. Given a morphism

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$

in $\mathrm{Vect}(X)$, if $\mathcal{E}$, $\mathcal{F}$ correspond to E, F, we get a map $\mathcal{E} \to \mathcal{F}$ given by $s\colon U \to E$ goes to $f \circ s$. We hence have a functor

$$\Phi\colon \left\{ \begin{array}{c} \text{geometric vector} \\ \text{bundles over } X \end{array} \right\} \to \left\{ \begin{array}{c} \text{full subcategory of} \\ \text{locally free sheaves on } X \end{array} \right\}$$

In the opposite direction, suppose $\mathcal{E}$ is a locally free sheaf on X, and define

$$V(\mathcal{E}) = \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(S^\bullet(\mathcal{E})) \xrightarrow{\pi} X.$$

If $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, then

$$S^\bullet(\mathcal{E})(U) \cong \mathcal{O}_U[t_1, \ldots, t_r],$$

hence $S^\bullet(\mathcal{E})$ is a reduced algebra. We then have an isomorphism

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr] & & U \times k^r \arrow[dl] \\ & U & \end{tikzcd}$$

which gives a vector space structure on the fibers of π (independent of the choice of $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$). Therefore, π is a geometric vector bundle on X.

Note that $\mathcal{E} \mapsto V(\mathcal{E})$ is a contravariant functor.

We want to work out what the sheaf of sections $\mathcal{F}$ of $V(\mathcal{E})$ is. Consider

$$\begin{tikzcd} V(\mathcal{E}) \arrow[d, "\pi"] & \\ X & U \arrow[l, hook'] \arrow[ul, dotted, bend right, "s"'] \end{tikzcd}$$

Then

$$\mathcal{F}(U) = \mathrm{Hom}_{\mathrm{Var}_X}(U, \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(S^\bullet(\mathcal{E}))) = \mathrm{Hom}_{\mathcal{O}_U\text{-alg}}(S^\bullet(\mathcal{E})|_U, \mathcal{O}_U).$$

By the universal property of $S^\bullet(\mathcal{E}|_U)$, this is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_U\text{-mod}}(\mathcal{E}|_U, \mathcal{O}_U) = \mathcal{E}^\vee(U).$$

Altogether, we see that the sheaf of section of $V(\mathcal{E})$ is isomorphic to $\mathcal{E}^\vee$.

It is now easy to see that the functor mapping E to the sheaf of sections is an equivalence of categories with inverse $\mathcal{E} \mapsto V(\mathcal{E}^\vee)$.

We previously constructed $\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})$ given a locally free sheaf $\mathcal{A}$. Next, we construct $\mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S})$ for a graded $\mathcal{O}_X$-algebra $\mathcal{S}$. Let X be an algebraic variety and $\mathcal{S}$ be an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra: an $\mathcal{O}_X$-algebra together with a decomposition

$$\mathcal{S} = \bigoplus_{m \geq 0} \mathcal{S}_m$$

such that $\mathcal{S}_i \cdot \mathcal{S}_j \subseteq \mathcal{S}_{i+1}$. We also assume that $\mathcal{S}$ is quasi-coherent, reduced, and locally generated by $\mathcal{S}_1$, which is a coherent sheaf, i.e. for any open subset $U \subseteq X$, $\mathcal{S}(U)$ is generated as an $\mathcal{O}_X(U)$-algebra by $\mathcal{S}_1(U)$ which is a finitely-generated $\mathcal{O}_X$-module.

Note that this condition implies that the map

$$\mathcal{O}_X \to \mathcal{S} \to \mathcal{S}_0$$

is surjective.

Given such an $\mathcal{S}$, for every affine open subset $U \subseteq X$, consider

$$\begin{tikzcd}
\mathrm{MaxProj}(\mathcal{S}(U)) \arrow[d] \arrow[dd, bend right=60, "\pi_U"'] \\
\mathrm{MaxSpec}(\mathcal{S}(U)) \arrow[d, hook] \\
U
\end{tikzcd}$$

where $\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{S}_0(U)) \to U$ is the closed immersion induced by $\mathcal{O}_X(U) \twoheadrightarrow \mathcal{S}_0(U)$.

One checks that if $V \subseteq U$ is an affine open subset, then the commutative diagram

$$\begin{tikzcd}
\mathrm{MaxProj}(\mathcal{S}(V)) \arrow[r] \arrow[d] & \mathrm{MaxProj}(\mathcal{S}(U)) \arrow[d] \\
V \arrow[r, hook] & U
\end{tikzcd}$$

is Cartesian, so we can glue these together to get

$$\pi \colon \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}) \to X.$$

Note that if $U \subseteq X$ is an affine open subset, then $\mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}(U))$ is separated, and hence π is a separated morphismand X is separated, so $\mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S})$ is separated.

Examples 1.5.3.

(1) *The blow-up of X along a coherent ideal.* If $\mathcal{I} \subseteq \mathcal{O}_X$ is a coherent dieal sheaf, we define

$$\mathcal{R}(\mathcal{I}) = \bigoplus_{m \geq 0} \mathcal{I}^m t^m \subseteq \mathcal{O}_X[t].$$

The *blow up of X along* $\mathcal{I}$ is $\mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{R}(\mathcal{I})) \to X$.

(2) *Projective bundles.* If $\mathcal{E}$ is a locally free sheaf on X, then consider

$$\mathcal{S} = \bigoplus_{m \geq 0} S^m(\mathcal{E})$$

and define the projective bundle associated to $\mathcal{E}$ to be

$$\mathbb{P}(\mathcal{E}) := \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}) \xrightarrow{\pi} X.$$

Note that if $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus n}$ where U is affine, then

$$\mathcal{S}|_U \cong \mathcal{O}_U[x_1, \ldots, x_n],$$

so $\pi^{-1}(U) \cong U \times \mathbb{P}^{n-1}$.

Definition 1.5.4. A *projective morphism* $f \colon Y \to X$ is a morphism such that there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}) \arrow[dl] \\ & X & \end{tikzcd}$$

It is clear that a projective morphism is proper.

1.6. **The cotangent sheaf.** We begin with the local case. Let R be a commutative ring and A be a commutative R-algebra.

Definition 1.6.1. If M is an A-module, an *R-derivation* $D \colon A \to M$ is a map such that

(1) D is R-linear,
(2) (Leibniz rule): $D(ab) = aD(b) + bD(a)$.

Remark 1.6.2. If D satisfies (2), then (1) is satisfies if and only if D is additive and $D = 0$ on $\mathrm{im}(R \to A)$.

We denote set of R-derivations $A \to M$ by

$$\mathrm{Der}_R(A, M) = \{R\text{-derivations } A \to M\} \subseteq \mathrm{Hom}_R(A, M).$$

We note that this is an A-submodule.

Given $u \in \mathrm{Hom}_A(M, N)$, we get a map

$$\mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A, N),$$
$$D \mapsto u \circ D,$$

so $\mathrm{Der}_R(A, -)$ is a covariant functor from A-mod to itself.

Proposition 1.6.3. *The functor* $\mathrm{Der}_R(A, -)$ *is* representable, *i.e. there exists an A-module $\Omega_{A/R}$ and an R-derivation $d = d_{A/R} \colon A \to \Omega_{A/R}$ that induces a bijection*

$$\mathrm{Hom}_A(\Omega_{A/R}, M) \to \mathrm{Der}_R(A, M)$$
$$\varphi \mapsto \varphi \circ d.$$

Proof. Take $\Omega_{A/R}$ to be the quotient of the free A-module with basis $\{d(a) \mid a in A\}$ by the following elements:

(1) $d(a_1 + a_2) - d(a_1) - d(a_2)$ for $a_1, a_2 \in A$,
(2) $d(ar) - rd(a)$ for $a \in A$, $r \in R$,
(3) $d(ab) - ad(b) - bd(a)$ for $a, b \in A$,

and define $d_{A/R}(a)$ to be the image of $d(a)$ in the quotient. It is easy to check that this satisfies the required universal property. □

Definition 1.6.4. The module $\Omega_{A/R}$ defined above is called the *module of Kähler differentials.*

Remark 1.6.5. The construction implies that $\{d_{A/R}(a) \mid a \in A\}$ generate $\Omega_{A/R}$. In fact, if $(a_i)_{i \in I}$ generate A as an R-algebra, then $\{d_{A/R}(a_i) \mid i \in I\}$ generate $\Omega_{A/R}$. This is because $d(a_{i_1}, \dots, a_{i_r})$ lies in the linear span of $d(a_{i_1}), \dots, d(a_{i_r})$ by the Leibniz rule.

In particular, if A is a finitely-generated R-algebra, then $\Omega_{A/R}$ is finitely-generated.

Examples 1.6.6.

(1) We have that $\Omega_{R/R} = 0$.
(2) We know that $\Omega_{R[x_1,\dots,x_n]/R}$ is generated by $dx_1, \dots, dx_n$. We claim that these are linearly independent. Indeed, consider

$$\begin{aligned} \partial_i \colon R[x_1, \dots, x_n] &\to R[x_1, \dots, x_n] \\ f &\mapsto \frac{\partial f}{\partial x_i}. \end{aligned}$$

This maps $x_j \mapsto 0$ for $j \neq i$ and $x_i \mapsto 1$, so the corresponding morphism

$$\Omega_{R[x_1,\dots,x_n]/R} \to R[x_1, \dots, x_n]$$

is given by $dx_j \mapsto 0$ for $j \neq i$ and $dx_i \mapsto 1$. This shows that $dx_1, \dots, dx_n$ are linearly independent.

Note that $df = \displaystyle\bigoplus_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$.

Proposition 1.6.7. *If $S \subseteq A$ is a multiplicative system, then we have a canonical isomorphism*

$$\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}.$$

Proof. If M is an $S^{-1}A$-module, we have a map

$$\mathrm{Der}_A(S^{-1}A, M) \to \mathrm{Der}_R(A, M)$$

induced by $A \to S^{-1}A$. We claim that this is an isomorphism, i.e. given an R-derivation $D \colon A \to M$, there is a unique extension $\widetilde{D} \colon S^{-1}A \to M$. This is given by the quotient rule:

$$\widetilde{D}\left(\frac{a}{s}\right) = \frac{1}{s} D(a) - \frac{a}{s^2} D(s).$$

It is easy to check this is well-defined and gives a derivation.

This means that for any $S^{-1}A$-module M,

$$\mathrm{Hom}_{S^{-1}A}(\Omega_{S^{-1}A/R}, M) \cong \mathrm{Hom}_R(\Omega_{A/R}, M) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}\Omega_A, M).$$

This is functorial in M, so it comes from an isomorphism $\Omega_{S^{-1}A/R} \cong S^{-1}\Omega_{A/R}$. ☐

Proposition 1.6.8. *Let A and B be finitely-generated B algebras. Then*

$$\Omega_{A\otimes_R B/R} \cong \Omega_{A/R} \otimes_R B.$$

Proof. We have a series of functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_{A\otimes_R B}(\Omega_{A\otimes_R B/R,M}) &\cong \mathrm{Der}_B(A\otimes_R B, M) \\ &\cong \mathrm{Der}_R(A, M) && \text{(induced by restriction)} \\ &\cong \mathrm{Hom}_A(\Omega_{A/R}, M) \\ &\cong \mathrm{Hom}_{A\otimes_R B}(\Omega_{A/R}\otimes_A (A\otimes_R B), M) \\ &\cong \mathrm{Hom}_{A\otimes_R B}(\Omega_{A/R}\otimes_R B, M) \end{aligned}$$

where last isomorphism is given by $\Omega_{A/R}\otimes_R B \cong \Omega_{A/R}\otimes_A (A\otimes_R B)$. ☐

Consider ring homomorphisms $R \xrightarrow{\varphi} A \xrightarrow{\psi} B$.

Proposition 1.6.9. *There exists an exact sequence*

$$\begin{array}{ccccccc} \Omega_{A/R}\otimes_A B & \xrightarrow{\quad\alpha\quad} & \Omega_{B/R} & \xrightarrow{\quad\beta\quad} & \Omega_{B/A} & \longrightarrow & 0 \\ d_{A/R}(a)\otimes b & \longrightarrow & b\cdot d_{B/R}(\psi(a)) & & & & \\ & & d_{B/R}(b) & \longrightarrow & d_{B/A}(b) & & \end{array}$$

Proof. The proof is given as a homework exercise. ☐

Proposition 1.6.10. *Suppose ψ is a surjective map with $\ker\psi = I$ (note that in this case, $\Omega_{B/A} = 0$). Then the sequence*

$$\begin{array}{ccccccc} I/I^2 & \xrightarrow{\quad\delta\quad} & \Omega_{A/R}\otimes_A B & \xrightarrow{\quad\alpha\quad} & \Omega_{B/R} & \longrightarrow & 0 \\ \overline{a} & \longrightarrow & d_{A/R}(a)\otimes 1 & & & & \end{array}$$

is exact.

Proof. The proof is given as a homework exercise. ☐

Remark 1.6.11. Suppose B is a finitely-generated R-algebra. Choose generators $b_1, \ldots, b_n \in B$, and suppose

$$\begin{aligned} A = R[x_1, \ldots, x_n] &\overset{\varphi}{\twoheadrightarrow} B \\ x_i &\mapsto b_i \end{aligned}$$

has $\ker\varphi = I$. Then by Proposition 1.6.10, we have an exact sequence

$$I/I^2 \xrightarrow{\delta} \underbrace{\Omega_{A/R} \otimes_A B}_{\bigoplus_{i=1}^n B dx_i} \xrightarrow{\alpha} \Omega_{B/R} \longrightarrow 0$$

and hence $\Omega_{B/R}$ is the quotient of the free module $\bigoplus_{i=1}^n Be_i$ by the relations:

$$f \in I \rightsquigarrow \sum_{i=1}^n \frac{\partial f}{\partial x_i}(b_1, \dots, b_n)e_i.$$

Remark 1.6.12. If $A \to S^{-1}A \to B$, then by Propositions 1.6.9 and 1.6.7

$$\underbrace{\Omega_{S^{-1}A/A}}_{=S^{-1}\Omega_{A/A}=0} \otimes B \longrightarrow \Omega_{B/A} \longrightarrow \Omega_{B/S^{-1}A} \longrightarrow 0$$

which shows that $\Omega_{B/A} \cong \Omega_{B/S^{-1}A}$.

Next, we want to define similar invariants associated to morphisms of algebraic varieties $f\colon X \to Y$. Explicitly, we want to glue the modules of differentials to get a quasi-coherent sheaf on X.

Lemma 1.6.13. *Let X be an algebraic variety. Suppose we have a map α that assigns to each affine open subset $U \subseteq X$, an $\mathcal{O}_X$-module $\alpha(U)$, together with restriction maps: for all affine open subsets $V \subseteq U$, we have a restriction map $\alpha(U) \to \alpha(V)$ which is a morphism of $\mathcal{O}_X(U)$-modules, which satisfy the usual compatibility condition, and if $V = D_U(f)$, then*

$$\alpha(U)_f \to \alpha(V)$$

is an isomorphism. Then there is a quasi-coherent sheaf $\mathcal{F}$ on X with isomorphisms for $U \subseteq X$ affine:

$$\mathcal{F}|_U \cong \widetilde{\alpha(U)},$$

compatible with restrictions, and $\mathcal{F}$ is unique up to isomorphism.

Proof. If $U \subseteq X$ is an affine open subset, consider $\mathcal{F}_U := \widetilde{\alpha(U)}$. If $U \supseteq V$, we have $\alpha(U) \to \alpha(V)$, which induces

$$\alpha(U) \otimes_{\mathcal{O}(U)} O(V) \to \alpha(V)$$

corresponding to the morphism of sheaves $\varphi_{U,V}\colon \mathcal{F}_U|_V \to \mathcal{F}|_V$.

If $U \supseteq V \supseteq W$, then $\varphi_{V,W} \circ \varphi_{U,V}|_W = \varphi_{U,W}$.

The assumptions also imply that $\varphi_{U,V}$ is an isomorphism if $V = D_U(f)$. In general, by covering V by principal affine open subsets, then $\varphi_{U,V}$ is an isomorphism for all $V \subseteq U$.

This implies that given any two affine open U_1, U_2, we have a canonical isomorphism

$$\mathcal{F}_{U_1}|_{U_1 \cap U_2} \cong \mathcal{F}_{U_2}|_{U_1 \cap U_2}$$

since they are both isomorphic to $\mathcal{F}|_{U_1 \cap U_2}$. We can glue the $\mathcal{F}_U$ together to get $\mathcal{F}$. $\square$

Remark 1.6.14. Given a morphism $f\colon X \to Y$, we get a similar statement if instead of using all affine open subsets of X, we only use those affine open subsets $U \subseteq X$ such that there is an affine open subset $V \subseteq Y$ such that $f(U) \subseteq V$.

Corollary 1.6.15. *In our geometric setting, let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be morphisms of affine varieties where g is an open immersion, then

$$\Omega_{\mathcal{O}(X)/\mathcal{O}(Y)} \cong \Omega_{\mathcal{O}(X)/\mathcal{O}(Z)}.$$

Proof. We have maps $\mathcal{O}(Z) \to \mathcal{O}(Y) \to \mathcal{O}(X)$ which given an exact sequence by Proposition 1.6.9

$$\Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)} \otimes \mathcal{O}(X) \longrightarrow \Omega_{\mathcal{O}(X)/\mathcal{O}(Z)} \longrightarrow \Omega_{\mathcal{O}(X)/\mathcal{O}(Y)} \longrightarrow 0$$

It is enough to show that $\Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)} = 0$. For any maximal ideal $\mathfrak{m}$ in $\mathcal{O}(Y)$, we need to show that

$$\Omega_{\mathcal{O}(Y)_{\mathfrak{m}}/\mathcal{O}(Z)} = (\Omega_{\mathcal{O}(Y)/\mathcal{O}(Z)})_{\mathfrak{m}} = 0$$

where the first equality follows from Proposition 1.6.7. By Remark 1.6.12,

$$\Omega_{\mathcal{O}(Y)_{\mathfrak{m}}/\mathcal{O}(Z)} \cong \Omega_{\mathcal{O}(Y)_{\mathfrak{m}}/\mathcal{O}(Z)_{\mathfrak{m}\cap\mathcal{O}(Z)}}$$

since $\mathcal{O}(Z)_{\mathfrak{m}\cap\mathcal{O}(Z)} \to \mathcal{O}(Y)_{\mathfrak{m}}$ is an isomorphism, since g is an open immersion. □

Suppose now $f\colon X \to Y$ is a morphism of algebraic varieties. For every affine open subsets $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, consider $\Omega_{\mathcal{O}(U)/\mathcal{O}(V)}$. By Corollary 1.6.15, this is independent of the choice of V.

If $U' \subseteq U \to V$, we have a map

$$\Omega_{\mathcal{O}(U)/\mathcal{O}(V)} \otimes_{\mathcal{O}(U)} \mathcal{O}(U') \to \Omega_{\mathcal{O}(U')/\mathcal{O}(U)}$$

which is an isomorphism if U' is a principal affine open subset. By Lemma 1.6.13, there exists a unique quasi-coherent sheaf $\Omega_{X/Y}$ such that for U, V as above,

$$\Omega_{X/Y}|_U \cong \widetilde{\Omega_{\mathcal{O}(U)/\mathcal{O}(V)}}.$$

Then $\Omega_{X/Y}$ is actually coherent.

Definition 1.6.16. For a morphism $f\colon X \to Y$ of algebraic varieties, $\Omega_{X/Y}$ is the *relative cotangent sheaf*. If Y is a point, we write Ω_X for $\Omega_{X/Y}$ and call it the *cotangent sheaf of* X. We call $T_X = \Omega_X^\vee$ the tangent sheaf.

Remark 1.6.17. For any $x \in X$, we have that

$$(\Omega_{X/Y})_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$$

by Remark 1.6.12 and Proposition 1.6.7.

The two Propositions 1.6.9 and 1.6.10 globalize as follows

- If $X \xrightarrow{f} Y \xrightarrow{g} Z$, we get an exact sequence of coherent sheaves on X:

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

- If f is a closed immersion with ideal $\mathcal{I}$, we get an exact sequence of coherent sheaves on X

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow 0$$

Definition 1.6.18. For a closed immersion $f\colon X \to Y$, $\mathcal{I}/\mathcal{I}^2$ is the *conormal sheaf of X in Y*, and

$$(\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \Omega_X)$$

is the *normal sheaf of X in Y*, denoted $N_{X/Y}$.

Proposition 1.6.19. *For every $x \in X$,*

$$(\Omega_X)_{(x)} \cong (T_xX)^\vee.$$

Proof. Recall that $T_xX = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, where $\mathfrak{m} \subseteq R = \mathcal{O}_{X,x}$ is the maximal ideal. Recall also that

$$(\Omega_X)_{(x)} = \Omega_{X,x}/\mathfrak{m}\cdot\Omega_{X,x},$$

so

$$\begin{aligned}
(\Omega_X)^\vee_{(x)} &= \operatorname{Hom}_k(\Omega_{X,x}/\mathfrak{m}\Omega_{X,x}, k) \\
&= \operatorname{Hom}_R(\Omega_{X,x}, k) && \text{as } k = R/\mathfrak{m} \\
&= \operatorname{Der}_k(R, k),
\end{aligned}$$

since $\Omega_{X,x} = \Omega_{R/k}$. We note note that

$$\operatorname{Der}_k(R,k) = \operatorname{Der}_k(R/\mathfrak{m}^2, k)$$

by the Leibniz rule, since $\mathfrak{m}\cdot k = 0$. Finally

$$R/\mathfrak{m}^2 = k + \mathfrak{m}/\mathfrak{m}^2$$

and it is easy to see that by restricting to $\mathfrak{m}/\mathfrak{m}^2$ we get that

$$\operatorname{Der}_k(R/\mathfrak{m}^2, k) \cong \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = T_xX.$$

This completes the proof by dualizing. □

Proposition 1.6.20. *A variety X is smooth if and only if Ω_X is locally free. In this case, Ω_X has rank n on an irreducible component of dimension n.*

Proof. We may assume that X is connected. If X is smooth, it is irreducible, and if $n = \dim X$, then $\dim_k T_xX = n$, so by Proposition 1.6.19, $\dim_k(\Omega_X)_{(x)} = n$ for any $x \in X$, so Ω_X is locally free of rank n.

Suppose conversely that Ω_X is locally free of rank n. Recall that $X_{\mathrm{sm}} \subseteq X$ is dense. Then every irreducible component of X has dimension n, because we can find an open subset of that component which is smooth. Hence $\dim(\mathcal{O}_{X,x}) = n$ for any $x \in X$. Since $\dim_k(\Omega_X)_{(x)} = n$ for all $x \in X$, this shows by Proposition 1.6.19 that X is smooth. □

Note that if X is smooth, then Ω_X is locally free, and hence $T_X = \Omega_X^\vee$ is also locally free.

Definition 1.6.21. The sheaf of *p-differentials* on X is defined by $\Omega_X^p = \bigwedge^p \Omega_X$. When X is smooth, this is locally free.

Definition 1.6.22. If X is irreducible of dimension n, $\omega_X = \Omega_X^n$ is the *canonical line bundle of X.*

Conjecture (Lipman-Zariski)**.** A variety X is smooth if and only if T_X is locally free.

This is known in many cases (but not all of them).

Example 1.6.23. If $X = \mathbb{P}^n$, we have a short exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

This will be discussed during the problem session.

Proposition 1.6.24. *If $Y \subseteq X$ is a subvariety defined by $\mathcal{I}$, with X and Y both smooth, then $\mathcal{I}/\mathcal{I}^2$ is locally free and we have a short exact sequence*

$$0 \longrightarrow T_Y \longrightarrow T_{X/Y} \longrightarrow N_{Y/X} \longrightarrow 0.$$

Proof. We may assume X and Y are both irreducible. Recall that for any $x \in X$, there is an affine open neighborhood U of x, there exist $f_1, \ldots, f_r \in \mathcal{O}(U)$ such that

$$I_{Y \cap U/U} = (f_1, \ldots, f_r)$$

where $r = \operatorname{codim}_X Y$. In this case,

$$\underbrace{\mathcal{O}(U)/(f_1, \ldots, f_r)}_{\mathcal{O}(U \cap Y)}[t_1, \ldots, t_r] \to \bigoplus_{m \geq 0} \frac{I^m_{Y \cap U/U}}{I^{m+1}_{Y \cap U/U}}$$

$$t_i \mapsto \overline{f_i} \in I/I^2$$

is an isomorphism. In particular,

$$I_{Y \cap U}/I^2_{Y \cap U/U} \cong \bigoplus_{i=1}^{r} \mathcal{O}(U \cap Y) t_i,$$

which is free. This shows that $\mathcal{I}/\mathcal{I}^2$ is locally free of rank equal to the codimension of Y in X.

We have an exact sequence

$$\begin{tikzcd}
\mathcal{I}/\mathcal{I}^2 \arrow[rr] \arrow[dr, two heads] & & \Omega_X|_Y \arrow[r, "\varphi"] & \Omega_Y \arrow[r] & 0 \\
& \ker(\varphi) \arrow[ur, hook] & & &
\end{tikzcd}$$

with Ω_Y locally free of rank $\dim Y$ and $\Omega_{X/Y}$ locally free of rank $\dim X$. Then $\ker(\varphi)$ is locally free of rank r, and

$$\mathcal{I}/\mathcal{I}^2 \to \ker(\varphi)$$

is a surjective morphism between locally free sheaves of the same rank, so it is an isomorphism. Therefore the above sequence is actually a short exact sequence. Dualizing it, we get the result. $\square$

2. Normal varieties and divisors

In this section, we will discuss divisors on algebraic varieties, which allow to study morphisms to projective spaces. We first start with a few sections on normal varieties.

2.1. **Normal varieties.**

We want to extend our definition of a normal variety to the case of varieties which may not be affine and possibly not irreducible.

Proposition 2.1.1. *Suppose A is a domain with fraction field K and $S \subseteq A$ is a multiplicative system. If $A' \subseteq K$ is the integral closure of A in K, then the integral closure of $S^{-1}A$ in K is $S^{-1}A'$.*

Proof. The proof is left as an exercise. $\square$

Proposition 2.1.2. *Given a variety X, the following are equivalent:*

1. *for any affine open subset $U \subseteq X$ and every connected component V of U, $\mathcal{O}(V)$ is a domain which is integrally closed (in its fraction field),*
2. *there exists an affine open cover of X by $U_1, \ldots, U_n$ such that each U_i is irreducible and $\mathcal{O}(U_i)$ is integrally closed.*
3. *for any irreducible closed subset $V \subseteq X$, $\mathcal{O}_{X,V}$ is an integrally closed domain,*
4. *for any $x \in X$, $\mathcal{O}_{X,x}$ is an integrally closed domain.*

Definition 2.1.3. A variety X is *normal* if the equivalent conditions in Proposition 2.1.2 hold.

Note that if X is irreducible and affine, this agrees with the previous definition.

Proof of Proposition 2.1.2. We see immediately that (1) implies (2) and (3) implies (4). Moreover, (2) implies (3) by Proposition 2.1.1.

It remains to show that (4) implies (1). Since $\mathcal{O}_{X,x}$ is a domain, every point x lies on a unique irreducible component, we may assume that X is irreducible. If $A = \mathcal{O}(U)$ is a domain with integral closed A', by assumption, we have that

$$A_{\mathfrak{m}} = A'_{\mathfrak{m}} \text{ for any maximal ideal } \mathfrak{m}$$

so $A = A'$. $\square$

Review of DVRs.

Definition 2.1.4. If K is a field, a *discrete valuation of K* is a surjective map

$$v \colon K \to \mathbb{Z} \cup \{\infty\}$$

such that

1. $v(a) = \infty$ if and only if $a = 0$,
2. $v(a + b) \geq \min\{v(a), v(b)\}$ for all a, b,
3. $v(ab) = v(a) + v(b)$.

Example 2.1.5. For $K = \mathbb{Q}$, we can let $v(p^n a) = n$ when $(a, p) = 1$ for a fixed prime p.

Proposition 2.1.6. *Given a domain R with fraction field K, the following are equivalent:*

1. *there exists a discrete valuation v on K such that $R = \{a \mid v(a) \geq 0\}$,*
2. *R is a local PID, not a field,*
3. *R is local with a principal maximal ideal, and Noetherian.*

Definition 2.1.7. A domain R is a *discrete valuation ring* (*DVR*) if the equivalent conditions in Proposition 2.1.6 hold.

Proof of Proposition 2.1.6. To show that (1) implies (2), $\mathfrak{m} = \{a \mid v(a) > 0\}$ is an ideal in R. If $a \in R \setminus \mathfrak{m}$, then $u^{-1} \in R$, so $\mathfrak{m}$ is the unique maximal ideal of R. If $I \neq 0$ is an ideal, choose $a \in I \setminus \{0\}$ such that $v(a)$ is minimal. If $b \in I$, $v(b) \geq v(a)$, then $\frac{b}{a} \in R$, so $I = (a)$.

Since (2) implies (3) is obvious, we only have to show that (3) implies (1). Suppose $\mathfrak{m} = (\pi)$ is the maximal ideal of R. If $a \in R$, by Krull's Intersection Theorem, there exists a unique j such that $a \in \mathfrak{m}^j \setminus \mathfrak{m}^{j+1}$. Then set $v(a) = j$. It is clear that $v(a+b) \geq \min\{v(a), v(b)\}$ and $v(a \cdot b) = v(a) + v(b)$, where the second assertion follows from $\mathfrak{m} = (\pi)$. This extends to K by $v(a/b) = v(a) - v(b)$, and one shows that $R = \{a \mid v(a) \geq 0\}$. □

Note that if $\mathfrak{m} = (\pi)$ is the maximal ideal in the local ring R, then every ideal in R is (0) or (π^m) for $m \geq 0$. Therefore, R has two prime ideals, (0) and $\mathfrak{m}$, which shows that $\dim R = 1$.

Suppose X is an algebraic variety and $V \subset X$ irreducible, closed, of dimension 1. Then $\mathcal{O}_{X,V}$ is a DVR if and only if X is smooth at V and by Problem 1 from Problem Set 3 we have that $V \cap X_{\mathrm{sm}} \neq \emptyset$.

Lemma 2.1.8. *If R is a Noetherian, integrally closed domain, $a \in R \setminus \{0\}$, $\mathfrak{p} \in \mathrm{Ass}_R(R/(a))$, then $R_\mathfrak{p}$ is a DVR (in particular,* $\mathrm{codim}(\mathfrak{p}) = 1$*).*

Proof. Replace R by $R_\mathfrak{p}$ to assume that R is local and $\mathfrak{p} = \mathfrak{m}$ is the unique maximal ideal. By hypothesis, there exists $b \notin (a)$ such that

$$\mathfrak{m} = \{u \in R \mid ub \in (a)\}.$$

For $\frac{b}{a} \in \mathrm{Frac}(R)$, we have that $\mathfrak{m} \cdot \frac{b}{a} \subseteq R$.

If $\mathfrak{m} \cdot \frac{b}{a} \subseteq \mathfrak{m}$, by the determinant trick, we get that $\frac{b}{a}$ is integral over R. Since R is integrally closed, $\frac{b}{a} \in R$. Then $\mathfrak{m} = R$, which contradicts $\mathfrak{m}$ being maximal.

Therefore, $\mathfrak{m} \cdot \frac{b}{a} = R$, which implies that $\frac{a}{b} \in \mathfrak{m}$. By the description of $\mathfrak{m}$ above, for any $u \in \mathfrak{m}$, $u\frac{b}{a} \in R$, so $u \in \left(\frac{a}{b}\right)$. This shows that $\mathfrak{m} = \left(\frac{a}{b}\right)$. Therefore, R is a DVR. □

Lemma 2.1.9. *Let R be a ring. Then the following are equivalent:*

1. *R is a DVR,*
2. *R is a local, Noetherian domain with* $\dim R = 1$*, which is integrally closed.*

Proof. Clearly, (1) implies (2) (note that since R is a PID, R is a UFD, so it is integrally closed).

To show that (2) implies (1), choose $a \in \mathfrak{m} \setminus \{0\}$. Then

$$(0) \notin \mathrm{Ass}_R(R/(a)) \neq \emptyset,$$

so $\mathfrak{m} \in \mathrm{Ass}_R(R/(a))$, since $\dim R = 1$. The result then follows from Lemma 2.1.8. $\square$

Proposition 2.1.10. *Let A be a Noetherian domain. Then A is integrally closed if and only if the following conditions hold:*

(1) *for any prime $\mathfrak{p}$ of codimension 1, $A_{\mathfrak{p}}$ is a DVR,*
(2) $A = \bigcap_{\operatorname{codim}\mathfrak{p}=1} A_{\mathfrak{p}}$.

Moreover, (2) *can be replaced by*

(2') *for any $a \in A \setminus \{0\}$ and any prime $\mathfrak{p} \in \mathrm{Ass}_R(R/(a))$,* $\operatorname{codim}(\mathfrak{p}) = 1$.

Proof. Suppose first that (1) and (2) hold. Then $A_{\mathfrak{p}}$ is integrally closed for all $\mathfrak{p}$ of codimension 1, and $A = \bigcap_{\operatorname{codim}\mathfrak{p}=1}$ implies that A is integrally closed.

If A is integrally closed, for any $\mathfrak{p}$ of codimension 1, $A_{\mathfrak{p}}$ is a DVR by Lemma 2.1.9, and (2') holds by Lemma 2.1.8.

The proof will be complete if we show that (2') implies (2). The '$\subseteq$' inclusion is immediate. Suppose $\frac{b}{a} \in \bigcap_{\operatorname{codim}\mathfrak{p}=1} A_{\mathfrak{p}}$. Consider a minimal primary decomposition

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r.$$

Then each $\mathfrak{q}_i$ is primary and $\mathrm{rad}(\mathfrak{q}_i) = \mathfrak{p}_i$ is prime. Then

$$\mathrm{Ass}_R(R/(a)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}.$$

By (2'), $\operatorname{codim}(\mathfrak{p}_i) = 1$ for all i, so $\frac{b}{a} \in A_{\mathfrak{p}_i}$. Then for any i, there exists $s_i \in A \setminus \mathfrak{p}_i$ such that $s_i b \in (a) \subseteq \mathfrak{q}_i$, so $b \in \mathfrak{q}_i$ since $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Therefore, $b \in (a)$ and hence $\frac{b}{a} \in A$. $\square$

2.2. Geometric properties of normal varieties.

Definition 2.2.1. An algebraic variety X is *smooth in codimension 1* if $\operatorname{codim}_X(X_{\mathrm{sing}}) \geq 2$.

If $Z \subseteq X$ is closed with irreducible components $Z_1, \ldots, Z_r$, $\operatorname{codim}_X(Z) = \min_i\{\operatorname{codim}_X(Z_i)\}$.

Note that X is smooth in codimension 1 if and only if for any irreducible closed subset $V \subseteq X$ of codimension 1, we have that $V \cap X \neq \emptyset$. This is equivalent to $\mathcal{O}_{X,V}$ being a regular ring. In particular, if X is a normal variety, then X is smooth in codimension 1 (since this holds for irreducible affine open subsets).

Proposition 2.2.2. *Let X be a normal variety. Then if $\mathcal{E}$ is a locally free sheaf on X and $U \subseteq X$ is an open subset such that* $\operatorname{codim}_X(X \setminus U) \geq 2$, *then the restriction map* $\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E})$ *is an isomorphism.*

Proof. It is enough to prove this when X is affine and irreducible and $\mathcal{E} = \mathcal{O}_X$. Indeed, choose an open cover $X = U_1 \cup \cdots \cup U_n$ by affine irreducible subsets U_i such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r_i}$. Then we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathcal{E}) \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{i=1}^{n} \Gamma(U_i,\mathcal{E}) \arrow[r] \arrow[d, "\beta"] & \bigoplus_{i,j=1}^{n} \Gamma(U_i\cap U_j,\mathcal{E}) \arrow[d, "\gamma"] \\
0 \arrow[r] & \Gamma(U,\mathcal{E}) \arrow[r] & \bigoplus_{i=1}^{n} \Gamma(U\cap U_i,\mathcal{E}) \arrow[r] & \bigoplus_{i,j=1}^{n} \Gamma(U\cap U_i\cap U_j,\mathcal{E})
\end{tikzcd}$$

with exact rows (by the sheaf axiom) and the vertical maps being restriction maps. The special case implies that β, γ are isomorphisms, so α is an isomorphism by Five Lemma.

Suppose that X is affine and $A = \mathcal{O}(X)$. The map $A = \mathcal{O}(X) \to \mathcal{O}(U)$ is injective since $U \subseteq X$ is dense (otherwise, $\mathrm{codim}_X(X \setminus U) = 0$). If $\varphi \in \mathcal{O}(U)$, then for any $\mathfrak{p}$ in A of codimension 1, $V(\mathfrak{p}) \cap U \neq \emptyset$, i.e. $\varphi \in A_{\mathfrak{p}}$, this completes the proof. ☐

Corollary 2.2.3. *Suppose X is an irreducible normal variety and $\varphi \in k(X)$ with domain U. Then every irreducible component of $X \setminus U$ has codimension 1.*

Proof. The proof is left as an exercise. ☐

Notation. Suppose X is an irreducible variety, smooth in codimension 1. If $V \subseteq X$ is irreducible, closed, of codimension 1, then $\mathcal{O}_{X,V}$ is a DVR. We write ord_V for the corresponding discrete valuation on $k(X)$.

We say that $\varphi \in k(X)$ has a *pole* along V if $\mathrm{ord}_V(\varphi) < 0$. This is equivalent to saying that φ^{-1} is defined in an open subset U with $U \cap V \neq \emptyset$ and $\varphi^{-1}|_{U\cap V} = 0$.

We say that $\varphi \in k(X)$ has a *pole of order* $m > 0$ if $\mathrm{ord}_V(\varphi) = -m$, and has a *zero of order* $m > 0$ if $\mathrm{ord}_V(\varphi) = m$.

Note that if X is normal and $\varphi \in k(X)$ with domain U, then

$$X \setminus U = \bigcup \{V \subseteq X \mid V \text{ irreducible, } \mathrm{codim}(V) = 1, \ \varphi \text{ has a pole along } V\}$$

.

Proposition 2.2.4. *Suppose that X is an irreducible variety, smooth in codimension 1.*

(1) *If $f\colon X \dashrightarrow Y$ is a rational map and Y is complete, $U = \mathrm{Dom}(f)$, then*

$$\mathrm{codim}_X(X \setminus U) \geq 2.$$

(2) *More generally, if $f\colon X \dashrightarrow Y$ is a rational map and $g\colon Y \to Z$ is proper such that $g \circ f$ is a morphism, $U = \mathrm{Dom}(f)$, then* $\mathrm{codim}_X(X \setminus U) \geq 2$.

Proof. It is enough to show (2). We may assume that f is dominant by replacing Y with a closed subvariety (hence Y is irreducible). By Chow's Lemma, there is a birational map $h\colon \widetilde{Y} \to Y$ with $\widetilde{Y}$ irreducible such that $g \circ h$ factors as in the diagram, with i a closed immersion, and p the projection:

$$
\begin{tikzcd}
 & & \widetilde{Y} \arrow[r, hook, "i"] \arrow[d, "h"] & Z \times \mathbb{P}^n \arrow[dll, bend left, "p"] \\
X \arrow[rr, dashed, "f"] \arrow[dr] & & Y \arrow[dl, "g"'] & \\
 & Z & &
\end{tikzcd}
$$

It is enough to prove the conclusion for $h^{-1} \circ f$. Moreover, it it enough to prove this for $i \circ (h^{-1} \circ f) = (g \circ f, u)$ for some $u \colon X \dashrightarrow \mathbb{P}^n$.

It is enough to prove that u is defined on the complement of a codim ≥ 2 subset. Hence it it enough to consider rational map $X \dashrightarrow \mathbb{P}^n$.

There is an open subset $U \subseteq X$ and functions $\varphi_0, \ldots, \varphi_n \in \mathcal{O}(U)$ such that f is defined on U, given by

$$x \mapsto [\varphi_0(x), \ldots, \varphi_n(x)].$$

We want to show that for any $V \subseteq X$ irreducible of codimension 1, $\operatorname{Dom}(f) \cap V \neq \emptyset$. Let j be such that $\operatorname{ord}_V(\varphi_j) = \min\{\operatorname{ord}_V(\varphi_i) \mid 0 \leq i \leq n\}$. Then $\operatorname{ord}_V \varphi_i/\varphi_j \geq 0$ for all j. Then there is $U \subseteq X$ open such that $U \cap V \neq \emptyset$ and $\frac{\varphi_i}{\varphi_j} \in \mathcal{O}(U)$. Then f can be defined on U by $[\varphi_0/\varphi_j, \ldots, \varphi_n/\varphi_j]$. $\square$

Theorem 2.2.5. *Let A be a domain which is an algebra of finite type over a field k. If $K = \operatorname{Frac}(A)$ and L/K is a finite field extension, the integral closure B of A in L is finite over A.*

Proof. Since A is Noetherian, it is enough to prove this when replacing L by a finite field extension.

Step 1. Reduce to the case when A is normal and L/K is separable.

By Noether Normalization Theorem, there exists $R \subseteq A$ such that $R \cong k[x_1, \ldots, x_n]$ and A/R is finite. After replacing A by R, we may assume that $A = k[x_1, \ldots, x_n]$. By enlarging L, we may assume that L/K is normal, $G = \operatorname{Gal}(L/K)$, and $K' = L^G \subseteq L$. We then have $K \subseteq K' \subseteq L$ with L/K separable and K'/K is purely inseparable. Let us show that the integral closure A' of A in K' is finite over A. If $K' \neq K$, $p = \operatorname{char}(K) > 0$ and for all $f \in K'$, there exists $e > 0$ such that

$$f^{p^e} \in K = k(x_1, \ldots, x_n).$$

Then there exists a finite extension k'/k such that

$$K' \subseteq k'\left(x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}\right)$$

for some e. Then A' is contained in the integral closure of $k[x_1, \ldots, x_n]$ in $k'\left(x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}\right)$. This integral closure is

$$k'\left[x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}\right]$$

which is finite over $k[x_1, \ldots, x_n]$. Therefore, A' is finite over A, and we have reduced to the case where A is normal and L/K is separable.

Step 2. Suppose A is normal and L/K is separable. By enlarging L, we may assume that L/K is Galois with $\mathrm{Gal}(L/K) = G = \{\sigma_1, \ldots, \sigma_d\}$. Choose a basis $u_1, \ldots, u_d$ for L/K. We may assume that $u_1, \ldots, u_d \in B$ (multiply each u_i by an element of L to make the polynomial for u_i monic).

Let

$$M = (\sigma_i(u_j)) \in M_d(B), \quad D = \det(M).$$

(1) If $D = 0$, then there exist $\lambda_1, \ldots, \lambda_d \in L$, not all 0, such that

$$\left(\sum_{i=1}^{d} \lambda_i \sigma_i\right)(u_j) = 0 \text{ for all } j.$$

Then we have that

$$\sum_{i=1}^{d} \lambda_i \sigma_i = 0,$$

which is a contradiction. Indeed, after reordering, we have that

$$\lambda_1 \sigma_1 + \cdots + \lambda_r \sigma_r = 0 \tag{1}$$

with $\lambda_i \neq 0$ and r minimal with this property. It is clear that $r \geq 2$. Then for $a, b \in L$,

$$\left(\sum_{i=1}^{r} \lambda_i \sigma_i\right)(ab) = 0,$$

and since this holds for any b

$$\sum_{i=1}^{r} \underbrace{\lambda_i \sigma_i(a)}_{\in L} \sigma_i = 0. \tag{2}$$

If a is such that $\sigma_1(a) \neq \sigma_2(a)$, then $(2) - \lambda_i \cdot (1)$ gives another relation for $\sigma_2, \ldots, \sigma_{r-1}$, contradicting the minimality of r.

(2) We may assume that $D \neq 0$. We then show that

$$B \subseteq \frac{1}{D^2} \sum_{i=1}^{d} A \cdot u_i.$$

Since the right hand side is finitely-generated over A, this will complete the proof.

Note that $D \in B$. Then $\sigma_i(D)$ is the determinant obtained from M by permuting the rows, so $\sigma_i(D) = \pm D$, and this shows that

$$\sigma_i(D^2) = D^2 \text{ for all } i.$$

Hence $D^2 \in K$.

Given any $b \in B$,

$$b = \sum_{j=1}^{d} \alpha_j u_j, \quad \alpha_i \in K.$$

We want to show that $D^2\alpha_j inA$. We then have that

$$B \ni \sigma_i(b) = \sum_{j=1}^{d} \alpha_j \sigma_i(u_j)$$

and this shows that

$$M \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \in B^n.$$

Multiplying by the adjoint of M, we see that $D \cdot \alpha_i \in B$, so $D^2\alpha_i \in B \cap K = A$, since A is normal.

This completes the proof. $\square$

Suppose X is an irreducible algebraic variety. We want to find a normal variety that dominates X. For an affine open subset $U \subseteq X$, let $\mathcal{A}(U)$ be the integral closure of $\mathcal{O}_X(U)$ in $k(X)$. If $U \subseteq V$ are affine, the inclusion $\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_X(U)$ gives an inclusion

$$\mathcal{A}(V) \hookrightarrow \mathcal{A}(U)$$

If $U = D_V(f)$, then

$$\mathcal{A}(V)_f \cong \mathcal{A}(U).$$

Then by Lemma 1.6.13, $\mathcal{A}$ can be extended to a quasicoherent $\mathcal{O}_X$-algebra. If $U \subseteq X$ is affine, $\mathcal{A}(U)$ is reduced and $\mathcal{A}(U)$ is finite over $\mathcal{O}_X(U)$, so $\mathcal{A}$ is a reduced coherent $\mathcal{O}_X$-algebra.

The normalization of X is

$$X^{\text{norm}} = \mathcal{M}\text{ax}\mathcal{S}\text{pec}(\mathcal{A}) \xrightarrow{\pi} X.$$

It is clear that

(1) X^{norm} is normal,
(2) π is finite,
(3) X^{norm} is irreducible,
(4) π is birational.

Universal property of normalization. Given a normal, irreducible variety Z and a dominant map $f\colon Z \to X$, there is a unique $g\colon Z \to X^{\text{norm}}$ such that $\pi \circ g = f$, i.e. the diagram

$$\begin{tikzcd} Z \arrow[r, dotted, "g"] \arrow[dr, "f"'] & X^{\text{norm}} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

commutes.

Proving this is left as a homework exercise.

Remark 2.2.6. If X has irreducible components $X_1, \ldots, X_r$, the normalization of X is

$$\coprod_{i=1}^{r} X_i^{\text{norm}} \to X.$$

Definition 2.2.7. A variety X is *locally factorial* if for any $x \in X$, the ring $\mathcal{O}_{X,x}$ is a UFD.

Note that this implies that $\mathcal{O}_{X,V}$ is a UFD for all irreducible closed $V \subseteq X$. It is actually very rare that for an affine variety X, $\mathcal{O}(X)$ is a UFD, but it does happen quite often that the local rings are UFDs.

Theorem 2.2.8. *If X is a smooth variety, then X is locally factorial (in particular, X is normal).*

The proof uses completions of local rings. For any $x \in X$:

$$\widehat{\mathcal{O}_{X,x}} = \varprojlim_{q \geq 1} \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^q$$

This records, roughly, what happens in a very small neighborhood of a point. One way to think about it is that $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^q$ records the first q coefficients of the Taylor polynomial, so the whole inverse limit is similar to a Taylor polynomials.

Example 2.2.9. We have that

$$\widehat{k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}} \cong k[\![x_1, \ldots, x_n]\!].$$

Suppose that X is an affine variety, $R = \mathcal{O}(X)$ and $Y \subseteq X$ is defined by I. Then we can set

$$\widehat{R} = \varprojlim R/I^n.$$

This is an algebraic analogue of a tubular neighborhood of Y inside X.

Proposition 2.2.10. *If $x \in X$ is a smooth point,* $\dim(\mathcal{O}_{X,x}) = d$, *then*

$$\widehat{\mathcal{O}_{X,x}} \cong k[\![t_1, \ldots, t_d]\!].$$

Proof. We show that if $\mathfrak{m} \subseteq \mathcal{O}_{X,x}$ is a maximal ideals, $u_1, \ldots, u_d \in \mathfrak{m}$ is a minimal system of generators, then

$$S = k[t_1, \ldots, t_d] \to \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$
$$t_i \mapsto \overline{u_i} \in \mathfrak{m}/\mathfrak{m}^2$$

is an isomorphism. Let $\mathfrak{n} = (t_1, \ldots, t_d) \subseteq S$ be the maximal ideal and let

$$S/\mathfrak{n}^i \xrightarrow{\varphi_i} R/\mathfrak{m}^i$$
$$\overline{t_i} \mapsto u_i \mod \mathfrak{m}^i$$

We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n}^i/\mathfrak{n}^{i+1} \arrow[r] \arrow[d, "\cong"] & S/\mathfrak{n}^{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"] & S/\mathfrak{n}^i \arrow[r] \arrow[d, "\varphi_i"] & 0 \\
0 \arrow[r] & \mathfrak{m}^i/\mathfrak{m}^{i+1} \arrow[r] & R/\mathfrak{m}^{i+1} \arrow[r] & R/\mathfrak{m}^i \arrow[r] & 0
\end{tikzcd}$$

Then by induction on n and the Five Lemma, we can show that φ_i is an isomorphism for all i. Then $\varprojlim \varphi_i$ given an isomorphism

$$k[\![t_1, \dots, t_d]\!] \xrightarrow{\cong} \widehat{\mathcal{O}_{X,x}},$$

completing the proof. ☐

Lemma 2.2.11. *Let R be a domain.*

(1) *If R is a UFD, then for any $a, b \in R$, the ideal*
$$aR : bR = \{h \in R \mid hb \in Ra\}$$
is principal.

(2) *The converse holds if R is Noetherian.*

Proof. If

$$a = u \cdot \prod_{i=1}^{r} \pi_i^{m_i}$$
$$b = v \cdot \prod_{i=1}^{r} \pi_i^{n_i}$$

for units u, v and irreducibles π_i, then

$$aR : bR = \left(\prod_{i=1}^{r} \pi_i^{\max\{0, m_i - n_i\}} \right).$$

Recall that a ring R is a UFD if and only if

(i) every nonzero non-invertible element is a product of irreducible elements,

(ii) uniqueness up to reordering and rescaling by invertible elements.

Note also that (i) always holds for Noetherian rings. Moreover, if (i) holds then (ii) holds if and only if every irreducible element is prime.

Therefore, we just need to show that every irreducible element π of R is prime. If $\pi | ab$, then

$$b \in (\pi) : (a) = (h),$$

so $\pi \in (h)$, and hence $\pi = hh'$. Therefore, either h is invertible, so $\pi | a$ or h' is invertible, so $b \in (\pi)$, and hence $\pi | b$. ☐

Lemma 2.2.12. *If $\widehat{\mathcal{O}_{X,x}}$ is a UFD, then $\mathcal{O}_{X,x}$ is a UFD.*

Proof. We have the map

$$\begin{aligned} \psi \colon \mathcal{O}_{X,x} &\to \widehat{\mathcal{O}_{X,x}} \\ a &\mapsto (a \mod \mathfrak{m}^n)_{n \geq 1} \end{aligned}$$

Note that $\widehat{\mathcal{O}_{X,x}}$ is local ring with maximal ideal $\mathfrak{m} \cdot \widehat{\mathcal{O}_{X,x}}$, and

$$\mathcal{O}_{X,x}/\mathfrak{m} \cong \widehat{\mathcal{O}_{X,x}}/\mathfrak{m} \cdot \widehat{\mathcal{O}_{X,x}}.$$

Then ψ is injective and flat.

Write $R = \mathcal{O}_{X,x}$ to simplify notation. By Lemma 2.2.11, it is enough to show that if $a, b \in R$, $J = aR : bR$ is principal.

We have the exact sequence

$$0 \longrightarrow J \longrightarrow R \xrightarrow{\cdot b} R/aR$$

and tensoring it with $\widehat{R}$, we obtain

$$0 \longrightarrow J\widehat{R} \longrightarrow \widehat{R} \xrightarrow{\cdot b} \widehat{R}/a\widehat{R}$$

by flatness. Note that $J\widehat{R} = a\widehat{R} : b\widehat{R}$ is principal since R is a UFD. We finally see that

$$\dim_k J\widehat{R}/J\mathfrak{m}\widehat{R} = 1$$

and

$$\widehat{J/\mathfrak{m}J} = J/\mathfrak{m}J \cong J/J\mathfrak{m} \otimes \widehat{R} \cong J\widehat{R}/J\mathfrak{m}\widehat{R}.$$

By Nakayama Lemma, this shows that J is principal. □

Proof of Theorem 2.2.8. By Proposition 2.2.10

$$\widehat{\mathcal{O}_{X,x}} \cong k[\![t_1, \ldots, t_d]\!]$$

We know that $k[\![t_1, \ldots, t_d]\!]$ is a UFD (see for example Zariski–Samuel). Then the result follows from Lemma 2.2.12. □

2.3. **Divisors.** We will next study the following picture

$$\begin{array}{ccc} \text{geometric subvarieties of codimension } 1 & \xleftrightarrow{\text{now}} & \text{line bundles} \\ \text{later}\updownarrow & & \\ \text{maps to projective spaces} & & \end{array}$$

2.4. **Weil divisors.**

Definition 2.4.1. Let X be an irreducible variety, smooth in codimension 1. A *prime divisor* on X is an irreducible closed subset $V \subset X$ of codimension 1. The *group of (Weil) divisors* is

$$\mathrm{Div}(X) = \text{free abelian group on the set of prime divisors,}$$

so a *divisor* $D \in \mathrm{Div}(X)$ can be written as

$$D = \sum_{i=1}^{r} n_i V_i \quad n_i \in \mathbb{Z},\ V_i \text{ prime divisors.}$$

A divisor D is *effective* if all the coefficients are nonnegative, $n_i \geq 0$. Write $D \leq E$ if $E - D$ is effective.

For $\varphi \in k(X)^*$, let

$$\operatorname{div}(\varphi) = \sum_{V \text{prime divisor}} \operatorname{ord}_V(\varphi) V \in \operatorname{Div}(X).$$

This is well-defined: suppose φ is defined on U and φ is invertible on $U' \subseteq U$. Then $\operatorname{ord}_V(\varphi) \neq 0$ implies that $V \subseteq X \setminus U'$ and there are only finitely many such V of codimension 1.

Note that $\operatorname{ord}_V(\varphi\psi) = \operatorname{ord}_V(\varphi) + \operatorname{ord}_V(\psi)$ for any $\varphi, \psi \neq 0$, so $\operatorname{div}\colon k(X)^* \to \operatorname{Div}(X)$ is a morphism of abelian groups.

Definition 2.4.2. A divisor $D \in \operatorname{Div}(X)$ is *principal* if $D = \operatorname{div}(\varphi)$ for some $\varphi \in k(X)^*$. The principal divisors form a subgroup

$$\operatorname{PDiv}(X) = \{\operatorname{div}(\varphi) \mid \varphi \in k(X)^*\} \subseteq \operatorname{Div}(X)$$

and the quotient

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{PDiv}(X)$$

is called the *class group* of X. We write $[D] \in \operatorname{Cl}(X)$ for the image of $D \in \operatorname{Div}(X)$ in the class group of X.

Remark 2.4.3. Consider X normal and $\varphi \in k(X)^*$. Then $\operatorname{div}(\varphi) \geq 0$ if and only if $\varphi \in \mathcal{O}(X)$ and $\operatorname{div}(\varphi) = 0$ if and only if $\varphi \in \mathcal{O}(X)^*$.

Proposition 2.4.4. *Let X be an affine irreducible normal variety. Then $\operatorname{Cl}(X) = 0$ if and only if $\mathcal{O}(X)$ is a UFD.*

Lemma 2.4.5. *Let A be a Noetherian domain. Then A is a UFD if and only if any prime $\mathfrak{p} \subseteq A$ of codimension 1 is principal.*

Proof. To show the 'only if' implication, choose $a \in \mathfrak{p} \setminus \{0\}$, and write

$$a = u_1 \dots u_r \text{ for } u_i \text{ irreducible}$$

and since $\mathfrak{p}$ is prime, $u_i \in \mathfrak{p}$ for some i. Then

$$(0) \subsetneq (u_i) \subseteq \mathfrak{p}$$

and (u_i) is prime since A is a UFD, so $\mathfrak{p} = (u_i)$ since $\mathfrak{p}$ has codimension 1.

Conversely, note that since A is Noetherian, it is enough to show that if π is irreducible, then (π) is prime. Let $\mathfrak{p}$ be a minimal prime containing (π). Then the Principal Ideal Theorem shows that $\operatorname{codim}(\mathfrak{p}) = 1$. By hypothesis, $\mathfrak{p} = (a)$ for some a, and $(\pi) \subseteq (a)$ shows that $\pi = a \cdot b$. Since π is irreducible, b is invertible, and hence $(\pi) = (a) = \mathfrak{p}$. This shows that (π) is prime. $\square$

Proof of Proposition 2.4.4. By definition, $\operatorname{Cl}(X) = 0$ if and only if for any prime ideal $\mathfrak{p}$ of codimension 1 in $\mathcal{O}(X)$, $V(\mathfrak{p})$ is principal, i.e. there exists $\varphi \in k(X)^*$ such that $\operatorname{div}(\varphi) = V(\mathfrak{p})$. This is equivalent to $\varphi \in \mathcal{O}(X)$ and $\varphi\mathcal{O}(X)_{\mathfrak{p}} = \mathfrak{p}\mathcal{O}(X)_{\mathfrak{p}}$ and $\varphi \notin \mathfrak{q}$ for $\mathfrak{q} \neq \mathfrak{p}$ of codimension 1.

If $\mathcal{O}(X)$ is a UFD, given $\mathfrak{p}$, choose φ such that $\mathfrak{p} = (\varphi)$ (by Lemma 2.4.5). Then the conditions above are clearly satisfied.

Conversely, suppose $\mathrm{Cl}(X) = 0$ and let $\mathfrak{p} \subseteq \mathcal{O}(X)$ be prime of codimension 1. Choose φ such that $V(\mathfrak{p}) = \mathrm{div}(\varphi)$, so $\varphi \in \mathfrak{p}$, $\mathrm{ord}_{V(\mathfrak{p})}(\varphi) = 1$. If $a \in \mathfrak{p}$, $\mathrm{div}(a/\varphi) \geq 0$ by assumption on φ. This means that $\frac{a}{\varphi} \in \mathcal{O}(X)$, and hence $\mathfrak{p} = (\varphi)$. Hence $\mathcal{O}(X)$ is a UFD by Lemma 2.4.5. □

By this Proposition, we know that for affine, irreducible, normal varieties $\mathrm{Cl}(X) = 0$ if and only if $\mathcal{O}(X)$ is a UFD. In general, the class group measures *how far* $\mathcal{O}(X)$ is from being a UFD. Note that this is essentially the same as the class group for number fields, which measures how far the ring of integers is from being a UFD.

Example 2.4.6. If $X = \mathbb{A}^n$, then Proposition 2.4.4 implies that $\mathrm{Cl}(X) = 0$.

Example 2.4.7. Let $X = \mathbb{P}^n$. Recall that if $V \subseteq \mathbb{P}^n$ is irreducible, closed, of codimension 1, the prime ideal corresponding to V is generated by 1 element $F \in S = k[x_0, \ldots, x_n]$, homogeneous of degree $d > 0$. Then we say that $\deg(V) = d$. This lets us define a group homomorphism:

$$\begin{aligned} \deg\colon\ \mathrm{Div}(\mathbb{P}^n) &\to \mathbb{Z} \\ \sum_{i=1}^{r} n_i V_i &\mapsto \sum_{i=1}^{r} n_i \deg(V_i) \end{aligned}$$

Note that 1 is the degree of a hyperplane, so this map is surjective.

We claim that if $\varphi \in k(\mathbb{P}^n)^*$, then $\deg(\mathrm{div}(\varphi)) = 0$. This will show that the degree map factors through the class group of $\mathbb{P}^n$.

We can write $\varphi = \frac{F}{G}$ for $F, G \in S$ homogeneous, nonzero, of the same degree. Since S is a UFD, write

$$F = c_F \cdot \prod_{i=1}^{r} F_i^{a_i}$$
$$G = c_G \cdot \prod_{j=1}^{s} G_j^{b_j}$$

for $a_i, b_j > 0$ and F_i, G_j irreducible. Then

$$\mathrm{div}(\varphi) = \sum_{i=1}^{r} a_i V(F_i) - \sum_{j=1}^{s} b_j V(G_j)$$

has degree

$$\sum a_i \deg(F_i) - \sum b_j \deg(G_j) = \deg F - \deg G = 0.$$

Hence we get a surjective map

$$\deg\colon\ \mathrm{Cl}(\mathbb{P}^n) \to \mathbb{Z}.$$

We claim that this map is also injective, and hence an isomorphism. Suppose $D = \sum_{i=1}^{r} n_i V(F_i)$ has degree 0. Taking

$$\varphi = \frac{\prod_{n_i > 0} F_i^{n_i}}{\prod_{n_i < 0} F_i^{-n_i}} \in k(X)^*,$$

we see that $\operatorname{div}(\varphi) = D$, so $D = 0$ in $\mathrm{Cl}(\mathbb{P}^n)$.

Definition 2.4.8. Two divisors D and E are *linearly equivalent* if $D - E$ is principal. We then write $D \sim E$.

Let X be normal and irreducible. For a divisor D on X, we will define a sheaf associated to it

$$\mathcal{O}_X(D) \subseteq \underline{k(X)} = \text{constant sheaf of rational functions.}$$

If $U \subseteq X$ is open,

$$\Gamma(U, \mathcal{O}_X(D)) = \{0\} \cup \{\varphi \in k(X)^* \mid \operatorname{div}(\varphi)|_U + D|_U \geq 0\}.$$

(Note that if $U \subseteq X$ and $E = \sum n_i V_i$ on X, $E|_U = \sum_{V_i \cap U \neq \emptyset} n_i(V_i \cap U)$ is a divisor on U.)

It is clear that $\mathcal{O}_X(D) \subseteq \underline{k(X)}$ is a subsheaf, which is in fact a sub $\mathcal{O}_X$-module.

Note that:

(1) if $D = 0$, $\mathcal{O}_X(D) = \mathcal{O}_X$,
(2) if $D \geq E$, $\mathcal{O}_X(E) \subseteq \mathcal{O}_X(D)$; in particular, if $E \leq 0$, then $\mathcal{O}_X(E) \subseteq \mathcal{O}_X$.

Proposition 2.4.9. *The sheaf $\mathcal{O}_X(D)$ associated to a divisor D is coherent, and the stalk at X is $k(X)$.*

Proof. We first show that it is quasicoherent. Suppose $U \subseteq X$ is an affine open subset, $f \in \mathcal{O}_X(U)$. The map

$$\Gamma(U, \mathcal{O}_X(D))_f \to \Gamma(D_U(f), \mathcal{O}_X(D))$$

is clearly injective, since $\mathcal{O}_X(D)$ is a subsheaf of $\underline{k(X)}$ and $k(X)$ is a domain. To show surjectivity, take $\varphi \in \Gamma(D_U(f), \mathcal{O}_X(D))$. Then

$$(\operatorname{div}(\varphi) + D)|_{D_U(f)} \geq 0.$$

We want to show that for some $m \geq 0$ such that

$$(\operatorname{div}(\varphi \cdot f^m) + D)|_U \geq 0.$$

Let $D' = (D + \operatorname{div}(\varphi))|_U$. Let $Z_1, \ldots, Z_r$ be the prime divisors in U where D' has negative coefficient. Then $Z_i \subseteq V(f)$, so $\operatorname{ord}_{Z_i}(f) \geq 1$, and hence if $m \gg 0$, $D' + (\operatorname{div}(f^m))_U \geq 0$. Therefore,

$$f^m \varphi \in \Gamma(U, \mathcal{O}_X(D)).$$

This proves that $\mathcal{O}_X(D)$ is quasicoherent. It remains to show that it is coherent. Let $U \subseteq X$ be affine. Let $Y_1, \ldots, Y_r$ be the prime divisors that appear in D with positive coefficients. Choose

$$g \in \prod_{i=1}^r I_U(Y_i).$$

If $m \gg 0$, $\operatorname{div}(g^m)|_U \geq D|_U$. Then

$$\Gamma(U, \mathcal{O}_X(D)) \subseteq \{\varphi \mid \operatorname{div}(\varphi \cdot g^m) \geq 0\} \cup \{0\} = \frac{1}{g^m}\mathcal{O}_X(U)$$

which is clearly finitely generated over $\mathcal{O}_X(U)$. This implies coherence.

If $D = \sum n_i V_i$ with $n_i \neq 0$, set

$$U = X \setminus \bigcup_i V_i.$$

Then $D|_U = 0$. Hence $\mathcal{O}_X(D)|_U = \mathcal{O}_U$. This implies the assertion about the stalk. □

Proposition 2.4.10. *For divisors $D, E \in \mathrm{Div}(X)$, $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ if and only if $D \sim E$.*

Proof. For the 'if' implication, suppose $D = E + \mathrm{div}(\alpha)$ for some α. Then $\varphi \in \Gamma(U, \mathcal{O}_X(D))$ if and only if

$$(\mathrm{div}(\varphi) + \mathrm{div}(\alpha) + E)|_U \geq 0,$$

which is equivalent to $\varphi\alpha \in \Gamma(U, \mathcal{O}_X(E))$. This gives an isomorphism $\mathcal{O}_X(D) \to \mathcal{O}_X(E)$ given on each open subset by multiplication by α. The converse implication will be proved in the problem session. □

Remark 2.4.11. There is a canonical isomorphism

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D+E)$$

induced by multiplication of rational functions.

Next, we describe the push-forward of Weil divisors.

Definition 2.4.12. If $f\colon X \to Y$ is a dominant of irreducible varieties with $\dim X = \dim Y$, then $k(Y) \hookrightarrow k(X)$ is finite, and we define the *degree of* f $\deg(f)$ as $[k(X) : k(Y)]$, the degree of this extension.

Definition 2.4.13. Suppose $f\colon X \to Y$ is a finite surjective morphism. We define the *push-forward* as

$$\begin{aligned} f_*\colon\ &\mathrm{Div}(X) \to \mathrm{Div}(Y) \\ &\sum_{i=1}^r n_i V_i \mapsto \sum_{i=1}^r n_i \cdot \deg(V_i/f(V_i)) f(V_i) \end{aligned}$$

Proposition 2.4.14. *Let $f\colon X \to Y$ be a finite surjective morphism of varieties which are smooth in codimension 1. Then*

$$f_*(\mathrm{div}(\varphi)) = \mathrm{div}(N_{k(X)/k(Y)}(\varphi)).$$

In particular, we get a map $f_*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$.

Proof. We need to show that for any prime divisor $W \subseteq Y$

$$\sum_{\substack{V \subseteq X \text{ prime divisor} \\ \text{such that } f(V) = W}} \mathrm{ord}_V(\varphi)[k(V) : k(W)] = \mathrm{ord}_W(N_{k(X)/k(Y)}(\varphi)).$$

Replace Y by U affine such that $U \cap W \neq \emptyset$ and X by $f^{-1}(U)$ to assume that X and Y are affine and $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X)$. We then get a finite injective map

$$\varphi A \hookrightarrow B$$

and if $\mathfrak{p} \subseteq A$ is the ideal corresponding to W, then

$$A_\mathfrak{p} \hookrightarrow B_\mathfrak{p}$$

is finite and injective. Note that $A_{\mathfrak{p}}$ is a DVR. The maximal ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ in $B_{\mathfrak{p}}$ are the localizations of the primes ocrresponding to the prime divisors V such that $f(V) = W$, and by assumption $(B_{\mathfrak{p}})_{\mathfrak{q}_i}$ is a DVR. We may assume $\varphi \in B$ (by writing it as a quotient of two functions in B). Then for $V_i = V(\mathfrak{p}_i)$

$$\operatorname{ord}_{V_i}(\varphi) = \ell((B_{\mathfrak{p}})_{\mathfrak{q}_i}/(\varphi)).$$

Then result then follows from Problem 2 from Problem Session 4. □

2.5. **Cartier divisors.** First, we discuss Cartier divisors on normal varieties. Let X be a normal variety and $\operatorname{Div}(X)$ be its group of divisors.

Definition 2.5.1. A divisor D is *locally principal* if there is an open cover

$$X = \bigcup_{i \in I} U_i$$

such that for any i, $D|_{U_i}$ is a principal divisors, i.e. there exists $\varphi_i \in k(X)^*$ such that $D|_{U_i} = \operatorname{div}(\varphi_i)|_{U_i}$.

A *Cartier divisor* is a locally principal divisor, and we write

$$\operatorname{Cart}(X) \subseteq \operatorname{Div}(X)$$

for the subgroup of Cartier divisors.

Proposition 2.5.2. *A divisor D on X is* locally principal *if and only if $\mathcal{O}_X(D)$ is a line bundle.*

Proof. If for $U \subseteq X$, $D|_U$ is principal, then $\mathcal{O}_X(D)|_U = \mathcal{O}_U(D|_U) \cong \mathcal{O}_U$, which shows that 'only if' implication. Conversely, if $\mathcal{O}_X(D)$ is a line bundle, we can cover X by open U_i such that

$$\mathcal{O}_{U_i}(D|U_i) = \mathcal{O}_X(D)|_{U_i} \cong \mathcal{O}_{U_i}.$$

Then by Proposition 2.4.10, $D|_{U_i}$ is principal. □

Proposition 2.5.3. *If $D, E \in \operatorname{Cart}(X)$, the map*

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E) \to \mathcal{O}_X(D + E)$$

is an isomorphism.

Proof. If $U \subseteq X$ is an open subset and $D|_U = \operatorname{div}(\varphi)|_U$, then for an open subset $V \subseteq U$

$$\Gamma(V, \mathcal{O}_X(D)) = \{\psi \mid \operatorname{div}(\psi)|_V + \operatorname{div}(\varphi)|V \geq 0\}.$$

Note that $\operatorname{div}(\psi)|_V + \operatorname{div}(\varphi)|V \geq 0$ if and only if $\varphi\psi \in \mathcal{O}_X(V)$. Therefore

$$\mathcal{O}_X(D)|_U = \frac{1}{\varphi}\mathcal{O}_U \subseteq k(X).$$

If $X = \bigcup_i U_i$ for affine open subsets $U_i \subseteq X$ such that

$$D|_{U_i} = \operatorname{div}(\varphi_i)|_{U_i}, \quad E|_{U_i} = \operatorname{div}(\psi_i)|_{U_i},$$

then

$$(D + E)|_{U_i} = \operatorname{div}(\varphi_i\psi_i)|_{U_i}.$$

Therefore, on U_i, the morphism above is the map

$$\frac{1}{\varphi_i}\mathcal{O}_X(U_i) \otimes_{\mathcal{O}_X(U_i)} \frac{1}{\psi_i}\mathcal{O}_X(U_i) \to \frac{1}{\varphi_i\psi_i}\mathcal{O}_X(U_i),$$

which is clearly an isomorphism. $\square$

Therefore, we have a group homomorphism

$$\begin{aligned} \mathrm{Cart}(X) &\to \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}_X(D) \end{aligned}$$

with kernel $\mathrm{PDiv}(X) \subseteq \mathrm{Cart}(X)$, and hence we get an injective map

$$\frac{\mathrm{Cart}(X)}{\mathrm{PDiv}(X)} \hookrightarrow \mathrm{Pic}(X).$$

We will see later that this is an isomorphism.

Remark 2.5.4. Arguing like in the proof of Proposition 2.4.4, we see that

$$\mathrm{Div}(X) = \mathrm{Cart}(X) \text{ if and only if } X \text{ is locally factorial.}$$

In particular, this is the case for smooth varieties.

Example 2.5.5. This implies that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Exercise. Show that if $H \subseteq \mathbb{P}^n$ is a hyperplane, then $\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$.

We now generalize the notion of Cartier divisors to all irreducible varieties. Suppose X is irreducible and let $k(X)$ be the field of rational functions on X. We will write $k(X)^*$ for the constant sheaf (previously denoted by $\underline{k(X)^*}$).

We have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \hookrightarrow k(X)^* \twoheadrightarrow k(X)^*/\mathcal{O}_X^* \longrightarrow 0.$$

Definition 2.5.6. The set of *Cartier divisors* is

$$\mathrm{Cart}(X) = \Gamma(X, k(X)^*/\mathcal{O}_X^*).$$

Explicitly, a *Cartier divisor* is given by an open cover $X = \bigcup_i U_i$ together with $\varphi_i \in k(X)^*$ such that for any i, j

$$\frac{\varphi_i}{\varphi_j} \in \mathcal{O}_X(U_i \cap U_j)^*.$$

Two Cartier divisors D, E given by such data are equal if, refining the covers to assume they are the same, when D is given by $(\varphi_i)_{i\in I}$ and E is given by $(\psi_i)_{i\in I}$, we have that

$$\frac{\varphi_i}{\psi_i} \in \mathcal{O}_X(U_i)^*.$$

Note that $D + E$ is given by $(\varphi_i \cdot \psi_i)_{i\in I}$.

We finally define the group of *principal Cartier divisors* as

$$\mathrm{PCart}(X) = \mathrm{im}(k(X)^* = \Gamma(X, k(X)^*) \to \Gamma(X, k(X)^*/\mathcal{O}_X^*)),$$

i.e. a divisor is *principal* if it comes from a global section.

Given any Cartier divisor D describe with respect to $X = \bigcup_i U_i$ by $(\varphi_i)_{i\in I}$, consider for each i, $\frac{1}{\varphi_i}\mathcal{O}_{U_i} \subseteq k(X)$.

Since for $\frac{\varphi_i}{\varphi_j} \in \mathcal{O}_X(U_i \cap U_j)^*$ we have

$$\frac{1}{\varphi_i}\mathcal{O}_{U_i\cap U_j} = \frac{1}{\varphi_j}\mathcal{O}_{U_i\cap U_j},$$

there is a unique subsheaf $\mathcal{O}_X(D) \subseteq k(X)$ such that

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{\varphi_i}\mathcal{O}_{U_i}.$$

Note that by definition $\mathcal{O}_X(D)$ is a line bundle.

As in the normal case, we hence get a map

$$\begin{aligned} \mathrm{Cart}(X) &\to \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}_X(D). \end{aligned}$$

Exercise.

(1) This is a group homomorphism: $\mathcal{O}_X(D) \otimes \mathcal{O}_X(E) \cong \mathcal{O}_X(D+E)$.
(2) $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if and only if $D \in \mathrm{PCart}(X)$.

Therefore, we get a map

$$\frac{\mathrm{Cart}(X)}{\mathrm{PCart}(X)} \hookrightarrow \mathrm{Pic}(X).$$

Proposition 2.5.7. *The map*

$$\frac{\mathrm{Cart}(X)}{\mathrm{PCart}(X)} \hookrightarrow \mathrm{Pic}(X).$$

is an isomorphism.

Proof. We need to show that if $\mathcal{L} \in \mathrm{Pic}(X)$, then there is a Cartier divisor D such that $\mathcal{O}_X(D) \cong \mathcal{L}$.

Choose an open cover $X = \bigcup_{i\in I} U_i$ and isomorphisms $\alpha_i \colon \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$ with transition functions

$$\alpha_i|_{U_i\cap U_j} \circ \alpha_j^{-1}|_{U_i\cap U_j} \colon \mathcal{O}_{U_i\cap U_j} \xrightarrow{\cong} \mathcal{O}_{U_i\cap U_j}$$

given by multiplication with some $\alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^*$ which satisfy:

(1) $\alpha_{i,i} = 1$,
(2) $\alpha_i, j\alpha_{j,k} = \alpha_{i,k}$ in $k(X)$ (since $U_i \cap U_j \cap U_0$ is dense in X, as X is irreducible)

Define $\varphi_i = \alpha_{i,i_0} \in (X)^*$ for all i and some i_0. Then

$$\frac{\varphi_i}{\varphi_j} = \frac{\alpha_{i,i_0}}{\alpha_{j,j_0}} = \alpha_{i,j} \in \mathcal{O}_X(U_i \cap U_j)^*.$$

Therefore, the φ_i define a divisor D. It is easy to see that $\mathcal{O}_X(D) \cong \mathcal{L}$ (the local isomorphisms $\mathcal{O}_X(D)|_{U_i} = \frac{1}{\varphi_i}\mathcal{O}_{U_i} \cong \mathcal{O}_{U_i} \cong \mathcal{L}_{U_i}$ glue together). ☐

We finally compare the two definitions of Cartier divisors. Suppose X is smooth in codimension 1 and D is a Cartier divisor on X described by $X = \bigcup_{i \in I} U_i$ and $\varphi_i \in k(X)^*$.

Consider $\operatorname{div}(\varphi)|_{U_i}$. Since $\frac{\varphi_i}{\varphi_j} \in \mathcal{O}_X^*(U_i \cap U_j)$,

$$\operatorname{div}(\varphi_i)|_{U_i \cap U_j} = \operatorname{div}(\varphi_j)|_{U_i \cap U_j}.$$

Therefore, there is a unique Weil divisor $\alpha(D)$ such that

$$\alpha(D)|_{U_i} = \operatorname{div}(\varphi_i)|_{U_i}.$$

We get a group homomorphism

$$\operatorname{Cart}(X) \to \operatorname{Div}(X).$$

If X is normal, this map is injective, since $\operatorname{div}(\varphi_i/\psi_i) = 0$ implies hat $\varphi_i/\psi_i \in \mathcal{O}_X(U_i)^*$ by normality.

Moreover, the image consists of the locally principal divisors on X.

Therefore, on normal varieties, we can identify Cartier divisors with locally principal Weil divisors, as we did in Definition 2.5.1.

Next, we will define the pull-back of Cartier divisors. Let $X \to Y$ be a dominant morphism of irreducible varieties and $\nu\colon k(Y) \to k(X)$ be the corresponding map. We define the *pull-back* map

$$f^*\colon \operatorname{Cart}(Y) \to \operatorname{Cart}(X).$$

For D described by an open cover $Y = \bigcup_i U_i$ with $\varphi \in k(U_i)^*$, we define $f^*(D)$ with respect to $X = \bigcup_{i \in I} f^{-1}(U_i)$ by $(\nu(\varphi_i))_{i \in I}$.

It is easy to see that

(1) this definition is independent of the presentation of D,
(2) f^* is a group homomorphism preserving PCart:

$$\begin{tikzcd} \operatorname{Cart}(Y) \arrow[r, "f^*"] & \operatorname{Cart}(X) \\ \operatorname{PCart}(Y) \arrow[r] \arrow[u, hook] & \operatorname{PCart}(X) \arrow[u, hook] \end{tikzcd}$$

and hence induces a commutative square

$$\begin{tikzcd} \frac{\operatorname{Cart}(Y)}{\operatorname{PCart}(Y)} \arrow[r, "f^*"] \arrow[d, "\cong"] & \frac{\operatorname{Cart}(X)}{\operatorname{PCart}(X)} \arrow[d, "\cong"] \\ \operatorname{Pic}(Y) \arrow[r, "f^*"] & \operatorname{Pic}(X). \end{tikzcd}$$

Fact 2.5.8. *Suppose $f\colon X \to Y$ be a finite surjective map of irreducible varieties smooth in codimension 1. For $D \in \operatorname{Cart}(Y)$,*

$$f_*(f^*(D)) = \deg(f) \cdot D$$

in $\operatorname{Div}(Y)$.

The proof of this will be a homework problem.

2.6. **Effective Cartier divisors.** We finally discuss effective Cartier divisors.

Definition 2.6.1. Let X be an irreducible variety and D be a Cartier divisor given with respect to $X = \bigcup_{i \in I} U_i$ by $(\varphi_i)_{i \in I}$. Then D is *effective* if $\varphi_i \in \mathcal{O}_X(U_i)$ for any $i \in I$ (this is independent of the presentation of D).

It is clear that if X is smooth in codimension and D is an effective divisor, then the corresponding Weil divisor is effective. The converse holds if X is normal.

We give an equivalent description of effective Cartier divisors.

Definition 2.6.2. A coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is *locally principal* if for any $x \in X$, there exists an open affine neighborhood U of x such that

$$\Gamma(U, \mathcal{I}) \subseteq \mathcal{O}_X(U)$$

is generated by a non-zero element.

Proposition 2.6.3. *There is a bijection between effective Cartier divisors on X and locally principal ideals in $\mathcal{O}_X$ given by*

$$D \mapsto \mathcal{O}_X(-D).$$

Proof. Suppose that D is described by $X = \bigcup_i U_i$ and $(\varphi_i)_{i \in I}$. Then

$$\mathcal{O}_X(-D)|_{U_i} = \mathcal{O}_X(U_i) \cdot \varphi_i \subseteq \mathcal{O}_X(U_i).$$

Conversely, if $\mathcal{I} \subseteq \mathcal{O}_X$ is a locally principal ideal, then there is an affine open cover $X = \bigcup_{i \in I} U_i$ such that

$$\Gamma(U_i, \mathcal{I}) = \beta_i \mathcal{O}_X(U_i)$$

and $\frac{\beta_i}{\beta_j} \in \mathcal{O}_X(U_i \cap U_j)^*$, so $(\beta_i)_{i \in I}$ defines an effective Cartier divisor.

It is finally easy to see that the two maps are inverse to each other. □

Definition 2.6.4. If D is an effective Cartier divisor, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{O}_X(-D) \longrightarrow 0$$

We define the *structure sheaf of D* as $\mathcal{O}_D = \mathcal{O}_X/\mathcal{O}_X(-D)$.

We define the *support of D* as $\operatorname{supp}(D) = V(\mathcal{O}_X(-D)) \subseteq X$, a closed subset of X.

Example 2.6.5. Suppose $\mathcal{I}$ is an ideal sheaf on X and $\pi \colon Y = \operatorname{Bl}_{\mathcal{I}} X \to X$ is the blow-up of X along $\mathcal{I}$. Since $\mathcal{I} \cdot \mathcal{O}_Y$ is locally principal, there exists an effective Cartier divisor E such that $\mathcal{I}\mathcal{O}_Y = \mathcal{O}_Y(-E)$. We then call E the *exceptional divisor*.

Exercise. If X is normal and D is an effective Cartier divisor, then $\mathcal{O}_X(-D)$ is a radical ideal if and only if all coefficients of D are 1.

In general, if D is an effective Cartier divisor, then we define $\operatorname{supp}(D) = V(\mathcal{O}_X(-D))$.

Effective Cartier divisors as zero-loci of sections of line bundles. Let X be an irreducible variety, $\mathcal{L}$ be line bundle, $s \in \Gamma(X, \mathcal{L})$ and $s \neq 0$. By tensoring the map

$$\begin{aligned} \mathcal{O}_X &\to \mathcal{L} \\ 1 &\mapsto s \end{aligned}$$

with $\mathcal{L}^{-1}$, we get a map $\alpha\colon \mathcal{L}^{-1} \to \mathcal{O}_X$ whose image is a coherent ideal $\mathcal{I}$ in $\mathcal{O}_X$.

We claim that this is an injective morphism and $\mathcal{I}$ is locally principal.

Suppose $U \subseteq X$ is an open affine subset such that $\mathcal{L}|_U \cong \mathcal{O}_U$ and s is sent to $f \in \mathcal{O}(U)$ via this isomorphism. This induces an isomorphism $\mathcal{L}^{-1}|_U \cong \mathcal{O}_U$ and $\alpha|_U$ gets identified with $\mathcal{O}_U \xrightarrow{\cdot f} \mathcal{O}_U$ which is clearly injective, and $\mathcal{I}|_U$ is generated by f.

Therefore, there is a Cartier divisor $Z(s)$ such that $\mathcal{I} = \mathcal{O}(-Z(s))$, called the *zero locus of* s.

Note that $x \in X$ lies in $Z(s)$ if and only if $s_{(x)} \in \mathcal{L}_{(x)}$ is 0.

Proposition 2.6.6.

(1) *By construction,* $\mathcal{O}(-Z(s)) \cong \mathcal{L}^{-1}$*, i.e.* $\mathcal{O}(Z(s)) \cong \mathcal{L}$.
(2) *If* $s' \in \Gamma(X, \mathcal{L}) \setminus \{0\}$*, then* $Z(s) = Z(s')$ *if and only if* $s = gs'$ *for some* $g \in \mathcal{O}(X)^*$.
(3) *If* D *is an effective divisor such that* $\mathcal{O}(D) \cong \mathcal{L}$*, then there is a section* $s \in \Gamma(X, \mathcal{L}) \setminus \{0\}$ *such that* $Z(s) = D$.

Proof. Parts (1) and (2) are immediate, so we just need to show (3). Suppose $X = \bigcup_{i \in I} U_i$ is an open cover such that D is described by $(\varphi_i)_{i \in I}$ for $\varphi_i \in \mathcal{O}_X(U_i)$. Then

$$\mathcal{O}(D)|_{U_i} = \frac{1}{\varphi_i} \mathcal{O}_{U_i} \subseteq k(X),$$

so $1 \in \Gamma(X, \mathcal{O}_X(D))$). Checking that the zero-locus of 1 is D is left as an exercise. Then we map this section to $\Gamma(X, \mathcal{L})$ via the isomorphism. □

Remark 2.6.7. Suppose X is complete. We will see later that $\Gamma(X, \mathcal{L})$ is a finite-dimensional vector space over k. Therefore,

$$\left\{ \begin{array}{c} D \text{ effective Cartier} \\ \text{divisor such that } \mathcal{O}(D) \cong \mathcal{L} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{projective space parametrizing} \\ \text{lines in } \Gamma(X, \mathcal{L}) \end{array} \right\}.$$

This is called the *linear system corresponding to* $\mathcal{L}$ and denoted $|\mathcal{L}|$.

3. Cohomology

3.1. **Derived functors.** Fix the category $\mathcal{C}$ to be $\mathcal{O}_X$-modules for some ringed space $(X, \mathcal{O}_X)$. In general, we could let $\mathcal{C}$ be any abelian category but we will stick to $\mathcal{C} = \mathcal{O}_X$-mod to simplify the exposition, since this is the only case we will be interested in.

Definition 3.1.1. A *complex of objects in* $\mathcal{C}$

$$A^\bullet : \cdots \to A^m \xrightarrow{d^m} A^{m+1} \to \cdots$$

is a collection $(A^m)_{m \in \mathbb{Z}^+}$ with maps $d^m\colon A^m \to A^{m+1}$ such that $d^m \circ d^{m-1} = 0$.

A *morphism of complexes* $u\colon A^\bullet \to B^\bullet$ is a sequence of maps $u^m\colon A^m \to B^m$ for all $m \in \mathbb{Z}^+$ such that $d \circ u^m = u^{m+1} \circ d$ for all $m \in \mathbb{Z}^+$.

Since the morphisms can be composed component-wise, complexes in $\mathcal{C}$ form a category. This category has kernels and cokernels, described componentwise, which make it into an abelian category.

Definition 3.1.2. If $A^\bullet$ is a complex, define for $i \in \mathbb{Z}$ the *ith cohomology functor* by letting

$$\mathcal{H}^i(A^\bullet) = \frac{\ker(A^i \to A^{i+1})}{\operatorname{im}(A^{i-1} \to A^i)} \in \mathcal{C}$$

and for $u\colon A^\bullet \to B^\bullet$, $\mathcal{H}^i(u)$ to be the natural map

$$\mathcal{H}^i(A^\bullet) \to \mathcal{H}^i(B^\bullet).$$

Proposition 3.1.3 (Long exact sequence in cohomology)**.** *Given an exact sequence of complexes*

$$0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0$$

there is a connecting map δ that makes the sequence

$$\cdots \longrightarrow \mathcal{H}^i(A^\bullet) \xrightarrow{\mathcal{H}^i(u)} \mathcal{H}^i(B^\bullet) \xrightarrow{\mathcal{H}^i(v)} \mathcal{H}^i(C^\bullet) \xrightarrow{\delta} \mathcal{H}^{i+1}(A^\bullet) \longrightarrow \cdots$$

exact. Moreover, this is functorial with respect to morphisms of exact sequences of complexes.

Sketch of proof. We first define δ. Given $s \in \Gamma(U, \mathcal{H}^i(C^\bullet))$ for an open neighborhood U of $x \in X$, we can find a lift $s'(x) \in \Gamma(U_x, \ker(C^i \to C^{i+1}))$ of $s|_{U(x)}$ where U_x is an open neighborhood of x. After passing to the smaller U_x, we may assume that $s'(x) = v(s''(x))$ for some $s''(x) \in \Gamma(U_x, \mathcal{B}^i)$, there exists $t(x) \in \Gamma(U_x, A^{i+1})$ such that $u(t(x)) = d(s''(x))$. It is easy to see that

$$t(x) \in \Gamma(U_x, \ker(A^{i+1} \to A^{i+2})).$$

The images $\overline{t(x)} \in \Gamma(U(x), \mathcal{H}^{i+1}(A^\bullet))$ glue together, giving $\delta(s)$.

To check exactness, pass to stalks and just deal with modules over a ring. This is left as an exercise. $\square$

Definition 3.1.4. Two morphisms of complexes $u, v\colon A^\bullet \to B^\bullet$ are *homotopic* ($u \approx v$), if there are map $\theta^i\colon A^i \to B^{i-1}$

$$\begin{tikzcd}
A^{i-1} \arrow[r, "d"] \arrow[d, shift left] \arrow[d, shift right] & A^i \arrow[r, "d"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, dotted, "\theta^i"] & A^{i+1} \arrow[d, shift left] \arrow[d, shift right] \arrow[dl, dotted, "\theta^{i+1}"] \\
B^{i-1} \arrow[r, "d"'] & B^i \arrow[r, "d"'] & B^{i+1}
\end{tikzcd}$$

such that $u^i - v^i = d \circ \theta^i + \theta^{i+1} \circ d$ for all i.

Note that if $u \approx v$ then $\mathcal{H}^i(u) = \mathcal{H}^i(v)$ for all i.

Definition 3.1.5. Let $\mathcal{A}$ be an abelian category. Then $Q \in \mathrm{Ob}\mathcal{A}$ is *injective* if $\mathrm{Hom}_{\mathcal{A}}(-, Q)$ is exact.

Exercise. If Q_i are injective objects, then $\prod_{i\in I} Q_i$ is injective.

Definition 3.1.6. We say that $\mathcal{A}$ has *enough injectives* if for any $A \in \mathrm{Ob}(\mathcal{A})$ there is an injective map

$$A \to Q$$

with Q injective.

Remark 3.1.7. Review Sheet 4 proves that the category of R-modules has enough injectives. We use this to show that the category of $\mathcal{O}_X$-modules also has enough injectives.

Proposition 3.1.8. *The category $\mathcal{O}_X$-mod has enough injectives.*

Proof. Suppose $x \in X$ and A is an $\mathcal{O}_{X,x}$-module. Define an $\mathcal{O}_X$-module $A(x)$ by

$$\Gamma(U, A(x)) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$

(with $\mathcal{O}(U)$ acting via $\mathcal{O}(U) \to \mathcal{O}_{X,x}$ for $x \in U$). If $\mathcal{F}$ is an $\mathcal{O}_X$-mod,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, A(x)) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, A).$$

(proving this is left as an exercise). Therefore, if A is an injective $\mathcal{O}_{X,x}$-module, $A(x)$ is an injective $\mathcal{O}_X$-module.

Given an $\mathcal{O}_X$-module $\mathcal{M}$, consider for each $x \in X$, and injective morphism

$$\mathcal{M}_x \hookrightarrow I^{(x)}$$

where $I^{(x)}$ is an injective $\mathcal{O}_{X,x}$-module. Then consider

$$\mathcal{M} \hookrightarrow \prod_{x\in X} (\mathcal{M}_x)(x) \hookrightarrow \prod_{x\in X} (I^{(x)})(x).$$

This gives an embedding of $\mathcal{M}$ in an injective $\mathcal{O}_X$-module. $\square$

Definition 3.1.9. 3.1.8 A *resolution* of $\mathcal{M} \in \mathrm{Ob}(\mathcal{C})$ is a complex $A^\bullet$ with $A^i = 0$ for $i < 0$ and with a morphism of complexes $\mathcal{M} \to \mathcal{A}^\bullet$ inducing an isomorphism in cohomology. Equivalently, $\mathcal{A}^\bullet$ is a *resolution* if

$$0 \longrightarrow \mathcal{M} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

is exact.

An *injective resolution* of $\mathcal{M}$ is a resolution $A^\bullet$ with all A^i injective.

Proposition 3.1.10.

(1) *Given any $\mathcal{M} \in \mathrm{Ob}(\mathcal{C})$, $\mathcal{M}$ has an injective resolution.*
(2) *Suppose we have*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha} & & & & & & \\ 0 & \longrightarrow & \mathcal{N} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & \cdots \end{array}$$

such that the top row is a resolution of $\mathcal{M}$ and the bottom row is a complex with all B^i injective, there is a morphism $u\colon A^\bullet \to B^\bullet$ such that

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d, "\alpha"] & A^\bullet \arrow[d] \\
\mathcal{N} \arrow[r] & B^\bullet
\end{tikzcd}$$

commutes.

(3) *If $v\colon A^\bullet \to B^\bullet$ is another such morphism, then $u \approx v$.*

Proof. For (1), by Proposition 3.1.8, there is an embedding $\mathcal{M} \hookrightarrow I^0$ with I^0 injective. Then apply Proposition 3.1.8 again to get and embedding

$$I^0/\mathcal{M} \hookrightarrow I^1 \text{ with } I^1 \text{ injective.}$$

This gives an exact sequence

$$0 \to \mathcal{M} \to I^0 \to I^1.$$

Repeating this, we obtain an injective resolution of $\mathcal{M}$.

For (2), we first get u^0:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, "\alpha"] & A^0 \arrow[r] \arrow[d, dotted, "u^0"] & A^1 \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{N} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

since the map $\mathcal{M} \hookrightarrow \mathcal{A}^0$ in injective and B^0 is an injective object. Then we have

$$\begin{tikzcd}
A^0/\mathcal{M} \arrow[r, hook] \arrow[d, "\overline{u^0}"] & A^1 \arrow[d, dotted, "u^1"] \\
\operatorname{coker}(\mathcal{N} \to B^0) \arrow[r] & B^1
\end{tikzcd}$$

where we get u^1 since B^1 is an injective object. Continuing this way, we get the chain map u.

For (3), suppose we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, "\alpha"] & A^0 \arrow[r] \arrow[d, shift right, "v^0"'] \arrow[d, shift left, "u^0"] & A^1 \arrow[r] \arrow[d, shift right, "v^1"'] \arrow[d, shift left, "u^1"] & \cdots \\
0 \arrow[r] & \mathcal{N} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

The map $u_0 - v_0$ induces a map

$$\begin{tikzcd}
A^0/\mathcal{M} \arrow[r, hook] \arrow[d] & A^1 \arrow[dl, dotted, "\theta^1"] \\
B^0 &
\end{tikzcd}$$

and since B^0 is injective we get a map θ^1 as in the diagram above such that $\theta^1 \circ d = u^0 - v^0$. Then $u^1 - v^1 - d \circ \theta^1$ vanishes on $\mathrm{im}(A^0 \to A^1)$ construction. Hence, it induces a map

$$\begin{tikzcd}
\mathrm{coker}(A^0 \to A^1) \arrow[r, hook] \arrow[d] & A^2 \arrow[dl, "\theta^2"] \\
B^1 &
\end{tikzcd}$$

such that $\theta^2 \circ d = u^1 - v^1$. Continuing this way, we get the desired homotopy showing $u \approx v$. □

Proposition 3.1.11 (Horseshoe Lemma)**.** *Given an exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

and an injective resolution $(\mathcal{I}')^\bullet$, $(\mathcal{I}'')^\bullet$ *for* $\mathcal{I}'$, $\mathcal{I}''$ *respectively, there is a commutative diagram of complexes*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{I}')^\bullet \arrow[r] & (\mathcal{I}')^\bullet \oplus (\mathcal{I}'')^\bullet \arrow[r] & (\mathcal{I}'')^\bullet \arrow[r] & 0.
\end{tikzcd}$$

In particular,

$$\mathcal{F} \to (\mathcal{I}')^\bullet \oplus (\mathcal{I}'')^\bullet$$

is an injective resolution of $\mathcal{F}$*.*

Sketch of proof. We will construct the first maps $\beta = (\beta_1, \beta_2)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d, hook, "\alpha"] & \mathcal{F} \arrow[r] \arrow[d, hook, dotted, "\beta"] & \mathcal{F}'' \arrow[r] \arrow[d, hook, "\gamma"] & 0 \\
0 \arrow[r] & (\mathcal{I}')^0 \arrow[r] & (\mathcal{I}')^0 \oplus (\mathcal{I}'')^0 \arrow[r] & (\mathcal{I}'')^0 \arrow[r] & 0.
\end{tikzcd}$$

where β_2 is the composition $\mathcal{F} \to \mathcal{F}'' \to (\mathcal{F}'')^0$ and β is the unique map making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[dl, dotted, "\beta_1"] \\
(\mathcal{I}')^0 &
\end{tikzcd}$$

commutative, which exists by injectivity of $(\mathcal{I}')^0$. The injectivity of β follows.

By the Snake Lemma, we then get an exact sequence

$$0 \longrightarrow \mathrm{coker}\,\alpha \longrightarrow \mathrm{coker}\,\beta \longrightarrow \mathrm{coker}\,\gamma \longrightarrow 0.$$

Repeat the above arugment to show the result. □

Right derived Functors. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor for two additive categories. We will usually consider $\mathcal{C}$ and $\mathcal{D}$ to be categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules.

Examples 3.1.12.

(1) Let $\mathcal{C}$ be the category of $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is a sheaf of R-algebras. Then
$$\mathcal{F} = \Gamma(X, -)\colon \mathcal{C} \to R\text{-mod}$$
is a left exact functor.

(2) Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then
$$f_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$$
is a left exact functor.

For example, if Y is a point, $\mathcal{O}_Y = R$, so we recover example 1.

(3) Consider $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of R-algebras and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to R - mod$$
is a left exact functor.

(4) Consider $(X, \mathcal{O}_X)$ and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$$
is a left exact functor.

The idea is that in general $\mathcal{F}$ is not exact, and we want to measure the failure of right exactness.

Definition 3.1.13. A *δ-functor* is given by a sequence of functors $(\mathcal{F}^i)_{i \geq 0}$ and for any short exact sequence
$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$
a *connecting homomorphism*
$$\mathcal{F}^i(\mathcal{M}'') \xrightarrow{\delta} \mathcal{F}^{i+1}(\mathcal{M}')$$
which is functorial with respect to morphisms of short exact sequences, and for every short exact sequence as above, we have a long exact sequence:
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}^0(\mathcal{M}') \arrow[r] & \mathcal{F}^0(\mathcal{M}) \arrow[r] & \mathcal{F}^0(\mathcal{M}'') \arrow[dll, "\delta"] \\ & \mathcal{F}'(\mathcal{M}') \arrow[r] & \mathcal{F}'(\mathcal{M}) \arrow[r] & \cdots \end{tikzcd}$$

A *morphism of δ-functors* $(\mathcal{F}^i)_{i\geq 0} \to (\mathcal{G}^i)_{i \geq 0}$ is a collection of functorial transformations $\mathcal{F}^i \to \mathcal{G}^i$ for all i such that any short exact sequence
$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$
the diagram

$$\begin{tikzcd} \mathcal{F}^i(\mathcal{M}'') \arrow[r, "\delta"] \arrow[d] & \mathcal{F}^{i+1}(\mathcal{M}') \arrow[d] \\ \mathcal{G}^i(\mathcal{M}'') \arrow[r, "\delta"] & \mathcal{G}^{i+1}(\mathcal{M}') \end{tikzcd}$$

commutes. (This implies that we get a morphism between the long exact sequences.)

Theorem 3.1.14. *Given a left exact functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, there exists a δ-functor $(R^i\mathcal{F})_{i\geq 0}$ such that*

(1) $R^0\mathcal{F} \cong \mathcal{F}$,
(2) $R^i\mathcal{F}(\mathcal{I}) = 0$ *for all injective objects* $\mathcal{I}$ *of* $\mathcal{C}$, $i \geq 1$.

Moreover, such a δ-functor is unique up to canonical isomorphism of δ-functors. In fact, given any δ-functor $(\mathcal{G}^i)_{i\geq 0}$ and a natural transformation $\mathcal{F} \xrightarrow{T} \mathcal{G}^0$, there is a unique morphism of δ-functors $(R^i\mathcal{F})_{i\geq 0} \to (\mathcal{G}^i)_{i\geq 0}$ which extends T for $i = 0$.

Proof. We begin by showing existence. Choose for each object $\mathcal{M}$ an injective resolution $\mathcal{M} \to \mathcal{I}^\bullet$ and define

$$R^i\mathcal{F}(\mathcal{M}) := \mathcal{H}^i(F(\mathcal{I}^\bullet)).$$

Given $u\colon \mathcal{M}_1 \to \mathcal{M}_2$, by Proposition 3.1.10, choose $\widetilde{u}\colon \mathcal{I}_1^\bullet \to \mathcal{I}_2^\bullet$ that makes the diagram

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r] \arrow[d] & \mathcal{M}_2 \arrow[d] \\ \mathcal{I}_1^\bullet \arrow[r, "\widetilde{u}"] & \mathcal{I}_2^\bullet \end{tikzcd}$$

commute and define

$$(R^i\mathcal{F})(u) = \mathcal{H}^i(\mathcal{F}(\widetilde{u})).$$

If $\widetilde{u'}$ is another such map then $\widetilde{u} \approx \widetilde{u'}$, so $R^i\mathcal{F}(u) = R^i\mathcal{F}(u')$.

It is easy to see that this is functorial (using independence of the choice of $\widetilde{u}$).

This also implies that if $(\mathcal{I}')^\bullet$ is another injective resolution of $\mathcal{M}$, we have an isomorphism

$$\mathcal{R}^i\mathcal{F}(\mathcal{M}) \cong \mathcal{H}^i(\mathcal{F}((\mathcal{I}^i)^\bullet)).$$

In particular, if $\mathcal{M}$ is injective, we can choose an injective resolution $0 \to \mathcal{M} \to \mathcal{M} \to 0$ of $\mathcal{M}$, and hence

$$R^i\mathcal{F}(\mathcal{M}) = 0 \text{ for } i \geq 1.$$

If

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots$$

is an injective resolution of $\mathcal{M}$ and $\mathcal{F}$ is left exact, then

$$0 \longrightarrow \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{I}^0) \longrightarrow \mathcal{F}(\mathcal{I}^1) \longrightarrow \cdots$$

is exact, so we get a functorial isomorphism $R^0\mathcal{F} \cong \mathcal{F}$.

We claim that we can define a *connecting homomorphism* in a functorial way. Given a short exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

and choosing resolutions $\mathcal{M}' \to (\mathcal{I}')^\bullet$, $\mathcal{M} \to (\mathcal{I})^\bullet$, $\mathcal{M}'' \to (\mathcal{I}'')^\bullet$, Proposition 3.1.11 gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathcal{I}')^\bullet \arrow[r] & (\mathcal{I}')^\bullet \oplus (\mathcal{I}'')^\bullet \arrow[r] & (\mathcal{I}'')^\bullet \arrow[r] & 0.
\end{tikzcd}$$

Since $\mathcal{F}$ is additive we have a short exact sequence of complexes:

$$0 \longrightarrow \mathcal{F}((\mathcal{I}')^\bullet) \longrightarrow \mathcal{F}((\mathcal{I}')^\bullet \oplus (\mathcal{I}'')^\bullet) \to \mathcal{F}((\mathcal{I}'')^\bullet) \longrightarrow 0.$$

By Proposition 3.1.3, we get a long exact sequence

$$R^i\mathcal{F}(\mathcal{M}') \longrightarrow \underbrace{\mathcal{H}^i(\mathcal{F}((\mathcal{I}')^\bullet \oplus (\mathcal{I}'')^\bullet))}_{R^i\mathcal{F}(\mathcal{M})} \longrightarrow R^i\mathcal{F}(\mathcal{M}'') \xrightarrow{\delta} R^{i+1}\mathcal{F}(\mathcal{M}').$$

This proves the existence.

To finish, it suffices to show that a δ-functor satisfying properties (1) and (2) satisfies the universal property.

Given any object $\mathcal{M}$ in $\mathcal{C}$, consider the injective resolution

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots$$

and truncate it to get a short exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{N} \longrightarrow 0$$

where $\mathcal{N} = \mathcal{I}^0/\mathcal{M}$. We then get a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}\mathcal{M} \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{I}^0) \arrow[r] \arrow[d] & \mathcal{F}(\mathcal{N}) \arrow[r] \arrow[d] & R^1\mathcal{F}(M) \arrow[r] \arrow[d, dotted] & R^1\mathcal{F}(\mathcal{I}^0) = 0 \\
0 \arrow[r] & \mathcal{G}^0\mathcal{M} \arrow[r] & \mathcal{G}^0(\mathcal{I}^0) \arrow[r] & \mathcal{G}^0(\mathcal{N}) \arrow[r] & \mathcal{G}^1(\mathcal{M}) &
\end{tikzcd}$$

where we have a unique map $R^1\mathcal{F}(\mathcal{M}) \to \mathcal{G}^1(\mathcal{M})$ such that the diagram commutes. It is easy to see this is a natural transformation.

For $i \geq 1$, we proceed by induction on i. Having constructed $R^i\mathcal{F}(\mathcal{N}) \to \mathcal{G}^i(\mathcal{N})$, we have the diagram

$$\begin{tikzcd} R^i\mathcal{F}(\mathcal{N}) \arrow[r, "\cong"] \arrow[d] & \mathcal{R}^{i+1}\mathcal{F}(\mathcal{M}) \\ \mathcal{G}^i(\mathcal{N}) \arrow[r] & \mathcal{G}^{i+1}(\mathcal{M}) \end{tikzcd}$$

and hence there is a unique map $\mathcal{R}^{i+1}\mathcal{F}(\mathcal{M}) \to \mathcal{G}^{i+1}(\mathcal{M})$ making this diagram commute. One can then check that this is a morphism of δ-functors, completing the proof. $\square$

Definition 3.1.15. The functor $R^i\mathcal{F}$ given by Theorem 3.1.14 is called the *ith derived functor of* $\mathcal{F}$.

In practice, it is better to compute $R^i\mathcal{F}$ using a resolution by F-acyclic objects, rather than injective objects.

Definition 3.1.16. An object $\mathcal{A}$ of $\mathcal{C}$ is *F-acyclic* if $R^i\mathcal{F}(A) = 0$ for $i \geq 1$.

For example, injective objects are F-acyclic.

Proposition 3.1.17. *If $\mathcal{M}$ is an object of $\mathcal{C}$ and we have a resolution $\mathcal{M} \to \mathcal{A}^\bullet$ with the objects $\mathcal{A}^p$ being $\mathcal{F}$-acyclic for all p, there is a canonical isomorphism*

$$R^i\mathcal{F}(\mathcal{M}) \cong \mathcal{H}^i(F(A^\bullet)).$$

Proof. The isomorphism for $i = 0$ follows by left-exactness of $\mathcal{F}$. We have an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{N} = \operatorname{coker}(\mathcal{M} \to \mathcal{A}^0) \longrightarrow 0,$$

which gives

$$\mathcal{F}(\mathcal{A}^0) \longrightarrow \mathcal{F}(\mathcal{N}) \longrightarrow R^1\mathcal{F}(\mathcal{M}) \longrightarrow R^1\mathcal{F}(A^0) = 0.$$

Thus

$$R^1\mathcal{F}(\mathcal{M}) = \operatorname{coker}(\mathcal{F}(\mathcal{A}^0) \to \mathcal{F}(\mathcal{N})) \cong \mathcal{H}^1(\mathcal{F}(\mathcal{A}^\bullet)).$$

because we have

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N} \arrow[r, hook] & \mathcal{A}^1 \arrow[r] & \mathcal{A}^2 \\ & \mathcal{A}^0 \arrow[u, two heads] \arrow[ur, "d^0"] & & \end{tikzcd}$$

so we get
$$\mathcal{F}(A^0) \to \mathcal{F}(\mathcal{N}) \hookrightarrow \mathcal{F}(\mathcal{A}^1)$$
and
$$\mathcal{H}^1(\mathcal{F}(\mathcal{A}^\bullet)) = \mathcal{F}(\mathcal{N})/\operatorname{im}(\mathcal{F}(\mathcal{A}^0) \to \mathcal{F}(\mathcal{N})) = \operatorname{coker}(\mathcal{F}(\mathcal{A}^0) \to \mathcal{F}(N)).$$
Also, $R^i\mathcal{F}(\mathcal{N}) \cong R^{i+1}F(\mathcal{M})$ for all $i \geq 1$, since N has an F-acyclic resolution
$$0 \to N \to \mathcal{A}^1 \to \mathcal{A}^2 \to \cdots .$$
Hence, if we know the assertion for i and $\mathcal{N}$, we get it for $i+1$ and $\mathcal{M}$. This completes the proof by induction. □

3.2. **Cohomology of sheaves.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{O}_X$ be a sheaf of R-algebras. The right derived functors of $\mathcal{F} = \Gamma(X, -)$ are the *sheaf cohomology*, written
$$H^i(X, -) = R^i\Gamma(X, -).$$
We have that

- $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$,
- for any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have a long exact sequence in cohomology
$$\cdots \longrightarrow H^i(\mathcal{F}') \longrightarrow H^i(\mathcal{F}) \longrightarrow H^i(\mathcal{F}'') \longrightarrow H^{i+1}(\mathcal{F}') \longrightarrow \cdots$$

Definition 3.2.1. An $\mathcal{O}_X$-module $\mathcal{F}$ on X is *flasque* (or *flabby*) if for any $U \subseteq X$ open, the restriction map
$$\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$$
is surjective.

Remark 3.2.2. Every $\mathcal{O}_X$-module has a *canonical* flasque resolution. For $\mathcal{M}$, define $\mathcal{E}^0(\mathcal{M})$ by $U \mapsto \prod_{x \in U} \mathcal{M}_x$ with restriction maps given by projection onto the corresponding component. Clearly, $\mathcal{E}$ is flasque and we have an injective morphism $\mathcal{M} \to \mathcal{E}^0$ given by
$$\Gamma(U, \mathcal{M}) \ni s \mapsto (s_x)_{x \in U}.$$
Then we define recursively for $i \geq 2$
$$\mathcal{E}^i(\mathcal{M}) = \mathcal{E}^0(\operatorname{coker}(\mathcal{E}^{i-2}(\mathcal{M}) \to \mathcal{E}^i(\mathcal{M})))$$
with $\mathcal{E}^0(\mathcal{M}) = \mathcal{M}$.

Proposition 3.2.3. *If we have a short exact sequence*
$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$
with $\mathcal{F}'$ flasque, then
$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$
is exact.

Proof. The proof is left as a homework exercise. □

Remark 3.2.4. If $\mathcal{F}$ is flasque, then $\mathcal{F}|_U$ is flasque for all $U \subseteq X$ open.

Corollary 3.2.5. *If we have a short exact sequence*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

with $\mathcal{F}'$ flasque, then $\mathcal{F}$ is flasque if and only if $\mathcal{F}''$ is flasque.

Proof. Proposition 3.2.3 and Remark 3.2.4 show that we have a commutative diagram with exact arrows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d, two heads] & \mathcal{F}(X) \arrow[r] \arrow[d, "\beta"] & \mathcal{F}''(X) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{F}''(U) \arrow[r] & 0
\end{tikzcd}$$

so by the Snake Lemma $\operatorname{coker}\beta \cong \operatorname{coker}\gamma$. □

Proposition 3.2.6. *If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque.*

Proof. The proof is left as a homework exercise. □

Proposition 3.2.7. *Every flasque $\mathcal{O}_X$-module is Γ-acyclic. In particular, if $\mathcal{M} \to \mathcal{A}^\bullet$ is a flasque resolution, then*

$$H^i(X, \mathcal{M}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{A}^\bullet)).$$

Proof. If A is flasque, consider

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I} \longrightarrow \mathcal{B} \longrightarrow 0$$

for an injective object $\mathcal{I}$. By Proposition 3.2.3, we have a short exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{A}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{B}) \longrightarrow 0$$

Hence $H^1(X, \mathcal{A}^\bullet) = 0$. For $i \geq 2$, the long exact sequence in cohomology shows that

$$H^i(X, \mathcal{A}) \cong H^{i-1}(X, \mathcal{B}).$$

Since $\mathcal{I}$ is injective, it is flasque by Proposition 3.2.6, and $\mathcal{A}$ is flasque, so by Corollary 3.2.5 $\mathcal{B}$ is flasque. This completes the proof by induction. □

We summarize what we have done so far and make a few comments. Suppose $(X, \mathcal{O}_X)$ is a ringed space where $\mathcal{O}_X$ is a sheaf of R-algebras. We then have a left exact functor $\Gamma(X, -)\colon \mathcal{O}_X\text{-mod} \to R\text{-mod}$ and its right derived functors are the *sheaf cohomology groups* $H^i(X, -)$.

Note that if R is an S-algebra with $\varphi\colon R \to S$, then we have a diagram

$$\begin{tikzcd} \mathcal{O}_X\text{-mod} \arrow[r, "\mathcal{F}=\Gamma(X,-)"] \arrow[dr, "\mathcal{G}=\Gamma(X,-)"'] & R\text{-mod} \arrow[d] \\ & S\text{-mod} \end{tikzcd}$$

and $R^i\mathcal{G} = \varphi \circ R^i\mathcal{F}$ by construction.

What if we change $\mathcal{O}_X$-modules to abelian groups? We have a diagram

$$\begin{tikzcd} \mathcal{O}_X\text{-mod} \arrow[r, "\psi"] \arrow[dr, "\mathcal{F}=\Gamma(X,-)"'] & \text{sheaves of abelian groups} \arrow[d, "\mathcal{G}=\Gamma(X,-)"] \\ & \text{Ab} \end{tikzcd}$$

Then $R^iG \circ \psi = R^i\mathcal{F}$, which follows from computing R^iF, R^iG by the canonical flasque resolution (Proposition 3.2.7).

Suppose $U \subseteq X$ is open and $i \hookrightarrow X$ is the inclusion map. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we can consider

$$H^i(U,\mathcal{F}) := H^i(U,\mathcal{F}|_U).$$

Then the functors $\{H^i(U,-)\}_{i\geq 0}$ and the derived functors of $\mathcal{F} \to \Gamma(U,\mathcal{F})$.

Note that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque (Proposition 3.2.6), so $\mathcal{F}|_U$ is flasque, and hence

$$H^i(U,\mathcal{F}|_U) = 0.$$

The natural transformation $\Gamma(X,-) \to \Gamma(U,-)$ extends to a morphism of δ-functors

$$(H^i(X,-) \to H^i(U,-))_{i\geq 0}.$$

This describe this explicitly, note that if $A \to \mathcal{I}^\bullet$ is an injective resolution, then we have a commuting square

$$\begin{tikzcd} \mathcal{H}^i(\Gamma(X,\mathcal{I}^\bullet)) \arrow[r, "\cong"] \arrow[d] & H^i(X,\mathcal{F}) \arrow[d] \\ \mathcal{H}^i(\Gamma(U,\mathcal{I}^\bullet)) \arrow[r, "\cong"] & H^i(U,\mathcal{F}) \end{tikzcd}$$

which is functorial with respect to inclusion of open subsets.

3.3. **Higher direct images.**

Let $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ be a morphism of ringed spaces. Then the functor

$$f_*\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod}$$

is left exact. Its derived functors are the *higher direct image functors*, R^if_*. Then

- $R^0f_* \cong f_*$,
- if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of $\mathcal{O}_X$-modules, then we have a long exact sequence

$$\begin{tikzcd}\cdots \arrow[r] & R^i f_*(\mathcal{F}') \arrow[r] & R^i f_*(\mathcal{F}) \arrow[r] & R^i f_*(\mathcal{F}'') \arrow[dll] \\ & R^{i+1} f_*(\mathcal{F}') \arrow[r] & \cdots \end{tikzcd}$$

in cohomology.

Definition 3.3.1. If f is as above, $U \subseteq Y$ is open and $\mathcal{F}$ is and $\mathcal{O}_X$-module, take

$$H^i(f^{-1}(U), \mathcal{F}).$$

If $V \subseteq U$, we have natural maps

$$H^i(f^{-1}(U), \mathcal{F}) \to H^i(f^{-1}(V), \mathcal{F})$$

which satisfy the usual compatibility condition. Note that $H^i(f^{-1}(U), \mathcal{F})$ is an $\mathcal{O}_X(f^{-1}(U))$-module, so it is an $\mathcal{O}_Y(U)$-module via $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. We therefore get a presheaf of $\mathcal{O}_Y$-modules denoted

$$\widetilde{R}^i f_*(\mathcal{F}).$$

Proposition 3.3.2. *We have a functorial isomorphism*

$$\widetilde{R}^i f_*(\mathcal{F})^+ \cong R^i f_*(\mathcal{F}).$$

Proof. We show that $(\widetilde{R}^i f_*(\mathcal{F})^+)_{i \geq 0}$ satisfy the universal property, so we actually have an isomorphism of δ-functors.

When $i = 0$, we have $\widetilde{R}^0 f_*(-)^+ = f_*(-)$.

If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is flasque (Proposition 3.2.6), so

$$H^i(f^{-1}(U), \mathcal{I}) = 0 \text{ for all } U \text{ open}, i \geq 1,$$

and hence

$$\widetilde{R}^i f_*(\mathcal{I})^+ = 0 \text{ for } i \geq 1.$$

Finally, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence, for any open set U, we get a long exact sequence

$$\cdots \longrightarrow H^i(f^{-1}(U), \mathcal{F}') \longrightarrow H^i(f^{-1}(U), \mathcal{F}) \longrightarrow \cdots$$

which by definition gives

$$(*) \qquad \cdots \longrightarrow \widetilde{R}^i f_*(\mathcal{F}') \longrightarrow \widetilde{R}^i f_*(\mathcal{F}) \longrightarrow \cdots$$

and taking the sheafification gives a sequence

$$\cdots \longrightarrow \widetilde{R}^i f_*(\mathcal{F}')^+ \longrightarrow \widetilde{R}^i f_*(\mathcal{F})^+ \longrightarrow \cdots$$

By passing to stalks and using exactness of sequence $(*)$, we get that this sequence is exact. Altogether $\left\{\widetilde{R}^i f_*(-)^+\right\}_{i\geq 0}$ satisfies the universal property of $\{R^i f_*(-)\}_{i\geq 0}$, so there is an isomorphism of δ-functors between them. $\square$

Corollary 3.3.3. *If f is as above and $\mathcal{M}$ is a flasque $\mathcal{O}_X$-module, $R^i f_*(\mathcal{M}) = 0$ for $i \geq 1$. Therefore, for any $\mathcal{O}_X$-module $\mathcal{A}$, if $\mathcal{A} \to \mathcal{M}^\bullet$ is a flasque resolution, then*

$$R^i f_*(\mathcal{A}) \cong \mathcal{H}^i(f_*(\mathcal{M}^\bullet)).$$

Proof. If $U \subseteq Y$ is open, $\mathcal{M}|_{f^{-1}(U)}$ is flasque, so $H^i(f^{-1}(U), \mathcal{M}) = 0$ for all $i \geq 1$, so $\widetilde{R}^i f_*(\mathcal{M}) = 0$ for all $i \geq 1$, and hence by Proposition 3.3.2, $R^i f_*(\mathcal{M}) = 0$ for all $i \geq 1$. $\square$

Proposition 3.3.4. *If $f\colon X \to Y$ is a morphism of algebraic varieties and $\mathcal{M}$ is a quasicoherent sheaf on X, then*

$$R^i f_*(\mathcal{M}) \text{ is quasicoherent for all } i \geq 1.$$

Moreover, if $U \subseteq Y$ is an affine open subset, then

$$\Gamma(U, R^i f_*(\mathcal{F})) \cong H^i(f^{-1}(U), \mathcal{F}).$$

Before we prove this result, we need another proposition.

Proposition 3.3.5. *If X is an affine algebraic variety and I is an injective $\mathcal{O}(X)$-module, then $\widetilde{I}$ is flasque.*

We will assume this result for now an delay the proof until later.

Corollary 3.3.6. *If X is an algebraic variety, then for any quasicoherent sheaf $\mathcal{F}$ on X, there is a quasicoherent flasque sheaf $\mathcal{E}$ with an injective map $\mathcal{F} \hookrightarrow \mathcal{E}$.*

Proof. Let $X = \bigcup_{i=1}^{r} U_i$ be an affine open cover. Note that $\mathcal{F}|_{U_i}$ is still quasicoherent. Let Q_i be an injective $\mathcal{O}(U_i)$-module such that there is an inclusion

$$\mathcal{F}(U_i) \hookrightarrow Q_i.$$

Then $\mathcal{F}|_{U_i} \hookrightarrow \widetilde{Q_i}$ and $\widetilde{Q_i}$ is flasque on U_i by Proposition 3.3.5. We then have

$$\mathcal{F} \hookrightarrow \bigoplus_{i=1}^{r} (\alpha_i)_*(\mathcal{F}|_{U_i}) \hookrightarrow \bigoplus_{i=1}^{r} (\alpha_i)_*(\widetilde{Q_i})$$

and the last sheaf is quasicoherent and flasque. $\square$

We can finally prove Proposition 3.3.4.

Proof of Proposition 3.3.4. By Proposition 3.3.5, there is a resolution

$$\mathcal{M} \to \mathcal{Q}^\bullet$$

with all $\mathcal{Q}^i$ quasicoherent and flasque. Then

$$R^i f_*(\mathcal{M}) \cong \mathcal{H}^i(f_*(\mathcal{Q}^\bullet))$$

and since $f_*(\mathcal{Q}^\bullet)$ is quasicoherent (as a pushforward of a quasicoherent sheaf), this shows $R^if_*(\mathcal{M})$ is quasicoherent.

If $U \subseteq Y$ is open an affine, then we have

$$\begin{aligned}
\Gamma(U, R^if_*(\mathcal{F})) &\cong \Gamma(U, \mathcal{H}^i(f_*(\mathcal{Q}^i))) \\
&\cong \mathcal{H}^i(\Gamma(U, f_*(\mathcal{Q}^\bullet))) && \text{since } \Gamma(U,-) \text{ exact on quasicoherent} \\
& && \text{sheaves on affine varieties} \\
&\cong \mathcal{H}^i(\Gamma(U, f_*(\mathcal{Q}^\bullet))) \\
&\cong \mathcal{H}^i(\Gamma(f^{-1}(U), \mathcal{Q}^\bullet)) \\
&\cong H^i(f^{-1}(U), \mathcal{M}).
\end{aligned}$$

This completes the proof. □

3.4. **Cohomology of quasicoherent sheaves on affine varieties.**

Theorem 3.4.1 (Serre). *If X is an algebraic variety, the following are equivalent*

(1) *X is affine,*
(2) *$H^i(X, \mathcal{F}) = 0$ for any $\mathcal{F}$ quasicoherent and $i \geq 1$,*
(3) *$H^1(X, \mathcal{I}) = 0$ for all coherent ideals sheaves $\mathcal{I} \subseteq \mathcal{O}_X$.*

Proof. We first show that (1) implies (2). If $\mathcal{F}$ is quasicoherent, there is a flasque resolution $\mathcal{F} \to \mathcal{Q}^\bullet$ such that $\mathcal{Q}^i$ is quasicoherent for all i by Corollary 3.3.6. Then

$$H^i(X, \mathcal{F}) \cong \mathcal{H}^i(\Gamma(X, Q^\bullet)) = 0,$$

since $\Gamma(X, -)$ is exact in the category of quasicoherent sheaves on affine varieties.

Note that (2) implies (3) is immediate, so it remains to show that (3) implies (1). For any $x \in X$, choose an affine open neighborhood U of x. Let $Z = \{x\} \cup (X \setminus U)$, which is closed in X, and let $\mathcal{I}_Z$ be the corresponding radical ideal sheaf. We then have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and the long exact sequence in cohomology gives

$$\Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{I}_Z) = 0.$$

Therefore, there exists $f \in \mathcal{O}_X(X)$ such that $f(x) \neq 0$ and $f|_{X \setminus U} = 0$. Then $x \in D_X(f) \subseteq U$, so $D_X(f) = D_U(f|_U)$, which is affine since U is affine.

Since X is quasicompact, there exist $f_1, \ldots, f_r \in \mathcal{O}_X(X)$ such that $X = \bigcup_{i=1}^r D_X(f_i)$ and each $D_X(f_i)$ is affine.

If we show that $\mathcal{O}_X(X) = (f_1, \ldots, f_r)$, then (by a result from a homework on Math 631) X is affine. To show this, consider the map

$$\begin{aligned}
\mathcal{O}_X^{\oplus r} &\xrightarrow{\varphi} \mathcal{O}_X \\
e_i &\mapsto f_i.
\end{aligned}$$

This is surjective, since on $D_X(f_i)$, f_i generates $\mathcal{O}_{D_X(f_i)}$. Let $\mathcal{F}$ be the kernel of φ. We have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

It is enough to show that $H^1(X,\mathcal{F})=0$. Indeed, this implies that

$$\Gamma(X,\mathcal{O}_X)^{\oplus r} \to \Gamma(X,\mathcal{O}_X)$$

is surjective, so $\Gamma(X,\mathcal{O}_X)=(f_1,\dots,f_r)$.

Let $\mathcal{E}_i \subseteq \mathcal{O}_X^{\oplus r}$ be generated by $e_1,\dots,e_i$. Then $\mathcal{E}_{i+1}/\mathcal{E}_I \cong \mathcal{O}_X$. Consider

$$0 \subseteq \mathcal{F}\cap\mathcal{E}_1 \subseteq \mathcal{F}\cap\mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{F}\cap\mathcal{E}_r = \mathcal{F},$$

which gives a short exact sequence

$$0 \longrightarrow \mathcal{F}\cap\mathcal{E}_i \longrightarrow \mathcal{F}\cap\mathcal{E}_{i+1} \longrightarrow \tfrac{\mathcal{F}\cap\mathcal{E}_{i+1}}{\mathcal{F}\cap\mathcal{E}_i} \longrightarrow 0.$$

Note that

$$\frac{\mathcal{F}\cap\mathcal{E}_{i+1}}{\mathcal{F}\cap\mathcal{E}_i} \hookrightarrow \mathcal{E}_{i+1}/\mathcal{E}_i \cong \mathcal{O}_X$$

is a coherent ideal, so

$$H^1\left(X, \frac{\mathcal{F}\cap\mathcal{E}_{i+1}}{\mathcal{F}\cap\mathcal{E}_i}\right)=0$$

by assumption. The long exact sequence in cohomology shows that

$$H^1(\mathcal{F}\cap\mathcal{E}_i)=0 \text{ implies } H^i(\mathcal{F}\cap\mathcal{E}_{i+1})=0.$$

Since $\mathcal{F}\cap\mathcal{E}_0=0$, by induction on i, we have that

$$H^1(X,\mathcal{F}\cap\mathcal{E}_i)=0 \text{ for all } i.$$

Taking $i=r$, this completes the proof. ☐

We finally give a sketch of the proof of Proposition 3.3.5.

Sketch of proof of Proposition 3.3.5. Let $A=\mathcal{O}(X)$ and Q be an injective A-module. We want to show that $\widetilde{Q}$ is flasque.

Step 1. Show that if $U=D_X(f)$ then

$$\underbrace{\Gamma(X,\widetilde{Q})}_{Q} \to\subseteq \underbrace{\Gamma(U,\widetilde{Q}))}_{Q_f} \text{ is surjective.}$$

Consider $\mathrm{Ann}(f)\subseteq \mathrm{Ann}(f^2)\subseteq\cdots$. As A is Noetherian, there is an r such that $\mathrm{Ann}(f^r)=\mathrm{Ann}(f^{r+1})=\cdots$. Consider $u\in Q_f$, $u=\frac{a}{f^s}$. Define a morphism

$$(f^{r+s}) \xrightarrow{\varphi} Q$$
$$f^{r+s}b \mapsto f^r ba.$$

This is well-defined since $f^{r+s}b=f^{r+s}b'$ implies that $f^rb=f^rb'$.

Since Q is injective, this can be extended to a map $\psi\colon A \to Q$ and let $v = \psi(1)$. Then $f^{r+s}v = f^r a$, and hence $\frac{a}{f^s} = \frac{v}{1}$ in Q_f.

For the other steps, see the official notes. □

3.5. **Soft sheaves on paracompact spaces.**

Definition 3.5.1. A topological space X is *paracompact* if the following conditions hold:

- Hausdorff,
- every open cover has a locally finite refinement.

It is easy to see that a closed subset of a paracompact space is paracompact.

Examples 3.5.2.

(1) Topological manifolds (which are assumed to be Hausdorff and have a countable basis of open subsets)
(2) Simplicial complexes
(3) CW complexes

The following result is always useful: if $X = \bigcup_i U_i$ is a locally finite open cover, then there is an open cover $X = \bigcup_i V_i$ such that $\overline{V_i} \subseteq U_i$.

A special case shows that if $F \subseteq U$ where F is an open subset and U is a closed subset (so $V \cup X \setminus F$ is an open cover), then there is an open set W such that $F \subseteq W \subseteq \overline{W} \subseteq V$. In other words, a paracompact space is *normal*.

Definition 3.5.3. Let X be a topological space. A sheaf $\mathcal{F}$ is *soft* if for any closed subset $Z \subseteq X$, $\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F})$ is surjective.

We recall a result from the problem session. If X is paracompact and $Z \subseteq X$ is closed, then for any $s \in \mathcal{F}(Z)$, there is an open subset U containing Z and $s_U \in \mathcal{F}(U)$ such that $s_U|_Z = s$. In particular, if $\mathcal{F}$ is flasque, then it is soft.

Lemma 3.5.4. *Suppose X is paracompact. If*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence with $\mathcal{F}'$ soft, then we have an exact sequence

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

Proof. We omit the proof here since this is similar to Problem 1 on Homework 6, but it can be found in the official notes. □

Corollary 3.5.5. *If X is paracompact and*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence with $\mathcal{F}'$ soft, then $\mathcal{F}$ is soft if and only if $\mathcal{F}''$ is soft.

Proof. If $Z \subseteq X$ is closed, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d, two heads, "\alpha"] & \mathcal{F}(X) \arrow[r] \arrow[d, "\beta"] & \mathcal{F}''(X) \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathcal{F}'(Z) \arrow[r] & \mathcal{F}(Z) \arrow[r] & \mathcal{F}''(Z) \arrow[r] & 0 \end{tikzcd}$$

with exact rows by Lemma 3.5.4 (note that $\mathcal{F}'|_Z$ is also soft and Z is also paracompact so the lemma applies). As α is surjective by hypothesis, the Snake Lemma shows that $\operatorname{coker}\beta \cong \operatorname{coker}\gamma$. $\square$

Proposition 3.5.6. *If X is paracompact and $\mathcal{F}$ is soft, then $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$. In particular, if $\mathcal{M}$ is any $\mathcal{O}_X$-module and $\mathcal{M} \to \mathcal{F}^\bullet$ is a resolution by soft $\mathcal{O}_X$-modules, then*

$$H^i(X, \mathcal{M}) \cong \mathcal{H}^i(\Gamma(X, \mathcal{F}^\bullet)).$$

Proof. Consider an embedding $\mathcal{F} \hookrightarrow \mathcal{A}$ into a flasque sheaf $\mathcal{A}$, and let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

be the corresponding short exact sequence. Then the long exact sequence in cohomology gives the exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] & \Gamma(X,\mathcal{A}) \arrow[r] & \Gamma(X,\mathcal{B}) \arrow[dll, out=0, in=180] \\ & H^1(X,\mathcal{F}) \arrow[r] & \underbrace{H^1(X,\mathcal{A}) = 0}_{\text{as } \mathcal{A} \text{ is flasque}}, \end{tikzcd}$$

$$0 = H^i(X, \mathcal{A}) \longrightarrow H^i(X, \mathcal{B}) \longrightarrow H^{i+1}(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{A}) = 0.$$

This shows that $H^i(X, B) \cong H^{i+1}(X, \mathcal{F})$ for $i \geq 1$. Since $\Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B})$ is surjective by Lemma 3.5.4, $H^1(X, \mathcal{F}) = 0$.

By Corollary 3.5.5, since $\mathcal{F}$ is soft and $\mathcal{A}$ is soft (since flasque), $\mathcal{B}$ is also soft. By induction, we see that $H^i(X, B) = 0$, so $H^{i+1}(X, \mathcal{F}) = 0$, which completes the proof. $\square$

3.6. **De Rham cohomology and sheaf cohomology.** Let X be a smooth manifold (in particular, since it is Hausdorff and has a countable basis, it is paracompact). Let

$$\mathcal{C}_X^\infty = \text{sheaf of smooth functions form } X \text{ to } \mathbb{R},$$
$$\mathcal{E}_X^p = \text{sheaf of smooth } p\text{-differential forms on } X.$$

Note that $\mathcal{E}_X^0 = \mathcal{C}_X^\infty$.

Let $d\colon \mathcal{E}_X^p \to \mathcal{E}_X^{p+1}$ be the exterior differential.

Recall that if $U \subseteq X$ is open with coordinates $x_1, \ldots, x_n$, any $\omega \in \Gamma(U, \mathcal{E}_X^p)$ can be written as

$$\omega = \sum_{|I|=p} f_I dx_I$$

where I is an ordered p-tuple $i_1 < \cdots < i_p$ and we write $dx_I = d_{i_1} \wedge \cdots \wedge dx_{i_p}$. Then

$$d\omega = \sum_{|I|=p} \left(\sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \right) \wedge dx_I.$$

Note that $d \circ d$, and hence we get the *de Rham complex*:

$$0 \longrightarrow \mathcal{E}_X^0(X) \longrightarrow \mathcal{E}_X^1(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_X^n(X) \longrightarrow 0$$

where $n = \dim(X)$. The *de Rham cohomology groups* are then defined as

$$H^p_{\mathrm{dR}}(X) = \mathcal{H}^p(\mathcal{E}_X^\bullet(X)),$$

which are $\mathbb{R}$-vector spaces.

Theorem 3.6.1. *We have a canonical isomorphism*

$$H^p_{\mathrm{dR}}(X) \cong H^p(X, \mathbb{R}),$$

where $\mathcal{R}$ is the constant sheaf.

Lemma 3.6.2. *Every $\mathcal{C}_X^\infty$-module $\mathcal{F}$ (for example, $\mathcal{E}_X^p$) is* soft.

Proof. Let $Z \subseteq X$ be closed and let $s \in \mathcal{F}(Z)$. We know that there is an open subset $U \supseteq Z$ and $s_U \in \mathcal{F}(U)$ such that $s_U|_Z = s$.

Choose open subset V_1, V_2 such that

$$Z \subseteq V_1 \subseteq \overline{V_1} \subseteq V_2 \subseteq \overline{V_2} \subseteq U.$$

By the smooth version of Urysohn's Lemma, there is a function $f \in \mathcal{C}_X^\infty(X)$ such that

$$f = \begin{cases} 1 \text{ on } \overline{V_1}, \\ 0 \text{ on } X \setminus \overline{V_2}. \end{cases}$$

Consider fs_U on U and 0 on $X \setminus \overline{V_2}$. They agree on $U \setminus \overline{V_2}$, so there exists $t \in \mathcal{F}(X)$ such that

$$t|_U = fs_U.$$

Then $t|_{V_1} = s_U|_{V_1}$, and hence $t|_Z = s$. □

Consider the complex

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{E}_X^0(X) \longrightarrow \mathcal{E}_X^1(X) \longrightarrow \cdots \longrightarrow \mathcal{E}_X^n(X) \longrightarrow 0.$$

The following lemma shows that this complex gives a resolution for $\mathbb{R}$ when $X = \mathbb{R}^n$.

Lemma 3.6.3 (Poincaré Lemma)**.** *For every $n \geq 0$, the complex*

$$\mathcal{E}^\bullet_{\mathbb{R}^n}(\mathcal{R}^n) : 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0_{\mathbb{R}^n}(\mathbb{R}^n) \longrightarrow \mathcal{E}^1_{\mathbb{R}^n}(\mathbb{R}^n) \longrightarrow \cdots \longrightarrow \mathcal{E}^n_X(\mathbb{R}^n) \longrightarrow 0.$$

is exact.

Proof. We use induction on $n \geq 0$. The $n = 0$ case is trivial. For $n \geq 0$, consider the maps

$$\begin{array}{ccccc} \mathbb{R}^{n-1} & \stackrel{i}{\hookrightarrow} & \mathbb{R}^n & \stackrel{\pi}{\twoheadrightarrow} & \mathbb{R}^{n-1} \\ & & (x_1, \dots, x_n) & \longmapsto & (x_2, \dots, x_n) \\ (x_2, \dots, x_n) & \longmapsto & (0, x_2, \dots, x_n) & & \end{array}$$

This gives maps

$$\mathcal{E}^\bullet_{\mathbb{R}^{n-1}}(\mathbb{R}^{n-1}) \xrightarrow{\pi^*} \mathcal{E}^\bullet_{\mathbb{R}^n}(\mathbb{R}^n) \xrightarrow{i^*} \mathcal{E}^\bullet_{\mathbb{R}^{n-1}}(\mathbb{R}^{n-1})$$

whose composition is the identity. To complete the proof by induction, it suffices to show that $\pi^* \circ i^* \approx 1_{\mathcal{E}^\bullet_{\mathbb{R}^n}(\mathbb{R}^n)}$. To define a differential

$$\begin{tikzcd} & \mathcal{E}^p(\mathbb{R}^n) \arrow[r, "d"] \arrow[dl, "\theta^p"] & \mathcal{E}^{p+1}(\mathbb{R}^n) \\ \mathcal{E}^{p-1}(\mathbb{R}^n) & & \end{tikzcd}$$

we use integration:

$$f dx_I \mapsto \begin{cases} 0 & \text{if } 1 \notin I, \\ \left(\int\limits_0^{x_1} f(t, x_2, \dots, x_n) dt \right) dx_{I'} & \text{if } I = \{1\} \cup I'. \end{cases}$$

The fact that this gives a homotopy as above is left as an exercise. For example, if $1 \notin I$, then

$$\begin{aligned} (\theta^{p+1} \circ d + d \circ \theta^p)(f dx_I) &= \theta^{p+1} \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \wedge dx_I \right) \\ &= \left(\int\limits_0^{x_1} \frac{\partial f}{\partial x_1}(t, x_2, \dots, x_n) dt \right) dx_I \\ &= (f(x_1, \dots, x_n) - f(0, x_2, \dots, x_n)) dx_I \qquad \text{by the FTC} \\ &= (\mathrm{id} - \pi^* \circ i^*)(f dx_I) \end{aligned}$$

The other case is a similar computation. ∎

Proof of Theorem 3.6.1. We have the following complex of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}^0_X \longrightarrow \cdots \longrightarrow \mathcal{E}^n_X \longrightarrow 0.$$

We know that if we take sections on U diffeomorphic to $\mathbb{R}^n$, then by Poincaré Lemma 3.6.3, we get an exact complex. Since every point has a basis of neighborhoods diffeomoprhic to $\mathbb{R}^n$, the above complex of sheaves is exact. By Lemma 3.6.2, $\mathbb{R} \to \mathcal{E}_X^\bullet$ is a soft resolution of $\mathbb{R}$, and the result follows from Proposition 3.5.6. ☐

We now move on to general topological spaces instead.

Theorem 3.6.4. *If X is a locally contractible topological space, which is paracompact, then for every commutative ring R and R-module A, we have an isomorphism*

$$H^p(X, \underline{A}) \cong \underbrace{H^p_{\text{sing}}(X, A)}_{\substack{\textit{singular cohomology} \\ \textit{with coefficients in } A}} .$$

Recall that X is a locally contractible space if every point in X has a basis of open neighborhoods which are contractible.

Remark 3.6.5. By a recent result, one can drop the *paracompactness* hypothesis.

Proof of Theorem 3.6.4. For every p, let Δ_p be the standard p-dimension simplex. Then a p-simplex in X is a continuous map $\Delta_p \to X$, and we set

$$C_p(X) = \text{free abelian group generated by } p\text{-simplices in } X$$

and we have the standard map $\partial\colon C_p(X) \to C_{p-1}(X)$ such that $\partial^2 = 0$. We let

$$C^p(X, A) = \operatorname{Hom}_{\mathbb{Z}}(C_p(X), A).$$

Then

$$H^p_{\text{sing}}(X, A) = \mathcal{H}^p(C^\bullet(X, A)).$$

Note that any map $f\colon X \to Y$ gives a chain map $C^\bullet(Y, A) \to C^\bullet(X, A)$.

For every p, let $\mathcal{C}^p_X$ be the presheaf that assigns to $X \supseteq U$, $C^p(U, A)$, and for $V \subseteq U$, the natural map $C^p(U, A) \to C^p(V, A)$ gives the restriction maps.

We get a complex

$$0 \longrightarrow \underline{A} \longrightarrow \mathcal{C}^0_X \longrightarrow \mathcal{C}^1_X \longrightarrow \cdots \longrightarrow \mathcal{C}^n_X \longrightarrow 0.$$

Note that $\mathcal{C}^p_X$ is not a sheaf: functions that agree on intersections can be glued, but far from uniquely. Hence let $\mathcal{S}^p_X = (\mathcal{C}^p_X)^+$.

Since X is locally contractible, each point has a basis of neighborhoods U such that the corresponding complex of sections of $\mathcal{C}^\bullet_X$ on U is exact. We then get an exact sequence

$$0 \longrightarrow \underline{A} \longrightarrow \mathcal{S}^0_X \longrightarrow \mathcal{S}^1_X \longrightarrow \cdots \longrightarrow \mathcal{S}^n_X \longrightarrow 0.$$

If every open subset of X is paracompact (for example, if X is a topological manifold), all $\mathcal{S}^p_X$ are flasque, since all maps $\mathcal{C}^p_X(U) \to \mathcal{S}^p_X(U)$ are surjective (this is by Problem 2 on Problem Set 6). In general, $\mathcal{S}^p_X$ are just soft, and this case is dealt with in the notes.

This shows that
$$H^p(X, \underline{A}) \cong \mathcal{H}^p(\mathcal{S}_X^\bullet(X)).$$
Consider
$$0 \longrightarrow V^\bullet(X) \longrightarrow \mathcal{C}_X^\bullet(X) \longrightarrow \mathcal{S}_X^\bullet(X) \longrightarrow 0.$$
It is enough to show that $\mathcal{H}^p(V^\bullet(X)) = 0$ for all p. Recall that
$$V^\bullet(X) = \varinjlim V_{\mathcal{U}}^\bullet(X)$$
where
$$V_{\mathcal{U}}^\bullet(X) = \left\{ \sigma \in C^p(X, A) \mid \begin{array}{c} \sigma \text{ vanishes on } p\text{-simplices in } X \\ \text{whose image is contained in some element of } \mathcal{U} \end{array} \right\}.$$
It is enough to show that $\mathcal{H}^p(V_{\mathcal{U}}^\bullet(X)) = 0$ for each $\mathcal{U}$.

A known fact from singular cohomology is that if $C_\bullet^{\mathcal{U}}$ is the free abelian group generated by the p-simplices in X whose image is contained in some element of $\mathcal{U}$, then
$$C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$$
is a homotopy equivalence. (This is proved using baricentric subdivision.)

This is still a homotopy equivalence after applying $\mathrm{Hom}_{\mathbb{Z}}(-, A)$. Then the exact sequence
$$0 \longrightarrow V_{\mathcal{U}}^\bullet(X) \longrightarrow C^\bullet(X, A) \longrightarrow \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X), A) \longrightarrow 0$$
shows that $\mathcal{H}^p(V_{\mathcal{U}}^\bullet(X)) = 0$ for all p. □

3.7. **Introduction to spectral sequences.** Let $K^\bullet$ be a complex in a category $\mathcal{C}$ (for example, the category of $\mathcal{O}_X$-modules for a ringed space $(X, \mathcal{O}_X)$). Consider a decreasing filtration $F_\bullet K^\bullet = (F_p K^\bullet)_{p \in \mathbb{Z}}$, i.e. a chain of subcomplexes:
$$K^\bullet \supseteq \cdots \supseteq F_p K^\bullet \supseteq F_{p+1} K^\bullet \supseteq .$$
This gives a filtration on the cohomology of $K^\bullet$ given by
$$F_p \mathcal{H}^n(K^\bullet) = \mathrm{im}(\mathcal{H}^n(F_p K^\bullet) \to \mathcal{H}^n(K^\bullet)).$$
We want a description of the quotients
$$\mathrm{gr}_p \mathcal{H}^n(K^\bullet) = \frac{F_p \mathcal{H}^n(K^\bullet)}{F_{p+1} \mathcal{H}^n(K^\bullet)}$$
in terms of some data coming from the successive quotients of $K^\bullet$.

This data is encoded by the *spectral sequence*. For $r \geq 0$, we denote by $(E_r^{p,q})_{p,q \in \mathbb{Z}}$ the rth page of the spectral sequence, where p is related to the filtration level and $p+q$ records the place in the complex. To define it, for $r \in \mathbb{Z}$ let
$$Z_r^{p,q} = \{u \in F_p K^{p+q} \mid d(u) \in F_{p+r} K^{p+q+1}\}$$
and set
$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})}.$$

Note that $Z_{(-1)}^{p,q} = Z_0^{p,q} = F_pK^{p+q}$. Also,

$$E_0^{p,q} = \frac{F_pK^{p+q}}{F_{p+1}K^{p+q}}$$

and there is a map

$$E_0^{p,q} \to E_0^{p,q+1}$$

induced by d. Similarly,

$$E_1^{p,q} = \frac{\{u \in F_pK^{p+q} \mid du \in F_{p+1}K^{p+q+1}\}}{F_{p+1}K^{p+q} + d(F_p(K^{p+q-1}))} = \mathcal{H}(E_0^{p-1,q} \to E_0^{p,q} \to E_0^{p,q+1}).$$

In general, d induces a map $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Proposition 3.7.1. *For each $r \geq 0$, there is a canonical isomorphism*

$$E_{r+1}^{p,q} = \mathcal{H}(E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1}).$$

Proof. The proof is left as an exercise. □

Definition 3.7.2. A filtration $F_\bullet K^\bullet$ on $K^\bullet$ is *pointwise finite* if for any n we have

$$F_pK^n = 0 \text{ for } p \gg 0,$$
$$F_pK^n = K^n \text{ for } p \ll 0.$$

Proposition 3.7.3. *If the filtration on $K^\bullet$ is pointwise finite, then for any $p, q \in \mathbb{Z}$, $E_r^{p,q}$ is eventually constant. We denote this value by $E_\infty^{p,q}$. Moreover, for all p, q we have that*

$$E_\infty^{p,q} \cong \mathrm{gr}_p\mathcal{H}^{p+q}(K^\bullet).$$

Proof. Fix p, q. We have that

$$Z_r^{p,q} = F_pK^{p+q} \cap \ker(d).$$

Consider the sequence

$$E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1}$$

whose cohomology gives $E_{r+1}^{p,q}$ by Proposition 3.7.1. For $r \gg 0$, $Z_r^{p+r,q-r+1} = 0$, since $F_pK^n = 0$ for $p \gg 0$. Similarly, $E_r^{p-r,q+r-1}$ for $r \gg 0$, since $Z_r^{p-r,q+r-1} = Z_{r-1}^{p-r+1,q+r-2}$. Therefore, taking the cohomology of the above sequence gives simply

$$E_{r+1}^{p,q} = E_r^{p,q} \text{ for } r \gg 0.$$

Moreover,

$$dZ_{r-1}^{p-r+1,q+r-2} = d(K^{p+q-1} \cap d(F_pK^{p+q})) = F_pK^{p+q} \cap \mathrm{im}(d).$$

It is easy to check that

$$\mathrm{gr}_p\mathcal{H}^{p+q}(K^\bullet) \cong \frac{F_pK^{p+q} \cap \ker(d)}{F_{p+1}K^{p+q} \cap \ker(d) + (F_pK^{p+q} \cap \mathrm{im}(d))},$$

which completes the proof. □

Definition 3.7.4. If the conclusion of Proposition 3.7.3, we write

$$E_r^{p,q} \Rightarrow_p \mathcal{H}^{p+q}(K^\bullet)$$

and say that the spectral sequence *converges* with respect to p to the cohomology of $K^\bullet$.

Definition 3.7.5. The spectral sequence *collapses at level* r_0 if $d_r = 0$ for $r \geq r_0$. In this case, $E_\infty = E_{r_0}$.

Suppose there is an a such that $E_{r_0}^{p,q} = 0$ unless $p = a$ and $r_0 \geq 1$. Then $E_\infty^{p,q} = 0$ if $p \neq a$ and $E_\infty^{p,q} = E_{r_0}^{p,q}$ for all p, q. This shows that

$$\mathcal{H}^n(K^\bullet) \cong E_{r_0}^{a,n-a} \text{ for all } n.$$

This way, we recover the cohomology of the complex from a spectral sequence.

Similarly, if there is a b such that $E_{r_0}^{p,q} = 0$ unless $q = b$, then $E_\infty^{p,q} = 0$ unless $q = b$, and

$$E_\infty^{p,b} = \begin{cases} E_{r_0}^{p,b} & \text{if } r_0 \geq 2, \\ E_2^{p,b} & \text{if } r_0 = 1. \end{cases}$$

In this case,

$$\mathcal{H}^n(K^\bullet) \cong E_{r_0}^{n-b,b}.$$

We now describe the spectral sequence of a double complex.

Definition 3.7.6. A *double complex* $A^{\bullet,\bullet}$ is a collection $(A^{p,q})_{p,q\in\mathbb{Z}}$ of objects together with morphisms

$$d_1 \colon A^{p,q} \to A^{p+1,q}$$

$$d_2 \colon A^{p,q} \to A^{p,q+1}$$

such that $0 = d_1 \circ d_1 = d_2 \circ d_2$ and $d_1 \circ d_2 = d_2 \circ d_1$.

In particular, both $A^{p,\bullet}$ and $A^{\bullet,q}$ are complexes all fixed p and q.

Definition 3.7.7. The *total complex* of $A^{\bullet,\bullet}$ is $K^\bullet = \operatorname{Tot}(A^{\bullet,\bullet})$ is defined by $K^n = \bigoplus_{i+j=n} A^{i,j}$ together with maps

$$d \colon K^n \to K^{n+1}$$

such that $d|_{A^{i,j}} = d_1 + (-1)^i d_2$.

It is easy to see that $d \circ d = 0$ so the total complex is indeed a complex.

We consider two filtration on $K^\bullet$:

$$F'_p K^n = \bigoplus_{\substack{i+j=n \\ i \geq p}} A^{i+j},$$

$$F''_p K^n = \bigoplus_{\substack{i+j=n \\ j \geq p}} A^{i+j}.$$

We will always assume that for any n, there are only finitely many p such that $A^{p,n-p} \neq 0$. Hence both filtrations are pointwise finite. This happens, for example, when $A^{p,q} = 0$ unless $p, q \geq 0$, i.e. for a *first quadrant double complex*.

In this case, Proposition 3.7.3 shows that the two spectral sequences associated to these two filtrations converge to $\mathcal{H}^n(K^\bullet)$.

Let us first consider the spectral sequence ${}'E_r^{p,q}$ with respect to the F' filtration. We have that ${}'E_0^{p,q} = A^{p,q}$ and the induced map $d_0\colon \underbrace{E_0^{p,q}}_{A^{p,q}} \to \underbrace{E_0^{p,q+1}}_{A^{p,q+1}}$ is given by $(-1)^i d_2$. This gives

$${}'E_1^{p,q} = H^q(A^{p,\bullet}).$$

The map induced by taking $\mathcal{H}^q$ of $A^{p,\bullet} \to A^{p+1,\bullet}$ gives

$$\underbrace{{}'E_1^{p,q}}_{\mathcal{H}^q(A^{p,\bullet})} \to \underbrace{{}'E_1^{p+1,q}}_{\mathcal{H}^q(A^{p+1,\bullet})}.$$

We then define

$${}'E_2^{p,q} = \mathcal{H}^p_{d_1}\mathcal{H}^q_{d_2}(A^{\bullet,\bullet}) = \mathcal{H}(\mathcal{H}^q(A^{p-1,\bullet}) \to \mathcal{H}^q(A^{p,\bullet}) \to \mathcal{H}^q(A^{p+1,\bullet})).$$

Similarly, for the F'' filtration we get

$$\begin{aligned} {}''E_0^{p,q} &= A^{q,p}, \\ {}''E_1^{p,q} &= \mathcal{H}^q(A^{\bullet,p}), \\ {}''E_2^{p,q} &= \mathcal{H}^p_{d_2}\mathcal{H}^q_{d_1}(A^{\bullet,\bullet}). \end{aligned}$$

3.8. **The Grothendieck spectral sequence.** Consider two left exact functors

$$\mathcal{C}_1 \xrightarrow{\mathcal{G}} \mathcal{C}_2 \xrightarrow{\mathcal{F}} \mathcal{C}_3.$$

Theorem 3.8.1. *Suppose for any injective object $\mathcal{I}$ of $\mathcal{C}_1$, $\mathcal{G}(\mathcal{I})$ is $\mathcal{F}$-acyclic. Then for any object $\mathcal{A}$ of $\mathcal{C}_1$, there is a spectral sequence $E_2^{p,q} = R^p\mathcal{F}(R^q\mathcal{G}(\mathcal{A}))$ and*

$$E_2^{p,q} \Rightarrow_p R^{p+q}(\mathcal{G} \circ \mathcal{F})(\mathcal{A}).$$

Example 3.8.2 (Leray Spectral Sequence)**.** A composition of morphisms of algebraic varieties

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

induces

$$\mathcal{O}_X\text{-mod} \xrightarrow{g_*} \mathcal{O}_Y\text{-mod} \xrightarrow{f_*} \mathcal{O}_Z\text{-mod}$$

If $\mathcal{I} \in \mathcal{O}_X$-mod is injective, it is flasque, and hence $g_*\mathcal{I}$ is flasque, i.e. f_*-acyclic. Then by Theorem 3.8.1, we get a spectral sequence

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathcal{G})) \Rightarrow_p R^{p+q}((f \circ g)_*\mathcal{F}).$$

In particular, if Z is a point, we see that for a morphism $g\colon X \to Y$ and and $\mathcal{O}_X$-module $\mathcal{F}$, we gave a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q g_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Example 3.8.3. Suppose g is affine (for example, if it is finite or a closed immersion). If $\mathcal{F}$ is quasicoherent on X, then $R^q f_*\mathcal{F} = 0$ for $q \geq 1$. If $U \subseteq Y$ is affine, then

$$\Gamma(U, R^q f_*\mathcal{F}) \cong H^q(\underbrace{f^{-1}(U)}_{\text{affine}}, \mathcal{F}) = 0 \text{ for } q \geq 1.$$

The Leray Spectral Sequence shows that

$$H^n(X, \mathcal{F}) = H^n(Y, g_*(\mathcal{F}))$$

for any quasicoherent sheaf $\mathcal{F}$ if g is affine.

Definition 3.8.4. Given a complex $C^\bullet$ bounded from below ($C^p = 0$ if $p \ll 0$), a *Cartan–Eilenberg resolution* of $C^\bullet$ is a double complex $A^{\bullet,\bullet}$ together with a morphism of complex $C^\bullet \to A^{\bullet,\bullet}$ such that

(1) there is a p_0 such that $A^{p,q} = 0$ if $p \leq p_0$ for any q; $A^{p,q=0}$ if $q < 0$ for any p,
(2) for any p, $C^p \to A^{p,0}$ is an injective resolution,
(3) for any p, $\ker(d^p) \to \ker(d_1^{p,\bullet})$ in an injective resolution,
(4) for any p, $\mathrm{im}(d^p) \to \mathrm{im}(d_1^{p,\bullet})$ is an injective resolution,
(5) for any p, $\mathcal{H}^p(C^\bullet) \to \mathcal{H}^p(A^{\bullet,0}) \to \mathcal{H}^p(A^{\bullet,1}) \to \cdots$ is an injective resolution.

Lemma 3.8.5. *In any category with enough injectives, any complex bounded from below has a Cartan-Eilenberg resolution.*

Proof. Fix p_0 such that $C^p = 0$ for $p \leq p_0$. For any p, we have two short exact sequences

$$0 \longrightarrow \mathrm{im}(d^{p-1}) \longrightarrow \ker(d^p) \longrightarrow \mathcal{H}^p(C^\bullet) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \ker(d^p) \longrightarrow C^p \longrightarrow \mathrm{im}(d^p) \longrightarrow 0 \tag{2}$$

For any p, choose injective resolutions

$$\mathcal{H}^p(C^\bullet) \to U^{p,\bullet}$$
$$\mathrm{im}(d^{p-1}) \to V^{p,\bullet}$$

such that $U^{p,\bullet} = V^{p,\bullet} = 0$ if $p \leq p_0$. By Horseshoe Lemma 3.1.11 applied to the exact sequence (1), we get an injective resolution $\ker(d^p) \to W^{p,\bullet}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{im}(d^{p-1}) \arrow[r] \arrow[d] & \ker(d^p) \arrow[r] \arrow[d] & \mathcal{H}^p(C^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V^{p,\bullet} \arrow[r] & W^{p,\bullet} \arrow[r] & U^{p,\bullet} \arrow[r] & 0
\end{tikzcd}$$

commutes. By Horseshoes Lemma 3.1.11 applied to the exact sequence (2), we get an injective resolution $C^p \to A^{p,\bullet}$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(d^p) \arrow[r] \arrow[d] & C^p \arrow[r] \arrow[d] & \mathrm{im}(d^p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W^{p,\bullet} \arrow[r] & A^{p,\bullet} \arrow[r] & V^{p+1,\bullet} \arrow[r] & 0
\end{tikzcd}$$

commutes. Putting these two together, we get morphisms of complexes

$$A^{p,\bullet} \twoheadrightarrow V^{p+1,\bullet} \hookrightarrow W^{p_1,\bullet} \hookrightarrow A^{p_1,\bullet},$$

making $A^{\bullet,\bullet}$ a double complex. $\square$

Suppose now that we have a complex $C^\bullet$ bounded from below of objects in a category $\mathcal{C}$ and $C^\bullet \to A^{\bullet,\bullet}$ is a Cartan–Eilenberg resolution. Let $\mathcal{G}\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor. The goal is to describe the spectral sequence associated to the double complex $\mathcal{G}(A^{\bullet,\bullet})$.

The first spectral sequence associated to this double complex is

$$'E_1^{p,q} = \mathcal{H}^q(\mathcal{G}(A^{p,\bullet})) = R^q\mathcal{G}(C^p)$$

with

$$'E_1^{p,q} \to \ 'E_1^{p+1,q}$$

induced by the map $C^p \to C^{p+1}$. This gives

$$E_2^{p,q} = \mathcal{H}^p(R^q\mathcal{G}(C^\bullet))$$

and

$$'E_1^{p,q} \Rightarrow_p \mathcal{H}^{p+q}(\mathrm{Tot}(\mathcal{G}(A^{\bullet,\bullet}))).$$

We show that the second exact sequence associate to the double complex $\mathcal{G}(A^{\bullet,\bullet})$ is

$$''E_1^{p,q} = \mathcal{H}^q(\mathcal{G}(A^{\bullet,p})).$$

Recall that for every p we have the two exact sequences

$$0 \longrightarrow \ker(d_1^{i,p}) \longrightarrow A^{i,p} \longrightarrow \mathrm{im}(d_1^{i,p}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{im}(d_1^{i-1,p}) \longrightarrow \ker(d_1^{i,p}) \longrightarrow \mathcal{H}^q(A^{\bullet,q}) \longrightarrow 0$$

which splits since $\ker(d_1^{i,p})$, $\mathrm{im}(d_1^{i-1,p})$ are injective. Hence the sequences stay exact after applying $\mathcal{G}$, and hence

$$''E_1^{p,q} = \mathcal{G}(\mathcal{H}^q(A^{\bullet,p})).$$

Since we know $\mathcal{H}^q(C^\bullet) \to \mathcal{H}^q(A^{\bullet,\bullet})$ is an injective resolution, we conclude that

$$''E_2^{p,q} = R^p\mathcal{G}(\mathcal{H}^q(C^\bullet)) \Rightarrow_p \mathcal{H}^{p,q}(\mathrm{Tot}(\mathcal{G}(A^{\bullet,\bullet}))).$$

Suppose that, in addition, all C^p are $\mathcal{G}$-acyclic. Then $'E_2^{p,q} = \mathcal{H}^q(R^q\mathcal{G}(C^\bullet))$ shows that $'E_2^{p,q} = 0$ if $q \neq 0$, and

$$\mathcal{H}^n(\mathrm{Tot}(\mathcal{G}(A^{\bullet,\bullet}))) \cong \mathcal{H}^n(\mathcal{G}(C^\bullet)).$$

Therefore, we have a spectral sequence

$$E_2^{p,q} = R^p\mathcal{G}(\mathcal{H}^q(C^\bullet)) \Rightarrow_p \mathcal{H}^n(\mathcal{G}(C^\bullet)).$$

Suppose now we are in the setting of Grothendieck spectral sequence 3.8.1:

$$\mathcal{C}_1 \xrightarrow{\mathcal{F}} \mathcal{C}_2 \xrightarrow{\mathcal{G}} \mathcal{C}_3$$

with $\mathcal{F}$ and $\mathcal{G}$ left exact, and $\mathcal{F}$ mapping injective objects to $\mathcal{G}$-acyclic objects. For $\mathcal{A} \in \mathrm{Ob}(\mathcal{C}_1)$, let $\mathcal{A} \to \mathcal{I}^\bullet$ be an injective resolution.

Consider the complex $\mathcal{F}(\mathcal{I}^\bullet)$. By assumption, all the terms are $\mathcal{G}$-acyclic, and hence the above discussion shows that the Grothendieck spectral sequence 3.8.1 becomes:

$$E_2^{p,q} = R^p\mathcal{G}(\mathcal{H}^q(\mathcal{F}(\mathcal{I}^\bullet))) = R^p\mathcal{G}(R^q\mathcal{F}(\mathcal{A})) \Rightarrow_p \mathcal{H}^{p+q}(\mathcal{G}(\mathcal{F}(\mathcal{I}^\bullet))) = R^{p+q}(\mathcal{G}\circ\mathcal{F})(\mathcal{A}).$$

3.9. **Čech cohomology.** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = (U_i)_{i \in I}$ be a finite open cover. Moreover, let $\mathcal{F}$ be an $\mathcal{O}_X$-module (or just a sheaf of abelian groups).

For $J \subseteq I$, write $U_J = \bigcap_{i \in J} U_i$, and by convention $U_\emptyset = X$. Choose an order on I. For $p \geq 0$, let

$$C^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{\substack{J \subseteq I \\ |J| = p+1}} \mathcal{F}(U_J).$$

For example,

$$C^0(U, \mathcal{F}) = \bigoplus_i \mathcal{F}(U_i),$$
$$C^1(U, \mathcal{F}) = \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j).$$

Define the map

$$C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^{p+1}(\mathcal{U}, \mathcal{F})$$
$$(s_J)_J \mapsto (s_{J'})_{J'}$$

where for $J' = \{j_0 < \cdots < j_{p+1}\}$ we set

$$s_{J'} = \sum_{q=0}^{p+1} (-1)^q s_{J' \setminus \{j_q\}}|_{U_{J'}}.$$

Exercise. Show that $d \circ d = 0$.

Definition 3.9.1. The *Čech complex* associated to $\mathcal{F}$ and the cover $\mathcal{U}$ is $C^\bullet(\mathcal{U}, \mathcal{F})$. The *Čech cohomology* is the cohomology of this complex

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \mathcal{H}^p(C^\bullet(\mathcal{U}, \mathcal{F})).$$

Note that by the sheaf axiom, $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Theorem 3.9.2. *If X is an algebraic variety, $\mathcal{U}$ is a finite affine open cover, and $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module, then there is a functorial isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \cong \mathcal{H}^p(X, \mathcal{F}).$$

Before we prove this theorem, we prove two lemmas. Let us first sheafify the above construction. For $J \subseteq I$, let $\alpha_J \colon U_J \hookrightarrow X$. Then set $\mathcal{F}_J = \mathcal{F}|_{U_J}$ and

$$\mathcal{C}^p = \mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{|J| = p+1} (\alpha_J)_* \mathcal{F}_J \quad \text{for } p \geq -1$$

with $\mathcal{C}^{-1} = \mathcal{F}$ by convention. Then

$$\Gamma(U, \mathcal{C}^\bullet) = \bigoplus_{|J| = p+1} \mathcal{F}(U \cap U_J)$$

and we have maps $d^p \colon \mathcal{C}^p \to \mathcal{C}^{p+1}$ defined by the same formulas as above. We then get a complex

$$0 \longrightarrow \mathcal{C}^{-1} = \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \cdots$$

and applying $\Gamma(X, -)$ recoves the previous complex $C^\bullet(\mathcal{U}, \mathcal{F})$:

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

Lemma 3.9.3. *The complex*

$$0 \longrightarrow \mathcal{C}^{-1} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \cdots$$

is an exact complex of sheaves.

Proof. We show that for any $x \in X$, the corresponding sequence of stalks at x is exact. By choosing $i_0 \in I$ such that $x \in U_{i_0}$, it is enough to show that for all open subsets $U \subseteq U_{i_0}$, the sequence

$$0 \longrightarrow \mathcal{C}^{-1}(U) \longrightarrow \mathcal{C}^0(U) \longrightarrow \mathcal{C}^1(U) \longrightarrow \cdots$$

is exact. In other words, we need to show that the identity map and the zero map on this complex are homotopic, id ≈ 0. Define

$$\theta^p \colon \mathcal{C}^p(U) = \bigoplus_{|J|=p+1} \mathcal{F}(U \cap U_J) \to \mathcal{C}^{p-1}(U)$$
$$(s_J)_J \mapsto (s_{J'})_{J'}$$

where

$$s_{J'} = \begin{cases} 0 & \text{if } i_0 \notin J' \\ (-1)^s s_{J' \setminus \{i_0\}} & \text{if } i_0 \in J' \text{ and } J' \text{ contains exactly } s \text{ elements } j' \text{ with } j' < i_0. \end{cases}$$

Note that $U \cap U_{J \setminus \{i_0\}} = U \cap U_J$.

Exercise. The maps $(\theta^p)_{p \geq 0}$ give a homotopy between id and 0. □

Lemma 3.9.4. *If U is an open affine subset of X, $j \colon U \hookrightarrow X$ is the inclusion map, and $\mathcal{F}$ is a quasicoherent sheaf on U, then $H^p(X, j_*(\mathcal{F})) = 0$ for $p \geq 1$.*

Proof. Since j is an affine map, this is a consequence of the Leray Spectral Sequence (Example 3.8.2):

$$H^p(X, j_*(\mathcal{F})) \cong H^p(U, \mathcal{F}) = 0$$

for $p \geq 1$ by vanishing of cohomology of quasicoherent sheaves on affine varieties (Theorem 3.4.1). □

Proof of Theorem 3.9.2. By Lemma 3.9.3 we have a resolution of $\mathcal{F}$ given by $\mathcal{F} \to \mathcal{C}^\bullet$. By Lemma 3.9.4, for any $\emptyset \neq J \subseteq I$, writing $\alpha_J \colon U_J \to X$ for the inclusion map, $(\alpha_J)_*\mathcal{F}_J$ is $\Gamma(X, -)$-acyclic. Therefore,

$$\begin{aligned} H^p(X, \mathcal{F}) &= \mathcal{H}^p(\Gamma(X, \mathcal{C}^\bullet)) && \text{by definition of sheaf cohomology} \\ &= \check{H}^p(\mathcal{U}, \mathcal{F}) && \text{by definition of Čech cohomology} \end{aligned}$$

completing the proof. □

Corollary 3.9.5. (1) *If any algebraic variety X, there is a d such that $H^i(X, \mathcal{F}) = 0$ for all $i \geq d$ and any quasicoherent sheaf $\mathcal{F}$ on X.*

(2) *Suppose $X = \operatorname{MaxProj}(S)$ and $n = \dim X$.*
- *If $\mathcal{F}$ is quasicoherent on X, then $H^i(X, \mathcal{F}) = 0$ for $i > n$.*
- *If $\mathcal{F}$ is cohreent on X, $\dim(\operatorname{supp}(\mathcal{F})) = r$, then $H^i(X, \mathcal{F}) = 0$ for $i > r$.*

Proof. For (1), take d such that there is a cover of X by d affine open subsets and use Čech cohomology 3.9.2.

For (2), if $Z \subseteq X$ is a closed subvariety of dimension r, there are affine open subsets $U_1, \ldots, U_{r+1}$ in X such that $Z \subseteq \bigcup_{i=1}^{r} U_i$. In fact, we can take $U_i = D_X^+(h_i)$, $h_i \in S_1$. For $Y = \operatorname{MaxSpec}(S_0)$, we have

$$\begin{tikzcd}
 & \mathbb{P}^N \\
X \arrow[ur, "f"] \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^N \times Y \arrow[u, "q"'] \arrow[d, "p"] \\
 & Y
\end{tikzcd}$$

If $\dim(f(Z)) \leq r$, then $f(Z) \cap \bigcap_{i=1}^{r+1} V(h_i) = \emptyset$ for suitable

$$h_i \in k[x_0, \ldots, x_n]_1 \subseteq (S_0[x_0, \ldots, x_n])_1 \to S_1.$$

This shows that $H^i(X, \mathcal{F}) = 0$ for $i > r$ using Čech cohomology to compute the sheaf cohomology (Theorem 3.9.2). $\square$

Remark 3.9.6 (Grothendieck)**.** If X is an algebraic variety of dimension n, then for any sheaf of abelian groups $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$ for $i > n$.

3.10. **Coherent sheaves on projective varieties.** Let $X = \operatorname{MaxProj}(S)$ where $S = \bigoplus_{i \geq 0} S_i$ is an $\mathbb{N}$-graded reduced k-algebra such that S_0 is finitely-generated over k and S is generated by S_1 as an S_0-algebra and S_1 is a finitely-generated S_0-module. In other words, there is a surjective map

$$S_0[x_0, \ldots, x_n] \twoheadrightarrow S$$

and hence we have a diagram

$$\begin{tikzcd}
 & Y = \operatorname{MaxSpec}(S_0) \\
X \arrow[ur] \arrow[r, hook] & Y \times \mathbb{P}^n = \mathbb{P}^n_Y \arrow[u]
\end{tikzcd}$$

Unsurprisingly, to define coherent sheaves on projective varieties, we need to introduce a grading.

Definition 3.10.1. A *graded S-module* is an S-module M with a decomposition
$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$
such that $S_i \cdot M_j \subseteq M_{i+j}$ for any i, j. Elements of M_i are called *homogeneous of degree i*.

A *morphism of graded modules* $f \colon M \to N$ is a morphism of modules such that $f(M_i) \subseteq N_i$ for any i.

Since we can compose these morphisms, this gives a category of graded S-modules.

If M is a graded S-module, a graded submodule $N \subseteq M$ is a submodule generated by homogeneous elements. Equivalently, the decomposition $M = \bigoplus_i M_i$ induces a decomposition $N = \bigoplus_i (N \cap M_i)$, so N is a graded module such that $N \hookrightarrow M$ is a graded module. Then the quotient
$$M/N = \bigoplus_{i \in \mathbb{Z}} (M_i/N \cap M_i)$$
is a graded module such that $M \twoheadrightarrow M/N$ is a graded morphism.

It is clear that if $f \colon M \to N$ is a morphism of graded modules, then $\ker(f) \subseteq M$ and $\operatorname{im}(f) \subseteq N$ are graded submodules. Using quotients, we construct $\operatorname{coker}(f)$.

Altogether, this shows that the category of graded S-modules is an abelian category.

Recall that on X we have a basis for the topology given by
$$D_X^+(f) \text{ with } f \text{ homogeneous },\ \deg(f) > 0.$$
Each of these is affine and
$$\Gamma(D_X^+(f), \mathcal{O}_X) = S_{(f)}.$$
Recall that, by definition, $S_{(f)} = (S_f)_0$, the 0-graded piece of S_f. We will similarly write $M_{(f)}$ for the 0-graded piece of M_f.

Suppose now M is a graded S-module. Given $D_X^+(f)$, consider $(M_f)_0$. Note that if $D_X^+(g) \subseteq D_X^+(f)$, we get canonical map $(M_f)_0 \to (M_g)_0$. Indeed, $V(g) \subseteq V(f)$, so $g \subseteq \sqrt{(f)}$, so by universal property of localization, we get a map
$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ M_f \arrow[rr] & & M_g \end{tikzcd}$$
which is graded, and hence gives a map $(M_f)_0 \to (M_g)_0$.

Lemma 3.10.2. *Given f and $f_1, \ldots, f_r \in S$ homogeneous of positive degree such that $D_X^+(f) = \bigcup_{i=1}^r D_X^+(f_i)$, then the following sequence*
$$0 \longrightarrow M_{(f)} \longrightarrow \bigoplus_{i=1}^r M_{(f_i)} \longrightarrow \bigoplus_{i<j} M_{(f_i f_j)}$$

Proof. This is similar to the corresponding assertion for affine varieties, so it is left as an exercise. (See the official notes for a solution.) $\square$

We conclude that there is an $\mathcal{O}_X$-module $\widetilde{M}$ such that

$$\Gamma(D_X^+(f), \widetilde{M}) = M_{(f)}.$$

(It remains to check the compatibility of the restriction maps but this is clear.)

Examples 3.10.3.

(1) Trivially, $\widetilde{S} = \mathcal{O}_X$.
(2) Given $m \in \mathbb{Z}$ and a graded module M, let $M(m)$ be M as an S-module, but $M(m)_i = M_{m+i}$. Then $M(m)$ is a graded module. Set

$$\widetilde{S(m)} = \mathcal{O}_X(m).$$

We claim that this is a line bundle. Since S is generated as an S_0-algebra by S_1, we can cover X by open subsets of the form $D_X^+(f)$ for $D_X(f) = 1$. Now, note that

$$\Gamma(D^+(X), \mathcal{O}_X(m)) = (S_f)_m \xrightarrow{\cong} (S_f)_0,$$
$$u \mapsto f^{-m}u.$$

(3) If $X = \mathbb{P}^n$, we recover the old $\mathcal{O}_{\mathbb{P}^n}(m)$. Indeed, the above isomorphism for $U_i = D_{\mathbb{P}^n}^+(x_i)$ becomes

$$\varphi_i \colon \mathcal{O}_{\mathbb{P}^n}(m)|_{U_i} \to \mathcal{O}_{\mathbb{P}^n}|_{U_i}$$

given by multiplication by $\frac{1}{x_i^m}$. Then the transition functions are given by $\varphi_i \circ \varphi_j^{-1} = \left(\frac{x_j}{x_i}\right)^m$

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n}(m)|_{U_i \cap U_j} \arrow[r, "\varphi_i|_{U_i \cap U_j}"] \arrow[dr, "\varphi_j|_{U_i \cap U_j}"'] & \mathcal{O}_{\mathbb{P}^n}|_{U_i \cap U_j} \\
& \mathcal{O}_{\mathbb{P}^n}|_{U_i \cap U_j} \arrow[u]
\end{tikzcd}$$

This agrees with the previously computed transition functions.

Notation. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, let

$$\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m).$$

We claim that $\widetilde{M}(m) = \widetilde{M(m)}$. Indeed, for $\deg(f) = 1$, we clearly have an isomorphism

$$\Gamma(D_X^+(f), \widetilde{M}(m)) = (M_f)_0 \otimes_{(S_f)_0} (S_f)_m \cong (M_f)_m = \Gamma(D_X^+(f), \widetilde{M(m)}),$$
$$\frac{u}{f^r} \otimes \frac{a}{f^s} \mapsto \frac{au}{f^{r+s}}.$$

For example, if $M = S(n)$, we have that

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).$$

Finally, the assignment $M \mapsto \widetilde{M}$ is functorial. If $\varphi\colon M \to N$ is a morphism of graded modules, we get $(M_f)_0 \to (N_f)_0$ for every homogeneous f of positive degree. These induce morphisms of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$.

Properties.

(1) This is an exact functor (since localization is exact).
(2) It commutes with arbitrary direct limits.

Proposition 3.10.4. *The $\mathcal{O}_X$-module $\widetilde{M}$ is quasicoherent. If M is finitely generated, $\widetilde{M}$ is coherent.*

Proof. Choose generators $(u_i)_{i\in I}$ homogeneous of $\deg u_i = a_i$ (finite if M is finitely-generated). Consider the map

$$\bigoplus_{i\in I} S(-a_i) \overset{\varphi}{\twoheadrightarrow} M,$$
$$e_i \mapsto u_i.$$

Repeat this for $\ker(\varphi)$ to get an exact sequence

$$\bigoplus_{j\in J} S(-b_j) \longrightarrow \bigoplus_{i\in I} S(-a_i) \longrightarrow M \longrightarrow 0.$$

Applying $\widetilde{\ }$, we get an exact sequence

$$\bigoplus_{j\in J} \mathcal{O}_X(-b_j) \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X(-a_i) \longrightarrow \widetilde{M} \longrightarrow 0.$$

Then $\widetilde{M}$ is quasicoherent (or coherent if M is finitely-generated) as a cokernel of a morphisms of quasicoherent (coherent) sheaves. ◻

Question. When is $\widetilde{M} = 0$?

It is enough to see when $(M_f)_0$ for $f \in S_1$ (since $\widetilde{M}$ is quasicoherent). Since $(M_f)_0 \cong (M_f)_m$ for any $m \in \mathbb{Z}$ by mapping $u \mapsto f^m u$, we see that this is equivalent to $M_f = 0$ for all $f \in S_1$.

Since S is generated over S_0 by S_1, we have $S_+ = \bigoplus_{i>0} S_i = (S_1)$.

Answer. This shows that $\widetilde{M} = 0$ if and only if for any $u \in M$, $(S_+)^N \cdot u = 0$ for some N.

If M is finitely-generated, this is equivalent to $(S_+^N) \cdot M = 0$ for some N.

Exercise. This is equivalent to $M_i = 0$ for $i \gg 0$.

We now define a functor in the opposite direction. For a quasicoherent sheaf $\mathcal{F}$ on X, let

$$\Gamma_*(\mathcal{F}) = \bigoplus_{m\in\mathbb{Z}} \Gamma(X, \mathcal{F}(m))$$

as a graded abelian group.

Remarks 3.10.5.

(1) For any M, we have a morphism
$$M \xrightarrow{\Phi_M} \Gamma(X, \widetilde{M}) = \bigoplus_{i\in\mathbb{Z}} \Gamma(X, \widetilde{M(i)}).$$
Indeed, take $u \in M_i$. For f homogeneous of positive degree, consider
$$\frac{u}{1} \in (M_f)_i = \Gamma(D_X^+(f), \widetilde{M(i)}).$$
These glue to give $\Phi_M(u) \in \Gamma(X, \widetilde{M(i)})$.
For $M = S$, this gives a map
$$\Phi_S \colon S \to \bigoplus_{i\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(i)).$$

(2) For any quasicoherent sheaf $\mathcal{F}$ on X, there is a map
$$\Gamma(X, \mathcal{O}_X(i)) \otimes \Gamma(X, \mathcal{F}(j)) \to \Gamma(X, \mathcal{F}(i+j))$$
given by tensor product of sections. For $\mathcal{F} = \mathcal{O}_X$, this makes $\Gamma_*(\mathcal{O}_X)$ a graded ring such that Φ_S is a graded homomorphism.
For any $\mathcal{F}$, this makes $\Gamma_*(\mathcal{F})$ a graded module over $\Gamma_*(\mathcal{O}_X)$, and hence a graded module over S via Φ_S.

(3) We get a functor
$$\mathrm{Qcoh}(X) \to \text{graded } S\text{-modules}.$$
Indeed, for a map $\mathcal{F} \to \mathcal{G}$, we get maps $\mathcal{F}(m) \to \mathcal{G}(m)$ for all m,which give a map $\Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{G})$.

Proposition 3.10.6. *For every $\mathcal{F}$, we have a canonical isomorphism*
$$\psi_{\mathcal{F}} \colon \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}.$$

Note that if $f \in S_+$, $D_X^+(f) = X \setminus V(f)$. If $f \in S_m$, we get a section in $\mathcal{O}_X(m)$, locally given by $\frac{f}{1}$. Then $V(f)$ is the zero locus of this section.

The following lemma is a generalization of this.

Lemma 3.10.7. *Suppose X is an algebraic variety, $\mathcal{F} \in \mathrm{Qcoh}(X)$, $s \in \Gamma(X, \mathcal{L})$, and $U = X \setminus V(s)$.*

(1) *If $t \in \Gamma(X, \mathcal{F})$ is a section such that such that $t|_U = 0$, then there is an N such that $s^N \cdot t \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^N)$ is 0.*
(2) *For any $t \in \Gamma(U, \mathcal{F})$, there is a q such that $s^q|_U \cdot t$ is the restriction of a section in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^q)$.*

Proof. Exercise. (Cover X by affine open subsets U_i such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. In this case, we know both assertion for $U_i \subseteq U \cap U_i$, so we just need to glue these.) □

Proof of Proposition 3.10.6. We need compatible maps for each $f \in S$, homogeneous of degree $d > 0$,
$$(\Gamma_*(\mathcal{F})_f)_0 \to \Gamma(D_X^+(f), \mathcal{F}).$$

An element of $(\Gamma_*(\mathcal{F})_f)_0$ can be written as

$$\frac{s}{f^m}$$

for $s \in \Gamma(X, \mathcal{F}(md))$, since $\frac{1}{f^m} \in \Gamma(D_X^+(f), \mathcal{O}_X(-md))$. We define

$$(\Gamma_*(\mathcal{F})_f)_0 \to \Gamma(D_X^+(f), \mathcal{F}),$$
$$\frac{s}{f^m} \mapsto \frac{1}{f^m} s|_{D_X^+(f)}.$$

These glue to give $\psi_{\mathcal{F}}$. Let us show that $\psi_{\mathcal{F}}$ is an isomorphism. It is enough to show it is an isomorphism on each $D_X^+(f)$.

For injectivity, suppose $\frac{s}{f^m} \mapsto 0$, so $s|_{D_X^+(f)} = 0$, since $\frac{1}{f^m} \neq 0$ on $D_X^+(f)$. Now, $D_X^+(f) = X \setminus V(f)$ and Lemma 3.10.7 shows that there is an N such that $f^N \cdot s = 0$ in $\Gamma(X, \mathcal{F}(md + Nd))$. Then

$$\frac{s}{f^m} = \frac{f^N s}{f^{m+N}} = 0$$

in $\Gamma_*(\mathcal{F})$.

For surjectivity, given $t \in \Gamma(D_X^+(f), \mathcal{F})$, Lemma 3.10.7 shows that there is a q such that $f|_{D_X^+(f)}^q \cdot t$ extends to $t' \in \Gamma(X, \mathcal{F}(qd))$. Then $t = \psi(t'/f^q)$, showing surjectivity. □

Corollary 3.10.8. *If $\mathcal{F} \in \mathrm{Coh}(X)$, there exists a finitely-generated S-module M such that $\widetilde{M} = \mathcal{F}$.*

Proof. By Proposition 3.10.6, there is an S-module N such that $\mathcal{F} \cong \widetilde{N}$. Choose a finite cover $X = \bigcup_{i=1}^{r} D_X^+(f_i)$ such that

$$(N_{f_i})_0 = \Gamma(D_X^+(f_i), \mathcal{F}) \text{ is finitely-generated over } (S_{f_i})_0.$$

Choose generators for each of these and let $M \subseteq N$ be the graded S-module generated by the numerators of these generators. Then

$$(N_{f_i})_0 \subseteq \Gamma(D_X^+(f_i), \widetilde{M}) \subseteq \Gamma(D_X^+(f_i), \widetilde{N})$$

for all i, and hence $\widetilde{M} = \widetilde{N}$. □

Remark 3.10.9. Given any M, we have $\Phi_M \colon M \to \Gamma_*(\widetilde{M})$. This gives a morphism

$$\widetilde{\Phi_M} \colon \widetilde{M} \to \widetilde{\Gamma_*(\widetilde{M})}$$

and by Proposition 3.10.6, we have an isomorphism

$$\Psi_{\widetilde{M}} \colon \widetilde{\Gamma_*(\widetilde{M})} \xrightarrow{\cong} \widetilde{M}.$$

It is easy to check that $\psi_{\widetilde{M}} \circ \widetilde{\Phi_M} = 1$, and hence $\widetilde{\Phi_M}$ is an isomorphism.

Exercise. Suppose $S \twoheadrightarrow T$ is a surjective graded homomorphism, inducing

$$i \colon \mathrm{MaxProj}(T) = X \hookrightarrow \mathrm{MaxProj}(S) = Y.$$

If N is a graded S-module, then $M = N \otimes_S T = N/IN$ where $I = \ker(S \to T)$ is a graded T-module. Then

$$i^*\widetilde{N} \cong \widetilde{M}.$$

In particular, $i^*\mathcal{O}_Y(m) \cong \mathcal{O}_X(m)$.

Remark 3.10.10. The construction of $\mathcal{O}(1)$ globalizes. Suppose T is any variety and $\mathcal{S} = \bigoplus_{m\geq 0} \mathcal{S}_m$ is a graded, reduced, quasicoherent $\mathcal{O}_T$-algebra such that $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent and $\mathcal{S}$ is generated by $\mathcal{S}_1$ over $\mathcal{S}_0$. Then we get a map π such that for any affine open U the diagram

$$\begin{tikzcd} X = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}) \arrow[d, "\pi"] & \mathrm{MaxProj}(\mathcal{S}(U)) \arrow[l, hook'] \arrow[d] \\ T & U \arrow[l, hook'] \end{tikzcd}$$

commutes. For each U, we have $\mathcal{O}_{\pi^{-1}(U)}(1)$ and these glue to give $\mathcal{O}_X(1)$.

We get a canonical morphism

$$\mathcal{S}_i \to \pi_*\mathcal{O}_X(i).$$

Example 3.10.11. Suppose T is irreducible and $\mathcal{I}$ is a coherent ideal on T. The blow-up along $\mathcal{I}$ was defined as

$$\begin{tikzcd} \widetilde{T} = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}\left(\bigoplus_{m\geq} \mathcal{I}^m\right) \arrow[d, "\pi"] \\ T \end{tikzcd}$$

Then $\mathcal{I} \cdot \mathcal{O}_{\widetilde{T}} = \mathcal{O}_{\widetilde{T}}(-E)$ for an effective Cartier divisor E.

Exercise. Check that $\mathcal{O}_{\widetilde{T}}(1) = \mathcal{O}_{\widetilde{T}}(-E)$.

To do this, write this down explicitly locally.

Let X be any variety and $\mathcal{F}$ be a quasicoherent sheaf. There is a canonical morphism $\Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X \to \mathcal{F}$ given on an open subset U by

$$\Gamma(X, \mathcal{F}) \otimes_k \mathcal{O}_X(U) \to \mathcal{F}(U)$$
$$\sum_{i=1}^r s_i \otimes f_i \mapsto f_i \cdot s_i|_U$$

where $f_i \in \mathcal{O}_X(U)$.

Definition 3.10.12. The quasicoherent sheaf $\mathcal{F}$ is *globally generated* if this map is surjective, i.e. for $x \in X$, $\mathcal{F}_x$ is generated over $\mathcal{O}_{X,x}$ by

$$\{s_x \mid s \in \Gamma(X, \mathcal{F})\}.$$

Definition 3.10.13. A line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ is *ample* if for any $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated for $m \gg 0$.

For example, if X is affine, every quasicoherent sheaf is globally generated. In particular, every line bundle is ample.

Proposition 3.10.14. *If $X = \operatorname{MaxProj}(S)$, $\mathcal{O}_X(1)$ is ample.*

Proof. Let $\mathcal{F} \in \operatorname{Coh}(X)$. Then there is a finitely-generated S-module M such that $\mathcal{F} \cong \widetilde{M}$. Let $u_1, \dots, u_n \in M$ be homogeneous generators with $d_i = \deg(u_i)$.

We show that if $d \geq \max d_i$, then $\mathcal{F} \otimes \mathcal{O}(d)$ is globally generated.

Let $S_+ = \bigoplus_{i>0} S_i$. Let $T \subseteq M$ be the submodule generated by $S_+^{d-d_i} \cdot u_i$ for $1 \leq i \leq n$. Then T is finitely-generated over S and it is generated by elements of degree d. Therefore, there is a surjective map

$$S(-d)^{\oplus q} \twoheadrightarrow T,$$

which gives a map $\mathcal{O}_X(-d)^{\oplus q} \twoheadrightarrow \widetilde{T}$. Twisting by $\mathcal{O}_X(d)$, we get

$$\mathcal{O}_X^{\oplus q} \twoheadrightarrow \widetilde{T}(d).$$

Hence $\widetilde{T}(d)$ is globally generated. For any i, $S_+^{d-d_i} \cdot u_i \subseteq T$, so there is an N such that $S_+^N \cdot (M/T) = 0$. Therefore, $\widetilde{T} = \widetilde{M}$. □

3.11. **Cohomology of coherent sheaves on projective varieties.** Let $X = \operatorname{MaxProj}(S)$.

Theorem 3.11.1 (Serre)**.** *If $\mathcal{F} \in \operatorname{Coh}(X)$,*

1. *$H^i(X, \mathcal{F})$ is a finitely-generated S_0-module for all i,*
2. *there exists $m_0(\mathcal{F})$ such that $H^i(X, \mathcal{F}(m)) = 0$ for $m \geq m_0(\mathcal{F})$, $i \geq 1$.*

Assume for now that this results holds if $S = S_0[x_0, \dots, x_n]$ (i.e. X is the product of an affine variety with $\mathbb{P}^n$) and $\mathcal{F} = \mathcal{O}(j)$ for some j. We will deal with this case later.

Proof. Choose a graded surjection

$$S_0[x_0, \dots, x_n] \twoheadrightarrow S$$

inducing

$$j \colon X \hookrightarrow Y = Z \times \mathbb{P}^n \text{ for } Z = \operatorname{MaxSpec}(S_0).$$

If $\mathcal{F} \in \operatorname{Coh}(X)$, $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m)) \cong H^i(Y, j_*(\mathcal{F} \otimes j^*\mathcal{O}_Y(m)))$, since we have noted before that $\mathcal{O}_X(m) = j^*\mathcal{O}_Y(m)$. By the projection formula,

$$j_ast(\mathcal{F} \otimes j^*\mathcal{O}_Y(m)) = j_*(\mathcal{F}) \otimes \mathcal{O}_Y(m).$$

Hence, we may assume that $S = S_0[x_0, \dots, x_n]$.

For $\mathcal{F} \in \operatorname{Coh}(X)$, since $\mathcal{O}_X(1)$ is ample by Proposition 3.10.14, there is a surjective map

$$\mathcal{O}_X^q \twoheadrightarrow \mathcal{F} \otimes \mathcal{O}(m)$$

(**exercise**: check why this is true). Suppose $\mathcal{F} \cong \widetilde{M}$ for a finitely-generated M. There is a surjection

$$\bigoplus_{i=1}^r S(-q_i) \twoheadrightarrow M.$$

We then have a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X(-q_i) \longrightarrow \mathcal{F} = \widetilde{M} \longrightarrow 0.$$

After tensoring with $\mathcal{O}_X(m)$ (which preserves exactness since $\mathcal{O}_X(m)$ is locally free) and taking the long exact sequence in cohomology, we get the exact sequence

$$(*) \qquad \bigoplus_{j=1}^{r} H^i(X, \mathcal{O}_X(m-q_j)) \longrightarrow H^i(X, \mathcal{F}(m)) \longrightarrow H^{i+1}(X, \mathcal{G}(m)).$$

We argue by decreasing induction. We know both assertions for H^i if $i > \dim$, since all cohomology groups vanish. For the inductive step, suppose we know $H^{i+1}(X, \mathcal{M})$ is finitely generated over S_0 and $H^{i+1}(X, \mathcal{M}(m)) = 0$ for $m \gg 0$ if $\mathcal{M}$ is coherent. The exact sequence $(*)$ for $m = 0$ together with the inductive hypothesis and what we assume about $\mathcal{O}(j)$, we conclude that $H^i(X, \mathcal{F})$ is finitely-generated over S_0.

Finally, for $m \gg 0$, the left term is 0 for $i \geq 1$ by what we assume about $\mathcal{O}(j)$, and the right term is 0 by the inductive hypothesis, and hence $H^i(X, \mathcal{F}(m)) = 0$ for all $m \gg 0$. $\square$

We still have to deal with the case when $S = S_0[x_0, \ldots, x_n]$ and $\mathcal{F} = \mathcal{O}(j)$ for some j. We prove a stronger result that allows to compute the cohomology explicitly in this case.

Theorem 3.11.2. *Let $X = \mathrm{MaxProj}(S)$ where $S = A[x_0, \ldots, x_n]$. Then*

1. *the canonical map $S \to \bigoplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{O}(m))$ is an isomorphism,*
2. *$H^i(X, \mathcal{O}(m)) = 0$ for all $1 \leq i \leq n-1$ and all m,*
3. *$H^n(X, \mathcal{O}(-n-1)) \cong A$ and we have for every m a canonical perfect pairing of finitely-generated free A-modules*

$$\Gamma(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \cong A$$

for all $m \in \mathbb{Z}$.

Note that this simplies that every $\mathcal{O}(j)$ on X satisfies the conclusion of Serre's Theorem 3.11.1.

Proof. We want to compute the cohomology groups using Cech cohomology (Theorem 3.9.2) with respect to the cover by $D_X^+(x_i)$ for $0 \leq i \leq n$. For $J \subseteq \{0, \ldots, n\}$, write $x_J = \prod_{i \in J} x_i$ and $U_J = \bigcap_{i \in J} U_i = D_X^+(x_J)$. We have that

$$C^p = \bigoplus_{\substack{J \subseteq \{0, \ldots, n\} \\ |J| = p+1}} \Gamma(U_J, \mathcal{O}(j))$$

and the Čech complex is

$$C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0.$$

Note that $\Gamma(U_J, \mathcal{O}(j)) = (S_{x_J})_j$ and, up to signs, the maps are given by inclusion. Let $C^{-1} = S_j \to \Gamma(X, \mathcal{O}(j)) \to C^0$. Altogether, we get a complex

$$C^\bullet : \qquad 0 \longrightarrow C^{-1} \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0.$$

The first two assertions (1), (2) are then equivalent to $\mathcal{H}^i(C^\bullet) = 0$ for all $i \leq n-1$. However, it is easier to start by dealing with assertion (3). Note that

$$H^n(X, \mathcal{O}(j)) = \operatorname{coker}\left(\bigoplus_{i=0}^{n} (S_{x_0 \ldots \hat{x}_i \ldots x_n})_j \to (S_{x_0 \ldots x_n})_j\right).$$

This is a free finitely generated A-module with basis $x^u = x_0^{u_0} \ldots x_n^{u_n}$, $u_i \leq -1$ and $u_0 + \cdots + u_n = j$.

It is clear that for $j \geq -n$, $H^n(X, \mathcal{O}(j)) = 0$ and for $j = -n-1$, $H^n(X, \mathcal{O}(-n-1)) \cong A$ with generator $(x_0 \ldots x_n)^{-1}$.

Note that if $s \in \Gamma(X, \mathcal{O}(j))$, this induces a map

$$\mathcal{O}(-n-1-j) \to \mathcal{O}(-n-1)$$

given by tensoring with s. Hence we get a map

$$H^n(X, \mathcal{O}_X(-n-1-j)) \to H^n(X, \mathcal{O}(-n-1)) \cong A.$$

Hence we get a bilinear map

$$\begin{tikzcd}
H^0(X, \mathcal{O}(j)) \times H^n(X, \mathcal{O}(-n-j-1)) \arrow[r] & H^n(X, \mathcal{O}(-n-1)) \cong A \\
S_j \times H^n(X, \mathcal{O}(-n-j-1)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where the diagonal map sends (x^u, x^v) to x^{u+v} if $u_i + v_i = -1$ for all i and $(x^u, x^v) \mapsto 0$ otherwise. It is easy to show that this is a perfect pairing.

Hence it is enough to show that $\mathcal{H}^i(C^\bullet) = 0$ for $i \leq n-1$. Recall that

$$C^p = \bigoplus_{\substack{J \subseteq \{0,\ldots,n\} \\ |J| = p+1}} (S_{x_J})_j = \bigoplus_{\substack{u \in \mathbb{Z}^{n+1} \\ u_0 + \cdots + u_n = j}} \bigoplus_{\substack{J_u \subseteq J \\ |J| = p+1}} A x^u e_J.$$

where $J_u = \{i \mid u_i < 0\}$ and $e_J \in C^p$ is the unit in S_{x_J}. We get a decomposition

$$C^\bullet = \bigoplus_{\substack{u \in \mathbb{Z}^{n+1} \\ u_0 + \cdots + u_n = j}} C_u^\bullet.$$

What if $J_u = \{0, \ldots, n\}$? Then $C_u^i = 0$ for $i \neq n$. We now show that if $J_u \neq \{0, \ldots, n\}$ then $C_u^\bullet$ is exact.

Note that up to a shift, this is the complex that computes the reduced simplicial cohomology with coefficients A for the full simplicial complex on $\{0, \ldots, n\} \setminus J_u$.)

We will write down a homotopy between id and 0 on $C_u^\bullet$. Fix $i_0 \in \{0,\dots,n\} \setminus J_u$. For $0 \le p \le n$, define

$$\begin{aligned}
\theta^p \colon C_u^p &\to C_u^{p-1} \\
x^u e_J &\mapsto \begin{cases} (-1)^{\ell-1} x^u e_{J\setminus\{i_0\}} & \text{if } J : j_1 < \cdots < j_\ell = i_0 < \cdots, \\ 0 & \text{if } i_0 \notin J. \end{cases} \\
J_u &\subseteq J \\
|J| &= p+1
\end{aligned}$$

Exercise. Check that the θ^p give a homotopy between $\mathrm{id}_{C_u^\bullet}$ and 0. $\square$

Recall that this also completes the proof of Serre's Theorem 3.11.1.

Corollary 3.11.3. *Suppose $X = \mathrm{MaxProj}(S)$. If M is a finitely-generated graded S-module,*

$$\Phi_M \colon M \to \Gamma_*(\widetilde{M}) = \bigoplus_{j\in\mathbb{Z}} \Gamma(X, \widetilde{M}(j)),$$

then $(\Phi_M)_j$ is an isomorphism for $j \gg 0$.

Proof. We may assume $S = A[x_0,\dots,x_n]$ by writing S as a quotient of a polynomial algebra. If $M = S(m)$, then Φ_M is an isomorphism by Theorem 3.11.2.

By choosing homogeneous generators for M, we get a short exact sequence

$$0 \longrightarrow Q \longrightarrow P = \bigoplus_{i=1}^r S(-m_i) \longrightarrow M \longrightarrow 0.$$

Then we get

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d, "\Phi_Q"] & P \arrow[r] \arrow[d, "\Phi_P"] & M \arrow[r] \arrow[d, "\Phi_M"] & 0 \\
0 \arrow[r] & \Gamma_*(\widetilde{Q}) \arrow[r] & \Gamma_*(\widetilde{P}) \arrow[r, "\alpha"] & \Gamma_*(\widetilde{M}) \arrow[r] & 0
\end{tikzcd}$$

By the above, we know that Φ_P is an isomorphism, and by Theorem 3.11.1, we see that $H^1(X, \widetilde{Q}(j)) = 0$ for $j \gg 0$, so α_j is surjective for $j \gg 0$, and hence $(\Phi_M)_j$ is surjective for $j \gg 0$. Finally, Snake Lemma applied to the above diagram gives the exact sequence

$$0 = \Phi_p \longrightarrow \ker \Phi_M \longrightarrow \underbrace{\operatorname{coker} \Phi_Q}_{\substack{=0 \text{ in degree } j\gg 0 \\ \text{by the above}}},$$

so $(\ker \Phi_M)_j = 0$ for $j \gg 0$. $\square$

Exercise. If $\mathcal{F}$ is a coherent sheaf on $X = \operatorname{MaxProj}(S)$, then for any m_0, $\Gamma_*(\mathcal{F})_{m_0} = \bigoplus_{j \geq m_0} \Gamma(X, \mathcal{F} \otimes \mathcal{O}(j))$ is a finitely-generated S-module.

Indeed, by Corollary 3.11.3, it is eventually isomorphic to $\mathcal{F}$, and the finitely many small degrees left are finitely-generated over degree 0.

Theorem 3.11.4. *If $f\colon X \to Y$ is a proper morphism and $\mathcal{F} \in \operatorname{Coh}(X)$, then*

$$R^p f_*(\mathcal{F}) \in \operatorname{Coh}(Y) \text{ for all } p.$$

Corollary 3.11.5. *If X is a complete variety and $\mathcal{F} \in \operatorname{Coh}(X)$, then $\dim_k H^i(X, \mathcal{F}) < \infty$.*

Proof. Apply Theorem 3.11.4 when Y is a point. $\square$

The idea of the proof is known as Grothendieck's *dÃ̃l'vissage*. We know the result for projective and we use Chow's Lemma to generalize it from that case.

Proof of Theorem 3.11.4. We know that $R^p f_*(\mathcal{F})$ is quasicoherent for all p. Hence it is enough to show that for any affine open subset $U \subseteq Y$, $H^p(f^{-1}(U), \mathcal{F})$ is a finitely-generated $\mathcal{O}_Y(U)$-module.

If f factors as

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & Y \times \mathbb{P}^n \arrow[d, "\text{proj}"] \\ & Y \end{tikzcd}$$

where i is a closed immersion, and $U \subseteq Y$ is affine, then

$$H^p(f^{-1}(U), \mathcal{F}) = H^p(U \times \mathbb{P}^n, i_*(\mathcal{F}))$$

is finitely-generated over $\mathcal{O}(U)$ by Serre's Theorem 3.11.1.

We will reduce to this case using Chow's Lemma.

Preparations.

(1) By Noetherian induction, we may assume that the theorem is true for every closed subvariety $X' \subsetneq X$ and the map $f|_{X'}\colon X' \to Y$, i.e. $FR^p(f|_{X'})_*(\mathcal{G})$ is coherent for any $\mathcal{G}$ coherent on X'.
(2) If there is an X' as above such that the ideal sheaf $\mathcal{I}_{X'}$ of X' in X satisfies $\mathcal{I}_{X'} \cdot \mathcal{F} = 0$, then writing $j\colon X' \hookrightarrow X$, $\mathcal{F} = j_*(\mathcal{F}')$ for some $\mathcal{F}' \in \operatorname{Coh}(X')$. Then
$$R^p f_*(\mathcal{F}) \cong R^p(f|_{X'})_*(\mathcal{F}')$$
is coherent.
(3) More generally, if $\operatorname{supp} \mathcal{F} \neq X$, we are done: take $X' = \operatorname{supp}(\mathcal{F})$ and choose r such that $\mathcal{I}_{X'}^r \mathcal{F} = 0$. Consider the filtration $0 \subseteq \mathcal{I}_{X'}^{r-1} \mathcal{F} \subseteq \cdots \subseteq \mathcal{I}_{X'} \mathcal{F} \subseteq \mathcal{F}$. We have a short exact sequence
$$0 \longrightarrow \mathcal{I}_{X'}^{j+1} \cdot \mathcal{F} \longrightarrow \mathcal{I}_{X'}^{j} \cdot \mathcal{F} \longrightarrow \underbrace{\mathcal{I}_{X'}^{j} \mathcal{F} / \mathcal{I}_{X'}^{j+1}}_{\text{annihilated by } \mathcal{I}_{X'}} \cdot \mathcal{F} \longrightarrow 0.$$

Arguing by descending induction on j, the long exact sequence for higher direct images shows that $R^p f_*(\mathcal{I}^j_{X'}\mathcal{F})$ is coherent for all p. For $j = 0$, this shows that $R^p f_*(\mathcal{F})$ is coherent.

(4) If $\mathcal{F}_1 \xrightarrow{\phi} \mathcal{F}_2$ is such that $\operatorname{supp}(\ker \phi) \neq X$, $\operatorname{supp}(\operatorname{coker} \phi) \neq X$, then $R^p f_*(\mathcal{F}_1)$ is coherent if and only if $R_p f_*(\mathcal{F}_2)$ is coherent.

Indeed, consider the show exact sequences

$$0 \longrightarrow \ker \phi \longrightarrow \mathcal{F}_1 \longrightarrow \operatorname{im} \phi \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{im} \phi \longrightarrow \mathcal{F}_2 \longrightarrow \operatorname{coker} \phi \longrightarrow 0.$$

By (3), using the long exact sequences for higher direct images, we see that $R^p f_*(\mathcal{F}_1)$ is coherent if and only if $R^p f_*(\operatorname{im}(\phi))$ is coherent if and only if $R^p f_*(\mathcal{F}_2)$ is coherent.

We can finally proceed with the proof. In the general case, by Chow's lemma, we have

$$W \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y$$

such that

- there are open dense subsets $U \subseteq X$, $V \subseteq W$ such that $g|_V \colon V \to U$ is an isomorphism,
- $f \circ g$ factors as

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr, "f\circ g"'] & Y \times \mathbb{P}^n \arrow[d] \\ & Y \end{tikzcd}$$

(in particular, the theorem holds for $f \circ g$).

Exercise. Morphisms that admit such factorizations include closed immersions, are closed under base change, and are closed under composition, and hence g also has such a factorization, and hence satisfies the theorem.

Note that we may assume in Chow's Lemma that $V = g^{-1}(U)$: simply replace U by $U \setminus g(W \setminus V)$.

Consider $\mathcal{F} \to g_* g^* \mathcal{F}$, which is an isomorphism over U. By (4), it is enough to show that $R^p f_*(g_*\mathcal{G})$ is coherent for any p, where $\mathcal{G} = g^*(\mathcal{F})$. (Since $\mathcal{F}$ is coherent, $\mathcal{G} = g^*\mathcal{F}$ is coherent, so $g_*\mathcal{G}$ is coherent on X, because g satisfies the theorem.)

Finally, we use the Leray spectral sequence:

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathcal{G})) \Rightarrow_p R^{p+q}(f \circ g)_*(\mathcal{G}).$$

We want to show that $E_2^{p,0}$ is coherent on Y.

We know that for $q \neq 0$, $E_2^{p,q}$ is coherent, since $R^q g_*(\mathcal{G})$ is coherent and supported on $Y \setminus U$ for $q \neq 0$. This shows that $E_r^{p,q}$ is coherent for all r and $q \neq 1$ as a subquotient of $E_2^{p,q}$.

Moreover, $E_\infty^{p,q}$ is a subquotient of $R^{p+q}(g \circ f)_*(\mathcal{G})$ which is coherent, and hence $E_\infty^{p,q}$ is coherent.

For $r \gg 0$, $E_\infty^{p,0} = E_r^{p,0}$ is coherent. To complete the proof, it is enough to show that if $E_{r+1}^{p,0}$ is coherent then $E_r^{p,0}$ is coherent for $r \geq 2$. For $r \geq 2$, we have maps

$$E_r^{p-r,r-1} \xrightarrow{d_r} E_r^{p,0} \xrightarrow{d_r} E_r^{p+r,-r+1} = 0$$

and $E_r^{p-r,r-1}$ is coherent since $r-1 \geq 1$. Then the exact sequence

$$\underbrace{E_r^{p-r,r-1}}_{\text{coherent}} \longrightarrow E_r^{p,0} \longrightarrow \underbrace{E_{r+1}^{p,0}}_{\text{coherent}} \longrightarrow 0$$

so $E_r^{p,0}$ is coherent. □

We have hence shown (Corollary 3.11.5) that if X is complete variety and $\mathcal{F} \in \mathrm{Coh}(X)$, then

$$h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F}) < \infty.$$

Definition 3.11.6. The *Euler-Poincaré characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(X,\ FF).$$

Note that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence, then the long exact sequence in cohomology shows that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Hence the Euler-Poincaré characteristic is additive in short exact sequences.

We can use this to define new invariants.

Examples 3.11.7.

(1) Let X be a complete variety of dimension n. The *arithmetic genus* is

$$p_a(X) = (-1)^n(\chi(\mathcal{O}_X) - 1).$$

For example, if X is a connected projective curve, then $p_a(X) = h^1(\mathcal{O}_X)$.

(2) Let X be a smooth connected complete variety. The *Hodge numbers* are:

$$h^{p,q}(X) = h^q(X, \Omega_X^p).$$

The *geometric genus* is

$$p_g(X) = h^{n,0} = h^0(X, \omega_X).$$

The *plurigenera* are $p_m(x) = h^0(X, \omega_X^{\otimes m})$.

Recall that we showed in Problem Session 9 that p_g is a birational invariant. In fact, more is true.

Theorem 3.11.8. *The geometric genus p_g, and more generally p_m and $h^{i,0}$ are birational invariants for smooth, connected, complete varieties.*

The proof of this can be found in the official notes.

Definition 3.11.9. An irreducible algebraic variety is *rational* if it is birational to some $\mathbb{P}^n$.

It is usually easy to see that a variety is rational, but hard to prove that it is not rational. One way to prove a variety is not rational is to define birational invariants (as above), compute them for the projective space and the variety and see that they are not equal.

An example of this was given in Problem Session 9.

Example 3.11.10. We have that $p_g(\mathbb{P}^{n-1}) = 0$. If $Y \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $d \geq n+1$, then $p_g(Y) \neq 0$, so Y is not rational.

4. Morphisms to $\mathbb{P}^n$

Problem 1. Given a variety X, describe morphisms $X \to \mathbb{P}^n$.

Let $S = k[x_0, \dots, x_n]$, $V = S_1 \cong \Gamma(\mathbb{P}^n, \underbrace{\mathcal{O}_{\mathbb{P}^n}(1)}_{\widetilde{S(1)}})$. Then

$$\phi\colon V \otimes_k \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n(1)}$$

is surjective, since $\displaystyle\bigcap_{i=0}^{n} V(x_i) = \emptyset$.

Suppose we have a map $f\colon Y \to \mathbb{P}^n$. Then $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is a line bundle on Y, and

$$V \otimes_k \mathcal{O}_Y \xrightarrow{f^*(\phi)} \mathcal{L}$$

is surjective.

Consider pairs $(\mathcal{L}, \alpha)$ on Y where $\mathcal{L}$ is a line bundle and

$$\alpha\colon V \otimes_k \mathcal{O}_Y \twoheadrightarrow \mathcal{L}.$$

(Giving such an α is equivalent to giving $V \to \Gamma(Y, \mathcal{L})$, i.e. giving $s_0, \dots, s_n \in \Gamma(Y, \mathcal{L})$ with $s_i = \alpha(x_i)$. Surjectivity is equivalent to $\bigcap_{i=0}^{n} V(s_i) = \emptyset$.)

Set $(\mathcal{L}, \alpha) \sim (\mathcal{L}', \alpha')$ if there is a $u\colon \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ such that $u \circ \alpha = \alpha'$.

Proposition 4.0.1. *We have a natural bijection*

$$\{\textit{morphisms } Y \to \mathbb{P}^n\} \to \{\textit{equivalence classes of pairs } (\mathcal{L}, \alpha) \textit{ as above}\}.$$

Proof. We have defined the map

$$(f\colon Y \to \mathbb{P}^n) \mapsto (f^*\mathcal{O}_{\mathbb{P}^n}(1), f^*(\phi)).$$

To define a map in the opposite direction, suppose we have $(\mathcal{L}, \alpha)$ and $\alpha\colon V \otimes_k \mathcal{O}_Y \twoheadrightarrow \mathcal{L}$ is given by $\alpha(x_i \otimes 1) = s_i \in \Gamma(Y, \mathcal{L})$. We want to define the map $f\colon Y \to \mathbb{P}^n$ by

$$y \mapsto [s_0(y) : \dots : s_n(y)]$$

and we do this locally.

Let $V_i = Y \setminus V(s_i) \subseteq Y$, which is an open subset. Since α is surjective, $Y = \bigcup_{i=0}^{n} V_i$. On V_i, $s_i \neq 0$, so it gives an isomorphism

$$\begin{aligned} \mathcal{O}_{V_i} &\to \mathcal{L}|_{V_i}, \\ 1 &\mapsto s_i|_{V_i}. \end{aligned}$$

Hence for all j, there exists a unique $a_{ij} \in \mathcal{O}_Y(V_i)$ such that $s_j|_{V_i} = a_{i,j} s_i|_{V_i}$. Define $f_i \colon V_i \to \mathbb{P}^n$ by

$$y \colon [a_{i,0}(y) : \cdots : a_{i,n}(y)] \in D^+_{\mathbb{P}^n}(x_i).$$

By uniqueness of the $a_{i,j}$, on $V_{i_1} \cap V_{i_2}$, we have that

$$a_{i,j} a_{i_2,i_1} = a_{i_2,j}$$

and $a_{i_2,i_1} \in \mathcal{O}(V_{i_1} \cap V_{i_2})^*$. Therefore

$$f_{i_1}|_{V_{i_1} \cap V_{i_2}} = f_{i_2}|_{V_{i_1} \cap V_{i_2}},$$

so we get a map $f \colon Y \to \mathbb{P}^n$ such that $f|_{V_i} = f_i$.

Exercise. Show that:

(1) f only depends on the isomorphism class of $(\mathcal{L}, \alpha)$,
(2) the two maps we defined are mutual inverses.

This completes the proof. □

Remark 4.0.2. Given a variety Y, let $\mathcal{P}^n(Y) = \{(\mathcal{L}, \alpha) \text{ as above}\}$. This is a contravariant functor: if $g \colon Z \to Y$, and $\alpha \colon V \otimes_k \mathcal{O}_Y \twoheadrightarrow \mathcal{L}$, then

$$\mathcal{P}^n(g)((\mathcal{L}, \alpha)) = (g^*\mathcal{L}, g^*(\alpha)).$$

We defined a natural transformation,

$$\mathrm{Hom}_{\mathrm{Var}/k}(-, \mathbb{P}^n) \to \mathbb{P}^n.$$

Proposition 4.0.1 shows that this is an isomorphism of functors, i.e. $\mathbb{P}^n$ represents the functor $\mathcal{P}^n$.

Remark 4.0.3. Let V be a finite-dimensional vector space over k and consider

$$S = \mathrm{Sym}^\bullet(V).$$

Then $V \cong S_1$. Note that after choosing a basis for V, this becomes the set up above.

Let $X = \mathrm{MaxProj}(S)$. Then $V \cong \Gamma(X, \mathcal{O}_X(1))$.

In this setting, Proposition 4.0.1 then says that the map

$$\{\text{morphisms } Y \to X\} \to \left\{ (\mathcal{L}, \alpha) \;\middle|\; \begin{array}{c} \mathcal{L} \text{ line bundle} \\ \alpha \colon V \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{L} \end{array} \right\} \Big/ \sim$$

is an isomorphism.

In particular, if Y is a point,

(the set underlying X) $= \{\text{morphisms } Y \to X\} \cong (\{\text{surjective maps } V \to k\}/\sim) = \{\text{hyperplanes in } V\}$.

We can hence define

$$\mathbb{P}(V) = \operatorname{MaxProj}(\operatorname{Sym}^\bullet(V)).$$

Note. If Y is a variety, $\mathcal{L}$ is a line bundle on Y and $\alpha\colon V \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{L}$, then Proposition 4.0.1 gives a map

$$\begin{aligned} f\colon Y &\to \mathbb{P}(V), \\ y &\mapsto \ker(V \to \mathcal{L}_{(y)}). \end{aligned}$$

Examples 4.0.4.

(1) Let $h\colon V \twoheadrightarrow W$ be a surjective k-linear map with $\dim_k V < \infty$. On $\mathbb{P}(W)$, we have $W \otimes_k \mathcal{O}_{\mathbb{P}(W)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(W)}(1)$ and the map $V \to W$ gives a surjective map

$$V \otimes_k \mathcal{O}_{\mathbb{P}(W)} \twoheadrightarrow W \otimes_k \mathcal{O}_{\mathbb{P}(W)}.$$

The composition

$$V \otimes_k \mathcal{O}_{\mathbb{P}(W)} \twoheadrightarrow W \otimes_k \mathcal{O}_{\mathbb{P}(W)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(W)}(1)$$

is surjective, and hence by Proposition 4.0.1, we get a map

$$\mathbb{P}(W) \to \mathbb{P}(V)$$

mapping $q\colon W \twoheadrightarrow k$ to $q \circ h$.

(2) For $\mathbb{P}(V)$, we have a map

$$V \otimes_k \mathcal{O}_{\mathbb{P}(V)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(V)}(1).$$

If $d \geq 1$, we get $S^d V \otimes_k \mathcal{O}_{\mathbb{P}(V)} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(V)}(d)$, which given

$$\mathbb{P}(V) \xrightarrow{f} \mathbb{P}(S^d V)$$

such that $f^*\mathcal{O}(1) \cong \mathcal{O}(d)$. This is the *Veronese embedding*.

(3) Consider $\mathbb{P}(V)$, $\mathbb{P}(W)$ and

$$\begin{tikzcd} & \mathbb{P}(V) \times \mathbb{P}(W) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{P}(V) & & \mathbb{P}(W) \end{tikzcd}$$

For $V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathbb{P}(V)}(1)$ and $W \otimes \mathcal{O}_{\mathbb{P}(W)} \to \mathcal{O}_{\mathbb{P}(W)}(1)$. On $\mathbb{P}(V) \times \mathbb{P}(W)$, we get

$$V \otimes W \otimes_k \mathcal{O}_{\mathbb{P}(V) \times \mathbb{P}(W)} \twoheadrightarrow \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}(W)}(1)$$

which gives

$$\mathbb{P}(V) \times \mathbb{P}(W) \xrightarrow{j} \mathbb{P}(V \otimes W)$$

such that $j^*\mathcal{O}(1) \cong \mathrm{pr}_1^* \mathcal{O}(1) \otimes \mathrm{pr}_2^* \mathcal{O}(1)$. This is the *Segre embedding*.

(4) Consider an injective linear map $W \hookrightarrow V$ and the corresponding exact sequence

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0.$$

This gives a map $\mathbb{P}(V/W) \hookrightarrow \mathbb{P}(V)$. Let $U = \mathbb{P}(V) \setminus \mathbb{P}(V/W)$. Then the composition

$$W \otimes_k \mathcal{O}_{\mathbb{P}(W)} \hookrightarrow V \otimes_k \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(1)$$

is surjective on U. Therefore, we get a map

$$\mathbb{P}(V) \setminus \mathbb{P}(V/W) \to \mathbb{P}(W),$$

which is given by $V \supset H \mapsto H \cap W \subset W$. This is the linear projection with center $\mathbb{P}(V/W)$.

Definition 4.0.5. A closed subvariety $Y \subseteq \mathbb{P}^n$ is

- *non-degenerate* if there is no hyperplane H such that $Y \subseteq H$ or, equivalently the map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(Y, \mathcal{O}_Y(1))$ is injective,
- *linearly normal* if $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(Y, \mathcal{O}_Y(1))$ is surjective,
- *projectively normal* if $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to \Gamma(Y, \mathcal{O}_Y(m))$ is surjective fro all $m \geq 1$.

Suppose Y is a variety, $\mathcal{L} \in \mathrm{Pic}(y)$, and $\alpha\colon V \otimes_k \mathcal{O}_Y \twoheadrightarrow \mathcal{L}$ corresponds to $\beta\colon V \to \Gamma(Y, \mathcal{L})$. Let $f\colon Y \to \mathbb{P}(V)$ be the map corresponding to α under Proposition 4.0.1.

If $v \in V \setminus \{0\}$, $\beta(v) = 0$ if and only if $f(Y)$ is contained in the hyperplane defined by v.

Hence $\overline{f(Y)}$ is non-degenerate if and only if β is injective.

In general, if we factor β as

$$\begin{tikzcd} V \arrow[rr, "\beta"] \arrow[dr] & & \Gamma(Y, \mathcal{L}) \\ & V/\ker\beta \arrow[ur, hook] & \end{tikzcd}$$

then f factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & \mathbb{P}(V) \\ & \mathbb{P}(V/\ker\beta) \arrow[ur, hook] & \end{tikzcd}$$

and $X \to \mathbb{P}(V/\ker\beta)$ has non-degenerate image closure.

Suppose Y is complete and $f\colon Y \hookrightarrow \mathbb{P}(V)$ is a closed embedding. Then, similarly, β is surjective if and only if $f(Y) \subseteq \mathbb{P}^n$ is linearly normal. As above, the triangle

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & \mathbb{P}(V) \\ & f(Y) \arrow[ur, hook] & \end{tikzcd}$$

corresponds to

$$\begin{tikzcd}
V = \Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \arrow[rr, "\beta"] \arrow[dr] & & \Gamma(Y, \mathcal{L}) \\
& \Gamma(f(Y), \mathcal{O}_{f(Y)}(1)) \arrow[ur] &
\end{tikzcd}$$

Remark 4.0.6. Suppose we have a variety Y, $\mathcal{L} \in \operatorname{Pic}(Y)$, and $\beta\colon V \to H^0(Y, \mathcal{L})$. Consider $V \otimes_k \mathcal{O}_X \to \mathcal{L}$. The image is $\mathcal{I} \otimes \mathcal{L}$ for some coherent ideal $\mathcal{I} \subset \mathcal{O}_Y$. Then

$$V(\mathcal{I}) = \bigcap_{s \in V} V(\beta(s)) \subseteq Y.$$

Note that $\beta \neq 0$ if and only if $V(\mathcal{I}) \neq Y$. In this case, we get a morphism $Y \setminus V(\mathcal{I}) \to \mathbb{P}(V)$.

4.1. **Linear systems.** Let X be an irreducible complete variety and $\mathcal{L} \in \operatorname{Pic} X$.

Definition 4.1.1. A *linear system* on X corresponding to V is a linear subspace in the projective space $\mathbb{P}(\Gamma(X, \ LL)^\vee)$ parametrizing lines in $\Gamma(X, \mathcal{L})$.

The *complete linear system* corresponding to $\mathcal{L}$ is $\mathbb{P}(\Gamma(X, \mathcal{L})^\vee)$ (this is empty if and only if $h^0(X, \mathcal{L}) = 0$).

Note that the complete linear system is in bijection with the set of effective Cartier divisors D on X such that $\mathcal{O}(D) \cong \mathcal{L}$.

Notation. A linear system corresponding to $\mathcal{L}$ given by $V \subseteq \Gamma(X, \mathcal{L})$ is denoted $|V|$.

Definition 4.1.2. The *base locus* of $|V|$ consists of

$$\operatorname{Bs}|V| = \bigcap_{D \in |V|} D = \bigcap_{s \in V} V(s).$$

If $\operatorname{Bs}|V| = \emptyset$, $|V|$ is *base-point free*.

If $|V|$ is base-point free, the map $V \otimes_k \mathcal{O}_X \to \mathcal{L}$ is surjective, so it gives a map $X \to \mathbb{P}(V)$. In general, we get a morphism

$$f\colon X \setminus \operatorname{Bs}(V) \to \mathbb{P}(V).$$

Note that an element $v \in V$ defines a hyperplane $H_v \subseteq \mathbb{P}(V)$ and an effective Cartier divisor D_v on X such that $f^* H_v = D_v|_{X \setminus \operatorname{Bs}|V|}$, i.e.

$$\mathcal{O}_{\mathbb{P}(V)}(-H_v) \cdot \mathcal{O}_{X \setminus \operatorname{Bs}(V)} = \mathcal{O}_X(-D_v)|_{X \setminus \operatorname{Bs}|V|}.$$

Next, consider a morphism $f\colon X \to \mathbb{P}(V)$ corresponding o $V \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{L}$. We want to decide when this is a closed embedding.

Assume that:

(1) X is **complete**,
(2) $V \subseteq \Gamma(X, \mathcal{L})$ (without loss of generality).

Definition 4.1.3.

(1) We say that V *separates the points* of X if for any $x \neq y$ in X, there is an $s \in V$ such that $s(x) = 0$, $s(y) \neq 0$.

(2) We say that V *separates the tangent vectors* on X if for any $x \in X$ and any $v \in T_xX \setminus \{0\}$, there is an $s \in V$ such that $s(x) = 0$, i.e. $s_x \in \mathfrak{m}_x\mathcal{L}_x$, and v does not vanish on $\overline{s_x} \in \mathfrak{m}\mathcal{L}_x/\mathfrak{m}^2\mathcal{L}_x \cong \mathfrak{m}_x/\mathfrak{m}_x^2$.

Intuitively, the second conditions says that the subvariety defined by s contains x but v is not a tangent vector to that subvariety.

Proposition 4.1.4. *Suppose X is complete and $\mathcal{L} \in \operatorname{Pic}(X)$. If $V \subseteq H^0(X, \mathcal{L})$ is a subspace such that $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L}$ is surjective, the corresponding morphism $f\colon X \to \mathbb{P}(V)$ is a closed immersion if and only if V separates the points of X and V separates the tangent vectors on V.*

Proof. Note that V separates the points of X if and only if for any $x, \neq y$ in X, there is a hyperplane H in V such that $f(x) \in H$, $f(y) \notin H$. This is equivalent to f being injective.

Moreover, V separates the tangent vectors on X if and only if for any $x \in X$, and $0 \neq v \in T_xX$, there is a hyperplane $H \subseteq \mathbb{P}(V)$ such that $H \ni f(x)$ and $df_x(v) \notin T_{f(x)}H$. This is equivalent to the property that for any $x \in X$, df_x is injective.

Clearly, if f is a closed immersion, then both of these conditions are satisfied. We just need to prove the converse, so suppose f is injective and df_x is injective for all $x \in X$. Since X is complete, f is closed, $Y = f(X) \subseteq \mathbb{P}(V)$ is a closed subvariety, and $g\colon X \to Y$ is a homeomorphism, where g is given by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & \mathbb{P}(V) \\ & Y = f(X) \arrow[u, hook] \end{tikzcd}$$

We just need to check that g is actually an isomorphism. It is enough to show that $\mathcal{O}_{\mathbb{P}(V)} \to f_*\mathcal{O}_X$ is surjective, since this implies that the canonical morphism $\mathcal{O}_Y \to g_*\mathcal{O}_X$ is an isomorphism, so g is an isomorphism, and hence f is a closed immersion.

Since f is proper, we know that $f_*\mathcal{O}_X$ is coherent. For $p \in X$, we get a morphism

$$\underbrace{\mathcal{O}_{\mathbb{P}(V),f(p)}}_{A} \to \underbrace{(f_*\mathcal{O}_X)_{f(p)}}_{B} = \mathcal{O}_{X,p}.$$

Since $f_*\mathcal{O}_X$ is coherent, $\varphi\colon (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a finite morphism of local rings (in particular, $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$).

Injectivity of $T_pX \to T_{f(p)}\mathbb{P}(V)$ is equivalent to surjectivity $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$, i.e. $\mathfrak{m}_B = \mathfrak{m}_A \cdot B + \mathfrak{m}_B^2$, which by Nakayama Lemma shows that $\mathfrak{m}_B = \mathfrak{m}_A \cdot B$. We know that $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism (since both are isomorphic to k). Hence $B = \varphi(A) + \mathfrak{m}_B = \varphi(A) + \mathfrak{m}_a \cdot B$ and B is a finitely-generated A-module, so by Nakayama Lemma $B = \varphi(A)$, showing that φ is surjective. $\square$

4.2. **Ample and very ample line bundles.** Why are ample line bundles useful?

(1) They are related to embeddings to projective spaces.

(2) They have "*positivity*": you can create globall sections and you can kill cohomology.
(3) We can use them to construct locally free resolutions. If $\mathcal{F} \in \mathrm{Coh}(X)$ and $\mathcal{L}$ is ample, then $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated for some m, so the map
$$H^0(X, \mathcal{F} \otimes \mathcal{L}^m) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^m.$$
Then we get an exact sequence
$$0 \longrightarrow \mathcal{G} \longrightarrow (\mathcal{L}^{-m})^{\oplus N} \longrightarrow \mathcal{F} \longrightarrow 0$$
and repeating the for $\mathcal{G}$ etc, we get a locally free resolution. We will see later that for smooth projective varieties, this resolution will actually be finite.

Exercise. For $\mathcal{L} \in \mathrm{Pic}(X)$ and any $m \geq 1$, $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is ample. (See Homework 10, Problem 1.)

Definition 4.2.1. Let $f\colon X \to Y$ be a morphism and $\mathcal{L} \in \mathrm{Pic}(X)$. Then $\mathcal{L}$ is *f-very ample* (or *very ample over* Y) if there is a locally closed immersion j in $\mathbb{P}^n_Y$ such that $\mathcal{L} \cong j^*(\mathcal{O}_{\mathbb{P}^n_Y}(1))$:
$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr] & & \mathbb{P}^n_Y = \mathbb{P}^n \times Y \arrow[dl] \\ & Y & \end{tikzcd}$$
If Y is a point we simply say that $\mathcal{L}$ is *very ample*.

Theorem 4.2.2. *Let $f\colon X \to Y$ be a morphism with Y affine and let $\mathcal{L} \in \mathrm{Pic}(X)$. Then $\mathcal{L}$ is ample if and only if there is an $m \geq 1$ such that $\mathcal{L}^m$ is very ample.*

Proof. The 'if' implication is clear, since we saw that $\mathcal{O}_{\mathbb{P}^n_Y}(1)$ is ample (since Y is affine), and hence its restriction to any locally closed subset is still ample (by Homework 10, Problem 2). Since there is an $m \geq 1$ such that $\mathcal{L}^m$ is ample, $\mathcal{L}$ is ample.

For the converse, suppose $\mathcal{L}$ is ample. For any $x \in X$, let $W \ni x$ be an affine open neighborhood such that $\mathcal{L}|_W \cong \mathcal{O}_W$ and consider $Y = X \setminus W$ with radical ideal sheaf $\mathcal{I}_Y$. Since $\mathcal{L}$ is ample, $\mathcal{I}_Y \otimes \mathcal{L}^m$ is globally generated for $m \gg 0$.

Since $(\mathcal{I}_Y)_x = \mathcal{O}_{X,x}$, there is a an $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^m) \subseteq \Gamma(X, \mathcal{L}^m)$ such that $s(x) \neq 0$. Consider $X \setminus V(s) \subseteq W$, since $s(y) = 0$ for all $y \in Y$. Then $x \in X \setminus V(s) = W \setminus V(s|_W)$, which is a principal affine open subset in W, and hence it is affine (and we may also replace s by s^q and this still holds).

By the above process for all $x \in X$, we get an open cover $X = U_1 \cup \cdot \cup U_r$ such that U_i is affine for all i and $s_i \in \Gamma(X, \mathcal{L}^{m_i})$ such that $U_i = X \setminus V(s_i)$. After replacing each m_i by a multiple, we may assume that $m_i = m$ for all i.

Since $\mathcal{O}(U_i)$ is a finitely-generated k-algebra, we may choose generators $a_{i,1}, \ldots, a_{i,q_i}$ of it as a $\mathcal{O}(Y)$-algebra.

By Lemma 3.10.7, for all i, j, if $q \gg 0$, $s_i^q|_{U_i} a_{i,j}$ is the restriction to U_i of $t_{ij} \in \Gamma(X, \mathcal{L}^{mq})$.

Fix one $q \gg 0$ that works for all i, j. Consider
$$k^{N+1} \otimes_k \mathcal{O}_X \to \mathcal{L}^{mq}$$

for $N+1 = r + \sum_{i=1}^{r} q_i$, which maps the element of the standard basis to $s_1^q, \ldots, s_r^q, t_{i,j}$ for all $1 \le i \le r$, $1 \le j \le q_i$. Note that this is surjective since

$$\bigcap_{i=1}^{r} V(s_i^q) = \emptyset.$$

We hence get a map $g\colon X \to \mathbb{P}^N = \mathrm{MaxProj}(k[x_i, y_{i,j}])$. Take $j\colon X \to \mathbb{P}_Y^N = Y \times \mathbb{P}^N$ given by $j = (f, g)$.

We claim that this is a closed embedding. Since $j^*\mathcal{O}_{\mathbb{P}_Y^N}(1) \cong \mathcal{L}^{mq}$, this suffices.

We have a diagram

$$\begin{tikzcd}
 & V = \bigoplus_{i=1}^{r} D^+(x_i) \arrow[dr, hook] & \\
X \arrow[ur, "j'"] \arrow[rr, "j"] \arrow[dr] & & \mathbb{P}_Y^N \arrow[dl] \\
 & Y &
\end{tikzcd}$$

Then $(j')^{-1}(D^+(x_i)) = U_i$ is affine and

$$\varphi\colon \mathcal{O}(D^+(x_i)) \to \mathcal{O}(U_i)$$

is surjective, since $\varphi\left(\frac{y_{i,j}}{x_i}\right)$ for $1 \le j \le q_i$ generate $\mathcal{O}(U_i)$ as an $\mathcal{O}(Y)$-algebra. This shows that j is a locally closed embedding, and hence completes the proof. $\square$

Corollary 4.2.3. *An algebraic variety X is quasiprojective if and only if there is an ample line bundle on X.*

Proof. Apply Theorem 4.2.2 when Y is a point. $\square$

Theorem 4.2.4. *Let $f\colon X \to Y$ be a proper morphism, Y be affine, and let $\mathcal{L} \in \operatorname{Pic} X$. Then the following are equivalent:*

(1) *$\mathcal{L}$ is ample,*
(2) *for any $\mathcal{F} \in \mathrm{Coh}(X)$, there is an m_0 such that $H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for all $i \ge 1$ and $m \ge m_0$.*

Proof. We will show that (2) is equivalent with $\mathcal{L}^m$ being ample over Y for some m, and apply Theorem 4.2.2 to complete the proof.

Exercise. Check that (1) implies (2) (this follows from Theorem 4.2.2 using the fact that on $\mathbb{P}_Y^n$ and any $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}_Y^n)$, there is an m_0 such that $H^i(\mathcal{F} \otimes \mathcal{O}(m)) = 0$ for all $i \ge 1$ and $m \ge m_0$).

For (2) implies (1), as in the previous proof, it is enough to show that for any $x \in X$, there is an m such that $s \in \Gamma(X, \mathcal{L}^m)$ such that $x \in X \setminus V(s)$ is affine. This is similar to the proof of Serre's Theorem 3.4.1.

For any $x \in X$, let $W \ni x$ be an affine open neighborhood such that $\mathcal{L}|_Q \cong \mathcal{O}_W$ and consider

$$Z = (X \setminus W) \cup \{x\}$$

with radical ideal $\mathcal{I}_Z$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and by the assumption, $H^1(\mathcal{I}_Z \otimes \mathcal{L}^m) = 0$ for $m \gg 0$, so the long exact sequence in cohomology shows that

$$\Gamma(X, \mathcal{L}^m) \to \Gamma(Z, \mathcal{L}^m|_Z)$$

is surjective. Then there is an $s \in \Gamma(X, \mathcal{L}^m)$ such that $s(x) \neq 0$ and $Z \subseteq V(s)$. As before, this shows that $x \in X \setminus V(s)$ is affine. □

Remark 4.2.5. In the official notes, there is a section about relative versions of ampleness and projective morphisms. This was omitted in class due to lack of time.

5. Ext and Tor functors

Let R be a commutative ring. Consider for $M \in R$-mod, the functor $\operatorname{Hom}_R(M, -)$. This is left-exact and the category has enough injectives, so we get right derived functors

$$\operatorname{Ext}^i_R(M, -) \colon R\text{-mod} \to R\text{-mod},$$

which satisfy the usual properties:

- $\operatorname{Ext}^0_R(M, -) \cong \operatorname{Hom}_R(M, -)$,
- if N is an injective R-module, then $\operatorname{Ext}^i_R(M, N) = 0$ for $i > 0$,
- for a short exact sequence, we get a long exact sequence of Ext modules.

Given $\varphi \colon M \to M'$, we get a natural transformation

$$\begin{aligned} \operatorname{Hom}(M', -) &\to \operatorname{Hom}(M, -) \\ \psi &\mapsto \psi \circ \varphi, \end{aligned}$$

which induces a natural transformation $\operatorname{Ext}^i_R(M', -) \to \operatorname{Ext}^i_R(M, -)$. In other words $\operatorname{Ext}^i_R(-, -)$ is a bifunctor.

Note that we can also derive with respect to the 1st variable. Consider the contravariant functor

$$\operatorname{Hom}_R(-, N) \colon R\text{-mod} \to R\text{-mod}$$

which is left exact. We can interpret it as a left exact covariant functor

$$\operatorname{Hom}_R(-, N) \colon (R\text{-mod})^{\mathrm{op}} \to R\text{-mod}.$$

Since R-mod has enough projective objects, we can construct the right derived functors of $\operatorname{Hom}_R(-, N)$, written $\overline{\operatorname{Ext}}^i_R(-, N)$. We temporarily write $\overline{\operatorname{Ext}}$, but we will soon see that these functor are naturally isomorphic to Ext.

How to compute it? Consider a projective resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

(Convention about upper/lower indexing: $C_i = C^{-i}$ and $\mathcal{H}_i = \mathcal{H}^{-i}$.) Then

$$\overline{\mathrm{Ext}}^i_R(M,N) \cong \mathcal{H}^i(\mathrm{Hom}_R(F_\bullet, N)).$$

As always:

- $\overline{\mathrm{Ext}}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$,
- if ${}_RM$ is a projective module, then $\overline{\mathrm{Ext}}^i_R(M,N) = 0$ for all $i \geq 1$,
- if $0 \to M' \to M \to M'' \to 0$ is exact, we get a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \overline{\mathrm{Ext}}^i_R(M'',N) \arrow[r] & \overline{\mathrm{Ext}}^i_R(M,N) \arrow[r] & \overline{\mathrm{Ext}}^i_R(M',N) \arrow[dll, out=-10, in=170] \\ & \overline{\mathrm{Ext}}^{i+1}_R(M'',N) \arrow[r] & \cdots \end{tikzcd}$$

Proposition 5.0.1. *We have a functorial isomorphism (in both variables)*

$$\mathrm{Ext}^i_R(M,N) \cong \overline{\mathrm{Ext}}^i_R(M,N).$$

Proof. It is enough to show that given a projective resolution $F_\bullet$ of M, we have a functorial isomorphism

$$\mathrm{Ext}^i_R(M,N) \cong \mathcal{H}^i(\mathrm{Hom}_R(F_\bullet, N)).$$

We check that the right hand side gives a δ-functor in N. Given an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

we get an exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}_R(F_\bullet, N') \longrightarrow \mathrm{Hom}_R(F_\bullet, N) \longrightarrow \mathrm{Hom}_R(F_\bullet, N'') \longrightarrow 0$$

since each F_i is projective. Then the long exact sequence in cohomology shows that

$$\{\mathcal{H}^i(\mathrm{Hom}_R(F_\bullet, -))\}_{i\geq 0}$$

is a δ-functor.

Since $F_1 \to F_0 \to M \to 0$ is exact and $\mathrm{Hom}_R(-,N)$ is left-exact, we see that

$$\mathcal{H}^0(\mathrm{Hom}_R(F_\bullet, N)) \cong \mathrm{Hom}_R(M,N).$$

Moreover, if N is injective, then $\mathrm{Hom}_R(-,N)$ is exact, which shows that

$$\mathcal{H}^i(\mathrm{Hom}_R(F_\bullet, N)) = 0$$

for $i > 0$.

By the universal property of δ-functors, we get a natural isomorphism of δ-functors, showing that:

$$\operatorname{Ext}^i_R(M,N) \cong \mathcal{H}^i(\operatorname{Hom}_R(F_\bullet, N)).$$

This completes the proof. $\square$

From now on, we will write $\overline{\operatorname{Ext}} = \operatorname{Ext}$ and use whichever interpretation is more useful.

Example 5.0.2. If $\varphi_a \colon N \to N$ is given by $u \mapsto au$ for some $a \in R$, then

$$\operatorname{Ext}^i_R(M,N) \xrightarrow{\operatorname{Ext}^i_R(M,\varphi_a)} \operatorname{Ext}^i_R(M,N)$$

is also given by mutliplication by a. Similarly in the first variable.

Proposition 5.0.3. *If R is Noetherian and M and N and finitely-generated R-modules, then*

(1) $\operatorname{Ext}^i_R(M,N)$ *is finitely-generated over R for all i,*
(2) *for any multiplicative system $S \subseteq R$,*
$$S^{-1}\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$$
as $S^{-1}R$-modules.

Proof. The idea is to compute Ext using a free resolution of M given by finitely-generated modules. $\square$

Definition 5.0.4. Let M be an R-module. The *projective dimension*, $\operatorname{pd}_R M$, is the smallest n such that there is a projective resolution

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

(if no such n exists, $\operatorname{pd}_R M = \infty$).

Proposition 5.0.5. *Let M be an R-module. Then the following are equivalent:*

(1) $\operatorname{pd}_R M \le n$,
(2) $\operatorname{Ext}^i_R(M,N) = 0$ *for all $i > n$ and all N,*
(3) $\operatorname{Ext}^{n+1}_R(M,N) = 0$ *for all N,*
(4) *for any exact sequence*
$$F_{n-1} \xrightarrow{\varphi} \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$
with F_i projective for all i, $\ker\varphi$ is projective.

Proof. It is clear that (2) implies (3) and (4) implies (1). Consider the exact sequence

$$0 \longrightarrow Q = \ker(\varphi) \longrightarrow F_{n-1} \xrightarrow{\varphi} \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with all F_i projective. Then we break this into short exact sequences and use $\operatorname{Ext}^i_R(F_j, N) = 0$ for all j and $i \ge 1$ to conclude that

$$(*) \qquad \operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i-n}_R(Q,N)$$

for all $i > n$. Indeed, the short exact sequence

$$0 \to M_1 \to F_0 \to M \to 0,$$

the long exact sequence shows that for $i \geq 1$:

$$\underbrace{\mathrm{Ext}^i_R(F_0, N)}_{=0} \longrightarrow \mathrm{Ext}^i_R(M, N) \longrightarrow \mathrm{Ext}^{i+1}_R(M, N) \longrightarrow \underbrace{\mathrm{Ext}^{i+1}_R(F_0, N)}_{=0}$$

and we iterate this to get $(*)$.

This shows that if (1) holds, then we have such a sequence with Q projective, so $\mathrm{Ext}^{i-n}_R(Q, N) = 0$ for all $i > n$, and hence (2) holds.

To prove (3) implies (4), it is enough to show that if $\mathrm{Ext}^1_R(Q, N) = 0$ for all N, then Q is projective. Choose a short exact sequence

$$0 \longrightarrow Q' \longrightarrow F \longrightarrow Q \longrightarrow 0$$

with F projective. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(Q, Q') \longrightarrow \mathrm{Hom}(F, Q') \longrightarrow \mathrm{Hom}(Q', Q') \longrightarrow \mathrm{Ext}^1(Q, Q') = 0,$$

and hence the sequence above is split. Since F is projective, this shows that Q is projective. □

Corollary 5.0.6. *If* $0 \to M' \to M \to M'' \to 0$ *is exact, then*

(1) $\mathrm{pd}_R M \geq \min\{\mathrm{pd}_R(M'), \mathrm{pd}_R(M'')\}$,
(2) $\mathrm{pd}_R M' \geq \min\{\mathrm{pd}_R(M), \mathrm{pd}_R(M'') - 1\}$,
(3) $\mathrm{pd}_R M'' \geq \min\{\mathrm{pd}_R(M), \mathrm{pd}_R(M') + 1\}$.

Proof. **Exercise**. Hint: use Proposition 5.0.5 and the long exact sequence. □

We now discuss the Tor modules. For an R-module M, the functor $M \otimes_R -$ is right exact. Since R-mod has enough projectives, we can construct the left derived functors $\mathrm{Tor}^R_i(M, -)$. As always, we have the following properties:

- $\mathrm{Tor}^R_0(M, -) \cong M \otimes_R -$,
- if N is projective, then $\mathrm{Tor}^R_i(M, N) = 0$ for $i > 0$,
- if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_i(M, N') \longrightarrow \mathrm{Tor}_i(M, N) \longrightarrow \mathrm{Tor}_i(M, N'') \longrightarrow \mathrm{Tor}_{i+1}(M', N) \longrightarrow \cdots$$

Moreover, given a morphism $M \to M'$, we get a natural transformation $\operatorname{Tor}_i(M,-) \to \operatorname{Tor}_i(M',-)$.

Note that by definition, $\operatorname{Tor}_i(M,N) \cong \mathcal{H}_i(M \otimes F_0)$ where $F_\bullet \to N$ is a projective resolution.

Proposition 5.0.7. *We have a functorial isomorphism*

$$\operatorname{Tor}_i(M,N) \cong \operatorname{Tor}_i(N,M),$$

i.e. if $G_\bullet \to M$ is a projective resolution, then

$$\operatorname{Tor}_i(M,N) \cong \mathcal{H}_i(G_\bullet \otimes N).$$

Proof. This is similar to the proof of Proposition 5.0.1, so we omit the proof here. □

Remarks 5.0.8.

(1) If $\varphi_a \colon N \to N$ is given by multiplication by $a \in R$, $\operatorname{Tor}_i(M,N)$ is again given by multiplication by a.
(2) If R is Noetherian, and M, N are R-modules, then $\operatorname{Tor}_i(M,N)$ is a finitely-generated R-module.
(3) If $S \subseteq R$ is a multiplicative system, then $S^{-1}\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}M, S^{-1}N)$.
(4) If $\operatorname{pd}_R M = n$, then $\operatorname{Tor}_i^R(M,N) = 0$ for $i > n$ (compute using a projective resolution of M of length n).

Proposition 5.0.9. *Let M be an R-module. Then the following are equivalent:*

(1) *M is a flat R-module,*
(2) *$\operatorname{Tor}_i^R(M,N) = 0$ for all $i > 0$ and all N,*
(3) *$\operatorname{Tor}_1^R(M,N) = 0$ for all N.*

Proof. To see that (1) implies (2), let $F_\bullet \to N$ be a projective resolution, and note that if M is flat, then tensoring with M is exact, so $\mathcal{H}_i(F_\bullet \otimes M) = 0$ for $i > 0$.

Since (2) implies (3) trivially, we just need to show that (3) implies (1). If

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

is exact, we get an exact sequence

$$0 = \operatorname{Tor}_1(M,N'') \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0$$

which shows that M is flat. □

Corollary 5.0.10. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence.*

(1) *If M' and M'' are flat, then M is flat.*
(2) *If M'' and M are flat, then M' is flat.*
(3) *If M'' is flat, then for any N, $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact.*

Proof. Use the long exact sequence for Tor and Proposition 5.0.9. □

Corollary 5.0.11. *If $(R, \mathfrak{m}, k)$ is a local Noetherian ring and ${}_R M$ is flat and finitely-generated, then M is free.*

Sketch of proof. As in the case when M was projective, choose a short exact sequence

$$0 \to N \to F \to M \to 0$$

where F is free and finitely-generated with a minimal set of generators. Then $N \subseteq \mathfrak{m} \cdot F$, and Corollary 5.0.10, we get a short exact sequence

$$0 \longrightarrow \underbrace{N \otimes k}_{N/\mathfrak{m}N} \longrightarrow \underbrace{F \otimes k}_{F/\mathfrak{m}F} \longrightarrow M \otimes k \longrightarrow 0$$

and $N/\mathfrak{m}N = 0$ shows by Nakayama Lemma that $N = 0$, and hence $M \cong F$. □

Finally, we discuss Ext and $\mathcal{E}xt$ for $\mathcal{O}_X$-modules. If $\mathcal{F}$ is and $\mathcal{O}_X$-module, we get two left exact functors:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod}$$
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}$$

which give right derived functors

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X(X)\text{-mod},$$
$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, -)\colon \mathcal{O}_X\text{-mod} \to \mathcal{O}_X\text{-mod}.$$

For example, $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = 0$ for $i > 0$, but $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G})$.

Proposition 5.0.12. *If $\mathcal{F}$ has a locally free resolution $\mathcal{P}_\bullet \to \mathcal{F}$, then*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}^i(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}_\bullet, \mathcal{G})).$$

Proof. The proof is the same as for Ext of R-modules (Proposition 5.0.1). □

Proposition 5.0.13. *For any open subset U of X, we have a canonical isomorphism*

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

We say that $\mathcal{E}xt$ are *local Ext* and Ext are *global Ext*.

6. Depth and Cohen-Macaulay rings

6.1. **Depth.** Let R be a Noetherian ring and M be a finitely-generated R-module.

The idea is to we start with notion of a nonzero divisor and extend this notion to a sequence of elements.

Definition 6.1.1. A sequence of elements $x_1, \ldots, x_n \in R$ ais an M*-regular sequence* if

(1) $(x_1, \ldots, x_n)M \neq M$,
(2) for any i, $1 \leq i \leq n$, x_i is a non zero-divisor on $M/(x_1, \ldots, x_{i-1})M$.

If $M = R$, this is just a *regular sequence*.

Remark 6.1.2. If $x_1, \dots, x_n$ is an M-regular sequence, and $S \subseteq R$ is a multiplicative system such that $(x_1, \dots, x_n) \cdot S^{-1}M \neq S^{-1}M$, then $\frac{x_1}{1}, \dots, \frac{x_n}{1}$ is an $S^{-1}M$-regular sequence.

Remark 6.1.3. If $x_1, \dots, x_n$ is an M-regular sequence, then $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$. Indeed, if $x_i \in (x_1, \dots, x_{i-1})$ and x_i is a non zero-divisor, then $M/(x_1, \dots, x_{i-1})M = 0$, contradicting (1).

Since R is Noetherian, every M-regular sequence is contained in a maximal one.

Similarly, if I is a fixed ideal, every M-regular sequence contained in I is part of a maximal one with the same property.

Note that if $IM \neq M$, an M-regular sequence $x_1, \dots, x_n \in I$ is maximal for such sequences contained in I if and only if for any $x \in I$, x is a zero-divisor on $M/(x_1, \dots, x_n)M$, which is equivalent to $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, \dots, x_n)M)$. Equivalently, $I \cdot u = 0$ for some nonzero $u \in M/(x_1, \dots, x_n)M$.

Definition 6.1.4. For an ideal $I \subseteq R$, the *depth of I with respect to M* is
$$\operatorname{depth}(I, M) = \min\left\{i \geq 0 \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\right\}$$
(and $+\infty$ if the set is empty).

If $(R, \mathfrak{m})$ is local, we write simply $\operatorname{depth}(M) = \operatorname{depth}(\mathfrak{m}, M)$.

Theorem 6.1.5.

(1) *If $IM = M$, then* $\operatorname{depth}(I, M) = \infty$.
(2) *If $IM \neq M$, then*$\operatorname{depth}(I, M)$ *is the length of any maximal M-regular sequence in I.*

Proof. For (1), suppose $IM = M$. It is enough to show that for any prime ideal $\mathfrak{p}$ in R,
$$\operatorname{Ext}^i_R(R/I, M)_{\mathfrak{p}} = 0.$$
Since Ext commutes with localization, we just need to show that
$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0.$$
If $I \subseteq \mathfrak{p}$, then $M_{\mathfrak{p}} = 0$. If $I \not\subseteq \mathfrak{p}$, then $R_{\mathfrak{p}}/IR_{\mathfrak{p}} = 0$. In both cases
$$\operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0.$$

For (2), suppose $IM \neq M$ and choose a maximal M-regular sequence $x_1, \dots, x_n$ in I. We show that $\operatorname{depth}(I, M) = n$ by induction on n.

For $n = 0$, there is an $u \in M \setminus \{0\}$ such that $Iu = 0$. Then we have a nonzero map
$$R/I \to M$$
$$\overline{1} \mapsto u.$$
This shows that $\operatorname{Hom}(R/I, M) \neq 0$, and hence $\operatorname{depth}(I, M) = 0$.

For the inductive step, suppose $n \geq 1$. Clearly, $x_2, \dots, x_n$ is a maximal M/x_1M-regular sequence in I. Note that
$$I(M/x_1M) = IM/x_1M \neq M/x_1M.$$
Consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/x_1M \longrightarrow 0.$$

Note that multiplication by x_1 is injective, since x_1 is a non zero-divisor. The long exact sequence gives

$$\operatorname{Ext}^i_R(R/I, M) \xrightarrow{\cdot x_i} \operatorname{Ext}^i_R(R/I, M) \longrightarrow \operatorname{Ext}^i_R(R/I, M/x_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(R/I, M),$$

but $x_i \in I$ implies that the multiplication by x_i map above is 0. We hence get short exact sequences

$$0 \longrightarrow \operatorname{Ext}^i_R(R/I, M) \longrightarrow \operatorname{Ext}^i_R(R/I, M/x_1M) \longrightarrow \operatorname{Ext}^{i+1}_R(R/I, M) \longrightarrow 0.$$

This shows that $\operatorname{Ext}^i_R(R/I, M/x_1M) = 0$ if and only if $\operatorname{Ext}^i_R(R/I, M) = 0$ and $\operatorname{Ext}^{i+1}_R(R/I, M) = 0$. Therefore,

$$\operatorname{depth}(I, M/x_1M) = \operatorname{depth}(I, M) - 1.$$

This completes the proof by induction, since $n - 1 = \operatorname{depth}(I, M/x_1M)$. □

Corollary 6.1.6.

(1) *We have that* $\operatorname{depth}(I, M) = \operatorname{depth}(\sqrt{I}, M)$*,*
(2) *If* $I \subseteq J$*, then* $\operatorname{depth}(I, M) \leq \operatorname{depth}(J, M)$*,*
(3) *We have that* $\operatorname{depth}(I, M) = \min\limits_{\substack{\mathfrak{p} \supseteq I \\ \textit{prime}}} \operatorname{depth}(M_{\mathfrak{p}})$*.*
(4) *If* $x \in I$ *is a non-zero-divisor on* M*,* $\operatorname{depth}(I, M/xM) = \operatorname{depth}(I, M) - 1$*,*
(5) *Recall that, by definition,* $\dim(M) = \dim(R/\operatorname{Ann}_R(M))$*. If* $(R, \mathfrak{m})$ *is local, then for every prime* $\mathfrak{p} \in \operatorname{Ass}(M)$*, for* $M \neq 0$*,*

$$\operatorname{depth}(M) \leq \dim(R/\mathfrak{p}).$$

In particular, $\operatorname{depth}(M) \leq \dim(M)$*.*

Proof. To see (1), note that $IM = M$ if and only if $\sqrt{I}M = M$. Otherwise, use the same proof to show that if $x_1, \ldots, x_n$ maximal M-regular sequence in I then $n = \operatorname{depth}(\sqrt{I}, M)$. The last step of the proof clearly works. For the $n = 0$ step, we needed $Iu = 0$ for some $u \in M \setminus \{0\}$, but this implies that $\sqrt{I}v = 0$ for some $v \in M \setminus \{0\}$.

Part (2) follows immediately from Theorem 6.1.5.

For (3), if $I \subseteq \mathfrak{p}$ is prime, then $\operatorname{depth}(I, M) \leq \operatorname{depth}(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \leq \operatorname{depth}(M_{\mathfrak{p}})$ by (2). We want to show this is achieved for some $\mathfrak{p}$. If $IM = M$, we are done. If $IM \neq M$ and $x_1, \ldots, x_n \in I$ is a maximal M-regular sequence, then $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M/(x_1, \ldots, x_n)M)$. Then $IR_{\mathfrak{p}} \subseteq \mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/(x_1, \ldots, x_n)M_{\mathfrak{p}})$. Hence $\frac{x_1}{1}, \ldots, \frac{x_n}{1}$ is a maximal $M_{\mathfrak{p}}$-regular sequence in $\mathfrak{p}R_{\mathfrak{p}}$, so n is the depth of $M_{\mathfrak{p}}$. This proves (3) by Theorem 6.1.5.

Part (4) is immediately from the proof of Theorem 6.1.5.

To prove (5), we induct on $n = \operatorname{depth}(M)$. If $n = 0$, this is clear. If $n \geq 1$, then there is a nonzero divisor $x \in \mathfrak{m}$. By (4), $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$. There exists $u \in M \setminus \{0\}$

such that $pu = 0$. By Krull's intersection theorem, we can write

$$u = x^j v \text{ for } v \notin xM.$$

Since $pu = 0$, $pv = 0$ since x is a non-zero-divisor on M. Then $\overline{v} \in M/xM$ is nonzero. Then $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Ass}(M/xM)$. Since x is a non-zero-divisor on M, $x \notin \mathfrak{p}$ and $x \in \mathfrak{q}$, we have that $\mathfrak{p} \neq \mathfrak{q}$. By induction, we have that

$$\operatorname{depth}(M) - 1 = \operatorname{depth}(M/xM) \leq \dim R/\mathfrak{q} \leq \dim(R/\mathfrak{p}) - 1.$$

This completes the proof. □

Example 6.1.7. Let X be an algebraic variety and $x \in X$ be a smooth point. Let $a_1, \ldots, a_n$ be a minimal system of generators of the maximal ideal in $\mathcal{O}_{X,x}$. Note that $n = \dim(\mathcal{O}_{X,x})$, since x is a smooth point. We saw that for any i with $1 \leq i \leq n$,

$$\mathcal{O}_{X,x}/(a_1, \ldots a_i)$$

is the local ring of a smooth variety. In particular, it is a domain. This implies that $a_1, \ldots, a_n$ is a regular sequence. Therefore:

$$\operatorname{depth}(\mathcal{O}_{X,x}) = n.$$

Proposition 6.1.8. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence, then*

$$\begin{aligned}
\operatorname{depth}(I, M) &\geq \min\{\operatorname{depth}(I, M'), \operatorname{depth}(I, M'')\},\\
\operatorname{depth}(I, M') &\geq \min\{\operatorname{depth}(I, M), \operatorname{depth}(I, M'') + 1\},\\
\operatorname{depth}(I, M'') &\geq \min\{\operatorname{depth}(I, M), \operatorname{depth}(I, M') - 1\}.
\end{aligned}$$

Proof. Use the definition and the long exact sequence. □

Theorem 6.1.9 (Auslander–Buchsbaum)**.** *If $(R, \mathfrak{m})$ is a local, Noetherian ring and $0 \neq M$ is a finitely-generated R-module with $\operatorname{pd}_R(M) < \infty$, then*

$$\operatorname{depth}(R) = \operatorname{depth}(M) + \operatorname{pd}_R(M).$$

Before we can prove the theorem, we begin with some preparations.

Suppose $u_1, \ldots, u_n \in M$ is a minimal system of generators and consider the map

$$\begin{aligned}
\varphi\colon F_0 = R^{\oplus n} &\twoheadrightarrow M\\
e_i &\mapsto u_i.
\end{aligned}$$

By minimality, $\ker(\varphi) \subseteq \mathfrak{m}F_0$. Repeat to get

$$F_1 \twoheadrightarrow \ker(\varphi)$$

using a minimal system of generators for $\ker(\varphi)$. Altogether, this gives a free resolution

$$\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0.$$

This is a *minimal* free resolution of M: $d_i(F_i) \subseteq \mathfrak{m} \cdot F_{i-1}$ for all $i \geq 1$ (so if we write d_i as a matrix with respect to a basis, all entries are in the maximal ideal).

Proposition 6.1.10. *If $F_\bullet \to M$ is a minimal free resolution,* $\operatorname{rank}(F_i) = \dim_k \operatorname{Tor}_i(k, M)$.

Proof. All maps in $F_\bullet \otimes_R R/\mathfrak{m}$ are 0, which shows that

$$\mathrm{Tor}_i(k, M) = F_i \otimes_R R/\mathfrak{m}$$

for all i. □

Corollary 6.1.11. *The following are equivalent for a local Noetherian ring* $(R, \mathfrak{m})$ *and finitely-generated* R*-module* M *of finite projective dimension:*

(1) $\mathrm{pd}_R(M) \leq q$,
(2) $\mathrm{Tor}_i(M, N) = 0$ *for all* $i \geq q + 1$ *and all* N,
(3) $\mathrm{Tor}_{q+1}(k, M) = 0$.

Proof. It is clear that (1) implies (2) and (2) implies (3). Finally, from Proposition 6.1.10 shows that(3) implies (1). □

Definition 6.1.12. The global dimension of R is

$$\mathrm{gldim}(R) := \sup\{\mathrm{pd}(M) \mid {}_R M \text{ finitely-generated}\}.$$

Corollary 6.1.13. *We have that* $\mathrm{gldim}(R) = \mathrm{pd}_R(k)$ *for a local ring* $(R, \mathfrak{m})$.

Proof. This follows from Corollary 6.1.11. □

Finally, we can proceed with the proof of the Auslander–Buchsbaum Theorem 6.1.9.

Proof of Theorem 6.1.9. Let $(R, \mathfrak{m})$ be a local Noetherian ring and $0 \neq M$ be a finitely-generated R-module with $\mathrm{pd}_R(M) < \infty$. We want to show that $\mathrm{depth}(R) = \mathrm{depth}(M) + \mathrm{pd}_R(M)$.

Let $n = \mathrm{depth}(R)$. Consider a minimal free resolution of M:

$$0 \longrightarrow F_q \xrightarrow{d_q} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0.$$

where $q = \mathrm{pd}_R(M)$ by Corollary 6.1.11.

Suppose $n = 0$. Then there exists $u \in R \setminus \{0\}$ such that $\mu = 0$. If $q \geq 1$, $d_q(u \cdot F_q) = 0$, contradicting the injectivitiy of d_q.

Hence $\mathrm{pd}_R M = 0$, so M is free and o $\mathrm{depth}(M) = \mathrm{depth}(R) = 0$.

Suppose now that $n > 0$. Additionally, assume for now that $\mathrm{depth}(M) > 0$. Then there is $x \in \mathfrak{m}$ which is a non-zero-divisor on both R and M. Then

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow R/(x) \longrightarrow 0$$

and tensoring with with M, the long exact sequence shows that $\mathrm{Tor}_i^R(R/(x), M) = 0$ for all $i \geq 1$.

Tensoring the minimal free resolution $F_\bullet \to M$ with $R/(x)$ is hence exact, so

$$F_\bullet \otimes R/(x) \to M/xM$$

is a free resolution of M/xM, which is also minimal.

Then $\operatorname{pd}_{R/(x)} M/xM = \operatorname{pd}_R M$. Since x is non-zero-divisor on both R, M, $\operatorname{depth}(M/xM) = \operatorname{depth} M - 1$ and $\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1$. This completes the proof by induction.

We are finally left with the case $\operatorname{depth}(R) = n > 0$ but $\operatorname{depth}(M) = 0$. Then M is not a free module. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with $N \neq 0$. We know that $\operatorname{pd}_R(N) = \operatorname{pd}_R(M) - 1$. Also, $\operatorname{depth}(N) = 1$ by Proposition 6.1.8.

Then we can apply the result for N to complete the proof. $\square$

Homework. Read about the Serre Condition SI (Review Sheet).

Remarks 6.1.14.

(1) Let R be a Noetherian ring, ${}_RM$ be finitely-generated, and let $\mathfrak{a} \subseteq R$ be such that $\mathfrak{a} \cdot M = 0$. Then
$$\operatorname{depth}(I, M) = \operatorname{depth}(I + \mathfrak{a}/\mathfrak{a}, M)$$
(using the description via regular sequences, Theorem 6.1.5).

(2) We showed that if $(R, \mathfrak{m})$ is a local Noetherian ring and M is a finitely-generated R-module, then
$$\operatorname{depth} M \leq \min_{\mathfrak{p} \in \operatorname{Ass}(M)} \leq \min \dim R/\mathfrak{p} \leq \dim M.$$
Similarly, one can show that for any Noetherian ring R and any ideal $\mathfrak{a} \subseteq R$,
$$\operatorname{depth}(\mathfrak{a}, R) \leq \operatorname{codim}(\mathfrak{a}).$$

6.2. **The Koszul complex.** Let R be a commutative ring and E be and R-module with a map $\varphi\colon R \to R$. The *Koszul complex* $K(\varphi) = K(\varphi)_\bullet$ given by

$$\cdots \longrightarrow K(\varphi)_p = \textstyle\bigwedge^p E \xrightarrow{d_p} K(\varphi)_{p-1} = \textstyle\bigwedge^{p-1} E \longrightarrow \cdots \longrightarrow E \xrightarrow{\varphi} R$$

where
$$d_p(e_1 \wedge \cdots \wedge e_p) = \sum_{i=1} (-1)^{i-1} \varphi(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_p.$$

It is easy to see that $d_p \circ d_{p+1} = 0$ for all $p \geq 1$, so this is actually a complex.

- We are only interested in the case where E is free of rank n. In this case $K(v\varphi)_p = 0$ for all $p > n$.
- For an R-module M, we set $K(\varphi, M) = K(\varphi) \otimes_R M$.
- For an n-tuple $\underline{x} = (x_1, \ldots, x_n) \in R^n$, we set
$$K(\underline{x}) = K(x_1, \ldots, x_n) = K(\varphi)$$

where

$$\begin{aligned} \varphi\colon R^n &\to R, \\ e_i &\mapsto x_i. \end{aligned}$$

For example, when $n = 1$, $K(x)$ is just the multiplication by x map $R \to R$.

Remark 6.2.1 (Functoriality)**.** Given a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "u"] & R \arrow[d, "="] \\ F \arrow[r, "\psi"] & R \end{tikzcd}$$

we get a morphism of complexes $K(\varphi) \to K(\psi)$ given by

$$K(\varphi)_p = \wedge^p E \xrightarrow{\bigwedge^p u} \wedge^p F = K(\psi)_p.$$

If u is an isomorphism, then $K(\varphi) \cong K(\psi)$.

Examples 6.2.2.

(1) If σ permutes $\{1, \ldots, n\}$, we get an isomorphism

$$K(x_1, \ldots, x_n) = K(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

(2) Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are minimal systems of generators for $\mathfrak{a} \subseteq R$, where $(R, \mathfrak{m})$ is local and Noetherian. Write

$$y_i = \sum_{j=1}^{n} a_{i,j} x_j.$$

Then we have a commutative diagram

$$\begin{tikzcd} & e'_i \arrow[r] & x_i \\ \sum_j a_{i,j} e'_j & R^n \arrow[r, "\varphi"] & R \arrow[d, "="] \\ e_i \arrow[u] & R^n \arrow[u] \arrow[r, "\psi"] & R \\ & e_i \arrow[r] & y_i \end{tikzcd}$$

Since $\det(a_{i,j}) \notin \mathfrak{m}$, u is an isomorphism, and we get $K(x_1, \ldots, x_n) \cong K(y_1, \ldots, y_n)$.

Remark 6.2.3. Note that $\mathcal{H}_0(K(\varphi, M)) = \mathrm{coker}(\varphi) \otimes_R M$.

Proposition 6.2.4. *If $a \in \mathrm{im}(\varphi)$, then multiplication by a on $K(\varphi)$ is homotopic to 0. In particular, if φ is surjective, then $\mathcal{H}_i(K(\varphi, M)) = 0$ for all i.*

Proof. We want to define maps θ_p

$$\begin{tikzcd} & \bigwedge^p E \arrow[r, "dp"] \arrow[dl, "\theta_p"'] & \bigwedge^{p-1} E \arrow[dl, "\theta_{p-1}"] \\ \bigwedge^{p+1} E \arrow[r, "dp"] & \bigwedge^p E & \end{tikzcd}$$

which give a homotopy between $\cdot a$ and 0.

Write $a = \varphi(e)$ for $e \in E$, and set

$$\theta_p(e_1 \wedge \cdots \wedge e_p) = e \wedge e_1 \wedge \cdots \wedge e_p.$$

It is then easy to check that $d \circ \theta + \theta \circ d = a \cdot \mathrm{id}$. $\square$

The next goal is to relate $K(x_1, \ldots, x_n; M)$ to $K(x_1, \ldots, x_{n-1}; M)$. more generally, suppose $\varphi \colon E \to R$ and consider for $a \in R$:

$$F = E \oplus R \xrightarrow{\quad\psi\quad} R$$

$$(e, \lambda) \longrightarrow \varphi(e) + \lambda a.$$

Note that

$$\bigwedge^p F \cong \bigwedge^p E \oplus \bigwedge^{p-1} E$$

via the map

$$\bigwedge^{p-1} E \to \bigwedge^p F$$
$$e_1 \wedge \cdots \wedge e_{p-1} \mapsto e_0 \wedge e_1 \wedge \cdots \wedge e_{p-1}$$

where $e_0 = (0, 1) \in F$. This gives an exact sequence

$$0 \longrightarrow K(\varphi) \longrightarrow K(\psi) \longrightarrow C_\bullet \longrightarrow 0$$

where $C_p = \wedge^{p-1} E$ with the differential is $-d_{K(\varphi)}$. The long exact sequence in cohomology gives

$$\mathcal{H}_p(K(\varphi)) \longrightarrow \mathcal{H}_p(K(\psi)) \longrightarrow \mathcal{H}_{p-1}(K(\varphi)) \xrightarrow{\cdot a} \mathcal{H}_{p-1}(K(\varphi))$$

Note that the short exact sequence above is split at each level, so tensoring with an R-module M gives a short exact sequence, and hence a similar result holds after tensoring with M.

Proposition 6.2.5. *Let R be a Noetherian ring and M be an R-module.*

(1) *If $x_1, \ldots, x_n$ is an M-regular sequence, then*

$$\mathcal{H}_i(K(\underline{x}, M)) = \begin{cases} 0 & \text{if } i \geq 1 \\ M/(x_1, \ldots, x_n)M & \text{if } i = 0. \end{cases}$$

(2) *If R is local, $x_i \in \mathfrak{m}$, $M \neq 0$, and $\mathcal{H}_i(K(\underline{x}, M)) = 0$ for $i > 0$, then $\underline{x}$ is an M-regular sequence.*

Proof. To prove (1), we use induction on n. For $n=1$, the Koszul complex is

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow 0$$

and the assertion is immediate. In general, consider the complexes $K(\underline{x}, M)$ and $K(\underline{x'}, M)$ with $\underline{x'} = x_1, \ldots, x_{n-1}$. Then the long exact sequence above gives

$$\mathcal{H}_i(K(\underline{x'}, M)) \longrightarrow \mathcal{H}_i(K(\underline{x}, M)) \longrightarrow \mathcal{H}_{i-1}(K(\underline{x'}, M)) \xrightarrow{\cdot x_n} \mathcal{H}_{i-1}(K(\underline{x'}, M)).$$

By the inductive hypothesis, $\mathcal{H}_j(K(\underline{x'}, M)) = 0$ for $j \geq 1$. It is hence clear that $H_i(K(\underline{x}, M)) = 0$ for $i \geq 2$. We also have a short exact sequence

$$0 \longrightarrow \mathcal{H}_1(K(\underline{x}, M)) \longrightarrow M/\underline{x'} \cdot M \xrightarrow{\cdot x_n} M/\underline{x'}M$$

but x_n is a non-zero-divisor on $M/\underline{x'}M$, so $\mathcal{H}_1(K(\underline{x}, M)) = 0$.

For (2), we again use induction on n. The $n=1$ case is immediate. In general, we use the same exact sequence as above. Since $\mathcal{H}_i(K(\underline{x}, M)) = 0$ for all $i > 0$, this shows that

$$\mathcal{H}_i(K(\underline{x'}, M)) = x_n \cdot \mathcal{H}_i(K(\underline{x'}, M))$$

for $i > 0$. Nakayama Lemma then shows that $\mathcal{H}_i(K(\underline{x'}, M)) = 0$ for all $i > 0$, and hence the inductive hypothesis shows that $x_1, \ldots, x_{n-1}$ is an M-regular sequence.

Also, $\mathcal{H}_1(K(\underline{x}, M)) = 0$, which shows that (using the exact sequence above again) x_n is a non-zero-divisor on $\mathcal{H}_0(K(\underline{x'}, M)) = M/\underline{x'}M$. Hence $x_1, \ldots, x_n$ is a regular sequence. □

Corollary 6.2.6. *Suppose $(R, \mathfrak{m})$ is a local Noetherian ring and ${}_RM$ is finitely-generated. Consider any M-regular sequence $x_1, \ldots, x_n$. Then*

1. *any permutation of it is a regular sequence,*
2. *$K(x_1, \ldots, x_n)$ gives a minimal free resolution of $R/(x_1, \ldots, x_n)$, so*

$$\mathrm{pd}_R\, R/(x_1, \ldots, x_n) = n.$$

Proof. Both assertions follow from Proposition 6.2.5. □

Example 6.2.7. Let X be an algebraic variety and $x \in X$ be a smooth point. Let $n = \dim(\mathcal{O}_{X,x})$, $R = \mathcal{O}_{X,x}$.

We saw that if $\mathfrak{m} = (u_1, \ldots, u_n)$ is a maximal ideal, then $u_1, \ldots, u_n$ is an $\mathcal{O}_{X,x}$-regular sequence. This shows that $\mathrm{pd}_R(R/\mathfrak{m}) = n$, so $\mathrm{gldim}(R) = n$.

The converse also holds by a Theorem due to Auslander-Buchsbaum-Serre. We do not prove this here.

If X is a smooth variety of dimension n, $\mathcal{F}$ is a coherent sheaf on X, and

$$\mathcal{E}_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

is an exact complex with $\mathcal{E}_i$ locally free, then $\ker(d_{n-1})$ is locally free.

For example, if X is smooth and quasiprojective, every coherent sheaf has a finite resolution by locally free sheaves.

How to glue the Koszul complex on algebraic varieties? Consider $s \in \Gamma(X, \mathcal{E}) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}^\vee, \mathcal{O}_X)$ where $\mathcal{E}$ is locally free. Then consider the complex

$$\cdots \longrightarrow \cdots \bigwedge^2(\mathcal{E}^\vee) \longrightarrow \mathcal{E}^\vee \xrightarrow{\ s\ } \mathcal{O}_X$$

with the map $\bigwedge^i(\mathcal{E}^\vee) \to \bigwedge^{i-1} \mathcal{E}^\vee$ is given by

$$\sum_{i=1} (-1)^{i-1} s(e_i) e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_p.$$

Note that if $U \subseteq X$ is an affine open subset, then the restriction of this complex to U is the Koszul complex for $\mathcal{E}^\vee(U) \xrightarrow{s|_U} \mathcal{O}_X(U)$.

Example 6.2.8. Let $X = \mathbb{P}(V)$ for $\dim_k V = n+1$. We have a surjective morphism

$$V \otimes_k \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(1)$$

and the Koszul complex of the map $V \otimes_k \mathcal{O}_X(-1) \twoheadrightarrow \mathcal{O}_X$ is

$$0 \longrightarrow \bigwedge^{n+1} V \otimes \mathcal{O}_X(-n-1) \longrightarrow \cdots \longrightarrow V \otimes \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

This is exact by Proposition 6.2.4.

Exercise. We have that

$$\ker\left(\bigwedge^p V \otimes \mathcal{O}(-p) \to \bigwedge^{p-1} V \otimes \mathcal{O}(-p+1)\right).$$

6.3. **Cohen–Macaulay modules.** Recall that if $(R, \mathfrak{m})$ is a local Noetherian ring and ${}_R M$ is a finitely-generated R-module, then $\operatorname{depth} M \leq \dim M$ (see Corollary 6.1.6).

Definition 6.3.1. Such a module M is *Cohen–Macaulay* if $\operatorname{depth} M = \dim M$. If R is not necessarily local, M is *Cohen–Macaulay* if $M_\mathfrak{m}$ is a Cohen–Macaulay $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ in $\operatorname{supp}(M)$.

A ring R is *Cohen–Macaulay* if it is Cohen–Macaulay over itself.

Definition 6.3.2. If X is an algebraic variety and $\mathcal{F}$ is coherent sheaf on X, then

(1) for $x \in X$, $\mathcal{F}$ is *Cohen–Macaulay* at $x \in X$ if $\mathcal{F}_x$ is a Cohen–Macaulay module over $\mathcal{O}_{X,x}$,
(2) $\mathcal{F}$ is *Cohen–Macaulay* if it is Cohen–Macaulay at all $x \in X$.

Finally, X is *Cohen–Macaulay* if $\mathcal{O}_X$ is *Cohen–Macaulay* over itself.

Examples 6.3.3.

(1) For a smooth variety X, X is Cohen–Macaulay. Indeed, we saw that $\operatorname{depth} \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{X,x}$ in Example 6.1.7.

(2) If X and Y are Cohen–Macaulay, then $X \times Y$ is Cohen–Macaulay (this is a homework problem).

Proposition 6.3.4. *Let $f\colon X \to Y$ be a finite, surjective moprhism of varieties and suppose Y is smooth. Then X is Cohen–Macaulay if and only if f is flat.*

Proof. Note that since f is finite, f is flat if and only if $f_*\mathcal{O}_X$ is locally free.

We may assume that X and Y are affine by choosing affine open covers for X and Y. Let $\varphi\colon A = \mathcal{O}(Y) \to \mathcal{O}(X) = B$ be the corrresponding morphism, and choose $y \in Y$ corresponding to $\mathfrak{m} \subseteq A$.

We will show later that $\operatorname{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \operatorname{depth}(\mathfrak{m}B_{\mathfrak{m}}, B_{\mathfrak{m}})$. Then

$$\operatorname{depth}(\mathfrak{m}B_{\mathfrak{m}}, B_{\mathfrak{m}}) = \min_{\substack{\mathfrak{n} \subseteq B \\ \text{maximal} \\ \mathfrak{n}B \subseteq \mathfrak{n}}} \operatorname{depth}(B_{\mathfrak{n}}).$$

by Corollary 6.1.6, and

$$\operatorname{depth}(B_{\mathfrak{n}}) \leq \dim(B_{\mathfrak{n}}) = \dim(A_{\mathfrak{m}}).$$

Altogether, this shows that $B_{\mathfrak{n}}$ is Cohen-Macaulay for all $\mathfrak{n}$ as above if and only if $\operatorname{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$. Since $y \in Y$ is a smooth point, $\operatorname{pd}_{A_{\mathfrak{m}}}(B_m) < \infty$, by Auslander-Buchsbaum 6.1.9,

$$\operatorname{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \underbrace{\operatorname{depth} A_{\mathfrak{m}}}_{= \dim A_{\mathfrak{m}}} - \operatorname{pd}_{A_{\mathfrak{m}}} B_{\mathfrak{m}}.$$

Hence $\operatorname{depth}(\mathfrak{m}A_{\mathfrak{m}}, B_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$ if and only if $B_{\mathfrak{m}}$ is projective (which is equivalent to flatness) over $A_{\mathfrak{m}}$. $\square$

Homework. Read about the *local flatness criterion* (Review Sheet).

Example 6.3.5. If X is an affine toric variety and it is normal, then X is Cohen–Macaulay.

Proposition 6.3.6. *A ring R is a Cohen–Macaulay ring if and only if for any ideal $\mathfrak{a} \subsetneq R$,* $\operatorname{depth}(\mathfrak{a}, R) = \operatorname{codim}(\mathfrak{a})$. *(In general, we just have '$\leq$').*

Proof. Note that:

$$\operatorname{depth}(\mathfrak{a}, R) = \min_{\substack{\mathfrak{p} \supseteq \mathfrak{a} \\ \text{prime}}} \operatorname{depth} R_{\mathfrak{p}} = \min_{\substack{\mathfrak{p} \supseteq \mathfrak{a} \\ \text{prime}}} \dim R_{\mathfrak{p}} = \operatorname{codim} \mathfrak{a}.$$

Exercise. Check the converse also holds. $\square$

Proposition 6.3.7. *Let X be a variety and $\mathcal{F} \in \operatorname{Coh}(X)$. If $\mathcal{I} \subseteq \mathcal{O}_X$ is a coherent ideal and $x \in V(\mathcal{I})$ is such that $\mathcal{I}_x$ is generated by an $\mathcal{F}_x$-regular sequence, then $\mathcal{F}$ is Cohen–Macaulay at x if and only if $\mathcal{F}/\mathcal{I}\cdot\mathcal{F}$ is Cohen–Macaulay at x.*

Proof. We may assume that X is an affine variety and (by induction on length of the regular sequence) that $\mathcal{I}_x = (f)$ for a non-zero-divisor f on $\mathcal{F}_x$. We saw that $\operatorname{depth}(\mathcal{F}_x/f\cdot\mathcal{F}_x) = \operatorname{depth}(\mathcal{F}_x) - 1$.

We have that $\operatorname{supp}(\mathcal{F}/\mathcal{I}\mathcal{F}) = \operatorname{supp}(\mathcal{F}) \cap V(\mathcal{I})$. Since f is a non-zero-divisor on $\mathcal{F}_x$, $V(f)$ does not contain any irreducible component of $\operatorname{supp}(\mathcal{F})$ passing through x. hence $\dim(\mathcal{F}_x/\mathcal{I}_x\mathcal{F}_x) = \dim \mathcal{F}_x - 1$. This completes the proof of (2). $\square$

Proposition 6.3.8 (Unmixedness)**.** *Let $\mathcal{F}$ be a coherent sheaf on X which is Cohen–Macauley at x. Then any associated subvarieties of $\mathcal{F}$ passing through x is an irreducible component of* $\operatorname{supp}(\mathcal{F})$*. Moveover, any 2 irreducible components of* $\operatorname{supp}(\mathcal{F})$ *passing through x have the same dimension.*

Proof. This follows from: if $(R, \mathfrak{m})$ is local Noetherian and ${}_R M$ is finitely-generated, then for any $\mathfrak{p} \in \operatorname{Ass}_R(M)$, $\operatorname{depth}(M) \leq \dim R/\mathfrak{p} \leq \dim M$. If M is Cohen–Macaulay, then $\dim(R/\mathfrak{p}) = \dim M$. (In particular, $\mathfrak{p}$ is minimal in $\operatorname{supp} M$). □

Proposition 6.3.9. *If $\mathcal{F}$ is a coherent sheaf on X which is Cohen–Macauley at x, then for any closed irreducible subvariety $Y \subseteq X$ containing x, $\mathcal{F}_Y$ is a Cohen–Macauley $\mathcal{O}_{X,Y}$-module.*

Sketch of proof. We may assume that X is affine, $A = \mathcal{O}(X)$, $M = \mathcal{F}(X)$, and $\mathfrak{p}$ is the prime ideal corresponding to Y, $\mathfrak{m}$ is the prime ideal corresponding to x.

We argue by induction on $r = \operatorname{depth}(M_\mathfrak{p})$. If $r = 0$, $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p})$, so $\mathfrak{p}A_\mathfrak{m} \in \operatorname{Ass}(M_\mathfrak{m})$. By Proposition 6.3.8, $\mathfrak{p}A_\mathfrak{m}$ is minimal in $\operatorname{supp}(M_\mathfrak{m})$, so $\dim(M_\mathfrak{p}) = 0$.

For the inductive step, take a non-zero-divisor on M in $\mathfrak{p}$ and use Proposition 6.3.7. □

Corollary 6.3.10. *If X is an affine variety which is Cohen–Macaulay, $\mathcal{O}(X)$ automatically satisfies Serre's condition (S_i):*

$$\operatorname{depth}(\mathcal{O}(X)_\mathfrak{p}) \geq \min\{i, \dim(\mathcal{O}(X)_\mathfrak{p})\}$$

for all $\mathfrak{p}$ prime. Hence (by Serre's normality condition[2]) a Cohen–Macaulay variety is normal if and only if it is smooth in codimension 1.

Definition 6.3.11. Let X be a variety. A coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ is locally a *complete intersection ideal* if for any $x \in V(\mathcal{I})$, $\mathcal{I}_x$ is generated by a regular sequence.

Example 6.3.12. Suppose X is Cohen–Macaulay. Then $\mathcal{I} \subseteq \mathcal{O}_X$ is locally a complete intersection ideal if and only if for any $x \in X$, $\mathcal{I}_x$ can be generated by $r = \operatorname{codim}(\mathcal{I}_x)$ elements.

Definition 6.3.13. A closed subvariety Y of X is regularly embedded if the corresponding radical ideal $\mathcal{I}_Y$ is locally a complete intersection ideal.

As an application, we present the classical result known as Bézout's Theorem.

Let $H_1, \ldots, H_n \subseteq \mathbb{P}^n$ be effected Cartier divisors on $\mathbb{P}^n$, $d_i = \deg(H_i)$ such that $Z = \bigcap_i H_i$ is 0-dimensional.

If $p \in Z$, and $f_i \in \mathcal{O}_{\mathbb{P}^n,p}$ is the image of a local equation of H_i, we define the *intersection multiplicity of $H_1, \ldots, H_n$ at p* as

$$i_p(H_1, \ldots, H_n) = \ell\left(\mathcal{O}_{\mathbb{P}^n,p}/(f_1, \ldots, f_n)\right).$$

Theorem 6.3.14 (Bézout)**.** *With the above definition:*

$$\sum_{p \in Z} i_p(H_1, \ldots, H_n) = \prod_{i=1}^n d_i.$$

[2]See Review Sheet 5.

Proof. For $1 \leq j \leq n$, let

$$\mathcal{F}_j = \mathcal{O}_{H_1} \otimes \cdots \otimes \mathcal{O}_{H_j} = \mathcal{O}_X/\mathcal{I}_j.$$

Note that

$$\operatorname{supp}(\mathcal{F}_j) = V(\mathcal{I}_j) = \bigcap_{i=1}^{j} \operatorname{supp}(H_i).$$

Each irreducible component has codimension $\geq j$. The hypothesis that $Z = \bigcap_{i=1}^{j} H_i$ is 0-dimensional forces each irreducible component to have codimension $= j$.

Since $X = \mathbb{P}^n$ is Cohen–Macaulay, so $\mathcal{I}_j$ is a locally complete intersection ideal. In particular, $\mathcal{F}_j$ is a Cohen–Macaulay sheaf. This shows that every associated subvariety of $\mathcal{F}_j$ is an irreducible component.

If $j \leq n-1$, H_{j+1} does not contain any associated subvariety of $\mathcal{F}_j$. We then have an exact sequence

$$0 \longrightarrow \mathcal{F}_j \otimes \mathcal{O}(-d_{j+1}) \longrightarrow \mathcal{F}_j \longrightarrow \mathcal{F}_{j+1} \longrightarrow 0$$

that comes from tensoring

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_{j+1}) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{H_{j+1}} \longrightarrow 0$$

with $\mathcal{F}_j$ (and exactness follows from the above condition).

Tensoring this with $\mathcal{O}_{\mathbb{P}^n}(m)$ and taking the Euler–Poincaré characteristic, χ, we see that

$$\mathcal{P}_{\mathcal{F}_{j+1}}(m) = \mathcal{P}_{\mathcal{F}_j}(m) - \mathcal{P}_{\mathcal{F}_j}(m - d_{j+1})$$

(where $\mathcal{P}$ denotes the Hilbert polynomial — see Problem Session 10). Recall that $\mathcal{P}_{\mathcal{F}}(m) = \frac{m^r}{r!}\deg(\mathcal{F}) +$ lower order terms, where $\dim \mathcal{F} = r$.

Therefore $\deg(\mathcal{F}_{j+1}) = \deg(\mathcal{F}_j) \cdot d_{j+1}$. Hence

$$\deg(\mathcal{F}_n) = \prod_{i=1}^{n} d_i.$$

Since $\mathcal{F}_n$ has 0-dimensional support,

$$\deg(\mathcal{F}_n) = \sum_{p \in \operatorname{supp} \mathcal{F}_n} \ell((\mathcal{F}_n)_p) = \sum_{p \in Z} i_p(H_1, \ldots, H_n).$$

This completes the proof. $\square$

7. More on flatness and smoothness

7.1. **More on flatness.** The easy case is when $f \colon X \to Y$ is finite and flat so $f_*\mathcal{O}_X$ is locally free. If Y is connected, then we define

$$\deg(f) = \operatorname{rank}(f_*\mathcal{O}_X).$$

If X and Y are irreducible, $f(X) = Y$ (being open and closed in Y). Then $\deg(f) = \deg(k(X)/k(Y))$.

Proposition 7.1.1. *Let $f\colon X \to Y$ be a finite and flat morphism. Then for any $y \in Y$,*

$$\sum_{p \in f^{-1}(y)} mult_p(f^{-1}(y)) = \deg(f),$$

where $\mathrm{mult}_p(f^{-1}(y)) =\ll ((\mathcal{O}_X/\mathfrak{m}_y\mathcal{O}_X)_\mathfrak{p})$ *where* $\mathfrak{m}_y$ *is the ideal sheaf corresponding to y in Y.*

Proof. We may assume that Y and X are affine, and $A = \mathcal{O}(Y) \hookrightarrow \mathcal{O}(X) = B$ makes B a free A-module of rank $r = \deg(f)$.

Taking $\mathfrak{m}_y \subseteq A$, we have that $B/\mathfrak{m}_yB$ has dimension r over $A/\mathfrak{m}_y = k$. Hence

$$r = \sum_{p \in f^{-1}(y)} \ell((B/\mathfrak{m}_yB)_\mathfrak{p}).$$

□

Definition 7.1.2. Let $f\colon X \to Y$ be a morphism and $\mathcal{F} \in \mathrm{Qcoh}(X)$. Then $\mathcal{F}$ is *flat over Y* if for all $x \in X$, $\mathcal{F}_x$ is flat over $\mathcal{O}_{Y,f(x)}$, or, equivalently, for all $U \subseteq X$, $V \subseteq Y$ affine open subsets such that $f(U) \subseteq V$, $\mathcal{O}(V) \to \mathcal{O}(U)$, $\mathcal{F}(U)$ is flat over $\mathcal{O}_Y(V)$.

We say that f is *flat* if $\mathcal{O}_X$ is flat over Y.

Suppose $X \hookrightarrow Y \times \mathbb{P}^n$ is a closed immersion, and consider the commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "p"] \\ & Y \end{tikzcd}$$

Let $\mathcal{F} \in \mathrm{Coh}(X)$. For any $y \in Y$, if $\mathfrak{m}_y$ is the maximal ideal in $\mathcal{O}_Y$ corresponding to y, then

$$\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathfrak{m}_y\mathcal{O}_X = \mathcal{F}/\mathfrak{m}_y\mathcal{F}.$$

We can consider this as a coherent sheaf on $\{y\} \times \mathbb{P}^n \cong \mathbb{P}^n$.

Theorem 7.1.3. *If $\mathcal{F}$ is flat over Y and Y is connected, then the Hilbert polynomial $\mathcal{P}_{\mathcal{F}_y}$ is independent of $y \in Y$.*

(The converse also holds, but we do not prove it here.)

Proof. We may assume that Y is affine. After replacing $\mathcal{F}$ by $i_*\mathcal{F}$, we may assume that $X = Y \times \mathbb{P}^n$.

Recall that $\mathcal{P}_{\mathcal{F}_y}(m) = \chi(\mathcal{F}_y(m))$ for all $m \in \mathbb{Z}$. Note that this is equal to $h^0(\mathcal{F}_y(m))$ for $m \gg 0$.

Let $A = \mathcal{O}(Y)$. Consider the Čech complex computing the cohomology of $\mathcal{F}(m)$ on $Y \times \mathbb{P}^n$ with respect to the cover by $D^+(x_i)$ for $0 \le i \le n$:

$$0 \longrightarrow \Gamma(X, \mathcal{F}(m)) \longrightarrow \bigoplus_i \underbrace{\Gamma(D^+(x_i), \mathcal{F}(m))}_{\text{flat } A\text{-module}} \longrightarrow \bigoplus_{i<j} \underbrace{\Gamma(D_X^+(x_i x_j), \mathcal{F}(m))}_{\text{flat } A\text{-module}} \longrightarrow \cdots$$

Since $H^i(X, \mathcal{F}(m)) = 0$ for $i > 0$ and $m \gg 0$, the above is an exact complex for $m \gg 0$.

This shows that $\Gamma(X, \mathcal{F}(m))$ is a flat A-module. It is also finitely-generated over A, so it is locally free of well-defined rank.

It is enough to show that for any $y \in Y$, if m is large enough $h^0(\mathcal{F}_y(m)) = \mathrm{rank}_A(\Gamma(X, \mathcal{F}(m)))$.

Let $\mathfrak{m}_y \subseteq A$ be a maximal ideal corresponding to y. By choosing generators of $\mathfrak{m}_y$, we get an exact sequence:

$$A^{\oplus N} \longrightarrow A \longrightarrow A/\mathfrak{m}_y \longrightarrow 0.$$

Take the corresponding sheaves, apply f^*, and tensor with $\mathcal{F}$:

$$\mathcal{F}^{\oplus N} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_y \longrightarrow 0.$$

If $m \gg 0$, the corresponding sequence

$$\Gamma(X, \mathcal{F}(m))^{\oplus N} \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}_y(m)) \longrightarrow 0$$

is exact. (In general, if $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact on $\mathbb{P}^n_A = X$, then $\Gamma(X, \mathcal{F}(m)) \to \Gamma(X, \mathcal{G}(m)) \to \Gamma(X, \mathcal{H}(m))$ is exact for $m \gg 0$.)

Tensoring this exact sequence with $A/\mathfrak{m}_y$, we get

$$\Gamma(X, \mathcal{F}(m)) \otimes_A A/\mathfrak{m}_y \cong \Gamma(X, \mathcal{F}_y(m)) = \Gamma(\{y\} \times \mathbb{P}^n, \mathcal{F}_y(m)).$$

This completes the proof. ☐

Corollary 7.1.4. *Suppose $X \hookrightarrow Y \times \mathbb{P}^n$ is a closed immersion, and we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "p"] \\ & Y \end{tikzcd}$$

Let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over Y. If Y is connected, then the map

$$y \mapsto \chi(\mathcal{F}_y)$$

is constant on Y, where $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X/\mathfrak{m}_y\mathcal{O}_X$.

7.2. Generic flatness.

Theorem 7.2.1 (Generic flatness, Grothendieck)**.** *Let $f\colon X \to Y$ be a morphism of algebraic varieties and $\mathcal{F} \in \mathrm{Coh}(X)$. Then there is an open dense subset $U \subseteq Y$ such that*

$$\mathcal{F}|_{f^{-1}(U)} \text{ is flat over } U.$$

Remark 7.2.2. Combining this with Theorem 7.1.3, if $X \hookrightarrow Y \times \mathbb{P}^n$ is a closed immersion, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{P}^n \arrow[d, "p"] \\ & Y \end{tikzcd}$$

then for any $\mathcal{F} \in \mathrm{Coh}(X)$, there is a finite set of polynomials that contains $\mathcal{P}_{\mathbb{F}_y}$ for any $y \in Y$.

We present a sketch fo the proof of Theorem 7.2.1. One can reduce to the case when X, Y are affine and Y is irreducible. Then the theorem follows from the following theorem.

Theorem 7.2.3 (Generic freeness)**.** *Let R be a domain and S be an R-algebra of finite type. If M is a finitely-generated S-module, then there is an $a \in R \setminus \{0\}$ such that M_a is free over R.*

Proof. We first reduce to the case where S is a domain and $M = S$.

(1) If we have an exact sequence
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
and we know the theorem for M' and M'', then we know the theorem for M.

(2) There is a filtration of M:
$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$
by S-modules such that
$$M_i/M_{i-1} \cong S/\mathfrak{p}_i \text{ for } \mathfrak{p}_i \text{ prime.}$$

Combining these two remarks, we may assume that S is a domain and $M = S$.

Case 1. There is a nonzero $a \in \ker(R \to S)$. Then $S_a = 0$ which is free.

Case 2. The map $R \hookrightarrow S$ is injective. Let $K = \mathrm{Frac}(R)$ and $n = \mathrm{trdeg}(S \otimes_R K/K)$. We proceed by induction on $n \geq 0$.

By Noether normalization, there are elements $a_1, \ldots, a_n \in S \otimes_R K$, algebraically independent over K, such that
$$K[a_1, \ldots, a_n] \hookrightarrow S \otimes_R K$$
is finite.

We may assume that $a_i \in S$ for all i. After replacing R by R_c for some $c \neq 0$ in R, we may also assume that
$$R[a_1, \ldots, a_n] \hookrightarrow S \text{ is finite.}$$

Now, we may replace S by $R[a_1, \ldots, a_n]$, and M is still S.

We filter M again so that the successive quotients are

$$R[a_1, \ldots, a_n]/\mathfrak{p}$$

for primes $\mathfrak{p}$.

If $\mathfrak{p} = 0$, $R[a_1, \ldots, a_n]$ is free over R, so we are done. If $\mathfrak{p} \neq 0$, then

$$\operatorname{trdeg}((R[a_1, \ldots, a_n]/\mathfrak{p}) \otimes K/K) < n.$$

Then we are done by induction. □

Recall the following properties of flat morphisms. If $f\colon X \to Y$ is flat, then

(1) if $W \subseteq Y$ is irreducible, closed, $f^{-1}(W) \neq \emptyset$, and Z is an irreducible component of $f^{-1}(W)$, then $\overline{f(Z)} = W$,
(2) if W and Z are as in (1), then $\operatorname{codim}_Y(W) = \operatorname{codim}_X(Z)$.

Remark 7.2.4. Note that (1) implies that every irreducible component of X dominates a (unique) irreducible component of Y. Moreover, if Y' is an irreducible component of Y such that $f^{-1}(Y') \neq \emptyset$, then all the irreducible components of $f^{-1}(Y')$ are irreducible components of X.

Exercise. Let $f\colon X \to Y$ be a flat morphism and $n \in \mathbb{Z}$. Then the following are equivalent:

(1) for any irreducible component X' of X, if Y' is an irreducible component of Y dominated by X', then
$$\dim X' = \dim Y' + n,$$
(2) for any irreducible closed subset $W \subseteq Y$ such that $f^{-1}(W) \neq \emptyset$, for any irreducible component V of $f^{-1}(W)$,
$$\dim V = \dim W + n,$$
(3) for any $y \in f(X)$, $f^{-1}(y)$ has pure dimension n.

Definition 7.2.5. If the equivalent conditions above hold, we say that f is flat of *relative dimension n*.

Proposition 7.2.6. *Suppose $f\colon X \to Y$ is a morphism such that Y is smooth, irreducible, of dimension n, and X is Cohen–Macaulay has pure dimension m. If all fibers of f have pure dimension $m - n$, then f is flat.*

To prove this proposition, we will use the following corollary of the local flatness criterion from Review Sheet 6.

Corollary 7.2.7. *Let $(A, \mathfrak{m}, k) \to (B, \mathfrak{n}, \ell)$ be a local morphism of Noetherian local rings and suppose M is a finitely-generated B-module. If $a_1, \ldots, a_n \in \mathfrak{m}$ is A-regular and M-regular, then M is flat over A if and only if $M/(a_1, \ldots, a_n)$ is flat over $A/(a_1, \ldots, a_n)$.*

Proof of 7.2.6. Let $x \in X$ and $y = f(x)$. Consider the map $\varphi\colon A = \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} = B$ induced by f. Write $\mathfrak{m}_A \subseteq A$ and $\mathfrak{m}_B \subseteq B$ for the maximal ideals.

Since Y is smooth, $\mathfrak{m}_A$ is generated by a regular sequence $a_1, \ldots, a_n$. Note that this is also a B-regular sequence:

$$\begin{aligned} \operatorname{depth}((a_1, \ldots, a_n)B, B) &= \operatorname{codim}((a_1, \ldots, a_n)B) && \text{as } B \text{ is Cohen–Macaulay} \\ &= \dim B - \dim(B/(a_1, \ldots, a_n)) \\ &= m - (m-n) \\ &= n. \end{aligned}$$

By Corollary 7.2.7, it is enough to show that $B/\mathfrak{m}_A B$ flat over$A/\mathfrak{m}_A$, which is clear, since $A/\mathfrak{m}_A$ is a field. This shows that B is flat over A, so φ is flat of relative dimension $m-n$. $\square$

7.3. **Generic smoothness.** In Math 631, we just discussed smooth morphisms between smooth varieties. We now discuss this notion in general.

Definition 7.3.1. A morphism $f \colon X \to Y$ is *smooth of relative dimension* r if

(1) f is flat, of relative dimension r,
(2) $\Omega_{X/Y}$ is locally free of rank r.

We say that f is *étale* if it is smooth, of relative dimension 0.

Note that when Y is a point, $X \to Y$ is smooth of relative dimension r if and only if X is smooth of pure dimension r.

Proposition 7.3.2. *A morphism $f \colon X \to Y$ is smooth of relative dimension r if and only if*

(1) *f is flat of relative dimension r,*
(2) *for any $y \in Y$ with corresponding radical ideal sheaf $\mathfrak{m}_y$, $\mathfrak{m}_y \cdot \mathcal{O}_X$ is a radical dieal sheaf, and $f^{-1}(y)$ is smooth.*

Proof. We may assume that X and Y are affine. Write $A = \mathcal{O}(Y)$, $B = \mathcal{O}(X)$, and $\mathfrak{m}_y \subseteq A$ for the ideal correponding to y. We have a map $A \to B$, which induces

$$k = \overline{A} = A/\mathfrak{m}_y \to \overline{B} = B/\mathfrak{m}_y B,$$

so

$$\Omega_{\overline{B}/\overline{A}} \cong \Omega_{B/A} \otimes_B \overline{B}.$$

Note that $\Omega_{X/Y}$ is locally free of rank r if and only if $\dim_k (\Omega_{X/Y})_{(x)} = r$.

For any $x \in f^{-1}(y)$, write $\mathfrak{n} \subseteq \overline{B}$ for the maximal ideal corresponding to x. Then

$$(\Omega_{X/Y})_{(x)} = \Omega_{B/A} \otimes_B \left(\overline{B}/\mathfrak{n}\right) = \Omega_{\overline{B}/\overline{A}} \otimes_{\overline{B}} \left(\overline{B}/\mathfrak{n}\right).$$

We saw that this is isomorphic to $\mathfrak{n}/\mathfrak{n}^2$. Since $f^{-1}(y)$ has pure dimension r, $\dim\left(\overline{B}_{\mathfrak{n}}\right) = r$.

Altogether, this shows that $(\Omega_{X/Y})_{(x)}$ has dimension r over k if and only if $\overline{B}_{\mathfrak{n}}$ is a regular local ring. This is equivalent to saying that $\mathfrak{m}_Y \mathcal{O}_X$ is radical at x and x is a smooth point of $f^{-1}(y)$. $\square$

Proposition 7.3.3. *Let $f \colon X \to Y$ be a morphism of smooth irreducible varieties and $\dim X = m$, $\dim Y = n$. Then the following are equivalent:*

(1) *f is smooth of relative dimension $m-n$,*
(2) *for any $x \in X$, $T_xX \xrightarrow{df_x} T_{f(x)}Y$ is surjective.*

Proof. Consider the exact sequence

$$\underbrace{f^*\Omega_Y}_{\substack{\text{locally free}\\ \text{rank } n}} \xrightarrow{\alpha} \underbrace{\Omega_X}_{\substack{\text{locally free}\\ \text{rank } m}} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

If (1) holds, then α is injective and it is in fact a morphism of vector bundles, so

$$f^*(\Omega_Y)_{(x)} \to (\Omega_X)_{(x)} \text{ is injective for all } x.$$

Hence $(df_x)^\vee$ is injective, which shows that df_x is surjective.

Conversely, the hypothesis implies that α is injective and $\Omega_{X/Y}$ is locally free of rank $m-n$. We already proved that (2) implies that each fiber has pure dimension $m-n$. Since X and Y are smooth, this shows that f is flat of relative dimension $m-n$. $\square$

Definition 7.3.4. A morphism $f\colon X \to Y$ is *unramified at x* if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ satisfies $\mathfrak{m}_{X,x} = \mathfrak{m}_{Y,y} \cdot \mathcal{O}_{X,x}$. A morphism is *unramified* if it is unramified at all points.

Exercise. For a morphism $f\colon X \to Y$, the following are equivalent:

(1) f is étale,
(2) f is flat and $\Omega_{X/Y} = 0$,
(3) f is flat and unramified,
(4) for any $x \in X$, $\widehat{\mathcal{O}_{Y,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$ is an isomorphism.

Almost all the theorems in the class except Hironaka's resolution of singularities hold in any characteristic. Even in the case of Hironaka's theorem, it is not known whether it holds in positive characteristic or not. The following theorem holds only in characteristic 0 and it (and its consequences) are known to fail in positive characteristic.

Theorem 7.3.5 (Generic Smoothness)**.** *Suppose* $\mathrm{char}(k) = 0$. *If $f\colon X \to Y$ is a morphism of irreducible varieties and X is smooth, then there is an open subset $U \subseteq Y$ such that $f^{-1}(U) \to U$ is smooth.*

Definition 7.3.6. A finite type field extension L/K is *separable* if there is a transcendence basis $a_1, \ldots, a_n \in L$ over K such that $L/K(a_1, \ldots, a_n)$ is a (finite) separable extension.

Example 7.3.7. In characteristic 0, any such extension is separable.

Lemma 7.3.8. *If L/K is a finite field extnesion which is separable, then*

$$\dim_L \Omega_{L/K} = \mathrm{trdeg}(L/K).$$

Proof. Choose a transcendence basis $a_1, \ldots, a_n$ of L/K such that $L/K(a_1, \ldots, a_n)$ is separable. By the primitive element theorem, there is a $b \in L$ such that $L = K(a_1, \ldots, a_n)(b)$. Let $f \in K(a_1, \ldots, a_n)[y]$ be a minimal polynomial of b over $K(a_1, \ldots, a_n)$. Note that $f'(b) \neq 0$ by separability.

After clearing denominators, there is a polynomial $g \in K[x_1, \dots, x_n, y]$ such that

$$g(a_1, \dots, a_n, b) = 0 \text{ and } \frac{\partial g}{\partial y}(a_1, \dots, a_n, b) \neq 0.$$

Then L is the field of fractions of

$$A = k[x_1, \dots, x_n, y]/(g).$$

Then $\Omega_{A/K}$ is the quotient

$$\frac{Adx_1 \oplus \cdots \oplus Adx_n \oplus Ady}{\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(\overline{x_1}, \dots, \overline{x_n}, \overline{y})dx_i + \frac{\partial g}{\partial y}(\overline{x_1}, \dots, \overline{x_n}, \overline{y})dy}.$$

Since $\frac{\partial g}{\partial y}(\overline{x_1}, \dots, \overline{x_n}, \overline{y}) \neq 0$, $\Omega_{A/K}$ is a free A-module of rank n, so

$$\dim_L \Omega_{L/K} = n,$$

completing the proof. $\square$

Proposition 7.3.9. *If $f \colon X \to Y$ is a dominant morphism of irreducible varieties such that $k(X)/k(Y)$ is separable then there is a non-empty open subset $V \subseteq X$ such that $V \to Y$ is smooth of relative dimension* $\dim X - \dim Y$.

Proof. After replacing X and Y by suitable open subsets, we may assume that X and Y are smooth.

By Lemma 7.3.8:

$$\dim_{k(X)} \Omega_{k(X)/k(Y)} = \operatorname{trdeg}(k(X)/k(Y)).$$

Note that

$$\Omega_{k(X)/k(Y)} = \left(\omega_{X/Y}\right)_X,$$

the stalk of the sheaf with respect to X. Therefore, there exists $U \subseteq X$ such that $\Omega_{X/Y}|_U$ is locally free of rank equal to $\dim X - \dim Y$.

This shows that for any $x \in U$, the map

$$T_x X \to T_{f(x)} Y$$

is surjective, and hence $U \to Y$ is smooth by Proposition 7.3.3. $\square$

Proposition 7.3.10. *Assume* $\operatorname{char}(k) = 0$ *and let $f \colon X \to Y$ be a morphism. For $r \geq 0$, consider:*

$$Z_r = \{x \in X \mid \operatorname{rank}(T_x X \to T_{f(x)} Y) \leq r\} \subseteq X.$$

Then every irreducible component of $\overline{f(Z_r)}$ has dimension at most r.

Proof. Let W be an irreducible component of $\overline{f(Z_r)}$ and let $Z \subseteq \overline{Z_r}$ be an irreducible component such that $\overline{f(Z)} \subseteq W$. Consider the commuting square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, hook] & Y \end{tikzcd}$$

By Proposition 7.3.9 (which applies because $\operatorname{char}(k) = 0$), there is an open subset $U \subseteq Z_{\mathrm{sm}} \cap g^{-1}(W_{\mathrm{sm}})$ such that for any $x \in U$, the map

$$dg_x \colon T_x Z \twoheadrightarrow T_{g(x)} W.$$

Taking tangent spaces of the square above, we get the square

$$\begin{tikzcd} T_x Z \arrow[r, hook] \arrow[d, two heads, "dg_x"'] & X \arrow[d, "df_x"] \\ T_{g(x)} W \arrow[r, hook] & T_{f(x)} Y \end{tikzcd}$$

If $x \in Z_r \cap U$, then $\dim T_{g(x)} W \leq r$, so $\dim W \leq r$. $\square$

Proof of Generic Smoothness Theorem 7.3.5. Suppose $\operatorname{char}(k) = 0$ and $f \colon X \to Y$ is a dominant map between irreducible varieties X and Y with X smooth. After replacing Y by an open subset, we may also assume that Y is smooth.

Taking $r = \dim Y - 1$ in Proposition 7.3.10, every irreducible component of $\overline{f(Z_r)}$ has dimension at most $\dim Y - 1$. Therefore $\overline{f(Z_r)} \neq Y$, and if $U = Y \setminus \overline{f(Z_r)}$, then for any $x \in f^{-1}(U)$, the map

$$T_x X \to T_{f(x)} Y$$

is surjective. Hence the map $f^{-1}(U) \to U$ is smooth. $\square$

8. Formal functions theorem

8.1. **Statement and consequences.** The goal is to understand for a proper morphism $f \colon X \to Y$ the stalk $R^i f_*(\mathcal{F})_y$ when $\mathcal{F} \in \operatorname{Coh}(X)$.

In the classical setting, we have the following theorem.

Theorem 8.1.1. *If $f \colon X \to Y$ is a proper continuous map (i.e. the preimages of compact sets are compact) between locally compact topological spaces, then for any $y \in Y$ there is an isomorphism $R^i f_*(\mathcal{F})_y \cong H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$ for any sheaf $\mathcal{F}$ of abelian groups on X.*

In the algebraic setting, the picture is more complicated. It can be summarized in two results.

(1) Formal function theorem: for proper morphisms, describes $\widehat{R^i f_*(\mathcal{F})}_y$ in terms of an inverse limit of sheaves supported on $f^{-1}(y)$.
(2) Base change theorems under certain conditions ($\mathcal{F}$ flat over Y and more):

$$R^i f_*(\mathcal{F})_{(y)} \cong H^i(X, \mathcal{F} \otimes \mathcal{O}_X / \mathfrak{m}_y \mathcal{O}_X).$$

We will focus on the first theorem. We begin with a formal statement.

Let $f \colon X \to Y$ be a proper morphism, $\mathcal{I} \subseteq \mathcal{O}_Y$ be a coherent ideal, and $\mathcal{F} \in \operatorname{Coh}(X)$. For $i \geq 0$, consider

$$\mathcal{F}_i = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{I}^{i+1} \mathcal{O}_X.$$

Note that $\mathrm{supp}(\mathcal{F}_i) \subseteq f^{-1}(V(\mathcal{I}))$. For $i \geq 0$, we have surjective maps $f \twoheadrightarrow \mathcal{F}_i$. Since f is proper, we get morphisms

$$R^q f_*(\mathcal{F}) \to R^q f_*(\mathcal{F}_i)$$

between coherent sheaves on Y and the codomain is annihilated by $\mathcal{I}^{i+1}$. Therefore, we have an induced map

$$R^q f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^{i+1} \to R^q f_*(\mathcal{F}_i).$$

Theorem 8.1.2 (Formal Functions theorem)**.** *These maps induce an isomorphism*

$$\varprojlim \left(R^q f_*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}^{i+1}\right) \to \varprojlim R^q f_*(\mathcal{F}_i).$$

Example 8.1.3. Suppose Y is affine and $R = \mathcal{O}(Y)$, $I = \mathcal{I}(Y)$. Then the Formal Functions Theorem 8.1.2 gives an isomorphism

$$\underbrace{\varprojlim H^q(X, \mathcal{F}) \otimes_R R/I^{i+1}}_{H^q(X,\mathcal{F})^\wedge} \to \varprojlim H^q(X, \mathcal{F}_i),$$

where $H^q(X, \mathcal{F})^\wedge$ is the completion with respect to I.

Example 8.1.4. Suppose $y \in Y$ is a point and let $\mathcal{I} = \mathfrak{m}_y$ be the radical ideal defining y. If $U \ni y$ is an affine open neighborhood, then the sections over U of the isomorphism in the Formal Functions Theorem 8.1.2 gives an isomorphism

$$\underbrace{H^q(f^{-1}(U), \mathcal{F})^\wedge}_{\cong (R^q f_*(\mathcal{F}))^\wedge_y} \to \underbrace{\varprojlim H^q(f^{-1}(U), \mathcal{F}_i|_{f^{-1}(U)})}_{\varprojlim H^q(X,\mathcal{F}_i)}.$$

Remark 8.1.5. The reason for the *formal functions* in the name is that the theorem can be restated as computing the global sections of a formal scheme. We do not do this here.

Corollary 8.1.6 (Zariski's Main Theorem)**.** *If $f\colon X \to Y$ is a proper morphism of algebraic varieties such that $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an isomorphism, then f has connected fibers.*

Proof. Let $y \in Y$. Suppose $W = f^{-1}(y) = W_1 \amalg W_2$ for W_1, W_2 open in W. We may assume that Y is affine. The Formal Functions Theorem 8.1.2 for $\mathcal{F} = \mathcal{O}_X$ and $q = 0$ shows that

$$(f_{*-}X)^\wedge_y \cong \varprojlim \Gamma(X, \mathcal{O}_{X_i})$$

where $\mathcal{O}_{X_i} = \mathcal{O}_X/\mathfrak{m}_y^{i+1}\mathcal{O}_X$. By assumption, $(f_*\mathcal{O}_X)^\wedge_y \cong \widehat{\mathcal{O}_{Y,y}}$.

Let $j\colon W \hookrightarrow X$ be the inclusion map. Since all $\mathcal{O}_{X_i}$ are supported on W, $\mathcal{O}_{X_i} = j_*(f^{-1}(\mathcal{O}_{X_i}))$ and $f^{-1}(\mathcal{O}_{X_i})$ is a sheaf on W. Then

$$\Gamma(X, \mathcal{O}_{X_i}) = A_i \times B_i$$

where $A_i = \Gamma(W_1, f^{-1}(\mathcal{O}_{X_i}))$, $B_i = \Gamma(W_2, j^{-1}(\mathcal{O}_{X_i}))$. Both A_i and B_i are both nonzero k-algebras. Taking $\varprojlim$, we see that

$$\varprojlim \Gamma(X, \mathcal{O}_{X_i}) \cong A \times B$$

for nonzero k-algebras $A = \varprojlim A_i$, $B = \varprojlim B_i$. They are non-zero since they contain a copy of k.

However, $\widehat{\mathcal{O}_{Y,y}}$ is a local ring, so it has no non-trivial decomposition as a product of rings, which is a contradiction. $\square$

Corollary 8.1.7. *Suppose that $f\colon X\to Y$ is a birational proper morphism between irreducible varieties with Y normal. Then $f_*\mathcal{O}_X=\mathcal{O}_Y$, and hence f has connected fibers by Zariski's Main Theorem 8.1.6.*

Proof. We show that $f_*\mathcal{O}_X=\mathcal{O}_Y$. We may assume that Y is affine. Let $U\subseteq X$ and $V\subseteq Y$ be open subsets such that f induces an isomorphism $U\to V$. Then the diagram commutes

$$\begin{tikzcd} \Gamma(Y,\mathcal{O}_Y) \arrow[r, hook] \arrow[d] & \Gamma(X,\mathcal{O}_X) \arrow[d, hook] \\ \Gamma(V,\mathcal{O}_Y) \arrow[r, "\cong"] & \Gamma(U,\mathcal{O}_X) \end{tikzcd}$$

so $\Gamma(Y,\mathcal{O}_Y)\hookrightarrow\Gamma(X,\mathcal{O}_X)$. We then have

$$\begin{tikzcd} \Gamma(Y,\mathcal{O}_Y) \arrow[rr, hook] \arrow[dr, hook] & & \Gamma(X,\mathcal{O}_X) \arrow[dl, hook'] \\ & k(Y)=k(X) & \end{tikzcd}$$

and $\Gamma(X,\mathcal{O}_X)$ is a finitely-generated $\Gamma(Y,\mathcal{O}_Y)$-module since f is proper. Hence $\Gamma(X,\mathcal{O}_X)$ is contained in the integral closure of $\Gamma(Y,\mathcal{O}_Y)$ in $k(Y)$, which is just $\Gamma(Y,\mathcal{O}_Y)$, since Y is normal. $\square$

Corollary 8.1.8 (Stein factorization)**.** *Given $f\colon X\to Y$ proper, there is a decomposition*

$$X\xrightarrow{g}Z\xrightarrow{u}Y$$

such that

- *$g_*\mathcal{O}_X=\mathcal{O}_Z$ (so g has connected fibers),*
- *u is finite.*

Exercise. Check that this is unique: if $X\xrightarrow{h}W\xrightarrow{v}Y$ is another such decomposition, then there is an isomorphism $\alpha\colon W\to Z$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & W \arrow[uu, "\alpha"] \arrow[ur] & \end{tikzcd}$$

commutes.

Proof. Let $\mathcal{A}=f_*\mathcal{O}_X$, which is a coherent $\mathcal{O}_Y$-algebra. If $U\subseteq Y$ is affine then $\mathcal{A}(U)=\mathcal{O}_X(f^{-1}(U))$ be a reduced ring, so $\mathcal{A}$ is a reduced $\mathcal{O}_X$-algebra. Let

$$Z=\mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A})\xrightarrow{u}Y.$$

Giving a morphism g

$$\begin{tikzcd} X \arrow[rr, dotted, "g"] \arrow[dr, "f"'] & & Z = \mathcal{M}\mathrm{ax}\mathcal{S}\mathrm{pec}(\mathcal{A}) \arrow[dl, "u"] \\ & Y & \end{tikzcd}$$

is the same as giving a morphism of $\mathcal{O}_X$-algebras $\mathcal{A} \to f_*\mathcal{O}_X$.

We claim that $g_*\mathcal{O}_X = \mathcal{O}_Z$. If $U \subseteq Y$ is open, then

$$u^{-1}(U) = \mathrm{MaxSpec}(\mathcal{O}_X(f^{-1}(U)))$$

and

$$\mathcal{O}(f^{-1}(U)) = \mathcal{A}(U) = \mathcal{O}(u^{-1}(U)) \to \mathcal{O}(g^{-1}(u^{-1}(U))) = \mathcal{O}(f^{-1}(U))$$

is the identity map. □

Corollary 8.1.9. *If $f\colon X \to Y$ is a proper morphism with finite fibers, then f is finite.*

Proof. Consider the Stein factorization:

$$X \xrightarrow{g} Z \xrightarrow{u} Y$$

with u finite and $g_*\mathcal{O}_X = \mathcal{O}_Z$. It is enough to show that g is an isomorphism. We know that the fibers of g are contained in the fibers of f, so g has finite fibers. Since g is proper, Zariski's Main Theorem 8.1.6 shows that g has connected fibers. Hence g is injective.

Since g is proper, it is closed, and hence it gives a homeomorphism onto a closed subvariety $Z' \subseteq Z$. Letting $j\colon Z' \hookrightarrow Z$ be the inclusion map, we then have

$$\begin{tikzcd} \mathcal{O}_Z \arrow[r] \arrow[rr, bend left, "\cong"] & j_*\mathcal{O}_{Z'} \arrow[r] & g_*\mathcal{O}_X \end{tikzcd}$$

so the first map $\mathcal{O}_Z \to j_*\mathcal{O}_{Z'}$ is injective. The kernel is the radical ideal corresponding to Z', so $Z' = Z$.

Since g is a homeomorphism onto Z such that $g_*\mathcal{O}_X = \mathcal{O}_Z$, g is an isomorphism. □

We give another corollary of the Formal Functions Theorem 8.1.2.

Corollary 8.1.10. *If $f\colon X \to Y$ is a proper morphism and $\mathcal{F} \in \mathrm{Coh}(X)$ such that $\dim(\mathrm{supp}(\mathcal{F}) \cap f^{-1}(y)) \leq n$ for all $y \in Y$, then*

$$R^q f_*(\mathcal{F}) = 0 \text{ for } q > n.$$

Proof. It is enough to show that $R^q f_*(\mathcal{F})_y = 0$ for all $y \in Y$ and $q > n$. In fact, since this module embeds into its completion, we just need to show that

$$R^q f_*(\mathcal{F})_y^\wedge = 0$$

for all $y \in Y$ and $q > n$.

By the Formal Functions Theorem 8.1.2, it is enough to show that

$$H^q(X, \mathcal{F}_i) = 0 \text{ for all } q > n.$$

Note that $\operatorname{supp}(\mathcal{F}_i) \subseteq \operatorname{supp}(\mathcal{F}) \cap f^{-1}(y)$. Since $\mathcal{F}_i$ has a finite filtration with quotients that are push-forwards from $\operatorname{supp}(\mathcal{F}) \cap f^{-1}(y)$, it is enough to show that if W is a complete n-dimensional variety and $\mathcal{F} \in \operatorname{Coh}(W)$, then $H^q(W, \mathcal{F}) = 0$ for $q > n$.

We proved this for projective varieties. To get the general case, we use Chow's Lemma to construct $\widetilde{W} \to W$ which is an isomorphism over dense open subsets, where $\overline{W}$ is a projective variety. The complete proof is in the official notes. □

Example 8.1.11. Recall that if $f \colon X \to \mathbb{P}^n$ is finite, then $f^*\mathcal{O}(1)$ is globally generated and ample. Conversely, if X is a projective variety and $\mathcal{L}$ is globally generated and ample, then we get a morphism $f \colon X \to \mathbb{P}^n$ such that $f^*\mathcal{O}(1) \cong \mathcal{L}$, and every such f is a finite morphism. Indeed, note that f is proper and has finite fibers: if $Z = f^{-1}(y)$, we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{P}^n \\ Z \arrow[u, hook] \arrow[r] & \{y\} \arrow[u, hook] \end{tikzcd}$$

so $\mathcal{L}|_Z \cong \mathcal{O}_Z$ and $\mathcal{L}|_Z$ is ample, so Z is finite. Therefore, by Corollary 8.1.9, f is finite.

8.2. **Proof of the Formal Functions Theorem 8.1.2.** Suppose X is a variety and $\mathcal{S}$ is an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra. We say that $\mathcal{S}$ satisfies $(*)$ if it is quasicoherent, reduced and locally generated over $\mathcal{S}_0$ by $\mathcal{S}_1$, and $\mathcal{S}_0$ and $\mathcal{S}_1$ are coherent $\mathcal{O}_X$-modules.

We suppose $\mathcal{S}$ satisfies $(*)$. Recall that we get a variety $\begin{tikzcd} \widetilde{X} = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S}) \arrow[d] \\ X \end{tikzcd}$. If $\mathcal{M}$ is a $\mathbb{Z}$-graded $\mathcal{S}$-module which is quasicoherent over $\mathcal{O}_X$, then we get a quasicoherent $\widetilde{\mathcal{M}}$ sheaf on $\widetilde{X}$. For example, $\widetilde{\mathcal{S}(1)} = \mathcal{O}_{\widetilde{X}}(1)$.

If $\mathcal{M}$ is locally finitely-generated over $\mathcal{S}$, $\widetilde{\mathcal{M}}$ is coherent.

Proposition 8.2.1. *Let $f \colon X \to Y$ be a proper morphism. Suppose $\mathcal{T}$ is an $\mathcal{O}_Y$-algebra that satisfies $(*)$ and $\mathcal{S}$ is an $\mathcal{O}_X$-algebra that satisfies $(*)$, and we have a surjective morphism $f^*\mathcal{T} \to \mathcal{S}$. If $\mathcal{M}$ is a graded $\mathcal{S}$-module on X which is quasicoherent over $\mathcal{O}_X$ and locally finitely generated over $\mathcal{S}$, then for all $q \geq 0$, $R^q f_*(\mathcal{M})$ is locally finitely-generated over $\mathcal{I}$.*

Note that cohomology commutes with arbitrary direct sums for quasicoherent sheaves (e.g. one can compute the cohomology using Čech cohomology).

Proof. By assumption, letting $\widetilde{X} = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{S})$, $\widetilde{Y} = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\mathcal{T})$, we have a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "u"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "v"] & Y \end{tikzcd}$$

such that $g^*\mathcal{O}_{\widetilde{Y}} \cong \mathcal{O}_{\widetilde{X}}(1)$. Let $\widetilde{\mathcal{M}}$ be the coherent sheaf on $\widetilde{X}$ corresponding to $\mathcal{M}$. Note that if $\mathcal{M} = \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_i$ then

$$R^q f_*(\mathcal{M}) = \bigoplus_{i\in\mathbb{Z}} R^q f_*(\mathcal{M}_i)$$

and each of $R^q f_*(\mathcal{M}_i)$ is a coherent $\mathcal{O}_Y$-module (since each component $\mathcal{M}_i$ is coherent). Since $\mathcal{M}$is locally finitely-generated over $\mathcal{S}$, $\mathcal{M}_i = 0$ for $i \ll 0$, so it is enough to show that

$$\bigoplus_{i\geq 0} R^q f_*(\mathcal{M}_i) \text{ is locally finitely-generated over } \mathcal{T}.$$

Let

$$\mathcal{P} = \bigoplus_{i\geq 0}(\widetilde{M} \otimes \mathcal{O}_{\widetilde{X}}(i)).$$

We use the two Leray spectral sequences corresponding to $f \circ u = v \circ g$ and $\mathcal{P}$.

First, we see that

$$E_2^{p,q} = R^p v_*(R^q g_*(\mathcal{P})) \Rightarrow_p R^{p+q}(v \circ g)_*(\mathcal{P})$$

and

$$R^q g_*(\mathcal{P}) = \bigoplus_{i\geq 0} R^q g_*(\widetilde{M} \otimes g^*\mathcal{O}_{\widetilde{Y}}(i)) \cong \bigoplus_{i\geq 0} \underbrace{R^q g_*(\widetilde{M})}_{\text{coherent } \mathcal{O}_{\widetilde{Y}}\text{-module}} \otimes \mathcal{O}_{\widetilde{Y}}(i).$$

For $p > 0$, $E_2^{p,q}$ is a coherent $\mathcal{O}_Y$-module (by asymptotic vanishing). For $p = 0$, we get a $\mathcal{T}$-module which is locally finitely-generated. Then the spectral sequence shows that

$$R^d(v \circ g)_*(\mathcal{P}) \text{ is locally finitely-generated over } \mathcal{T}$$

for all d.

The second spectral sequence is

$$\overline{E}_2^{p,q} = R^p f_*(R^q u_*(\mathcal{P})) \Rightarrow_p R^{p+q}(f \circ u)_*(\mathcal{P}).$$

We have that

$$\overline{E}_2^{p,q} = R^p f_* \Bigg(\underbrace{\bigoplus_{i\geq 0} R^q u_*(\widetilde{M} \otimes \mathcal{O}_{\widetilde{X}}(1))}_{\substack{\text{coherent } \mathcal{O}_X\text{-module for } q>0 \\ \bigoplus_{i\geq 0} \mathcal{M}_i \text{ in all but finitely} \\ \text{many degrees for } q=0}} \Bigg).$$

Hence it is enough to show that $\overline{E}_2^{p,q}$ is locally finitely-generated over $\mathcal{T}$ for all p. We know that $\overline{E}_2^{p,q}$ is a coherent $\mathcal{O}_Y$-module (in particular, it is locally finitely generated over $\mathcal{T}$) and $\overline{E}_\infty^{p,q}$ is locally finitely-generated over $\mathcal{T}$ for all p, q (since it is a subquotient of $R^{p+q}(v\circ g)_*(\mathcal{P})$). Since $\overline{E}_r^{p,0} = \overline{E}_\infty^{p,0}$ for all $r \gg 0$, it is enough to show that for $r \geq 2$ if $\overline{E}_{r+1}^{p,0}$ is locally finitely-generated over $\mathcal{T}$, then $\overline{E}_r^{p,0}$ is locally finitely-generated over $\mathcal{T}$. We have the sequence

$$\overline{E}_r^{p-r,r-1} \xrightarrow{\varphi} \overline{E}_r^{p,0} \longrightarrow \overline{E}_r^{p+r,1-r} = 0.$$

We know $\overline{E}_{r+1}^{p,0} = \operatorname{coker}(\varphi)$ is locally finitely-generated over $\mathcal{T}$ and $\overline{E}_r^{p-r,r-1}$ is locally finitely-generated over $\mathcal{T}$, since it is a subquotient of $\overline{E}_2^{p-r,r-1}$ for $r-1>0$. This shows that $\overline{E}_r^{p,0}$ is locally finitely-generated over $\mathcal{T}$. $\square$

Proof of the Formal Functions Theorem 8.1.2. Recall that $f\colon X \to Y$ is a proper morphism, $\mathcal{F}$ is a coherent sheaf on X, $\mathcal{I}$ is a coherent ideal on Y, and $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{O}_X/\mathcal{I}^{i+1}\mathcal{O}_X$. We want to show that

$$\varprojlim R^q f_*(\mathcal{F})/\mathcal{I}^{i+1} R^q f_*(\mathcal{F}) \to \varprojlim R^q f_*(\mathcal{F}_i).$$

We may assume that Y is affine, $R = \mathcal{O}(Y)$, $I = \mathcal{I}(Y)$. Letting $T = \bigoplus_{n\geq 0} I^n$,

$$\mathcal{T} = \widetilde{T} = \bigoplus_{n\geq 0} \mathcal{I}^n,$$

$$\mathcal{S} = \bigoplus_{n\geq 0} \mathcal{I}^n \mathcal{O}_X$$

satisfy the condition $(*)$. The module $\mathcal{M} = \bigoplus_{n\geq 0} \mathcal{I}^{n+1}\mathcal{F}$ is quasicoherent over $\mathcal{O}_X$ and locally finitely-generated over $\mathcal{S}$. Then Proposition 8.2.1 shows that

$$N^{(q)} = \bigoplus_{n\geq 0} H^q(X, \mathcal{I}^{n+1}\mathcal{F})$$

is a finitely-generated T-module. We want to show that

$$\varprojlim H^q(X,\mathcal{F})/I^{i+1}H^q(X,\mathcal{F}) \stackrel{\cong}{\to} \varprojlim H^q(X,\mathcal{F}_i).$$

We have the exact sequence

$$0 \longrightarrow \mathcal{I}^{i+1}\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_i \longrightarrow 0$$

and the long exact sequence in cohomology gives

$$0 \longrightarrow A_i \longrightarrow H^q(X,\mathcal{F}) \longrightarrow H^q(X,\mathcal{F}_i) \longrightarrow B_i \longrightarrow 0$$

where

$$A_i = \operatorname{im}(H^q(\mathcal{I}^{i+1}\mathcal{F}) \to H^q(\mathcal{F})),$$
$$B_i = \operatorname{im}(H^q(\mathcal{F}_i) \to H^{q+1}(\mathcal{I}^{i+1}(\mathcal{F}))).$$

Since taking $\varprojlim$ is left-exact, we get an exact sequence

$$0 \longrightarrow \varprojlim (H^q(X,\mathcal{F})/A_i) \longrightarrow \varprojlim (H^q(X,\mathcal{F}_i)) \longrightarrow \varprojlim B_i.$$

We just need to show that $\varprojlim B_i = 0$ and $\varprojlim(H^q(X,\mathcal{F})/A_i) \cong \varprojlim(H^q(X,\mathcal{F})/I^{i+1}H^q(X,\mathcal{F}))$. If $f \in T_m = I^m$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^{i+1}\mathcal{F} \arrow[r] \arrow[d, "\cdot f"] & \mathcal{F} \arrow[r] \arrow[d, "\cdot f"] & \mathcal{F}_i \arrow[r] \arrow[d, "\cdot f"] & 0 \\
0 \arrow[r] & \mathcal{I}^{i+m+1}\mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_{m+i} \arrow[r] & 0
\end{tikzcd}$$

which gives a commuting square

$$\begin{tikzcd}
H^q(\mathcal{F}_i) \arrow[r] \arrow[d] & H^{q+1}(\mathcal{I}^{i+1}\mathcal{F}) \arrow[d] \\
H^q(\mathcal{F}_{m+1}) \arrow[r] & H^{q+1}(\mathcal{I}^{i+m+1}\mathcal{F}).
\end{tikzcd}$$

This gives a map $B_i \to B_{m+i}$ and $\bigoplus_{i\geq 0} B_i$ is a T-module, a submodule of N^{q+1}, so it is finitely-generated over T. Therefore, there is an i_1 such that $B_{m+i_1} = T_m B_{i_1}$ for all $m \geq 0$.

Exercise. For any $f \in T_m = I^m$, if $B_{i+m} \to B_i$ is the structural map of the inverse system, then the composition

$$B_i \xrightarrow{\cdot f} B_{i+m} \longrightarrow B_i$$

is the usual multiplication by f on B_i.

Recall that $H^q(X,\mathcal{F}_i) \twoheadrightarrow B_i$ and $H^q(X,\mathcal{F}_i)$ is annihilated by I^{i+1}, so B_i is annihilated by I^{i+1}.

We claim that if $m \geq i_1 + 1$, then the map $B_{i+m} \to B_i$ is 0. This is clear from the above, since element in B_{i+m} lies in $T_{i+m-i_1} \cdot B_{i_1}$.

Altogether, we conclude that $\varprojlim B_i = 0$.

The argument showing that $\varprojlim(H^q(X,\mathcal{F})/A_i) \cong \varprojlim(H^q(X,\mathcal{F})/I^{i+1}H^q(X,\mathcal{F}))$ is similar, so we omit this here. (It can be found in the official notes.) $\square$

9. Serre duality

Serre duality is the analogue of Poincaré duality in the algebraic setting. It can be stated for complete varieties, but we only discuss it here for projective varieties.

9.1. **Preliminaries.**

Definition 9.1.1. Let X be a projective variety of dimension n. We say that *Serre duality holds for X* if there exists $\omega_X^0 \in \mathrm{Coh}(X)$ such that there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \cong H^{n-i}$$

for any coherent sheaf $\mathcal{F}$ on X and any $i \geq 0$.

Note that both sides are contravariant δ-functors $\mathrm{Coh}(X) \to \mathrm{Vect}_k$. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence and $\omega_X^0 \to \mathcal{I}^\bullet$ is an injective reslution, we get a short exact sequence of complexes

$$0 \longrightarrow \mathrm{Hom}(\mathcal{F}'', \mathcal{I}^\bullet) \longrightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{I}^\bullet) \longrightarrow \mathrm{Hom}(\mathcal{F}', \mathcal{I}^\bullet) \longrightarrow 0$$

so taking long exact sequence in cohomology shows that the left hand side is a δ-functor. The right hand side is a δ-functor since $H^i(X, -) = 0$ for $i > n$. For Serre duality, we actually require an isomorphism of δ-functors.

Definition 9.1.2. A contravariant δ-functor $(F_i)_{i \geq 0}$ is *co-effaceable* if for any A and $i > 0$, there is a surjection $B \twoheadrightarrow A$ such that $F_i(B) = 0$.

Lemma 9.1.3. *Any co-effaceable δ-functor $F = (F_i)_{i \geq 0}$ is universal, i.e. for any contravariable δ-functor $G = (G_i)_{i \geq 0}$, the natural map*

$$\mathrm{Hom}_{\delta\text{-}funct}(F, G) \to \mathrm{Hom}_{funct}(F_0, G_0)$$

is an isomorphism.

Proof. Given $\alpha_0 \colon F_0 \to G_0$, construct $\alpha_i \colon F_i \to G_i$ by recursion. Suppose we have α_i. Given any A, consider a short exact sequence

$$0 \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow 0$$

such that $F_{i+1}(B) = 0$. Then we have a diagram

$$\begin{tikzcd}
F_i(B) \arrow[r] \arrow[d, "\alpha_i^B"] & F_i(C) \arrow[r] \arrow[d, "\alpha_i^C"] & F_{i+1}(A) \arrow[r] \arrow[d, dotted] & F_{i+1}(B) = 0 \\
G_i(B) \arrow[r] & G_i(C) \arrow[r] & G_{i+1}(A) &
\end{tikzcd}$$

where we get the unique dotted map $F_{i+1}(A) \to G_{i+1}(A)$ by the universal property of the cokernel.

Exercise. Check all the details to finish this proof. $\square$

9.2. **Examples of Serre duality.** Note that in our setting, for every ω_X^0, the δ-functor $\{\mathrm{Ext}^i_{\mathcal{O}_X}(-, \omega_X^0)\}_{i \geq 0}$ is co-effaceable. Given any $\mathcal{F}$ and an ample line bundle $\mathcal{L}$ on X, there is a surjection

$$\left(\mathcal{L}^{-q}\right)^{\oplus ?} \twoheadrightarrow \mathcal{F}$$

for all $q \gg 0$. Moreover, note that

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{L}^{-q}, \omega_X^0) \cong H^i(\omega_X^0 \otimes \mathcal{L}^q) = 0 \text{ for } i > 0 \text{ if } q \gg 0.$$

This shows that Serre duality holds for X if and only if

(1) there is an ω_X^0 such that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) \cong H^n(X, \mathcal{F})^\vee$ for all $\mathcal{F} \in \mathrm{Coh}(X)$,

(2) $\{H^{n-1}(-)^\vee\}_{i\geq 0}$ is co-effaceable.

Definition 9.2.1. A *dualizing sheaf* on X is a coherent sheaf ω_X^0 that represents the function $\mathcal{F} \mapsto H^n(X, \mathcal{F})^\vee$.

Proposition 9.2.2. *Serre duality holds for $\mathbb{P}^n$ with $\omega^0_{\mathbb{P}^n} = \omega_{\mathbb{P}^n}$.*

Proof. Condition (2) holds: for any coherent sheaf there is a surjective morphisms
$$\mathcal{O}(-q)^{\oplus ?} \twoheadrightarrow \mathcal{F}$$
for $q \gg 0$, and
$$H^{n-i}(\mathbb{P}^n, \mathcal{O}(-q)) = 0 \text{ for } i > 0 \text{ and } q > 0.$$

We show that condition (1) holds. We showed that $H^n(\mathbb{P}^n, \omega_{\mathbb{P}^n}) \cong k$. For any $\mathcal{F} \in \mathrm{Coh}(X)$, we hence have a map
$$\mathrm{Hom}(\mathcal{F}, \omega_{\mathbb{P}^n}) \to \mathrm{Hom}(H^n(\mathcal{F}), H^n(\omega)) = H^n(\mathcal{F})^\vee$$
and this is an isomorphism if $\mathcal{F} = \mathcal{O}(m)$ for some $m \in \mathbb{Z}$. Every $\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^n)$ has a presentation
$$\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$
where $\mathcal{E}_0$, $\mathcal{E}_1$ are direct sums of line bundles.

Since both $\mathrm{Hom}(-, \omega_{\mathbb{P}^n})$ and $H^n(-)^\vee$ are left exact, this completes the proof. $\square$

For a general projective variety, proving the Serre duality amount to proving the two conditions (1) and (2) separately. We begin with (1): finding a dualizing sheaf.

Proposition 9.2.3. *If $X \subseteq \mathbb{P}^N$ is a closed subvariety of dimension n and $r = N - n$, then $\omega^0_X = \mathrm{Ext}^r_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ is a dualizing sheaf on X.*

To simplify notation, we will sometimes write $\mathbb{P}$ for $\mathbb{P}^N$.

Proof. Note that $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = 0$ for $i < r$. The stalk at $x \in X$ is
$$\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \omega_{\mathbb{P},x})$$
and $\omega_{\mathbb{P},x} = \mathcal{O}_{\mathbb{P},x}$. Vanishing for $i < r$ is equivalent to $\mathrm{codim}(I_{X,x}) = \mathrm{depth}(I_{X,x}, \mathcal{O}_{X,x}) \geq r$. This is clear.

If $\mathcal{F} \in \mathrm{Coh}(X)$,
$$H^n(X, \mathcal{F})^\vee \cong H^n(\mathbb{P}^n, \mathcal{F})^\vee \cong \mathrm{Ext}^n_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \omega_{\mathbb{P}}).$$
Choosing an injective resolution on $\mathbb{P}^N$
$$0 \longrightarrow \omega_{\mathbb{P}^n} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots,$$
we get an exact sequence
$$(*) \qquad 0 \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{I}^0) \longrightarrow \cdots \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{I}^{r-1}) \longrightarrow Q \longrightarrow 0$$
where $Q = \mathrm{im}(\mathcal{H}om(\mathcal{O}_X, \mathcal{I}^{r-1}) \to \mathcal{H}om(\mathcal{O}_X, \mathcal{I}^r))$, and we have

$$0 \longrightarrow Q \longrightarrow \mathcal{H}om(\mathcal{O}_X, \mathcal{I}^r) \xrightarrow{\alpha} \mathcal{H}om(\mathcal{O}_X, \mathcal{I}^{r+1})$$

Since $(*)$ is an exact complex, $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{I}^j)$ is an injective $\mathcal{O}_X$-module.

In particular, $(*)$ is split exact and Q is an injective $\mathcal{O}_X$-module. Then $\omega_X^0 = \ker(\alpha)/Q$ and

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^0) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \ker(\alpha)/Q).$$

We have the complex

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{I}^{r-1}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{I}^{r}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{I}^{r+1})$$

and $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{I}^r) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{I}^r))$, so the cohomology is

$$\frac{\mathrm{Hom}(\mathcal{F}, \ker(\alpha))}{\mathrm{Hom}(\mathcal{F}, Q)} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \frac{\ker(\alpha)}{Q}).$$

Since Q is injective, the short exact sequence

$$0 \longrightarrow Q \longrightarrow \ker(\alpha) \longrightarrow \ker(\alpha)/Q \longrightarrow 0$$

is split. □

Remark 9.2.4. If Serre duality holds on X,

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X^0) \cong H^{n-1}(\mathcal{F})^\vee.$$

If $\mathcal{F}$ is locally free,

$$H^i(\omega_X^0 \otimes \mathcal{F}^\vee) \cong H^{n-i}(\mathcal{F})^\vee.$$

This is the form of Serre duality that we usually apply it in.

Theorem 9.2.5. *Let $X \subseteq \mathbb{P}^N$ be a closed subvariety of dimension n. Then the following are equivalent:*

1. *X satisfies Serre duality,*
2. *X is Cohen–Macaulay of pure dimension,*
3. *for any locally free sheaf $\mathcal{E}$ on X, $H^i(X, \mathcal{E}(-q)) = 0$ for all $i < n$ and $q \gg 0$.*

We will need the following lemma in the proof.

Lemma 9.2.6. *Let $x \in X$ be a smooth point and M be a finitely-generated $\mathcal{O}_{X,x}$-module. Then $\mathrm{pd}_{\mathcal{O}_{X,x}} M \leq r$ if and only if $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(M, \mathcal{O}_{x,x}) = 0$ for $i > r$.*

We omit the proof of the lemma for now and prove the theorem.

Proof of Theorem 9.2.5. By Proposition 9.2.3, X satisfies Serre duality if and only if the δ-functor $\{H^{n-i}(-)^\vee\}_{i\geq 0}$ is co-effaceable.

To see that (1) implies (3), note that Serre duality shows that

$$H^i(X, \mathcal{E}(-q)) \cong H^{n-i}(X, \omega_X^0 \otimes \mathcal{E}^\vee(q)) = 0$$

for $i < n$ and $q \gg 0$. Conversely, to show (3) implies (1), use that for every $\mathcal{F}$, there is a surjective map $\mathcal{O}(-q)^{\oplus ?} \twoheadrightarrow \mathcal{F}$.

We just need to show that (2) is equivalent to (3). Recall that X is Cohen–Macaulay of pure dimension n if and only if $\operatorname{depth} \mathcal{O}_{X,x} \geq n$ for all $x \in X$. Note that $\operatorname{depth} \mathcal{O}_{X,x} = \operatorname{depth}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x}$ and by Theorem 6.1.9

$$\operatorname{depth}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x} = N - \operatorname{pd}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x}.$$

By Lemma 9.2.6, $\operatorname{pd}_{\mathcal{O}_{\mathbb{P},x}} \mathcal{O}_{X,x} \leq r$ if and only if $\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{P},x}}(\mathcal{O}_{X,x}, \mathcal{O}_{\mathbb{P},x}) = 0$ for $i > r$ for all $x \in X$.

Altogether, we conclude that (2) is equivalent to

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}}) = 0$$

for $i > r$. Recall that $\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{E}) \cong \mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}}) \otimes \mathcal{E}$ for a locally free sheaf $\mathcal{E}$.

Recall that by Homework 11, Problem 3, we have a spectral sequence that gives for any coherent sheaf $\mathcal{F}$

$$E_2^{p,q} = H^p(\mathbb{P}^N, \mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \mathcal{F}(j))) \Rightarrow_p \operatorname{Ext}^{p+q}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \mathcal{F}(j)).$$

For $j \gg 0$ and $p > 0$, $E_2^{p,q} = 0$ and

$$\mathcal{E}xt^q_{\mathcal{O}^{\mathbb{P}^N}}(\mathcal{O}_X, \mathcal{F}) \otimes \mathcal{O}(j) \text{ is globally generated}$$

and

$$\operatorname{Ext}^m_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{F}(j)) \cong \Gamma(\mathbb{P}^N, \mathcal{E}xt^m_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \mathcal{F}(j))).$$

by Homework 11.

Take $\mathcal{F} = \omega_{\mathbb{P}^N}$.

We show that (3) implies (2). It is enough to show that (3) implies that

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(j)) = 0$$

for $j \gg 0$. By the above argument, to show this, it is enough to show that

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(j)) = 0.$$

By Serre duality on $\mathbb{P}^N$ (Proposition 9.2.2),

$$\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \omega_{\mathbb{P}}(j)) = H^{N-i}(\mathbb{P}^N, \mathcal{O}_X(-j))^\vee = H^{N-i}(X, \mathcal{O}_X(-j))$$

for $N - i < N - r = n$. This is 0 by condition (3).

The proof that (2) implies (3) is similar. We already showed that (2) is equivalent to

$$\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}}) = 0$$

for $i > r$. If (2) holds, then $\mathcal{E}xt^i_{\mathbb{P}}(\mathcal{E}^\vee, \mathcal{O}_{\mathbb{P}}) = 0$ for all $i > r$ and locally free sheaves $\mathcal{E}$. Reversing the above arugment, we conclude that

$$H^{n-i}(X, \mathcal{E}(-q)) = 0$$

for $i < n$ and $q \gg 0$. $\square$

While this shows that Serre duality holds for Cohen–Macaulay projective varieties of pure dimension, the dualizing sheaf ω_X^0 is in general hard to understand. However, in the case of smooth varieties, it is actually equal to the canonical bundle.

Proposition 9.2.7. *If $X \hookrightarrow \mathbb{P}^n$ is smooth, then $\omega_X^0 \cong \omega_X$.*

The ideal of the proof is to locally write down $X = Z(s)$ where $s \in \Gamma(X, \mathcal{E})$ is a regular section of a vector bundle, and then use the Koszul complex to compute $\mathcal{E}xt^n_{\mathcal{O}_\mathbb{P}}(\mathcal{O}_X, \omega_\mathbb{P})$. We omit this here, but it can be found in the official notes.

Remark 9.2.8. By the same argument as in the proof of Theorem 9.2.5 shows that if $\operatorname{depth}(\mathcal{O}_{X,x}) \geq 2$ for all $x \in X$ then for any locally free sheaf $\mathcal{E}$ on X:

$$H^1(X, \mathcal{E}(-q)) = 0$$

for $q \gg 0$.

Corollary 9.2.9. *If X is an irreducible, normal, projective variety and $\dim X \geq 2$, and D is an effective Cartier divisor such that $\mathcal{O}(D)$ is ample, then $\operatorname{supp}(D)$ is connected.*

Proof. Choose $m \geq 1$ such that $\mathcal{O}(mD)$ is very ample. Since $\operatorname{depth} \mathcal{O}_{X,x} \geq 2$ for all $x \ inX$, $H^1(X, \mathcal{O}_X(-qmD)) = 0$ for $q \gg 0$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{O}(-qmD) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{qmD} \longrightarrow 0$$

and the long exact sequence in cohomology gives

$$k = H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{qmD}) \to H^1(X, \mathcal{O}(-qmD)) = 0,$$

so $H^0(X, \mathcal{O}_{qmD}) = k$. Hence $\operatorname{supp}(D)$ is connected. □

10. Algebraic curves

A *curve* is an irreducible variety of dimension 1.

Recall that if X is a smooth projective curve, then Serre duality implies that if $\mathcal{E}$ is a locally free sheaf on X then

$$H^i(X, \mathcal{E})^\vee \cong H^{1-i}(X, \omega_X \otimes \mathcal{E}^\vee).$$

Example 10.0.1. If X is a complete curve, $p_a(X) = (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1) = h^1(X, \mathcal{O}_X)$. If X is smooth, Serre duality shows that $h^1(X, \mathcal{O}_X) = h^0(X, \omega_X)$, and hence

$$p_a(X) = p_g(X).$$

In this case, we simply call this invariant the *genus* of X.

Let X be a smooth, projective curve. If $D = \sum_{i=1}^r a_i P_i$ is a divisor, the *degree* of D is $\deg(D) = \sum_{i=1}^r a_i$. Note that $\deg(D + E) = \deg(D) + \deg(E)$.

10.1. **Riemann–Roch Theorem.**

Theorem 10.1.1 (Riemann–Roch)**.** *If X is a smooth projective curve of genus g and D is a divisor on X, then*

$$\chi(\mathcal{O}_X(D)) = \deg(D) - g + 1.$$

Proof. If $D = 0$,

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - g.$$

Given any D, by adding or subtracting a point finitely many times, we get to 0. It is hence enough to show that for any D and any $P \in X$, the formula holds for D if and only if it holds for $E = D - P$. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

and tensoring with $\mathcal{O}_X(D)$, we get

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(D) \longrightarrow \underbrace{\mathcal{O}_X(D) \otimes \mathcal{O}_P}_{\cong \mathcal{O}_P} \longrightarrow 0.$$

Taking Euler–Poincaré characteristic, we see that

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(E)) + \chi(\mathcal{O}_P) = \chi(\mathcal{O}_X(E)) + 1.$$

Since $\deg(D) = \deg(E) + 1$, this completes the proof. $\square$

Corollary 10.1.2. *If $D \sim E$, $\deg(D) = \deg(E)$.*

Proof. This is because the left hand side in Riemann–Roch Theorem 10.1.1 only depends on the line bundle. $\square$

Definition 10.1.3. If $\mathcal{L} \in \mathrm{Pic}(X)$, choose D such that $\mathcal{L} \cong \mathcal{O}_X(D)$, and define the *degree of* $\mathcal{L}$ by $\deg \mathcal{L} = \deg D$. (Note that this is well-defined by Corollary 10.1.2). This gives a group homomorphism

$$\deg\colon\ \mathrm{Pic}(X) \to \mathbb{Z}.$$

Example 10.1.4. If X is a smooth projective curve in $\mathbb{P}^n$, then $\deg(X) = \deg(\mathcal{O}_X(1))$. Indeed, the Hilbert polynomial of $\mathcal{O}_X$ satisfies

$$P_X(m) = \chi(\mathcal{O}_X(m)) = \deg(\mathcal{O}_X(m)) - g + 1 = m \cdot \deg(\mathcal{O}_X(1)) - g + 1$$

by the Riemann–Roch Theorem 10.1.1 where g is the genus of X.

Remark 10.1.5. Let X be a smooth projective curve of genus g. If $\mathcal{L} \in \mathrm{Pic}(X)$ satisfies $h^0(\mathcal{L}) \geq 1$, then $\deg(\mathcal{L}) \geq 0$ with equality if and only if $\mathcal{L} = \mathcal{O}_X$. To see this, choose $D \in |\mathcal{L}|$; it is clear that $\deg(D) \geq 0$ and equality holds if and only if $D = 0$.

Corollary 10.1.6. *For a smooth projective curve of genus g,* $\deg(\omega_X) = 2g - 2$.

Proof. Apply Riemann–Roch Theorem 10.1.1 with $\mathcal{O}_X(D) = \omega_X$:

$$\deg(\omega_X) - g + 1 = \chi(\omega_X) = h^0(\omega_X) - h^1(\omega_X) = h^1(\mathcal{O}_X) - h^0(\mathcal{O}_X) = g - 1$$

by Serre duality. $\square$

Corollary 10.1.7. *If $\mathcal{L} \in \mathrm{Pic}(X)$ such that $h^1(\mathcal{L}) > 0$, then $\deg(\mathcal{L}) \leq 2g-2$ with equality if and only if $\mathcal{L} \cong \omega_X$.*

Proof. By Serre duality, $h^1(\mathcal{L}) \cong h^0(\omega_X \otimes \mathcal{L}^{-1}) > 0$. By Remark 10.1.5, $\deg(\omega_X \otimes \mathcal{L}^{-1}) \geq 0$ with equality if and only if $\omega_X \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$ and

$$\deg(\omega_X \otimes \mathcal{L}^{-1}) = \deg(\omega_X) - \deg(\mathcal{L}) = (2g-2) - \deg(\mathcal{L}).$$

This completes the proof. $\square$

Proposition 10.1.8. *If $\mathcal{L} \in \mathrm{Pic}(X)$, then*

(1) *$\mathcal{L}$ is globally generated if and only if*
$$h^0(X, \underbrace{\mathcal{L}(-P)}_{\mathcal{L}\otimes\mathcal{O}_X(-P)}) = h^0(\mathcal{L}) - 1$$
for all $P \in X$,

(2) *$\mathcal{L}$ is very ample if and only if $h^0(X, \mathcal{L}(-P-Q)) = h^0(\mathcal{L}) - 2$ for any $P, Q \in X$ (not necessary distinct).*

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

and tensoring with $\mathcal{L}$ we get

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow \underbrace{\mathcal{L} \otimes \mathcal{O}_P}_{\mathcal{L}|_P} \longrightarrow 0.$$

Taking cohomology, we get

$$0 \longrightarrow H^0(X, \mathcal{L}(-P)) \longrightarrow H^0(\mathcal{L}) \longrightarrow \underbrace{\mathcal{L}|_P}_{\cong k} \cong k$$

$$s \longrightarrow s(P).$$

Therefore, $H^0(X, \mathcal{L}(-P)) = \{s \in H^0(X, \mathcal{L}) \mid s(P) = 0\}$, so either $h^0(X, \mathcal{L}(-P)) = h^0(\mathcal{L}) - 1$ (when $\mathcal{L}$ is globally generate at P) or $h^0(X, \mathcal{L}(-P)) = h^0(\mathcal{L})$ (when $P \in \mathrm{Bs}(\mathcal{L})$).

This proves (1). Also note that (2) implies (1), so we may assume that $\mathcal{L}$ is globally generated.

We know that in this case, $\mathcal{L}$ is very ample if and only if $\mathcal{L}$ separates points and $\mathcal{L}$ separates tangent directions at every point. Hence it is enough to show that

(α) For $P \neq Q$, $\mathcal{L}$ separates P, Q if and only if $h^0(\mathcal{L}(-P-Q)) = h^0(\mathcal{L}) - 2$.
(β) For any P, $\mathcal{L}$ separates tangent directions at P if and only if $h^0(\mathcal{L}(-2P)) = h^0(\mathcal{L}) - 2$.

To show (α), note that $\mathcal{L}$ separates P and Q if and only if $H^0(X, \mathcal{L}(-P)) \cap H^0(X, \mathcal{L}(-Q))$ has dimension 2. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P-Q) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\{P,Q\}} \longrightarrow 0.$$

Tensoring with $\mathcal{L}$ and taking H^0, we have that

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{L}(-P-Q)) & \longrightarrow & H^0(\mathcal{L}) & \longrightarrow & \mathcal{L}_{(P)} \oplus \mathcal{L}_{(Q)} \\ & & & & s & \longrightarrow & (s(P), s(Q)), \end{array}$$

so $H^0(\mathcal{L}(-P-Q)) = H^0(\mathcal{L}(-P)) \cap H^0(\mathcal{L}(-Q))$.

To show (β), note that $\mathcal{L}$ separates tangent directions at P if and only if there is an $s \in H^0(\mathcal{L})$ such that $s(P) = 0$ (i.e. $s_P \in \mathfrak{m}_P\mathcal{L}_P$), but $s_P \notin \mathfrak{m}_P^2\mathcal{L}_P$. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{2P} \longrightarrow 0.$$

Tensoring with $\mathcal{L}$, we get

$$0 \longrightarrow \mathcal{L}(-2P) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_{2P} = \mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P.$$

Hence

$$H^0(X, \mathcal{L}(-2P)) = \{s \in H^0(\mathcal{L}) \mid s_P \in \mathfrak{m}_P^2\mathcal{L}_P\}.$$

Hence $\mathcal{L}$ separates tangent directions at P if and only if $H^0(X, \mathcal{L}(-2P))$ is a hyperplane in $H^0(X, \mathcal{L}(-2P))$. □

Corollary 10.1.9. *Suppose X is a smooth, projective curve of genus g. For $\mathcal{L} \in \operatorname{Pic}(X)$,*

(1) *if $\deg(\mathcal{L}) \geq 2g$, then $\mathcal{L}$ is globally generated,*
(2) *if $\deg(\mathcal{L}) \geq 2g+1$, then $\mathcal{L}$ is very ample.*

Proof. We prove the first assertion, the second is analogous. If $\deg(\mathcal{L}) \geq 2g-1$, $h^0(\mathcal{L}) = \deg(\mathcal{L}) - g + 1$ since $h^1(\mathcal{L}) = 0$. Since $\deg(\mathcal{L}) \geq 2g$, for any $P \in X$, $\deg(\mathcal{L}(-P)) \geq 2g-1$, so $h^0(\mathcal{L}(-P)) = \deg(\mathcal{L}(-P)) - g + 1 = h^0(\mathcal{L}) - 1$. Applying (1) of Proposition 10.1.8 (1) gives the result. The proof of (2) is similar. □

Corollary 10.1.10. *If $\mathcal{L} \in \operatorname{Pic}(X)$, $\mathcal{L}$ is ample if and only if $\deg(\mathcal{L}) > 0$.*

Proof. If $\mathcal{L}$ is ample, there is an m such that $\mathcal{L}^m$ is very ample, giving $X \hookrightarrow \mathbb{P}^N$ such that $\mathcal{O}_X(1) \cong \mathcal{L}^m$. Then $0 < \deg(X) = \deg(\mathcal{O}_X(1)) = m \cdot \deg(\mathcal{L})$, so $\deg(\mathcal{L}) > 0$.

Conversely, if $\deg(\mathcal{L}) > 0$, then for any m such that $m \cdot \deg(\mathcal{L}) \geq 2g+1$, Corollary 10.1.9 shows that $\mathcal{L}^m$ is very ample. □

10.2. **Classification of curves.** Genus is the fundamental invariant for curve classification.

(1) Suppose $g = 0$. A smooth projective curve has genus 0 if and only if $X \cong \mathbb{P}^1$. We know that the genus of $\mathbb{P}^1$ is 0, so we just need to prove the converse. If $\mathcal{L} = \mathcal{O}_X(P)$ has degree 1 (which is at least $2g + 1$), $\mathcal{L}$ is very ample, and $h^0(\mathcal{L}) = \deg \mathcal{L} - g + 1$. We then have an embedding $X \hookrightarrow \mathbb{P}^1$ which muse be an isomorphism.
(2) Suppose $g = 1$. In this case, we call X and *elliptic curve*. Since $\deg(\omega_X) = 2g-2 = 0$, $h^0(\omega_X) = 1$, we see that $\omega_X \cong \mathcal{O}_X$ in this case.
If $\mathcal{L} \in \mathrm{Pic}(X)$, $\deg(\mathcal{L}) = 3 = 2g+1$, $\mathcal{L}$ is very ample. Note that $h^0(\mathcal{L}) = 3-1+1 = 3$. Then we get an embedding $X \hookrightarrow \mathbb{P}^2$, which makes X a plane curve of degree 3. Conversely, if $X \subseteq \mathbb{P}^2$ is a smooth curve of degree 3, $\omega_X \cong \omega_{\mathbb{P}^2} \otimes \mathcal{O}(X)|_X \cong \mathcal{O}_X$, so it is an elliptic curve.
(3) The case $g \geq 2$ is called the *general case*. In this case, ω_X is ample.
The following fact will be proved next time.

Fact 10.2.1. *For any $g \geq 2$, there is a smooth projective curve of genus g.*

For example, if $X \subseteq \mathbb{P}^2$ is smooth of degree $d \geq 4$, then $g = \binom{d-1}{2}$.
Exercise. If $X \subseteq \mathbb{P}^n$ is smooth and is a smooth intersection of hyperplanes of degree $d_1, \ldots, d_{n-1}$, then

$$g = d_1 \ldots d_{n-1}(d_1 + \cdots + d_{n-1} - n - 1).$$

However, it is clear we cannot get all genera this way.

10.3. **Morphisms between algebraic curves.** Any rational map $X \dashrightarrow Y$ where X is a smooth curve and Y is a complete variety is a morphism. Every morphism $X \to Y$ between complete curves which is not constant is finite (since it is proper, with finite fibers).

Proposition 10.3.1. *Any birational map $X \dashrightarrow Y$ where X and Y are smooth complete curves is an isomorphism. Moreover, every smooth complete curve is projective.*

Proof. If $\varphi \colon X \to Y$ is birational, then both φ and φ^{-1} are morphisms by the above remark, so φ is an isomorphism.

Suppose that X is smooth and complete. Let $U \subseteq X$ be affine, and let $\overline{U}$ be the closure in some $\mathbb{P}^N$, and let $\widetilde{U} \to \overline{U}$ be the normalization. Since $\overline{U}$ is projective, $\widetilde{U}$ is also projective, so $\widetilde{U}$ is a smooth projective curve birational to X. Since $X \cong \widetilde{U}$ by the first assertion, X is projective. ☐

Remark 10.3.2. We have an equivalence of categories:

$$\left\{ \begin{array}{c} \text{smooth projective curves} \\ \text{and} \\ \text{non-constant morphisms} \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{field extensions } K/k \\ \text{of finite type} \\ \text{and transcendence degree 1} \end{array} \right\}$$
$$X \mapsto k(X).$$

Note that given K, there is a smooth projective curve X such that $k(X) \cong K$ (choose an affine curve U with $k(U) \cong K$, take closrue in some $\mathbb{P}^N$, and then normalize).

Let $f\colon X \to Y$ be a finite morphism between smooth, projective curves. Note that f is flat. Given $P \in X$, define the *ramification index* $e_P(f)$ as follows: we have a map of local rings $(u) \subseteq \mathcal{O}_{Y,f(P)} \xrightarrow{\varphi} \mathcal{O}_{X,P} \supseteq (v)$, where (u) and (v) are maximal ideals, and note that $\varphi(u) = v^e w$ for an invertible element w; then $e_p(f) = e$. Note that this is the multiplicity of $f^{-1}(f(P))$ at P, which we defined as

$$\ell(\mathcal{O}_{X,P}/\mathfrak{m}_{f(P)}\mathcal{O}_{X,P}).$$

Since f is finite and flat, for any $Q \in Y$:

$$\sum_{P \in f^{-1}(Q)} e_P(f) = \deg(f).$$

Remark 10.3.3. Note that $e_P(f) = 1$ if and only if $df_p T_p X \to T_{f(P)} Y$ is an isomorphism.

Remark 10.3.4. For any $Q \in Y$, the pullback of the divisor Q is $f^*(Q) = \sum_{P \in f^{-1}(Q)} e_P(f)P$.

This gives the following formula: for any divisor D on Y

$$\deg(f^*(D)) = \deg(f) \cdot \deg(D).$$

Example 10.3.5. Suppose X is a smooth projective curve of genus ≥ 1. Fix $P \in X$. Then

$$h^0(X, \mathcal{O}_X(P)) = 1.$$

Otherwise, there is a divisor $D \geq 0$ such that $D \neq P$ and $D \sim P$. Then $\deg(D) = 1$, so $D = Q$ for a point $Q \neq P$. Since $P \sim Q$, there is a function $\varphi \in k(X)^*$ such that $\operatorname{div}(\varphi) = P - Q$. In this case, φ corresponds to a morphism

$$f\colon X \to \mathbb{P}^1$$

and $\operatorname{div}(\varphi) = f^*(0) - f^*(\infty)$. Hence $f^*(0) = P$, so $\deg(f) = 1$, so f is birational, and hence na isomorphism. This contradicts the fact that the genus is at least 1.

Example 10.3.6. Suppose X is a smooth, projective, genus $g > 0$. Then $\mathcal{O}_X(P)$ is ample (since it has degree > 0). However, $|\mathcal{O}_X(P)| = \{P\}$, so $\mathcal{O}_X(P)$ is not globally generated.

Let $f\colon X \to Y$ be a finite morphism between smooth projective curves. Suppose f is separable. We have the canonical exact sequence

$$\underbrace{f^*\Omega_Y}_{f^*\omega_Y} \xrightarrow{\alpha} \underbrace{\Omega_X}_{\omega_X} \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

Since f is separable, there is an open subset $U \subseteq X$ such that $f|_U\colon U \to Y$ is smooth of relative dimension 0, in which case $\alpha|_U$ is an isomorphism. Note that α corresponds to

$$s \in \Gamma(X, \omega_X \otimes f^*\omega_Y^{-1})$$

and $s|_U \neq 0$, so $s \neq 0$. We define Ram_f to be the effective divisor associated to s. Note that

$$\mathcal{O}_X(\mathrm{Ram}_f) \cong \omega_X \otimes f^*\omega_Y^{-1}.$$

The short exact sequence above shows that

$$\Omega_{X/Y} \cong f^*\omega_Y \otimes \mathcal{O}_{\mathrm{Ram}(f)}.$$

Suppose for simplicity that the characteristic of k is 0.

Theorem 10.3.7 (Riemann–Hurwitz). *We have that*

$$\mathrm{Ram}(f) = \sum_{P \in X} (e_P(f) - 1)P.$$

In particular, $2g_X - 2 = \deg(f) \cdot (2g_Y - 2) + \sum_{P \in X}(e_P(f) - 1)$.

11. Intersection numbers of line bundles

11.1. General theory.

Theorem 11.1.1 (Snapper). *Let X be a complete algebraic variety and $\mathcal{F} \in \mathrm{Coh}(X)$, $\mathcal{L}_1, \ldots, \mathcal{L}_r \in \mathrm{Pic}(X)$. Then there is a polynomial $P \in \mathbb{Q}[x_1, \ldots, x_r]$ of total degree $\leq$ $\dim(\mathrm{supp}(\mathcal{F}))$ such that*

$$P(m_1, \ldots, m_r) = \chi(\mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r} \otimes \mathcal{F}).$$

Lemma 11.1.2. *Let X be an algebraic variety and $\mathcal{F} \in \mathrm{Coh}(X)$. Then there is a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F}_i/\mathcal{F}_{i-1})$ is the radical ideal of an irreducible closed subset of X.

Proof. We proceed by Noetherian induction on $\mathrm{supp}(\mathcal{F})$. Let $\mathcal{I} = \sqrt{\mathrm{Ann}(\mathcal{F})}$. We have a filtration

$$0 = \mathcal{I}^m\mathcal{F} \subset \mathcal{I}^{m-1} \subset \cdots \subset \mathcal{I}\mathcal{F} \subset \mathcal{F}$$

such that $\mathcal{I}$ annihilates each successive quotient.

If $V(\mathcal{I}) \neq X$, by induction, we have a filtration on each successive quotient, so we get a filtration on $\mathcal{F}$.

Suppose $V(\mathcal{I}) = X$, i.e. $\mathcal{I} = 0$. Let $X_1, \ldots, X_s$ be irreducible components of X. If $s = 1$, we are done. If $s > 1$ and $\mathcal{I}_j$ is the radical ideal corresponding to X_j, then

$$\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_s = 0.$$

Consider the filtration $0 \subset \mathcal{I}_1\mathcal{F} \subset \mathcal{F}$. Then $\mathcal{I}_1\mathcal{F}$ is annihilated by $\mathcal{I}_2 \cap \cdots \cap \mathcal{I}_s \neq \emptyset$ and $\mathcal{F}/\mathcal{I}_1\mathcal{F}$ is annihilated by $\mathcal{I}_1 \neq 0$. Therefore, they both have filtrations by induction, and so does $\mathcal{F}$. $\square$

Proof of Theorem 11.1.1. Argue by induction on $d = \dim(\mathrm{supp}(\mathcal{F}))$. The case $d = -1$ is trivial (where, by convention, $\dim(\emptyset) = -1$, $\deg(0) = -1$).

Assume X is projective. If we have a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

and for two of $\mathcal{F}', \mathcal{F}, \mathcal{F}''$, the function given by χ is a polynomial of degree $\leq d$, then the same is true for the third one.

Since X is projective, we may write $\mathcal{L}_1 \cong \mathcal{M}_1 \otimes \mathcal{M}_2^{-1}$ for very ample line bundle $\mathcal{M}_1$, $\mathcal{M}_2$. Choose $A \in |\mathcal{M}_1|$, $B \in |\mathcal{M}_2|$ such that A and B contain no associated variety of $\mathcal{F}$. We have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1^{m_1} \otimes \mathcal{F} \otimes \mathcal{O}(-A) \arrow[r] \arrow[d, "="] & \mathcal{L}_1^{m_1} \otimes \mathcal{F} \arrow[r] & \mathcal{F}_1^{m_1} \otimes \mathcal{F} \otimes \mathcal{O}_A \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{L}_1^{m_1-1} \otimes \mathcal{F} \otimes \mathcal{O}(-B) \arrow[r] & \mathcal{L}_1^{m_1-1} \otimes \mathcal{F} \arrow[r] & \mathcal{L}_1^{m_1-1} \otimes \mathcal{F} \otimes \mathcal{O}_B \arrow[r] & 0
\end{tikzcd}$$

Tensoring with $\mathcal{L}_2^{m_2} \otimes \cdots \otimes \mathcal{L}_r^{m_r}$ and taking the Euler–Poincaré characteristic, we obtain

$$\chi(\mathcal{F} \otimes \mathcal{F}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}) - \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1-1} \otimes \mathcal{L}_2^{m_2} \otimes \cdots \otimes \mathcal{L}_r^{m_r})$$

is equal to

$$\chi(\mathcal{F} \otimes \mathcal{O}_A \otimes \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}) - \chi(\mathcal{F} \otimes \mathcal{O}_B \otimes \mathcal{L}_1^{m_1-1} \otimes \mathcal{L}_2^{m_2} \otimes \cdots \otimes \mathcal{L}_r^{m_r}).$$

Note that $\dim(\operatorname{supp}(\mathcal{F}\otimes\mathcal{O}_A)), \dim(\operatorname{supp}(\mathcal{F}\otimes\mathcal{O}_B)) \le d-1$. By induction, the above difference is a polynomial function of total degree $d-1$.

A similar property holds with respect to the other variables. Therefore,

$$(m_1, \dots, m_r) \mapsto \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r})$$

is a polynomial of total degree at most d.

Exercise. Deduce the case when X is a complete variety using Chow's lemma. (See official notes for solution.) □

Definition 11.1.3 (Intersection number)**.** If X is a complete variety, $\mathcal{F} \in \operatorname{Coh}(X)$, $\mathcal{L}_1, \dots, \mathcal{L}_r \in \operatorname{Pic}(X)$ such that $\dim \operatorname{supp}(\mathcal{F}) \le r$, then $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r; \mathcal{F})$ is the coefficient of $x_1, \dots, x_r$ in the polynomial $P(x_1, \dots, x_r)$ such that

$$P(m_1, \dots, m_r) = \chi(\mathcal{F} \otimes \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}).$$

If $\mathcal{F} = \mathcal{O}_Y$ where Y is a subvariety of X, we write $(\mathcal{L}_1 \cdot \cdots \mathcal{L}_r \cdot Y)$ and if $Y = X$, we write $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r)$.

Note that $(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r \cdot Y) = (\mathcal{L}_1|_Y \cdot \cdots \cdot \mathcal{L}_r|_Y)$.

If $\mathcal{L}_1 = \cdots = \mathcal{L}_r = \mathcal{L}$, write $(\mathcal{L}^r; \mathcal{F})$ or $(\mathcal{L}^r)$ if $\mathcal{F} = \mathcal{O}_X$.

If X is irreducible and $D_1, \dots, D_r$ are Cartier divisors on X, then

$$(D_1 \cdot \cdots \cdot D_r) = (\mathcal{O}_X(D_1) \cdot \cdots \cdot \mathcal{O}_X(D_r)).$$

Lemma 11.1.4. *If $P(x_1, \dots, x_r) \in R[x_1, \dots, x_r]$ of degree $\le r$, the coefficient of $x_1 \dots x_r$ in P is*

$$\sum_{I \subseteq \{1, \dots, r\}} (-1)^{|I|} P(\delta_{I,1}, \dots, \delta_{I,r})$$

where

$$\delta_{I,j} = \begin{cases} 0 & \text{if } j \notin I, \\ -1 & \text{if } j \in I. \end{cases}$$

Proof. We proceed by induction on r. For $r = 1$, the assertion is clear. For the inductive step, consider

$$Q(X_1, \dots, x_{r-1}) = P(x_1, \dots, x_{r-1}, 0) - P(x_1, \dots, x_{x-1}, -1).$$

Then the coefficient of $x_1 \dots x_{r-1}$ in Q is the coefficient of $x_1 \dots x_r$ in P and $\deg(Q) \leq r-1$. $\square$

In particular, the lemma shows that:

$$(*) \qquad (\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) = \sum_{I \subseteq \{1,\dots,r\}} (-1)^{|I|} \chi\left(\mathcal{F} \otimes \bigotimes_{j \in I} \mathcal{L}_j^{-1}\right).$$

We note a few basic properties implied by this observation.

Proposition 11.1.5. *Let $\mathcal{L}_1, \dots \mathcal{L}_r \in \operatorname{Pic}(X)$ and $\dim(\operatorname{supp}(\mathcal{F})) = d \leq r$. Then:*

(1) *$(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) \in \mathbb{Z}$ and it is 0 if $d < r$,*
(2) *$(\mathcal{L}_1, \dots, \mathcal{L}_r) \mapsto (\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F})$ is a multilinear symmetric pairing on $\operatorname{Pic}(X)^{\oplus r}$,*
(3) *if $Y_1, \dots, Y_r$ are irreducible components of $\operatorname{supp}(\mathcal{F})$ of dimension r, then*

$$(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) = \sum_{i=1}^r \ell_{\mathcal{O}_{X;Y_i}}(\mathcal{F}_{Y_i}) \cdot (\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r \cdot Y_i),$$

(4) *(projection formula): if $f \colon X \to Y$ is a surjective morphism between complete varieties and $\mathcal{L}_i = f^*(\mathcal{M}_i)$, then*

$$(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r) = \begin{cases} 0 & \text{if } \dim Y < \dim X, \\ \deg(f) \cdot (\mathcal{M}_1 \cdot \cdots \cdot \mathcal{M}_r) & \text{if } \dim Y = \dim X. \end{cases}$$

(5) *if X is irreducible and $\mathcal{L}_r = \mathcal{O}_X(D)$ for an effective Cartier divisor, which does not contain any associated variety of $\mathcal{F}$, then*

$$(\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) = (\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_{r-1} ; \mathcal{F} \otimes \mathcal{O}_D).$$

Proof. Part (1) clearly follows from $(*)$. In (2), symmetry is clear and to see multilinearity, note that

$$((\mathcal{L}_1' \otimes \mathcal{L}_1'') \cdot \mathcal{L}_2 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) - (\mathcal{L}_1' \cdot \mathcal{L}_2 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) - (\mathcal{L}_1'' \cdot \mathcal{L}_2 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F})$$

by $(*)$ is equal to

$$-(\mathcal{L}_1' \cdot \mathcal{L}_1'' \cdot \mathcal{L}_2 \cdot \cdots \cdot \mathcal{L}_r ; \mathcal{F}) = 0$$

by (1).

For (3), note that both sides of the equality are additive in short exact sequence. By Lemma 11.1.2, we may assume that X is irreducible and $\operatorname{Ann}(\mathcal{F}) = 0$. Moreover, if we have a map $\mathcal{F} \to \mathcal{G}$ which is an isomorphism on some open subset of X, the property holds for $\mathcal{F}$ if and only if it holds for $\mathcal{G}$. We can clearly reduce to the case when X is projective.

If X is projective, there is a morphism $\varphi \colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}(D)$ where D is an effective ample divisor, $\mathcal{F} \otimes \mathcal{O}(D)$ is globally generated, and φ is an isomorphism on some open subset of X. Therefore, we may assume that $\mathcal{F}$ is globally generated.

If $m = \ell_{k(X)}(\mathcal{F}_X)$ and $s_1, \dots, s_m \in \Gamma(X, \mathcal{F})$ are general, then the map

$$\begin{aligned} \mathcal{O}_X^{\oplus m} &\to \mathcal{F} \\ e_i &\to s_i \end{aligned}$$

is an isomorphism on some open subset, so we may assume that $\mathcal{F} = \mathcal{O}_X^{\oplus m}$. In this case, the assertion is clear.

For (4), we use the Leray Spectral Sequence 3.8.2. We see that for any $\mathcal{L} \in \mathrm{Coh}(X)$,

$$\chi(X, \mathcal{L}) = \sum_{i \geq 0} (-1)^i \chi(Y, R^i f_*(\mathcal{L})).$$

If $\mathcal{L} = f^*\mathcal{M}$, then $R^i f_*(\mathcal{L}) = \mathcal{M} \otimes R^i f_*(\mathcal{O}_X)$. Then $(*)$ shows that

$$(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r) = \sum_{i \geq 0} (-1)^i (\mathcal{M}_1 \cdot \dots \cdot \mathcal{M}_r; R^i f_*(\mathcal{O}_X)).$$

This is clearly 0 if $\dim Y < \dim X$.

Assume that X and Y are irreducible. If $\dim Y = \dim X$, then there is an open subset $U \subseteq Y$ such that $f^{-1}(U) \to U$ is a finite morphism. Then

$$\dim(\mathrm{supp}(R^i f_*(\mathcal{O}_X))) < \dim Y \leq r$$

for $i \geq 1$. Moreover,

$$\dim_{k(X)}(R^0 f * (\mathcal{O}_X))_X = \deg(f),$$

so the assertion in (4) follows from (3).

To show (5), compute the left hand side using $(*)$ by considering the two cases $r \notin I$ and $r \in I$. Then $(\mathcal{L}_1 \cdot \dots \cdot \mathcal{L}_r; \mathcal{F})$ is equal to

$$\sum_{I \subseteq \{0, \dots, r-1\}} (-1)^{|I|} \chi\left(\mathcal{F} \otimes \bigotimes_{j \in I} \mathcal{L}_j^{-1}\right) + \sum_{I \subseteq \{0, \dots, r-1\}} (-1)^{|I|+1} \chi\left(\mathcal{F} \otimes \mathcal{O}_X(-D) \otimes \bigotimes_{j \in I} \mathcal{L}_j^{-1}\right)$$

which can be written as

$$\sum_{I \subseteq \{0, \dots, r-1\}} (-1)^{|I|} \chi\left(\mathcal{F} \otimes \mathcal{O}_D \otimes \bigotimes_{j \in I} \mathcal{L}_j^{-1}\right),$$

completing the proof. $\square$

Remark 11.1.6. Suppose X is a Cohen–Macaulay complete irreducible variety of dimension n and $D_1, \dots, D_n$ are effective Cartier divisors such that $\mathrm{codim}_X(D_1 \cap \dots \cap D_n) = i$. Then for any $x \in D_1 \cap \dots \cap D_i$, the equation of $D_1, \dots, D_i$ at x form a regular sequence. Then

$$(D_1 \cdot \dots \cdot D_r) = (D_2 \cdot \dots \cdot D_r; \mathcal{O}_{D_1}) = \dots = h^0(X, \mathcal{O}_{D_1} \otimes \dots \otimes \mathcal{O}_{D_r}).$$

(Note that by Proposition 11.1.5, $(\mathcal{O}_X(D); \mathcal{F}) = h^0(\mathcal{F} \otimes \mathcal{O}_D)$.)

If all the intersection points are smooth points of X and and all the D_i, and all the intersections are transversal, then

$$(D_1 \cdot \dots \cdot D_n) = \#(D_1 \cap \dots \cap D_n).$$

Exercise. Check that properties (1)–(5) in Proposition 11.1.5 uniquely characterize the intersection numbers.

Remark 11.1.7. If $Q \in R[x]$ has degree $\leq d$, $P(x_1, \dots, x_r) = Q(x_1 + \cdots + x_r)$ has total degree $\leq d$ and the coefficient of $x_1 \dots x_r$ is $d!$ times the coefficient of x^d in Q.

Suppose $\mathcal{L} \in \mathrm{Pic}(X)$ where X is a complete variety of dimension d. By Snapper's Theorem 11.1.1, there is a polynomial $Q \in \mathbb{Q}[x]$, $q(M) = \chi(\mathcal{L}^m)$ with $\deg Q \leq d$.

Then we obtain the *asymptotic Riemann-Roch formula*:

$$\chi(\mathcal{L}^m) = \frac{(\mathcal{L}^d)}{d!} m^d + \text{lower order terms.}$$

Example 11.1.8. Suppose $\mathcal{L}$ is very ample and gives an embedding $X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{O}_X(1) \cong \mathcal{L}$. Then the polynomial Q defined in Remark 11.1.7 is the Hilbert polynomial of X, and

$$(\mathcal{L}^{\dim X}) = \deg(X).$$

In particular, $(\mathcal{L}^{\dim X}) > 0$. More generally, this shows that for any ample line bundle $\mathcal{L}$ on X and any subvariety $Y \subseteq X$,

$$(\mathcal{L}^{\dim Y}; Y) > 0.$$

11.2. **Intersection numbers for curves and surfaces.** Let X be an irreducible complete curve. Note that $(\mathcal{O}_X(P)) = h^0(\mathcal{O}_P) = 1$ by Proposition 11.1.5 (5). What is $(\mathcal{L})$ in general?

- If X is smooth, $(\mathcal{L}) = \deg(\mathcal{L})$. (Both sides are additive, so it is enough to show this when $\mathcal{L} = \mathcal{O}_X(P)$.)
- In general, consider the normalization $\widetilde{X} \xrightarrow{f} X$. By Proposition 11.1.5 (4), $(\mathcal{L}) = (f^*(\mathcal{L})) = \deg(f^*(\mathcal{L}))$. We the define

  $$\deg(\mathcal{L}) = \deg(f^*(\mathcal{L})).$$

  Then $(*)$ implies that

  $$(\mathcal{L}) = \chi(\mathcal{O}_X) = \chi(\mathcal{L}^{-1}),$$

  so

  $$-\deg \mathcal{L} = \chi(\mathcal{O}_X) - \chi(\mathcal{L}).$$

  The formula

  $$\chi(\mathcal{L}) = \deg(\mathcal{L}) + \chi(\mathcal{O}_X)$$

  is *Riemann–Roch for singular curves.*

From now on, assume X is a smooth projective surface. If

$$D = \sum_i a_i D_i, \quad E = \sum_j b_j E_j,$$

then

$$(D \cdot E) = \sum_{,j} a_i b_j \deg(\mathcal{O}(D_i)|_{E_j}).$$

Theorem 11.2.1 (Adjunction formula)**.** *Let $C \subseteq X$ be an irreducible curve. Then*

$$2p_a(C) - 2 = (C^2) + (C \cdot K_X)$$

where K_X is any divisor with corresponding line bundle ω_X.

Proof. If C is smooth, the right hand side is

$$(C \cdot (C + K_X)) = \deg(\underbrace{(\mathcal{O}_X(K_X + C))|_C}_{\omega_C}).$$

This is equal to the left hand side by Riemann–Roch Theorem 10.1.1.

If C is singular, we argue similarly. We showed that

$$\omega_C^0 \cong \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X).$$

Using the locally free resolution

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

we see that

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \omega_X) \cong \omega_X(C)|_C.$$

The fact that $\deg(\omega_C^0) = 2p_a(C) - 2$ follows from the singular version of Riemann–Roch. □

Example 11.2.2. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Then $\operatorname{Pic} X$ is generated by $L_1 = \mathrm{pr}_1^*(P)$, $L_2 = \mathrm{pr}_2^*(Q)$. Then

$$(L_1^2) = 0 = (L_2^2),$$
$$(L_1 \cdot L_2) = 1.$$

If $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is an irreducible curve C, we say that C has type (a, b) if $C \sim aL_1 + bL_2$. Note that

$$a = (C \cdot L_2) \geq 0, b = (C \cdot L_1) \geq 0.$$

By Adjuction formula 11.2.1,

$$2p_a(C) - 2 = ((aL_1 + bL_2) \cdot ((a-2)L_1 + (b-2)L_2)) = a(b-2) + b(a-2).$$

Therefore,

$$p_a(C) = (a-1)(b-1)$$

is the genus of the curve.

In particular, we can obtain curves of arbitrary genus this way.

Theorem 11.2.3 (Riemann-Roch for surfaces)**.** *Let D be a divisor on X. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D \cdot (D - K_X)).$$

Proof. Consider $\mathcal{L}_1, \mathcal{L}_2$ on X. Then

$$(\mathcal{L}_1 \cdot \mathcal{L}_2) = (\mathcal{L}_1^{-1} \cdot \mathcal{L}_2^{-1}) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}_1) - \chi(\mathcal{L}_2) + \chi(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

In this case, take $\mathcal{L}_1 = \mathcal{O}_X(D)$, $\mathcal{L}_2 = \omega_X \otimes \mathcal{O}_X(-D)$. Then

$$(D \cdot (K_X - D)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(D)) - \chi(\omega_X \otimes \mathcal{O}_X(-D)) + \chi(\omega_X).$$

By Serre duality,

$$\chi(\omega_X \otimes \mathcal{O}_X(-D)) = \chi(\mathcal{O}_X(D)),$$
$$\chi(\omega_X) = \chi(\mathcal{O}_X),$$

so

$$2(\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X)) = (D \cdot (D - K_X)),$$

giving the result. □

The following exercise is an application of intersection theory.

Exercise. Suppose $\mathcal{L} \in \operatorname{Pic}(X)$ has $(\mathcal{L}^2) > 0$. Then there exists $C > 0$ such that either $h^0(X, \mathcal{L}^m) > C \cdot m^2$ or $h^0(X, \mathcal{L}^{-m}) > C \cdot m^2$ for $m \gg 0$.

12. Introduction to birational geometry

12.1. **Preliminaries.** Let $f\colon X \to Y$ be a proper birational morphism with Y normal.

Proposition 12.1.1. *If* $U = \operatorname{Dom}(f^{-1})$, *then*

(1) $f^{-1}(U) \to U$ *is an isomorphism,*
(2) *for any* $y \in Y \setminus U$, $f^{-1}(y)$ *is connected of dimension* ≥ 1,
(3) $\operatorname{codim}_Y(Y \setminus U) \geq 2$.

Proof. Since f is proper, $W = \{y \in Y \mid f^{-1}(y) \text{ is finite}\}$ is open in Y. Since f is dominant, $f^{-1}(y) \neq \emptyset$ for all $y \in Y$.

Since Y is normal and f is proper and birational, $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism, so $f^{-1}(y)$ is connected for all $y \in Y$ by Zariski's Main Theorem 8.1.6.

Consider $g = f|_{f^{-1}(W)}\colon f^{-1}(W) \to W$. It is surjective, injective, homeomorphic, and induces an isomorphism $\mathcal{O}_W \to g_*\mathcal{O}_{f^{-1}(W)}$, so g is an isomorphism.

For any $y \in Y \setminus W$, $\dim f^{-1}(y) \geq 1$ and $f^{-1}(y)$ is connected. Clearly, $U = \operatorname{dom}(f^{-1}) \supseteq W$. Since $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is an isomorphism, $U \subseteq W$, so $U = W$.

Finally, part (3) follows from the result that if we have a diagram

$$\begin{tikzcd} & Z_1 \arrow[d, "\text{proper}"] \\ Z_2 \arrow[ur, dotted, "\varphi"] \arrow[r] & S \end{tikzcd}$$

with Z_2 normal, then $\operatorname{codim}_{Z_2}(z_2 \setminus \operatorname{dom}(\varphi)) \geq 2$. □

Definition 12.1.2. With the notation of Proposition 12.1.1, $f^{-1}(Y \setminus U)$ is the *exceptional locus* of f, written $\operatorname{Exc}(f)$.

A prime divisor $E \subseteq X$ is *exceptional* if $E \subseteq \operatorname{Exc}(f)$ (equivalently, $\dim(f(E)) < \dim E$).

If D is a prime divisor in Y, the *strict transform* of D is $\widetilde{D} = \overline{f^{-1}(U \cap D)}$. This is extended by linearity to Weil divisors.

Proposition 12.1.3. *If* $f\colon X \to Y$ *is a proper birational morphism between smooth varieties, then there exists a divisor* $K_{X/Y}$ *on* X *(the* relative canonical divisor*) such that*

$$\begin{aligned} \operatorname{supp}(K_{X/Y}) &= \operatorname{Exc}(f), \\ \mathcal{O}_X(K_{X/Y}) &= \omega_X \otimes f^*(\omega_Y)^{-1}. \end{aligned}$$

Proof. Recall that we have an exact sequence

$$f^*\Omega_Y \xrightarrow{\varphi} \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

If $U = \mathrm{dom}(f^{-1})$, $\varphi|_{f^{-1}(U)}$ is an isomorphism, so φ is injective. Note that

$$\det(\varphi)\colon f^*\omega_Y \to \omega_X$$

corresponds to

$$s \in \Gamma(X, \omega_X \otimes f^*(\omega_Y)^{-1})$$

(by twisting by $f^*\omega_Y^{-1}$). Since $s|_{f^{-1}(U)} \neq 0$, $s \neq 0$. Let

$$K_{X/Y} = \text{effective divisor corresponding to } s.$$

Then by definition

$$\mathcal{O}_X(K_{X/Y}) \cong \omega_X \otimes f^*(\omega_Y)^{-1}.$$

We just need to check that $\mathrm{supp}(K_{X/Y}) = \mathrm{Exc}(f)$. If $x \in \mathrm{supp}(K_{X/Y})$, $x \in \mathrm{Exc}(f)$, since s has no zeros on $f^{-1}(U)$.

Conversely, if $x \notin \mathrm{supp}(K_{X/Y})$, there is an open set $V \ni x$ such that $V \cap \mathrm{supp}(K_{X/Y}) = \emptyset$. For any $x \in V$,

$$T_{f(x)}Y^* \xrightarrow{\cong} T_X X^*$$

so $f|_V$ is étale, so $f|_V$ has finite fibers. This shows that $V \cap \mathrm{Exc}(f) = \emptyset$, so $x \notin \mathrm{Exc}(f)$. The uniqueness statement follows from the more general Lemma 12.1.4. □

Lemma 12.1.4. *If $f\colon X \to Y$ is proper, birational morphism such that X and Y are normal, and D is a Weil divisor supported on* $\mathrm{Exc}(f)$ *such that $D \sim 0$, then $D = 0$.*

Proof. Since $D \sim 0$, there exists $\varphi \in k(X)$, $\varphi \neq 0$ such that $\mathrm{div}_X(\varphi) = D$. If E is a prime divisor on Y, $\mathrm{ord}_E(\varphi) = \mathrm{ord}_{\widetilde{E}}(\varphi) = 0$ since $\widetilde{E}$ does not show up in D. Therefore, $\mathrm{div}_Y(\varphi) = 0$, so $\varphi \in \mathcal{O}_Y^*(Y)$, and hence $\varphi \in \mathcal{O}_X^*(X)$. This shows that $D = 0$. □

12.2. **Birational maps.**

Definition 12.2.1. Let $\varphi\colon X \dashrightarrow Y$ be a rational map between irreducible varieties and let $U \subseteq X$ be an open subset such that φ is represented by $f\colon U \to Y$. Then

$$\Gamma_f = \{(x, f(x)) \mid x \in U\} \subseteq U \times Y$$

is closed, and we define the graph of φ by

$$\Gamma_\varphi = \overline{\Gamma_f} \subseteq X \times Y.$$

We check that this definition is independent of the choice of f. Consider the diagram

$$\begin{tikzcd} & \Gamma_f \arrow[r, hook] \arrow[dl, "\cong"'] & \Gamma_\varphi \arrow[rr, hook] \arrow[dl, "p"] \arrow[dr, "q"] & & X \times Y \\ U \arrow[r, hook] & X & & Y & \end{tikzcd}$$

We have that $\Gamma_\varphi \cap U \times Y = \Gamma_f$ and hence p is birational and $\varphi = q \circ p^{-1}$. In particular, φ is birational if and only if q is birational.

Suppose we have morphisms
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$
(i.e. φ is a rational map of varieties over S) such that X and Y are proper over S. Then $\Gamma_\varphi \subseteq X \times_S Y$ is closed, so both p and q are closed.

Definition 12.2.2. If φ as above and $T \subseteq X$ is closed, then the *image* of T under φ is defined as
$$\varphi(T) = q(p^{-1}(T)) \subseteq Y.$$

Then Proposition 12.1.1 gives the following result.

Corollary 12.2.3. *Let*
$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$
be a rational map of varieties over S which are proper over S and X be normal. If $x \notin \mathrm{dom}(\varphi)$, then $\varphi(x)$ is connected of dimension at least 1.

Proof. Consider
$$\begin{tikzcd} & \Gamma_\varphi \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dotted] & & Y. \end{tikzcd}$$
Since $x \notin \mathrm{dom}(\varphi)$, $x \notin \mathrm{dom}(p^{-1})$. By Proposition 12.1.1, $p^{-1}(x)$ is connected of dimension at least 1. Since $q|_{p^{-1}(x)}$ is a closed immersion, this gives the result. $\square$

We state a few simple properties, which are left as exercises. Their proofs can be found in the official notes.

(1) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper birational maps and X, Y, Z are normal, then
$$\mathrm{Exc}(g \circ f) = \mathrm{Exc}(f) \cup f^{-1}\,\mathrm{Exc}(g).$$

(2) If $\varphi\colon X \dashrightarrow Y$ is a birational map, $\Gamma_{\varphi^{-1}}$ corresponds to Γ_φ via the isomorphism
$$X \times Y \cong Y \times X$$
$$(x, y) \mapsto (y, x).$$

(3) Suppose $X \overset{\varphi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$ be rational maps with φ dominant. Then
$$\Gamma_{\psi\circ\varphi} \cong \Gamma_\varphi \times_Y \Gamma_\psi.$$

Moreover, for any closed subset $T \subseteq X$, $(\psi \circ \varphi)(T) \subseteq \psi(\varphi(T))$ with equality if ψ is a morphism.

12.3. **Smooth blow-ups.** Recall that if X is an irreducible variety, $\mathcal{I} \neq 0$ is an ideal of $\mathcal{O}_X$, then

$$\widetilde{X} = \operatorname{Bl}_{\mathcal{I}} X = \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}\left(\bigoplus_{m\geq 0} \mathcal{I}^m\right) \xrightarrow{f} X$$

is the blow up of X along $\mathcal{I}$. There is an effective Cartier divisor E on $\widetilde{X}$ such that $\mathcal{I}\otimes\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E) = \mathcal{O}_{\widetilde{X}}(1)$.

Proposition 12.3.1 (Universal Property of Blow-ups)**.** *If Y is an irreducible variety, given $g\colon Y \to X$ such that $\mathcal{I}\cdot\mathcal{O}_Y$ is locally principal, there exists a unique $h\colon Y \to \widetilde{X}$ such that $f\circ h = g$:*

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[ur, dotted, "h"] \arrow[r, "g"'] & X. \end{tikzcd}$$

We omit the proof of this result here. It will not be useful for us, because checking that $\mathcal{I}\cdot\mathcal{O}_Y$ becomes locally principal is very difficult in practice. Instead, we will restrict our attention to the case when X is smooth and $\mathcal{I}$ is an ideal defining a smooth subvariety Z of codimension r.

In general,

$$f^{-1}(V(\mathcal{I})) \cong \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}\left(\left(\bigoplus_{m\geq 0}\mathcal{I}^m/\mathcal{I}^{m+1}\right)_{\mathrm{red}}\right).$$

If X and Z are smooth, $\bigoplus_{m\geq 0}\mathcal{I}^m/\mathcal{I}^{m+1} \cong \operatorname{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2)$ and $\mathcal{I}/\mathcal{I}^2$ is locally free. In this case,

$$\begin{tikzcd} f^{-1}(V(\mathcal{I})) = E \cong \mathcal{M}\mathrm{ax}\mathcal{P}\mathrm{roj}(\operatorname{Sym}^\bullet(\mathcal{I}/\mathcal{I}^2)) \arrow[d] \\ Z \end{tikzcd}$$

is a projective bundle.

In particular, E is a smooth variety and $\widetilde{X}$ is a smooth variety. Then $\mathcal{O}_{\widetilde{X}}(E)|_E \cong \mathcal{O}_E(-1)$.

Example 12.3.2. Suppose $Z = \{P\}$ is a point. Then $E \cong \mathbb{P}^{n-1}$ where $n = \dim X$. Then

$$(E^n) = (\mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{n-1}) = (-1)^{n-1}.$$

12.4. **Picard group of a smooth blow-up.** Suppose X and Z are smooth with $\operatorname{codim}_X(Z) = r \geq 2$. We have the diagram:

$$\begin{tikzcd} f^{-1}(U) \arrow[r, hook] \arrow[d, "\cong"] & \widetilde{X} \arrow[d, "f"] & E \arrow[l, hook'] \arrow[d] \\ U = X\setminus Z \arrow[r, hook] & X & Z. \arrow[l, hook'] \end{tikzcd}$$

Note that

$$\begin{aligned} \operatorname{Pic}(X) &\xrightarrow{\cong} \operatorname{Pic}(U) \\ \mathcal{L} &\mapsto \mathcal{L}|_U \end{aligned}$$

and we have that

$$\begin{array}{l} \mathbb{Z} \longrightarrow \operatorname{Pic}(\widetilde{X}) \longrightarrow \operatorname{Pic}(f^{-1}(U)) \longrightarrow 0 \\ 1 \longrightarrow \mathcal{O}_X(E). \end{array}$$

The morphism

$$\begin{aligned} \operatorname{Pic} X \oplus \mathbb{Z} &\to \operatorname{Pic}(\widetilde{X}) \\ (\mathcal{L}, m) &\mapsto f^*(\mathcal{L}) \otimes \mathcal{O}_{\widetilde{X}}(mE) \end{aligned}$$

is surjective.

We claim that it is also injective. Suppose $f^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}(mE) \cong \mathcal{O}_{\widetilde{X}}$. Choose a curve $\mathbb{C} \subseteq f^{-1}(y)$, $y \in Z$. Then

$$\mathcal{O}_C \cong f^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}(mE)|_C \cong \mathcal{O}_C(-m)$$

implies that $m = 0$. Then $f^*\mathcal{L} \cong \mathcal{O}_{\widetilde{X}}$, so

$$\mathcal{L} \cong \mathcal{L} \otimes f_*\mathcal{O}_{\widetilde{X}} = f_*(f^*(\mathcal{L})) \cong f_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X.$$

Therefore, $\operatorname{Pic}(\widetilde{X}) \cong \operatorname{Pic}(X) \oplus \mathbb{Z}$.

Let us compute $K_{\widetilde{X}/X}$. We know that $K_{\widetilde{X}/X} = aE$, so we just need to find a.

Definition 12.4.1. If X is a smooth variety, $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ form a *system of coordinates* if $dx_1, \ldots, dx_n$ give an isomorphism

$$\Omega_U \cong \mathcal{O}_U^{\oplus n}.$$

Equivalently, the morphism $\varphi = (x_1, \ldots, x_n) \colon U \to \mathbb{A}^n$ induces an isomorphism $\varphi^*\Omega_{\mathbb{A}^n} \to \Omega_U$, i.e. it is étale.

Algebraically, if $p \in X$ and $x_1, \ldots, x_n \in \mathcal{O}_{X,p}$ are a regular system of parameters, there is an open neighborhood $U \ni p$ such that $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ form a system of generators.

In our case, $Z \subseteq X$ is smooth, so for any $p \in Z$ there is an open neighborhood $U \ni p$ and a system of coordinates $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ such that $Z \cap U = (x_1, \ldots, x_r = 0)$. We have a closed immersion

$$f^{-1}(U) = \widetilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$$

and $\widetilde{U}$ is defined by $x_i y_j = x_j y_i$ for $1 \leq i \leq j \leq r$.

We have charts $V_1, \ldots, V_r$ on $\widetilde{U}$ where V_i is defined by $y_i \neq 0$ and coordinates $u_1, \ldots, u_n$ on V_i such that

$$x_i = u_i, \quad x_j = u_j \text{ for } j > r, \quad x_j = u_i u_j \text{ for } 1 \leq j \leq r, j \neq i.$$

Here $u_j = \frac{y_j}{y_i}$ on V_i.

In these coordinates, note that $E|_{V_j} = (u_i = 0)$, since the ideal defining Z is $(x_1 = \cdots = x_r = 0)$.

For simplicity, assume that $i = 1$ so that

$$x_1 = u_1, \quad x_j = u_1 u_j \text{ for } 2 \leq j \leq r, \quad x_j = u_j \text{ for } j > r.$$

The map $\varphi\colon f^*\Omega_U \to \Omega_{\widetilde{U}}$ satisfies

$$\begin{aligned} f^*(dx_1) &= du_1, && \\ f^*(dx_j) &= d(u_1u_j) = u_1du_j + u_jdu_1 && \text{for } 2 \leq j \leq r, \\ f^*(dx_j) &= du_j && \text{for } j > r. \end{aligned}$$

We know that $a = \operatorname{ord}_E(\det(\varphi))$, so just need to compute $\det(\varphi)$. Under $\det(\varphi)$:

$$f^*(dx_1) \wedge \cdots \wedge f^*(dx_n) \mapsto du_1 \wedge (u_1du_2 + u_2du_1) \wedge \cdots = u_1^{r-1}du_1 \wedge du_2 \wedge \cdots \wedge u_n.$$

The conclusion is that

$$K_{\widetilde{X}/X} = (r-1)E.$$

Therefore,

$$\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}((r-1)E).$$

Suppose X is a smooth projective surface and $\begin{array}{c} \widetilde{X} = \operatorname{Bl}_p X \\ \big\downarrow f \\ X \end{array}$ is the blow-up at p.

Then $(E^2) = (-1)$, $(f^*D_1 \cdot f^*D_2) = (D_1 \cdot D_2)$, and $(f^*D \cdot E) = 0$.

Suppose now that D is an effective divisor on X. Then

$$f^*D = \widetilde{D} + \alpha E$$

for some constant α.

Exercise. The constant α is $\operatorname{mult}_p D$ defined in coordinates as follows: if around p the equation for D is h, $\operatorname{mult}_p D = \max\{r \mid h \in \mathfrak{m}_p^r\}$.

In particular, note that $\operatorname{mult}_p D = 0$ if and only if $p \notin \operatorname{supp}(D)$. In this case it is clear that $f^*D = \widetilde{D}$. Moreover, $\operatorname{mult}_p D = 1$ if and only if $p \in D$ and D is smooth at p.

Question. Given an irreducible curve $C \subseteq X$, what is $p_a(\widetilde{C})$ in terms of $p_a(C)$?

By the adjuction formula 11.2.1,

$$\begin{aligned} 2p_a(C) - 2 &= (C \cdot (K_X + C)), \\ 2p_a(\widetilde{C}) - 2 &= (\widetilde{C} \cdot (K_{\widetilde{X}} + \widetilde{C})). \end{aligned}$$

We have that $f^*C = \widetilde{C} + mE$ where $m = \mathrm{mult}_p(C)$ and $K_{\widetilde{X}} \sim f^*K_X + E$. Therefore,

$$\begin{aligned} 2p_a(\widetilde{C}) - 2 &= (f^*(C) - mE) \cdot (f^*(K_X + C) + (1-m)E) \\ &= (C \cdot (K_X + C)) - m(1-m)(-1) \\ &= 2p_a(C) - 2 - m(m-1). \end{aligned}$$

Hence

$$p_a(\widetilde{C}) = p_a(C) - \frac{m(m-1)}{2}.$$

Conclusion. If $p \in C_{\mathrm{sing}}$, then $m \geq 2$ and $0 \leq p_a(\widetilde{C}) < p_a(C)$. This implies that if $C \subseteq X$ is an irreducible curve, then after blowing up singular points finitely many times, we get a smooth curve.

This gives a *resolution of singularities* which is easier to compute than the general result of Hironaka.

Theorem 12.4.2. *If $f\colon X \to Y$ is a proper birational morphism of smooth surfaces, f decomposes as a composition of blow-ups of points on smooth surfaces.*