MATH 678: PERIODS OF AUTOMORPHIC FORMS

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Contents

1. Overview and classical statements

Let F be a number field. We first state a theorem which may not not be fully clear right now but we will work towards understanding and proving it throughout the class.

Theorem 1.1 (Waldspurger)**.** *Let $T \subseteq \mathrm{GL}_2$ be a non-split torus. Let $\phi = \bigotimes \phi_v \in \pi$, where π is an automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. Then*

$$\left| \int_{T(F)\backslash T(\mathbb{A})} \phi(h)dh \right|^2 = c \cdot L(1/2, \pi) \cdot L(1/2, \pi \otimes \eta) \cdot \prod_v \mathcal{L}_v(\phi_v, \phi_v)$$

where the constant c is independent of π and $\mathcal{L}_v(\phi_v, \phi_v)$ are local periods.

The rough schedule is as follows:

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- 3–4 classes on classical theory,
- 11 classes on local theory: GL_1, GL_2, quaternion algebras, Jacquet–Langlands, local Waldspurger formula,
- 11 classes on global theory: automorphic forms/Siegel–Weil formulas/proof of the Waldspurger formula 1.1.

Today, we will work on reformulating Waldspurger's formula 1.1 in classical forms.

1.1. **Dirichlet characters.** Let $N \in \mathbb{N}$. A *Dirichlet character* is a group homomorphism

$$\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times.$$

A character χ is *primitive* if it does not factor through

$$\begin{array}{ccc} (\mathbb{Z}/N\mathbb{Z})^\times & \longrightarrow & \mathbb{C}^\times \\ & \searrow \quad \nearrow & \\ & (\mathbb{Z}/N'\mathbb{Z})^\times & \end{array}$$

for any $N' < N$, $N'|N$.

We extend it $\mathbb{Z}$ by setting

$$\chi(n) = \begin{cases} \chi(n \mod N) & \text{if } (n, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We then set

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} \text{ for } s \in \mathbb{C}.$$

(1) This converges to a holomorphic function for $\mathrm{Re}(s) > 1$.
(2) If $\chi = \mathbb{1}$, $L(\chi, s) = \zeta(s)$.
(3) We have an Euler product $L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1}$.

Theorem 1.2. *Let $\chi\colon (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}^\times$ be primitive. Define*

$$\nu_\chi = \frac{1 - \chi(-1)}{2},$$
$$\tau(\chi) = \sum_{x \mod q} \chi(x) e^{\frac{2\pi i x}{q}},$$
$$\epsilon(\chi) = i^{-\nu_x} \frac{\tau(\chi)}{\sqrt{q}}.$$

It is a simple exercise to check that $|\epsilon(\chi)| = 1$.

Set $\Lambda(\chi, s) = \pi^{-(s+\nu_x)/2}\Gamma((s + \nu_x)/2)L(\chi, s)$. Then $\Lambda(\chi, s)$ has a meromorphic continuation to $\mathbb{C}$ and

$$\Lambda(\chi, s) = \epsilon(\chi)q^{1/2-s}\Lambda(\chi^{-1}, 1 - s).$$

1.2. **Automorphic functions.** We follow [Bum97, Chapter 1] here. All the details we omit can be found there.

Let $\mathcal{H} = \{x + iy \mid y > 0\} \subseteq \mathbb{C}$ be the *upper half plane*. Let $G = \mathrm{SL}_2(\mathbb{R})$ act on $\mathcal{H}$ via

$$gz = \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This extends to an action on function $f\colon \mathcal{H} \to \mathbb{C}$ via $gf(z) = f(gz)$.

The *Laplace operator* $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on $\mathbb{C}^\infty(\mathcal{H})$ has the property:

$$\Delta(g \cdot f) = g \cdot (\Delta f) \quad \text{for } g \in G.$$

In fact, it generates the algebra of G-invariant differential operators.

Definition 1.3. A *Maass form* is a function $f \in C^\infty(\mathcal{H})$ which is:

- an eigenfunction of Δ, i.e. $\Delta f = (1/4 - \nu^2)f$ for some ν,
- $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z}) \subseteq G = \mathrm{SL}_2(\mathbb{R})$,
- $f(x + iy) = O(y^N)$ for some N as $y \to \infty$.

Let f be a Maass form. Then $f(z+1) = f(z)$ because $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Define the *K-Bessel function* for $\nu \in \mathbb{C}$ as:

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^\nu \frac{dt}{t}$$

Then

$$\Delta W_\nu = (1/4 - \nu^2) W_\nu$$

where $W_\nu(z) = 2\sqrt{y} K_\nu(2\pi y) e^{2\ piix}$ is the *Whittaker function*.

In general, for $f \in C^\infty(\mathcal{H})$, we have that

$$f(z) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty (f, W_{it}(r\cdot))_{\mathcal{H}} W_{it}(rz) t \sinh(\pi t) dt \frac{dr}{r},$$

where $(-,-)_{\mathcal{H}}$ is a scalar product on $\mathcal{H}$. Therefore, the *Whittaker function* give general Fourier expansions.

For a Maass form f, we have a simpler *Fourier expansion*:

$$f(z) = a_0(y) + \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n W_\nu(nz),$$

where

$$a_n = \int_0^1 f(x+iy) e^{2\pi i n x} dx.$$

This is analogous to the *q-expansion* of holomorphic modular forms.

Definition 1.4. The L-function of a Maass form f is defined as

$$L(f,s)=\sum_{n\geq 1}\frac{a_n}{n^s}.$$

Since $a_n=O(\sqrt{n})$, this series converges for $\mathrm{Re}(s)>\frac{3}{2}$.

Definition 1.5. A Maass form f is a *cusp form* if $a_0=0$. It is *even/odd* if $f(-\overline{z})=\pm f(z)$.

Proposition 1.6. *Assume that f is even/odd. We have that*

$$\Lambda(f,s)=\pi^{-s}\Gamma\left(\frac{s+\epsilon+\nu}{2}\right)\Gamma\left(\frac{s+\epsilon-\nu}{2}\right)L(s,f)$$

where $\epsilon=\begin{cases}0 & \textit{if even,}\\ -1 & \textit{if odd.}\end{cases}$

This function has meromorphic continuation to $\mathbb{C}$ and functional equation:

$$\Lambda(f,s)=(-1)^{\epsilon}\Lambda(f,1-s).$$

It is holomorphic if f is a cusp form.

Sketch of proof. Recall that

$$f(z)=\sum_{n\geq 0}a_nW_\nu(nz).$$

Note the following properties:

(1) when $y>4$, $|K_\nu(y)|\leq e^{-y/2}K_{\mathrm{Re}(\nu)}(2)$,
(2) for $s\gg 0$, $\int\limits_0^\infty K_\nu(y)y^s\frac{dy}{y}=2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)$.

Assume that f is even for simplicity. Look at $\int\limits_0^\infty f(iy)y^{s-1/2}dy$. By convergence at ∞ and $f(i/y)=f(iy)$, the same is true at 0. By evaluating the above integral term by term, we have that

$$\int_0^\infty f(iy)y^{s-1/2}dy=\frac{1}{2}\Lambda(f,s).$$

This proves the above claims. □

Remark 1.7. Note that we have a resemblance to Waldspurger's Theorem 1.1 here already. Integrating a modular form over a torus (here, from 0 to ∞) gives an L-function.

1.3. **Hecke operators.** Let p be a prime. Define for a function $f\colon\mathcal{H}\to\mathbb{C}$:

$$T_p(f)=\begin{pmatrix}p&0\\0&1\end{pmatrix}f+\sum_{b \mod p}\begin{pmatrix}1&b\\0&p\end{pmatrix}f$$

where again

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}f(z)=f\left(\frac{az+b}{cz+d}\right).$$

Then T_p commute with each other and with Δ. Then the collection (T_p) acts on the space of Maass forms with fixed eigenvalue.

Definition 1.8. A *Maass eigenform* is a Maass form which is a simultaneous eigenform of all T_p.

If f is such a form and $f \neq 0$, then $a_1 \neq 0$ because all the other Fourier coefficients are given in terms of a_1.

Assume without loss of generality that $a_1 = 1$. Then

$$L(s, f) = \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1}$$

for $\mathrm{Re}(s) > \frac{3}{2}$ and $a_p f = T_p(f)$.

1.4. Twists.

Let χ be a Dirichlet character and f be a Maass form. Then

$$L(f \otimes \chi, s) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s} = \prod_p (1 - \chi(p) a_p p^{-s} + p^{-2s})^{-1}$$

with the last equality only valid if f is a Hecke eigenform.

1.5. Waldspurger's formula for Maass forms.

Let $D \in \mathbb{Z}$ be negative, squarefree, except for possible 4 as a factor. A *Heegner point* is **the** unique $z \in \mathcal{H}$ such that

$$az^2 + bz + c = 0,\ b^2 - 4ac = D,\ a, b, c \in \mathbb{Z},\ (a, b, c) = 1,\ a > 0.$$

Let $\mathcal{H}(D)$ be the set of such Heegner points. Note that if $z \in \mathcal{H}(D)$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z}) = \Gamma$, then $\gamma z \in \mathcal{H}(D)$. We will see next time that the number of orbits of this action is finite.

We can finally state a classical version of Waldspurger's formula.

Theorem 1.9 (Waldspurger, classical version I)**.** *Let f be a normalized ($a_1 = 1$), Maass, even, eigenform. Then*

$$L(f, 1/2) \cdot L(f \otimes \eta_D, 1/2) = \frac{1}{2^?} \frac{1}{\sqrt{D}} \left| \sum_{z \in \Gamma \backslash \mathcal{H}(D)} f(z) \right|^2.$$

Here, η_D is a Dirichlet character associated to $\mathbb{Q}(\sqrt{D})$.

Another (somewhat more complicated) classical version of this formula will be stated in Theorem 1.27.

We will define the character η_D carefully next time.

Next time, we will see and explicit Maass form, and prove this theorem in that case. We will also see some applications of this formula.

1.6. **Number theoretic background.** Number fields are finite extensions of $\mathbb{Q}$. For an irreducible polynomial $P \in \mathbb{Q}[x]$,

$$K = \frac{\mathbb{Q}[x]}{P(x)}.$$

Let $\mathcal{O} = \mathcal{O}_K \subseteq K$ be the *ring of integers*, which is a free $\mathbb{Z}$-module of rank $d = [K : \mathbb{Q}]$. For any $x \in K$, $x \in \mathcal{O}_K$ if and only if $x^k + \sum a_i x^i = 0$ for some $a_i \in \mathbb{Z}$.

Unique prime factorization fails usually in $\mathcal{O}_K$. Let $\mathcal{O}_K^\times \subseteq \mathcal{O}_K$ be the subset of invertible elements, called *units*. An element $a \in \mathcal{O}_K$ is prime if and only if when $a = bc$ for $b, c \in \mathcal{O}_K$ then b or c is a unit.

Clearly, $21 = 3 \cdot 7 = (1 - 2\sqrt{-5})(1 + 2\sqrt{-5})$ inside $\mathbb{Q}(\sqrt{-5})$ are two decompositions of 21 into prime ideals, so unique factorization fails here.

A *fractional ideal* is an $\mathcal{O}$-submodule of K of *finite type* (or equivently, such that $aI \subseteq \mathcal{O}_K$ for some $a \in K^\times$).

For example, for $K = \mathbb{Q}$, $q\mathbb{Z}$ for any $q \in \mathbb{Q}^\times$ is a fractional ideal, but $\mathbb{Z}_{(p)}$ is not.

Let $F(\mathcal{O}_K)$ be the set of fraction ideals. For $I, J \in F(\mathcal{O}_K)$,

$$I \cdot J = \left\{ \sum a_i b_i \mid a_i \in I, b_i \in J \right\} \in F(\mathcal{O}_K).$$

Defining

$$I^{-1} = \{a \in K \mid aI \subseteq \mathcal{O}_K\}$$

makes $F(\mathcal{O}_K)$ into a group with identity element $\mathcal{O}_K$.

An ideal $\mathfrak{p} \subseteq \mathcal{O}$ is *prime* if $\mathcal{O}/\mathfrak{p}$ is an integral domain.

Theorem 1.10. *Any ideal $I \in F(\mathcal{O}_K)$ can be uniquely written as*

$$I = \prod_{\mathfrak{p} \text{ prime}} \mathfrak{p}^{n_\mathfrak{p}}$$

with $n_\mathfrak{p} \in \mathbb{Z}$ and $n_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$.

Ideals of the type $\langle x \rangle := x \cdot \mathcal{O}$ for $x \in K^\times$ are called *principal*. Consider the subgroup:

$$P(\mathcal{O}_K) = \text{group of principal ideals} \subseteq F(\mathcal{O}_K)$$

and the quotient

$$\mathrm{Cl}(K) = \frac{F(\mathcal{O}_K)}{P(\mathcal{O}_K)}$$

called the *class group*.

It is a standard theorem in algebraic number theory that the class group is finite.

Let $\mathfrak{p}$ be prime. The norm of $\mathfrak{p}$ is

$$N_K(\mathfrak{p}) = \#\left(\mathcal{O}/\mathfrak{p}\right).$$

This extends to a homomorphism

$$N_K \colon F(\mathcal{O}) \to \mathbb{Q}^\times.$$

For a principal ideal,

$$N_K(\langle x\rangle) = |N(x)|$$

where

$$N(x) = \text{determinant of the map } a \mapsto ax \text{ from } K \text{ to } K.$$

We can define for any number field K, the *Dedekind zeta function*

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a}\subseteq\mathcal{O} \\ \text{ideals}}} \frac{1}{N(\mathfrak{a})^s} \text{ for } \operatorname{Re}(s) > 1.$$

It has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p} \text{ prime}} (1 - N_K(\mathfrak{p})^{-s})^{-1}.$$

Adding Euler factors *at infinity*:

$$\Lambda_K(s) = \pi^{-r_1\cdot s/2}(2\pi)^{-r_2\cdot s}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_K(s),$$

one can prove that this *completed* zeta function has a functional equation and analytic continuation:

$$\Lambda_K(s) = |D_K|^{1/2-s}\Lambda_K(1-s).$$

Here, r_1 is the number of $\sigma\colon K \hookrightarrow \mathbb{R}$ and r_2 is the number of proper $\sigma\colon K \hookrightarrow \mathbb{C}$ (up to conjugation) so that $r_1 + 2r_2 = d = [K : \mathbb{Q}]$. Moreover, for any basis $\omega_1, \ldots, \omega_d$ of $\mathcal{O}_K$, the *discriminant* of K is:

$$D_K = (\det(\sigma_i(\omega_j))_{i,j=1,\ldots,d})^2.$$

Proposition 1.11 (Class Number Formula)**.** *The residue of $\Lambda_K(s)$ at $s = 1$ is*

$$\frac{2^{r_1}\#\mathrm{Cl}(K)\cdot R_K}{w_K\sqrt{|D_K|}}$$

where w_k is the number of roots of units in K and writing $\mathcal{O}_K^\times \cong \{$roots of unity$\} \times \mathbb{Z}^{r_1+r_2-1}$, R_K is the volume of the lattice $\mathbb{Z}^{r_1+r_2-1}$.

We now discuss quadratic fields. Suppose $K = \mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Z} \setminus \{0, 1\}$, square free. Then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & d \equiv 2,3\ (4) \\ \mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] & d \equiv 1\ (4) \end{cases}$$

and

$$D_K = \begin{cases} 4d & d \equiv 2,3\ (4) \\ d & d \equiv 1\ (4) \end{cases}.$$

We may hence write concisely

$$\mathcal{O}_K = \mathbb{Z}\left[\frac{D_K + \sqrt{D_K}}{2}\right].$$

Let χ be a primitive Dirichlet character of conductor q. Suppose $\chi^2 = 1$ (so χ is a quadratic character). Then

$$\Lambda_{\mathbb{Q}}(s)\Lambda(\chi, s) = \Lambda_{\mathbb{Q}(\sqrt{\chi(-1)q})}(s).$$

If K is quadratic of discriminant D, there is a unique primitive Dirichlet character η_K of conductor D such that

$$\Lambda_{\mathbb{Q}}(s)\Lambda(\eta_K, s) = \Lambda_K(s).$$

(A similar result holds for higher degree extension, but we do not discuss this here.)

Concretely,

$$\eta_K(p) = \begin{cases} 0 & \text{if } p \text{ ramifies}, i.e.\ \langle p \rangle = \mathfrak{q}^2 \text{ for } \mathfrak{q} \subseteq \mathcal{O}_K, \\ 1 & \text{if } p \text{ splits}, i.e.\ \langle p \rangle = \mathfrak{q}_1\mathfrak{q}_2 \text{ for } \mathfrak{q}_1 \neq \mathfrak{q}_2, \\ -1 & \text{if } p \text{ is inert (remains prime).} \end{cases} .$$

1.7. **Classical Waldspurger's formula for Eisenstein series.** Recall the statement of Theorem 1.9.

Let f be a Maass cusp eigenform and $K/\mathbb{Q}$ be imaginary quadratic. Then

$$\left| \sum_{\tau \in \mathcal{H}(D)/\operatorname{SL}_2(\mathbb{Z})} f(\tau) \right|^2 = \sqrt{|D|} \cdot \Lambda(f, 1/2) \cdot \Lambda(f \otimes \eta_K, 1/2).$$

Here, $\tau \in \mathcal{H}(D)$ satisfies

$$a\tau^2 + b\tau + c = 0, \quad b^2 - 4ac = D, \quad a, b, c \in \mathbb{Z}.$$

What are some examples of Maass forms? The first examples are *Eisenstein series.* For $\lambda \in \mathbb{C}$, define

$$E(z, \lambda) = \frac{1}{2}\pi^{-(\lambda+1/2)}\Gamma(\lambda + 1/2) \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^{y+1/2}}{|m + nz|^{2\lambda+1}}$$

for $z = x + iy \in \mathcal{H}$. Note that

$$\Delta y^{\lambda+1/2} = (\lambda^2 - 1/4)y^{\lambda+1/2}.$$

Moreover,

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^{y+1/2}}{|m + nz|^{2\lambda+1}} = \zeta(2\lambda) \sum_{\gamma\in\Gamma_\infty\setminus\operatorname{SL}_2(\mathbb{Z})} \operatorname{Im}(\gamma z)^{\lambda+1/2}$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

We can hence deduce that $E(z, \lambda)$ is a Δ-eigenform since Δ commutes with the Γ-action.

Theorem 1.12.

(1) *The function $E(z, \lambda)$ has a meromorphic contuation to $\lambda \in \mathbb{C}$.*
(2) *We have that $\Delta E(\cdot, \lambda) = (\lambda^2 - 1/4)E(\cdot, \lambda)$.*
(3) *The Eisenstein series has a Fourier expansion*

$$E(z, \lambda) = a_0(y, \lambda) + \sum_{n\neq 0} a_n(\lambda)W_\lambda(nz)$$

for

$$a_0(y,\lambda) = \Lambda_{\mathbb{Q}}(2\lambda+1)y^{\lambda+1/2} + \Lambda_{\mathbb{Q}}(1-2y)y^{-\lambda+1/2}$$
$$a_n(\lambda) = |n|^{\lambda}\sigma_{-2\lambda}(|n|) \quad \text{and} \quad \sigma_\nu(m) = \sum_{d|m} d^\nu.$$

One can also check that $E(z,\lambda)$ is a Hecke eigenform with T_p-eigenvalue $p^\lambda + p^{-\lambda}$. Indeed, the L-function has a decomposition

$$\begin{aligned} L(E(\cdot,\lambda),s) &= \sum \frac{a_n(\lambda)}{n^s} \\ &= \zeta_{\mathbb{Q}}(s+\lambda)\zeta_{\mathbb{Q}}(s-\lambda) \\ &= \prod_p (1-(p^\lambda+p^{-\lambda})p^{-s}+p^{-2s})^{-1}. \end{aligned}$$

The completed L-function is

$$\Lambda(E(\cdot,\lambda),s) = \Lambda(s+\lambda)\Lambda(s-\lambda).$$

The following should be thought of as the easiest instance of Waldspurger's formula 1.1.

Theorem 1.13. *Let $K/\mathbb{Q}$ be imaginary quadratic of discriminant D. Then*

$$\sum_{\tau\in\mathcal{H}(D)/\Gamma} E(\tau,\lambda) = \frac{|\mathcal{O}_K^\times|}{2}\sqrt{|D|}^{1/2+\lambda}\Lambda_K(\lambda+1/2).$$

Exercise. For $\lambda \in i\mathbb{R}$, Theorem 1.13 implies the analog of Theorem 1.9 for Eisenstein series.

Remark 1.14. When $K = \mathbb{Q}(i)$, $\mathcal{H}(D)/\Gamma$ consist of just one element, i. Then the theorem is that

$$E(i,\lambda) = \frac{|\mathcal{O}_K^\times|}{2}\sqrt{|D|}^{1/2+\lambda}\Lambda_{\mathbb{Q}(i)}(\lambda+1/2)$$

which one can see directly from the definition of $E(\tau,\lambda)$.

1.8. **Heegner points.** Let D be a negative square-free number except for a possible factor of 4. Consider quadratic forms

$$q(x,y) = ax^2+bxy+cy^2, \quad a,b,c\in\mathbb{Z},\ a>0, \quad b^2-4ac = D.$$

Recall that a Heegner point is the unique $\tau \in \mathcal{H}$ such that $q(\tau,1)=0$.

The group Γ acts on such quadratic forms: for $\gamma\in\Gamma$, $(x,y)\mapsto q(\gamma x,\gamma y)$ is a new quadratic forms with such property. Gauss noticed there is a *composition law* on such quadratic forms, up to Γ-action. This is explained by the following theorem.

Theorem 1.15. *Let K be an imaginary quadratic field of discriminant D. Let $\tau\in\mathcal{H}(D)$ be a solution of $ax^2+bx+c=0$ as above. Let $\mathfrak{a}\in \mathrm{Cl}(K)$ correspond to the $\mathcal{O}_K$-module $\mathbb{Z}[1,\tau]$. This induces a bijection*

$$\begin{aligned} \mathcal{H}(D)/\operatorname{SL}_2(\mathbb{Z}) &\cong \mathrm{Cl}(K) \\ \tau &\mapsto \mathbb{Z}[1,\tau] = I(\tau). \end{aligned}$$

If τ' is a solution of $ax^2-bx+c=0$, then $\mathbb{Z}[\tau']$ represents the inverse of $\mathbb{Z}[\tau]$ in $\mathrm{Cl}(K)$.

Note that Theorem 1.13 implies the Class Number Formula 1.11. To prove this, take the residue at $\lambda = 1/2$ of both sides and compute that

$$\mathrm{Res}_{\lambda=1/2} E(\tau, \lambda) = 1/2.$$

1.9. **Proof of Theorem 1.13.**

Lemma 1.16. *We have that* $N(\mathbb{Z}[1,\tau]) = \frac{1}{a} = \frac{2\mathrm{Im}(\tau)}{\sqrt{|D|}}$.

Proof. Since $\tau = \frac{-b+\sqrt{D}}{2a}$ and $\mathcal{O}_K = \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$, we have that

$$\mathbb{Z}[1, a\tau] = \mathcal{O}_K,$$

which proves the first equality. The second equality is a simple computation. □

Proposition 1.17. *Let* $\mathcal{A} \in \mathrm{Cl}(K)$ *correspond to* $\mathbb{Z}[1,\tau]$. *The map*

$$\{\mathfrak{b} \in \mathcal{A}^{-1} \mid \mathfrak{b} \subseteq \mathcal{O}_K\} \to \mathbb{Z}[1,\tau] \setminus \{0\}/\mathcal{O}_K^\times$$
$$\mathfrak{b} \mapsto x \text{ such that } \langle x \rangle = \mathfrak{b} \cdot \mathbb{Z}[1,\tau]$$

is a bijection.

Proof. Suppose $\mathfrak{b}$ is integral and $\mathfrak{b}\mathbb{Z}[1,\tau] = \langle x \rangle$ for some $x \in K^\times$. Since $\mathfrak{b}$ is integral, $x \in \mathbb{Z}[1,\tau]$ and it is well-defined up to a unit.

Conversely, suppose $x = n + m\tau \in \mathbb{Z}[1,\tau] \setminus \{0\}$. Then $\langle x \rangle = \mathbb{Z}[1,\tau] \cdot \mathfrak{b}$ for some ideal $\mathfrak{b}$. We claim that $\mathfrak{b}$ is integral. By Lemma 1.16,

$$\mathbb{Z}[1,\tau] \cdot \mathbb{Z}[a, a\tau'] = \mathcal{O}_K$$

where τ' is as in Theorem 1.15. Then

$$\mathbb{Z}[a, a\tau']\langle x \rangle = \mathfrak{b}$$

is clearly integral and $\mathfrak{b} \mapsto x$, as required. □

Recall that

$$E(z,\lambda) = \frac{1}{2}\pi^{-(\lambda+1/2)}\Gamma(\lambda+1/2) \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^{\lambda+1/2}}{|m+nz|^{2\lambda+1}} \quad \text{for } z = x+iy \in \mathcal{H}.$$

We will mostly be concerned with evaluating the infinite sum.

Let $\tau \in \Gamma\backslash\mathcal{H}(D)$. Then

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\mathrm{Im}(\tau)^{\lambda+1/2}}{|m+n\tau|^{2\lambda+1}} = |\mathcal{O}_K^\times| \sum_{\substack{\mathfrak{b}\in\mathcal{A}^{-1} \\ \mathfrak{b}\subseteq\mathcal{O}_K}} \left|\frac{|d_K|}{2N(b)}\right|^{1/2+\lambda}$$

and summing over all τ, we get

$$\sum_{\tau} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\mathrm{Im}(\tau)^{\lambda+1/2}}{|m+n\tau|^{2\lambda+1}} = |\mathcal{O}_K^\times| \left(\frac{\sqrt{|d_K|}}{2}\right)^{\lambda+1/2} \zeta_K(\lambda+1/2).$$

Altogether this proves Theorem 1.13.

1.10. **Gauss' class number problem.** Let $h(D) = |\mathrm{Cl}(K)|$. If generalized Riemann hypothesis (GRH) is true, then one can show that

$$h(D) \gg \frac{|D|}{\log |D|}.$$

In particular, there are supposed to be finitely many D with $h(D) = 1$, assuming (GRH) is true.

Now assume that the Riemann hypothesis (RH) is false. We still prove that there are finitely many D with $h(D) = 1$. Let $\mathrm{Re}(\lambda_0) > 0$ be such that $\Lambda_{\mathbb{Q}}(\lambda_0 + 1/2) = 0$. We plug $\lambda_0 + 1/2$ into Waldspurger's formula. Then

$$\sum_{\tau \in \Gamma \backslash \mathcal{H}(D)} E(\tau, \lambda_0) = 0.$$

Recall that

$$E(z, \lambda) = \Lambda_{\mathbb{Q}}(2\lambda + 1) y^{\lambda + 1/2} + \Lambda_{\mathbb{Q}}(1 - 2\lambda) y^{-\lambda + 1/2} + O(y^{-N}) \text{ for any } N > 0.$$

Assume that $h(D) = 1$. Pick $\tau = \frac{-b+\sqrt{D}}{2}$, where $b = 0, 1$ such that $b \equiv D$ (4). Then

$$0 = \frac{E(\tau, \lambda_0)}{\sqrt{|D|}} = \underbrace{\Lambda_{\mathbb{Q}}(2\lambda_0 + 1)}_{\neq 0} |D|^{\lambda_0/2} + \Lambda_{\mathbb{Q}}(1 - 2\lambda_0) |D|^{-\lambda_0/2} + O(|D|^{-n}).$$

This can only be true for finitely many D.

1.11. **Quaternion algebras.** We will next study how Theorem 1.9 relates to Waldspurger's formula 1.1. We begin with a quick introduction to quaternion algebras.

Let F be a field of characteristic not equal to 2. For $a, b \in F^\times$, there is a unique unitary associative F-algebra of dimension 4 over F, called $\left(\frac{a,b}{F}\right)$ with basis $1, i, j, ij$ such that $i^2 = a$, $j^2 = b$, $ij = -ji$.

Definition 1.18. A *quaternion algebra* over F is an algebra isomorphic to $\left(\frac{a,b}{F}\right)$.

Proposition 1.19. *The following are equivalent for an F-algebra D of dimension 4:*

(1) *D is quaternion,*
(2) *$F = Z(D)$ the center of D and D is simple (has no 2-sided ideals),*
(3) *$D \otimes_F \overline{F} = M_2(\overline{F})$.*

Fact 1.20. *A quaternion algebra is either $M_2(F)$ (*split*) or a division algebra (*non-split*).*

Let F be $\mathbb{Q}_p$ or $\mathbb{R}$. Then there is a unique non-split quaternion algbera. We can take $\left(\frac{a,p}{\mathbb{Q}_p}\right)$ for any $a \in (F^\times)^2$ when $F = \mathbb{Q}_p$, and the Hamilton quaternions $\left(\frac{-1,-1}{\mathbb{R}}\right)$ when $F = \mathbb{R}$.

Definition 1.21. A *prime* is a prime of $\mathbb{Z}$ or ∞. If v is a prime, then

$$\mathbb{Q}_v = \begin{cases} \mathbb{Q}_p & \text{if } v = p, \\ \mathbb{R} & \text{if } v = \infty. \end{cases}$$

Theorem 1.22. *Let D be a quaternion algebra over $\mathbb{Q}$. The set of* primes *such that $D\otimes_{\mathbb{Q}}\mathbb{Q}_v$ is non-split is finite and of even cardinality. If S is any finite set of primes of even cardinality, there is a unique quaternion algebra $D/\mathbb{Q}$ that does not split precisely for $v\in S$.*

Definition 1.23. An *order* $\mathcal{O}\subseteq D$ in a quaternion algebra D over $\mathbb{Q}$ is a free $\mathbb{Z}$-module of rank 4 such that $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Q}=D$ which is also a subring.

There is always a maximal order, we fix it from now on. When $D=M_2(\mathbb{Q})$, we may take $M_2(\mathbb{Z})$ as the maximal order (it is unique in this case but not in the non-split case).

Example 1.24. The following quaternion algebra is non-split at $\{\infty,p\}$:

- when $p=2$, $\left(\frac{-1,-1}{\mathbb{Q}}\right)$,
- when $p\equiv 3\ (4)$, $\left(\frac{-1,-p}{\mathbb{Q}}\right)$,
- when $p\equiv 5\ (8)$, $\left(\frac{-2,-p}{\mathbb{Q}}\right)$.

Definition 1.25. A quaternion algebra is *definite* if it is non-split at ∞.

Suppose D is definite. Let d be the product of primes of $\mathbb{Z}$ such that D is non-split at p. Fix a maximal order $\mathcal{O}\subseteq D$. Let

$$\hat{\mathbb{Z}}=\prod_{p<\infty}\mathbb{Z}_p.$$

Write $D_f=D\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}$ and $\mathcal{O}_p=\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Z}_p$, a maximal order in $D_p=D\otimes\mathbb{Q}_p$. Then

$$D_f^\times=\left\{x\in\prod_p D_p^\times\ \middle|\ x\in\mathcal{O}_p^\times \text{ for almost all } p\right\}.$$

Define the class group of an order $\mathcal{O}$ as

$$\mathrm{Cl}(\mathcal{O})=\Delta D^\times\backslash D_f^\times/\prod_p\mathcal{O}_p^\times.$$

Here $\Delta\colon D^\times\to D_f^\times$ is the diagonal embedding.

This set finite when D is a definite quaternion algebra. It is in bijection with to the class group of $\mathcal{O}$ defined classically. One should think of it as the *0-dimensional analog* of the modular curve in this case.

Define the set of *weight 2 modular forms* as

$$S_2^D(1)=\{f\colon\mathrm{Cl}(\mathcal{O})\to\mathbb{C}\}.$$

A priori these are just functions on a finite set, but the interesting structure comes from the *Hecke action*.

Take any p coprime to d. Then $D_p^\times\cong\mathrm{GL}_2(\mathbb{Q}_p)$ and we may fix the isomorphism which takes $\mathcal{O}_p^\times$ to $\mathrm{GL}_2(\mathbb{Z}_p)$. For $g\in\mathrm{GL}_2(\mathbb{Q}_p)$, $f\in S_2^D(1)$, define $gf(x)=\sum_i f(xg_i)$ where:

$$\mathrm{GL}_2(\mathbb{Z}_p)g\,\mathrm{GL}_2(\mathbb{Z}_p)=\coprod_i g_i\,\mathrm{GL}_2(\mathbb{Z}_p).$$

This gives an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $S_2^D(1)$. Let T_p be the endomorphism of $S_2^D(1)$ corresponding to $g = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Obviously, these actions commute for different primes p (because we act only at one adelic place at a time).

On $\mathrm{Cl}(\mathcal{O})$, there is a counting measure. Let $S_2^D(1)^{\mathrm{new}} \subseteq S_2^D(1)$ consist of funtions such that

$$\int\limits_{\mathrm{Cl}(\mathcal{O})} f = 0,$$

i.e. the sum of the values of f is 0.

Theorem 1.26 (Jacquet–Langlands)**.** *There is an isomorphism of Hecke modules*

$$S_2^D(1)^{\mathrm{new}} \cong S_2(\Gamma_0(d))^{\mathrm{new}}.$$

1.12. **Another Waldspurger's formula.** Let $K/\mathbb{Q}$ be an imaginary quadratic field. Let $p|d$ be such that p is inert in K. Then $K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field.

One can embed K into D so that $K \cap \mathcal{O} = \mathcal{O}_K$. Then

$$\mathrm{Cl}(\mathcal{O}_K) \cong K^\times \backslash \hat{K}^\times / \hat{\mathcal{O}}_K^\times \hookrightarrow \mathrm{Cl}(\mathcal{O}).$$

For $\phi \in S_2^D(1)$, let

$$P_K(\phi) = \int\limits_{\mathrm{Cl}(O_K)} \phi(t) dt$$

and

$$(\phi, \phi) = \int\limits_{\mathrm{Cl}(\mathcal{O})} \phi(t)\overline{\phi(t)} dt.$$

For $f \in S_2(d)^{\mathrm{new}}$,

$$(f, f)_{\mathrm{Pet}} = 8\pi^2 \int_{\Gamma_0(d)\backslash\mathcal{H}} f(z)\overline{f(z)} dx dy.$$

Waldspurger's theorem 1.1 may then be restated as follows.

Theorem 1.27 (Waldspurger, classical version II)**.** *Suppose $f \mapsto \phi$ via Jacquet–Langlands correspondence 1.26. Then*

$$\frac{L(f,1)L(f\otimes\eta_K,1)}{(f,f)_{Pet}} = \frac{1}{|d_K|}\frac{|\mathcal{P}_K(\phi)|^2}{(\phi,\phi)}.$$

Remark 1.28. This is the restatement of Theorem 1.1 without reference to automorphic representation. Such a statement is always possible via a Jacquet–Langlands transfer a unique quaternion algebra.

Example 1.29. We sketch the Jacquet–Langlands correspondence 1.26 in a particular case.

Let $d = 11$. Then $\mathrm{Cl}(\mathcal{O})$ has 2 elements and $S_2^D(1)^{\mathrm{new}}$ has 1 element. For $p \neq 11$, the eigenvalue of T_p is

$$a_p = \frac{Q_p - Q'_p}{4}$$

where Q and Q' are two quadratic forms and Q_p, Q'_p are the number of ways p can be represented by these quadratic forms.

Let F be a quadratic form over $\mathbb{Z}$. Define

$$\theta_F(q) = \sum_{n\geq 0} F_n q^n \in M_2(\Gamma_0(11)).$$

Then $\frac{\theta_Q - \theta'_Q}{4}$ is the Jacquet–Langlands transfer of the unique form in $S_2^D(1)$.

2. Tate's thesis

We will start the automorphic theory by recalling the results of Tate's thesis. We provide brief justifications. The complete proofs can be found in [CF10] and in notes by Stephen Kudla http://u.cs.biu.ac.il/~reznikov/courses/kudla-1.pdf

2.1. **Non-archimedean case.**

Let F be a local, non-archimedean field pf characteristic 0, i.e. $F/\mathbb{Q}_p$ is a finite extension. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers of F. It is a local ring where the maximal ideal $\mathfrak{p} \subseteq \mathcal{O}$ is principal. Fix a *uniformizer* ϖ, i.e. $\mathfrak{p} = \langle \varpi \rangle$. Then (topologically) $F^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}^\times$ via the valuation map

$$v \colon F^\times \to \mathbb{Z}.$$

Define the norm

$$|x| = q^{-v(x)}$$

where $q = |\mathcal{O}/\mathfrak{p}|$. This is the *right* normalization of the norm: if μ is a Haar measure on F, then $\mu(xA) = |x|\mu(A)$.

Let $\hat{F} = \{\psi \colon F \to \mathbb{C}^\times \text{ continuous}\}$. If $\psi \in \hat{F}$, there is a smallest ℓ such that $\psi|_{\mathfrak{p}^\ell} \equiv 1$. We call $\ell = n(\psi)$ the *level* of ψ.

Fix $\psi \in \hat{F} \setminus \{0\}$. For $a \in F$, define $\psi_a(x) = \psi(ax)$. Then

$$\begin{aligned} F &\to \hat{F}, \\ a &\mapsto \psi_a. \end{aligned}$$

Then $n(\psi_a) = n(\psi) - v(a)$.

We check that there exists $\psi \in \hat{F} \setminus \{0\}$. Fix $\lambda \colon \mathbb{Q}_p \to \mathbb{R}/\mathbb{Z}$ such that $\lambda(x) - x \in \mathbb{Z}_p$. Define

$$\psi(x) = e^{2\pi i \lambda(\mathrm{Tr}_{F/\mathbb{Q}_p}(x))}$$

We now discuss Fourier transforms. Fix a Haar measure $\mu = dx$ on F and $\psi \in \hat{F}$. Let

$$\mathcal{S}(F) = C_c^\infty(F) = \text{compactly supported continuous functions } \Phi \colon F \to \mathbb{C},$$

called the *Schwartz function.* We define for $\Phi \in \mathcal{S}(F)$, the *Fourier transform*

$$\hat{\Phi}(x) = \int_F \Phi(y)\psi(xy)\,dy.$$

The subspace $\mathcal{S}(F) \subseteq L^2(F)$ inherits the inner product and the Fourier transform is an isometry $\mathcal{S}(F) \to \mathcal{S}(F)$. Moreover:

$$\hat{\hat{\Phi}}(x) = c\Phi(-x)$$

where

$$c(\mu, \psi) = \mu(\mathcal{O})^2 q^{-n(\psi)}.$$

Moreover,

- $\displaystyle\int_{\mathfrak{p}^j} \psi(x)dx = \begin{cases} \mu(\mathfrak{p}^j) = q^{-j}\mu(\mathcal{O}) & \text{if } \mathfrak{p}^j \subseteq \mathfrak{p}^{n(\psi)} \\ 0 & \text{otherise} \end{cases},$
- $\displaystyle\int (f(ax))dx = |a|^{-1} \int_F f(x)dx,$
- $\displaystyle\int_{\mathfrak{p}^j} |x|^s dx = \sum_{i \geq j} \int_{\mathfrak{p}^i \setminus \mathfrak{p}^{i+1}} |x|^s dx = \mu(\mathcal{O}) \sum_{i \geq j} q^{-is}(q^{-i} - q^{-i-1}) = \nu(\mathcal{O})(1 - q^{-1}) \frac{q^{-is}}{1 - q^s},$
  convergent for $\mathrm{Re}(s) > 0$ with a meromorphic continuation to $\mathbb{C}$ and simple pole at $s = 0$.

There is a unique μ_ψ such that $c(\mu, \psi) = 1$ and $\mu_\psi(\mathcal{O}) = q^{n(\psi)/2}$. If we change ψ to ψ_a, then

$$\mu_{\psi_a} = |a|^{1/2}\mu_\psi.$$

A character of $F^\times$ is a continuous map $\omega\colon F^\times \to \mathbb{C}^\times$. One example is

$$x \mapsto |x|^s \text{ for any } s \in \mathbb{C}.$$

These characters are *unramified.* Unramified characters are trivial on $\mathcal{O}^\times$ (by definition). Characters non-trivial on $\mathcal{O}^\times$ are called *ramified.*

A character of $F^\times$ is *unitary* if $|\omega| = 1$. If $\omega\colon \mathcal{O}^\times \to \mathbb{C}^\times$, it is automatically unitary, and there exists a largest n such that ω is trivial on $1 + \mathfrak{p}^n\mathcal{O} \subseteq \mathcal{O}^\times$.

Roughly, any character is a product of an unramified character and a character of $\mathcal{O}^\times$.

We now discuss L-functions. Fix $d^\times x$, a Haar measure on $F^\times$. By uniqueness, there is a constant c such that $d^\times x = c \cdot \frac{dx}{|x|}$. For fixed ψ, we fix $dx = d\mu_\psi$.

For an integer $m \in \mathbb{Z}$ and $\Phi \in \mathcal{S}(F)$, define

$$Z_m = \int_{\varpi^m \mathcal{O}^\times} \Phi(t)\omega(t)d^\times t.$$

We note that $Z_m = 0$ for $m \ll 0$. Set

$$Z(\Phi, \omega, X) = \sum_{m \in \mathbb{Z}} Z_m \cdot X^m \in \mathbb{C}(\!(X)\!)$$

- If ω is ramified or $\Phi \in C_c^\infty(F^\times)$, $Z_m = 0$ for $m \gg 0$, and hence
$$Z(\Phi, \omega, X) \in \mathbb{C}[X, X^{-1}].$$
- Define the *local zeta function* as:
$$\zeta(\Phi, \omega, s) = \int_{F^\times} \Phi(t)\omega(t)|t|^s d^\times t = Z(\Phi, \omega, q^{-s}) \quad \text{forRe}(s) \gg 0.$$

This has a meromorphic continuation with

- a simple pole at s such that $\omega(\varpi)q^{-s} = 1$ when ω is unramified,
- no poles if ω is ramified.

Consider
$$Z(\omega) = \text{span of } Z(\Phi, \omega, X) \subseteq \mathbb{C}(\!(X)\!).$$
We claim that $Z(\omega)$ is a $\mathbb{C}[X, X^{-1}]$ is a submodule. If we set $a\Phi(x) = \Phi(a^{-1}x)$,
$$Z(a\Phi, \omega, X) = \omega(a)X^{\nu(a)}Z(\Phi, \omega, X).$$

Hence
$$X \cdot Z(\Phi, \omega, X) = Z(\varpi\Phi, \omega, X)/\omega(\varpi).$$

Proposition 2.1. *We have that*
$$Z(\omega) = P_\omega(X)^{-1}\mathbb{C}[X^{-1}, X]$$
where
$$P_\omega(X) = \begin{cases} 1 - \omega(\varpi)X & \text{if } \omega \text{ is unramified} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. We know that plugging in elements $\Phi \in C_c^\infty(F^\times)$, we get elements of $\mathbb{C}[X^{-1}, X]$ and plugging in $\mathbb{1}_{\varpi^m\mathcal{O}^\times}$ for $m \gg 0$ gives non-zero constant. Hence
$$\mathbb{C}[X^{-1}, X] \subseteq Z(\omega).$$
Since $\mathcal{S}(F)$ decomposes as a sum of $C_c^\infty(F^\times)$ and $\mathbb{1}_{\mathcal{O}}$, computing that
$$\zeta(\mathbb{1}_{\mathcal{O}}, \omega, X) = \mu(\mathcal{O}^\times)P_\omega(X)^{-1}$$
implies the result. □

Theorem 2.2 (Weil)**.** *There is a rational function $c(\omega, \psi, X) \in \mathbb{C}(X)$ such that*
$$Z\left(\hat{\Phi}, \omega^{-1}, \frac{1}{qX}\right) = c(\omega, \psi, X)Z(\Phi, \omega, X).$$

Proof. Consider the set of distributions:
$$\mathcal{V} = \{\Lambda \colon \mathcal{S}(F) \to \mathbb{C}(\!(X)\!) \mid \text{linear and } \Lambda(a\Phi) = \omega(a)X^{\nu(a)}\Lambda(\Phi)\}.$$
Note that

(1) $Z(\Phi, \omega, X) \in \mathcal{V}$,
(2) $Z(\hat{\Phi}, \omega, 1/qX) \in \mathcal{V}$.

It is enough to show that $\mathcal{V}$ is 1-dimensional over $\mathbb{C}(X)$. In other words, it is enough to show that there exists $\Phi \in S(F)$ such that $\Lambda \in \mathcal{V}$ is determined by $\Lambda(\Phi)$.

Let $\Phi = \mathbb{1}_{U_n}$ where we write $U_n = 1 + \varpi^n \mathcal{O}^\times$ and n is such that $\omega|_{1+\omega^n\mathcal{O}} = 1$. Let $\Lambda \in \mathcal{V}$ be such that

$$\Lambda(\Phi) = 0.$$

We want to show that $\Lambda = 0$. Note that

$$\Lambda(a\mathbb{1}_{U_n}) = \Lambda(\mathbb{1}_{U_k})$$

for $a \in U_n$ and $k \geq n$. Note that

$$U_n = \coprod_{\xi \in \mathcal{O}/\mathfrak{p}} (1 + \xi\mathfrak{p}^n) U_{n+1},$$

and hence

$$0 = \Lambda(\mathbb{1}_{U_n}) = \sum \Lambda((1 + \xi\mathfrak{p}^n)\mathbb{1}_{U_{n+1}}) = q\Lambda(U_{n+1}).$$

Hence

$$0 = \Lambda(\mathbb{1}_{U_n}) = q^k \Lambda(\mathbb{1}_{U_{n+k}}).$$

Now, functions in $C_c^\infty(F^\times)$ are linear combinations of translates of $\mathbb{1}_{U_{n+k}}$. This shows that

$$\lambda|_{C_c^\infty(F^\times)} = 0.$$

Finally,

$$\Lambda(\Phi) = c \cdot \Phi(0)$$

and we must have $c = 0$ by the above. Hence $\Lambda = 0$ as required. $\square$

We define

$$\zeta(\Phi, \omega, s) = Z(\Phi, \omega, q^{-s})$$

$$L(\omega, s) = P_\omega(q^{-s})^{-1}$$

and

$$\epsilon(\omega, s, \psi) = c(\omega, \psi, q^{-s}) \frac{L(\omega, s)}{L(\omega', 1-s)}.$$

We list some properties of ϵ-factors:

- $\epsilon(\omega, s, \psi) \cdot \epsilon(\omega^{-1}, s, \psi) = \omega(-1)$,
- $\epsilon(\omega, s, \psi) = q^{(1/2-s)n(\psi,\omega)}\epsilon(\omega, 1/2, \psi)$ for some $n(\psi, \omega) \in \mathbb{Z}$,
- $\epsilon(\omega, s, \psi_a) = |a|^{s-1/2}\omega(a)\epsilon(\omega, s, \psi)$,
- $\epsilon(\omega, s, \psi) = q^{s-1/2}\omega(\varpi)^{-1}$ for ω unramified.

2.2. **The archimedean case.** Let $F = \mathbb{R}$ or $\mathbb{C}$. Then $\mathcal{S}(F)$ is the *Schwartz space*. It is a *Fréchet space* where we assume that

$$\|f\|_{\alpha,\beta} = \sup_{x \in F} |x^\alpha D^\beta f(x)|$$

is bounded for any α, β. For example $p(x)e^{-x^2}$ for any polynomial x belongs to $S(F)$.

A distribution $T \colon \mathcal{S}(F) \to \mathbb{C}$ is *tempered* if it is *continuous*. We can define

$$T_\phi(f) = \int_F f\phi dx$$

if ϕ is bounded by a polynomial or $e^x \cos(e^x)$ for example. Write $\mathcal{S}'(F)$ for the space of tempered distribution and define D^α by the requirement

$$\langle D^\alpha T, f\rangle = (-1)^{|\alpha|}\langle T, D^\alpha f\rangle$$

and $\hat{T}$ by the requirement

$$\langle \hat{T}, f\rangle = \langle T, \hat{f}\rangle.$$

Let dx be the Lebesgue measure and $\psi(x) = e^{2\ pii\, \mathrm{Tr}_{F/\mathbb{R}}(x)}$. For

- $F = \mathbb{R}$, $\omega(x) = x^{-a}$, $a = 0, 1$,
$$L(s, \omega) = \pi^{-s/2}\Gamma(s/2),$$
- $F = \mathbb{C}$, $\omega(z) = z^{-a}\overline{z}^{-b}$ for $a, b \in \mathbb{Z}$, $\min(a, b) = 0$,
$$L(s, \omega) = (2\pi)^{1-s}\Gamma(s).$$

For any $\tilde{\omega} \colon F^\times \to \mathbb{C}$, $\tilde{\omega} = |\cdot|^\lambda \cdot \omega$, and

$$L(\tilde{\omega}, s) = L(\omega, s + \lambda).$$

Considering $F = \{0\} \cup F^\times$, we arrive at considering:

$$S'(\omega) = \{\Lambda \in \mathcal{S}'(F) \mid a\Lambda = \omega(a)\Lambda\}.$$

Moreover, cass

$$\langle a \cdot \Lambda, f\rangle = \langle \Lambda, a^{-1}f\rangle.$$

Then $\dim \mathcal{S}'(\omega) = 1$. Define

$$\zeta(s, \omega, f) = \int_{F^\times} f(x)\omega(x)|x|^s d^\times x$$

and for

$$\zeta_0(s, \omega) = L(s, \omega)^{-1}\zeta(s, \omega)$$

we have that

$$\zeta_0(\widehat{1 - s, \omega}^{-1}) = \epsilon(s, \omega, \psi)\zeta_0(s, \omega).$$

When $F = \mathbb{R}$, $\epsilon(s, \omega, \psi) = i^a$ and $F = \mathbb{C}$, $\epsilon(s, \omega, \psi) = i^{\max(a,b)}$.

The details of this can be found in Stephen Kudla's notes.

3. Representations of totally disconnected groups

A topological group G is *totally disconnected* if it is

- Hausdorff,
- every open subset $U \subseteq G$, containing 1, contains an open-compact subgroup of G,
- for any open compact subgroup $K \subseteq G$, G/K is countable.

Remark 3.1. Throughout this section, unless otherwise specified, K is an open compact subgroup.

Example 3.2. For a finite extension F of $\mathbb{Q}_p$, both $(F, +)$ and $(F^\times, \times)$ are totally disconnected. The basis of neighborhood of 1 in each case is $\mathfrak{p}^n\mathcal{O}$ and $1 + \mathfrak{p}^n\mathcal{O}$. More generally, $\mathrm{GL}_n(F)$ is totally disconnected.

Let G be a totally disconnected topological group. A *representation* (π, V) is a $\mathbb{C}$-vector space V together with a homomorphism $\pi \colon G \to \mathrm{Aut}(V)$.

Sometimes (usually):

- we say V is a representation,
- for $v \in V$, $gv = \pi(g)v$.

Let V be a representation of G and $H \subseteq G$ be a subgroup. Then

$$V^H = \{v \in V \mid hv = v \text{ for all } h \in H\}.$$

Definition 3.3. A representation V is *smooth* if

$$V = \bigcup_{\substack{K \subseteq G \\ \text{open compact}}} V^K.$$

Let $\mathrm{Rep}(G)$ be the *category of smooth representations.*

One check that that V is smooth if and only if $G \times V \to V$ is continuous for the discrete topology on V.

If V is a representation of G, set

$$V^\infty = \bigcup_{\substack{K \subseteq G \\ \text{open compact}}} V^K.$$

If $\mathrm{Rep}^{\mathrm{any}}(G)$ is the category of all representations, this defines a functor:

$$\mathrm{Rep}^{\mathrm{any}}(G) \to \mathrm{Rep}(G)$$
$$V \mapsto V^\infty.$$

We check that $V^\infty \subseteq V$ is a G-stable subspace: any $v \in V^\infty$ belongs to V^K for some K, and then $gv \in V^{gKg^{-1}}$; moreover the conjugate of an open compact is still open compact.

Lemma 3.4. *The functor:*

$$\mathrm{Rep}^{any}(G) \to \mathrm{Rep}(G)$$
$$V \mapsto V^\infty$$

is left-exact.

Proof. Suppose $0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3$ is exact. We want to see if

$$0 \longrightarrow V_1^\infty \longrightarrow V_2^\infty \longrightarrow V_3^\infty$$

is still exact. The first map is still clearly surjective. We check exactness at V_2^∞. Suppose $v_2 \in V_2^\infty$ is sent to 0. By exactness of the first sequences, $f_2(v_2) = 0$ implies that $f_1(v_1) = v_2$ for some $v_1 \in V_1$. Supposing $v_2 \in V_2^K$, we see that

$$kf_1(v_1) = kv_2 = v_2 = f_1(kv_1).$$

Hence $kv_1 = v_1$ by injectivity of f_1, showing that $v_1 \in V^K \subseteq V_1^\infty$. □

Definition 3.5. A representation V is *irreducible* if 0 and V are the only G-stable subspaces.

Example 3.6. Any continuous holomorphism $G \to \mathbb{C}^\times$ is a smooth representation.

Example 3.7. Let G be compact and totally disconnected and V be a smooth irreducible representation of G. For $v \in V \setminus 0$, $v \in V^K$ for some $K \subseteq G$. Let $\{g_i\}$ be a finite set of representatives for G/K. Then $\operatorname{span}\{g_i(v)\} \subseteq V$ is a subrepresentation, so equality holds by irreducibility of V. This shows that V is finite-dimensional.

Let $K' = \bigcap g_i K g_i^{-1}$. This is an open compact which acts trivially and is a normal subgroup of G. Therefore, V is a representation of G/K', a finite group.

Representations of G may fail to be semisimple. Here is an equivalent definition of semisimplicity.

Proposition 3.8. *Suppose G is any totally disconnected group and V is a smooth representation of G. The following are equivalent:*

(1) *V is a sum of irreducible subspaces,*
(2) *V is the direct sum of irreducible subspaces,*
(3) *any invariant subspace of V has an invariant complement.*

A representation satisfies any (and hence all) of these conditions is called *semisimple*.

Proof. We first show that (1) implies (2). Let $\bigcup_{i \in I} U_i \subseteq V$ be irreducible such that $V = \sum U_i$. Let $\mathcal{I}$ be the set of subsets $J \subseteq I$ such that $\sum_{i \in J} U_i$ is direct. By Zorn's Lemma, this shows (2).

We now show that (2) implies (3). Let $V = \bigoplus_{i \in I} W_i$ and W be an invariant subspace. Let $\mathcal{I}$ be the set of subsets $J \subseteq I$ such that $W \cap \sum_{i \in J} U_i = 0$. By Zorn's Lemma, there is a maximal element, which gives the complement.

For (3) implies (1), let $V_0 = \sum_{U_i}$ for U_i irreducible. Then $V = V_0 \oplus W$ and if $W \neq 0$, we can find $0 \subseteq W_1 \subseteq W_2 \subseteq W$ such that W_2/W_1 is irreducible. Then $V = V_0 \oplus W_1 \oplus U$. Then the projection $V \to U$ sends W_2 ot an irreducible subspace of U, which is a contradiction. □

Lemma 3.9. *Let V be a smooth representation and $K \subseteq G$ be an open compact. Then V is K-semisimple.*

Proof. Let $v \in V$. Then $v \in V^{K_0} \subseteq V^{K_0 \cap K}$ for some K_0. Let

$$K' = \bigcap_{g \in K/K_0 \cap K} g(K_0 \cap K)g^{-1}.$$

The span of $\{kv \mid k \in K\}$ is a finite dimensional representation of K/K'. Then v is in the sum of irreducible representations of K/K', and hence also of K. By Proposition 3.8, this completes the proof. □

Let V be a smooth representation of G and $K \subseteq G$ be open compact. Let $\hat{K}$ be the set of equivalence classes of irreducible smooth representations of K.

For $\varrho \in \hat{K}$, let

$$V^\varrho = \sum_{\substack{W \subseteq V \\ W \cong \varrho \text{ as } K\text{-reps}}} W.$$

Proposition 3.10. *Let V be smooth.*

(1) *We have that $V = \bigoplus_{\varrho \in \hat{K}} V^\varrho$ as K-representations.*
(2) *If $f \colon V \to W$ is a G-morphism, then $f(V^\varrho) \subseteq W^\varrho$ and, in fact, $W^\varrho \cap f(V) = f(V^\varrho)$.*

Proof. Write $V = \bigoplus U_i$ for irreducible representations U_i of K. Then define

$$U(\varrho) = \sum_{U_i \cong \varrho} U_i \subseteq V^\varrho.$$

Let $W \subseteq V$ be such that $W \cong \varrho$ as K-representations. Suppose $W \not\subseteq U(\varrho)$. Then $V = \bigoplus_{\varrho} U(\varrho) \to W$ is a K-epimorphism with $U(\varrho') \to W$ non-trivial for some $\varrho' \neq \varrho$. This contradicts $W \cong \varrho$, and hence shows that $U(\varrho) = V^\varrho$.

Part (2) follows easily from part (1). □

Corollary 3.11. *Let V, W, U be smooth G-representations. The sequence*

$$V \xrightarrow{f_1} W \xrightarrow{f_2} U \qquad \text{is exact}$$

if and only if

$$V^K \xrightarrow{f_1} W^K \xrightarrow{f_2} U^K \qquad \text{is exact for all } K.$$

Proof. The 'if' implication is clear. For the 'only if' implication, note that the kernel of $W^K \to U^K$ is $f_1(V) \cap W^K = f_1(V^K)$ by Proposition 3.10 (2). □

Definition 3.12. Let $H \subseteq G$ any subgroup and V be a representation of G. We define

$$V(H) = \operatorname{span}\{v - hv \mid h \in H\}.$$

Corollary 3.13. *Let V be smooth and $K \subseteq G$. Then*

(1) $V(K) = \bigoplus_{\varrho \neq 1} V^{\varrho}$,
(2) $V = V(K) \oplus V^K$,
(3) *$V(K)$ is the unique K-complement of V^K.*

Proof. Note that (3) implies both (1) and (2). It is hence enough to prove (3). Let

$$W \oplus V^K = V$$

be any K-decomposition and $f\colon V \to V^K$ be a K-morphism with $W = \ker f$.

We have that $V(K) \subseteq W$ and want to show equality. We know that W is K-invariant and semisimple. For any non-trivial, irreducible $U \subseteq W$:

$$U(K) = \operatorname{span}(u - ku \mid u \in U, k \in K) = U \subseteq V(K).$$

This completes the proof of (3) and hence of the corollary. ☐

3.1. Induced representations.

Definition 3.14. Let $H \subseteq G$ be closed and (σ, W) be a smooth representation of H. Define the *smooth induction* $\operatorname{Ind}_H^G(\sigma)$ to be the space of $f\colon G \to W$ such that

- $f(hg) = \sigma(h)f(g)$,
- $f(gk) = f(g)$ for all $k \in K$ where K is some open compact (dependent on f).

If we drop the second assumption, we write $I_H^G(\sigma)$.

Both $I_H^G(\sigma)$ and $\operatorname{Ind}_H^G(\sigma)$ are representations of G via $gf(x) = f(xg)$ and

$$I_H^G(\sigma)^\infty = \operatorname{Ind}_H^G(\sigma).$$

Proposition 3.15 (Frobenius reciprocity)**.** *Let $\pi \in \operatorname{Rep}(G)$, $\sigma \in \operatorname{Rep}(H)$. Then there is a functorial isomorphism*

$$\begin{aligned} \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G(\sigma)) &\cong \operatorname{Hom}_H(\operatorname{Res}_H^G \pi, \sigma), \\ \Phi &\mapsto (v \mapsto \Phi(v)(1)), \\ (v \mapsto (g \mapsto F(gv))) &\hookleftarrow F. \end{aligned}$$

In other words, the isomorphism is induced by $\alpha\colon \operatorname{Ind}_H^G(\sigma) \to \sigma$ sending $f \mapsto f(1)$.

Proposition 3.16. *The functor $\operatorname{Ind}_H^G\colon \operatorname{Rep}(H) \to \operatorname{Rep}(G)$ is exact.*

Proof. Since $\operatorname{Ind}_H^G = I_H^{G,\infty}$, it is immediately left-exact. To check right-exactness, take $W \longrightarrow U \longrightarrow 0$ exact and check that $\operatorname{Ind}_H^G(W) \longrightarrow \operatorname{Ind}_H^G(W) \longrightarrow 0$ is exact. Pick a K-invariant $\phi \in \operatorname{Ind}_H^G(U)$. Then for all $g \in G$,

$$\phi(g) \in U^{H \cap gKg^{-1}}$$

and $W^{H \cap gKg^{-1}} \longrightarrow U^{H \cap gKg^{-1}} \longrightarrow 0$ is exact with $Wg \mapsto \phi(g)$. Letting $\Phi \in \operatorname{Ind}_H^G(W)$ be $\Phi(g) = Wg$, this is the desired preimage of ϕ. ☐

Lemma 3.17 (Schur's Lemma)**.** *If V is an irreducible G-representation,* $\operatorname{End}_G(V) = \mathbb{C}$.

Proof. For $\phi \in \mathrm{End}_G(V)$, $\ker\phi$ and $\mathrm{im}\,\phi$ are both G-invariant, so any $\phi \neq 0$ is invertible. Now, $\mathrm{End}_G(V)$ is a complex division algebra. If $v \in V \setminus \{0\}$, gv spans V, so any ϕ is determined by $\phi(v)$. Since G/K is countable, V has a countable basis, so $\mathrm{End}_G(V)$ has a countable basis over $\mathbb{C}$.

Let $\phi \in \mathrm{End}_G(V) \setminus \mathbb{C}$. Then $\mathbb{C}(\phi)$ is transcendental over $\mathbb{C}$ and

$$\left\{ \frac{1}{\phi - a} \;\middle|\; a \in \mathbb{C} \right\} \subseteq \mathbb{C}(\phi)$$

is an uncountable set of linearly independent vectors. This contradiction shows that $\mathrm{End}_G(V) = \mathbb{C}$. □

Definition 3.18. A representation is *admissible* if $\dim V^K < \infty$ for all K.

If $G = G(F)$ for $F/\mathbb{Q}_p$ finite extension, then any smooth, irreducible representation is admissible. Another way to prove Schur's lemma would be to use this fact.

Corollary 3.19. *If (π, V) is irreducible, there exists a smooth map $\omega_\pi \colon Z \to \mathbb{C}^\times$ where Z is the center of G such that*

$$\pi(z)v = \omega_\pi(z)v.$$

Proof. By Schur's Lemma 3.17, get $\omega_\pi \colon Z \to \mathbb{C}^\times$ a character such that $\pi(zg) = \omega_\pi(z)\pi(g)$. Restricting to $K \cap Z$ such that $V^K \neq 0$ shows that ω_π is trivial on $K \cap Z$. This shows ω_π is smooth. □

Definition 3.20. We call ω_π the *central character* of π.

We finally discuss duality in this general setting. Let V be a smooth representation of G. Consider

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

with a G-action by $\langle g\lambda, v\rangle = \langle \lambda, g^{-1}v\rangle$ where $\lambda \in V^*$, $v \in V$.

However, there is no reason for V^* to be smooth. In fact, if V is infinite-dimensional, V^* is never smooth. We hence define $V^\vee = (V^\infty)^\infty$.

Proposition 3.21. *The map $V^\vee \to (V^K)^*$ given by $\lambda \mapsto \lambda|_{V^K}$ defines an isomorphism $(V^K)^* \cong (V^\vee)^K$.*

Proof. Recall that $V = V^K \oplus V(K)$ by Corollary 3.13, where $V(K)$ is the space on $v - kv$. Then $\lambda \in (V^K)^*$ extends to V and we get an element of $V^\vee$ which is K-invariant. This defines an inverse to the defined map. □

Let $\delta \colon V \to (V^\vee)^\vee$ be defined by $\langle \delta(v), \lambda\rangle = \langle \lambda, v\rangle$. It is a G-equivariant map.

Proposition 3.22. *The map δ is an isomorphism if and only if V is admissible.*

Proof. Indeed, δ is an isomorphism if and only if the map

$$V^K \to [(V^\vee)^\vee]^K = [(V^K)^*]^*$$

is an isomorphism. This holds only if V^K is finite-dimensional. □

We used in the proof the general fact that $W \hookrightarrow W^{**}$ is an isomorphism if and only if $\dim W < \infty$.

Proposition 3.23. *The functor $V \mapsto V^\vee$ is an exact functor* $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$.

Proof. Suppose $0 \to V_1 \to V_2 \to V_3 \to 0$ is exact. Then $0 \to V_1^K \to V_2^K \to V_3^K \to 0$ is exact by Corollary 3.11, which implies that $0 \to (V_1^K)^* \to (V_2^K)^* \to (V_3^K)^* \to 0$ is exact. This shows that for any K,

$$0 \to (V_3^\vee)^K \to (V_2^\vee)^K \to (V_1^\vee)^K \to 0.$$

Again, by Corollary 3.11, this shows that $0 \to V_3^\vee \to V_2^\vee \to V_1^\vee \to 0$ is exact. □

4. Representation theory of $G = \mathrm{GL}_2(F)$ for $F/\mathbb{Q}_p$ finite

Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O} \subseteq F$ and maximal ideal $\langle \varpi \rangle = \mathfrak{p} \subseteq \mathcal{O}$. Any element $x \in F^\times$ can be written as $\varpi^n \cdot u$ for some $u \in \mathcal{O}^\times$. Then $|x| = q^{-n}$ where $q = |\mathcal{O}/\mathfrak{p}|$.

Let

$$G = \mathrm{GL}_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \middle| ad - bc \neq 0 \right\}$$
$$K = \mathrm{GL}_2(\mathcal{O}), = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(O) \middle| ad - bc \in \mathcal{O}^\times \right\}.$$

Then K is open and compact. In fact, it is a maximal open compact subset of G (up to conjugation) because:

$$(M_2(\mathcal{O}) \cap G) \cap (M_2(\mathcal{O}) \cap G)^{-1}.$$

Lemma 4.1 (Cartan decomposition)**.** *Let $\Lambda \cong \mathbb{Z}^2$ be*

$$\Lambda = \left\{ \begin{pmatrix} \varpi^{n_1} & 0 \\ 0 & \varpi^{n_2} \end{pmatrix} \middle| n_i \in \mathbb{Z} \right\}$$

and $\Lambda^+ \subseteq \Lambda$ be defined by requiring that $n_1 \geq n_2$. Then

$$G = K\Lambda^+ K.$$

Consider the *Borel subgroup*:

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \right\},$$

the *torus*

$$T = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right\} \subseteq B,$$

and the *nilpotent subgroup*

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \subseteq B.$$

Then N is normal in B. Hence T acts on N by conjugation:

$$t \cdot n = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_2^{-1} \end{pmatrix} = \begin{pmatrix} 1 & (t_1/t_2)x \\ 0 & 1 \end{pmatrix}.$$

Note that $T \cap N = \{1\}$, so $B = N \rtimes T$ and the sequence
$$1 \to N \to B \to T \to 1$$
is exact. In particular, any representation of T will lift to B.

Consider $N =\supseteq N_i = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathfrak{p}^i \right\}$, a compact subgroup. Then $N_i \subseteq N_{i-1}$ and
$$N = \bigcup_i N_i.$$

Lemma 4.2 (Iwasawa decomposition)**.** *We have a decomposition $G = BK$.*

Definition 4.3. The congruence subgroup of level $\mathfrak{p}^i$ is defined as
$$K_i = \{g \in \mathrm{GL}_2(\mathcal{O}) \mid g \equiv 1 \mod \mathfrak{p}^i\}.$$
It is a compact subgroup of G and gives a basis of neighborhoods of 1.

We now discuss putting a measure on G. There is a general theorem that asserts such a measure exists.

Theorem 4.4 (Haar's Theorem)**.** *Let G be a locally compact Hausdorff group. Then G has a unique (up to scaling) non-zero left-invariant ($\mu(xA) = \mu(A)$) Radon measure.*

Definition 4.5. Any such measure is called a *Haar measure* on G.

We want to define a Haar measure on G explicitly.

Consider the following space of *test functions*:
$$C_c^\infty(G) = \{f \colon G \to \mathbb{C} \mid \text{compactly supported, locally constant}\}.$$
Any $f \in C_c^\infty(G)$ can be written as
$$f = \sum_j c_j \mathbb{1}_{g_j K_i} \text{ for some } i, g_j \in G/K_i.$$
In that case, we define
$$\mu(f) = \int_G f(g)dg = \frac{1}{[K : K_i]} \sum_j c_j.$$
This is a left-invariant measure. It is the unique Haar measure such that $\mu(K) = 1$.

We clarify the meaning of left-invariant.

Definition 4.6. For $x \in G$, define $L_x \colon C_c^\infty(G) \to C_c^\infty(G)$ by $L_x f(g) = f(x^{-1}g)$ and, similarly, $R_x f(g) = f(gx)$. Then μ is *left-invariant* if $\mu(L_x f) = \mu(f)$ and *right-invariant* if $\mu(R_x f) = \mu(f)$.

Lemma 4.7. *The measure μ is also right-invariant.*

Proof. We know that $\mu(L_x f) = \mu$. For $x \in G$, $f \mapsto \mu(R_x f)$ defines a left-invariant measure, which has to be $\mu(f)\Delta(x)$ by uniqueness in Haar's Theorem 4.4. We know that $\Delta(xy) = \Delta(x)\Delta(y)$. Therefore, Δ is trivial on $[G, G] = \mathrm{SL}_2(F)$, and hence on $Z\,\mathrm{SL}_2(F)$, which is a finite index subgroup of GL_2. Therefore, the image of Δ is a finite subgroup of $\mathbb{R}_>^\times$, showing that Δ is equal to 1. ∎

We now proceed to discussing the representation theory of $\mathrm{GL}_2(F)$. We start with gathering a few *random facts.*

Proposition 4.8. *Let V any representation of a group G. If it is finitely-generated then it has an irreducible quotient.*

Proof. Let $E \subseteq V$ be finite-dimensional subspace generating V. Let $E_{\max} \subseteq E$ be the maximal subspace such that $E_{\max} = E \cap U$ for a proper invariant subspace $U \subseteq V$. Let $\mathcal{U}$ be the set of all $U \subseteq V$ such that $U \cap E = E_{\max}$. By Zorn's Lemma (which should be called Kuratowski–Zorn's Lemma for historical reasons), there exists a maximal subspace $U_{\max}$ such that $V/U_{\max}$ is irreducible. □

Proposition 4.9. *If V is a finitely-generated smooth representation of G, then it is a finitely generated B-representation.*

Proof. Recall that $G = BK$ and since V is smooth, we can find a generating subspace $E \subseteq V^{K_i}$ for some i. Then $(K/K_i)E$ is a finite set which generates B as a B-module. □

Proposition 4.10. *Let V be an irreducible, smooth, finite-dimensional representation. Then $V \cong \mathbb{C}$ where the action on $\mathbb{C}$ given by $g \mapsto \chi(\det g)$ for a smooth character $\chi\colon F^\times \to \mathbb{C}^\times$.*

Remark 4.11. Smooth characters $\chi\colon G \to \mathbb{C}^\times$ are smooth representations. Since $\mathrm{SL}_2(F) = [G, G]$, such characters have to factor through the quotient so $\chi(g) = \chi'(\det g)$ for some smooth $\chi'\colon F^\times \to \mathbb{C}^\times$.

Proof. Suppose $V \subseteq V^{K_i}$ for some i. Then $N_i = K_i \cap N$ acts trivially. If $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then $tnt^{-1} = n_c \in N_i$ for some $t \in T$. Then

$$\pi(n)v = \pi(t^{-1})\pi(n_c)\pi(t)v = \pi(t^{-1})\pi(t)v = v \quad \text{for } v \in V.$$

This shows that N acts trivially. In the same way, we can show that $\overline{N} = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}$ actrs trivially.

Exercise. Show that $\mathrm{SL}_2(F)$ is generated by N and $\overline{N}$.

Since SL_2 acts trivially and π is an irreducible representation of $F^\times$, Schur's Lemma 3.17 shows that it is 1-dimensional. □

We have hence shown that there are no finite-dimensional smooth irreducible representations of G, except for the 1-dimensional ones. To get interesting representations, we need to go to infinite-dimensional representations. The question is: how do we produce examples of infinite-dimensional representations?

Help comes from the notion of induced representations. Recall that for $H \subseteq G$ closed and $\sigma \in \mathrm{Rep}(H)$, we define

$$\mathrm{Ind}_H^G(\sigma) = \{f\colon G \to \sigma \mid f(hg) = \sigma(h)f(g)\}.$$

Consider the particular case $H = T \subseteq G = \mathrm{GL}_2$. Let $\chi\colon \to \mathbb{C}^\times$ be any character, i.e.

$$\chi\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \chi_1(t_1)\chi_2(t_2)$$

for characters $\chi_i\colon F^\times \to \mathbb{C}^\times$. We may pull χ back to a character $\chi\colon B \to \mathbb{C}^\times$.

Consider the character $\delta_B\colon T \to \mathbb{C}^\times$ given by

$$\delta_B\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = |t_1/t_2|$$

and pull it back to $\delta_b\colon B \to \mathbb{C}^\times$.

Definition 4.12. For a character χ of T, define the *automorphic induction* of χ to be

$$I(\chi) = \mathrm{Ind}_B^G(\chi \cdot \delta_B^{1/2}).$$

Fact 4.13. *The representation $I(\chi)$ is admissible.*

Proof. By Iwasawa decomposition 4.2, $G = B \cdot K$, so $f \in I(\chi)$ is determined by $f|_K$, and $\dim I(\chi)^K \leq [K : K_i]$. □

We now define the *Jacquet functor.*

Set

$$V(N) = \mathrm{span}(v - \pi(n)v \mid n \in N).$$

Let $V[N_i] = \left\{ v \in V \;\middle|\; \int_{N_i} \pi(n) v dn = 0 \right\}$ for each i. We claim that

$$V[N_i] \subseteq V[N_{i-1}].$$

Indeed, for $v \in V[N_i]$,

$$\int_{N_{i-1}} \pi(n) v dn = \sum_{\xi \in \mathfrak{p}^{i-1}/\mathfrak{p}^i} \int_{N_i} \pi(n(\xi)n) v dn = \sum_{\xi \in \mathfrak{p}^{i-1}/\mathfrak{p}^i} \pi(n(\xi)) \int_{N_i} \pi(n) v dn = 0$$

where we write $n(\xi) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Lemma 4.14. *We have that $V(N) = \bigcup_{i \in \mathbb{Z}} V[N_i]$.*

Definition 4.15. Let (V, π) be a smooth representation of $G = \mathrm{GL}_2$. The *Jacquet module* associated to V is

$$V_N = V/V(N).$$

The functor $\mathrm{Rep}(G) \to \mathrm{Reg}(G)$ given by $V \mapsto V_N$ is called the *Jacquet functor.*

Proposition 4.16. *The Jacquet functor is exact.*

Proof. Suppose

$$0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \longrightarrow 0$$

is exact. We want to show that

$$0 \longrightarrow V_{1,N} \xrightarrow{f_1} V_{2,N} \xrightarrow{f_2} V_{3,N} \longrightarrow 0$$

is still exact. The only non-trivial part is showing injectivity of f_1. Suppose $f_1(v_1) \in V_2(N)$. It is enough to show that $v_1 \in V_1(N)$. We have that for some i:

$$0 = \int_{N_i} n f_1(v_1) dn = f_1 \left(\int_{N_i} n v_1 dn \right).$$

Since f_1 is injective, this shows that $\int_{N_i} n v_1 dn = 0$, and hence $v_i \in V_1(N)$. □

Proposition 4.17 (Frobenius reciprocity)**.** *For $\pi \in \mathrm{Rep}(G)$, $\chi \colon T \to \mathbb{C}^\times$,*

$$\mathrm{Hom}_G(\pi, I(\chi)) \cong \mathrm{Hom}_T(\pi_N, \chi\delta_B^{1/2})$$

Proof. This follows from Frobenius reciprocity 3.15 and noting that

$$\mathrm{Hom}_B(\pi, \chi\delta^{1/2}) \cong \mathrm{Hom}_T(\pi_N, \chi\delta^{1/2})$$

since N acts trivially on $\chi\delta^{1/2}$. □

Proposition 4.18. *Let (V, π) be a smooth irreducible representation such that $V_N \neq 0$. Then V is a submodule of $I(\chi)$ for some χ.*

Proof. Since V is finitely generated as a G-module, by Proposition 4.9, V is a finitely-generated B-module. Hence V_N is a finitely-generated T-module, and hence V_N has an irreducible quotient by Proposition 4.8. This shows that

$$\mathrm{Hom}_T(V_n, \delta^{1/2}\chi) \neq 0$$

for some χ. By Frobenius reciprocity 4.17, this completes the proof. □

4.1. **Supercuspidal representations.** We have seen that we can realize a lot of representations of G as submodules of $I(\chi)$ for some χ. The remaining representations are called *supercuspidal.*

Definition 4.19. A smooth representation (π, V) is *supercuspidal* if $V_N = 0$.

Definition 4.20. For a smooth representation (π, V), the *matrix coefficient* associated to $v \in V$, $\lambda \in V^\vee$ is defined as

$$\varphi_{v,\lambda}(g) = \langle \lambda, \pi(g)v \rangle.$$

Definition 4.21. A function $f \colon G \to \mathbb{C}$ has *compact support modulo H* if $\mathrm{supp}(f) \subseteq H \cdot C$ where $C \subseteq G$ is a compact subset.

Proposition 4.22. *Matrix coefficients of irreducible supercuspidals are compactly supported modulo the center Z.*

Proof. Let $v \in V$ and $\lambda \in V^\vee$. Let $U = \mathrm{span}(Kv)$, $U' = \mathrm{span}(K\lambda)$. Since $V_N = 0$, there exists $i \in \mathbb{N}$ such that

$$U \subseteq V[N_{-n}], \quad U' \subseteq (V^\vee)^{K_i} \subseteq (V^\vee)^{N_i}.$$

Recall the Cartan decomposition 4.1:

$$G = K\left(\bigcup_{a\geq 0} t_a\right) K\cdot Z \quad \text{for } t_a = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a}\end{pmatrix}.$$

Using this, it is enough to show that there is an a_0 such that for $a > a_0$,

$$\varphi_{v,\lambda}(t_a) = 0 \text{ for all } v \in U, \lambda \in U'.$$

Let $a \in \mathbb{N}$. Then for the above i:

$$\begin{aligned}
\mathrm{vol}(N_i)\varphi_{v,\lambda}(t_a) &= \mathrm{vol}(N_i)\langle \lambda, \pi(t_a)v\rangle \\
&= \int_{N_i} \langle \pi^\vee(n)\lambda, \pi(t_a)\rangle dn && \text{because } N_i \text{ acts trivially on } U' \\
&= \int_{N_i} \langle \lambda, \pi(n)\pi(t_a)v\rangle dn \\
&= \int_{N_i} \langle \lambda, \pi(t_a)\pi(t_a^{-1}nt_a)v\rangle dn \\
&= |\varpi|^{2a} \int_{N_{i-2a}} \langle \lambda, \pi(t_a)\pi(n)v\rangle d_n \\
&= 0 && \text{if } i-2a < -i.
\end{aligned}$$

Therefore, we may take $a_0 = i$. This shows that

$$\mathrm{supp}(\varphi_{v,\lambda}) \subseteq K \bigcup_{a=0}^{i} \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a}\end{pmatrix} K\cdot Z,$$

as required. □

In fact, the converse is also true. We will prove this soon.

Proposition 4.23. *Suppose V is irreducible, and supercuspidal, and U is a smooth representation with a non-trivial homomorphism $P\colon U \to V$. Then there exists $S\colon V \to U$ such that $P\circ S = \mathrm{id}_V$.*

Proof. Let $\omega_\pi\colon Z \to \mathbb{C}^\times$ be the central character of π. Identifying Z with $F^\times$, we may write

$$\omega_\pi(\varpi^k u) = \omega(u)|\varpi^k|^s \quad \text{for some } s \in \mathbb{C}.$$

Therefore, we may replace π by $\pi \otimes |\det g|^{-s/2}$ to assume that the central character of π is unitary (i.e. valued in $S^1 \subseteq \mathbb{C}^\times$).

Let $v \in V\setminus\{0\}$ and consider

$$\varphi_{v,\lambda}(g) = \langle \lambda, \pi(g)v\rangle$$

for $\lambda \in V^\vee \setminus \{0\}$. Then $\varphi_{v,\lambda}$ is non-zero and we may define a map $i\colon V^\vee \to V$ given by

$$i(\lambda) = \int_{Z\backslash G} \overline{\varphi_{v,\lambda}}(g)\pi(g)v\, dg$$

This map is G-equivariant and sesqui-linear (i.e. $i(a\lambda) = \overline{a}i(\lambda)$). Note that, because V is supercuspidal, this integral is actually a finite sum.

The map i is injective:

$$\langle \lambda, i(\lambda) \rangle = \int\limits_{Z\backslash G} |\varphi_{v,\lambda}|^2 \neq 0.$$

Since V is irreducible, this automatically shows that i is also surjective. In particular, this shows that $V^\vee$ is admissible.

For U as in the statement, pick $u \in U$ such that $p(u) = v$. Define $j\colon V^\vee \to U$ by

$$j(\lambda) = \int\limits_{Z\backslash G} \overline{\varphi_{v,\lambda}}(g)\pi(g)u\,dg$$

which is clearly sesqui-linear an G-equivariant. The composition

$$S = j \circ i^{-1}\colon V \to U$$

is G-equivariant and linear.

Finally, $P \circ S\colon V \to V$ is G-equivariant and linear. By Schur's Lemma 3.17, it is constant. Checking that

$$(P \circ S)(v) = \int\limits_{Z\backslash G} \overline{\varphi_{v,i^{-1}v}}(g)\pi(g)P(u)dg = i(i^{-1}(v)) = v$$

shows that $P \circ S = \mathrm{id}_V$, as required. □

Corollary 4.24. *If V is irreducible and supercupsidal, then V^{K_i} is finite-dimensional.*

Proof. We know that $i\colon (V^\vee)^{K_i} \cong V^{K_i}$ and $(V^\vee)^{K_i} \cong (V^{K_i})^*$. For cardinality reasons, the resulting isomorphism $(V^{K_i})^* \cong V^{K_i}$ is only possible if V^{K_i} is finite-dimensional. □

We present an alternative proof of this corollary, because we will use similar ideas later.

Alternative proof of Corollary 4.24. Let $K = K_i$. Consider the map $P_K\colon V \to V^K$ given by

$$P_K(v) = \frac{1}{\mathrm{vol}(K)} \int\limits_K kv\,dk \quad \text{for } v \in V^K.$$

This is the projection onto the V^K-component of $V = V(K) \oplus V^K$. For $v \in V^K \setminus \{0\}$, the image $P_K((G/K)v)$ spans V^K. Let $S \subseteq G$ be such that $P_K(Sv)$ is a basis of V^K.

Take $\lambda \in (V^\vee)^K = (V^K)^*$ such that

$$\varphi_{v,\lambda}(g) = \langle \lambda, \pi(g)v \rangle = 1$$

for $g \in S$. Then

$$SZ/Z \subseteq \mathrm{supp}(\varphi_{v,\lambda})/Z,$$

showing that SZ/Z is compact. Moreover sK are disjoint (because $P_K(Sv)$ is a basis), and hence SZ/Z is discrete. This shows that SZ/Z is finite. □

Let us summarize the results. We defined $(V,\pi) \in \mathrm{Rep}(G)$ to be *supercuspidal* if $\pi_N = 0$. This is equivalent to saying that for all $v \in V$,

$$\int_{N_i} \pi(n)v\,dn = 0 \quad \text{for } i \ll 0.$$

We showed in Proposition 4.22 that the matrix coefficients

$$g \mapsto \langle \pi(g)\lambda, v\rangle \quad \text{for } v \in V, \lambda \in V^\vee$$

are compactly supported modulo center. We also showed in Proposition 4.23 that if V is irreducible, then any surjection $U \to V$ has a splitting.

We then showed in Corollary 4.24 that irreducible cuspidal representations π are admissible. Therefore, $\pi^\vee$ is also admissible, and $(\pi^K)^* \cong (\pi^\vee)^K$.

The converse of Proposition 4.22 is also true.

Proposition 4.25. *Let $\pi \in \mathrm{Rep}(G)$ be irreducible such that its matrix coefficients are compactly supported modulo Z. Then π is supercuspidal.*

Proof. Let $K_i = \{g \in \mathrm{GL}_2(\mathcal{O}) \mid g \equiv 1 \mod \mathfrak{p}^i\}$. Let $v \in V^{K_i}$. We want to show that

$$\int_{N_j} \pi(n)vdn = 0$$

for some j. Recall that we defined a projection

$$p_K \colon V \to V_K, \quad v \mapsto \frac{1}{\mathrm{vol}(K)} \int_K kv\,dk.$$

Let $S_v = \{g \in G \mid P_{K_i}(\pi(g^{-1})v) \neq 0\}$. We claim that S_v is compactly-supported modulo Z.

Recall that π is admissible by the same proof as the proof of Corollary 4.24 (we only used that the matrix coefficients are compactly supported modulo center to prove it). Let $\{\lambda\}$ be a basis of $(V^\vee)^{K_i}$ (which is finite-dimensional). Then

$$S_v \subseteq \bigcup_{\{\lambda\}} \mathrm{supp}(\varphi_{v,\lambda})$$

because

$$\langle P_{K_i}(\pi(g^{-1})v), \lambda\rangle = \langle \pi(g^{-1})v, \lambda\rangle = \varphi_{v,\lambda}(g).$$

This shows that S_v is compact modulo Z. Therefore,

$$P_{K_i}(\pi(t_a)v) = 0 \quad \text{for } a \gg 0$$

because the sequence $t_a = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a} \end{pmatrix}$ escapes any compact subset.

Defining $N_0 = \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} = N(\mathcal{O})$, $\overline{N}_i = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^i & 1 \end{pmatrix}$, $T_i = \begin{pmatrix} 1+\mathfrak{p}^i & 0 \\ 0 & 1+\mathfrak{p}^i \end{pmatrix} \subseteq T$, a matrix manipulation shows that

$$K_i = N(\mathcal{O})T_i\overline{N}_i$$

(the proof of this can be found in [Bum97]). For $a \gg 0$,

$$\begin{aligned} 0 &= \pi(t_a^{-1})P_{K_i}(\pi(t_a)v) \\ &= \int\limits_{K_i} \pi(t_a^{-1}kt_a)v\,dk \\ &= \int\limits_{N(\mathcal{O})}\int\limits_{T_i}\int\limits_{\overline{N}_i} \pi(t_a^{-1}nt_i\overline{n_i}t_a)v\,d\overline{n}_i\,dt_i\,dn \\ &= \int\limits_{N(\mathcal{O})}\int\limits_{T_i}\int\limits_{\overline{N}_i} \pi(n^{t_a}t_i^{t_a}\overline{n}_i^{t_a})v\,d\overline{n}_i\,dt_i\,dn \qquad \text{where } g^{t_a} = t_a^{-1}gt_a. \end{aligned}$$

Now,

$$\overline{n}_i^{t_a} = \begin{pmatrix} \varpi^{-a} & 0 \\ 0 & \varpi^a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathfrak{p}^i & 1 \end{pmatrix}\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varpi^{2a}\mathfrak{p}^i & 1 \end{pmatrix} \subseteq K_i,$$

$$t_i^{t_a} = t_i \in K_i,$$

$$t_a^{-1}N_it_a = N_{i-2a}.$$

Up to some volume factors, this integral hence becomes equal to

$$\int_{N(\mathcal{O})} \pi(n^{t_a})v\,d_n$$

and, after a change of variables $\varpi^{2i}x \mapsto x$, we get

$$|\varpi|^{2a} \cdot \int_{N_{-2a}} \pi(n)v\,dn.$$

Since this is equal to 0, we have shown that π is supercuspidal. □

4.2. **Compact induction and Haar measures.** Consider a closed subgroup $H \subseteq G$ and consider $\sigma \in \mathrm{Rep}(H)$. We defined

$$\mathrm{Ind}_H^G(\sigma) = (\{f \colon G \to \sigma \mid f(hg) = \sigma(h)f(g)\})^\infty,$$

where the smooth vectors are taken under the action $g \cdot f(x) = f(xg)$.

Definition 4.26. We define the *compact induction* as

$$\text{c-Ind}_H^G(\sigma) = \{f \in \mathrm{Ind}_H^G(\sigma) \mid f \text{ compactly supported modulo } H\} \in \mathrm{Rep}(G).$$

Note that $\mathrm{Ind}_B^G(\chi) = \text{c-Ind}_B^G(\chi)$, since $B\backslash G$ is compact.

Lemma 4.27. *The functor* c-Ind$_H^G$ *is exact.*

We already proved in Proposition 3.16 that Ind_H^G is exact. The proof of this is similar, so we omit it here.

Consider the function

$$\begin{aligned} \alpha \colon \sigma &\to \text{c-Ind}_H^G(\sigma) \\ v &\mapsto f_v \end{aligned}$$

where

$$f_v(g) = \begin{cases} \sigma(g)v & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

This only works when H is open, which we will suppose from now on. Thus, H is open and closed in what follows.

Lemma 4.28.

(1) *The map α is an H-isomorphism onto $f \in \mathrm{Ind}_H^G$ supported on H.*
(2) *Let $\mathcal{B}$ be a basis of σ and fix representatives for G/H. Then $\{g \cdot f_v \mid v \in \mathcal{B}, g \in G/H\}$ is a basis of* $\text{c-Ind}_H^G(\sigma)$*.*

While Ind_H^G was the a adjoint of restriction, c-Ind_H^G is a right adjoint.

Proposition 4.29 (Frobenius reciprocity 2)**.** *Let H be open and $\sigma \in \mathrm{Rep}(H)$, $\pi \in R(G)$. Then*

$$\mathrm{Hom}_G(\text{c-Ind}_H^G(\sigma), \pi) \cong \mathrm{Hom}_H(\sigma, \pi)$$

via $\Phi \mapsto \Phi \circ \alpha$.

Proof. The inverse is given by sending $\Psi \in \mathrm{Hom}_H(\sigma, \pi)$ to $(f_w \mapsto \Psi(w))$, which extends to a G-homomorphism. □

We now discuss Haar measures. For a compact open $K \subseteq G$, we define

$${}^K C_c^\infty = \text{c-Ind}_K^G(\mathbb{C}),$$

where $\mathbb{C}$ is the trivial representation of K.

Lemma 4.30. *The dimension of $\mathrm{Hom}_G({}^K C_c^\infty(G), \mathbb{C})$ is 1.*

Proof. By Frobenius reciprocity 2 4.29, $\mathrm{Hom}_G({}^K C_c^\infty(G), \mathbb{C}) = \mathrm{Hom}_K(\mathbb{C}, \mathbb{C}) = 1$. □

Let $I_K \in \mathrm{Hom}({}^K C_c^\infty(G), \mathbb{C})$ be given by

$$I_K(\mathbb{1}_{K \cdot g}) = 1$$

for all $g \in K \backslash G$. Note that

$$C_c^\infty(G) = \bigcup_{K_n}^{K_n} C_c^\infty(G)$$

where K_n is a neighborhood basis of 1 such that $K_n \supseteq K_{n+1}$. Define $I_n = \frac{1}{[K_i : K_n]} I_{K_n}$ so that $I_{n+1}|^{K_n} C_c^\infty(G) = I_n$. This defines an element

$$\mathrm{Hom}_G(C_c^\infty(G), \mathbb{C})$$

and hence a right invariant Haar measure. It is unique (up to a scalar), because its restrictions to ${}^K C_c^\infty(G)$ is unique by Lemma 4.30.

Fix a μ-Haar measure on G. For any $g_0 \in G$, there is a positive number $\delta_G(g_0) \in \mathbb{R}_{>0}$ such that

$$\delta_G(g_0) \int f(g_0 g)\, dg = \int f(g)\, dg$$

for any $f \in C_c^\infty(G)$, by uniqueness of Haar measure.

The function $\delta_G \colon G \to \mathbb{R}_{>0}^\times$ is

- independent of the choice of μ,
- a character,
- smooth (take $f = \mathbb{1}_K$).

Example 4.31.

(1) For any abelian group, δ_G is trivial.
(2) When $G = \mathrm{GL}_2$, $\delta_G = 1$, because $[G, G] \cdot Z$ has finite index in G.
(3) Consider $G = B = N \rtimes T$. Given an additive measure dx on F, we have a Haar measure $\frac{dx}{|x|}$ on $F^\times$. Since $N \cong F$, we get a measure on N. Since $T \cong (F^\times)^2$, we also get a measure on T. We may then define

$$\int_B f(b)\, db = \int_N \int_T f(nt)\, dt\, dn$$

which is a Haar measure. Then for $t_0 = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$,

$$\int_B f(t_0 b)\, db = \int_N \int_T f(t_0 nt)\, dt\, dn = \int_N \int_T f((t_0 n t_0^{-1})(t_0 t))\, dt\, dn = |t_1/t_2|^{-1} \int_N \int_T f(nt)\, dt\, dn$$

because

$$t_0 n t_0^{-1} = \begin{pmatrix} 1 & t_1/t_2 x \\ 0 & 1 \end{pmatrix}.$$

This shows that $\delta_B(tn) = \delta_B(nt) = |t_1/t_2|$.

Let $H \subseteq G$ be a closed subgroup. Let $\delta_{H\backslash G}$ to be the character δ_H/δ_G on H. For any $\theta \colon H \to \mathbb{C}^\times$, we have $\text{c-Ind}_H^G(\theta) = C_c^\infty(H\backslash G, \theta)$, where the right hand side are f such that $f(hg) = \theta(h) f(g)$.

Theorem 4.32. *We have that* $\dim \mathrm{Hom}_G(C_c^\infty(H\backslash G, \theta), \mathbb{C}) = \begin{cases} 0 & \text{if } \theta \neq \delta_{H\backslash G}, \\ 1 & \text{if } \theta = \delta_{H\backslash G}. \end{cases}$ *In particular, if $\delta_H = \delta_G = 1$ (e.g. $G = \mathrm{GL}_2$, $H = K$), then there is a G-invariant measure on $H\backslash G$.*

We write $\int_{H\backslash G} f(g)\, dg$ for the integral in $\mathrm{Hom}_G(C_c^\infty(H\backslash G, \theta), \mathbb{C})$ when $\theta = \delta_{H\backslash G}$. Then

$$\int_G f(g)\, dg = \int_{H\backslash G} \left[\int_H f(hg) \delta_{H\backslash G}^{-1}(h)\, dh \right] dg$$

for $f \in C_c^\infty(G)$.

Theorem 4.33 (Duality)**.** *Suppose $H \subseteq G$ is closed and $\sigma \in \mathrm{Rep}(H)$. Then*

$$(\text{c-Ind}_H^G \sigma)^\vee \cong \mathrm{Ind}_H^G(\delta_{H\backslash G} \otimes \sigma^\vee).$$

We will prove this shortly.

Corollary 4.34. *We have that* $I_B^G(\chi)^\vee = I_B^G(\chi^{-1})$.

Proof. Recall that $I_B^G(\chi) = \text{c-Ind}_B^G(\delta_B^{1/2}\chi)$. We have that

$$\begin{aligned} I_B^G(\chi)^\vee &= (\text{c-Ind}_B^G(\delta_B^{1/2}\chi))^\vee \\ &= \text{Ind}_B^G(\delta_B \cdot (\delta_B^{1/2}\chi)^{-1}) \\ &= \text{Ind}_B^G(\delta_B^{1/2}\chi^{-1}) \\ &= I_B^G(\chi^{-1}), \end{aligned}$$

as required. □

Sketch of proof of Duality Theorem 4.33. Let $\langle\cdot,\cdot\rangle\colon \text{Ind}_H^G(\delta_{H\backslash G}\otimes\sigma^\vee)\times \text{c-Ind}_H^G(\sigma)\to\mathbb{C}$ by

$$\langle\Phi,\varphi\rangle = \int_{H\backslash G}\langle\Phi(g),\varphi(g)\rangle_\sigma dg$$

where $\langle\cdot,\cdot\rangle_\sigma\colon \sigma^\vee\to\sigma\to\mathbb{C}$ is the canonical pairing. This is well-defined and $\langle g\Phi, g\varphi\rangle = \langle\Phi,\varphi\rangle$. This gives a G-homomorphism

$$\text{Ind}_H^G(\delta_{H\backslash G}\otimes\sigma^\vee)\to(\text{c-Ind}_H^G(\sigma))^\vee.$$

Checking that this is an isomorphism is skipped. □

4.3. **Classification of representations of** GL_2**.** We want to classify irreducible smooth representations of GL_2. The goal is to prove the following theorem.

Theorem 4.35. *Let* π *is an irreducible smooth representation of* GL_2*. Then one of the following statements is true:*

- π *is supercuspidal (i.e.* $\pi_N = 0$ *or the matrix coefficients are compactly supported modulo* Z*),*
- $\pi\cong I_B^G(\chi)$*,*
- $\pi\cong\eta(\det g)$*, a 1-dimensional representation,*
- π *is an infinite-dimensional submodule/quotient of some* $I_B^G(\chi)$*.*

The last class of representations are called *Steinberg*.

To prove this theorem, we just have to analyze $I_B(\chi)$. We need to know when it is irreducible or reducible and what is its composition series in general.

Proposition 4.36. *Let* V *be an irreducible subquotient of* $I(\chi)$*, then* $V_N\neq 0$*.*

Proof. If it were, it would embed: by Proposition 4.23, if $W\subseteq I(\chi)$, then $W\to V\to 0$ splits, so we get a non-trivial map $V\to W\hookrightarrow I(\chi)$.

By Frobenius reciprocity 3.15,

$$0\neq\text{Hom}_G(V,I(\chi))\cong\text{Hom}_T(V_N,\chi\delta^{1/2}) = 0.$$

This is a contradiction. □

Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N_G(T)$. In fact, w and T generate $N_G(T)$.

Proposition 4.37 (Bruhat decomposition)**.** *We have a disjoint union $G = B \cup BwN$.*

Proof. We know that B consists of elements $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and a simple computation shows that BwN consists of elements $\begin{pmatrix} * & * \\ * \neq 0 & * \end{pmatrix}$. □

If $\chi = (\chi_!, \chi_2)$ is a character of T, define $\chi^w = (\chi_2, \chi_1)$. Note that

$$\chi^w(t) = \chi(wtw^{-1}).$$

We will understand the representations $I(\chi)$ through understanding their Jacquet modules $I(\chi)_N$.

Proposition 4.38. *There is an exact sequence of T-modules:*

$$0 \longrightarrow \delta^{1/2}\chi^w \longrightarrow I(\chi)_N \longrightarrow \delta^{1/2}\chi \longrightarrow 0.$$

Proof. Let $I_0 \subseteq I(\chi)$ be a B-submodule consisting of $f \in I(\chi)$ supported on BwN. Define

$$\begin{aligned} \alpha \colon I(\chi) &\to \delta^{1/2}\chi \\ f &\mapsto f(1). \end{aligned}$$

This is a B-homomorphism (by definition of induction from a Borel). We then have an exact sequence

$$0 \longrightarrow I_0 \longrightarrow I(\chi) \longrightarrow \delta^{1/2}\chi \longrightarrow 0$$

of B-modules. Since the Jacquet functor is exact, we get an exact sequence

$$0 \longrightarrow I_{0,N} \longrightarrow I(\chi)_N \longrightarrow \delta^{1/2}\chi \longrightarrow 0$$

of B-modules. We want to show that $I_{0,N} \cong \delta^{1/2}\chi^w$. Let

$$\begin{aligned} \alpha_w \colon I_0 &\to \delta^{1/2}\chi^w, \\ f &\mapsto \int_N f(wn)\, dn. \end{aligned}$$

Note that, writing $t^w = wtw^{-1}$,

$$\int_N f(wnt)\, dn = \delta(t) \int_N f(wtu)\, dn = \delta(t)\delta^{1/2}(t^w)\chi(t^w) \int_N f(wn)dn = \delta^{1/2}(t)\chi^w(t) \int_N (wn)\, dn.$$

Therefore, the map α_w is indeed well-defined and clearly surjective. What is the kernel of α_w? It is:

$$I_0(N) = \left\{ v \in I_0 \;\middle|\; \int\limits_{N_i} nv = 0 \text{ for some } i \right\}.$$

Therefore,

$$I_0/I_0(N) \cong \delta^{1/2}\chi^w.$$

This completes the proof. □

Corollary 4.39. *The composition series of $I(\chi)$ is at most of length 2.*

Proof. Indeed, if $0 \subseteq V_1 \subseteq \cdots \subseteq V_k = I(\chi)$, then $V_{i,N} \neq 0$ by Proposition 4.36. This shows that $\mathrm{length}(I(\chi)) \leq \mathrm{length}(I(\chi)_N) = 2$ by Proposition 4.38. □

Lemma 4.40. *If $\chi \neq \chi^w$ (i.e. $\chi_1 \neq \chi_2$), then $I(\chi)_N$ is semisimple. If $\chi = \chi^w$, then it is not.*

Lemma 4.41. *If $\chi \neq \chi^w$, then α_w defined in the proof of Proposition 4.38 extends to $I(\chi)$. If $\chi = \chi^w$, then it does not.*

Lemma 4.40 implies Lemma 4.41 when $\chi \neq \chi^w$.

Proof. Indeed, by Proposition 4.40, $I(\chi)_N$ is semisimple, so the map $I_{0,N} \to \delta^{1/2}\chi^w$ extends to $I(\chi)_N$ (trivially on the complementary piece). Then

$$\mathrm{Hom}_T(I(\chi)_N, \delta^{1/2}\chi^w) = \mathrm{Hom}_B(I(\chi), \delta^{1/2}\chi^w) \cong \mathrm{Hom}_G(I(\chi), I(\chi^w)).$$

This completes the proof. □

In fact, it gives an intertwining operator $A_w \in \mathrm{Hom}_G(I(\chi), I(\chi^w))$ defined by the formula

$$A_w(f)(g) = \int_N f(wng)\, dn$$

which converges if $s_1 > s_2$ (where $\chi_i = |\cdot|^{s_i} \cdot \mu_i$) and has meromorphic continuation.

Similarly, Lemma 4.41 implies Lemma 4.40 when $\chi = \chi^w$. We will see this next.

Proposition 4.42. *Let ϱ be a 2-dimensional representation of $T = F^\times \times F^\times$. Then either*

$$\varrho \cong \left(t \mapsto \begin{pmatrix} \xi(t) & 0 \\ 0 & \xi'(t) \end{pmatrix} \right)$$

or ϱ has a unique irreducible subspace $\mathbb{C}\xi$ with $\varrho/\mathbb{C}\xi \cong \mathbb{C}\xi$, and then

$$\varrho \cong \left(t \mapsto \xi(t) \begin{pmatrix} 1 & \nu(t) \\ 0 & 1 \end{pmatrix} \right)$$

where $\nu\colon T(F)/T(\mathcal{O}) \to \mathbb{C}_+$ is a non-trivial homomorphism, and $\xi, \xi\colon T \to \mathbb{C}^\times$ are characters.

Proof. Any representation has an irreducible quotient. By dimension considerations, we have a short exact sequence

$$0 \longrightarrow \xi \longrightarrow V \xrightarrow{\Psi} \xi' \longrightarrow 0$$

for characters $\xi, \xi' \colon T \to \mathbb{C}^\times$. Let $v \neq 0$ be such that $t \cdot v = \xi(t)v$, so that ξ is the representation $\mathbb{C}v \subseteq V$. Let $w \in V \setminus \mathbb{C}v$. Then

$$\Psi(t \cdot w - \xi'(t)w) = \Psi(t \cdot w) - \xi'(t)\Psi(w) = \xi'(t)\Psi(w) - \xi'(t)\Psi(w) = 0.$$

Hence

$$t \cdot w = \xi'(t)w + \lambda(t)v$$

for some $\lambda(t) \in \mathbb{C}$. Setting $t = t_1 \cdot t_2$, we see that

$$\lambda(t_1t_2) = \xi(t_1)\lambda(t_2) + \xi'(t_2)\lambda(t_1) = \xi(t_2)\lambda(t_2) + \xi'(t_1)\lambda(t_2).$$

Since $\lambda(t_1t_2) = \lambda(t_2t_1)$, this gives us the following equation

$$\lambda(t_1)[\xi(t_1) - \xi'(t_1)] = \lambda(t_1)[\xi(t_2) - \xi'(t_2)]. \tag{1}$$

If $\xi \neq \xi'$, $\xi(t_1) \neq \xi'(t_2)$ for some t_1, and hence for any t:

$$\lambda(t) = c(\xi(t) - \xi'(t))$$

for some constant c. This shows that ϱ is diagonal in the basis v, $w - cv$.

If $\xi = \xi'$, we claim that $\lambda \cdot \xi^{-1} \colon T \to \mathbb{C}_+$ is a homomorphism. This follows from equation (1). Finally, $\lambda \cdot \xi^{-1}|_{T(\mathcal{O})}$ is clearly trivial. We let $\nu = \lambda \cdot \xi^{-1}$.

If $\nu = 0$, then $\lambda = 0$, so the representation is semisimple. Else, v, w is a basis such that

$$\varrho \cong \left(t \mapsto \xi(t) \begin{pmatrix} 1 & \nu(t) \\ 0 & 1 \end{pmatrix} \right).$$

This completes the proof. □

We will write $N(\xi)$ for any of the non-semisimple representation associated to ξ.

Lemma 4.43. *If $\alpha_w \in \mathrm{Hom}_B(I_0(\chi), \delta^{1/2}\chi^w)$ does not extend to $\mathrm{Hom}_B(I(\chi), \delta^{1/2}\chi^w)$.*

Corollary 4.44. *We have that $I(\chi)_N = N(\chi_1)$, which is not semisimple.*

Corollary 4.45. *If $\chi = \chi^w$, $I(\chi)$ is irreducible.*

Proof. We have that $\chi = \begin{pmatrix} \chi_0 & \\ & \chi_0 \end{pmatrix}$. Assume that χ_0 is unitary (by twisting by $\eta(\det g)$).

Then $I(\chi)$ is unitary: if $f_1, f_2 \in I(\chi)$, then

$$f_1 \cdot \overline{f_2} \in C_c^\infty(B \backslash G, \delta_B)$$

so by Theorem 4.32, the following integral exists

$$\langle f_1, f_2 \rangle = \int_{B \backslash G} f_1(g)\overline{f_2(g)}\, dg$$

and defines a Hermitian product.

Any unitary representation is semisimple: the orthogonal complement defines the necessary splitting. Hence $I(\chi)$ is semisimple. Finally,

$$\dim \mathrm{Hom}_G(I(\chi), I(\chi)) = \dim \mathrm{Hom}_T(I(\chi)_N, \delta^{1/2}\chi) = 1.$$

By the converse to Schur's Lemma (which holds for semisimple representations), this shows that $I(\chi)$ is irreducible. ☐

The next goal is to show that $f \mapsto \int_N (wn)dn$ extends to

$$\mathrm{Hom}_B(I(\chi), \delta^{1/2}\chi^w)$$

uniquely if $\chi \neq \chi^w$, and does not extend if $\chi = \chi^w$.

We first go back to a statement in the GL(1) case which similar and is easier to understand.

Let $\alpha_\eta(f) = \int\limits_{F^\times} f(t)\eta^{-1}(t)\frac{dt}{|t|}$ for $f \in C_c^\infty(F^\times)$, $\eta\colon F^\times \to \mathbb{C}^\times$. Then

$$\alpha_\eta \in \mathrm{Hom}_{F^\times}(C_c^\infty(F^\times), \eta).$$

Lemma 4.46. *This map α_η extends uniquely to $\mathrm{Hom}_{F^\times}(C_c^\infty(F), \eta)$ if and only if $\eta \neq 1$.*

Proof. Suppose α_η extends. For $f \in C_c^\infty(F)$ and look at $f' = f - \varpi \cdot f$. Clearly, $f'(0) = 0$, and hence $f' \in C_c^\infty(F^\times)$. Then

$$\alpha_\eta(f') = \int_{F^\times} [f - \varpi \cdot f]\eta^{-1} d^\times t = \alpha_\eta(f) - \alpha_\eta(\varpi f) = [1 - \eta(\varpi)]\alpha_\eta(f).$$

We observe that if $\eta(\varpi) = 1$ for any uniformizer ϖ, then $\eta = 1$. This is not possible, since $= \int_{F^\times}[f - \varpi \cdot f]\eta^{-1}d^\times t \neq 0$ for $f = \mathbb{1}_{\mathcal{O}}$.

Hence $\eta(\varpi) \neq 1$, and

$$\alpha_\eta(f) = [1 - \eta(\varpi)]^{-1} \int [f - \varpi f] d^\times t$$

exists and is unique. ☐

In our case, we look at

$$\alpha_w(\chi)(f) = \int\limits_N f(wn)\, dn$$

for f supported on BwN.

Consider $t_\varpi = \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$ and set for $f \in I(\chi)$

$$f' = f - (\delta^{1/2}\chi)^{-1}(t_\varpi)[t_\varpi \cdot f].$$

Then $f'(0) = 0$ and $f' \in I_0(\chi)$. Once again, the extension has to be of the form

$$\alpha_w(f) = [1 - (\chi^w/\chi)(t_\varpi)]^{-1} \int\limits_N f'(wn)\, dn.$$

Overall, we have shown Lemma 4.41.

Proposition 4.47. *The representation $I(\chi)$ is irreducible if and only if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$. If $\chi_1/\chi_2 = |\cdot|^{-1/2}$, then $I(\chi)$ has a 1-dimensional submodule, and if $\chi_1/\chi_2 = |\cdot|^{1/2}$, this has a 1-dimensional subquotient.*

Definition 4.48. The unique irreducible infinite-dimensional quotient/submodule of $I(\chi)$ when $\chi_1/\chi_2 = |\cdot|^{\pm 1/2}$ is called a *Steinberg representation.* In other words, if $\chi_{\mathrm{sp}} = \delta^{-1/2}\begin{pmatrix}\chi_0 & 0\\ 0 & \chi_0\end{pmatrix}$, then $g \mapsto \chi_0(\det g) \in I(\chi)$, and

$$\mathrm{St}(\chi_0) = \frac{I(\chi_{\mathrm{sp}})}{\chi_0(\det(\cdot))}.$$

Suppose $\chi \neq \chi^w$. We have that

$$\dim \mathrm{Hom}_G(I(\chi), I(\chi)) = \mathrm{Hom}_T(I(\chi)_N, \chi\delta^{1/2}) = 1.$$

We produced

$$A_w(\chi)\colon I(\chi) \to I(\chi^w)$$

and hence we get a map

$$A_w(\chi^w) \circ A_w(\chi)\colon I(\chi) \to I(\chi)$$

which is a number $c(\chi)$.

We summarize the classification so far:

(1) supercupsidal,
(2) 1-dimensional $\chi_0(\det g)$,
(3) $\mathrm{St}(\chi_0) = I\left(\delta^{-1/2}\begin{pmatrix}\chi_0 & \\ & \chi_0\end{pmatrix}\right)/\chi_0(\det(\cdot))$,
(4) if $\chi \neq \delta^{\pm 1/2}\begin{pmatrix}\chi_0 & \\ & \chi_0\end{pmatrix}$, then $I(\chi)$ is irreducible and isomorphic to $I(\chi^w)$.

When $\chi = \chi^w$, we know this. When $\chi \neq \chi^w$, we defined

$$\alpha_w(\chi) \in \mathrm{Hom}_B(I(\chi), \delta^{1/2}\chi^w) \cong \mathrm{Hom}_G(I(\chi), I(\chi^w)) \ni A_w(\chi)$$

which is an intertwiner. Recall that

$$A_w(\chi)(f)(g) = \int_N f(wng)\, dn.$$

Composing $A_w(\chi)$ with $A_w(\chi^w)$, we get a map

$$A_w(\chi^w) \circ A_w(\chi)\colon I(\chi) \to I(\chi)$$

which is a constant $c(\chi) \in \mathbb{C}$.

Lemma 4.49. *If $\chi_1 \neq \chi_2$, $I(\chi)$ is reducible if and only if $c(\chi) = 0$.*

Proof. We noted that $A_w(\chi) \neq 0$ when $I(\chi)$ is irreducible, so $c(\chi) \neq 0$. If $I(\chi)$ is reducible and $V \subseteq I(\chi)$ is a submodule, then $V_N \cong \delta^{1/2}\chi$ by Frobenius reciprocity 3.15. Again, by Frobenius reciprocity 4.29,

$$\mathrm{Hom}(v, I(\chi^w)) = 0,$$

so $\ker(A_w(\chi)) \supseteq V$ and

$$U = A_w(I(\chi)/V) \subseteq I(\chi^w).$$

Again, $U_N \cong \delta^{1/2}\chi^w$. Then

$$U \subseteq \ker A_w(\chi^w),$$

showing that $c(\chi) = 0$. □

The constants $c(\chi)$ is actually related to L-functions. This is called the Langlands–Shahidi method.

Here are the goals for the next few weeks:

- classify representations up to supercupsidals (almost done),
- Langlands correspondence $\mathrm{Rep}(G) \cong \{\phi\colon \mathrm{Gal}(\overline{F}/F) \to {}^L G\}$,
- L-functions, ϵ-factors,
- models: Whittaker model/Kirilov model,
- harmonic analysis: studying invariant distrributions,
- classify square-integrable represntations of G (or even unitary representations):
  - square-integrable means matrix-coefficients are square-integrable on $Z\backslash G$,
  - supercuspidals are square-integrable,
  - for GL_2, Steinberg is square-inegrable and no other is,
- classify all unitary representations of GL_2.

5. Square integrable functions

For $G = \mathrm{GL}_2$, recall that

$$\mathrm{St}(\chi_0) = \text{irreducible quotient of } \mathrm{Ind}_B^G \begin{pmatrix} \chi_0 & 0 \\ 0 & \chi_0 \end{pmatrix}.$$

Note that $\mathrm{St}(\chi_0)_N = \chi_0 \delta$.

Proposition 5.1. *For χ_0 unitary, the matrix coefficients of $\mathrm{St}(\chi_0)$ are square-integrable mod Z.*

Proof. We may assume that χ_0 is trivial. Let

$$G^0 = \{g \in G \mid \det g \in \mathcal{O}^\times\} = \coprod_{a \geq 0} K\lambda_a K \quad \text{for } \lambda = \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^{-a} \end{pmatrix}.$$

Then $[G^0 Z : G] = 2$, so it is enough to show that $\mathrm{St}(1)$ is G^0-square-integrable.

Let $v_0 \in V = \mathrm{St}(1)$, $\lambda_0 \in V^\vee$. Let U be the K-span of v_0 and $\tilde{U}$ be the K-span of λ_0. For $u \in U$, $\tilde{u} \in \tilde{U}$, we want to show that

$$\varphi(g) = \langle \tilde{u}, \pi(g) u \rangle$$

is square-integrable.

We now use Lemma 5.2, which we prove below:

$$\begin{aligned} \int_{G^0} |\varphi(a)|^2 \, dg &= \sum_{a \geq 0} \int_{K\lambda_a K} |\varphi(g)|^2 dg \\ &\leq C_1 + C_2 \sum_{a \geq 0} \mathrm{vol}(K\lambda_a K) \delta(\lambda_a)^2 \end{aligned}$$

where C_1 are the terms up to i and C_2 is a bound for $\phi(k\lambda_i k')$ (where i is fixed).

Note that $\delta(\lambda_a) = |\varpi^{2a}| = q^{-2a}$ which will converge. We need to compute the volume of $K\lambda_a K$. Note that

$$\mathrm{vol}(K\lambda_a K) = \mathrm{vol}(K\lambda_a K\lambda_a^{-1})$$

We have a coset decomposition

$$K\lambda_a K\lambda_a^{-1} = \coprod_{g\in K\cap\lambda_a K\lambda_a^{-1}\backslash K} [\lambda_a K\lambda_a^{-1}]g.$$

The Haar measure is normalized so that $\mathrm{vol}(\lambda_a^{-1}K\lambda_a) = 1$, so

$$\mathrm{vol}(K\lambda_a K) = \#\left(K\cap\lambda_a K\lambda_a^{-1}\backslash K\right).$$

Finally, elements of $\lambda_a K\lambda_a^{-1}$ are of the form

$$\begin{pmatrix}\varpi^a & 0\\ 0 & \varpi^{-a}\end{pmatrix}\begin{pmatrix}x & y\\ z & v\end{pmatrix}\begin{pmatrix}\varpi^{-a} & 0\\ 0 & \varpi^{a}\end{pmatrix} = \begin{pmatrix}x & \varpi^{2a}y\\ \varpi^{-2a}z & v\end{pmatrix}.$$

Therefore:

$$K'_{2a} = \lambda_a K\lambda_a^{-1}\cap K = \left\{\begin{pmatrix}x & y\\ z & v\end{pmatrix}\in K \;\middle|\; z\in\mathfrak{p}^{2a}\right\}.$$

Note that

$$K'_i\backslash K \cong \frac{\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^i)}{B(\mathcal{O}/\mathfrak{p}^i)}.$$

Recall that

$$K_i = \ker(\mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^i)).$$

We compute the number of elements:

- $|B(\mathcal{O}/\mathfrak{p}^i)| = |(\mathcal{O}/\mathfrak{p}^i)^\times|^2\cdot|\mathcal{O}/\mathfrak{p}^i| = (q^i - q^{i-1})^2q^i$,
- $K_1\backslash K \cong \mathrm{GL}_2(\mathcal{O}/\mathfrak{p})$, so its order is $(q^2-q)(q^2-1)$,
- $K_{i+1}\backslash K'_i$ has order q^4,
- $|K_i\backslash K| = |\,\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^i)| = q^{4(i-1)}(q^2-q)(q^2-1)$.

Therefore:

$$\mathrm{vol}(K\lambda_a K) = \#\left(K\cap\lambda_a K\lambda_a^{-1}\backslash K\right) = q^i\frac{q+1}{q} = c_q\delta(\lambda_i)^{-1}.$$

where $i = 2a$.

Finally, this shows that

$$\int_{G^0}|\varphi(a)|^2dg \le C_1 + C'_2\sum_{a\ge 0}\delta(\lambda_a)^{-1}\delta(\lambda_a)^2 = C_1 + C'_2\sum_{a\ge 0}q^{-2a}.$$

This completes the proof. □

Lemma 5.2. *There exists $i \geq 0$ (depending on U and $\tilde{U}$) such that for all $a \geq 0$, the following equality holds:*

$$\phi(k\lambda_{a+i}k') = \delta(\lambda_a)\phi(k\lambda_i k').$$

Proof. Let $u_a = \pi(\lambda_a)u - \delta(\lambda_a)u$. The above statement is equivalent to showin that there is an i such that

$$\langle\tilde{u}, \pi(\lambda_i)u_a\rangle = 0.$$

Observe the following: $u_1 \in V(N) = \ker(\psi\colon V \to V_N)$, because $V_N = \delta$, so

$$\psi(u_1) = \delta(\lambda_1)\psi(u) - \delta(\lambda_1)\psi(u) = 0.$$

Let i be such that for all $u \in U$, $u_1 \in V(N_{-i})$ and $\tilde{u} \in (V^\vee)^{K_i}$; then

$$\langle \tilde{u}, \pi(\lambda_i)u_a\rangle = \int_{N_i} \langle \pi^\vee \tilde{u}, \pi(\lambda_i)u_a\rangle dn = |\varpi|^{2i} \int_{N_{-i}} \langle \tilde{u}, \pi(\lambda_i)\pi(u)u_a\rangle dn = 0,$$

because

$$\pi(\lambda_1)V(N_{-i}) \subseteq V(-i+1) \subseteq V(N_{-i})$$
$$u_a = \pi(\lambda_1)u_{a-1} + \delta(\lambda_{a-1})u_1 \in V(N_{-i}).$$

This completes the proof. □

6. Hecke algebras and distributions

Let $\mathcal{H}(G) = C_c^\infty(G) = \mathcal{H}$ and $\mathcal{H}(G)_K = C_c^\infty(K\backslash G/K) = \mathcal{H}_K$. Note that

$$\mathcal{H} = \bigcup \mathcal{H}_K.$$

Then $\mathcal{H}$ is a $\mathbb{C}$-algebra under convolution:

$$\phi_1 * \phi_2(x) = \int_G \phi_1(xg^{-1})\phi_2(g)\, dg$$

and $\mathcal{H}_K$ is a subalgebra with $\frac{1}{\mathrm{vol}(K)}\mathbb{1}_K$ as the unit.

If (π, V) is smooth and $f \in \mathcal{H}$, then

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg,$$

which defines a representation of the $\mathbb{C}$-algebra algebra $\mathcal{H}$ on V. For $f \in \mathcal{H}_K$, $\pi(f)V \subseteq V^K$.

Theorem 6.1. *For π smooth, the following are equivalent:*

- *π is irreducible,*
- *V is $\mathcal{H}$-irreducible,*
- *V^K is either 0 or $\mathcal{H}_K$-irreducible.*

Theorem 6.2. *If π_1, π_2 are smooth, irreducible, admissible representations, then*

$$\pi_1 \cong \pi_2 \text{ if and only if } \pi_1^K \cong \pi_2^K \text{ as } \mathcal{H}_K\text{-modules for all } K.$$

If π is admissible, we have a distribution

$$\chi_\pi\colon \mathcal{H} \to \mathbb{C}$$

defined by

$$\chi_\pi(f) = \mathrm{Tr}(\pi(f)).$$

Indeed, if $T\colon V \to V$ such that $\mathrm{im}(T) = U$ is finite-dimensional, then if $U \subseteq W \subseteq V$ is such that $T(W) = U$ and W is finite-dimensional, then we can define

$$\mathrm{Tr}(T) = \mathrm{Tr}(T|_W)$$

is independent of W. Since $\pi(f)V \subseteq V^K$ for $f \in \mathcal{H}_K$ and V^K is finite-dimensional, we may define the trace this way.

Theorem 6.3. *If π_1, π_2 are irreducible admissible, then*

$$\pi_1 \cong \pi_2 \text{ if and only if } \chi_{\pi_1} = \chi_{\pi_2}.$$

Proposition 6.4. *Suppose π is admissible. For $g \in G$ and $f \in \mathcal{H}_K$, define $f^g(x) = f(gxg^{-1})$. Then $\chi_\pi(f) = \chi_\pi(f^g)$.*

Proof. Note that $\pi(f^g) = \pi(g)^{-1}\pi(f)\pi(g)$. We have

$$V^{K\cap g^{-1}Kg} \xrightarrow{\pi(g)} V^{K\cap gKg^{-1}} \xrightarrow{\pi(f)} V^{K\cap g \cap Kg^{-1}} \xrightarrow{\pi(g)^{-1}} V^{K\cap g^{-1}Kg}.$$

and these are maps between finite-dimensional vector spaces where we know that trace is invariant under conjugation. □

Theorem 6.5. *Suppose (π, V) is an irreducible representation of GL_n.*

(1) *Let $\pi_1(g) = \pi((g^t)^{-1})$. Then $\pi_1 \cong \pi^\vee$.*
(2) *For $n = 2$, let $\pi_2 = \pi \otimes \omega_\pi^{-1}$ be given by $\pi_2(g) = \pi(g)\omega_\pi^{-1}(\det g)$. Then $\pi_2 \cong \pi^\vee$.*

Proof. We show that (1) implies (2). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$g^{-t} = \begin{pmatrix} \det g & 0 \\ 0 & \det g \end{pmatrix}^{-1} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} \det g & 0 \\ 0 & \det g \end{pmatrix}^{-1} w^{-1}gw$$

for $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore, (1) implies that

$$\pi^\vee = \omega_\pi^{-1} \otimes \pi^w = \omega_\pi^{-1} \otimes \pi$$

which is the assertion of (2).

We hence have to prove (1). For $f \in \mathcal{H}$, define

$$f'(g) = f(g^{-1}), \quad f''(g) = f(g^{-t}).$$

By definition, $\chi_{\pi_1}(f) = \chi_\pi(f'')$ and $\chi_{\pi^\vee}(f) = \chi_\pi(f')$. We need to show that $\chi_\pi(f'') = \chi_\pi(f')$. The theorem now follows from Theorem 6.6 below. □

Theorem 6.6 (Gelfand, Kazhdan)**.** *Any conjugation-invariant distribution on $G = \mathrm{GL}_n$ is transpose-invariant.*

Let X be a totally disconnected topological space.

Lemma 6.7. *If $Y \subseteq X$ is compact, then every cover of X can be refined so that it covers Y by disjoint subsets.*

Proof. Let $\{U_i\}_{i=1,\dots,n}$ be a cover of Y and set $U_i' = U_i \setminus \bigcup_{j<i} U_j$. Then U_i' is still open, and hence U_i' does the job. □

Lemma 6.8. *If $Y \subseteq X$ is closed, then*

$$0 \longrightarrow S(X \setminus Y) \longrightarrow S(X) \longrightarrow S(Y) \longrightarrow 0$$

is exact, where $S(X) = C_c^\infty(X)$.

Proof. The only non-trivial part is proving surjectivity of $f \mapsto f|_Y$, but this follows from the previous lemma (by considering the compact set $Y \cap \operatorname{supp}(f)$). □

Definition 6.9. An $S(X)$-module M is *smooth* if for $m \in M$, there is $U \subseteq X$ such that $\mathbb{1}_U \cdot m = m$.

For a point $x \in X$, let $\mathfrak{m}_x = S(X \setminus \{x\})$. This is a maximal ideal by the lemma, because

$$S(X)/\mathfrak{m}_x = S(X)/S(X \setminus \{x\}) \cong S(\{x\}) \cong \mathbb{C}.$$

Lemma 6.10. *Suppose M is smooth. Then $m \in \mathfrak{m}_x M$ if and only if for any compact open U containing x, small enough,* $\mathbb{1}_U m = 0$.

Proof. Note that m is a linear combination of $\mathbb{1}_V m'$ for $V \not\ni x$, $m' \in M$, so any U such that $x \in U$ and $U \cap V = \emptyset$, we have that

$$\mathbb{1}_U \cdot \mathbb{1}_V m' = 0 \cdot m' = 0.$$

Conversely, assume that $\mathbb{1}_U m = 0$ for some U open compact. Let V be an open compact such that $\mathbb{1}_V m = m$ (by smoothness of M).

If $x \notin V$, then we are done. Otherwise, for $x \in U \subseteq V$ small enough, we have

$$\mathbb{1}_U m = 0,$$

so

$$m = \mathbb{1}_V m = \mathbb{1}_{V \setminus U} m + \mathbb{1}_U m = \mathbb{1}_{V \setminus U} m$$

is in $\mathfrak{m}_x M$. □

Lemma 6.11. *Let M be smooth. Then:*

$$\bigcap_{x \in X} \mathfrak{m}_x M = 0.$$

Proof. Let $m \in \bigcap_{x \in X} \mathfrak{m}_x M$. Since M is smooth, there is a V such that $\mathbb{1}_V m = m$. For all $x \in V$, there exists $x \in U_x \subseteq V$ such that for all $x \in U_x' \subseteq U_x$, $\mathbb{1}_{U_x'} m = 0$. In fact, if $U \subseteq U_x$ is any open compact subset, then

$$0 = \mathbb{1}_{U_x} m = \mathbb{1}_U m + \mathbb{1}_{U_x \setminus U} m,$$

so $\mathbb{1}_U m = 0$.

The cover U_x of V can be refined into a disjoint cover U_i by Lemma 6.7. Therefore,

$$m = \mathbb{1}_V m = \sum_i \mathbb{1}_{U_i} m = 0,$$

completing the proof. □

Definition 6.12. Define the *stalk* of M at x to be $M_x = M/\mathfrak{m}_x M$.

Corollary 6.13. *If M is smooth such that $M_x = 0$ for all x, then $M = 0$.*

Lemma 6.14. *If* $0 \to M' \xrightarrow{f} M \to M'' \to 0$ *be an exact sequence of smooth* $S(X)$*-modules. Then* $0 \to M'_x \to M_x \to M''_x \to 0$ *is also exact.*

Proof. Note that $M_x = M \otimes (S(x)/\mathfrak{m}_x)$. Since tensoring is right-exact, we just need to check injectivity. Let $m' \in M'$ be such that $m' \mapsto \mathfrak{m}_x M$. Then $\mathbb{1}_{U_x} f(m') = 0$ in M. □

Lemma 6.15. *Let* $Y \to X$ *be a continuous map of totally disconnected spaces. Then* $S(Y)$ *is a smooth* $S(X)$*-module with the action given by* $\psi \cdot \psi(y) = \pi(f(y)) \cdot \phi(y)$*. Moreover,*

$$S(Y)_x = S(Y_x)$$

for $x \in X$ *where* $Y_x = f^{-1}(x)$*.*

Proof. Smoothness of easy. For the second statement, note that we have an exact sequence

$$0 \to S(Y \setminus Y_x) \to S(Y) \to S(Y_x) \to 0$$

and $S(Y \setminus Y_x) = \mathfrak{m}_x S(Y)$. □

Suppose we have a group G acting on Y. We define the analog of Jacquet modules in this general context.

Proposition 6.16. *Let* $f\colon Y \to X$ *be* G*-invariant, i.e.* $f(gy) = f(y)$ *for all* g*. Let*

$$\begin{aligned} S(Y)(G) &= \text{span of } \phi - g\cdot\phi \quad \text{for all } g \in G \text{ and } \phi \in S(Y) \\ S(Y)_G &= S(Y)/S(Y)(G). \end{aligned}$$

Then $S(Y)(G)$ *is a smooth* $S(X)$*-module and hence* $S(Y)_G$ *is a smooth* $S(X)$*-module and*

$$S(Y)_{G,x} \cong S(Y_x)_G.$$

Fix $\psi\colon F \to \mathbb{C}^\times$ be a non-trivial additive a character. Recall that

$$F \cong N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Definition 6.17. Let (π, V) be an irreducible representation of $G = \mathrm{GL}_2$. A *Whittaker functional* is a non-zero element of $\mathrm{Hom}_N(\pi, \psi)$, i.e. $\ell\colon V \to \mathbb{C}$ satisfying $\ell(\pi(n)v) = \psi(n)\ell(v)$.

Theorem 6.18. *Let* π *be irreducible. Then* $\dim \mathrm{Hom}_N(\pi, \psi) \le 1$*.*

In other words, a Whittaker functional is unique (up to $\mathbb{C}^\times$) if it exists.

Let $H = N \times N$ acting on GL_2 by $(n_1, n_2) \cdot g = n_1 g n_2^{-1}$. Let $\psi\colon H \to \mathbb{C}^\times$ be given by

$$\eta(n_1, n_2) = \psi^{-1}(n_1)\psi(n_2).$$

If G acts on Y, then G acts on $S(Y)$ by $g \cdot f(x) = f(g^{-1}x)$, and G acts on $D(Y) = S(Y)^*$ via $D(f) = D(g^{-1} \cdot f)$.

If $\eta\colon G \to \mathbb{C}^\times$ is a character, $D \in D(Y)$ is called *η-equivariant* if $g \cdot D = \eta(g)D$. When η is the trivial character, D is called *invariant*.

Let θ be an automorphism of GL_2 (as a variety) defined by $\theta(x) = wx^tw$ where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Explicitly,

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Theorem 6.19. *An H-η-equivariant distribution on GL_2 is θ-invariant.*

Meta-theorem. Any G-η-equivariant distribution of Y is also θ-equivariant where G acts on Y, $\theta\colon Y \to Y$ is an automorphism, and $\eta\colon G \to \mathbb{C}^\times$ is a character.

The two cases which we are interested in are:

(1) $G = \mathrm{GL}_2$ acting on $Y = \mathrm{GL}_2$ by conjugation, $\eta = 1$, $\theta(y) = y^t$ (Theorem 6.6),
(2) $G = N \times N$ acting on $Y = \mathrm{GL}_2$ as above, $\eta\colon N \to \mathbb{C}^\times$ is non-trivial, and $\theta(y) = wy^tw$ (Theorem 6.19).

Let G act on Y, $\eta\colon G \to \mathbb{C}^\times$ be a character. Recall that $D \in D(Y)$ is η-equivariant if $g \cdot D = \eta(g)D$.

Let

$$S(Y)(G, \eta) = \text{span of } g \cdot \phi - \eta^{-1}(g)\phi \quad \text{for } \phi \in S(Y),\ g \in G.$$

Then D is η-equivariant if and only if it factors through

$$S(Y)_{G,\eta} = S(Y)/S(Y)(G, \eta).$$

Lemma 6.20. *Let $H \subseteq G$ be a closed subgroup. If H, G are unimodular, there is a left-invariant measure dg on G/H. Then*

$$\dim S(G/H)_{G,\eta} = \begin{cases} 1 & \text{if } \eta|_H = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The functional

$$f \mapsto \int\limits_{G/H} \eta(g) f(g)\, dg$$

is well-defined if and only if $\eta|_H = 1$. If $\eta|_H = 1$, uniqueness of Haar measure implies the result. Otherwise

$$\eta(h)D(f) = D(h \cdot f) = D(f)$$

implies that $D = 0$. □

Let $\theta\colon Y \to Y$ be such that $\theta^2 = 1$. Suppose there exists an automorphism $i \in \mathrm{Aut}(G)$ such that $i^2 = 1$ and $\theta(g \cdot y) = i(g)\theta(y)$. Assume also that

$$\eta(i(g)) = \eta(g).$$

Lemma 6.21. *Let $y \in Y$ be an element such that the orbit $Gy \subseteq Y$ is θ-stable and the stabilizer $G_y = \{g \in G \mid gy = y\}$ is unimodular. Then the distribution $\mathcal{O}_y$ given by*

$$f \mapsto \int\limits_{G/G_y} f(g \cdot y)\, dg$$

is θ-invariant.

Here, $\mathcal{O}_y$ is an *orbital integral*.

Proof. The distribution

$$f \mapsto \theta_y(f^\theta) = \int_{G/G_y} f(\theta(g \cdot y))dg$$

is G-invariant. Indeed,

$$\begin{aligned}\mathcal{O}_y((g \cdot f)^\theta) &= \int_{G/G_y} (g_0 f)(\theta g \cdot y)\, dg \\ &= \int_{G/G_y} f(g_0^{-1}\theta(g \cdot y))\, dg \\ &= \int_{G/G_y} f(i(g_0)gy)\, dg \\ &= \mathcal{O}_Y(f^\theta).\end{aligned}$$

By uniqueness of Haar measure, there is a constant $c_{\mathcal{O}}$ such that

$$\mathcal{O}_y(f^\theta) = c_{\mathcal{O}}\mathcal{O}_y(f).$$

Since θ is an involution, $c_{\mathcal{O}}^2 = 1$. Moreover, $c_{\mathcal{O}} > 0$, so $c_{\mathcal{O}} = 1$. □

We consider our two cases:

(1) $G = \mathrm{GL}_2$ acting on $Y = \mathrm{GL}_2$ by $g \cdot y = gyg^{-1}$:

$$\theta(g \cdot y) = g^{-t}y^t g = i(g)\theta(y)$$

for $i(g) = g^{-t}$,

(2) $G = N \times N$ acting on GL_2:

$$\theta((n_1, n_2) \cdot y) = \theta(n_1 y n_2^{-1}) = \theta(n_2)^{-1}\theta(y)\theta(n_1) = n_2^{-1}\theta(y)n_1$$

since $\theta|_N = 1$, so $i(n_1, n_2) = (n_2^{-1}, n_1^{-1})$.

Lemma 6.22. *Let $y \in Y$ be such that $\theta(y) = y$, G_y is unimodular, and η be such that $\eta(g) = \eta(i(g))$, $\eta|_{G_y} = 1$. Then*

$$f \mapsto \mathcal{O}_{y,\eta}(f) = \int_{G/G_y} \eta(g)f(g \cdot y)\, dg$$

is θ-invariant.

Proof. We have that

$$\begin{aligned}\mathcal{O}_{y,\eta}(f^\theta) &= \int_{G/G_y} \eta(g) f(\theta(g\cdot y))\,dg \\ &= \int_{G/G_y} \eta(g) f(i(g)y)\,dg \\ &= \int_{G/G_y} \eta(i(g)) f(i(g)y)\,dg \\ &= \theta_{y,\eta}(f).\end{aligned}$$

In the last equality, we use Lemma 6 where $i\colon G/G_y \to G/G_y$ is treated as θ. ☐

Therefore, η-equivariant distributions are θ-equivariant **on orbits**. We still need to prove this on the whole variety.

Note that θ stabilizes the space $S(Y)(G,\eta)$. Indeed,

$$\theta(g\cdot\phi) = i(g)\cdot\theta(\phi)$$

and hence

$$\theta(g\cdot\phi - \eta^{-1}(g)\phi) = i(g)\theta(\phi) - \eta(i(g))^{-1}\theta(\phi).$$

To say that all η-equivariant distributions are θ-stable is to say θ acts trivially on $S(Y)_{G,\eta} = S(Y)/S(Y)(G,\eta)$.

The following proposition and lemma allow us to use the statements we proved on orbits to deduce similar statements for the whole space.

Proposition 6.23. *Let $f\colon Y\to X$ be continuous. Suppose f is G-invariant where G is a unimodular group acting on G. Then*

(1) *$S(Y)_G$ is a smooth $S(X)$-module,*
(2) *$S(Y)_{G,\eta}$ is a smooth $S(X)$-module.*

Assume moreover that G acts transitively on all $Y_x = f^{-1}(x)$, $x\in X$ with unimodular stabilizers. Let G_x be the conjugacy class of stabilizers of points in Y_x. Then

(1) *$S(Y_x)_G = S(Y_x)/S(Y_x)(G)$ is 1-dimensional,*
(2) *$S(Y_x)_{G,\eta}$ is 1-dimensional if $\eta|_{G_x} = 1$ (i.e. η is trivial on stabilizers of points in Y_x); otherwise, it is 0 if $\eta|_{G_{y_0}} \neq 1$ for some y_0.*

A fiber Y_x is *good* if $S(Y_x)_{G,\eta}$ is 1-dimensional, i.e. $\eta|_{G_x} = 1$.

Lemma 6.24. *Suppose moreover that*

(1) *f is θ-invariant if $\eta = 1$,*
(2) *for all good Y_x, there is $y\in Y_x$ such that $\theta(y) = y$; in particular f is θ-invariant on Y_x.*

Then θ acts trivially on $S(Y_x)_{G,\eta}$ and hence on $S(Y)_{G,\eta}$. Therefore, every G-η-equivariant distribution is θ-stable.

The tricky part is constructing a map $f\colon Y \to X$ as above. One has conjugation-invariant maps, for example: trace, determinant, characteristic polynomial, but G will not act transitively on the fibers of these maps. However, there is at least a large locus on which this holds.

Theorem 6.25. *Every conjugation-invariant distribution on $Y = \mathrm{GL}_2$ is θ-invariant for $\theta(y) = y^t$.*

Proof. Let Z be the center of GL_2. This is a closed subspace of GL_2, so we have a short exact sequence

$$0 \longrightarrow S(\mathrm{GL}_2 \setminus Z) \longrightarrow S(\mathrm{GL}_2) \longrightarrow S(Z) \longrightarrow 0$$

whose dual is a short exact sequence

$$0 \longrightarrow D(Z) \longrightarrow S(\mathrm{GL}_2) \longrightarrow S(\mathrm{GL}_2 \setminus Z) \longrightarrow 0.$$

Note that Z is θ-invariant and G-invariant. We write $Y = \mathrm{GL}_2 \setminus Z$. For $D \in D(\mathrm{GL}_2)$ which is G-invariant, let $\overline{D} \in D(\mathrm{GL}_2 \setminus Z)$.

We note the following:

(1) any $D \in D(Z)$ is G and θ-invariant,
(2) if $\overline{D}^\theta = \overline{D}$, then $D \setminus D^\theta$ has image zero in $D(\mathrm{GL}_2 \setminus Z)$, so $D - D^\theta$ is supported on Z, which implies that $D - D^\theta = 0$ by (1).

We hence just need to prove that any θ-invariant distribution on $Y = \mathrm{GL}_2 \setminus Z$ is G-invariant.

We want to apply Proposition 6.23 to get this result. Let $Y = \mathrm{GL}_2 \setminus Z$, $X = F^\times \times F$, and $f\colon Y \to X$ be given by

$$f(y) = (\det y, \mathrm{tr}(y)).$$

Any $y \in \mathrm{GL}_2 \setminus Z$ is conjugate to

- $y = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$, $G_y = ZN$,
- $y = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a \neq b$, $G_y = T$,
- *elliptic elements* such that $G_y \cong E^\times$ for E/F quadratic extension.

Hence, clearly, G acts on Y_x transitively for any $x \in X$ and the stabilizer G_x is unimodular. Combining Proposition 6.23 and Lemma 6.24, this completes the proof. □

Theorem 6.26. *Let $G = N \times N$ acting on GL_2 by $(n_1, n_2) \cdot y = n_1 y n_2^{-1}$. Let $\eta(n_1, n_2) = \psi(n_1^{-1} \cdot n_2)$ and $\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$.*

If D is a G-equivariant distribution on GL_2, then it is θ-equivariant.

Proof. Recall the Bruhat decomposition $\mathrm{GL}_2 = B \cup BwN$. We then have a short exact sequence:

$$0 \longrightarrow D(B) \longrightarrow D(GL_2) \longrightarrow D(BwN) \longrightarrow 0$$

As above, it is enough to study distributions on B and BwN separately.

- Let $Y = B$, $X = F^\times \times F^\times$, and
$$f\colon Y \to X$$
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto (a, d).$$
This map is G-equivariant. For $y = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, we have that
$$G_y = \left\{ \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} \;\middle|\; ax_2 = dx_1 \right\}.$$
Note that $\eta|_{G_y} = 1$ if and only if $a = d$. These elements are θ-stable. This is exactly the assumption of Lemma 6.24.
- Let $Y = BwN$, $X = F^\times \times F^\times$, and
$$f\colon Y \to X$$
$$y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, \det(y)/c).$$
In this case, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is G-conjugate to $\begin{pmatrix} 0 & -\frac{\det y}{c} \\ c & 0 \end{pmatrix}$. The assumptions of Lemma 6.24 are again satisfied.

This completes the proof. □

We have hence also completed the proof of Theorem 6.5.

We now apply this to prove the uniqueness of Whittaker functionals (Theorem 6.18). Recall the definition and the statement of the theorem

Definition 6.27. Let (π, V) be an irreducible representation of $G = \mathrm{GL}_2$. A *Whittaker functional* is a non-zero element of $\mathrm{Hom}_N(\pi, \psi)$, i.e. $\ell\colon V \to \mathbb{C}$ satisfying $\ell(\pi(n)v) = \psi(n)\ell(v)$.

Theorem 6.28. *Let π be irreducible. Then* $\dim \mathrm{Hom}_N(\pi, \psi) \leq 1$.

Proof. Consider a representation π' defined by $\pi'(g) = \pi((g^i)^{-1})$ where $i(g) = wg^tw$. Then $\pi' \cong \pi^\vee$. In particular, there is a pairing
$$\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{C}.$$
such that $\langle \pi(g)v, \pi'(g)w \rangle = \langle v, w \rangle$, so
$$\langle \pi(g)v, w \rangle = \langle v, \pi'(g^{-1})w \rangle = \langle v, \pi(g^i)w \rangle.$$

If $\Lambda\colon V \to \mathbb{C}$ is smooth, we get $[\Lambda] \in V$ such that $\Lambda(v) = \langle v, [\Lambda]\rangle$. Consider any functional $\Lambda\colon V \to \mathbb{C}$ and any $\phi \in \mathcal{H} = S(G)$. While $\Lambda \in V^*$ may not be smooth, we may define

$$\Lambda * \phi(v) = \Lambda(\pi(\phi)(v)),$$

so $\Lambda * \phi \in V^\vee$, so it makes sense to consider

$$[\Lambda * \phi] \in V.$$

Assuming Lemma 6.29 below, for two functional Λ_1, Λ_2, we see that $[\Lambda_1 * \phi] = c[\Lambda_2 * \phi]$ by Schur's Lemma for a fixed constant c. Then $\Lambda_1(\pi(\phi)\xi) = c \cdot \Lambda_2(\pi(\phi)\xi)$, so $\Lambda_1 = c\Lambda_2$.

The proof is hence complete, if we prove the lemma below. This will require more work and will, in particular, use Theorem 6.26. □

Lemma 6.29. *If Λ_1, Λ_2 are Whittaker functional, then the map*

$$T\colon V \to V$$
$$[\Lambda_1 * \phi] \mapsto [\Lambda_2 * \phi]$$

is well-defined and G-equivariant.

Proof. We give a sketch of a proof.

(1) The map $\mathcal{H} \to (V, \pi')$ given by $\phi \mapsto [\Lambda_1 * \phi]$ is G-equivariant.
(2) One can check that $\Lambda * (\phi_1 * \phi_2) = (\Lambda * \phi_1) * \phi_2$, where

$$\phi_1 * \phi_2(g) = \int_G \phi_1(h)\phi_2(h^{-1}g)\, dg.$$

(3) If $L\colon V \to \mathbb{C}$ is smooth, then $[L * \phi] = \pi(\phi^i)[L]$. Here, $i(g) = wg^t w$.
(4) If $L\colon V \to \mathbb{C}$ is a Whittaker functional, then

$$[\Lambda * \lambda(u)\phi] = \psi(u)[\Lambda * \phi]$$

where $\lambda(u)\phi(x) = \phi(u^{-1}x)$ is the left regular action. Indeed, note that

$$\langle \xi, [\Lambda * \lambda(u)\phi]\rangle = \int_G (\Lambda(\pi(g)\xi)\phi(u^{-1}g))\, dg.$$

In general, the above assertions should all be verified by evaluating at vectors.

Therefore, we get a distribution $D(\phi) = \Lambda_2([\Lambda_1 * \phi])$ for $\phi \in \mathcal{H}$. Then D is $N \times N$-η-equivariant:

$$D(\lambda(n_1)\rho(n_2)\phi) = \phi(n_1)\phi(n_2)^{-1}D(\phi).$$

Here, $\rho(g)\phi(x) = \phi(xg)$ is the right regular action. By Theorem 6.26, such a distribution is automatically i-invariant, i.e. $D(\phi) = D(\phi^i)$.

Any $v \in V$ can be written as $[\Lambda_1 * \phi]$ for some ϕ by (1). We need to show that it is well-defined, i.e. $\Lambda_1 * \phi = 0$ implies that $\Lambda_2 * \phi = 0$.

(1) We see that $\Lambda_1 * \rho(g)\phi = 0$ since $[\Lambda_1 * \rho(g)\phi] = \pi'(g)[\Lambda_1 * \phi] = 0$ by (1) above.

(2) In particular, since $D(\phi) = 0$, $D(\rho(g)\phi) = 0$ for all g. Since D is i-invariant, we have that

$$\begin{aligned} D(\lambda(g)\phi^i) &= D((\lambda(g)\phi^i)^i) \\ &= D(\rho(g')\phi) \\ &= 0. \end{aligned}$$

(3) For any $\sigma \in \mathcal{H}$ we see that:

$$\begin{aligned} 0 &= \int_G \sigma(g) D(\lambda(g)\phi^i)\, dg \\ &= D(\sigma * \phi^i) \end{aligned}$$

(4) Finally:

$$\begin{aligned} 0 &= D(\sigma * \phi^i) \\ &= \Lambda_2([\Lambda * (\sigma * \phi^i)]) \\ &= \Lambda_2([(\Lambda * \sigma) * \phi^i]) && \text{by (2) above} \\ &= \Lambda_2(\pi(\phi)[\Lambda_1 * \sigma]) && \text{by (3) above} \\ &= (\Lambda_2 * \phi)([\Lambda_1 * \sigma]) \end{aligned}$$

Since any vector is of the form $[\Lambda_1 * \sigma]$ for some σ, this shows that $\Lambda_2 * \phi = 0$. □

6.1. **The Whittaker model.** We know there is a canonical bijection

$$\mathrm{Hom}_N(\pi, \psi) = \mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G(\psi)) = \mathrm{Hom}_G(\pi, C^\infty(N\backslash G, \psi))$$

given by Frobenius reciprocity.

A *model* of a representation is an embedding of it into a *nice space*. Uniqueness of Whittaker functionals (Theorem 6.18) says that there is a unique embedding

$$\pi \hookrightarrow C^\infty(N\backslash G, \psi)$$

if it exists. We call this the *Whittaker model* of π.

We now discuss existence of Whittaker models. First, recall that given a non-trivial character $\psi\colon F \to \mathbb{C}^\times$, any other character is given by

$$\psi_a(x) = \psi(a \cdot x)$$

for $a \in F$. If $\ell \in \mathrm{Hom}_N(V, \psi)$, then ℓ_a defined by

$$\ell_a(v) = \ell\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is in $\mathrm{Hom}_N(V, \psi_a)$ for any $a \in F^\times$.

We write $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in F^\times \right\} \cong F^\times$.

Recall from Tate's thesis that $S(F) = C_c^\infty(F)$ and we have

- $f_1 * f_2(x) = \int_F f_1(y) f_x(x-y)\, dy$,

- if $\hat{f}(x) = \int\limits_F f(y)\psi(xy)dy$, then $\hat{\hat{f}}(x) = f(-x)$ by choosing dy to be ψ-self-dual,
- $\widehat{f_1 * f_2} = -\hat{f}_1 \cdot \hat{f}_2$.

Theorem 6.30. *We have that* $\operatorname{Hom}_N(\pi, \psi) \neq 0$ *if* π *is irreducible and infinite-dimensional.*

We work towards proving this theorem.

We introduce a structure of a smooth $S(F)$-module on V (where $S(F)$ is a ring under multiplication).

- Since V is a smooth N-representation, so $S(F) = S(N)$-acts where we use that $F \cong N$. Explicitly, for $f \in S(F)$,
$$\rho(f)v = \int\limits_N f(n)\pi(n)v\,dn.$$
Note that $\varrho(f_1 * f_2)v = \rho(f_1)[\rho(f_2)v]$.
- The $S(F)$-module structure is defined by
$$\hat{\rho}(f)v = \rho(\hat{f})v.$$
Because $\widehat{f_1 \cdot f_2} = \hat{f}_1 \cdot \hat{f}_2$, we have that
$$\hat{\rho}(f_1 \cdot f_2)v = \hat{\rho}(f_1)(\rho(f_2)v)$$
as required.
- We check that this module structure is smooth. Recall that an $S(X)$-module M is smooth if for all $m \in M$, there is an open compact $U \subseteq X$ such that $\mathbb{1}_U \cdot m = m$. Recall that
$$\widehat{\mathbb{1}_{\mathfrak{p}^j}} = \mathrm{vol}(\mathfrak{p}^j)\mathbb{1}_{\mathfrak{p}^{n(\psi)-j}}$$
where $n(\psi)$ is the level of ψ, i.e. the smallest n such that $\psi|_{\mathfrak{p}^{n(\psi)}} = 1$. For $v \in V$, $\rho(\mathbb{1}_{\mathfrak{p}^j})v = \mathrm{vol}(\mathfrak{p}^j)v$ for $j \gg 0$ by smoothness of V. Then
$$\hat{\rho}(\mathbb{1}_{\mathfrak{p}^{-j}})v = \rho(\mathbb{1}_{\mathfrak{p}^{n(\psi)+j}})v \cdot \mathrm{vol}(\mathfrak{p}^{-j}) = \mathrm{vol}(\mathfrak{p}^{-j})\mathrm{vol}(\mathfrak{p}^{n(\psi)+j})v = v.$$
This shows that V is a smooth $S(F)$-module.

Recall that $V(N)$ is the span of $v - \pi(n)$ for all $n \in N$, and $V(N) = \bigcup\limits_i V(N_{-i})$. The Jacquet module was defined as $V/V(N)$. We define a twisted version of this. First, $V(N, \psi)$ is the span of $\psi(n)v - \pi(n)v$ for all $n \in N$, and letting
$$V(N_{-i}, \psi) = \left\{ v \;\middle|\; \int\limits_{N_{-i}} \overline{\psi(n)}\pi(n)v\,dn = 0 \right\}$$
one can show that
$$V(N, \psi) = \bigcup_i V(N_{-n}).$$
The *twisted Jacquet module* is
$$V_{N,\psi} = V/V(N, \psi).$$

We want to show that $V_{N,\psi} \neq 0$ and we already know that $\dim V_{N,\psi} \leq 1$.

Lemma 6.31. *Let $a \in F$. Then $\mathfrak{m}_a V = V(N, \psi_{-a})$, where $\mathfrak{m}_a = S(F \setminus a)$.*

Proof. We know that $v \in \mathfrak{m}_a V$ if and only if $\hat{\rho}(\mathbb{1}_{a+\mathfrak{p}^j})v = 0$ for $j \gg 0$ by Lemma 6.10. We have that

$$\begin{aligned}\hat{\rho}(\mathbb{1}_{a+\mathfrak{p}^j})v \\ &= \rho(\hat{\mathbb{1}_{a+\mathfrak{p}^j}})v \\ &= \rho(\psi_a \hat{\mathbb{1}_{\mathfrak{p}^j}})v \\ &= \mathrm{vol}(\mathfrak{p}^j)\rho(\psi_a \mathbb{1}_{\mathfrak{p}^{n(\psi)-j}}).\end{aligned}$$

This is 0 if and only if $v \in V(N_{n(\psi)-j}, \psi_{-a})$. □

Corollary 6.32. *Theorem 6.30 holds, i.e. π has a Whittaker model.*

Proof. Otherwise, $V(N, \psi_a) = 0$ for all $a \in F^\times$. Then $V_a = 0$ for all $a \in F^\times$. Therefore, $V = V_0 = V_N$, but $\dim V_N \leq 2$, contradicting infinite-dimensionality of V. □

7. L-functions

The next goal is to talk about L-functions. We first talk about another model of representations of GL_2, called the *Kirillov model*.

7.1. Kirillov model.

Let π be irreducible and infinite-dimensional. Recall that

$$\mathrm{Hom}_N(\pi, \psi) \cong \mathrm{Hom}_g(\pi, \mathrm{Ind}_N^G(\psi)) = C^\infty(N \backslash G, \psi)$$

and we can identify π with its image in $C^\infty(N\backslash G, \psi)$, called the *Whittaker model* and written $\mathcal{W}_\pi \subseteq C^\infty(N\backslash G, \psi)$.

For $W \in \mathcal{W}$, let

$$\phi_W(a) = \phi_W(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} W(e)$$

for $a \in F^\times$.

Lemma 7.1. *The map $\mathcal{W} \to C^\in(F^\times)$ given by $W \mapsto \phi_W$ is injective.*

Proof. If $\phi_W(a) = 0$ for all $a \in F^\times$, we need to show that $W = 0$. We show that if $\phi(a) = 0$, then $W \in V(N, \psi_a)$. For $W' = \psi_a(n)W - \pi(n)W \in \mathcal{W}(N, \psi) \subseteq \mathcal{W}$, we have that $\phi_{W'}(a) = 0$. Therefore, $W \mapsto \phi_W$ factors through $W(N, \psi_a)$. Since $\dim \frac{\mathcal{W}}{\mathcal{W}(N,\psi_a)} = 1$, this shows that $W \in V(N, \psi_a)$.

Therefore, $W \in V(N, \psi_a)$ for all $a \in F^\times$ and the same holds for $\pi(n)W \in V(N, \psi_a)$. Hence $W - \pi(n)W \in V(N, \psi_a)$ for all $a \in F$. Since $W - \pi(n)W$ is zero on all the stalks, it must be 0, so $W = \pi(n)W$ for all $n \in N$.

This is impossible unless $W = 0$, because N and any $\gamma \in \mathrm{SL}_2 \backslash B$ generates SL_2. □

Definition 7.2. The *Kirillov model* $\mathcal{K}_\pi$ of π is the image of $\mathcal{W}_\pi$ in $C^\infty(F^\times)$.

We want to describe the action of G on $\mathcal{K}_\pi$ explicitly. We clearly have that

$$\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\phi(x) = \phi(ax) \tag{2}$$

$$\pi\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\phi(x) = \psi(nx)\phi(x). \tag{3}$$

We want to know how elements which have a non-zero left bottom entry act. First, we note that not all the functions belong to $\mathcal{K}_\pi$. Since $\pi\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\phi = \phi$ for $|n| \ll 1$, so

$$\phi(y) = \psi(y) \cdot \psi(ny).$$

For $|y| \gg 1$, $\psi(ny) \neq 1$, $\phi(y) = 0$.

The key is to understand the behavior of $\phi \in \mathcal{K}_\pi$ around 0.

Proposition 7.3. *Any $\phi \in \mathcal{K}_\pi$ has compact support on F. Moreover, $\mathcal{K}_\pi(N) = C_c^\infty(F^\times)$.*

Proof. The first assertion has been proved in the above discussion, so we only need to prove that $\mathcal{K}_\pi(N) = C_c^\infty(F^\times)$. Recall that $V(N)$ is spanned by $\pi(n)\phi - \phi$. We have that

$$[\pi(n)\phi - \phi]\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \psi(na)\phi(a) - \phi(a) = 0$$

for $|a| \ll 1$. This shows that $\mathcal{K}_\pi(N) \subseteq C_c^\infty(F^\times)$.

Let $B_1 = \begin{pmatrix} a & n \\ 0 & 1 \end{pmatrix}$ act on $C_c^\infty(F^\times)$ by equations (2) and (3). We claim that this action is irreducible.

Let $U \subseteq C_c^\infty(F^\times)$ be an invariant subspace. Let $a \in F^\times$. We want to show that $\mathbb{1}_{a+\mathfrak{p}^j} \subseteq U$ for $j \gg 0$. We can assume that $\phi(a) \neq 0$. Then let

$$\phi_1(x) = \int_N f(n)\pi(n)\phi(x)\,dn = \int_N f(n)\psi(nx)\phi(x)\,dn = \hat{f}(x)\phi(x).$$

If $f \in S(N)$, we can localize the support of ϕ_1 around a by choosing appropriate f. ☐

Recall that structure of $V_N = V/V(N)$. Recall the notation

$$A = \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} \,\middle|\, a \in F^\times \right\} \subseteq T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\}$$

$$\chi = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix},$$

$$\chi^w = \begin{pmatrix} \chi_2 & \\ & \chi_1 \end{pmatrix}$$

$$\delta\begin{pmatrix} a & \\ & b \end{pmatrix} = |a/b|.$$

Then the possibilities are:

- $V_N = 0$ for π supercuspidal,

- $V_N = \delta^{1/2}\chi \oplus \delta^{1/2}\chi^w$ as T-modules if $V = I(\chi)$, $\chi \neq \chi^w$, $\chi \neq \delta^{\pm 1/2}$; in this case, as A-modules, $V_n \cong |\cdot|^{1/2}\chi_1 \oplus |\cdot|^{1/2}\chi_2$,
- $V_N = \delta^{1/2}\chi \begin{pmatrix} 1 & \nu(y) \\ 0 & 1 \end{pmatrix}$ if $\chi = \chi^w$, $V = I(\chi)$; in this case, as A-modules, $V_N \cong |\cdot|^{1/2}\chi_1 \begin{pmatrix} 1 & \nu(\cdot) \\ 0 & 1 \end{pmatrix}$, where $\nu\colon F^\times \to \mathbb{Q}_+$ is the valuation,
- $V_N = \delta\chi$ where V is a quotient of $I(\chi\delta^{-1/2})$; as A-modules, $V_N \cong |\cdot| \cdot \chi_1$.

We know that $\mathcal{K}_\pi(N) = C_c^\infty(F^\times)$ and the Jacquet module is $\mathcal{K}_\pi/\mathcal{K}_\pi(N) = (\mathcal{K}_\pi)_N$. Therefore, the Jacquet module will tell us exactly how the functions before around 0.

Lemma 7.4. *Let $\phi \in \mathcal{K}_\pi$ such that ϕ is mapped to an A-eigenvector in $\mathcal{K}_\pi$ with eigenvalue χ. Then $\phi(x) = C\chi(x)$ for $|x| \ll 1$ for some $C \in \mathbb{C}$.*

Proof. Let $t_0 \in \varpi\mathcal{O}^\times \subseteq A$. Then

$$\pi(t_0)\phi - \chi(t_0)\phi \in V(N)$$

so $\phi(t_0 u) = \chi(t_0)\phi(u)$ for $|u| \ll 1$ (where how small $|u|$ is depends on t_0). By smoothness and compactness of $\varpi\mathcal{O}^\times$, $\phi(tu) = \chi(t)\phi(u)$ for $|u| \ll 1$, $t \in \mathfrak{p} \setminus \mathfrak{p}^2$. Iterating this, we see that $\phi(tu) = \chi(t)\phi(u)$, $t \in \mathfrak{p} \setminus 0$, $|u| \ll 1$.

Fix u. Then tu runs over $\mathfrak{p}^j$ for some j when $t \in \mathfrak{p}$. Then $\phi(tn) = \chi(tn)[\phi(n)\chi(n)^{-1}]$. This completes the proof. □

Theorem 7.5. *The Kirillov model consists of smooth functions $\phi \in C^\infty(F^\times)$ such that $\phi(y) = 0$ for $|y| \gg 0$ and*

1. *$\phi \in C_c^\infty(F^\times)$ when π is supercuspidal,*
2. *$\phi(t) = C_1\chi_1|\cdot|^{1/2} + C_2|\cdot|^{1/2}\chi_2$ for $|t| \ll 1$ when $\pi \cong I(\chi)$, $\chi \neq \chi^w$ and $\chi \neq \delta^{\pm\frac{1}{2}}$*
3. *$\phi(t) = C_1|\cdot|^{1/2}\chi_1 + C_2\nu(t)\chi_1(t)|\cdot|^{1/2}$ for $|t| \ll 1$ when $\pi \cong I(\chi)$, $\chi = \chi^w$,*
4. *$\phi(t) = C_1|\cdot|\chi_1$ for $|t| \ll 1$ when π is Steinberg, i.e. a quotient of $I(\chi\delta^{-1/2})$, $\chi_1 = \chi_2$.*

7.2. L-functions and ϵ-factors.

Definition 7.6. (1) If π is supercuspidal, set $L(s,\pi) = 1$.

(2) If $\pi = I(\chi)$ is irreducible, $L(s,\pi) = (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1} = L(s,\chi_1)L(s,\chi_2)$. Here, $\chi = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$ and $\alpha_i = \begin{cases} \chi_i(\varpi) & \text{if } \chi_i \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases}$

(3) If π is Steinberg, i.e. a quotient of $I(\delta^{-\frac{1}{2}}) \otimes \chi_1$, then

$$L(s,\pi) = (1-\alpha_1 q^{-\frac{1}{2}-s})^{-1} = L(s+\frac{1}{2},\chi_1).$$

(4) If $\pi \cong \chi \circ \det$ is 1-dimensional, then

$$L(s,\pi) = L(s+\frac{1}{2},\chi)L(s-\frac{1}{2},\chi).$$

Proposition 7.7. *Let π be infinite-dimensional and $\mathcal{K}$ be its Kirillov model. Define its zeta-function to be*

$$Z(\phi, s) = \int\limits_{F^\times} \phi(y)|y|^{s-\frac{1}{2}} d^\times y,$$

which converges for $\mathrm{Re}(s) \gg 0$ *and have meromorphic continuation given by*

$$Z(\phi, s) = L(s, \pi) \cdot P(q^{-s})$$

for $P \in \mathbb{C}[x]$. *More generally, for any character* $\xi \colon F^\times \to \mathbb{C}^\times$, *we have an integral*

$$Z(s, \phi, \xi) = \int\limits_{F^\times} \phi(y)\xi(y)|y|^{s-\frac{1}{2}} d^\times y$$

which is equal to

$$Z(s, \phi, \xi) = L(s, \pi \otimes \xi) P_\phi(q^{-s}).$$

Proof. This follows from the description of ϕ from Theorem 7.5. The computations are similar to Tate's thesis, so we omit them here. □

Theorem 7.8. *There is a factor* $\gamma = \gamma(s, \pi, \xi, \psi)$ *such that*

$$\gamma(s, \pi, \xi, \psi) Z(s, \phi, \xi) = Z(1-s, \pi(w)\phi, \omega_\pi^{-1}\xi^{-1})$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. Look at $L_1(\phi) = Z(s, \phi, \xi)$ and $L_2(\phi) \in Z(1-s, \pi(w)\phi, \omega_\pi^{-1}\xi^{-1})$. These both define functionals on $\mathcal{K}$. We have that

$$L_i\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi\right) = \xi(a)^{-1}|a|^{\frac{1}{2}-s} L_i(\phi). \tag{4}$$

Indeed:

$$L_i\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi\right) = \int\limits_{F^\times} \phi(ay)|y|^{s-\frac{1}{2}}\xi(a)\, d^\times y$$

and the result follows by a change of variables. For L_2, (dropping π from the notation)

$$\begin{aligned}
L_2\left(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi\right) &= \int\limits_{F^\times} \left(w\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\phi(y)|y|^{\frac{1}{2}-s}(\xi\omega)^{-1}(y)\, d^\times y \\
&= \int\limits_{F^\times} \begin{pmatrix} 1 & \\ & a \end{pmatrix}(w\phi(y)) \cdot |y|^{\frac{1}{2}-s}(\xi\omega)^{-1}(y)\, d^\times y \\
&= \int\limits_{F^\times} \begin{pmatrix} a & \\ & a \end{pmatrix}\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}(w\phi(y)) \cdot |y|^{\frac{1}{2}-s}(\xi\omega)^{-1}(y)\, d^\times y \\
&= \int\limits_{F^\times} (w\phi)(a^{-1}y)|y|^{\frac{1}{2}-s}(\xi\omega)^{-1}(y)\omega(a)\, d^\times y
\end{aligned}$$

which proves equation (4) via the change of variables $a^{-1}y = y$.

We now claim that for π irreducible, $\chi\colon F^\times \to \mathbb{C}^\times$, the dimension of the space of functionals $L\colon V \to \mathbb{C}$ such that

$$L\left(\pi\begin{pmatrix} a & \\ & 1\end{pmatrix} v\right) = \chi(s)|a|^s L(v)$$

is at most 1, except for possibly 2 values of s mod $\frac{2\pi i}{\log q}$.

Let L_1, L_2 be two such. Restricting L_i to $C_c^\infty(F^\times) \subseteq \mathcal{K}$, we have that

$$L_i(f) = c_i \cdot \int\limits_{F^\times} f(x)\chi^{-1}(x)|x|^{-s} d^\times x \quad \text{for } f \in C_c^\infty(F^\times)$$

by uniqueness of Haar measure. Hence some nontrivial linear combination $d_1L_1 + d_2L_2$ is zero on $C_c^\infty(F^\times) = V(N)$, i.e. $d_1L_1 + d_2L_2$ factors though $V_N = V/V(N)$. Now, we know what V_N is explicitly in each case. Using this to complete the proof is left as an exercise. ☐

Definition 7.9. Define

$$\epsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\pi^\vee, 1-s)}$$

for π infinite-dimensional, and

$$\epsilon(\pi, s, \psi) = \epsilon(\chi, s - \frac{1}{2}, \psi)\epsilon(\chi, s + \frac{1}{2}, \psi)$$

for $\pi = \chi \circ \det$.

We have that $\epsilon(\pi, s, \psi)\epsilon(\pi^\vee, 1-s, \psi) = \omega_\pi(-1)$ and

$$\epsilon(\pi, s, \psi) = q^{n(\pi,\psi)(\frac{1}{2}-s)}\epsilon(\pi, \frac{1}{2}, \psi)$$

where $n(\pi, \psi) \in \mathbb{N}$.

Here is a different approach: define for $\Phi \in S(M_{2\times 2}(F))$ and $f(v) = \langle \lambda, \pi(g)v\rangle$ where $v \in v$, $\lambda \in V^\vee$:

$$\zeta(\Phi, f, s) = \int\limits_G \Phi(g)f(g)|\det g|^{s+\frac{1}{2}}\, dg.$$

Then the idea is to proceed as above.

7.3. **Unitarity.**

Definition 7.10. A smooth (irreducible) representation (π, V) is *unitary* if there exists a pairing $\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{C}$ such that

- $\langle gv, gw\rangle = \langle v, w\rangle$,
- linear in first variable, sesquilinear in the second variable,
- non-degenerate,
- $\langle v, w\rangle = \overline{\langle w, v\rangle}$,
- $\langle v, v\rangle > 0$ for $v \neq 0$.

We make a few immediate observations.

(1) If π is irreducible and unitarym then ω_π is unitary,

(2) If $g \mapsto \langle \lambda, \pi(g)v \rangle$ is square-integrable,

$$\langle u, v \rangle = \int_{Z\backslash G} \langle \lambda, \pi(g)u \rangle \overline{\langle \lambda, \pi(g)v \rangle}\, dg$$

works.

(3) The representation $I(\chi)$ is unitary if χ is unitary with

$$\langle f_1, f_2 \rangle = \int_K f_1(k)\overline{f_2(k)}\, dk = \int_{P\backslash G} f_1(g)\overline{f_2(g)}\, dg.$$

(4) When else is $I(\chi)$ unitary?

The Ramanujam conjecture implies that these are the only automorphic unitary representations.

Lemma 7.11. *If $I(\chi)$ is unitary but χ is not unitary, then $\overline{\chi} = (\chi^{-1})w$, i.e. $\chi_1 = \overline{\chi_2}^{-1}$.*

Proof. We have a map

$$I(\chi) \to I(\overline{\chi})$$
$$f \mapsto \overline{f}$$

which is G-equivariant and sesquilinear. Let $\langle \cdot, \cdot \rangle$ be a Hermitian form on $I(\chi)$. Define for $f_1 \in I(\chi)$, $f_2 \in I(\overline{\chi})$,

$$\langle f_1, f_2 \rangle = \langle f_1, \overline{f_2} \rangle.$$

This is bilinear, non-degenerate, and G-invariant, and hence

$$I(\overline{\chi}) \cong I(\chi)^\vee \cong I(\chi^{-1}).$$

Therefore, $\overline{\chi} = \chi^{-1}$ (unitary) or $\overline{\chi} = (\chi^{-1})^w$. □

Suppose $I(\chi)$ is unitary and χ is not unitary. Write $\chi_1 = \xi \cdot |\cdot|^s$ for s real and ξ unitary. Then $\chi_2 = \xi \cdot |\cdot|^{-s}$. Twisting by ξ, suppose

$$\pi = I(\chi) = I(s) = I\begin{pmatrix} |\cdot|^s & \\ & |\cdot|^{-s} \end{pmatrix}.$$

Recall the intertwining operator

$$\Pi(s)\colon I(s) \to I(-s)$$

(which is G-equivariant) given by

$$\Pi(s)f(g) = \int_N f(wng)\, dn.$$

This converges in this case.

Let $K = \mathrm{GL}_2(\mathcal{O})$. Then $I(s)^K$ is 1-dimensional, spanned by $f_s(bk) = \delta^{\frac{1}{2}+s}(b)$. We have that

$$\Pi(s)f_s = \frac{L(2s)}{L(2s+1)} f_{-s}$$

where $L(s) = (1 - q^{-s})^{-1}$.

Set $\Pi^*(s) = \frac{L(2s+1)}{L(2s)}\Pi(s)$ so that

$$\Pi^* f_s = f_{-s}.$$

For $f_1, f_2 \in I(s)$, define

$$\langle f_1, f_2 \rangle = \int_{P\backslash G} (\Pi^*(s)f_1)(g)\overline{f_2(g)}\, dg.$$

- This is non-degenerate: for $-\frac{1}{2} < s < \frac{1}{2}$, no invariant subspaces, so $\langle f_s, f_s \rangle = 1$.
- Defining $\langle f_1, f_2\rangle_2 = \overline{\langle f_1, f_2\rangle}$. we see that $\langle f_1, f_2 \rangle = c\langle f_2, f_1\rangle$ by a similar argument to the one above. The previous point then implies that $c = 1$.

8. Weil representation

The goal for this class was to cover the proof of the Waldspurger formula. In hindsight, this seems to ambitious. What do we still need to do?

- supercuspidals
- Jacquet–Langlands,
- Siegel–Weil formula,
- Weil representations.

It would take too much time to cover the Siegel–Weil formula. However, talking above the Weil representation will naturally lead us to discussing supercuspidals and the Jacquet–Langlands correspondence. Therefore, it is natural to talk about the Weil representation.

The next two lectures were missed due to travel. I will add the missed lectures to the notes later on.

8.1. Heisenberg group.

Let F be a local field. Let W be a $2n$-dimensional symplectic space with symplectic form

$$\langle \cdot, \cdot \rangle \colon W \times W \to \mathbb{C}.$$

For example, we can take

$$\langle u, w \rangle = u^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} w.$$

Definition 8.1. The *Heisenberg group* is $H(W) = W \times F$ with multiplication given by

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2,\ t_1 + t_2 + \frac{\langle w_1, w_2 \rangle}{2} \right)$$

and inverse given by $(w, t)^{-1} = (-w, -t)$.

Note that the center of $H(W)$ is $\{0\} \times F$, so a character $\psi \colon F \to \mathbb{C}^\times$ gives a character of $Z(H(W))$.

Theorem 8.2 (Stone–von Neumann)**.** *The Heisenberg group $H(W)$ has a unique (unitary) smooth irreducible representation with central character Ψ.*

We do not prove this theorem but instead discuss its applications.

Fix a maximal isotropic subspace $V_1 \subseteq W$ ($\dim V_1 = n$, $\langle v, w\rangle = 0$ for $v, w \in V_1$).

Then $A = V_1 \times F$ is an abelian subgroup of $H(W)$, so we can extent ψ to A by $\psi(v,t) = \psi(t)$. Consider the representation of $H(W)$:

$$\text{c-Ind}_A^{H(W)}(\psi) = \{f\colon H(W) \to \mathbb{C} \mid f(ax) = \psi(a)f(x) \text{ for } x \in A\}.$$

This is a smooth irreducible representation of $H(W)$ with central character Ψ, but we do not check this.

Let $V_2 \subseteq W$ be another maximal isotropic subspace of W, dual to V_1. Then

$$W = V_1 \oplus V_2.$$

Consider

$$\text{c-Ind}_A^{H(W)} \xrightarrow{\cong} S(V_2),$$
$$f \mapsto f|_{V_2}.$$

Therefore, $H(W)$ acts on $S(V_2)$ by $g \cdot (f|_{V_2}) = (g \cdot f)|_{V_2}$.

We want to describe this representation of $H(W)$ more explicitly. Let $f \in \text{c-Ind}_A^{H(W)}$ and write $\phi = f|_{V_2}$. Let $g = (v_1, v_2, t) \in V_1 \oplus V_2 \times F = H(W)$. For $v \in V_2$:

$$\begin{aligned}(g \cdot \phi)(v) &= f((0, v, 0)(v_1, v_2, t)) \\ &= f\left(v_1, v + v_2, t + \frac{1}{2}\langle v, v_1\rangle\right) \\ &= \Psi\left(t + \langle v, v_1\rangle + \frac{\langle v_2, v_1\rangle}{2}\right)\phi(v + v_2)\end{aligned}$$

because

$$\left(v_1, v + v_2, t + \frac{1}{2}\langle v, v_1\rangle\right) = \left(v_1, 0, t + \langle v, v_1\rangle + \frac{\langle v_2, v_1\rangle}{2}\right)(0, v + v_2, 0).$$

This is an explicit formula for the action of $H(W)$ on $S(V_2)$. We could have started with this formula directly, but it would be a little out of the blue.

Recall that $\text{Sp}(W) \subseteq \text{GL}(W)$ is the set of $g \in \text{GL}(W)$ such that $\langle gv, gw\rangle = \langle v, w\rangle$. Then $\text{Sp}(W)$ acts on $H(W)$ by

$$g(w,t) = (gw, t).$$

Indeed:

$$g[(w,t)(w',t')] = (g(w + w'), t + t' + \langle w, w'\rangle/2) = (gw, t) \cdot (gw', t).$$

Let $\omega\colon H(W) \to \text{Aut}(S(V_2))$ be the Stone–von Neumann representation defined above. For each $g \in \text{Sp}(W)$, define $\omega^g(h) = \omega(g \cdot h)$. By Stone–von Neumann theorem 8.2, $\omega \cong \omega^g$, so there is an intertwining operator $\omega_1(g) \in \text{Aut}(S(V_2))$ such that

$$\omega_1(g) \cdot \omega(h) = \omega(gh) \cdot \omega_1(g).$$

One can check that

$$\omega_1(g_1 g_2) = \sigma(g_1 g_2)\omega_1(g_1)\omega_1(g_2).$$

Altogether, we get a (projective) representation

$$\omega_1 \colon \operatorname{Sp}(W) \to \operatorname{PGL}(S(V_2)).$$

Definition 8.3. The projective representation $\operatorname{Sp}(W) \to \operatorname{PGL}(S(V_2))$ is the local *Weil representation.*

Let V be an n-dimensional F-space with a symmetric bilinear form $B \colon V \times V \to \mathbb{C}$. In a basis, it is given by a matrix Φ such that $\Phi^t = \Phi$.

Consider $W = V \times V$ with symplectic structure given by

$$\langle v_1 + v_2, w_1 + w_2 \rangle = B(v_1, w_2) - B(v_1, w_1) = (v_1, v_2) \begin{pmatrix} 0 & \Phi \\ -\Phi & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

The two components $V_1 = V \times \{0\}$ and $V_2 = \{0\} \times V$ of W are complementary isotropic spaces as considered above. The action π of $H(W)$ on $S(V)$ simplifies to

$$\pi(v_1 + v_2, t)\phi(v) = \psi\left(t + B(v, v_1) - \frac{B(v_1, v_2)}{2}\right)\phi(v + v_2).$$

We have an embedding

$$\operatorname{SL}_2(F) \hookrightarrow \operatorname{Sp}(W)$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_n & cI_n \\ bI_n & dI_n \end{pmatrix}.$$

Therefore, we get a representation

$$\omega \colon \operatorname{SL}_2(F) \to \operatorname{PGL}(S(V))$$

such that

$$\omega(g)\phi(h) = \pi(gh)\omega(g).$$

We describe this explicitly on the generators of SL_2.

- For $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \operatorname{SL}_2(F)$, define

$$\omega(g)\phi(v) = \psi((1/2)xB(v, v))\phi(v).$$

  We check the defining property for $h = (v_1, v_2, t)$: $gh = (v_1 + xv_2, v_2, t)$ and

$$\omega(g)\pi(h)\phi(v) = \psi((1/2)xB(v, v))\psi(t - B(v, v_1) - B(v_1, v_2)/2)\phi(v + v_2),$$
$$\pi(gh)\omega(g)\phi(v) = \psi(t - B(v_1, v_1 + xv_2) - (1/2)B(v_1 + xv_2, v_2))\psi((1/2)xB(v + v_2, v_2))\phi(v + v_2).$$

- For $g = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in \operatorname{SL}_2(F)$, define

$$\omega(g) \cdot \phi(v) = |a|^{n/2}\phi(av).$$

  We leave checking the defining property as an exercise.

- For $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \operatorname{SL}_2(F)$, define

$$\omega(g)\phi(v) = \hat{\phi}(v) = \int_V \phi(u)\psi(B(u, v))\, du.$$

We check the defining property:

$$\omega(g)\pi(h)\phi(v) = \int_V \psi(t - B(u, v_1) - (1/2)B(v_1, v_2))\phi(u + v_2)\psi(B(u, v))\, du,$$

$$\pi(gh)\omega(g)\phi(v) = \psi(t - B(v, v_2) + (1/2)B(v_1, v_2)) \cdot \int_V \phi(u)\psi(B(u, v - v_1))\, du.$$

For $\phi_1, \phi_2 \in S(V)$, we define

$$\phi_1 * \phi_2(v) = \int_V \phi_1(u)\phi_2(v - u)\, du.$$

If we define this convolution with respect to the same measure as the Fourier transform, we have that

$$\widehat{\phi_1 * \phi_2} = \hat{\phi}_1 \cdot \hat{\phi}_2.$$

The key was to consider the function $F_B(v) = \psi((1/2)B(v, v))$. For $F = \mathbb{R}$, this is the Gaussian $e^{i\pi x^2}$. Note that this is **not** a Schwartz function.

Lemma 8.4.

(1) $\Phi * F_B \in S(V)$.
(2) *Rhere exists* $\gamma(B) \in S^1$ *such that*

$$\widehat{\Phi * F_B} = \gamma(B)\hat{\Phi} \cdot F_{-B}.$$

(3) $\widehat{\Phi * F_{aB}} = |a|^{-n/2}\gamma(aB)\hat{\Phi} \cdot F_{-a^{-1}B}$.

Proof. For (1), we have

$$\begin{aligned}\Phi * F_B(v) &= \int_V \Phi(u)\psi((1/2)B(v - u, v - u))\, du \\ &= F_B(v) \cdot \int_V \Phi(u)F_B(u)\psi(B(u, -v))\, du \\ &= F_B(v) \cdot \widehat{\Phi \cdot F_B}(-v).\end{aligned}$$

The proof replies on the following identity: for $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have that

$$w_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} w_1.$$

There is a constant $\gamma(B)$ such that

$$\omega(\text{LHS})\Phi = \gamma(B)\omega(\text{RHS})\Phi.$$

For the right hand side, we have that

$$\omega(\text{RHS})\Phi(v) = \psi(-(1/2)B(v, v))\hat{\Phi}(v) = F_{-B}(v)\hat{\Phi}(v).$$

For the left hand side, note that

$$\omega\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} w_1 \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right)\Phi(v)$$

is equal to

$$\psi((1/2)B(v,v))\int_V \psi((1/2)B(u,u))\Phi(-u)\psi(B(v,u))\,du.$$

Bu the proof of (1), we see that this is equal to $\Phi * F_B$. Therefore,

$$\omega(\text{LHS})\Phi(v) = \widehat{\Phi * F_B}.$$

To prove (3), one has to just check the normalization of the two measures. □

Remark 8.5. We will later show that $\gamma(B)$ is an 8th root of unity (for a local field). In general, it may be complicated to work it out explicitly.

Note that any B corresponds to a quadratic form $q(v) = B(v,v)$. To go back, we may define

$$B(v,w) = \frac{1}{2}(q(v+w) - q(v) - q(w)).$$

We let $\gamma(q) = \gamma(B)$.

Define $(V_1, q_1) \oplus (V_2, q_2) = (V_1 \oplus V_2, q_1 + q_2)$.

We then have the Grothendieck group $\hat{W}(F)$.

Quadratic forms are diagonalizable, i.e. we may write $QF(a_1, \dots, a_n) = \langle a_1\rangle \oplus \cdots \oplus \langle a_n\rangle$.

One can check that

- $\gamma(q_1 \oplus q_2) = \gamma(q_1)\gamma(q_2)$,
- $\gamma(q)^{-1} = \gamma(-q)$,
- $\gamma(QF(1,-1)) = 1$

Lemma 8.6. *For a large enough lattice $L \subseteq V$, we have that*

$$\gamma(B) = \int_L F_B(v)dv.$$

Proof. Let L' be the dual lattice, given by

$$L' \in \{v \in V \mid \Psi(B(L,v)) = 1\}.$$

We have that

$$\mathbb{1}_{L'} * F_B(v) = \int_{L'} \Psi((1/2)B(u-v,u-v))\,du = F_B(v)\text{vol}(L')\mathbb{1}_L(v).$$

Take Fourier transform and evaluate at 0. The left hand side gives

$$\gamma(B)\cdot \text{vol}(L').$$

The right hand side gives

$$\operatorname{vol}(L')\int\limits_{L} F_B(v)\,dv,$$

which completes the proof. ∎

Recall that:

$$\left(\frac{a,b}{F}\right)$$

is the quaternion algebra $\langle 1,i,j,k\rangle$ such that $i^2=a$, $j^2=b$, $ij=k$, $ji=-ij$. Then

$$Z = x + yi + zj + wk$$
$$Z^* = x - (yi + zj + wk).$$

The *reduced norm* is

$$F \ni \nu(Z) = Z\cdot Z^* = x^2 - ay^2 - bz^2 + abw^2.$$

The *Hilbert symbol* is defined as

$$(a,b) = \begin{cases} 1 & \text{if } \left(\frac{a,b}{F}\right) \cong M_2(F), \\ 1 & \text{if } \left(\frac{a,b}{F}\right) \not\cong M_2(F). \end{cases}$$

Lemma 8.7. *The Hilbert symbol $(a,b)=1$ if and only if $\nu(Z)=0$ has a non-trivial solution. This is equivalent to a being a norm from $F(\sqrt{b})$.*

Proof. If $b\in (F^\times)^2$, then $\nu(Z)=0$ has an easy solution.

If $b\notin (F^\times)^2$,

$$N_{K/F}(x+\sqrt{b}y) = x^2 - by^2.$$

Then $\nu(Z)$ if and only if $N_{K/F}(x+\sqrt{b}z) = aN_{K/F}(y+\sqrt{b}w)$ if and only if $a\in N_{K/F}(K^\times)$. ∎

Lemma 8.8. *We have that:*

- $(a,b)=(b,a)$,
- $(at^2,b)=(a,b)$,
- $(aa',b)=(a,b)(a',b)$,
- $(a,-a)=1$.

Let $A=\left(\frac{a,b}{F}\right)$ be the 4-dimensional quadratic space. It is isomorphic to

$$QF(1,-a,-b,ab).$$

Theorem 8.9. *We have that $\gamma(A)=(a,b)$.*

Proof. If A is split, then $q\begin{pmatrix} a & b \\ c & d\end{pmatrix} = ab-cd \cong QF(1,-1)\oplus QF(-1,1)$. Hence $\gamma(A) = 1\cdot 1 = 1$.

Assume A is a division algebra. If $F=\mathbb{R}$, $A\cong\left(\frac{-1,-1}{F}\right)\cong QF(1,1,1,1)$, so

$$\gamma(A) = \gamma(QF(1))^4 = -1,$$

because $\psi(x) = e^{-2\pi i x}$ and one can compute explicitly that $\gamma(1) = e^{-\pi i/4}$.

Take a lattice $L \subseteq A$. To show that $\gamma(A) = -1$, it is enough to show that $\gamma(A) = -1$. We have that

$$\int_L F_B = \int_L \psi(\nu(z))\, dz = \int_L \psi(\nu(z))|\nu(z)|^2 d^\times z.$$

Take $L = \nu^{-1}(\varpi^{-N}\mathcal{O})$. Then (up to scaling the Haar measure)

$$\int_L F_B = \int_{\varpi^{-N}\mathcal{O}} \psi(x)|x|^2 d^\times x = \int_{\varpi^{-N}\mathcal{O}} \psi(x)|x| dx = \sum_{j \geq -N} q^{-j} \int_{\mathfrak{p}^i \setminus \mathfrak{p}^{i+1}} \psi(x)\, dx.$$

Now,

$$\int_{\mathfrak{p}_j} \psi(x)\, dx = \begin{cases} \mathrm{vol}(\mathfrak{p}^j) = q^{-j} & \mathfrak{p}^j \subseteq \mathfrak{p}^{n(\psi)}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\int_{\mathfrak{p}^j \setminus \mathfrak{p}^{j+1}} \psi(x)\, dx = \begin{cases} \mathrm{vol}(\mathfrak{p}^j \setminus \mathfrak{p}^{j+1}) = q^j(1-q^{-1}) & \text{for } j \geq n(\psi) \\ -\mathrm{vol}(\mathfrak{p}^{n(\psi)}) = -q^{n(\psi)} & \text{for } j+1 = n(\psi) \\ 0 & \text{for } j < n(\psi) - 1. \end{cases}$$

Hence

$$\int_L F_b = -q^{1-2n(\psi)} + (1-q^{-1})q^{-2n(\psi)} \sum_{j=0}^{\infty} q^{-2j} = -q^{1-2n(\psi)}(1+q^{-1})^{-1},$$

which is < 0. □

Proposition 8.10. *Suppose* $(V, \beta) \cong QF(a_1, \ldots, a_n)$ *and* n *is even. If* $\Delta = (-1)^{n/2} a_1 \ldots a_n$, *then*

$$\gamma(a\beta) = (\Delta, a)\gamma(\beta).$$

Proof. We have that

$$\gamma(\beta) = \prod_i \gamma(QF(a_{2i}, a_{2i+1}))$$

$$\gamma(a\beta) = \prod_i \gamma(QF(aa_{2i}, aa_{2i+1}))$$

$$(\Delta, a) = \prod_i (-a_{2i}a_{2i+1}, a).$$

It is hence enough to show that

$$\gamma(QF(aa_1, aa_2)) = (-a_1a_2, a)\gamma(QF(a_1, a_2)).$$

Recall that

$$\gamma(QF)(1, -a_1, -a_2, a_1a_2) = (a_1, a_2).$$

Hence

$$\gamma(QF(1, a_1a_2)) = (a_1, a_2)\gamma(QF(a_1, a_2)).$$

Using this formula for a_1, a_2 and aa_1, aa_2, we just need to show that:

$$\gamma(QF(1, a^2a_1a_2))(aa_1, aa_2)^{-1} = (-a_1a_2, a)\gamma(QF(1, a_1a_2))(a_1, a_2)^{-1}.$$

Since the quadratic forms are equivalent, we just need to show that

$$(a_1, a_2) = (aa_1, aa_2)(-a_1a_2, a).$$

The right hand side is equal to

$$(a, a)(a, a_1)(a, a_2)(a_1, a_2)(-1, a)(a_1, a)(a_2, a) = (-a, a)(a_1, a_2) = (a_1, a_2),$$

completing the proof. □

Suppose V, B is even-dimensional and set $\chi(a) = (\Delta, a)$. Then the action of

- $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ by $\omega(g)\Phi(v) = \psi((1/2)xB(v, v))\Phi(v)$,
- $g = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ by $\omega(g)\Phi(v) = \chi(a)|a|^{n/2}\Phi(av)$,
- $g = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ by $\omega(g)\Phi(v) = \gamma(B)\widehat{\Phi}(v)$

define a true representation of SL_2 on $S(V)$.

Write

$$t(y) = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We need to check that ω preserves the relations between these elements.

(1) The relations $t(y_1)t(y_2) = y(y_1y_2)$, $n(x_1)n(x_2) = n(x_1 + x_2)$, $t(y)n(z)t(y^{-1}) = n(y^2)$ are all clear.
(2) The relation $wt(y)w^{-1} = t(-y^{-1})$ can be checked as follows:

$$\phi(-y^{-1})\chi(-y^{-1})|y|^{n/2} = \omega(\mathrm{RHS})\phi(v)$$

$$\omega(\mathrm{LHS})\phi(v) = \gamma(B)^2|y|^{n/2}\chi(y)(\widehat{v \mapsto \hat{\Phi}(yv)}) = \gamma(B)^2\chi(y)|y|^{n/2-n}\hat{\hat{\Phi}}(y^{-1}v).$$

We hence just need to show that $\gamma(B)^2 = \chi(-1)$. To show this, we may assume that B is 2-dimensional, and $B \cong QF(a_1, a_2)$. Then

$$\gamma(QF(a_1, a_2))^2 = (-a_1a_2, -1) = \gamma(QF(1, 1, a_1a_2, a_1a_2)),$$

so

$$(a_1, a_2)\gamma(QF(1, a_1a_2))^2 = \gamma(QF(1, a_1a_2))^2.$$

For n even, the formulas above define a representation ω of $\mathrm{SL}_2(F)$ on $S(V)$. On the other hand, there is an action of $O(V)$ on $S(V)$ by
$$k \cdot \phi(v) = \phi(k^{-1} \cdot v).$$
We call this action ω_2.

The $SL_2(F)$ and $O(V)$-actions commute. We consider the three cases above separately:

- for $g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$,
$$\omega(g)\omega_2(k)\phi(v) = \psi((1/2)xB(v,v))\phi(k^{-1}v)$$
and
$$\omega_2(k)\omega(g)\phi(v) = \psi((1/2)xB(k^{-1}v, k^{-1}v))\phi(k^{-1}v)$$
which are equal since k preserves B.
- $g = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ is similar,
- $g = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ is similar.

We hence have a representation ω_∞ of $\mathrm{SL}_2 \times O(V)$. A representation π_1 of SL_2 corresponds to a representation of π_2 if $\omega_\infty \twoheadrightarrow \pi_1 \boxtimes \pi_2$. This is the *theta correspondence*.

8.2. **Quaternion algebras.** Let $V = K$ where either

- K/F is a quadratic extension and x^* is Galois conjugation,
- K is a quaternionic division algebra and x^* is the standard involution
$$(x + iz + jv + kt)^* = x - (iz + jv + kt)$$

equipped with a bilinear form
$$B(x,y) = \mathrm{Tr}(x \cdot y^*)$$
so that the associated quadratic form is
$$\frac{1}{2}B(x,x) = \nu(x).$$

We mostly consider the second case, when K is a quaternion algebra. More details about the first case are in [Bum97]. The different between the two cases is that:

- $\nu(K^\times) \subseteq F^\times$ has index 2 if K is a field,
- $\nu(K^\times) = F^\times$ if K is a quaternion algebra.

Some statements in the first case would have to specifc that an element is the norm etc. In either case, we set $K' = \ker \nu$.

The second case will give us the Jacquet–Langlands functoriality while the second case would be base change functoriality from $\mathrm{GL}_{1,K}$ to $\mathrm{GL}_{2,F}$.

We need to first extend everything from SL_2 to GL_2.

For $a \in F^\times$, define for $g \in \mathrm{SL}_2$

$$\omega_a(g) = \omega\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\right).$$

Let ϱ be the right regular representation of $K^\times$ on $S(K)$.

Lemma 8.11. *Let $b \in K^\times$ be such that $\nu(b) = a$. Then*

$$\omega_a(g)\varrho(b) = \varrho(b)\omega(g).$$

In particular, $\varrho|_{K'}$ commutes with ω.

Let Ω be an irreducible representation of $K^\times$ acting on a finite-dimensional vector space U. Then

$$S(K,\Omega) = \{f \in S(K) \otimes U \mid (\varrho(b)f)(x) = \Omega^{-1}(b) \cdot f(x) \text{ for } x \in K \text{ and } b \in K'\}$$

is $\mathrm{SL}_2(F)$-stable.

We let $|b|_K = \begin{cases} |\nu(b)|_F & K \text{ field,} \\ |\nu(b)|_F^2 & K \text{ quaternion algebra.} \end{cases}$

Lemma 8.12. *The representation ω on $S(K,\Omega)$ extends to*

$$G_+ = \{g \in \mathrm{GL}_2(F) \mid \det(g) \in N(K^\times)\}$$

via

$$\omega\begin{pmatrix} a & \\ & 1 \end{pmatrix}\phi(v) = |b|_K^{1/2}\Omega(b)\phi(vb)$$

where $a = \nu(b)$.

Proof. Since $G = \mathrm{GL}_2 = \mathrm{SL}_2 \rtimes \left\{\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right\}$, proving this amounts to checking that

$$\omega\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}\right) = \omega\begin{pmatrix} a & \\ & 1 \end{pmatrix}\omega(g)\omega\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}.$$

This is simple for the first two cases, so we only check it when $g = w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have that

$$\begin{pmatrix} a & \\ & 1 \end{pmatrix} w \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} w.$$

The left hand side is hence

$$\omega\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}\omega(w)\phi(v) = |a|^2[\omega(w)\phi](av) = -|a|^2\int_K \phi(u)\psi(\mathrm{Tr}(u^*av))\,du.$$

The right hand side is

$$|b|_K^{1/2}\Omega(b)\left[\omega(w)\omega\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}\phi(vb)\right] = -|b|_K^{1/2}\Omega(b)\left[\int_K \omega\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}\phi(u)\psi(\mathrm{Tr}(u^*vb))\,du\right]$$
$$= -|b|_K^{1/2}|b|_K^{-1/2}\Omega(b)\int_K \Omega(b^{-1})\phi(ub^{-1})\psi(\mathrm{Tr}(u^*vb))\,du$$
$$= -|b|_K\int_K \phi(u)\psi(\mathrm{Tr}[b^*u^*vb])\,du,$$

remembering that $\chi(a) = 1$ and $\gamma(B) = -1$. □

Theorem 8.13. *The representation* ω *on* $S(K,\Omega)$ *is smooth, admissible and isomorphic to* $\dim\Omega$ *copies of:*

(1) *supercuspidal representations of* $G = \mathrm{GL}_2$ *if* Ω *is not a character,*
(2) $\mathrm{St}\begin{pmatrix}\chi|\cdot|^{1/2} & \\ & \chi|\cdot|^{-1/2}\end{pmatrix}$ *if* $\Omega(b) = \chi(\nu(b))$.

Proof. We first check smoothness. Consider that $K(\mathfrak{a}) = \{g \in \mathrm{GL}_2(\mathcal{O}) \mid g \equiv 1 \mod \mathfrak{a}\}$. We have a decomposition:

$$K(\mathfrak{a}) = N_-(\mathfrak{a})T(\mathfrak{a})N(\mathfrak{a}).$$

If ϕ is stable by $N(\mathfrak{a})$ and $T(\mathfrak{a})$, then $\omega(w_1)g$ is fixed by $N(\mathfrak{a})$ if and only if ϕ is fixed by $N_-(\mathfrak{a})$. It is hence enough to show that ϕ is stable by $N(\mathfrak{a})$ and $T(\mathfrak{a})$. We omit this here.

For admissibility, it is enough to show that $\dim V^{N(\mathfrak{a})} \cap V^{N_-(\mathfrak{a})} < \infty$. We have that:

- if $\phi \in V^{N(\mathfrak{a})}$, $\mathrm{supp}(\phi) \subseteq \nu^{-1}(\mathfrak{a}')$ for some $\mathfrak{a}'$,
- if $\phi \in V^{N_-(\mathfrak{a})}$, $\mathrm{supp}(\hat{\phi}) \subseteq \nu^{-1}(\mathfrak{a}')$ for some $\mathfrak{a}'$.

These two can hold simultaneously only for a finite number of functions.

Recall that $B_1(F) = \left\{\begin{pmatrix} a & x \\ 0 & 1\end{pmatrix}\right\}$ acts on $S(F^\times, U)$ and we have

$$S(F^\times, U) \to S(K,\Omega)$$
$$\phi \mapsto \Phi_\phi(b) = |b|_K^{-1/2}\Omega(b)^{-1}\phi(\nu(b)) \text{ and } \Phi_\phi(0) = 0,$$

intertwining the $B_1(F)$-action. One should think of $S(F^\times, U)$ as realizing $\dim U$ copies of the Kirillov model.

Suppose Ω is not a character. Then $\Phi(0) = 0$ for $\Phi \in S(K,\Phi)$. Hence the above map is a bijection, with inverse $\Phi \mapsto \phi_\Phi$.

For $\Phi \in S(K,U)$, define

$$Z(s,\Omega,\Phi) = \int_{K^\times} |b|_K^{(s+1/2)/2}\Omega(b)\Phi(b)\,db$$

so that

$$Z(1-s,\Omega^{-1},\hat{\Phi}) = -\gamma(s,\Omega,\psi)Z(s,\Omega,\Phi)$$

for all Φ. Moreover, for $\phi \in S(F^\times, U)$, we defined

$$Z(s,\phi,\xi) = \int\limits_{F^\times} \phi(y)|y|^{s-1/2}\xi(y)\, d^\times y$$

This is related to the above Z-function as

$$Z(s,\phi_\Phi,\xi) = Z(1-s,\Omega\otimes\xi,\Phi).$$

For Φ, we get functions ϕ_Φ, and $\phi' = \phi_{\omega(w)\Phi}$. Then

$$Z(1-s,\varphi',\xi^{-1}\chi_\Omega^{-1}) = -Z(-s,\Omega^{-1}\otimes\chi^{-1},\hat{\Phi}).$$

If ϕ takes values in $U_0 \subseteq U$, then so does ϕ'. Therefore, $S(F^\times, U_0)$ is G-stable. If U_0 is 1-dimensional, G acting on $S(F^\times, U_0)$ is supercuspidal and irreducible. Finally, $\gamma(s,\pi_0,\xi,\psi) = \gamma(s,\Omega\otimes\xi,\psi)$

Moreover, if we have two representations π_1, π_2 of GL_2 with the same central character such that $\gamma(\pi_1\otimes\xi) = \gamma(\pi_2\otimes\xi)$ for all ξ, then $\pi_1 \cong \pi_2$. □

9. Jacquet–Langlands correspondence

9.1. **Distribution characters.** Let $G = D^\times$ where D is a quaternion algebra over a local (p-adic) field F.

If π is a smooth representation of G, we defined the distribution character

$$\chi_\pi(f) = \mathrm{Tr}(\pi(f)),$$

where $f \in S(G) = C_c^\infty(G)$ and $\pi(f)\colon \pi \to \pi$ is the operator given by integration.

Theorem 9.1 (Harish–Chandra)**.** *There is a unique smooth function θ_π on $G_{\mathrm{reg}} \subseteq G(F)$, locally integrable on $G(F)$, such that*

$$\chi_\pi(f) = \int\limits_G \theta_\pi(g)f(g)\, dg.$$

Here, in general:

$$G_{\mathrm{reg}} = \{\Delta \neq 0\}$$

where Δ is some function. In our case:

$$\Delta(g) = 4 - \frac{\mathrm{Tr}(g)^2}{\nu(g)}.$$

Lemma 9.2. *If π is essentially L^2, then there exists $d_\pi \in \mathbb{R}_{>0}$ (the formal dimension) such that*

$$\int\limits_{G/Z} \langle\pi(g)v,\lambda\rangle\langle\pi(g^{-1})v',\lambda'\rangle\, dg = d_\pi^{-1}\langle v,\lambda'\rangle\langle v',\lambda\rangle$$

for all $v, v' \in \pi$, $\lambda, \lambda' \in \pi^\vee$.

- If π is supercuspidal and we choose v, λ so that $\langle v, \lambda\rangle = d_\pi$,
$$\chi_\pi(f) = \int_{G/Z}\int_G f(h)\langle \pi(g^{-1}hg)v, \lambda\rangle\, dh\, dg.$$
and changing the order of integration gives a formula for θ_π.
- If $\pi = I_B^G(\chi)$,
$$\theta_\pi(t) = \begin{cases} 0 & \text{if } t \text{ is elliptic,} \\ |D(t)|^{-1/2}(\chi(t)+\chi^w(t)) & \text{if } t \text{ is conjugate to an element of } T. \end{cases}$$

Suppose D is a quaternion algebra over F. An extension K of F *splits* D if $D\otimes_F K \cong M_2(K)$.

The following lemma is true in general (even if F is not local).

Lemma 9.3. *Let K/F be a quadratic étale extension. A field K splits D if and only if $K \hookrightarrow D$ (an F-algebra homomorphism).*

If F is a local field and K/F is even, then K splits D. Hence any quadratic extension K/F embeds into D.

Let π be a smooth irreducible representation of G. Consider $C(\pi) \subseteq C^\infty(G)$ generated by $\langle \pi(g)v, \lambda\rangle$. Define the zeta function:
$$\zeta(\Phi, f, s) = \int_G \Phi(g) f(g) |\nu(g)|^{s+1/2}\, dg$$
for $\Phi \in S(G)$, $f \in C(\pi)$.

(1) The functions ζ generate an ideal in $\mathbb{C}(q^s)$ of the form $\frac{1}{P(q^{-s})}\mathbb{C}[q^s, q^{-s}]$ and we define
$$L(\pi, s) = \frac{1}{P(q^{-s})}.$$
Moreover, there is a functional equation
$$\zeta(\hat{\Phi}, f^\vee, 1-s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, s)$$
for $f^\vee(g) = f(g^{-1})$. We then define
$$\epsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\pi^\vee, 1-s)}.$$
We have that
$$\epsilon(\pi, s, \psi) = q^{n(1/2-s)}\epsilon(\pi, 1/2, \psi)$$
and there is a formula:
$$\epsilon(\pi, s, \psi)\epsilon(\pi^\vee, 1-s, \psi) = \omega_\pi(-1).$$

There are maps
$$\mathrm{GL}_2, D^\times \to X(F) = F^\times \times F$$
$$g \mapsto (\nu(g), \mathrm{Tr}(g)).$$
We say that $\gamma \in \mathrm{GL}_2$ *corresponds to* $\gamma' \in D^\times$ if they have the same image in $X(F)$.

Theorem 9.4 (Jacquet–Langlands)**.** *There is a unique bijection*

$$\begin{matrix}\textit{irreducible smooth representations of } D^\times \\ \textit{with central character } \chi\end{matrix} \leftrightarrow \begin{matrix}\textit{irreducible smooth essentially square-integrable} \\ \textit{representations of } \mathrm{GL}_2 \\ \textit{with central character } \chi\end{matrix}$$

such that π corresponds to π' if and only if

$$\theta_\pi(\gamma) = -\theta_{\pi'}(\gamma')$$

for all γ corresponding to γ'.

Moreover,

- $L(s, \pi' \otimes \xi) = L(s, \pi \otimes \xi)$,
- $\epsilon(s, \pi' \otimes \xi, \psi) = -\epsilon(s, \pi \otimes \xi, \psi)$.

Suppose $\pi' = \chi$, a 1-dimensional representation of $D^\times$. Then $\theta_\pi(\gamma') = \chi(\nu(\gamma'))$. On the other hand, we have the Steinberg representation $\pi = \mathrm{St}(\chi)$ of GL_2, which sits in the short exact sequence

$$0 \longrightarrow \mathrm{St}(\chi) \longrightarrow I(\chi\delta^{1/2}) \longrightarrow \chi \longrightarrow 0.$$

Therefore,

$$\theta_\pi + \chi = \theta_{I(\chi\delta^{1/2})}.$$

Since $\theta_{I(\chi\delta^{1/2})}(\gamma) = 0$ for elliptic elements γ,

$$\theta_\pi(\gamma) = -\chi(\det(\gamma)).$$

This shows that $\pi' = \chi$ corresponds to $\pi = \mathrm{St}(\chi)$ via the Jacquet–Langlands correspondence.

10. Global situation

Let $\pi = \bigotimes_v \pi_v$ be a representation of $G(\mathbb{A}) = \prod_v{}' G(F_v)$.

Definition 10.1. The *period* of $\phi \in \pi$ is

$$\mathcal{P}_{H,\xi}(\phi) = \int_{H(F)\backslash H(\mathbb{A})} \phi(h)\xi^{-1}(h)\, dh.$$

for $H \subseteq G$ and a character ξ of H.

For example:

- When $G = \mathrm{GL}_2$ or $D^\times$ and H is an elliptic torus, then $P_{H,\xi}$ is a Waldspurger period.
- When $G = \mathrm{GL}_2$ and $H = N$, $\xi = \psi$ is non-trivial, then $P_{H,\xi}$ is the Whittaker coefficient.

Note that

$$P_{H,\xi}(h_0\phi) = \xi(h_0)P_{H,\xi}(\phi),$$

so

$$P_{H,\xi} \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \xi).$$

It is hence natural to study locally:

$$\mathrm{Hom}_{H(F_v)}(\pi_v, \xi_v).$$

Let E/F be a quadratic extension of local field. Let $T \subseteq D^\times, \mathrm{GL}_2$ be a torus such that $T(F) \cong E^\times$. Let $\eta_{E/F}$ be the unique non-trivial character of $F^\times/N_{E/F}(E^\times)$.

We consider $G = D^\times$ or GL_2.

Proposition 10.2 (Waldspurger)**.** *Let π be a smooth irreducible representation of G. Then $\dim_T(\pi, \mathbb{C}) \leq 1$. It is zero unless ω_π is trivial on $N_{E/F}(E^\times)$, i.e. $\omega_\pi = \eta_{E/F}^k$.*

Let $\epsilon(\pi) = \epsilon(1/2, \pi, \psi)$ for fixed ψ. Recall that

$$\epsilon(\pi)\epsilon(\pi^\vee) = \omega_\pi(-1).$$

Recall that $\pi^\vee \cong \pi \otimes \omega_\pi^{-1}$.

Theorem 10.3 (Tunnell–Saito)**.** *Let π be a representation of G which is $D^\times$ or GL_2. Assume that $\omega_\pi = \eta_{E/F}^k$. Then*

$$\mathrm{Hom}_T(\pi, \mathbb{C}) = 1$$

if and only if

$$\epsilon(\pi)\epsilon(\pi \otimes \eta_{E/F}) = \begin{cases} \eta_{E/F}(-1) & \text{if } G = \mathrm{GL}_2, \\ -\eta_{E/F}(-1) & \text{if } G = D^\times. \end{cases}$$

We give some context for this statement.

Base change functoriality: an irreducible representation π of GL_2/F given a representation π_E of GL_2/E. Then:

$$\epsilon(\pi_E) = \epsilon(\pi)\epsilon(\pi \otimes \eta_{E/F})$$
$$\omega_{\pi_E}(x) = \omega_\pi(N_{E/F}(x)).$$

If $\omega_\pi = \eta_{E/F}^k$, $\omega_{\pi_E} = 1$, so $\pi_E \cong \pi_E^\vee$. Hence

$$\epsilon(\pi_E)^2 = 1.$$

Note that, in particular, if π corresponds to π' via the Jacquet–Langlands correspondence, then

$$\dim_T(\pi, \mathbb{C}) + \dim_T(\pi', \mathbb{C}) = 1.$$

Moreover, this statement holds even for representations $\pi = I_B^G \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$. Since these do not transfer to the quaternion algebra, the statement amounts to

$$\dim_T(\pi, \mathbb{C}) = 1.$$

We have that

$$\epsilon(\pi)\epsilon(\pi \otimes \eta_{E/F}) = \epsilon(\chi_1)\epsilon(\chi_2)\epsilon(\chi_1\eta)\epsilon(\chi_2\eta).$$

We know that $\chi_1 \cdot \chi_2 = \eta_{E/F}^k$ and we use the above formula in the two cases.

- if $k=0$, $\chi_1^{-1}=\chi_1^\vee=\chi_2$, so
$$\epsilon(\pi)\epsilon(\pi\otimes\eta_{E/F})=\chi_1(-1)\chi_1(-1)\eta(-1)=\eta(-1).$$
- If $k=1$, $\chi_1^\vee=\chi_2\eta_{E/F}$ and $\chi_2^\vee=\chi_1\eta_{E/F}$, so
$$\epsilon(\pi)\epsilon(\pi\otimes\eta_{E/F})=\chi_1(-1)\chi_1(-1)\eta(-1)=\eta(-1).$$

By the Tunnell–Saito Theorem 10.3, this shows that $\mathrm{Hom}_T(\pi,\mathbb{C})=1$.

11. Global representation theory

We will give a crash course on global representation theory. This is treated more thoroughly in [Bum97].

Let F be a finite extension of $\mathbb{Q}$ and $\mathcal{O}$ be the ring of integers of F. For a place v of F, F_v is the completion of v, and $\mathcal{O}_v$ are the integers of F_v when v is finite.

We did not study the case $F_v=\mathbb{R}$ or $F_v=\mathbb{C}$. This can also be found in [Bum97].

Let $\hat{\mathcal{O}}=\prod_{v<\infty}\mathcal{O}_v$, $\mathbb{A}^f=\hat{\mathcal{O}}\otimes_{\mathcal{O}}F$, $\mathbb{A}^\infty=\prod_{v\mid\infty}F_v$, and
$$\mathbb{A}=\mathbb{A}^\infty\times\mathbb{A}^f$$
be the *adéles*. The diagonal embedding $F\hookrightarrow\mathbb{A}$ has discrete image and $F\backslash\mathbb{A}$ is compact.

The *ideles* are $\mathbb{A}^\times=\mathrm{GL}_1(\mathbb{A})$, the invertible elements of $\mathbb{A}$, with the topology coming from the algebraic group structure, not from the subspace topology. The norm
$$|\cdot|\colon \mathbb{A}^\times\to\mathbb{R}_{>0}$$
$$x\mapsto|x|=\prod_v|x_v|_v.$$

Letting $\mathbb{A}^1=\ker(|\cdot|)$, the product formula implies that $F^\times\subseteq\mathbb{A}^1$. In fact, $F^\times\subseteq\mathbb{A}^1$ is discrete and $F^\times\backslash\mathbb{A}^1$ is compact.

For any reductive group over F (e.g. GL_n or $D^\times$), we may consider $G(\mathbb{A})$ and $G(F)\subseteq G(\mathbb{A})$ is discrete. We will write
$$[G]=G(F)\backslash G(\mathbb{A}).$$

Moreover, $G(\mathbb{A})$ is unimodular. If Z_G is the center, $Z_G(\mathbb{A})G(F)\backslash G(\mathbb{A})$ has finite measure. It is compact if and only if G does not have unipotent subgroups. For example, when $G=D^\times$, this quotient is always compact.

Let $\omega\colon F^\times\backslash\mathbb{A}^\times\to\mathbb{C}^\times$ be an automorphic character. Suppose it is unitary.

We will assume $G=\mathrm{GL}_n$, GL_2, or $D^\times$ for a division algebra D/F. In any of these cases, $Z_G\cong\mathrm{GL}_1$. Then consider
$$L^2([G],\omega)=\left\{f\colon G(F)\backslash G(\mathbb{A})\to\mathbb{C}\;\middle|\; f(gz)=\omega(z)f(g),\quad \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})}|f(g)|^2\,dg<\infty\right\}.$$
This admits an action of $G(\mathbb{A})$ by $gf(x)=f(xg)$.

Let $N \subseteq G$ be unipotent:

- $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for GL_2,
- $\begin{pmatrix} I_k & X \\ 0 & I_{n-k} \end{pmatrix}$ for $X \in M_{k\times(n-k)}$ for GL_n,
- $N = \{e\}$ for $D^\times$.

We let

$$\phi_N(x) = \int_{[N]} \phi(nx)\,dn.$$

Definition 11.1. An element $\phi \in L^2([G,\omega])$ is *cuspidal* if $\phi_N = 0$ for all N. Denote by

$$L^2_0([G],\omega)$$

the closure of the space of cuspidals.

Theorem 11.2 (Piatetsky-Shapiro)**.** *The action $G(\mathbb{A})$ on $L^2_0([G],\omega)$ decomposes with finite multiplicities.*

Definition 11.3. A function $\phi\colon [G] \to \mathbb{C}$ such that $\phi(zg) = \omega(z)\phi(g)$ is an *automorphic form on G* if

(1) ϕ is smooth,
(2) for a maximal compact K of $G(\mathbb{A})$, $K = \prod_v K_v$, ϕ is K-finite,
(3) there is an action of $X \in \mathfrak{g} = \mathrm{Lie}(G(\mathbb{A}^\infty))$ via $X\phi(g) = \frac{d}{dt} f(ge^{tX})_{t=0}$, which extends to an action of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ (the universal enveloping algebra); if $\mathcal{Z}$ is the center of $\mathcal{U}$, we assume that ϕ is $\mathcal{Z}$-finite,
(4) moderate growth.

Let $\mathcal{A}([G],\omega)$ be the set of automorphic forms.

There is almost an action $G(\mathbb{A})$ on $\mathcal{A}([G],\omega)$, but not quite. For v infinite, $G(F_v)$ does not really act, but $(\mathfrak{g}_\infty, K)$ does act.

Definition 11.4. An *automorphic representation* is an irreducible subquotient of $\mathcal{A}([G])$.

Given irreducible representations (π_v, V_v) of $G(F_v)$ and a choice of vector $\xi_v^0 \in V_v^{K_v}$ for almost all v, we can define $V = \bigotimes_v' V_v$ and an action of $G(\mathbb{A})$ on V. We write

$$\pi = \bigotimes_v{}' \pi_v$$

for this representation.

Definition 11.5. A representation V of $G(\mathbb{A})$ is *admissible* if

- every $v \in V$ is K-finite,
- for any irreducible finite-dimensional representation ϱ of $K = \prod_v K_v$, $V(\varrho)$ is finite-dimensional.

Theorem 11.6. *Any irreducible admissible representation decomposes as* $\pi = \bigotimes_v' \pi_v$ *for* π_v *smooth and irreducible.*

Theorem 11.7. *Automorphic representations are admissible.*

Theorem 11.8. *If* π *is an irreducible constituent of* $L^2_0([G], \omega)$*, the action of* $G(\mathbb{A})$ *on smooth* K*-finite vectors in* π *is admissible.*

Theorem 11.9. *Any* $\pi \subseteq L^2_0(G)$ *for* $G = \mathrm{GL}_n, \mathrm{GL}_2, D^\times$ *has multiplicity 1 and if* π, π' *are both constituents of* $L^2_0(G)$ *such that* $\pi_v \cong \pi'_v$ *for almost all* v*, then* $\pi \cong \pi'$.

Let $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}$ be a non-trivial additive character. For $\phi\colon [G] \to \mathbb{C}$, let

$$W_\psi(\phi, g) = W_\psi(g) = \int\limits_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)}.$$

This is a *Whittaker functional*; indeed $W_\psi(-, 1) \in \mathrm{Hom}_{N(\mathbb{A})}(\pi, \psi)$.

Note that:

$$\phi(g) = \phi_N(g) + \sum_{a \in F^\times} W_{\psi_a}(g) = \phi_N(g) + \sum_{\gamma \in \begin{pmatrix} F^\times & 0 \\ 0 & 1 \end{pmatrix}} W_\psi(\gamma g).$$

Let $\pi = \bigotimes_v' \pi_v$ be an automorphic representation of G. Then we define

$$L(s, \pi) = \prod_v L(s, \pi_v)$$
$$\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v, \psi_v).$$

Theorem 11.10. *The* L*-function* $L(s, \pi)$ *converges for* $\mathrm{Re}(s) \gg 0$ *and has a holomorphic continuation to* $\mathbb{C}$ *if* π *is cuspidal. It has a functional equation:*

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi^\vee).$$

Assume π is cuspidal. Then

$$Z(\phi, s) = \int\limits_{F^\times\backslash\mathbb{A}^\times} \phi\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-1/2}\, d^\times a$$
$$= \int\limits_{\mathbb{A}^\times} W_\psi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-1/2}\, d^\times a.$$

Fix an isomorphism $\pi \cong \bigotimes_v' \pi_v$ and assume $\phi = \bigotimes \phi_v$ is a pure tensor. Then

$$W_\psi(g) = \prod_v W_v(g_v)$$

where W_v is the Whittaker functions

$$W_v \in \operatorname{Hom}(\pi_v, C^\infty(N(F_v)\backslash G(F_v), \psi_v)).$$

Finally, $W_v \begin{pmatrix} a_v & \\ & 1 \end{pmatrix}$ are the functions that defined the Kirillov model. Hence

$$\begin{aligned} Z(\phi, s) &= \int_{F^\times\backslash\mathbb{A}^\times} \phi\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-1/2}\, d^\times a \\ &= \int_{\mathbb{A}^\times} W_\psi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^{s-1/2}\, d^\times a \\ &= \prod_v \int_{F_v^\times} W_v \begin{pmatrix} a_v & \\ & 1 \end{pmatrix} |a_v|^{s-1/2}\, d^\times a_v \\ &= \prod_v L(s, \pi_v). \end{aligned}$$

Consider an automorphic representation σ of GL_2, i.e.

$$\sigma \subseteq \mathcal{A}_0(\mathrm{GL}_2) \subseteq L_0^2([G]).$$

We discuss the Whittaker functionals further.

Fix a decomposition $\sigma \cong \bigotimes_v' \sigma_v$. We can talk about *pure tensors* $\phi = \bigotimes_v' \phi_v$ such that $\phi_v = \phi_v^0$ for almost all v, where ϕ_v^0 is a fixed K_v-fixed vector.

The *Whittaker model* of ϕ is

$$W_\psi(\phi, g) = \int_{[N]} \phi(ng)\psi(n)\, dn,$$

a non-zero linear functional. To decompose it: there are $W_v \colon \sigma_v \hookrightarrow C^\infty(N\backslash G, \overline{\psi_v})$ such that $W_v(\phi_v^0)(1) = 1$ for almost all v, and for any $\phi' = \bigotimes_v' \phi_v'$ we have

$$W_\psi(\phi', g) = \prod_v W_v(\phi_v')(g_v) = \prod_v W_v(g_v).$$

How to construct W_v? Take $\phi = \bigotimes_v' \phi_v$ such that $W_\phi(1) = 1$ and define

$$W_v(\phi_v') = \phi^v \otimes \phi_v'$$

where

$$\phi^v = {\bigotimes_{w \neq v}}' \phi_w.$$

Now, suppose $G = B^\times$, where B is a quaternion algebra. Let π be a unitary automorphic representation of G and consider $\pi^\vee$. There is a natural pairing, called the *Petersson inner product*:

$$\langle \phi, \phi' \rangle_{\mathrm{Pet}} = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi(g)\phi'(g)\, dg.$$

Indeed, since π is unitary, $\pi^\vee = \{\overline{\phi} \mid \phi \in \pi\}$. This is the unique pairing $\pi \times \pi^\vee \to \mathbb{C}$, up to scalar.

Fix decompositions $\pi \cong \bigotimes \pi_v$, $\pi^\vee \cong \bigotimes \pi_v^\vee$. We can then fix $\langle -, - \rangle_v \colon \pi_v \times \pi_v^\vee \to \mathbb{C}$ such that

$$\langle \phi, \phi' \rangle = \prod_v \langle \phi_v, \phi'_v \rangle_v$$

for pure tensors ϕ, ϕ'. These define

$$\begin{aligned} F \colon \pi \otimes \pi^\vee &\to \mathbb{C} \\ \phi \otimes \phi' &\mapsto \langle \phi, \phi' \rangle_{\mathrm{Pet}} \\ F_v \colon \pi_v \otimes \pi_v^\vee &\to \mathbb{C} \\ \phi_v \otimes \phi'_v &\mapsto \langle \phi_v, \phi'_v \rangle_v \end{aligned}$$

such that

$$F = \prod_v F_v.$$

Let E/F be a quadratic extension such that $E \hookrightarrow B$ and let $H = E^\times \hookrightarrow G$. Consider a character $\chi \colon [H] \to \mathbb{C}$ such that $\chi \cdot \omega_\pi|_{\mathbb{A}^\times} = 1$. Define

$$\mathcal{P}_\chi(\phi) = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\chi(h)\, dh \qquad \text{for } \phi \in \pi$$

$$\mathcal{P}_{\chi^{-1}}(\phi') = \int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi'(h)\chi^{-1}(h)\, dh \qquad \text{for } \phi' \in \pi^\vee.$$

Theorem 11.11 (Waldspurger). *For $\phi \in \pi$, $\phi' \in \pi^\vee$, we have that*

$$P_\chi(\phi) P_{\chi^{-1}}(\phi') = \frac{1}{8} \frac{\zeta_F(2) L(1/2, \sigma_E \otimes \chi)}{L(1, \eta_{E/F})^2 L(1, \pi, \mathrm{ad})} \prod_v \alpha_v(\phi_v, \phi'_v),$$

where

- *π corresponds to σ of GL_2 via Jacquet–Langlands, and σ_E is the representation σ_E of GL_2/E,*
- *$L(1, \sigma, \mathrm{ad})$ is the* adjoint representation *of σ, i.e. the representation associated to the adjoint representation of ${}^L G$,*
- $\alpha_v(\phi_v, \phi'_v) = \int_{Z(F_v)\backslash H(F_v)} \langle \pi(h_v)\phi_v, \phi'_v \rangle_v \, dh_v \cdot \dfrac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{\zeta_v(2) L_v(1/2, \pi_{E_v} \otimes \eta_v)}$.

Lemma 11.12. *If χ_v is unramified, and we choose dh_v so that $\mathrm{vol}(Z(F_v)\backslash H(F_v)) = 1$, then*

$$\alpha_v(\phi_v^0, (\phi'_v)^0) = 1.$$

Let V be an F-vector space of dimension $2n$. We saw the Weil representation of $\mathrm{SL}_2(F_v) \times O(V(F_v))$ acting on $S(V(F_v))$. Here, $\psi \colon F\backslash\mathbb{A} \to \mathbb{C}^\times$ is the fixed character and ψ_v is its component at v.

One can extend this representation to a representation ω_{ψ_v} of $\mathrm{GL}_2(F_v) \times \mathrm{GO}(F_v)$ acting on $S(V(F_v) \times F_v^\times)$.

We define

$$S(V(\mathbb{A}) \times \mathbb{A}^\times) = \bigotimes_v{}' S(V(F_v) \times F_v^\times),$$

where the restriction is with respect to $\mathbb{1}_{V(\mathcal{O}_v)\times\mathcal{O}_v^\times}$. This is a fixed vector of ω_{ψ_v}, so we obtain a representation

$$\omega_\psi = \bigotimes_v{}' \omega_{\psi_v}$$

of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(V(\mathbb{A}))$ acting on the above space.

For $\phi \in S(V(\mathbb{A}) \times \mathbb{A}^\times)$, the *theta series*

$$\theta(g, h, \phi) = \sum_{(\xi,u)\in V(F)\times F^\times} (\omega(g, h)\Phi)(\xi, u).$$

This is an automorphic form on $[\mathrm{GL}_2] \times [\mathrm{GO}(V)]$ (Weil proves this).

Moreover, we define the *Eisenstein series*

$$E(s, g, \phi) = \sum_{\gamma\in B(F)\backslash \mathrm{GL}_2(F)} \delta_B(\gamma g)^s(\omega(\gamma g)\phi)(0, u)$$

where $\delta_B\begin{pmatrix} a & x \\ & b\end{pmatrix} = |a/b|^{1/2}$.

Theorem 11.13 (Siegel–Weil formula)**.** *We have that*

$$E(s, g, \phi)|_{s=0} = \kappa \int_{[\mathrm{SO}(V)]} \theta(g, h, \phi)\, dh,$$

where κ is 1 or 2.

Consider $V = B$, a 4-dimensional vector space. There is an exceptional isomorphism, given by an exact sequence

$$1 \longrightarrow F^\times \longrightarrow B^\times \times B^\times \rtimes \mathbb{Z}/2 \longrightarrow \mathrm{GO}(V) \longrightarrow 1.$$

The *Shimizu lift*: for $\varphi \in \sigma$, define

$$\theta(\Phi \otimes \varphi)(h) = \int_{[\mathrm{GL}_2]} \varphi(g)\theta(g, h, \Phi)\, dg \in \pi \otimes \pi^\vee$$

where $h \in B^\times \times B^\times(\mathbb{A})$. This is an explicit realization of the Jacquet-Langlands correspondence.

We now consider

$$\mathcal{P}_\chi \otimes \mathcal{P}_{\chi^{-1}}(\theta(\Phi \otimes \varphi)) = \int_{Z(\mathbb{A})\,\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g) \int_{[H]} \chi(t) \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \theta(g, (tt_2, t_2), \Phi)\, dt_2\, dt\, dg.$$

Writing $B = E \oplus E_j$, we we can identify $Z\backslash H$ in $B^\times \times B^\times$ with $Z\backslash H \cong \mathrm{SO}(Ej)$. Hence the torus integration is as if we were integerating over $\mathrm{SO}(Ej)$. Therefore, using Siefel–Weil formula:

$$\mathcal{P}_\chi \otimes \mathcal{P}_{\chi^{-1}}(\theta(\Phi \otimes \varphi)) = \int_{Z(\mathbb{A})\,\mathrm{GL}_2(F)\backslash\,\mathrm{GL}_2(\mathbb{A})} \varphi(g)\ \underbrace{I(s=0,\chi,\Phi,g)}_{\text{mixed Eisenstein-theta}}\ dg$$

$$= \prod_v \int_{Z\backslash H} \chi_v(t_v) \int_{N\backslash\,\mathrm{GL}_2} \delta^s(g) W_v(g)\omega(g)\Phi_v(t^{-1},\nu(t))\, dg\, dt \quad \text{by unfolding methods}$$

$$= (*)L((s+1)/2,\pi_E)\prod_v \int_{Z\backslash H} \chi(t)F_v(\pi(t)\theta_v(\Phi_v \otimes \varphi_v)).$$

Another computation shows that

$$F(\theta(\Phi\otimes\varphi)) = \frac{1}{L(1,\pi,\mathrm{ad})}\prod_v \int_{N\backslash\,\mathrm{GL}_2} W_v(g)\Phi_v()\, db$$

This is where the adjoint L-value shows up.

References

[Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997, doi:10.1017/CBO9780511609572, https://doi.org/10.1017/CBO9780511609572. MR 1431508

[CF10] J. W. S. Cassels and A. Fröhlich (eds.), *Algebraic number theory*, London Mathematical Society, London, 2010, Papers from the conference held at the University of Sussex, Brighton, September 1–17, 1965, Including a list of errata. MR 3618860