MATH 679: AUTOMORPHIC FORMS

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These are notes from Math 679 taught by Professor Tasho Kaletha in Winter 2019, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from September 26, 2019. Check for the latest version of these notes at

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1. Introduction: Review of modular forms

1.1. **Automorphy and functional equations of L-functions.** Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half plane. It has an action of the Lie group $\mathrm{SL}_2(\mathbb{R})$ of 2×2 matrices with $\mathbb{R}$-coefficients and determinant 1 by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}.$$

Definition 1.1. A *holomorphic modular form* of weifght $k \in \mathbb{Z}$ is a holomorphic function

$$f\colon \mathbb{H} \to \mathbb{C}$$

Date: September 26, 2019.

satisfying

- $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,
- f is bounded on domains of the form $\{z \in \mathbb{H} \mid \mathrm{Im}(z) > C > 0\}$.

The second condition is equivalent to f being holomorphic at $i\infty$.

Remark 1.2. The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

In particular, taking $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we get

$$f(z+1) = f(z),$$

so f has a Fourier expansion

$$\sum_{n \in \mathbb{Z}} a_n q^n \quad \text{for } q = e^{2\pi i z}.$$

Then the second condition is also equivalent to saying

$$a_n = 0 \text{ for } n < 0.$$

By Remark 1.2, the first condition is equivalent to

$$f(z+1) = f(z) \text{ and } f\left(-\frac{1}{z}\right) = (-z)^k f(z).$$

How can we generalize this?

(1) We can replace $\mathrm{SL}_2(\mathbb{Z})$ by a subgroup (usually of finite index).

Definition 1.3. For $N \in \mathbb{N}$, let

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})).$$

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is *congruence* if it contains $\Gamma(N)$ for some N.

Example 1.4. The subgroup $\Gamma(2)$ is generated by

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\Delta \subseteq \Gamma(2)$ be the subgroup generated by these. Notice that Δ is normal in $\mathrm{SL}_2(\mathbb{Z})$. Consider $\Gamma(2)/\Delta$. An element of $\Gamma(2)/\Delta$ can be represented as an alternating word in $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Compute to check that the smallest length of a word contained in $\Gamma(2)$ has length 6 and it is equal to

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \Delta.$$

Example 1.5. The subgroup Γ generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ called the *theta group.*

(2) We can allow half-integral weight. On the slices complex place $\mathbb{C} \setminus \mathbb{R}_{<0}$. We have an analytic map $z \mapsto \sqrt{z}$ determined by $\operatorname{Re}(\sqrt{z}) > 0$.

This can be extended non-continuously by $\sqrt{-r} = i\sqrt{r}$ for $r \in \mathbb{R}_{>0}$.

We can try to define, for $k \in \mathbb{Z}$, a modular form of weight $\frac{k}{2}$ by requiring that

$$f(\gamma z) = (cz + d)^{k/2} f(z).$$

We will write $\nu(\gamma, z) = (cz + d)$.

Remark 1.6. We have that $\nu(\gamma\delta, z) = \nu(\gamma, \delta z)\nu(\delta, z)$. This is no longer true for $\nu(\gamma, z)^{\frac{1}{2}}$. It is only true up to a $\pm$ sign.

Therefore, we introduce a *multiplicator system*:

$$\mu\colon \Gamma \to \mu_\infty(\mathbb{C})$$

such that

$$\mu(\gamma) \cdot \nu(\gamma, z)^{\frac{k}{2}}$$

has the desired behavior.

Example 1.7. The theta multiplicator system is defined by

$$\mu \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = 1, \ \mu \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = e^{\pi i/4}.$$

Example 1.8. The *Jacobi theta function*

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}$$

is a modular forms of weight $\frac{1}{2}$ for Γ the theta group.

This converges locally uniformly. Take a domain $\{\operatorname{Im}(z) > C > 0\}$. If $z = x + iy$ is in this domain, then

$$\left| e^{\pi i n^2 z} \right| = \left| e^{\pi i n^2 x} \cdot e^{-\pi n^2 y} \right| < e^{-\pi n^2 C}$$

is bounded uniformly.

To check automorphy, note that $\gamma = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is obvious. The case $\gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is interesting.

Fourier analysis. Let $f\colon \mathbb{R} \to \mathbb{C}$ and $\hat{f} = \int\limits_{\mathbb{R}} f(x) e^{2\pi i x y} dx$. We have the *inversion formula*

$$\hat{\hat{f}} = f(-x)$$

and the *Poisson summation formula*

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

The *Gaussian* $f(x) = e^{-\pi x^2}$ is its own Fourier transform:

$$\begin{aligned}\hat{f}(y) &= \int\limits_{\mathbb{R}} e^{-\pi x^2 + 2\pi i x y} dx \\ &= \int\limits_{\mathbb{R}} e^{-\pi(x^2 - 2ixy + (iy)^2)} e^{-\pi y^2} dx \\ &= e^{-\pi y^2} \int\limits_{\mathbb{R}} e^{-\pi(x-iy)^2} dx.\end{aligned}$$

We claim that the last integral is equal to 1.

(a) The contour integral of $e^{-\pi z^2}$ over the region R which is a rectangle with corner $-T, T, iy - T, iy + T$ is 0 because the function is holomorphic. Therefore, letting $T \to \infty$,

$$\int_{\mathbb{R}} e^{-\pi(x-iy)^2} dx = \int_{\mathbb{R}} e^{-\pi x^2} dx.$$

(b) Rescale to see that

$$\int_{\mathbb{R}} e^{-\pi x^2} dx = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2} dx.$$

(c) Take square to see that

$$\begin{aligned}\left(\int e^{-x^2} dx\right)^2 &= \int e^{-x^2 - y^2} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\varphi \qquad \text{changing to polar coordinates} \\ &= 2\pi \left[\frac{e^{-r^2}}{-2}\right]_0^{\infty} \\ &= \pi.\end{aligned}$$

Let $f_t(x) = f(xt^{1/2})$ for $t \in \mathbb{R}_{>0}$. Note that

$$\hat{f}_t(y) = \frac{1}{t^{1/2}} \hat{f}(y/t) = \frac{1}{t^{1/2}} f(y/t)$$

We want to show that

$$\theta\left(-\frac{1}{z}\right) = e^{\pi i/y} \cdot (-z)^{1/2} \theta(z) = \left(\frac{z}{i}\right)^{1/2} \theta(z).$$

Since $\frac{z}{i}$ is in the right half plane where the square root is analytic, both sides are analytic functions, so it is enough to take $z = it$, $t \in \mathbb{R}_{>0}$. Then

$$\begin{aligned}
\theta(it) &= \sum_{n\in\mathbb{Z}} f_t(n) \\
&= \sum_{n\in\mathbb{Z}} \hat{f}_t(n) && \text{Poisson summation} \\
&= \frac{1}{t^{1/2}} \sum_{n\in\mathbb{Z}} f_t(n/t) \\
&= \frac{1}{t^{1/2}} \theta\left(\frac{i}{t}\right).
\end{aligned}$$

Recall that the *Riemann zeta* function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

We want to show

(1) ζ has an analytic continuation to $\mathbb{C}$ with a pole at $s = 1$,
(2) ζ obeys a functional equation.

Note that ζ is incomplete. We have the Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

In hindsight, we are missing an Euler factor for $p = \infty$. Define

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}$$

is the Γ-function, defined for $\operatorname{Re}(s) > 0$.

Remark 1.9. This is a "*Fourier transform*" on $(\mathbb{R}^\times, \cdot)$ in place of $(\mathbb{R}, +)$. Indeed, normally we integrate the function against the character $x \mapsto e^{2\pi i x y}$ of $(\mathbb{R}, +)$ with respect to the addition-invariant measure dt. Here, we integrate against the character $t \mapsto t^s$ of $(\mathbb{R}^\times, \cdot)$ with respect to the multiplication-invariant measure $\frac{dt}{t}$. This is called a *Mellin transform.*

We have

$$\begin{aligned}
\theta(z) &= \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \\
&= 1 + 2\underbrace{\sum_{n=1}^{\infty} e^{\pi i n^2 z}}_{w(z)}.
\end{aligned}$$

Let us apply the Mellin transform to $t \mapsto w(it)$:

$$\begin{aligned}\int_0^\infty w(it)t^s\frac{dt}{t} &= \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 t}t^s\frac{dt}{t}\\ &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t}t^s\frac{dt}{t}\\ &= \sum_{n=1}^\infty \int_0^\infty e^{-t}\left(\frac{t}{\pi n^2}\right)^s\frac{dt}{t}\\ &= \sum_{n=1}^\infty \frac{1}{(\pi n^2)^s}\int_0^\infty e^{-t}t^s\frac{dt}{t}\\ &= \pi^{-s}\zeta(2s)\Gamma(s)\\ &= \Lambda(2s).\end{aligned}$$

From the automorphy of θ, we get

$$\begin{aligned}w\left(-\frac{1}{z}\right) &= \frac{1}{2}\left(\theta\left(-\frac{1}{z}\right)-1\right)\\ &= \frac{1}{2}\left(\left(\frac{z}{i}\right)^{1/2}\theta(z)-1\right)\\ &= (-iz)^{1/2}w(z)+\frac{1}{2}(-iz)^{1/2}-\frac{1}{2}.\end{aligned}$$

Therefore,

$$w\left(\frac{i}{t}\right) = t^{1/2}w(it)+\frac{1}{z}t^{1/2}-\frac{1}{2}.$$

Using this, we get that

$$\begin{aligned}\Lambda(2s) &= \int_0^\infty w(it)t^s\frac{dt}{t}\\ &= \int_1^\infty w(it)t^s\frac{dt}{t}+\int_0^1 w(it)t^s\frac{dt}{t}\\ &= \int_1^\infty w(it)t^s\frac{dt}{t}+\int_1^\infty w\left(\frac{i}{t}\right)t^{-s}\frac{dt}{t}\\ &= \int_1^\infty w(it)(t^s+t^{-s+\frac{1}{2}})\frac{dt}{t}+\frac{1}{2}\int_1^\infty (t^{-s+1/2}-t^{-s})\frac{dt}{t}.\end{aligned}$$

We consider these two summands separately. The second summand is

$$\frac{1}{2}\int_1^\infty t^{-1/2-s}-t^{-1-s}dt = \frac{1}{2}\left[\frac{t^{1/2-s}}{\frac{1}{2}-s}\right]_1^\infty - \frac{1}{2}\left[\frac{t^{-s}}{-s}\right]_1^\infty = \frac{1}{1-2s}+\frac{1}{2s} = \frac{1}{2s(1-2s)}.$$

Let us now consider the first summand. We claim that $|w(it)|$ decays exponentially as $t \to \infty$. Recall that writing

$$\theta(z) = \sum_{n=0}^\infty a_n q^n,$$

we get that

$$w(z) = \frac{1}{2}\sum_{n=1}^{\infty} a_n q^n = q\frac{1}{2}\underbrace{\sum_{n=0}^{\infty} a_{n+1} q^n}_{\text{bounded around } 0}.$$

Therefore, the first integral converges for any s, and gives an analytic function $f(s)$, invariant under $s \mapsto \frac{1}{2} - s$

Altogether, we conclude that

(1) $\Lambda(2s) = \frac{1}{1-2s} + \frac{1}{2s} + \underbrace{f(s)}_{\text{analytic}}$
(2) $\Lambda(1-s) = \Lambda(s)$.

Morals.

(1) The automorphy of θ implies the analytic continuation and functional equation of ζ. This generalizes.
(2) Fourier transforms and Poisson summation are important for automorphy, hence for L-functions.

The theory of L-functions for automorphic forms generalizes to connected reductive groups. In this course, we will not emphasize the group theory, but the methods in appropriate examples:

(1) $G = \mathrm{GL}_1$ (Tate's thesis); this will be a warm up,
(2) $G = \mathrm{GL}_2$ (Jacquet–Langlands theory); this is the true goal of the course — this is the simplest non-abelian reductive group. We will study representations of the Lie group $\mathrm{GL}_2(\mathbb{R})$, the p-adic Lie group $\mathrm{GL}_2(\mathbb{Q}_p)$, and finally we will study representations of the group $\mathrm{GL}_2(\mathbb{A})$.

The argument we gave proving the analytic continuation and functional equation for ζ using the automorphy of θ was generalized by Hecke.

Theorem 1.10 (Hecke)**.** *Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers such that $a_n = O(n^c)$ for some $c > 0$. Fix a positive even integer k. Consider*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z},$$
$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$
$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s).$$

Then the following are equivalent:

(1) $\Phi(s) + \frac{a_0}{s} + \frac{(-1)^{k/2}a_0}{k-s}$ *is entire, bounded in vertical strips, and satisfies the functional equation*
$$\Phi(s) = (-1)^{k/2}\Phi(k-s),$$
(2) f *is a modular form of weight k and level 1 (i.e. for the full modular group).*

The proof of this theorem is entirely analogous to the argument above, so we do not reproduce it here.

Remark 1.11. The implication "(2) implies (1)" generalizes fairly directly to higher level, but one has to pay attention to the normalizing factors.

The implication "(1) implies (2)" also generalizes, but since $\Gamma(N)$, or $\Gamma_0(N)$, has many more generators, it is not enough to consider $L(f,s)$ (i.e. $\phi(s)$). We need all χ-twists for Dirichlet characters χ. This leads to *Weil's converse theorem.*

1.2. **Translation to automorphic forms.**

Today, we will set up the language for the rest of the course. In other words, we will translate the notion of modular forms to a form which we will deal with for the rest of the class.

Fact 1.12. *The action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$ *is transitive and*

$$\mathrm{Stab}(i, \mathrm{SL}_2(\mathbb{R})) = \mathrm{SO}(2).$$

Proof. If $z = x+iy$, then $\begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} z = iy$ and $\begin{bmatrix} y^{-1/2} & 0 \\ 0 & -y^{1/2} \end{bmatrix} iy = i$. This proves transitivity and calculating the stabilizer is immediate. □

Corollary 1.13. *We have the (canonical) isomorphism*

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) &\to \mathbb{H}, \\ g &\mapsto g \cdot i. \end{aligned}$$

In particular, from a modular form f, we obtain a function

$$\begin{aligned} \mathrm{SL}_2(\mathbb{R}) &\to \mathbb{C} \\ g &\mapsto f(gi) \end{aligned}$$

which is invariant on the right by $\mathrm{SO}(2)$.

Recall that we defined $\nu(g,z) = (cz+d)$ if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define

$$\phi_f(g) = f(g\cdot i)\nu(g,i)^{-k},$$

where k is the weight of f.

Now, $\phi_f\colon \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ has the following properties:

(1) it is invariant under Γ on the left,
(2) under the action of $\mathrm{SO}(2)$ on the right, we have the following: writing $e(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$,
$$\phi(ge(\varphi)) = e^{ik\varphi}\phi(g),$$
i.e. ϕ transforms by a character under the action of $\mathrm{SO}(2)$.
(3) growth condition,
(4) ϕ satisfies a differential equation.

We will only discuss conditions (3) and (4) later, when we have the structure theory of $\mathrm{SL}_2(\mathbb{R})$. Since we will first discuss the theory for GL(1), we will not need this until we reach the theory for GL(2).

Remark 1.14. This is essentially the definition of and *automorphic form* on $\mathrm{SL}_2(\mathbb{R})$. In fact, we will slightly weaken condition (2) to allow for other objects such a Maass forms (which are not holomorphic).

We now translate things to the adelic setting. We briefly recall what adeles are.

Consider the field $\mathbb{Q}$ of rational numbers.

- We have the archimedean absolute value
$$|\cdot|_\infty \colon \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$
and the completion $\mathbb{Q}$ is $\mathbb{R}$, a locally compact, connected, topological field.
- For each prime number p, we have the p-adic absolute value
$$|\cdot|_p \colon \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$
given by $|x|_p = p^{-r}$ where $x = p^r \frac{a}{b}$ for $a, b \in \mathbb{Z}$ such that p does not divide a or b. We write $r = \mathrm{ord}_p(x)$. Completing $\mathbb{Q}$ with respect to $|\cdot|_p$ gives $\mathbb{Q}_p$, a locally compact, totally d isconnected, topological field. It has a compact open subring $\mathbb{Z}_p \subseteq \mathbb{Q}_p$.

The adeles are defined as
$$\mathbb{A} = \mathbb{R} \times \mathbb{A}^{\mathrm{fin}}$$
where the *finite adeles* are
$$\mathbb{A}^{\mathrm{fin}} = \prod_p{}' \mathbb{Q}_p = \left\{ (x_p) \in \prod_p \mathbb{Q}_p \ \mid \mid x_p \in \mathbb{Z}_p \text{ for almost all } p \right\}.$$
In other words:
$$\mathbb{A}^{\mathrm{fin}} = \varinjlim_S \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$$
where S runs over finite sets of places.

Then $\mathbb{A}$ is a locally compact topological ring. The topology is the $\varinjlim$ topology of the product topology. Explicitly, it is generated by sets of the form
$$\prod_{p \geq \infty} U_p$$
where $U_p \subseteq \mathbb{Q}_p$ is open and $U_p = \mathbb{Z}_p$ for almost all p. Here, we are using the notation $\mathbb{Q}_\infty = \mathbb{R}$.

Remark 1.15. The diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$ is discrete.

Fact 1.16. *We have that*
$$\mathrm{SL}_2(\mathbb{A}) = \prod_{p \leq \infty}{}' \mathrm{SL}_2(\mathbb{Q}_p)$$
where the prime means restricted direct product *again, and the restriction is with respect to* $\mathrm{SL}_2(\mathbb{Z}_p)$

Proposition 1.17.

(1) *The embedding* $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{SL}_2(\mathbb{A})$ *has discrete image.*
(2) *We have that*

$$\mathrm{SL}_2(\mathbb{Q}) \Big\backslash \mathrm{SL}_2(\mathbb{A}) \Big/ \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}),$$

i.e. we can thing of automorphic forms as functions on the left hand side.
(3) *Let* $N = \prod_p p^{r_p}$. *Define*

$$K_{p,r} = \ker(\mathrm{SL}_2(\mathbb{Z}_p) \to \mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})).$$

Then

$$\mathrm{SL}_2(\mathbb{Q}) \Big\backslash \mathrm{SL}_2(\mathbb{A}) \Big/ \prod_{p<\infty} K_{p,r_p} \cong \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{R}),$$

Remark 1.18. For a general simply connected algebraic group, the analogous theorem is called the *strong approximation theorem*. This is harder to prove.

Remark 1.19. We will write G for SL_2.

Proof of Proposition 1.17. For (1), note that $\mathbb{Q} \hookrightarrow \mathbb{A}$ is discrete so $\mathbb{Q}^n \hookrightarrow \mathbb{A}^n$ is discrete, and hence

$$X(\mathbb{Q}) \hookrightarrow X(\mathbb{A})$$

is discrete for any affine variety X.

For (2), fix a prime p. Given $g \in \mathrm{SL}_2(\mathbb{Q}_p)$, there exists $\gamma_p \in \mathrm{SL}_2(\mathbb{Q})$ such that

(1) $\gamma \cdot g \in \mathrm{SL}_2(\mathbb{Z}_p)$,
(2) $\gamma \in \mathrm{SL}_2(\mathbb{Z}_\ell)$ for all $\ell \neq p$.

This is left as an exercise.

Given $g = (g_p) \in G(\mathbb{A})$, there is a finite set S of primes such that $g_p \in G(\mathbb{Z}_p)$ for $p \notin S$. Choose γ_p as above for each $p \in S$. Then

$$\prod_{p \in S} \gamma_p g \in G(\mathbb{R}) \prod_{p<\infty} G(\mathbb{Z}_p).$$

Thus $G(\mathbb{R})$ meets each coset in

$$\mathrm{SL}_2(\mathbb{Q}) \Big\backslash \mathrm{SL}_2(\mathbb{A}) \Big/ \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p) \cong \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}).$$

Two elements of $G(\mathbb{R})$ represent the same coset if and only if their different lies in

$$G(\mathbb{Q}) \cup \prod_{p<\infty} G(\mathbb{Z}_p) = G(\mathbb{Z}),$$

since $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$.

Part (3) is similar and hence left as an exercise. □

Remark 1.20. The group $K_{p,r}$ is compact and open in $G(\mathbb{Q}_p)$. For $r = 0$, it is maximal such.

Hence $K = K_\infty \times \prod_{p<\infty} K_p \subseteq G(\mathbb{A})$ is maximal compact, where $K_\infty = \mathrm{SO}(2)$ and $K_p = K_{p,0}$.

Moreover, $K_N = K_\infty \times \prod_{p<\infty} K_{p,r_p} \subseteq G(\mathbb{A})$ is compact and has finite index in K.

Corollary 1.21. *Extending ϕ to a function $\phi\colon \mathrm{SL}_2(\mathbb{A}) \to \mathbb{C}$ via Proposition 1.17 gives a function with the following properties:*

1. *left-invariant under* $\mathrm{SL}_2(\mathbb{Q})$,
2. *K-finite on the right (i.e. the span of the orbit of ϕ under K is finite-dimensional),*
3. *growth condition,*
4. *differential equation.*

Proof. To prove (2), note that K_∞ acts on ϕ by a character. If f is full level, then the span of Kf is 1-dimensional. If f has higher level, the span will have higher dimension because ϕ is only invariant under a finite index subgroup of $\prod_{p<\infty} K_p$. ☐

Definition 1.22. An *automorphic form* on $\mathrm{SL}_2(\mathbb{A})$ is the automorphic form is a function $\mathrm{SL}_(\mathbb{A}) \to \mathbb{C}$ satisfying conditions (1)–(4) of Corollary 1.21.

Remark 1.23. We may switch from SL_2 to GL_2 using the identification

$$\mathrm{SL}_2(\mathbb{Q}) \Big\backslash \mathrm{SL}_2(\mathbb{A}) \Big/ \prod_{p<\infty} \mathrm{SL}_2(\mathbb{Z}_p) = Z(\mathbb{A})\,\mathrm{GL}_2(\mathbb{Q}) \Big\backslash \mathrm{GL}_2(\mathbb{A}) \Big/ \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p),$$

where $Z = Z(\mathrm{GL}_2) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right\}$. For modular forms, the automorphy factor for GL(2) is

$$\nu(g, z) = (cz + d) \cdot \det(g)^{1/2}.$$

For now on, $G = \mathrm{GL}_2$.

1.3. **Hecke operators.** Recall that for any prime p, we have the operator

$$T(p)f(z) = p^{k-1} \sum_{\substack{a\geq 1,\ ad=p \\ 0\leq b<d}} d^{-k} f\left(\frac{az+b}{d}\right).$$

These play an important role:

- they are self-adjoint with respect to the *Petersson scalar product* for $(p, N) = 1$, so we can expect eigenfunctions for them,
- if $f = \sum_{n=1}^{\infty} a_n q^n$ is a cusp form, normalized such that $a_1 = 1$, which is an eigenform, then $T(p)f = a_p f$.

Thus, the Fourier coefficients may be thought of as eigenvalues of the Hecke operators.

In the adelic language, the Hecke operators has a nicer description.

Definition 1.24. For a prime p, define for $\phi\colon G(\mathbb{A}) \to \mathbb{C}$, $g \in G(\mathbb{A})$,

$$h \in G(\mathbb{Q}_p) \hookrightarrow G(\mathbb{A}) \ni h_p = (1, 1, \ldots, 1, h, 1, \ldots),$$

the *Hecke operator* as the integral:

$$\widetilde{T}(p)\phi(g) = \int_{K_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_p} \phi(g \cdot h_p) dh.$$

Remark 1.25. Here, dh is the Haar measure on $G(\mathbb{Q}_p)$, normalized so that the compact open subgroup K_p has measure 1. We will discuss this later in the class.

Fact 1.26. *For $(p, N) = 1$, we have that*

$$p^{k/2-1} \cdot \widetilde{T}(p)\phi_f = \phi_{T(p)f}.$$

Here, f is a modular form of weight k for the Hecke group $\Gamma_0(N)$.

Proof. First, express the double coset as a union of single cosets:

$$K_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_p = \bigcup_{t=0}^{p-1} \overbrace{\begin{bmatrix} p & -t \\ 0 & 1 \end{bmatrix}}^{\xi_t} K_p \cup \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}}^{\xi_p} K_p.$$

Checking this is left as an exercise. Then

$$\widetilde{T}(p)\phi_f(g) = \sum_{t=0}^{p} \phi_f(g \cdot \xi_{t,p}),$$

because ϕ_f is right K_p-invariant. Write

$$g = \gamma \cdot g_\infty \cdot k$$

for $\gamma \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})^+$, $k \in K$. Consider $\gamma g_\infty k \cdot \xi_{t,p}$. Note that

$$k_p \cdot \xi_{t,p} \in K_p \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} K_p$$

so there exist t' and k'_p such that

$$k_p \cdot \xi_{t,p} = \xi_{t',p} \cdot k'_p.$$

We avoid computing t' based on t by the observation: if $t_1 \neq t_2$, then $t'_1 \neq t'_2$. Therefore, we get

$$\widetilde{T}(p)\phi_f(g) = \sum_{t=0}^{p} \phi_f(\gamma \cdot g_\infty \cdot \xi_{t,p} \cdot k')$$

and we may write

$$\gamma \cdot g_\infty \cdot \xi_{t,p} \cdot k' = \underbrace{(\gamma\xi_t)}_{\in G(\mathbb{Q})} \cdot \underbrace{(\xi_{t,\infty}^{-1} \cdot g_\infty)}_{\in G_\infty^+} \cdot \underbrace{k'_p \prod_{\ell \neq p} \xi_{t,\ell}^{-1} k_\ell}_{\in K^{\mathrm{fin}}}$$

and hence

$$\widetilde{T}(p)\phi_f(g) = \sum_{t=0}^{p} \underbrace{f(\xi_{t,\infty}^{-1} g_\infty \cdot i)}_{f(\xi_{t,\infty}^{-1}\cdot z)} \cdot \underbrace{\nu(\xi_{t,\infty}^{-1} g_\infty, i)^{-k}}_{\nu(\xi_{t,\infty}^{-1}, z)^{-k}\nu(g_\infty, i)^{-k}} .$$

For $0 \le t < p$, we get

$$f(\xi_{t,\infty}^{-1} z) = f\left(\frac{z+t}{p}\right)$$
$$\nu(\xi_{t,\infty}^{-1}, z) = p^{-1/2}.$$

For $t = p$, we get

$$f(\xi_{t,\infty}^{-1} z) = f(pz)$$
$$\nu(\xi_{t,\infty}^{-1}, z) = p^{1/2}.$$

Putting all of this together gives the result. □

Fact 1.27. *If f is cuspidal, then*

$$\phi_f \in L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})).$$

The right hand side has a unitary action of $G(\mathbb{A})$. If f is a cuspidal, new eigenform, then the subrepresentation it spans is an irreducible subrepresentation. Conversely, one can recover from any irreducible subrepresentation of $G(\mathbb{A})$ in this Hilbert space such a vector ϕ.

Therefore, this reduces the question of studying automorphic forms to studying the irreducible constituents of this representation.

Theorem 1.28. *If π is an irreducible representation of $G(\mathbb{A})$, then*

$$\pi = \bigotimes{}' \pi_p$$

where π_p is an irreducible representation of $G(\mathbb{Q}_p)$.

Therefore, we need to study irreducible representations of the groups $G(\mathbb{Q}_p)$.

2. Overview of harmonic analysis on LCA groups

We will soon develop the theory for GL_1, following Tate's thesis, but we start by discussing harmonic analysis on LCA groups. The reference for this is Hewitt–Ross.

2.1. LCA groups.

Definition 2.1. An *LCA group* is an abelian topological group G, whose topology is locally compact (i.e. Hausdorff and every points has a compact neighborhood).

Example 2.2. The groups $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \cdot)$, $(\mathbb{Q}_p, +)$, $(\mathbb{Q}_p^\times, \cdot)$, $(\mathbb{A}, +)$, $(\mathbb{A}^\times, \cdot)$ are all LCA.

Definition 2.3. We write LCA for the category of locally compact abelian groups with continuous group homomorphism.

Recall that a morphism f in any category is

- *monic* if and only if $f \cdot g_1 = g \circ g_2$ implies $g_1 = g_2$,
- *epic* if and only if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

In LCA:

- monic is equivalent to injective,
- epic is equivalent to having dense image.

Example 2.4. Let $r \in \mathbb{R}$ be irrational. Consider

$$\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$$
$$t \mapsto (t, rt)$$

This is monic and epic, but it is not an isomorphism.

Therefore, LCA is not an abelian category. It is only additive. We can make it into an *exact category*, i.e. we will declare which monics and epics are *admissible*.

Lemma 2.5. *Let G be LCA and $H \subseteq G$ be a closed subgroup. Then H is LCA and G/H is also LCA. Conversely, if $H \subseteq G$ is a topological subgroup, which is LCA, then H is closed.*

Recall that a continuous map $f\colon X \to Y$ of topological space is an *immersion* if it is injective and a homeomorphism onto its image. Moreover, f is an *open/closed* immersion if f is an immersion and it is open/closed; equivalently, if f is an immersion and $f(X)$ is open/closed.

A map $f\colon X \to Y$ is a *quotient map* if f is surjective and open.

Definition 2.6. A *admissible monic* is a closed immersion. An *admissible epic* is a quotient.

An sequence $1 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 1$ is *exact* in LCA if

- it is exact in Ab,
- f is an admissible monic,
- g is an admissible epic.

Theorem 2.7 (Open mapping)**.** *Let $f\colon G \to H$ be a surjective morphism of LCA. Assume G is a countable union of compact sets. Then f is open.*

2.2. **Pontryagin duality.** Let G be LCA.

Definition 2.8. A *character* of G is an LCA morphism $\chi\colon G \to \mathbb{C}^\times$. We say that χ is unitary if $\chi(G) \subseteq S^1$. The *dual group* is

$$\hat{G} = \{\chi\colon G \to S^1\}$$

with compact open topology $U(K, V) = \{\chi \mid \chi(K) \subseteq V\}$ where K is a compact in G and V is an open in S^1.

Proposition 2.9. *The group $\hat{G}$ is LCA.*

Definition 2.10. If $f\colon G \to H$ is a morphism, define $\hat{f}\colon \hat{H} \to \hat{G}$ by

$$\hat{f}(\chi) = \chi \circ f.$$

Remark 2.11. If V is a finite-dimensional vector space, then $V \cong V^*$ non-canonically, but there is a canonical isomorphism

$$\mathrm{ev}\colon V \to V^{**}$$

such that $\langle \mathrm{ev}(v), \xi \rangle = \langle v, \xi \rangle$.

Theorem 2.12 (Pontryagin duality)**.** *The map $G \to \hat{G}$ is an exact self-equivalence of LCA. Moreover, $\mathrm{ev}\colon G \to \hat{\hat{G}}$ is an isomorphism from the functor $\mathrm{id}_{\mathrm{LCA}}$ to the functor $G \mapsto \hat{\hat{G}}$.*

In particular, if $f\colon H \to G$ is a closed immersion, then $\hat{f}\colon \hat{G} \to \hat{H}$ is surjective, so every unitary character of H extends to a unitary character of G.

Remark 2.13. An LCA group G is compact if and only if its dual $\hat{G}$ is discrete.

Remark 2.14. An isomorphism $G \to \hat{G}$ is the same as a non-degenerate bi-character

$$\langle \cdot, \cdot \rangle\colon G \times G \to S^1.$$

Examples 2.15.

- The dual of $\mathbb{Z}$ is S^1 by definition.
- The dual of S^1 is $\mathbb{Z}$ by duality.
- For $G = \mathbb{R}$, the bi-character $(x, y) \mapsto e^{ixy}$ is non-degenerate, so the dual of $\mathbb{R}$ is isomorphic to $\mathbb{R}$. I.e. every unitary character of $\mathbb{R}$ is of the form $x \mapsto e^{ixy}$ for $y \in R$. Moreover, every character $\mathbb{R} \to \mathbb{C}^\times$ is of the form $x \mapsto e^{sx}$ for $s \in \mathbb{C}$.
- Every character of $\mathbb{R}_{>0}$ is of the form $x \mapsto x^s$ for $s \in \mathbb{C}$. It is unitary if and only if $s \in i\mathbb{R}$.
- Every character of $\mathbb{R}^\times$ is of the form $x \mapsto |x|^s \cdot \mathrm{sgn}(x)^m$ for $s \in \mathbb{C}$ and $m \in \mathbb{Z}/2\mathbb{Z}$.
- Every character of $\mathbb{C}^\times$ is of the form $z \mapsto z^a \overline{z}^b$ for $a, b \in \mathbb{C}$, $a - b \in \mathbb{Z}$.

Exercise. Prove this without using Potryagin duality.

Definition 2.16. Given a closed subgroup $H \subseteq G$, define

$$H^\perp = \{\chi \in \hat{G} \mid \langle h, x \rangle = 1 \text{ for all } h \in H\}.$$

Given a closed subgroup $H' \subseteq \hat{G}$, define

$$^\perp H' = \{g \in G \mid \langle g, \chi \rangle \text{ for all } x \in H'\}.$$

Proposition 2.17. *The isomorphism $G \to \hat{\hat{G}}$ identifies H with $^\perp(H^\perp)$.*

Proposition 2.18. *If $H \subseteq G$ is a closed subgroup, then*

$$\widehat{G/H} \cong H^\perp$$

canonically. In other words, the short exact sequences

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

$$1 \longrightarrow H^\perp \longrightarrow \hat{G} \longrightarrow \hat{G}/H^\perp \longrightarrow 1$$

are dual to each other.

2.3. **Haar measure.** Let G be a locally compact group.

Definition 2.19. A *Haar measure* on G is a Borel measure μ such that it is

- left-invariant,
- countably additive,
- finite on compact sets,
- inner regular: if $S \subseteq G$ is measurable, then
$$\mu(S) = \lim\{\mu(K) \mid K \subseteq S \text{ compact}\},$$
- outer regular: if $S \subseteq G$ is measurable, then
$$\mu(S) = \lim\{\mu(O) \mid S \subseteq O \text{ open}\}.$$

Theorem 2.20 (Haar)**.** *A Haar measure exists and is unique up to* $\mathbb{R}_{>0}$.

Remark 2.21. In general, no canonical normalization. In some special cases, there are standard normalizations:

- if G is compact, we can take $\mu(G) = 1$,
- if G is discrete, we take the counting measure $\mu(\{1\}) = 1$.

In practice, there will always be a natural way to choose the normalization.

Proposition 2.22. *Let* $1 \to A \to B \to C \to 1$ *be an exact sequence and* da, db *be Haar measures on* A *and* B. *There is a unique Haar measure* $dc = \frac{db}{da}$ *on* C *such that*
$$\int_B f(b) fb = \int_C \int_A f(ac) da dc.$$

2.4. **Fourier transforms.** Let G be LCA and dg be a Haar measure on G. We have the spaces $L^p(G)$, independent of the choice of dg.

Definition 2.23. For $f \in L^1(G)$, define the *Fourier transform* $\hat{f}_{dg} \colon \hat{G} \to \mathbb{C}$ by
$$\hat{f}_{dg}(\xi) = \int_G f(g) \langle f, \xi \rangle dg.$$

We will often omit the dg from the notation.

Proposition 2.24 (Riemann-Lebesgue)**.** *The function* $\hat{f}$ *is continuous and vanishes towards infinity.*

Theorem 2.25 (Inversion theorem)**.** *There exists a unique Haar measure* $d\xi$ *on* $\hat{G}$ *such that if* $f \in L^1(G)$ *and* $\hat{f} \in L^1(\hat{G})$, *then*
$$(\hat{f}_{dg})^{\wedge}_{d\xi}(x) = f(-x).$$

Definition 2.26. The Haar measure $d\xi$ is the *dual measure* to dg.

Exercise. If G is compact and $\mathrm{vol}(G, dg) = 1$, then the dual measure $d\xi$ on $\hat{G}$ is the counting measure.

Theorem 2.27 (Plancharet theorem). *If $f \in L^2(G) \cap L^1(G)$ and $\hat{f} \in L^2(\hat{G})$, and*

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} |\hat{f}(\xi)|^2 d\xi.$$

The Fourier transform extends to an isometric isomorphism

$$L^2(G) \to L^2(\hat{G}).$$

Theorem 2.28 (Poisson summation). *Let $1 \to A \to B \to C \to 1$ be an exact sequence in LCA. Fix Haar measures da, db on A and B. Let $dc = \frac{db}{da}$, $d\hat{b}$ be dual to db and $d\hat{a}$ be dual to da. Let $f \in L^1(B)$ be such that*

1. $\hat{f}|_{A^\perp} \in L^1(A^\perp)$,
2. $a \mapsto f(a+b)$ *is in* $L^1(A)$ *for any* $b \in B$,
3. $b \mapsto \int_A f(a+b)da$ *is continuous.*

Then

$$\int_A f(a)da = \int_{\hat{C}} \hat{f}(\hat{c})d\hat{c},$$

where $\hat{C} = A^\perp$ and the measure $d\hat{c}$ is the dual of dc (or, equivalently, the unique measure such that $d\hat{a} = \frac{d\hat{b}}{d\hat{c}}$).

Proof. Define

$$F(c) = \int_A f(a+b)da$$

if $b \mapsto c$. Then, by Fourier inversion,

$$\int_A f(a)da = F(0) = \int_{\hat{C}} \hat{F}(\hat{c})d\hat{c}.$$

Now,

$$\begin{aligned} \hat{F}(\hat{c}) &= \int_C F(c)\langle c, \hat{c}\rangle dc \\ &= \int_C \int_A f(a+c)\langle c, \hat{c}\rangle da dc \\ &= \int_C \int_A f(a+c)\langle a+c, \hat{c}\rangle da dc \\ &= \int_B f(b)\langle b, \hat{c}\rangle db \\ &= \hat{f}(\hat{c}). \end{aligned}$$

This completes the proof. □

2.5. **Self-duality.** Let B be LCA, $j \colon B \to \hat{B}$ be an isomorphism, given by the non-degenerate bi-character $\langle \cdot, \cdot \rangle \colon B \times B \to S^1$. We have

$$\hat{j} \colon \hat{\hat{B}} \cong B \to \hat{B}.$$

Exercise. Show that

(1) $\hat{\hat{j}} = j$,
(2) $\hat{j} = j$ if and only if $\langle \cdot, \cdot \rangle$ is symmetric.

For any $f \in L^1(G)$, we get a function

$$\hat{f}_j = \hat{f} \circ j \colon G \to \mathbb{C}.$$

Again, we will omit the j from the notation occasionally. Explicitly,

$$\hat{f}_j(b) = \int_B f(b')\langle b', b\rangle db'.$$

Definition 2.29. The measure db is called *self-dual* if $j_*(db)$ is the measure on $\hat{B}$ dual to db.

Note that this definition depends on a choice of bi-character.

Exercise. A self-dual measure exists an is unique.

Exercise. Assume $\hat{j} = j$ and db is self-dual. Then

$$(\hat{f}_j)^\wedge_j(b) = f(-b).$$

Lemma 2.30. *Let $A \subseteq B$ be closed. Assume that j identifies A with $A^\perp$. For any Haar measure da on A, $j_*(da)$ is the Haar measure on $A^\perp = \hat{C}$ dual to $dc = \frac{db}{da}$, where db is self-dual.*

Proof. Both $j_*(da)$ and $d\hat{c}$ are Haar measures, so $j_*(da) = kd\hat{c}$ for some $k \in \mathbb{R}_{>0}$. By Poisson summation 2.28,

$$\int_A f(a)da = \int_{\hat{C}} \hat{f}(\hat{c})d\hat{c} = k\int_A \hat{f}_j(a)da.$$

Applying this equation twice, we get that

$$\int_A f(a)da = k^2 \int_a (\hat{f}_j)^\wedge_j(a)da = k^2 \int_a f(-a)da = k^2 \int_A f(a)da.$$

Hence $k^2 = 1$, so $k = 1$. □

Corollary 2.31 (Possion summation)**.** *For any sufficiently nice function $f \colon B \to \mathbb{C}$ (see Theorem 2.28), we have*

$$\int_A f(a)da = \int_A \hat{f}(a)da,$$

where $\hat{f} \colon B \to \mathbb{C}$ is formed with respect to the self-dual measure on B.

Fact 2.32. *Let G be a compact group and $\chi \colon G \to \mathbb{C}^\times$ is non-trivial. Then*

$$\int_A \chi(g)dg = 0.$$

This is analogous to the statement for finite groups that:

$$\sum_{g\in G} \chi(g) = 0.$$

The proof is also analogous.

3. Harmonic analysis on local fields and adeles

Recall that

$$\mathbb{Z}_p = \varprojlim_k \mathbb{Z}/p^k\mathbb{Z}, \quad \mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Z}_p\left[p^{-1}\right].$$

Then $x \in \mathbb{Z}_p$ can be written as

$$x = \sum_{k=0}^{\infty} a_k p^k$$

and $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{k>-N} a_n p^k$$

for $a_k \in \{0, \dots, p-1\}$.

Fact 3.1.

(1) *We have that* $\mathbb{Q}_p = \mathbb{Z}_p + \frac{1}{p^\infty}\mathbb{Z}$ *and* $\frac{1}{p^\infty}\mathbb{Z} \cap \mathbb{Z}_p = \mathbb{Z}$.
(2) *The map*

$$\begin{aligned} \mathbb{Q}_p/\mathbb{Z}_p &\to \mathbb{Q}/\mathbb{Z} \\ x = y + z &\mapsto z \end{aligned}$$

is an injective group homomorphism, inverse to $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}_p/\mathbb{Z}_\mathfrak{p}$ *induced by* $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$. *These identify* $\mathbb{Q}_p/\mathbb{Z}_p$ *with the p-power torsion gorup of* $\mathbb{Q}/\mathbb{Z}$.

Definition 3.2. Define the character $\psi_p \colon \mathbb{Q}_p/\mathbb{Z}_p \to S^1$ as the composition of $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$ with

$$\begin{aligned} \mathbb{Q}/\mathbb{Z} &\to S^1 \\ x &\mapsto e^{2\pi i x}. \end{aligned}$$

Fact 3.3. *Let* $\psi \colon \mathbb{Q}_p \to S^1$ *be any character. There exists* k *such that* $\psi(p^k\mathbb{Z}) = 1$.

Proof. Take an open neighborhood $U \subseteq S^1$ not contained in a subgroup of S^1. Then $\psi^{-1}(U)$ is an open subset of $\mathbb{Q}_p$, containing the subgroup $\psi^{-1}\{1\}$. This open subset contains an open subgroup $p^k\mathbb{Z}_p$ for large enough k, whose image in contained in U. Therefore, the image of $p^k\mathbb{Z}_p$ is $\{1\}$. □

Definition 3.4. The *level* of ψ is the smallest k such that $\psi(p^k\mathbb{Z}) = 1$.

Lemma 3.5. *The bi-character*

$$\begin{aligned} \mathbb{Q}_p \times \mathbb{Q}_p &\to S^1 \\ (x, y) &\mapsto \psi_p(x \cdot y) \end{aligned}$$

is non-degenerate. Under the isomorphism $\mathbb{Q}_p \to \hat{\mathbb{Q}}_p$, *we have that* $\mathbb{Z}_p^\perp = \mathbb{Z}_p$.

Proof. Injectivity and continuity of $\mathbb{Q}_p \to \hat{\mathbb{Q}}_p$ are clear. Once surjectivity is proved, openness will follow from the open mapping theorem ($\mathbb{Q}_p = \bigcup (1/p^k)\mathbb{Z}_p$).

We show surjectivity. Given $\psi\colon \mathbb{Q}_p \to S^1$, want $y \in \mathbb{Q}_p$ such that $\psi(x) = \psi_p(xy)$. Let k be the level of ψ. Then on $p^{k-1}\mathbb{Z}_p$, we have that

$$\psi(x) = \psi_p(u_0 p^{-k} x)$$

for some $u_1 \in \mathbb{Z}_p^\times$, well-defined up to $1 + p\mathbb{Z}_p$.

Now, look at $p^{k-1}\mathbb{Z}_p$. Get $u_2 \in \mathbb{Z}_p^\times$ such that $u_2 \equiv u_1 \mod 1 + p\mathbb{Z}_p$ and

$$\psi(x) = \psi_p(u_2 p^{-k} x)$$

for all $x \in p^{k-2}\mathbb{Z}_p$. Inductively, get a sequence (u_n), $u_n \in \mathbb{Z}_p^\times$, $u_n \equiv u_{n-1} \mod 1 + p^n\mathbb{Z}_p$. This sequence is Cauchy, so it converges to $u \in \mathbb{Z}_p^\times$ such that $\psi(x) = \psi_p(p^{-k}ux)$ for all $x \in \mathbb{Q}_p$. □

Recall that

$$\mathbb{A}^{\text{fin}} = \prod_{p<\infty}{}' \mathbb{Q}_p$$

contains

$$\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Z}_p.$$

Moreover, $\mathbb{A} = \mathbb{R} \times \mathbb{A}^{\text{fin}}$. The topology is generated by

$$\prod_{p\in\mathcal{S}} U_p \times \prod_{p\notin S} \mathbb{Z}_p$$

for finite sets S of places and $U_p \subseteq \mathbb{Q}_p$ open.

Remark 3.6. The sequence $x + n = (1, 1, 1, \dots, 1, 0, 0, \dots)$ with n ones coverges to $1 \in \mathbb{A}$.

Fact 3.7. *The map $\mathbb{Q} \to \mathbb{A}$ has discrete image and $\mathbb{A}/\mathbb{Q}$ is compact.*

Proof. Consider the following open set $U = (-0.5, 0.5) \times \prod_p \mathbb{Z}_p$. Take $q \in \mathbb{Q} \cap U$. Then $q_p \in \mathbb{Z}_p$ for all $< \infty$ implies that $q \in \mathbb{Z}$. Then $q_\infty \in (-0.5, 0.5)$ implies that $q = 0$. This proves discreteness.

Now, consider the compact set $[-0.5, 0.5] \times \prod_p \mathbb{Z}_p$. Then

$$\mathbb{Q} + W = \mathbb{A}.$$

Thus restriction of $\mathbb{A} \to \mathbb{A}/\mathbb{Q}$ to W is surjective and hence $\mathbb{A}/\mathbb{Q}$ is compact. □

Remark 3.8. The embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$ is a closed immersion, if $\mathbb{Q}$ is given the discrete topology. From now on, we will always give $\mathbb{Q}$ the discrete topology.

Corollary 3.9. *The embedding $\mathbb{Q} \hookrightarrow \mathbb{A}^{\text{fin}}$ induces an isomorphism*

$$\mathbb{Q}/\mathbb{Z} \xrightarrow{\cong} \mathbb{A}^{\text{fin}}/\hat{\mathbb{Z}},$$

which translates the p-primary decomposition of $\mathbb{Q}/\mathbb{Z}$ into the canonical decomposition

$$\mathbb{A}^{\text{fin}}/\hat{\mathbb{Z}} = \prod_{p<\infty} \mathbb{Q}_p/\mathbb{Z}_p.$$

This gives

- a character $\psi_{\mathrm{fin}}\colon \mathbb{A}^{\mathrm{fin}}/\hat{\mathbb{Z}} \to S^1$ via the diagram

$$\begin{array}{ccc} \mathbb{A}^{\mathrm{fin}}/\hat{\mathbb{Z}} & \xrightarrow{\psi_{\mathrm{fin}}} & S^1 \\ & \searrow \quad \nearrow_{e^{2\pi i x}} & \\ & \mathbb{Q}/\mathbb{Z} & \end{array}$$

- $\psi_{\mathrm{fin}}|_{\mathbb{Q}_p} = \psi$ defined before.

Define

$$\begin{aligned} \psi_\infty\colon \mathbb{R}/\mathbb{Z} &\to S^1 \\ x &\mapsto e^{-2\pi i x}. \end{aligned}$$

Note the minus sign.

Then define $\psi_{\mathbb{A}} = (\psi_\infty, \psi_{\mathrm{fin}})\colon \mathbb{A} \to S^1$.

Lemma 3.10.

(1) *The character $\psi_{\mathbb{A}}$ is trivial on $\mathbb{Q}$.*
(2) *The bi-character*

$$\begin{aligned} \mathbb{A} \times \mathbb{A} &\to S^1 \\ (x, y) &\mapsto \psi_{\mathbb{A}}(x, y) \end{aligned}$$

is non-degenerate.
(3) *Under this character, $\mathbb{Q}^\perp = \mathbb{Q}$.*

Proof. Part (1) is immediate from construction and the sign convention. As in the local case, it is enough to prove surjectivity: given $\psi\colon \mathbb{A} \to S^1$, there is a $y \in \mathbb{A}$ such that $\psi(x) = \psi_{\mathbb{A}}(xy)$.

For each p, let $y_p \in \mathbb{Q}_p$ be such that $\psi|_{\mathbb{Q}_p}(x_p) = \psi_p(x_p y_p)$. Note that ψ is trivial on a subgroup of the form $\prod_{p \notin S} \mathbb{Z}_p$. Therefore, $y_p \in \mathbb{Z}_p$ for all $p \notin S$. Therefore, $y \in \mathbb{A}$ and this completes the proof of (2).

Consider $\mathbb{Q}^\perp \subseteq \mathbb{A}$.

- Since $\mathbb{A}/\mathbb{Q}$ is compact, $\mathbb{Q}^\perp$ is discrete.
- By (1), $\mathbb{Q} \subseteq \mathbb{Q}^\perp$. Therefore, $\mathbb{Q}^\perp/\mathbb{Q}$ is a discrete subgroup of the compact $\mathbb{A}/\mathbb{Q}$, hence finite.
- By definition, $\mathbb{Q}^\perp$ is invariant under multiplication by $\mathbb{Q}$, and hence it is a $\mathbb{Q}$-vector space.

Hence $\mathbb{Q}^\perp/\mathbb{Q} = \{1\}$, proving (3). □

Exercise. We have the following self-dual exact sequences

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q} \longrightarrow 0.$$

Additive Haar measures.

Fact 3.11. *The Lebesgue measure on $\mathbb{R}$ is Haar and self-dual with respect to ψ_∞.*

Fact 3.12. *Let dx_p be the Haar measure on $\mathbb{Q}_p$, self dual with respect to ψ_p. Then*

(1) *if $f = \mathbb{1}_{\mathbb{Z}_p}$, then $\hat{f} = f$,*
(2) $\operatorname{vol}(\mathbb{Z}_p, dx_p) = 1$.

Proof. We compute for $f = \mathbb{1}_{\mathbb{Z}_p}$ and any Haar measure dy:

$$\begin{aligned}\hat{f}(x) &= \int_{\mathbb{Q}_p} \mathbb{1}_{\mathbb{Z}_p}(y)\psi_p(xy)dy \\ &= \int_{\mathbb{Z}_p} \psi_p(xy)dy \\ &= \begin{cases} \operatorname{vol}(\mathbb{Z}_p, dx) & \text{for } x \in \mathbb{Z}_p \\ 0 & \text{for } x \notin \mathbb{Z}_p \end{cases}\end{aligned}$$

In the second case, we note that

$$\int_{\mathbb{Z}_p} \psi_p(xy)dy = \int_{x\cdot\mathbb{Z}_p} \psi_p(y)d(x^{-1}y) = 0$$

Thus:

$$\hat{f} = \operatorname{vol}(\mathbb{Z}_p, dx) \cdot f$$

and

$$\hat{\hat{f}} = \operatorname{vol}(\mathbb{Z}_p, dx)^2 \cdot f.$$

If dx is self dual, then $\hat{\hat{f}} = f$, proving both statements. $\square$

Remark 3.13. Recall that for the Gaussian $f = e^{-\pi x^2}$ on $\mathbb{R}$ satisfies $\hat{f} = f$. Therefore, $\mathbb{1}_{\mathbb{Z}_p}$ is a $\mathbb{Q}_p$-analog of the Gaussian.

Remark 3.14. Note that we can by-pass Haar's existence theorem. Indeed, if we can reverse engineer the measure from what we know: if $\operatorname{vol}(\mathbb{Z}_p, dx) = 1$, $\operatorname{vol}(p^{-k}\mathbb{Z}_p, dx) = p^k$. Now, any open subset of $\mathbb{Q}_p$ may be written as a union of translates of $p^{-k}\mathbb{Z}_p$, and we may have defined the measure this way.

Exercise. Suppose $p \leq \infty$. For any $a \in \mathbb{Q}_p^\times$, $d(ax) = |a|_p dx$.

Fact 3.15. *Let dx_p be a Haar measure on $\mathbb{Q}_p$. Assume $\mathrm{vol}(\mathbb{Z}_p, dx_p) = 1$ for almost all p. Then*

$$dx = \bigotimes dx_p$$

is a measure on $\mathbb{A}$. If each dx_p is ψ_p-self-dual, then dx is $\psi_{\mathbb{A}}$-self-dual.

Corollary 3.16. *We have that $\mathrm{vol}(\mathbb{A}/\mathbb{Q}, da/dq) = 1$ where da is self-dual and dq is the counting measure.*

Proof. Let $\delta_0 \colon \mathbb{Q} \to \mathbb{C}$ be the delta function. Then

$$\hat{\delta}_0(\hat{q}) = \int_{\mathbb{Q}} \delta_0(q)\langle q, \hat{q}\rangle dq = 1.$$

By Fourier inversion, this shows that

$$\mathrm{vol}(\hat{\mathbb{Q}}) = \int_{\hat{\mathbb{Q}}} \hat{\delta}_0(\hat{q}) d\hat{q} = \delta_0(0) = 1.$$

Consider the short exact sequence $0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q} \longrightarrow 0$. Then Lemma 2.30 says that $d\hat{q}$ is the quotient of da by $j_*(dq)$, which is the counting measure. This completes the proof. □

Recall that there is an adelic norm $|\cdot|_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{R}$.

Fact 3.17. *We have that $d(ax) = |a|_{\mathbb{A}} dx$.*

Schwartz functions.

Definition 3.18.

(1) The space $\mathcal{S}(\mathbb{R})$ of *Schwartz functions* consists of smooth functions $f \colon \mathbb{R} \to \mathbb{C}$ such that
$$\left| x^p \cdot f^{(q)}(x) \right| \to 0 \text{ as } x \to \pm\infty.$$
(2) For $p < \infty$, we set $\mathcal{S}(\mathbb{Q}_p) = \mathcal{C}_c^\infty(\mathbb{Q}_p)$, the space of locally constant compactly supported functions $\mathbb{Q}_p \to \mathbb{C}$ (*Schwartz–Bruhat functions*).
(3) $\mathcal{S}(\mathbb{A}) = \bigotimes_{p \leq \infty}' \mathcal{S}(\mathbb{Q}_p)$, i.e. the vector space generated by $f = \bigotimes_p f_p$ for $f_p \in \mathcal{S}(\mathbb{Q}_p)$ such that $f_p = \mathbb{1}_{\mathbb{Z}_p}$ for almost all p. We call these generators *factorizable.*
Explicitly, $f(a) = \prod_p f_p(a_p)$ is well-defined, since almost all factors are 1.

Fact 3.19. *For $p \leq \infty$, Fourier transforms gives a bijection $\mathcal{S}(\mathbb{Q}_p) \to \mathcal{S}(\mathbb{Q}_p)$.*

Proof. For $p < \infty$, use $\hat{\mathbb{1}}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p}$ and the fact that $f \in \mathcal{C}_c^\infty(\mathbb{Q}_p)$ is a finite linear combination of additive and multiplicative transforms of $\mathbb{1}_{\mathbb{Z}_p}$.

For $p = \infty$, use $\widehat{f'}(x) = x\hat{f}(x)$ and $\widehat{xf(x)} = (\hat{f})'(x)$. □

Fact 3.20. *If $f = \bigotimes_p f_p \in \mathcal{S}(\mathbb{A})$ then $\hat{f} = \bigotimes_p \hat{f}_p$.*

Corollary 3.21. *Fourier transform is a bijection $\mathcal{S}(\mathbb{A}) \to \mathcal{S}(\mathbb{A})$.*

We can apply the Poisson summation formula 2.28 to the case of adeles.

Corollary 3.22 (Adelic Poisson summation)**.** *For $f \in \mathcal{S}(\mathbb{A})$, we have that*

$$\sum_{q\in\mathbb{Q}} f(q) = \sum_{q\in\mathbb{Q}} \hat{f}(q).$$

Corollary 3.23. *For $f \in S(\mathbb{A})$ and $a \in \mathbb{A}^\times$,*

$$\sum_{q\in\mathbb{Q}} f(aq) = |a|_{\mathbb{A}}^{-1} \sum_{q\in\mathbb{Q}} \hat{f}(a^{-1}q).$$

Proof. This is immediate: if $f^a(x) = f(ax)$, then $\hat{f}^a(x) = |a|_{\mathbb{A}}^{-1}\hat{f}(a^{-1}x)$. □

Remark 3.24. Note that $\hat{f}$ is formed with respect to the fixed character $\psi_{\mathbb{A}}$. Any other character $\psi\colon \mathbb{A}/\mathbb{Q} \to S^1$ is of the form $\psi(x) = \psi_{\mathbb{A}}(a \cdot x)$ for some $a \in \mathbb{Q}$.

So the $\psi_{\mathbb{A}}$-self-dual measure is also the ψ-self-dual measure (by the product formula). Also, the Poisson summation formula still holds.

Multiplicative Haar measures.

Fact 3.25. *Let dx be a Haar measure on $\mathbb{Q}_p$. Then $|x|_p^{-1}dx$ is a Haar measure on $\mathbb{Q}_p^\times$. (Here, $p \le \infty$.)*

Proof. Note that the measure dx on $\mathbb{Q}_p$ restricted to $\mathbb{Q}_p^\times$ is still a measure. However, it is not invariant. The factor of $|x|_p^{-1}$ makes it invariant. □

The case of $\mathbb{A}^\times$ is more subtle. This is because the topology of $\mathbb{A}^\times$ is not the subspace topology:

- $\mathbb{A}^\times$ is not closed in $\mathbb{A}$,
- inversion is not continuous.

This is what we do instead. We recognize that:

$$\mathbb{A}^\times = \mathbb{R}^\times \prod_{p<\infty}{}' [\mathbb{Q}_p^\times, \mathbb{Z}_p^\times]$$

(the restricted direct product of $\mathbb{Q}_p^\times$ with respect to $\mathbb{Z}_p^\times$). Therefore, we take the topology to be generated by

$$\prod_{p\in S} U_p \times \prod_{p\in S} \mathbb{Z}_p^\times \qquad \text{for } U_p \subseteq \mathbb{Q}_p^\times \text{ open, and } S \text{ finite.}$$

Fact 3.26. *If $d^\times x_p$ is a Haar measure on $\mathbb{Q}_p^\times$ such that $\mathrm{vol}(\mathbb{Z}_p^\times, d^\times x_p) = 1$ for almost all p, then*

$$d^\times x = \bigotimes d^\times x_p$$

is a Haar measure on $\mathbb{A}^\times$.

Fact 3.27. *If $d^\times x_p = |x|_p^{-1} dx_p$ for dx_p the ψ_p-self-dual measure, then*

$$\mathrm{vol}(\mathbb{Z}_p^\times, |x|_p^{-1}dx_p) = 1 - \frac{1}{p}.$$

Remark 3.28. Note that
$$\mathrm{vol}([-0.5,0.5]\times\prod_p \mathbb{Z}_p^\times, \bigotimes |x|_p^{-1}dx_p) = \prod_{p<\infty}\left(1-\frac{1}{p}\right) = \zeta(1)^{-1} = 0.$$

Definition 3.29. Define $d^\times x_p = \frac{p}{p-1}|x|_p^{-1}dx_p$.

Multiplicative characters.

Definition 3.30. For $p<\infty$, a character $\chi\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is called *unramified* if $\chi|_{\mathbb{Z}_p^\times} = \mathbb{1}$.

We have a short exact sequence
$$1 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{Q}_p^\times \xrightarrow{\mathrm{val}} \mathbb{Z} \longrightarrow 0.$$

Therefore, unramified characters are characters on $\mathbb{Z}$. More precisely, unramified characters χ are of the form $\chi(x) = |x|_p^s = e^{s\cdot\log|x|_p}$ for some $s\in\mathbb{C}$. Note that this s is unique up to
$$\frac{2\pi i}{\log(p)}\mathbb{Z}.$$
Therefore, the set of unramified characters of $\mathbb{Q}_p^\times$ has the structure of a Riemann surface, namely
$$\mathbb{C}\Big/\frac{2\pi i}{\log(p)}\mathbb{Z}.$$

Since the group of unramified characters acts freely on the group of all characters, we get a structure of a Riemann surface on the latter.

Recall: $|\cdot|_{\mathbb{A}}\colon \mathbb{A}^\times \to \mathbb{R}_{>0}$. Define
$$\mathbb{A}^1 = \ker(|\cdot|).$$
By the product formula, $\mathbb{Q}^\times \subseteq \mathbb{A}^1$.

Fact 3.31. *The group $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^1$ and $\mathbb{A}^1/\mathbb{Q}^\times$ is compact.*

The proof is left as an exercise.

Definition 3.32. A *Hecke character* is a character $\mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$.

Example 3.33. For any $s\in\mathbb{C}$, $x\mapsto |x|_{\mathbb{A}}^s$ is a Hecke character.

We again get a Riemann surfact structure on all Hecke characters (by letting this group of characters act on the set of all characters).

4. Tate's thesis

Local and global zeta integrals.

Definition 4.1. Let $p\le\infty$, $\chi\colon \mathbb{Q}_p^\times\to\mathbb{C}^\times$, $f\in\mathcal{S}(\mathbb{Q}_p)$. Define
$$Z_p(f,\chi) = \int_{\mathbb{Q}_p^\times} f(x)\chi(x)d^\times x_p.$$

Definition 4.2. Let χ be a Hecke character, $f \in S(\mathbb{A})$. Define

$$Z(f,\chi) = \int_{\mathbb{A}^\times} f(x)\chi(x)d^\times x.$$

Remark 4.3. We can study meromorphy on each component of the Riemann surfact of characters.

Definition 4.4. Assume χ is unitary. Then define

$$Z_p(s,f,\chi) = Z_p(f,\chi\cdot|\cdot|_p^s),$$
$$Z(s,f,\chi) = Z(f,\chi\cdot|\cdot|_{\mathbb{A}}^s).$$

Fact 4.5. *In both the local and the global setting, $Z(f,\chi)$ is linear in f and*

$$Z(f^a,\chi) = \chi(a)^{-1}Z(f,\chi),$$

where $f^a(x) = f(ax)$.

Lemma 4.6.

(1) *The integral defining $Z_p(s,f,\chi)$ converges for* $\mathrm{Re}(s) > 0$.
(2) *If there is a neighborhood U of 0 such that $f|_U = 0$, then the integral defining $Z_p(s,f,\chi)$ converges for all values of s, and defines an entire function.*
(3) *For $p < \infty$ and for* $\mathrm{Re}(s) > 0$,

$$Z_p(s,\mathbb{1}_{\mathbb{Z}_p},\chi) = \begin{cases} (1-\chi(p)p^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 0 & \text{if } \chi \text{ is ramified.} \end{cases}$$

Proof. Split the integration domain into $|x|_p > 1$ and $|x|_p \le 1$. On $|x|_p > 1$, the Schwartzness of f dominates the behavior of $|x|_p^s$ so the integral converges absolutely to a holomorphic function.

Now, consider the domain $|x|_p \le 1$. If $f|_U = 0$ for some open set U around 0, then the integration is over $\{x \mid |x|_p \le 1\} \setminus U$, compact in $\mathbb{Q}_p^\times$,so the result converges absolutely to a holomorphic function. This proves (2).

If there is no such U, we have $|f(x)|$ is continuous on $|x| \le 1$, and hence is it bounded. Therefore, it is enough to study

$$\int_{|x|_p\le 1} |x|_p^s d^\times x.$$

For $p < \infty$, this integral is

$$\begin{aligned}\sum_{k\ge 0}\int_{p^k\mathbb{Z}_p^\times} |x|_p^s d^\times x &= \sum_{k\ge 0}\int_{\mathbb{Z}_p^\times} |p^k x|_p^s d^\times x \\ &= \sum_{k\ge 0} p^{-ks}\cdot \underbrace{\int_{\mathbb{Z}_p^\times} 1\cdot d^\times x}_{=1}.\end{aligned}$$

This series converges when $\mathrm{Re}(s) > 0$.

For $p = \infty$, we have

$$\int_{-1}^{1} |t|_\infty^s \frac{dt}{|t|_\infty}.$$

Consider

$$\int_0^1 t^{s-1} dt,$$

which converges for $\mathrm{Re}(s) > 0$. This completes the proof of (2).

For (3), as above, we compute

$$\begin{aligned} Z_p(s, \mathbb{1}_{\mathbb{Z}_p}, \chi) &= \sum_{k \geq 0} \int_{p^k \mathbb{Z}_p^\times} \chi(x) |x|_p^s d^\times x \\ &= \sum_{k \geq 0} \int_{\mathbb{Z}_p^\times} \chi(p^k x) |p^k x|_p^s d^\times x \\ &= \sum_{k \geq 0} \chi(p)^k p^{-ks} \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x \\ &= \left(\sum_{k \geq 0} \left(\chi(p) p^{-s} \right)^k \right) \cdot \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x \\ &= (1 - \chi(p) p^{-s})^{-1} \cdot \int_{\mathbb{Z}_p^\times} \chi(x) d^\times x \qquad \text{for } \mathrm{Re}(s) > 0. \end{aligned}$$

Finally,

$$\int_{\mathbb{Z}_p^\times} \chi(x) d^\times x = \begin{cases} 1 & \text{if } \chi \text{ is unramified,} \\ 0 & \text{if } \chi \text{ is ramified.} \end{cases}$$

This completes the proof. □

Corollary 4.7. *The integral defining $Z(s, f, \chi)$ converges for* $\mathrm{Re}(s) > 1$ *and satisfies for a factorizable function $f = \bigotimes_p f_p$:*

$$Z(s, f, \chi) = \prod_{p \leq \infty} Z_p(s, f_p, \chi_p).$$

Proof. For convergence, we may also assume that $f = \bigotimes_p f_p$ by linearity. By definition of $d^\times x = \bigotimes d^\times x_p$, the integral is a product of local integrals. When does this product converge? By the previous lemma, all local integrals converge for $\mathrm{Re}(s) > 0$, and almost all equal $(1 - \chi(p) p^{-s})^{-1}$, so the convergence of the Euler product for ζ implies the claim. □

Analytic continuation of Z_p.

Lemma 4.8. *Let $p < \infty$.*

(1) *The zeta integral $Z_p(s, f, \chi)$ has a meromorphic continuation to $\mathbb{C}$ as a rational function in p^{-s}.*

(2) *If χ is unramified, $Z_p(f,\chi)$ has a simple pole at $\chi = \mathbb{1}$ and*

$$Z_p(f,\chi)(1-\chi(p))$$

is entire.

(3) *If χ is ramified, $Z_p(s,f,\chi)$ is entire.*

Proof. Let $g = f - f(0)\mathbb{1}_{\mathbb{Z}_p}$. Then $g \in \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)$. By Lemma 4.6, $Z_p(s,g,\chi)$ is entire for any χ and we have explicit formulas for $\mathbb{1}_{\mathbb{Z}_p}$. When χ is ramified, we have

$$Z_p(f,\chi) = Z_p(g,\chi).$$

When χ is unramified, we have

$$Z_p(s,g,\chi) = Z_p(s,f,\chi) - f(0)(1-\chi(p)p^{-s})^{-1}.$$

This proves the result. □

Now, consider the case $p = \infty$. Recall that Landau notation: for $f, g\colon \mathbb{R} \to \mathbb{R}$, we have

- $f = O_{x\to c}(g)$ if and only if $\frac{f}{g}$ is bounded as $x \to c$,
- $f = o_{x\to c}(g)$ if and only if $\frac{f}{g} \to 0$ as $x \to c$.

Fact 4.9. *For $f \in \mathcal{C}^\infty(\mathbb{R})$, we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R(x)$$

with $R(x) = O(x^{n+1})$.

Proof. By Taylor's theorem:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + o(x^n).$$

Therefore,

$$f(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(0)}{k!} x^k + o(x^{n+1})$$

proves the result. □

Lemma 4.10. *The zeta integral $Z_\infty(s,f,\chi)$ has meromorphic continuation to $\mathbb{C}$ with (at most) simple poles at even non-positive integers when $\chi(x) = 1$ and odd non-positive integers when $\chi(x) = \operatorname{sgn}(x)$. The residue at $s = -n$ is $\frac{f^{(n)}(0)}{n!}$.*

Proof. As last time, we split the integral into domain $|x| \le 1$ and $|x| > 1$. The integral over $|x| > 1$ is absolutely convergent to an analytic function in s, so we just need to consider the region $|x| \le 1$.

Decompose $f = f_e + f_0$ into a sum of even and odd function, and consider f_r and f_0 separately. We may hence assume f is even or odd.

The integral is 0 unless f is odd when $\chi = \mathrm{sgn}$ and f is even when $\chi = \mathrm{triv}$. Assume this is the case (by decomposing f into a sum). We are now studying the integral:

$$\int_0^1 f(x)x^s \frac{dx}{x}.$$

We use Fact 4.9 to get

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \underbrace{\int_0^1 x^{k+s}\frac{dx}{x}}_{=\frac{1}{k+s}} + \underbrace{\int_0^1 R(x)x^s\frac{dx}{x}}_{\substack{\text{absolutely convergent}\\ \text{and analytic}\\ \text{for } \mathrm{Re}(s)>-n}} .$$

This completes the proof. □

Invariant distributions.

Definition 4.11. Let $p < \infty$. A *distribution* on $\mathbb{Q}_p$ is any linear functional on $\mathcal{C}_c^\infty(\mathbb{Q}_p)$.

Definition 4.12. A *distribution* on $\mathbb{R}$ is a continuous linear functional on $\mathcal{C}_c^\infty(\mathbb{R})$.

Remark 4.13. Note that we did not define a topology on $\mathcal{C}_c^\infty(\mathbb{R})$ and we will avoid this for now. Instead, we will refine this definition slightly.

Definition 4.14. A *tempered distribution* on $\mathbb{R}$ is a continuous linear functional on $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is topologized with respect to seminorms:

$$\|\varphi\|_{a,b} = \sup_{x\in\mathbb{R}} |x^a \varphi^{(b)}(x)|.$$

Fact 4.15. *If V is a $\mathbb{C}$-vector space, topologized with respect to a family A of semi-norms and $\lambda\colon V \to \mathbb{C}$ is a continuous linear functional, then there exist $B \subseteq A$ finite and $C > 0$ such that*

$$|\lambda(v)| \leq C \cdot \max_{p\in B} p(v).$$

Proof. Recall that a basis of neighborhoods of $x \in V$ is given by

$$U_{B,\delta} = \{y \in V \mid p(x-y) < \delta \text{ for all } p \in B\}$$

for $B \subseteq A$ finite and $\delta > 0$.

Let now $\lambda\colon V \to \mathbb{C}$ be continuous. Choose $\varepsilon > 0$. There exist B, δ such that $\lambda(U_{U,\delta}(0)) \subseteq (-\varepsilon, \epsilon)$.

Let $v \in V$ be arbitrary, define

$$\|v\|_B = \max_{p\in B} p(v).$$

If $\|v\|_B = 0$, then $zv \in U_{B,\delta}(0)$ for all $z \in \mathbb{C}^\times$. Therefore, $|\lambda(v)| \leq |z|^{-1}\varepsilon$ for all $z \in \mathbb{C}^\times$. This shows that $\lambda(v) = 0$.

Otherwise, $\frac{\delta}{\|v\|_B} \cdot v \in U_{B,\delta}(0)$, so

$$\lambda(v) \leq \frac{\varepsilon}{\delta}\|v\|_B.$$

This prove the fact with $C = \frac{\varepsilon}{\delta}$. □

Let $\mathcal{D}(\mathbb{Q}_p)$ be the space of (tempered) distributions. On this space, we have

- a $\mathbb{C}$-vector space structure,
- for $p = \infty$, we have differentiation given by $\langle \lambda, f \rangle = -\langle \lambda, \partial f \rangle$,
- Fourier transform given by $\langle \hat{\lambda}, f \rangle = \langle \lambda, \hat{f} \rangle$,
- an action of $\mathbb{Q}_p^\times$ by $\langle \lambda^a, f \rangle = \langle \lambda, f^{a^{-1}} \rangle$ where $f^a(x) = f(ax)$.

Definition 4.16. Let $\chi \colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$, The space of χ-*eigendistributions* is

$$\mathcal{D}(\mathbb{Q}_p)^\chi = \{\lambda \in \mathcal{D}(\mathbb{Q}_p) \mid \lambda^a = \chi(a)\lambda\}.$$

Fact 4.17. *If $\lambda = \mathcal{D}(\mathbb{Q}_p)^\chi$, then $\hat{\lambda} \in \mathcal{D}(\mathbb{Q}_p)_{\chi^{-1}\cdot|\cdot|}$.*

Proof. Check that $\hat{f^{r^{-1}}}(x) = |r| \cdot \hat{f}(rx)$. Then trace through the definitions this to get the result. □

Example 4.18. The distribution $Z_p(\chi) \colon f \mapsto Z_p(f, \chi)$ is an element of $\mathcal{D}(\mathbb{Q}_p)^\chi$, provided that it is defined at χ.

Theorem 4.19. *The dimension of $\mathcal{D}(\mathbb{Q}_p)^\chi$ over $\mathbb{C}$ is 1.*

Proof. **Step 1.** Consider $\mathcal{C}_c^\infty(\mathbb{Q}_p^\times) \subseteq \mathcal{S}(\mathbb{Q}_p)$. This dualizes to

$$0 \longrightarrow \mathcal{D}(\mathbb{Q}_p)_0 \longrightarrow \mathcal{D}(\mathbb{Q}_p) \longrightarrow \mathcal{C}_0^\infty(\mathbb{Q}_p^\times)' \longrightarrow 0.$$

Here, the $'$ denotes the dual space. The space $\mathcal{D}(\mathbb{Q}_p)_0$ is defined to be the kernel and its elements are called *distributions supported at 0*. Taking χ-eigenspaces, we get

$$0 \longrightarrow \mathcal{D}(\mathbb{Q}_p)_0^\chi \longrightarrow \mathcal{D}(\mathbb{Q}_p)^\chi \longrightarrow \mathcal{C}_0^\infty(\mathbb{Q}_p^\times)'^\chi.$$

Step 2. We claim that $\dim \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)'^\chi = 1$.

Note that for $\chi_1, \chi_2 \colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we have an isomorphism

$$\mathcal{C}_c^\infty(\mathbb{Q}_p^\times)'^{\chi_1} \to \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)'^{\chi_2}$$
$$\lambda_1 \mapsto \lambda_2$$

where $\lambda_2(f) = \lambda_1(f \cdot \chi_1 \cdot \chi_2^{-1})$. It is hence enough to consider $\mathcal{C}_c^\infty(\mathbb{Q}_p^\times)'^{\mathbb{1}}$. By the existence and uniqueness of the Haar measure, this proves the claim.

Step 3. Suppose $p < \infty$. We claim that $\mathcal{D}(\mathbb{Q}_p)_0 = \mathbb{C}\delta_0$. In particular,

$$\mathcal{D}(\mathbb{Q}_p)_0^\chi = \begin{cases} \mathbb{C} \cdot \delta_0 & \chi = \mathbb{1}, \\ \{0\} & \chi \neq \mathbb{1}. \end{cases}$$

To prove this claim, let $\lambda \in \mathcal{D}(\mathbb{Q}_p)_0$. For $f \in \mathcal{C}_c^\infty(\mathbb{Q}_p)$, we have that

$$f - f(0)\mathbb{1}_{\mathbb{Z}_p} \in \mathcal{C}_c^\infty(\mathbb{Q}_p^\times),$$

so $\lambda(f - f(0)\mathbb{1}_{\mathbb{Z}_p}) = 0$, showing that $\lambda(f) = f(0) \cdot \lambda(\mathbb{1}_{\mathbb{Z}_p})$. Then note that $f(0) = \delta_0(f)$ and $C = \lambda(\mathbb{1}_{\mathbb{Z}_p})$ is a constant. Moreover, $\delta_0(f^r) = \delta_0(f)$, showing that $C = 1$.

Step 4. We are done for $p < \infty$, $\chi \neq \mathbb{1}$.

Indeed, steps 1, 2, and 3 show that $\mathcal{D}(\mathbb{Q}_p)^\chi \leq 1$. But $0 \neq z(\chi) \in \mathcal{D}(\mathbb{Q}_p)^\chi$.

Step 5. Consider $p < \infty$ and $\chi = \mathbb{1}$.

Steps 1,2, and 3 show that $\dim \mathcal{D}(\mathbb{Q}_p)^\chi \leq 2$. We need to show that

$$\mu = \int_{\mathbb{Q}_p^\times} f(x) d^\times x$$

does not lift to $\mathcal{D}(\mathbb{Q}_p)^{\mathbb{1}}$. Observe that

$$\lambda(f) = \int_{\mathbb{Q}_p^\times} f(x) - f(0)\mathbb{1}_{\mathbb{Z}_p} d^\times x$$

lifts μ. More formally, $f \mapsto f - f(0)\mathbb{1}_{\mathbb{Z}_p}$ is a section of

$$\mathcal{C}_c^\infty(\mathbb{Q}_p) \to \mathcal{C}_c^\infty(\mathbb{Q}_p^\times)$$

and we are using its dual.

Any lift of μ is of the form $\lambda + a\delta_0$, and will be unramified if and only if λ is invariant. Now, $\lambda(\mathbb{1}_{\mathbb{Z}_p}) = 0$. Moreover,

$$\lambda^p(\mathbb{1}_{\mathbb{Z}_p}) = \lambda(\mathbb{1}_{\mathbb{Z}_p}^{p^{-1}}) = \lambda(\mathbb{1}_{p\mathbb{Z}_p}).$$

Now, $-\mu(\mathbb{1}_{\mathbb{Z}_p^\times}) = -1$, showing that the measure is not invariant.

Step 6. Suppose $p = \infty$. We claim that $\mathcal{D}(\mathbb{R})_0 = \bigoplus_{k \geq 0} \mathbb{C}\partial^k \delta_0$. In particular,

$$\mathcal{D}(\mathbb{R})_0^\chi = \begin{cases} \mathbb{C} \cdot \partial^k \delta_0 & \chi(x) = x^{-k}, \\ \{0\} & \chi(x) \neq x^{-k}. \end{cases}$$

Let $\lambda \in \mathcal{D}(\mathbb{R})_0$. By Fact 4.15, there are $C > 0$ and $a \in \mathbb{Z}_{\geq 0}$ such that

$$|\lambda(f)| \leq C \max_{b+c \leq a} \sup_{x \in \mathbb{R}} |x^b f^{(c)}(x)|.$$

By Fact 4.9,

$$f(x) = \sum_{k=0}^{a} \frac{f^{(k)}(0)}{k!} x^k + R(x)$$

where $R(x) = O(x^{a+1})$. Then

$$\lambda(f) = \sum_{k=0}^{a} f^{(k)}(0) \cdot \frac{\lambda(x^k)}{k!} + \lambda(R).$$

Note that $f^{(k)}(0) = \partial^k \delta_0 f$ by definition. We just need to show $\lambda(R) = 0$. First,

$$\langle (\partial^k \delta_0)^r, f \rangle = (-1)^k \frac{d^k}{dx^k}\bigg|_{x=0} f(r^{-1}x).$$

Therefore,

$$\langle \partial^k \delta_0, f \rangle = (-1)^k r^{-k} \left(\frac{d^k}{dx^k}\bigg|_{=0} f \right) (r^{-1}x).$$

Let $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a bump function such that

$$\xi|_{(-0.5,0.5)} = 1, \ \xi|_{(-1,1)^c} = 0.$$

Then $R - \xi^{\varepsilon^{-1}} \cdot R \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$, so

$$\lambda(R) = \lambda(\xi^{\varepsilon^{-1}} R) \leq C \max_{b+c\leq a} \sup_{x\in\mathbb{R}} |x^b \partial^c (\xi^{\varepsilon^{-1}} R)(x)|.$$

Using the Leibnitz rule, we can show that this is bounded by $C \cdot \varepsilon^{a+1-c} < C \cdot \varepsilon$.

Step 7. We are done unless $\chi(x) = x^{-k}$. Indeed, then $\mathcal{D}(\mathbb{Q}_p)^\chi \leq 1$, but $Z(s,\chi)$ is a non-trivial element.

Step 8. Consider the case $\chi(x) = x^{-k}$. We need to show that

$$f \mapsto \int_{\mathbb{R}^\times} f(x) x^{-k} \frac{dx}{|x|}$$

on $\mathcal{C}_c^\infty(\mathbb{R}^\times)$ does not lift to $\mathcal{D}(\mathbb{R})^\chi$.

See Remark 4.20 for a general strategy. We treat the case $k = 0$, the other cases are similar. We have

$$f \mapsto 2 \int_0^\infty f_e(x) \frac{dx}{x},$$

where $f_e(x)$ is the even part of $f(x)$. We have the lift λ to $\mathcal{D}(\mathbb{R})$ given by

$$f \mapsto 2 \int_0^1 (f_e(x) - f_e(0)) \frac{dx}{x} + 2 \int_1^\infty f_e(x) dx.$$

Compute

$$\lambda^{r^{-1}}(f) = \int_0^f (f_e(x) - f_e(0)) \frac{dx}{x} + \int_r^\infty f_e(x) \frac{dx}{x}.$$

Therefore,

$$\lambda^{r^{-1}}(f) - \lambda(f) = -f(0) \underbrace{\int_1^r \frac{dx}{x}}_{\neq 0}.$$

Therefore,

$$\lambda^{r^{-1}} = \lambda + C\delta_0,$$

where $C \neq 0$. □

Remark 4.20. General strategy employed in Step 8 of the proof. Suppose we have some integral $Z(f,s)$, defined at some s but potentially at some poles. We want to define a distribution associated to this integral. We can then take the *finite part* at a pole $s = s_0$, $\text{f.p.}_{s=s_0} Z(s,f)$, which is the constant term of the Laurent expansion of $Z(s,f)$ at $s = s_0$.

Remark 4.21. We introduced

$$Z_p(s,\chi,f) = \int_{\mathbb{Q}_p^\times} f(x)\chi(x)|x|_p^s d^\times x$$

for $f \in S(\mathbb{Q}_p)$. This looks like a Fourier transform. First, note that $\chi(x)|x|_p^s$ is not unitary, but we could consider the Fourier transform of $f(x)|x|_p^\times$ evaluated at the character $\chi(x)$.

However, this is the Fourier transform on $\mathbb{Q}_p^\times$, so this only makes sense if $f(x)$ vanishes at $x = 0$.

One could hence think about it as extending the Fourier transform around $x = 0$. However, we really want to think of $|x|_p^s$ as part of the character, not part of the function. Therefore, it is useful to call these zeta integrals, not just an extension of Fourier transforms.

Canonical basis for $\mathcal{D}(\mathbb{Q}_p)^\chi$. First, consider the case $p < \infty$.

- If χ is ramified, define
$$Z_p^0(s, \chi) = Z_p(s, \chi) \in \mathcal{D}(\mathbb{Q}_p)^{\chi \cdot |\cdot|^s}.$$
- If χ is unramified, define
$$Z_p^0(s, \chi)(f) = Z_p(s, \chi)(f - f^{p^{-1}}).$$
Since $f - f^{p^{-1}}$ is 0 at $x = 0$, this function is entire in s.

Fact 4.22. *For $p < \infty$ and χ unramified, $Z_p^0(s, \chi)$ is "natural" in the sense that*
$$Z_p^0(s, \chi)(\mathbb{1}_{\mathbb{Z}_p}) = 1.$$

Proof. We have that
$$\begin{aligned} Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) - Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) &= Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) - \chi(p)p^{-s} Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) \\ &= Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) - \chi(p)p^{-s} Z_p(s,\chi)(\mathbb{1}_{\mathbb{Z}_p}) \\ &= (1 - \chi(p)p^{-s})^{-1} + \chi(p)p^{-s}(1 - \chi(p)p^{-s})^{-1} \quad \text{by Lemma 4.6} \\ &= 1 \end{aligned}$$
This completes the proof. □

Now, consider $p = \infty$. In this case, a character is unramified if it is independent of the sign of $x \in \mathbb{R}^\times$. Take $f^0(x) = e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$ and demand that
$$Z_\infty^0(s, 1)(f^0) = 1.$$
Then
$$Z_\infty^0(s, 1/x)(xf^0) = 1.$$

Local L-factors.

Definition 4.23. Let $p \leq \infty$ and $\chi \colon \mathbb{Q}_p^\times \to S^1$. The *$L$-factor* $L_p(s, \chi)$ is defined as
$$\frac{Z_p(s, \chi)}{Z_p^0(s, \chi)}.$$

In other words, $L_p(s, \chi)$ is the *change of basis* between the basis $Z_p(s, \chi)$ and $Z_p^0(s, \chi)$ of $\mathcal{D}(\mathbb{Q}_p)^\chi$.

Lemma 4.24.

(1) *For $p < \infty$, χ ramified, $L_p(s, \chi) = 1$.*
(2) *For $p < \infty$, χ unramified, $L_p(s, \chi) = Z_p(s, \chi)(\mathbb{1}_{\mathbb{Z}_p}) = (1 - \chi(p)p^{-s})^{-1}$.*

(3) *For* $p=\infty$, $\chi(x)=1$ *or* $\chi(x)=\frac{1}{x}$, *then*

$$L_p(s,\chi)=Z_\infty(s,\chi)(f^0_\chi)=\pi^{-s/2}\Gamma(s/2)$$

where $f^0_{\mathbb{1}}(x)=f^0$, $f^0_\chi=xf^0$ *for* $\chi(x)=\frac{1}{x}$.

(4) *If* $p<\infty$, *then* $L_p(s,\chi)$ *is a generator for the* $\mathbb{C}[p^s,p^{-s}]$*-submodule of* $\mathbb{C}(p^s)$ *given by*

$$\{Z_p(s,\chi)(f)\mid f\in\mathcal{S}(\mathbb{Q}_p)\}.$$

(5) *For any* p,

$$Z^0_p(s,\chi)=\frac{Z_p(s,\chi)}{L_p(s,\chi)}$$

is entire.

Proof. Parts (1) and (2) are clear. For part (3), we compute

$$\begin{aligned}\int_{\mathbb{R}^\times}e^{-\pi x^2}|x|^s\frac{dx}{|x|}&=2\int_0^\infty e^{-\pi x^2}x^s\frac{dx}{x}\\&=\int_0^\infty e^{-\pi x}x^{s/2}\frac{dx}{x}\\&=\pi^{-s/2}\int_0^\infty e^{-x}x^{s/2}\frac{dx}{x}.\end{aligned}$$

Parts (4) and (5) amount to proving that $Z^0_p(s,\chi)$ is entire. For $p<\infty$, this is Lemma 4.6. For $p=\infty$, this follows from (3) and Lemma 4.10. □

Local ϵ-factors and local functional equation. Recall that for $\lambda\in\mathcal{D}(\mathbb{Q}_p)^\chi$, $\hat{\lambda}\in\mathcal{D}(\mathbb{Q}_p)^{\chi^{-1}\cdot|\cdot|}$.

Note that $Z_p(s,\chi),Z^0_p(s,\chi)\in\mathcal{D}(\mathbb{Q}_p)^\chi$.

Definition 4.25. Define

(1) *the* ϵ*-factor* as $\epsilon_p(s,\chi,\psi_p)=\dfrac{\hat{Z}^0_p(1-s,\chi^{-1})}{Z^0_p(s,\chi),}$

(2) *the* γ*-factor* as $\gamma_p(s,\chi,\psi_p)=\dfrac{\hat{Z}_p(1-s,\chi^{-1})}{Z_p(s,\chi)}$.

In other words, $\epsilon_p(s,\chi,\psi_p)$ is the *change of basis* between the basis $\hat{Z}^0_p(1-s,\chi^{-1})$ and $Z^0_p(s,\chi)$ and similarly for $\gamma_p(s,\chi,\psi_p)$.

Remark 4.26. Recall that $\psi_p\colon\mathbb{Q}_p\to\mathbb{C}^\times$ is a non-trivial additive character. It is used to define $\hat{f}$. We had a natural such character, that we denoted by ψ_p. From now on, we will call it ψ^0_p, and ψ_p will be any such.

So far, we specified dx to be the Lebesgue measure when $p=\infty$ and normalized so that $\mathrm{vol}(\mathbb{Z}_p)=1$ when $p<\infty$. We then showed that dx is ψ^0_p-self-dual. Form now on, we take $dx=dx_\psi$, the ψ-self-dual measure.

Fact 4.27.

(1) *The ϵ-factor $\epsilon_p(s,\chi,\psi_p)$ is entire.*
(2) *We have*

$$\gamma_p(s,\chi,\psi_p) = \frac{L_p(1-s,\chi^{-1})}{L_p(s,\chi)}\epsilon_p(s,\chi,\psi_p).$$

(3) *Local functional equation:*

$$Z_p(1-s,\hat{f},\chi^{-1}) = \gamma_p(s,\chi,\psi_p)Z_p(s,f,\chi),$$
$$Z_p^0(1-s,\hat{f},\chi^{-1}) = \epsilon_p(s,\chi,\psi_p)Z_p^0(s,f,\chi).$$

Proof. This is immediate from the definitions. □

Lemma 4.28. *We have that*

(1) $\epsilon_p(s,\chi|\cdot|^t,\psi_p) = \epsilon_p(s+t,\chi,\psi_p)$,
(2) $\epsilon_p(s,\chi,\psi_p^a) = |a|_p^{s-\frac{1}{2}}\chi(a)\epsilon_p(s,\chi,\psi_p)$.

Proof. Part (1) follows from the definition. For part (2), write $\hat{f}_{\psi,dx}$ for the Fourier transform formed with respect to the character ψ and measure dx. Then

$$\hat{f}_{\psi^a,dx} = (\hat{f}_{\psi,dx})^a,$$
$$\hat{f}_{\psi,|a|dx} = |a|\hat{f}_{\psi,dx}.$$

Write $\hat{f}_\psi = \hat{f}_{\psi,dx_\psi}$. Then

$$dx_{\psi^a} = |a|_p^{1/2}dx_\psi,$$
$$\hat{f}_{\psi^a} = |a|_p^{1/2}(\hat{f}_\chi)^a.$$

Compute

$$\frac{Z_p^0(1-s,\chi^{-1},dx_{\psi^a})(\hat{f}_{\psi^a})}{Z_p^0(s,\chi,dx_{\psi^a})(f)} = \frac{Z_p^0(1-s,\chi^{-1},dx_\psi)(|a|_p^{1/2}\hat{f}_\psi^a)}{Z_p^0(s,\chi,dx_\psi)(f)} = |a|_p^{1/2}\chi(a)|a|_p^{s-1}\epsilon_p(s,\chi,\psi).$$

This completes the proof of (2). □

Lemma 4.29. *The factor $\epsilon_p(s,\chi,\psi_p)$ can be computed explicitly as follows:*

(1) *when $p=\infty$,*

$$\epsilon_\infty(s,\mathbb{1},\psi_\infty^0) = 1$$
$$\epsilon_\infty(s,\frac{1}{x},\psi_\infty^0) = i,$$

(2) *when $p<\infty$, χ is unramified, $\epsilon_p(s,\chi,\psi_p^0)=1$,*
(3) *when $p<\infty$, χ is ramified and specifically trivial on U_p^c, non-trivial on U_p^{c-1} where $U_p^c = 1+p^c\mathbb{Z}_p$ when $c>0$ and $U_p^0=\mathbb{Z}_p^\times$, then*

$$\epsilon_p(s,\chi,\psi_p^0) = \chi(p^c)p^{c(s-\frac{1}{2})}g(\chi,\psi_p^0)$$

where the Gauss sum

$$g(\chi,\psi_p^0) = p^{-\frac{1}{2}c}\int_{\mathbb{Z}_p^\times}\chi(y)^{-1}\psi_p^0(p^{-c}y)dy$$

is a square root of $\chi(-1)$. *(It is a subtle thing to determine which square root it is.)*

Proof. We can plug in any test function into $\hat{Z}_p^0(1-s,\chi^{-1})/Z_p^0(s,\chi)$ (since this is a constant). For (1), we plug in f_χ^0. We check $\hat{f}_{\mathbb{1}}^0 = f_1^0 \mathbb{1}$ In the other case,

$$\hat{f}_\chi^0 = \widehat{xf_{\mathbb{1}}^0} = -2\pi i \frac{d}{dx}\hat{f}_{\mathbb{1}}^0 = -2\pi i \frac{d}{dx} f_{\mathbb{1}}^0 = ixf_{\mathbb{1}}^0 = if_\chi^0.$$

This proves the result, given that $Z_\infty^0(s,\chi^{-1}) = Z_\infty^0(s+2,\chi)$.

In (2), take $f^0 = \mathbb{1}_{\mathbb{Z}_p}$ and use $\hat{f}^0 = f^0$.

Finally, in (3), take

$$f(x) = \begin{cases} \chi(x)^{-1}, & x \in \mathbb{Z}_p^\times, \\ 0, & \text{otherwise.} \end{cases}$$

Then $Z_p^0(s,\chi)(f) = Z_p(s,\chi)(f) = 1$. Recall that

$$\hat{f}(y) = \int_{\mathbb{Z}_p^\times} \chi(y)^{-1}\psi(xy)dy.$$

Note that dy is the additive Haar measure but on $\mathbb{Z}_p^\times$ gives a multiplicative Haar measure. We claim that $\hat{f}$ is supported on $p^{-c}\mathbb{Z}_p$. Write $x = p^{-k}x_0$ for $x_0 \in \mathbb{Z}_p^\times$. If $k > c$, then

$$\hat{f}(x) = \int_{y\in\mathbb{Z}_p^\times/(1+p^c\mathbb{Z}_p)} \chi(y)^{-1} \int_{z\in 1+p^c\mathbb{Z}_p} \psi(xyz)dzdy.$$

Write $z = 1 + p^c z_0$ for $z_0 \in \mathbb{Z}_p$. Then

$$\psi(xyz) = \psi(xz(1+p^cz_0))\psi(xy)\cdot\psi(x_0yz_0p^{c-k}).$$

Therefore:

$$\begin{aligned}\hat{f}(x) &= \int_{y\in\mathbb{Z}_p^\times/(1+p^c\mathbb{Z}_p)} \chi(y)^{-1}\psi(xy) \underbrace{\int_{z_0\in\mathbb{Z}_p} \psi(x_0yp^{c-k}z_0)dz_0}_{=0} dy \\ &= 0,\end{aligned}$$

because $x_0y \in U_p^+$, $p^{c-k} \notin \mathbb{Z}_p$, so the character

$$z_0 \mapsto \psi(x_0yp^{c-k}z_0)$$

is non-trivial on $\mathbb{Z}_p$.

If $k < c$, then

$$\hat{f}(x) = \int_{y\in\mathbb{Z}_p^\times/U_p^{c-1}} \int_{z\in U_p^{c-1}} \psi(xyz)\chi(yz)^{-1}dzdy,$$

and $\psi(xyz) = \psi(xy)\psi(x_0 y p^{c-1-k} z_0)$ where $x_0 y p^{c-1-k} z_0 \in \mathbb{Z}_p$. Therefore,

$$\hat{f}(x) = \int_{y \in \mathbb{Z}_p^\times / U_p^{c-1}} \psi(xy)\chi(y)^{-1} \underbrace{\int_{z \in U_p^{c-1}} \chi(z)^{-1} dz}_{=0} \, dy.$$

Finally, if $k = c$, then

$$\begin{aligned} \hat{f}(x) &= \int_{\mathbb{Z}_p^\times} \chi(y)^{-1}\psi(p^{-c}x_0 y) dy \\ &= \underbrace{\chi(x_0)}_{\overline{f(p^{-c}c)}} \underbrace{\int_{\mathbb{Z}_p^\times} \chi(y)^{-1}\psi(p^{-c}y) dy}_{\text{call this } G}. \end{aligned}$$

Therefore, for all $x \in \mathbb{Q}_p$,

$$\hat{f}(x) = \overline{f(p^{-c}x)} \cdot G \tag{1}$$

(Here, G is the unnormalized Gauss sum.)

We can use Fourier inversion on f, we get that

$$G^2 = p^c \chi(-1).$$

Indeed,

$$\overline{f}(x) = \begin{cases} \chi(x), & x \in \mathbb{Z}_p^\times, \\ 0, & \text{otherwise.} \end{cases}$$

and $\chi^{-1}(x)$ also has conductor p^c if $\chi(x)$ does. Therefore, the formula (1) applies to $\overline{f}$ and yields:

$$\hat{\overline{f}}(x) = f(p^{-c}x) \cdot G. \tag{2}$$

Finally, we get that for $x \in \mathbb{Z}_p^\times$

$$\begin{aligned} f(-x) &= \hat{\hat{f}}(x) && \text{Fourier inversion} \\ &= \widehat{\overline{f(p^{-c}x)}} \cdot G && \text{equation (1)} \\ &= \widehat{\overline{f}^{p^{-c}}}(x) \cdot G \\ &= p^{-c}\hat{\overline{f}}(p^c x) \cdot G && \widehat{f^a} = |a|^{-1}(\hat{f})^{a^{-1}} \\ &= p^{-c} \cdot f(x) \cdot G^2 && \text{equation 2} \\ &= p^{-c} \cdot G^2 \cdot \chi(-1)^{-1} f(-x) && \text{as } f(x) = \chi(x) \text{ for } x \in \mathbb{Z}_p^\times. \end{aligned}$$

This shows that

$$G^2 = p^c \chi(-1)$$

and so $g = p^{-c/2} G$ satisfies $g^2 = \chi(-1)$.

Moreover,

$$\epsilon(s,\chi,\psi_p^0) = Z_p(1-s,\chi^{-1})(\hat{f}) = Z_p(1-s,\chi^{-1})(G\cdot\overline{f^{p^{-c}}}) = G\cdot\chi(p^c)p^{c(s-1)}\underbrace{Z_p(1-s,\chi^{-1},\bar{f})}_{=1}.$$

This completes the proof. □

Analytic continuation and functional equation of the global zeta integral. We define the *global zeta integral* to be

$$Z(s,\chi,f) = \int_{\mathbb{A}^\times} f(x)\chi(x)|x|^s d^\times x.$$

Theorem 4.30.

(1) *The function $Z(s,\chi,f)$ is meromorphic on all $s\in\mathbb{C}$.*
(2) *If $\chi(x) = |x|^t$, then $Z(s,\chi,f)$ has a simple pole at $s=-t$ with residue $-f(0)\cdot\mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times,d^\times x)$, and at $s=1-t$ with residue $\hat{f}(0)\cdot\mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times,d^\times x)$.*[1]
(3) *For all other s, $Z(s,\chi,f)$ is holomorphic.*
(4) *The functional equation holds:*

$$Z(s,\chi,f) = Z(1-s,\chi^{-1},\hat{f}).$$

Proof. We integrate in stages, according to

$$1 \longrightarrow \mathbb{A}^1 \longrightarrow \mathbb{A}^\times \xrightarrow{|\cdot|} \mathbb{R}_{>0} \longrightarrow 1$$

and obtain

$$\int_{r\in\mathbb{R}_{>0}} |r|^s\chi(r)\int_{a\in\mathbb{A}^1}\chi(a)f(ra)d^\times a d^\times r.$$

Depending on s, $|r|^s$ explodes as $r\to\infty$ or $r\to 0$. Since $f\in\mathcal{S}(\mathbb{A})$, the explosion for $r\to\infty$ is controlled.

We split the integral over $(0,\infty)$ into $(0,1)\cup(1,\infty)$, and worry about $(0,1)$. To treat $(0,1)$, we split further according to

$$1 \longrightarrow \mathbb{Q}^\times \longrightarrow \mathbb{A}^1 \longrightarrow \mathbb{A}^1/\mathbb{Q}^\times \longrightarrow 1$$

and get

$$\int_{r=0}^{r=1} |r|^s\chi(r)\int_{\overline{a}\in\mathbb{A}^1/\mathbb{Q}^\times}\chi(\overline{a})\sum_{q\in\mathbb{Q}^\times} f(rq\overline{a})d^\times\overline{a}d^\times r.$$

[1]We normalized the Haar measure $d^\times x$ so that the volumes are 1. However, if you replace $\mathbb{Q}$ with another number field, these volumes will give a non-trivial contribution.

By (the Corollary to) Poisson summation formula 3.23, this is equal to

$$\int_{r=0}^{r=1} |r|^s \chi(r) \int_{\overline{a}\in\mathbb{A}^1/\mathbb{Q}^\times} \chi(\overline{a}) \left(f(0) + r^{-1} \left(\hat{f}(0) + \sum_{q\in\mathbb{Q}^\times} \hat{f}(r^{-1}\overline{a}^{-1}q) \right) \right) d^\times \overline{a} d^\times r.$$

This is equal to

$$\begin{aligned} &\int_{r=0}^{r=1} |r|^{s-1}\chi(r) \int_{\overline{a}\in\mathbb{A}^1/\mathbb{Q}^\times} \chi(\overline{a}) \left(\sum_{q\in\mathbb{Q}^\times} \hat{f}(r^{-1}\overline{a}^{-1}q) \right) d^\times \overline{a} d^\times r \\ &\qquad - f(0) \int_{r=0}^{r=1} r^s \chi(r) \int_{\overline{a}\in\mathbb{A}^1\backslash\mathbb{Q}^\times} \chi(\overline{a}) d^\times a d^\times r \\ &\qquad + \hat{f}(0) \int_{r=0}^{r=1} r^{s-1}\chi(r) \int_{\overline{a}\in\mathbb{A}^1\backslash\mathbb{Q}^\times} \chi(\overline{a}) d^\times a d^\times r. \end{aligned}$$

In the first time, substitute $r \mapsto r^{-1}$ and $\overline{a} \mapsto \overline{a}^{-1}$ to get

$$\int_{r=1}^{\infty} r^{1-s}\chi(r)^{-1} \int_{\overline{a}\in\mathbb{A}^1/\mathbb{Q}^\times} \chi(\overline{a})^{-1} \sum_{q\in\mathbb{Q}^\times} \hat{f}(r\overline{a}q) d^\times \overline{a} d^\times r.$$

The inner integral in the second line is equal to 0 unless $\chi|_{\mathbb{A}^1} = 1$, i.e. $\chi(x) = |x|^t$, in which case it is equal to $\mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times)$, and we get

$$-\mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times) \cdot f(0) \cdot \underbrace{\int_{r=0}^{1} r^{s+t} d^\times r}_{\frac{1}{s+t}}.$$

Similarly, the third line is equal to

$$\hat{f}(0) \cdot \mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times) \cdot \frac{1}{s-1+t}.$$

We collect everything together to get that $Z(s, f, \chi)$ is equal to

$$\begin{aligned} &\int_{r=1}^{\infty} \left(r^s \int_{a\in\mathbb{A}^1} \chi(ra) f(ra) d^\times a \right) + \left(r^{1-s} \int_{a\in\mathbb{A}^1} \chi^{-1}(ra) \hat{f}(ra) d^\times a \right) d^\times r \\ &\qquad - \mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times) \cdot f(0) \frac{1}{s+t} + \mathrm{vol}(\mathbb{A}^1/\mathbb{Q}^\times) \cdot \hat{f}(0) \frac{1}{s-1+t}. \end{aligned}$$

The first line is holomorphic, and the entire expression is invariant under $s \mapsto 1-s$, $\chi \mapsto \chi^{-1}$, $f \mapsto \hat{f}$. □

Adelic eigendistributions. Recall that:

- a tempered distribution on $\mathbb{R}$ is a continuous linear functional $\mathcal{S}(R) \to \mathbb{C}$.
- a (tempered) distribution on $\mathbb{Q}_p$ is any linear function $\mathcal{S}(\mathbb{Q}_p) = \mathcal{C}_c^\infty(\mathbb{Q}_p) \to \mathbb{C}$.

There is a canonical topology on $\mathcal{S}(\mathbb{Q}_p)$ for $p < \infty$. Note that

$$\mathcal{C}_c^\infty(\mathbb{Q}_p) = \varinjlim_{H_p' \subseteq H_p} \mathcal{C}(H_p/H_p')$$

where $H_p' \subseteq H_p$ run over lattices on $\mathbb{Q}_p$. Since H_p/H_p' is finite, so $\mathcal{C}(H_p/H_p')$ is a finite-dimensional $\mathbb{C}$-vector space, so it has a canonical topology. We get a colimit topology on $\mathcal{C}_c^\infty(\mathbb{Q}_p)$. With respect to this topology, every linear functional is continuous.

Recall that

$$\mathcal{S}(\mathbb{A}) = \bigotimes_{p \leq \infty}{}' \mathcal{S}(\mathbb{Q}_p).$$

Equivalently,

$$\mathcal{S}(\mathbb{A}) = \varinjlim_{\substack{H' \subseteq H \subseteq \mathbb{A}_{\text{fin}} \\ \hat{\mathbb{Z}}\text{-lattices}}} \mathcal{S}(\mathbb{R} \times (H/H')).$$

We have isomorphisms

$$\mathcal{S}(\mathbb{R} \times H/H') \xrightarrow{\cong} \prod_{x \in H/H'} \mathcal{S}(\mathbb{R}) \xrightarrow{\cong} \mathcal{S}(\mathbb{R}) \otimes \mathcal{C}(H/H')$$

$$f \longrightarrow (f(-,x))_{x \in H/H'}$$

$$(f_x)_{x \in H/H'} \longrightarrow \textstyle\sum_{x \in H/H'} f_x \otimes \delta_x$$

Put the product topology of $\prod_{x \in H/H'} \mathcal{S}(\mathbb{R})$ on $\mathcal{S}(\mathbb{R} \times (H/H'))$, and take the colimit topology on $\mathcal{S}(\mathbb{A})$.

Definition 4.31. A *tempered distribution* on $\mathbb{A}$ is a continuous linear function on $\mathcal{S}(\mathbb{A})$. The set of tempered distributions is denote $\mathcal{D}(\mathbb{A})$.

Lemma 4.32. *Let $(\lambda_p)_{p \leq \infty}$ be a collection of $\lambda_p \in \mathcal{D}(\mathbb{Q}_p)$ such that $\lambda_p(f_p^0) = 1$ for almost all p. Define $\lambda \colon \mathcal{S}(\mathbb{A}) \to \mathbb{C}$ by*

$$\lambda(\otimes f_p) = \prod_{p \leq \infty} \lambda_p(f_p).$$

Then $\lambda \in \mathcal{D}(\mathbb{A})$ and any element of $\lambda \in \mathcal{D}(\mathbb{A})$ is of this form.

Proof. To show that $\lambda = \otimes \lambda_p$ is continuous, restrict to

$$\mathcal{S}(\mathbb{R} \times (H/H')) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}(H/H').$$

Then $\lambda = \lambda_\infty \otimes \lambda_{\text{fin}}$, and λ_{fin} must be a linear combination of δ-distributions. So in terms of

$$\mathcal{S}(\mathbb{R} \times (H/H')) = \prod \mathcal{S}(\mathbb{R})$$

we see that λ is

$$\sum_{x \in H/H'} c_x \lambda_\infty$$

for $c_x \in \mathbb{C}$, so it is continuous.

Let $\lambda \in \mathcal{D}(\mathbb{A})$ be arbitrary non-zero. Let $f = \bigotimes f_p$ such that $\lambda(f) = 1$. For each p, define $f^p = \bigotimes_{q \neq p} f_q$. Define

$$\lambda_p(g_p) = \lambda(g_p \otimes f^p).$$

Then $\lambda_p \in \mathcal{D}(\mathbb{Q}_p)$ for $p < \infty$, and for $p = \infty$, we just need to check continuity. There are lattices $H' \subseteq H \subseteq \mathbb{A}_{\text{fin}}$ such that

$$f^\infty \in \mathcal{C}(H/H').$$

The map

$$\begin{aligned} S(\mathbb{R}) &\to \mathcal{S}(\mathbb{R}) \otimes \mathcal{C}(H/H') \\ g_\infty &\mapsto g_\infty \otimes f^\infty \end{aligned}$$

is continuous, and hence so is λ_∞.

Moreover, $\lambda_p(f_p) = \lambda(f) = 1$. For almost all p, $f_p = f_p^0$, and thus (λ_p) satisfies the assumption.

Exercise. Check that $\lambda = \bigotimes \lambda_p$. □

Corollary 4.33. *Let $\chi\colon \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be a Hecke character. Then*

$$\mathcal{D}(\mathbb{A})^\chi = \bigotimes_{p \leq \infty} \mathcal{D}(\mathbb{Q}_p)^{\chi_p}$$

is one-dimensional.

In particular, we have $Z(s,\chi), Z^0(s,\chi) \in \mathcal{D}(\mathbb{A})^\chi$, and

$$\begin{aligned} Z(s,\chi) &= \bigotimes_p Z_p(s,\chi_p), \\ Z^0(s,\chi) &= \bigotimes_p Z_p^0(s,\chi_p). \end{aligned}$$

Definition 4.34. We define the *completed L-function* and the *global ϵ-factor* as:

$$\begin{aligned} \Lambda(s,\chi) &= Z(s,\chi)/Z^0(s,\chi), \\ \epsilon(s,\chi) &= \hat{Z}^0(1-s,\chi^{-1})/Z^0(s,\chi). \end{aligned}$$

Fact 4.35. *We have that:*

(1)

$$\begin{aligned} \Lambda(s,\chi) &= \prod_p L_p(s,\chi_p), \\ \epsilon(s,\chi) &= \prod_p \epsilon_p(s,\chi_p,\psi_p). \end{aligned}$$

(2) *$\epsilon(s,\chi)$ does not depend on ψ,*

(3) $\epsilon(s,\chi)$ *is entire,*
(4) $\Lambda(s,\chi)$ *has simple poles at* $s=-t$ *and* $s=1-t$ *when* $\chi=|\cdot|_{\mathbb{A}}^t$.

Proof. For (2), note that ψ_p is defined up to a rational unit and scaling ψ_p by an element of $\mathbb{Q}^\times$ multiplies by a factor.

For (3), note that $Z^0(s,\chi)$ is entire and has no zeros.

Part (4) follows simply from Theorem 4.30. □

Theorem 4.36 (Functional equation)**.** *We have that*

$$\epsilon(s,\chi)\Lambda(1-s,\chi^{-1})=\Lambda(s,\chi).$$

Proof. Immediate from Theorem 4.30. □

This completes Tate's thesis.

5. Artin L-functions

Review of algebraic number theory. Let K be a number field (finite extension of $\mathbb{Q}$). Let $\mathcal{O}_K\subseteq K$ be the ring of integers (integral closure of $\mathbb{Z}$ in K). It is a *Dedekind domain.* Any prime ideal $\mathfrak{p}\subseteq\mathcal{O}_K$ is maximal and we define

$$k(\mathfrak{p})=\mathcal{O}_K/\mathfrak{p},\quad N(\mathfrak{p})=|k(\mathfrak{p})|.$$

Being an Dedekind domain then implies that any ideal $I\subseteq\mathcal{O}_K$ can be written (uniquely) as

$$I=\prod_{\mathfrak{p}}\mathfrak{p}^{n_{\mathfrak{p}}}.$$

If L/K is any finite extension and $\mathfrak{p}_K\subseteq\mathcal{O}_K$, then

$$\mathfrak{p}_K\cdot\mathcal{O}_L=\prod_{\mathfrak{p}_L}\mathfrak{p}_L^{n_{\mathfrak{p}_L}}.$$

We say that $\mathfrak{p}_L|\mathfrak{p}_K$ if $n_{\mathfrak{p}_L}>0$, i.e. $\mathfrak{p}_K\subseteq\mathfrak{p}_L$.

If L/K is Galois with Galois group G, then G acts transitively on $\{\mathfrak{p}_L|\mathfrak{p}_K\}$. Hence:

(1) In $\mathcal{O}_L$, $\mathfrak{p}_K=\mathfrak{p}_{L,1}^e\cdot\dots\cdot\mathfrak{p}_{L,k}^e$ and we say that $\mathfrak{p}_K$ ramified if $e>1$; we call e the *ramification degree.*
(2) The finite fields $k(\mathfrak{p}_L)$ for $\mathfrak{p}_L|\mathfrak{p}_K$ are all isomorphic, of degree f, called the *inertial degree.*
(3) With the above notation, $[L:K]=e\cdot f\cdot k$.
(4) Let $G_{\mathfrak{p}_L}$ be the stabilizer of $\mathfrak{p}_L|\mathfrak{p}_K$. These subgroups of G are all conjugate. The action of $\mathbb{G}_{\mathfrak{p}_L}$ on $k(\mathfrak{p}_L)$ gives a short exact sequence

$$1\longrightarrow I(\mathfrak{p}_L)\longrightarrow G_{\mathfrak{p}_L}\longrightarrow \mathrm{Gal}(k(\mathfrak{p}_L)/k(\mathfrak{p}_K))\longrightarrow 1.$$

Here, $I(\mathfrak{p}_L)$ is the *inertia group*, which it has degree e. Moreover, $\mathrm{Gal}(k(\mathfrak{p}_L)/k(\mathfrak{p}_K))$ is canonically generated by the *Frobenius element* $\mathrm{Frob}_{\mathfrak{p}_L}$.

We can form the completion $L_{\mathfrak{p}_L}$ (either topologically, with respect to the valuation given by $\mathfrak{p}_L$, or algebraically, as the quotient ring of $\mathcal{O}_{L,\mathfrak{p}_L}$ where $\mathcal{O}_{L,\mathfrak{p}_L} = \varprojlim \mathcal{O}_L/\mathfrak{p}_L^n)$. The action $G_{\mathfrak{p}_L}$ on L extends to a continuous action of $G_{\mathfrak{p}_L}$ on $L_{\mathfrak{p}_L}$, and gives

$$G_{\mathfrak{p}_L} \cong \mathrm{Gal}(L_{\mathfrak{p}_L}/K_{\mathfrak{p}_K}).$$

Any prime $\mathfrak{p}_K$ of K is called a *finite place of* K. An *infinite place of* K is either $j_K \colon K \hookrightarrow \mathbb{R}$ (a *real place*) or a pair $(j_K, \overline{j_k})$ with $j_K \colon K \hookrightarrow \mathbb{C}$ (a *complex place*).

If j_L is an infinite place of L, we say $j_L | j_K$ if and only if $j_L|_K = j_K$. Note that G acts on $\{j_L | j_K\}$ transitively and

$$G_{j_L} \cong \mathrm{Gal}(L_{j_L}/K_{j_K}).$$

Notation. If v is a place,

$$K_v \cong \begin{cases} K_{\mathfrak{p}_K} & v \leftrightarrow \mathfrak{p}_K, \\ \mathbb{R} & v \leftrightarrow j_K \text{ real}, \\ \mathbb{C} & v \leftrightarrow (j_K, \overline{j_K}) \text{ complex.} \end{cases}$$

Then

$$G_{v_L} \cong \mathrm{Gal}(L_{v_L}/K_{v_K}).$$

Artin L-functions. Let K be a number field. Let $G_K = \mathrm{Gal}(\overline{K}/K)$ and consider a representation $\rho \colon G_k \to \mathrm{GL}(V)$ for a finite-dimensional $\mathbb{Q}$-vector-space V.

Since $\mathrm{GL}(V)$ has no small subgroups, there is a finite Galois extension E/K such that ρ factors as

$$\rho \colon G_K/G_E \to \mathrm{GL}(V).$$

For an unramified prime $\mathfrak{p} \subseteq \mathcal{O}_K$, define

$$L_{\mathfrak{p}}(s, \rho) = \det(I - N(\mathfrak{p})^{-s} \rho(\mathrm{Frob}_{\mathfrak{P}}))^{-1}$$

where $\mathfrak{P} \subseteq \mathcal{O}_E$ and $\mathfrak{P}|\mathfrak{p}$. Note that this is independent of the choice of $\mathfrak{P}$, because $\mathrm{Frob}_{\mathfrak{P}}$ is well-defined up to conjugation and we are taking the determinant.

If $\rho = \chi$ is a character and $K = \mathbb{Q}$, then this gives

$$(1 - p^{-s}\chi(p))^{-1}$$

which agrees with our previous definition of a local L-factor.

When $\mathfrak{p}$ is (possibly) ramified, we define

$$L_{\mathfrak{p}}(s, \rho) = \det(I - N(\mathfrak{p})^{-s} \rho(\mathrm{Frob}_{\mathfrak{P}}) \mid V^{I(\mathfrak{P})})^{-1}.$$

This agrees with the previous definition when $I(\mathfrak{P}) = \{1\}$, i.e. $\mathfrak{p}$ is unramified.

The *global Artin L-function* is defined to be

$$L(s, \rho) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} L_{\mathfrak{p}}(s, \rho).$$

Lemma 5.1. *Each local factor is analytic for* $\mathrm{Re}(s) > 0$ *and the product converges when* $\mathrm{Re}(s) < 1$.

Proof. Note that $L_{\mathfrak{p}}(s,\rho)^{-1} = 0$ if and only if there is an eigenvalue of $\rho(\mathrm{Frob}_{\mathfrak{p}})$ equal to $N(\mathfrak{p})^s$. But ρ is unitary with respect to a suitable scalar product on V, so all eigenvalues have absolute value 1.

Let $z_1(\mathfrak{p}),\dots,z_n(\mathfrak{p})$ be the eigenvalues. Then

$$L(s,\rho) = \prod_{\mathfrak{p}} \prod_{i=1}^{n} (1 - N(\mathfrak{p})^{-s} z_i).$$

Fact. If $0 \le x_n < 1$, then

$$\prod (1-x_n) \text{ converges if and only if } \sum x_n \text{ converges.}$$

Therefore, we want to know if

$$\sum_{\mathfrak{p}} N(\mathfrak{p})^{-s} \underbrace{\sum_{i=1}^{n} z_i(\mathfrak{p})}_{\le n}$$

converges. Note that $\#\{\mathfrak{p}|p\} \le [K:\mathbb{Q}]$ and $N(\mathfrak{p})^{-s} \le p^{-s}$. This proves convergence by the convergence of the Riemann zeta function. □

Fact 5.2. *The local L-factor $L_{\mathfrak{p}}(s,\rho)$ depends only on $\rho|_{G_{\mathfrak{p}}}$ for one $\mathfrak{P}|p$.*

Therefore, we can simply define for K local field and $\rho\colon G_K \to \mathrm{GL}(V)$:

$$L(s,\rho) := \det(I - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_{\mathfrak{P}}) \mid V^{I(\mathfrak{P})})^{-1}.$$

Then the global L-function is

$$L(s,\rho) = \prod_{\mathfrak{p}} L(s,\rho|_{G_{\mathfrak{p}}}).$$

Fact 5.3. *The function $L(s,\rho)$ is independent of the choice of E.*

Proof. If $E'/E/K$ and $\mathfrak{P}'|\mathfrak{P}|\mathfrak{p}$, then

$$\begin{array}{ccccc} I(\mathfrak{P}') & \hookrightarrow & \mathrm{Gal}(E'/K) & & \mathrm{Frob}_{\mathfrak{P}'} \\ \downarrow & & \downarrow & & \downarrow \\ I(\mathfrak{P}) & \hookrightarrow & \mathrm{Gal}(E/K) & & \mathrm{Frob}_{\mathfrak{P}}. \end{array}$$

This completes the proof. □

Fact 5.4. *We have that $L(s,\rho_1\oplus\rho_2) = L(s,\rho_1)\cdot L(s,\rho_2)$.*

Corollary 5.5. *The L-fucntion $L(s,\rho)$ makes sense for virtual representation ρ, i.e. a formal $\mathbb{Z}$-linear combination of irreducible representations of G_K.*

Completed Artin L-functions. Given a finite Galois extension L/K of archimedean local fields, $G_{L/K}$ is $\{1\}$ or $\mathbb{Z}/2\mathbb{Z}$. Since the Artin L-function is supposed to be additive, we just need to define the local L-factor at infinite for the trivial representations and the non-trivial representation of $\mathbb{Z}/2\mathbb{Z}$.

Define

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2),$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

For an irreducible representation ρ of $\mathrm{Gal}(L/K)$, we define

$$L(s,\rho) = \begin{cases} \Gamma_{\mathbb{R}}(s) & K = \mathbb{R}, \rho = 1, \\ \Gamma_{\mathbb{R}}(s+1) & K = \mathbb{R}, \rho = \mathrm{sgn}, \\ \Gamma_{\mathbb{C}}(s) & K = \mathbb{C}. \end{cases}$$

By definition, $L(s,\rho)$ is additive and independent of L.

Definition 5.6. The *completed Artin L-function* is

$$\Lambda(s,\rho) = \prod_v L(s, \rho|_{G_v}).$$

Corollary 5.7. *The completed Artin L-function Λ is additive and independent of E.*

Review of representation theory. If $H \subseteq G$ are finite groups and $\rho_H \colon H \to \mathrm{GL}(V)$ is a finite-dimensional $\mathbb{C}$-representation, then

$$\mathrm{Ind}_H^G \rho_H = \{f \colon G \to V \mid f(hg) = \rho_H(h) f(g)\}.$$

This is a representation of G via the action by right translation.

Fact 5.8 (Transitivity)**.** *If $H_1 \subseteq H_2 \subseteq G$, then*

$$\mathrm{Ind}_{H_1,H_2}^G \rho_{H_1} = \mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} \rho_{H_1}$$

We also have $\mathrm{Res}_H^G \pi$ for π a representation of G. We have a H-representation homomorphism:

$$\begin{aligned} \mathrm{ev}_1 \colon \mathrm{Res}_H^G \mathrm{Ind}_H^G V &\to V \\ f &\mapsto f(1) \end{aligned}$$

Fact 5.9 (Frobenius reciprocity)**.** *We have natural isomorphism*

$$\begin{aligned} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) &\xrightarrow{\cong} \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \rho) \\ F &\mapsto \mathrm{ev}_1 \circ F. \end{aligned}$$

Fact 5.10 (Mackey formula)**.** *Consider two subgroups $H_1, H_2 \subseteq G$ and a finite-dimensional $\mathbb{C}$-representation $\rho\colon H_1 \to \mathrm{GL}(V)$. Then*

$$\begin{aligned}\mathrm{Res}^G_{H_2} \mathrm{Ind}^G_{H_1} \rho &= \bigoplus_{g \in H_2\backslash G/H_1} \mathrm{Ind}^{H_1}_{gH_1g^{-1}\cap H_2} \mathrm{Res}^{gH_1g^{-1}}_{gH_1g^{-1}\cap H_2}(\rho^g)\\ &= \bigoplus_{g\in H_1\backslash G/H_2} \left(\mathrm{Ind}^{gH_2g^{-1}}_{H_1\cap gH_2g^{-1}} \mathrm{Res}^{H_1}_{H_1\cap gH_2g^{-1}} \rho\right)^{g^{-1}},\end{aligned}$$

where ρ^g is the representation of gH_1g^{-1} given by $\rho \circ \mathrm{Ad}(g^{-1})$ and

$$\begin{aligned}\mathrm{Ad}(g)\colon H_1 &\to gH_1g^{-1},\\ x &\mapsto gxg^{-1}.\end{aligned}$$

Proof. Exercise. □

Theorem 5.11 (Brauer induction)**.** *The Grothendieck group of virtual representations of G is generated by the elements $\mathrm{Ind}^G_H \chi$, where $H \subseteq G$ is a subgroup and χ is a 1-dimensional representation of H.*

Inductivity of Artin L-functions. Recall that

$$\begin{aligned}L(s,\rho) &= \prod_{v \text{ finite}} L(s, \mathrm{Res}^{G_K}_{G_{K_{\dot v}}} \rho)\\ \Lambda(s,\rho) &= \prod_{v \text{ all places}} L(s, \mathrm{Res}^{G_K}_{G_{K_{\dot v}}} \rho),\end{aligned}$$

where $\dot v$ is any place of E above v and the representation ρ factors through the finite extension E.

Proposition 5.12. *Let L/K be a finite Galois extension of local fields or number fields and let $\rho_L\colon G_L \to \mathrm{GL}(V)$ be a finite-dimensional $\mathbb{C}$-representation. Then*

$$L(s, \mathrm{Ind}^{G_K}_{G_L} \rho_L) = L(s, \rho_L).$$

In the global case, also

$$\Lambda(s, \mathrm{Ind}^{G_K}_{G_L} \rho_L) = \Lambda(s, \rho_L).$$

Proof. **Step 1:** Reduce the global statement to the local statement. Fix a large Galois extension E/L and write G_K, G_L for $\mathrm{Gal}(E/K)$ and $\mathrm{Gal}(E/L)$. Moreover, write

$$\rho_K = \mathrm{Ind}^{\mathrm{Gal}(E/K)}_{\mathrm{Gal}(E/L)} \rho_L.$$

It is enough to show that for a given v, a place of K,

$$L(s, \mathrm{Res}^{G_K}_{G_{K,\dot v}} \rho_K) = \prod_{w|v} L(s, \mathrm{Res}^{G_L}_{G_{L,\dot w}} \rho_L).$$

Indeed, one can compare the Euler factors on both sides. As above $\dot v$ and $\dot w$ are any places of E above v and w.

By Mackey formula 5.10,

$$\operatorname{Res}^{G_K}_{G_{K,\dot v}} \operatorname{Ind}^{G_K}_{G_L} \rho_L = \bigoplus_{g\in G_L\backslash G_K/G_{K,\dot v}} \left(\operatorname{Ind}^{G_{K,g\dot v}}_{G_{K,g\dot v}\cap G_L} \operatorname{Res}^{G_L}_{G_{K,g\dot v}\cap G_L} \rho_L\right)^{g^{-1}}.$$

Moreover, $G_{K,g\dot v}\cap G_L = G_{L,g\dot v}$ and

$$\begin{aligned} G_L\backslash G_K/G_{K,\dot v} &\to \{w \mid v\} \\ g &\mapsto p(g\dot v) \end{aligned}$$

where p is the restriction map $p\colon \{\text{places of } E\} \to \{\text{places of } L\}$. We are now done by additivity and inductivity in the local case.

Step 2: Local archimedean case. The only possibility is

$$K=\mathbb{R},\ L=\mathbb{C}, \rho_L = 1,$$

and

$$\rho_K = \operatorname{Ind} 1_{G_L} = 1_{G_K}\oplus \operatorname{sgn}.$$

Then

$$L(s,\rho) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \pi^{-s/2}\Gamma(s/2)\pi^{-s/2-1/2}\Gamma(s/2+1/2).$$

By the duplication formula ($\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$) for the Gamma function, we get that

$$L(s,\rho) = \Gamma_{\mathbb{C}}(s).$$

Step 3: Local non-archimedean case. Let L/K be a finite extension. Let $K'\subseteq L$ be the maximal unramified subextension. The $L/K'/K$ is a tower where L/K' is totally ramified and K'/K is unramified. By transitivity of induction 5.8, it is enough to deal with each case separately.

Step 3a: L/K is totally ramified. Let $I_K \subseteq G_K$ be the inertia subgroup. Then

$$I_K\backslash G_K/G_L = \{1\},$$

so the product is trivial.

We hence have an isomorphism

$$\begin{aligned} I_L\backslash G_L &\to I_K\backslash G_K, \\ \operatorname{Frob}_L &\mapsto \operatorname{Frob}_K. \end{aligned}$$

Then $\operatorname{Res}^{G_K}_{I_K}\operatorname{Ind}^{G_K}_{G_L}\rho_L = \operatorname{Ind}^{I_K}_{I_L}\operatorname{Res}^{G_L}_{I_L}\rho_L$ and hence by Frobenius reciprocity 5.9, $\rho_K^{I_K} = \rho_L^{I_L}$. Since the Frobenii match, this shows that the L-factors match.

Step 3b: L/K is unramified. Then $I_L = I_K \subseteq G_L \subseteq G_K$. Thus G_L is normal in G_K and G_K/G_L is cyclic of order $f=[L:K]=[k_L:k_K]$, generated by the image of $\operatorname{Frob}_K\in G_K/I_K$. We have that $\operatorname{Frob}_L = \operatorname{Frob}_K^f$. By Mackey formula 5.10:

$$\begin{aligned} \operatorname{Res}^{G_K}_{I_K}\operatorname{Ind}^{G_K}_{G_L}\rho_L &= \bigoplus_{g\in G_L\backslash G_K/I_K}\left(\operatorname{Ind}^{gI_Kg^{-1}}_{gI_Kg^{-1}\cap G_L}\operatorname{Res}^{G_L}_{gI_Kg^{-1}\cap G}\rho_L\right)^{g^{-1}} \\ &= \bigoplus_{g\in G_K/G_L}\left(\operatorname{Ind}^{I_K}_{I_L}\operatorname{Res}^{G_L}_{I_L}\rho_L\right)^{g^{-1}}. \end{aligned}$$

Let $A \in \mathrm{GL}(\rho_L)$ be $\rho_L(\mathrm{Frob}_L)$. Then $\rho_K(\mathrm{Frob}_K)$ be given by

$$B = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \\ & & & 1 \\ A & & & 0 \end{bmatrix} \in M_{f\times f}(\mathrm{End}(V_L^{I_L}))$$

We want to compute $\det(I - t\cdot B)$ where $t = q_K^{-s}$. Let $R \subseteq \mathrm{End}(V_L^{I_L})$ be generated by scalars and A.

Formula. If $R \subseteq M_{n\times n}(F)$ is a commutative subgring, and $B \in \mathrm{Mat}_{m\times m}(R) \subseteq \mathrm{Mat}_{nm\times nm}(F)$, then

$$\det{}_F(B) = \det{}_F(\det{}_R(B)).$$

We apply this to $I - t\cdot B$:

$$\begin{bmatrix} I & -tI & & \\ & I & -tI & \\ & & & \\ & & & -tI \\ -tA & & & I \end{bmatrix}.$$

We have that

$$\det{}_R(I - tB) = I + (-1)^{f-1}(-t)^f A = I - t^f A.$$

By the formula above

$$\det{}_F(I - tB) = \det{}_F(I - t^f A).$$

We use the fact that $\mathrm{Frob}_L = \mathrm{Frob}_K^f$ to conclude that

$$L(s, \mathrm{Ind}_{G_L}^{G_K} \rho_L) = L(s, \rho_L).$$

This completes the proof. □

Review of class field theory. Let K be local non-archimedean with Galois group G_K. We have the Weil group $W_K \subseteq G_K$ given by the diagram

$$\begin{array}{ccc} W_K & \hookrightarrow & G_K \\ \downarrow & & \downarrow\!\!\downarrow \\ \mathbb{Z} & \hookrightarrow & \hat{\mathbb{Z}} \end{array}$$

but not with the subspace topology but rather with $I_K \subseteq W_K$ open.

Theorem 5.13 (Local Class Field theory)**.**

(1) *There is a canonical isomorphism*

$$W_K^{\mathrm{ab}} \xrightarrow{\cong} K^\times$$

mapping inertia to $\mathcal{O}_K^\times$ and any Frobenius element a uniformizer.

(2) *For a finite Galois extension L/K, we have two diagrams:*

$$\begin{array}{ccc} W_K^{\mathrm{ab}} & \longrightarrow & K^\times \\ {\scriptstyle \mathrm{incl}^{\mathrm{ab}}}\uparrow & & \uparrow{\scriptstyle N_{L/K}} \\ W_L^{\mathrm{ab}} & \longrightarrow & L^\times \end{array} \qquad \begin{array}{ccc} W_K^{\mathrm{ab}} & \longrightarrow & K^\times \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow \\ W_L^{\mathrm{ab}} & \longrightarrow & L^\times \end{array}$$

(3) *If L/K is abelian, the first diagram gives*

$$K^\times / N_{L/K}(L^\times) \to G_{L/K}.$$

For any finite extnesion L/K, we have the the *relative Weil group* $W_{L/K} = W_K/[W_L, W_L]$. Then

$$1 \longrightarrow L^\times \longrightarrow W_{L/K} \longrightarrow G_{L/K} \longrightarrow 1$$

and conversely $W_K = \varprojlim_L W_{L/K}$.

In the archimedean case, there is also a relative Weil group which fits in the above short exact sequence.

Let K be a number field. There exists a Weil group W_K equipped with a map $W_K \to G_K$ satisfying the following properties.

Theorem 5.14.

(1) *There exists an isomorphism*

$$W_K^{\mathrm{ab}} \to \mathbb{A}^\times / K^\times = C_K.$$

(2) *For any finite Galois extension L/K, we have two diagrams*

$$\begin{array}{ccc} W_K^{\mathrm{ab}} & \longrightarrow & C_K \\ {\scriptstyle \mathrm{incl}^{\mathrm{ab}}}\uparrow & & \uparrow{\scriptstyle N_{L/K}} \\ W_L^{\mathrm{ab}} & \longrightarrow & C_L \end{array} \qquad \begin{array}{ccc} W_K^{\mathrm{ab}} & \longrightarrow & C_K \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow \\ C_L & \longrightarrow & L^\times \end{array}$$

(3) *When L/K is abelian, we have an isomorphism*

$$C_K / N_{L/K}(C_L) \cong G_{L/K}.$$

Construction of the Weil groups in the global case:

(1) Identify a canonical class in $H^2(G_{L/K}, C_L)$.
(2) Obtain the short exact sequence

$$1 \longrightarrow C_L \longrightarrow W_{L/K} \longrightarrow G_{L/K} \longrightarrow 1.$$

(3) Define $W_K = \varprojlim W_{L/K}$.

Notation. When K is local, we set $C_K = K^\times$.

Theorem 5.15. *If $\dot{v}$ is a place of $\overline{K}$, then we have a diagram*

$$\begin{array}{ccc} W_K^{\text{ab}} & \longrightarrow & C_K \\ \uparrow & & \uparrow \\ W_{K_{\dot{v}}}^{\text{ab}} & \longrightarrow & C_{K_{\dot{v}}}. \end{array}$$

Fact 5.16. *Every finite-dimensional representation of W_K factors through $W_{L/K}$ for some L/K finite. It extends to G_K if and only if its image is finite.*

Remark 5.17. If ρ is an irreducible representation of W_K (primitive if K is a number field — not induced from a subgroup), then $\rho = \rho_1 \otimes \chi$ where ρ_1 has finite image and χ is a character.

Remark 5.18. We can define L-functions for representations of W_K. Everything goes through, with convergence in some right half-plane.

Abelian case.

Proposition 5.19. *Let K be local or global and $\rho\colon G_K \to \mathbb{C}^\times$ be a 1-dimensional representation, $\chi\colon C_K \to \mathbb{C}^t imes$ the corresponding character. Then*

$$L(s, \rho) = L(s, \chi).$$

Proof. Due to compatibility of local and global Artin reciprocity maps, we are reduce to the local case. Since the local map $I_K \to \mathcal{O}_K^\times$ sends Frobenius to the uniformizer, we are done. □

Corollary 5.20. *Let K be a number field and $\rho\colon G_K \to \mathbb{C}^\times$. Then $L(s, \rho)$ has meromorphic continuation and $\Lambda(s, \rho)$ satisfies the functional equation.*

Meromorphic continuation of non-abelian Artin L-functions and the Artin conjecture.

Theorem 5.21. *Let K be a number field and $\rho\colon G_K \to \mathrm{GL}(V)$ be a representation. Then $L(s, \rho)$ has meromorphic continuation and $\Lambda(s, \rho)$ satisfies a funtional equation.*

Proof. By Brauer Induction Theorem 5.11,

$$\rho = \sum u_i \rho_i$$

for $u_i \in \mathbb{Z}$, $\rho_i = \mathrm{Ind}_{H_i}^{G_K} \chi_i$ where $\chi_i\colon H_i \to \mathbb{C}^\times$ and $H_i \subseteq G_K$. By inductivity 5.12 and additivity 5.7, we have that

$$L(s, \rho) = \prod L(s, \chi_i)^{n_i}$$

and similarly for Λ. □

Conjecture (Artin)**.** If ρ is irreducible and not the trivial representation, then $L(s, \rho)$ is entire.

Remark 5.22. In the proof, the n_i can be negative, so a zero of $L(s, \chi_i)$ will contribute to a pole of $L(s, \rho)$. The Artin conjecture says that these poles cancel out to give a holomorphic function.

Note that we have not defined an ϵ-factor which is to appear in the functional equation. We may simply define it as

$$\epsilon(s,\rho) = \frac{\Lambda(s,\rho)}{\Lambda(1-s,\rho^\vee)}.$$

This makes it meromorphic, inductive, and additive. By inductivity and additivity, it is entire and has no zeros, since this is true in the 1-dimensional case.

Let

$$\epsilon(s,\rho) = |\epsilon(s,\rho)| \cdot W(\rho)$$

where $W(\rho)$ is the *root number*. Each of these pieces is inductive and additive, and we want to describe them separately. The absolute value is described using the theory of *Artin conductors*. The root number is harder to understand but we will still describe it as best we can.

Filtrations of the Galois group of a local field. Let L/K be a finite extension of non-archimedean local fields, $G = \mathrm{Gal}(L/K)$.

Definition 5.23. The *lower numbering* higher ramification groups are

$$G_i = \{\sigma \in G \mid \sigma|_{\mathcal{O}_L/\mathfrak{p}_L^{i+1}} = 1\}$$

where $G_{-1} = G$ by definition.

Note that G_0 is the inertia group. We can write

$$G_i = \{\sigma \in G \mid \iota_G(\sigma) > i\}$$

where

$$\iota_G(\sigma) = \max\{v_L(\sigma(x) - x) \mid x \in \mathcal{O}_L\}.$$

Fact 5.24.

(1) *For $\sigma, \tau \in G$, $\iota_G(\sigma\tau\sigma^{-1}) = \iota_G(\tau)$.*
(2) *If $H \subseteq G$, $\iota_H = \iota_G|_H$.*

In particular, $H_i = G_i \cap H$.

Definition 5.25.

(1) For $r \in \mathbb{R}$, $r \geq -1$. define $G_r = G_{\lceil r \rceil}$.
(2) Define $\varphi(u) = \int_0^u [G_0 : G_t]^{-1} dt$, $\psi = \varphi^{-1}$.
(3) Define the *upper numbering* higher ramification groups to be $G^u = G_{\psi(u)}$.

Remark 5.26. Note that $\varphi(-1) = -1$, $\varphi(0) = 0$, $\varphi(m) = \sum_{i=1}^m \frac{g_i}{g_0}$ for $g_i = \#G_i$.

Fact 5.27. *Let $N \subseteq G$ be normal.*

(1) *We have that $\iota_{G/N}(\overline{\sigma}) = e_{L/L^N}^{-1} \sum_{\sigma \mapsto \overline{\sigma}} \iota_G(\sigma)$.*
(2) *Therefore,*

$$(G/N)^t = G^t N/N.$$

Theorem 5.28 (Hasse-Arf). *If L/K is abelian, then the jumps in the upper numbering are integers.*

Theorem 5.29. *The local Artin reciprocity maps are*

$$G_K^{\mathrm{ab},t} \cong U_K^t \subseteq \mathcal{O}_K^\times.$$

Review of representation theorem. Let G be a finite group.

(1) Let (ρ, V) be a representation of G. Then the *character* of ρ is define to be
$$\rho_\rho(g) = \mathrm{tr}\rho(g)$$
(a class function). The map $(\rho, V) \mapsto \chi_\rho$ is injective.
(2) Each representation is a direct sum of irreducibles.
(3) The characters of irreducible representations form an orthonormal basis of the space of class functions with respect to
$$(f_1, f_1)_G = \frac{1}{\#G}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

We have the monoid of isomorphism classes of rerpesentations of G with $\oplus$. Take the abelian group generated by this monoid; this is called the (*Grothendieck*) group of *virtual representations*. This group is isomorphism to

$$\mathbb{Z}^{\#(\text{conjugacy classes})}.$$

This group embeds into the group of complex-valued class functions.

If π, σ are representations and σ is irreducible, then

$$(\chi_\pi, \chi_\sigma) = \dim \mathrm{Hom}_G(\sigma, \pi).$$

If $H \subseteq G$ is a subgroup and π is a representation of G, σ a representation of H, then

$$(\chi_\pi, \chi_{\mathrm{Ind}_H^G \sigma} G)_G = (\chi_{\mathrm{Res}_H^G \pi}, \chi_\sigma)_H.$$

In particular, $\mathrm{Ind}_H^G 1$ contains every irreducible representation ρ of G with multiplicity $\dim \rho^H$.

Review of discriminants. Let L/K be a finite extension of local number fields. We have a symmetric bilinear form

$$L \otimes L \to K,$$
$$x \otimes y \mapsto \mathrm{tr}_{L/K}(xy).$$

Definition 5.30.

(1) The *complementary module* of $\mathcal{O}_L$ is
$$\mathcal{O}_L^* = \{x \in L \mid (x, y) \subseteq \mathcal{O}_K \text{ for all } y \in \mathcal{O}_L\}.$$
(2) Then $\mathcal{D}_{L/K} = (\mathcal{O}_L^*)^{-1}$ is the *different* (as ideal of $\mathcal{O}_L$).
(3) Finally, the *discriminant* is $\mathfrak{d}_{L/K} = N_{L/K}(\mathcal{D}_{L/K})$.

Theorem 5.31.

(1) *For a tower $M/L/K$,*
$$\mathcal{D}_{M/K} = \mathcal{D}_{M/L}\mathcal{D}_{L/K}$$
$$\mathfrak{d}_{M/K} = \mathfrak{d}_{L/K}^{[M:L]} N_{L/K}\mathfrak{d}_{M/L}.$$
(2) *For L/K an extension of number fields,*
$$\mathcal{D}_{L/K} = \prod_{\mathfrak{p}} \mathcal{D}_{L_{\mathfrak{P}}/K_{\mathfrak{p}}}$$
and similarly for $\mathfrak{d}$.
(3) *If $\mathcal{O}_L/\mathcal{O}_K$ is generated by $\alpha \in \mathcal{O}_L$ and g be the minimal polynomial ofα. Then $\mathcal{D}_{L/K}$ is generated by $g'(\alpha)$.*
(4) *The discriminant $\mathfrak{d}_{L/K}$ is generated by the discriminants of all bases of L/K contained in $\mathcal{O}_L$.*
(5) *A prime $\mathfrak{P} \subseteq \mathcal{O}_L$ ramifies if and only if $\mathfrak{P}|\mathcal{D}_{L/K}$. A prime $\mathfrak{p} \subseteq \mathcal{O}_K$ ramifies if and only if $\mathfrak{p}|\mathfrak{d}_{L/K}$.*

Fact 5.32. *Let $\psi\colon \mathbb{A}_K/K \to \mathbb{C}^\times$. Let u_v be such that $\psi|_{\mathfrak{p}_v^{u_v}} = 1$ but $\psi_{\mathfrak{p}_v^{u_v-1}} \neq 1$. Then*
$$\prod_v q_v^{-n_v}$$
is independent of ψ and equals $\mathfrak{d}_{K/\mathbb{Q}}$.

Proof. Since the dual of $\mathbb{A}_K/K$ is K, the independence is immediate from the product formula. We can use
$$\psi = \psi^0 \circ \operatorname{tr}_{K/\mathbb{Q}}.$$
Then $\mathfrak{p}_v^{-n_v} = \mathcal{D}_{K_v/\mathbb{Q}_p}$. Then
$$\mathfrak{d}_{K/\mathbb{Q}} = \prod_v \mathfrak{d}_{K_v/\mathbb{Q}_p} = \prod_v \#(\mathcal{O}_{K_v}/\mathfrak{d}_{K_v/\mathbb{Q}_p}) = \prod_v q_v^{-n_v}.$$
This completes the proof. □

The Artin conductor. Let L/K be a Galois extension of local non-archimedean fields. Define
$$a_G(\sigma) = \begin{cases} -f\iota_G(\sigma) & \sigma \neq 1 \\ f\sum_{\tau\neq 1}\iota_G(\tau) & \sigma = 1 \end{cases}$$

Fact 5.33. *The function a_G is a class function and $a_G(\sigma^{-1}) = a_G(\sigma)$.*

Therefore,
$$a_G = \sum_{\rho\in\operatorname{Irr}(G)} f(\rho)\cdot\chi_\rho$$
for $f(\rho) = (a_G, \chi_\rho)_G$. The main goal is to show that $f(\rho) \in \mathbb{Z}_{\geq 0}$.

Lemma 5.34. *We have that $a_G(1) = f\cdot \operatorname{ord}(\mathcal{D}_{L/K})$.*

Proof. Choose a generator α of $\mathcal{O}_L$ over $\mathcal{O}_K$. Let g be the minimal polynomial of α. Then $\mathcal{D}_{L/K}$ is generated by

$$g'(\alpha) = \prod_{\substack{\tau \in G \\ \tau \neq 1}} (\tau\alpha - \alpha).$$

Then

$$\begin{aligned} \mathrm{ord}_L(\mathcal{D}_{L/K}) &= \sum_{\substack{\tau \in G \\ \tau \neq 1}} \mathrm{ord}_L(\tau\alpha - \alpha) \\ &= \sum_{\tau \neq 1} \iota_G(\tau). \end{aligned}$$

This completes the proof. □

It is immediate from the definition of a_G that $(a_G, 1_G)_G = 0$.

Lemma 5.35. *For i, let $u_i = \mathrm{Ind}_1^{G_i} 1 - 1$ be the augmentation representation. Then*

$$a_G = \sum_{i=0}^{\infty} [G_0 : G_i]^{-1} \mathrm{Ind}_{G_i}^G u_i.$$

Proof. **Case 1**: $\sigma \in G$, $\sigma \neq 1$.

Let k be unique such that $\sigma \in G_k \setminus G_{k+1}$. Then

$$a_G(\sigma) = -f(k+1).$$

On the other hand,

$$\mathrm{Ind}_{G_i}^G u_i = \mathrm{Ind}_1^G 1 - \mathrm{Ind}_{G_i}^G 1.$$

Then the character of $\mathrm{Ind}_{G_i}^G u_i$ is g/g_i if $i \leq k$ and 0 otherwise.

Case 2: $\sigma = 1$.

We have $\sum\limits_{\sigma \in G} a_G = 0$. This is equivalent to $(a_G, 1_G)_G = 0$. It is hence enough to show that

$$(\mathrm{RHS}, 1_G)_G = 0.$$

This is true because

$$(\mathrm{Ind}_{G_i}^G u_i, 1_G)_G = (u_i, 1_{G_i})_{G_i} = 0$$

by Frobenius reciprocity. □

Proposition 5.36. *If ρ has dimension 1, then let j be maximal such that $\rho|_{G_j} \neq 1$. Then*

$$f(\rho) = \varphi(j) + 1,$$

and this is an integer.

Proof. Using Lemma 5.35,

$$\begin{aligned} f(\rho) &= (a_G, \chi_\rho)_G \\ &= \sum_{i=0}^{\infty} [G_i : G_0]^{-1} \underbrace{(u_i, \chi_\rho|_{G_i})_{G_i}}_{\begin{cases} 1 & i \le j \\ 0 & i > j \end{cases}} \\ &= \sum_{i=0}^{j} (g_i/g_i)^{-1} \\ &= \varphi(j) + 1. \end{aligned}$$

We just have to check that $\varphi(j)$ is an integer. Let $K' = L^{\ker(\rho)}$ and $r = \varphi(j)$. Then

$$\mathrm{Gal}(K'/K)^t = \mathrm{im}(G^t).$$

We have that $\mathrm{Gal}(K'/K)^t \neq 0$ but $\mathrm{Gal}(K'/K)^{t+\epsilon} = 0$ for $\epsilon > 0$. Therefore, t is a jump in the upper numbering for the abelian extension K'/K, so the Hasse–Arf Theorem 5.28, t is an integer. ☐

Lemma 5.37. *Let $H \subseteq G$ be a subgroup and let $r_H = \mathrm{Ind}_1^H 1$. Let $K' = L^H$. Then*

$$a_G|_H = \mathrm{ord}_K(\mathfrak{d}_{K'/K}) \cdot r_H + f_{K'/K} a_H.$$

Proof. Let $\sigma \in H$. Recall that $\iota_G|_H = \iota_H$.

Case 1: $\sigma \neq 1$. Then

$$\begin{aligned} a_G(\sigma) &= -f_{L/K}\iota_G(\sigma) \\ &= -f_{L/K'} f_{K'/K} \iota_H(\sigma) \\ &= f_{K'/K} a_H(\sigma) \end{aligned}$$

by definition, while $r_H(\sigma) = 0$.

Case 2: $\sigma = 1$. Then

$$\begin{aligned} r_H(\sigma) &= [H : 1] \\ &= e_{L/K'} \cdot f_{L/L'}. \end{aligned}$$

We then have that:

$$\begin{aligned} \mathrm{RHS}(1) &= \mathrm{ord}_K(\partial_{K'/K}) e_{L/K} f_{L/K'} + f_{K'/K} f_{L/K'} \mathrm{ord}_L(\mathcal{D}(L/K')) && \text{by Lemma 5.34} \\ &= \mathrm{ord}_K(N_{K'/L}\mathcal{D}_{K'/K}) e_{L/K'} f_{L/K'} + f_{L/K} \mathrm{ord}_L(\mathcal{D}_{L/K'}) \\ &= \mathrm{ord}_{K'}(\mathcal{D}_{K'/K}) f_{K'/K} e_{L/K} f_{L/K'} + f_{L/K} \mathrm{ord}_L(\mathcal{D}_{L/K'}) \\ &= f_{L/K} \mathrm{ord}_L(\mathcal{D}_{K'/K}) + f_{L/K} \mathrm{ord}_L(\mathcal{D}_{L/K'}) \\ &= f_{L/K} \mathrm{ord}_L(\mathcal{D}_{L/K}) \\ &= a_G(1) && \text{by Lemma 5.34} \end{aligned}$$

This gives the result. ☐

Theorem 5.38. *For any irreducible representation ρ of G, $f(\rho) \in \mathbb{Z}_{\geq 0}$.*

Proof. Using Brauer induction 5.11, we know that

$$\rho = \sum_i u_i \operatorname{Ind}_{H_i}^G \rho_i$$

for $u_i \in \mathbb{Z}$ and 1-dimensional ρ_i. Then

$$\begin{aligned} f(\rho) &= (a_G, \chi_\rho)_G \\ &= \sum_i u_i (a_G|_{H_i}, \rho_i)_{H_i} \\ &= \sum_i u_i (\operatorname{ord}(\mathfrak{d}_{K_i/K})(r_{H_i}, \rho_i)_{H_i} + f_{K_i/K}(a_{H_i}, \rho_i)_{H_i}) \qquad \text{Lemma 5.37} \end{aligned}$$

Finally, $(r_{H_i}, \rho_i) \in \mathbb{Z}$ and $(a_{H_i}, \rho_i)_{H_i} \in \mathbb{Z}$ by Proposition 5.36.

On the other hand, by Lemma 5.35, $g_0 \cdot a_G$ is a representation, so $f(\rho) \in \mathbb{Q}_{\geq 0}$. This completes the proof. □

Remark 5.39. This theorem is equivalent to showing that a_G is a representation of G, called the *Artin representation*. There is no explicit construction of this representation.

Definition 5.40. The *Artin conductor* $f(\rho)$ of a (virtual) representation ρ is defined as $(a_G, \chi_\rho)_G$.

Fact 5.41. *If ρ is unramified, then $f(\rho) = 0$.*

Proof. Use Proposition 5.36. □

Proposition 5.42. *The Artin conductor $f(\rho)$ is compatible with inflation: if $M/L/K$ is a tower then*

$$\rho \colon G_{L/K} \to \operatorname{GL}(V)$$

satisfies

$$f_{M/K}(\rho) = f_{L/K}(\rho).$$

Proof. We have that

$$\begin{aligned} a_{G_{L/K}}(\sigma') &= f_{L/K} e_{M/L}^{-1} d_{M/K}^{-1} \sum_{\sigma \mapsto \sigma'} a_{G_{M/K}}(\sigma) \\ &= [M : L]^{-1} \sum_{\sigma \mapsto \sigma'} a_{G_{M/K}}(\sigma). \end{aligned}$$

Then

$$(a_{G_{M/K}}, \chi_\rho)_{G_{M/L}} = (a_{G_{L/K}}, \rho_\rho)_{G_{L/K}},$$

completing the proof. □

Let L/K be a finite Galois extension of number fields and $\rho \colon G_{L/K} \to \operatorname{GL}(V)$ be a complex representation.

Definition 5.43. The *Artin conductor* $f(\rho) = f_{L/K}(\rho)$ of ρ is

$$\prod_{\mathfrak{p}} \mathfrak{p}^{f(\rho|_{G_{L/K,\mathfrak{P}}})}$$

for any choice of $\mathfrak{P}|\mathfrak{p}$.

Remark 5.44. The conductor is independent on the choice of $\mathfrak{P}|\mathfrak{p}$, because the resulting representations are isomorphic for different choices. Also, note that $f(\rho|_{G_{L/K,\mathfrak{P}}}) = 0$ for almost all $\mathfrak{p}$ so this product is well-defined.

Fact 5.45. *The Artin conductor f is additive and compatible with inflation.*

Proposition 5.46. *The Artin conductor f is compatible with induction as follows: for a tower $M/L/K$, $\rho_L\colon G_{M/L} \to \mathrm{GL}(V)$, we have that*

$$f_{M/K}(\mathrm{Ind}_{G_{M/L}}^{G_{M/K}} \rho_L) = \mathfrak{d}_{L/K}^{\dim(\rho_L)} \cdot N_{L/K}(f_{M/L}(\rho_L)).$$

Proof. The proof is left as an exercise. Hint: use Mackey formula 5.10 and established results. □

Definition 5.47. Let L/K be a finite Galois extension of number fields and $\rho\colon G_{L/K} \to \mathrm{GL}(V)$ be a representation. Define

$$c_{L/K}(\rho) = \mathfrak{d}_{K/\mathbb{Q}}^{\dim(\rho)} N_{K/\mathbb{Q}}(f_{L/K}(\rho))$$

as an ideal of $\mathbb{Z}$; equivalently, as a positive integer.

Corollary 5.48. *The function $c_{L/K}(\rho)$ is compatible with addition, inflation, and induction.*

Proof. This is immediate from the above results. □

From now on, we may refer to $c_{L/K}(\rho)$ as simply $c_K(\rho)$, because it is independent of the choice of L.

Theorem 5.49. *The absolute value of the epsilon factor $|\epsilon(s,\rho)| = c_K(\rho)^{\frac{1}{2}-s}$.*

Proof. Both sides are compatible with induction and additive. By Brauer induction 5.11, this reduces to the case $\dim \rho = 1$. Both sides are products of local factors. For the left side, we need to choose a non-trivial additive character $\psi_K\colon \mathbb{A}_K/K \to \mathbb{C}^\times$ to write

$$\epsilon(s,\rho) = \prod_v \epsilon(s,\rho_v,\psi_{K_v}).$$

We take $\psi_K^0 = \psi_{\mathbb{Q}}^0 \circ \mathrm{tr}_{K/\mathbb{Q}}$. Now it is enough to compare $|\epsilon(s,\rho_v,\psi_{K_v}^0)|$ with

$$\left(\mathfrak{p}_{K_v/\mathbb{Q}_p} \cdot N_{K_v/\mathbb{Q}_p}\mathfrak{p}_v^{f(\rho_v)}\right)^{\frac{1}{2}-s}.$$

Note that when $v|\infty$, both sides are 1.

To lighten the notation, we write $K = K_v$ for any v, a local non-archimedean field. Recall that we have computed the local ϵ-factors in Lemma 4.29 for a special choice of character (in the case $K = \mathbb{Q}_p$, but this generalizes easily).

Let n be such that ψ_K^0 is trivial on $\mathfrak{p}_K^n$, not on $\mathfrak{p}_K^{n-1}$. Then $\psi'_K(x) = \psi_K^0(\pi^n x)$ is of level 0, and

$$|\epsilon(s,\rho,\psi_K^0)| = q^{n(s-1/2)}|\epsilon(s,\rho,\psi'_K)|$$

by Lemma 4.28. By Fact 5.32, $\mathfrak{d}_{K/\mathbb{Q}_p} = q^{-n}$. Thus we are left with showing that:

$$|\epsilon(s,\rho,\psi'_K)| = (N_{K/\mathbb{Q}_p}(\mathfrak{p}^{f(\mathfrak{p})}))^{\frac{1}{2}-s}.$$

We can finally use Lemma 4.29. There are two cases:

(1) When ρ is unramified, $\epsilon(s,\rho,\psi'_K)=1,\ f(\rho)=0$.
(2) When ρ is ramified, let c be smallest so that $\rho|_{U_K^c}$ is trivial. Then

$$\epsilon(s,\rho,\psi'_K)=q_K^{c(1/2-s)}\rho(\pi^c)g(\rho,\psi'_K).$$

At the same time, Proposition 5.36 shows that

$$f(\rho)=c.$$

Finally, we use the local Artin map $G_K^{\mathrm{ab},c}\cong U_K^c$ to get the result.

This completes the proof. □

The root number. The root number $W(\rho)$ is a root of unity, independent of s. We will define it soon.

Aside. Let G be a finite group and $K_0(G)$ be the Grothendieck group of virtual representations. We have a map

$$\dim\colon K_0(G)\to\mathbb{Z}.$$

If $H\subseteq G$,

$$K_0(H)\ \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}}\ K_0(G).$$

Theorem 5.50 (Deligne/Langlands)**.**

(1) *Let K be a local field, ψ_K be a non-trivial character $K\to\mathbb{C}^\times$. For any finite extension L/K, $\psi_L=\psi_K\circ\mathrm{tr}_{L/K}$. There is exactly one assignment*

$$(L/K,\rho_L\colon G_L\to\mathrm{GL}(V))\mapsto\epsilon(\rho_L,\psi_L)\in\mathbb{C}$$

such that
  (a) $\epsilon(\rho_L\oplus\rho'_L,\psi_L)=\epsilon(\rho_L,\psi_L)\cdot\epsilon(\rho'_L,\psi_L)$,
  (b) *if ρ_L has dimension 1, then*

$$\epsilon(\rho_L,\psi_L)=\epsilon(1/2,\chi,\psi_L),$$

  where $\chi\colon L^\times\to\mathbb{C}^\times$ corresponds to ρ_L under the local Artin map,
  (c) *ϵ is additive in degree 0.*

(2) *Let K be a global field, $\psi_K\colon\mathbb{A}_K/K\to\mathbb{C}^\times$ be a non-trivial character. Then*

$$\epsilon\left(\frac{1}{2},\rho\right)=W(\rho)=\prod_v\epsilon(\rho_v,\psi_{K_v}).$$

Definition 5.51. Let L/K be an extension of local fields. Define

$$\lambda(L/K,\psi_K)=\epsilon(\mathrm{Ind}_{G_L}^{G_K}1_L,\psi_K)=\frac{\epsilon(\mathrm{Ind}_{G_L}^{G_K}1_L,\psi_K)}{\epsilon(1_L,\psi_L)}.$$

Fact 5.52. *For any $\rho_L\colon G_L\to\mathrm{GL}(V)$,*

$$\epsilon(\mathrm{Ind}_{G_L}^{G_K}\rho_L,\psi_K)=\lambda(L/K,\psi_K)^{\dim\rho_L}\epsilon(\rho_L,\psi_L).$$

Remark 5.53. The proof of the Theorem is hard. Langlands takes around 250 pages to do it purely locally with some very difficult computations. Deligne's idea is to use the existence of the global epsilon factor and specialize it to deduce something about the local epsilon factors. This turns out to be much easier.

6. Non-abelian class field theory?

Class field theory was the crucial input for describing the 1-dimensional Artin representations in terms of Hecke characters.

What about representations ρ of the Galois group that do not factor through an abelian quotient?

Conjecture (Langlands)**.** Let K be a number field. Let ρ be an irreducible n-dimensional complex representation of G_K. Then there exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ such that

$$L(s,\rho) = L(s,\pi).$$

What is a cuspidal automorphic representation? We have

$$\mathrm{GL}_n(\mathbb{A}_K) \text{ — a locally compact group with Haar measure}$$

which contains

$$\mathrm{GL}_n(K) \text{ — a discrete subspace with the counting measure.}$$

We can hence form

$$\mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathbb{A}_K),$$

which is a coset space with a measure. This is the non-abelian (n-dimensional) version of $\mathbb{A}_K^\times/K^\times$.

Consider

$$L^2(\mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathbb{A}_K)),$$

which is a Hilbert space with $\mathrm{GL}_n(\mathbb{A}_K)$-action.

Definition 6.1. An *automorphic representation* is a "constituent" of

$$L^2(\mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathbb{A}_K)).$$

Caution. This L^2-space is **not the direct sum** of irreducibles. Therefore, it is not clear what "constituent" means.

Instead of defining constituents in general, we will reduce to a simpler case.

Note that $Z(\mathrm{GL}_n) = \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_n$ acts on GL_n. This gives an action of $\mathbb{A}_K^\times/K^\times$ on L^2. Using Fourier theory, we can decompose the space

$$L^2(\mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathbb{A}_K)) = \int_\psi^\oplus L^2(\mathrm{GL}_n(K)/\,\mathrm{GL}_n(\mathbb{A}_K),\psi)d\psi$$

as a *direct integral.*

We deal with $L^2(\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathbb{A}_K),\psi)$. This still does not decompose discretely, but there is a subspace

$$L^2_0(\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathbb{A}_K),\psi)\subseteq L^2(\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathbb{A}_K),\psi)$$

of *cusp forms.*

Theorem 6.2. *The space $L^2_0(\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathbb{A}_K),\psi)$ decomposes as a direct sum of irreducible representations with finite multiplicities.*

Remark 6.3. It is another big theorem that for GL_n the multiplicities must in fact be one. This fails for other reductive groups (while the stated theorem is still true).

Definition 6.4. A *cuspidal automorphic representation* is an irreducible constituent of

$$L^2_0(-,\psi).$$

Remark 6.5. More generally, we have a decomposition

$$L^2(-,\psi)=\bigoplus_{P\text{ parabolic}} L^2_{P,\psi}.$$

When $P=G$, $L^2_0\subseteq L^2_{G,\psi}$ is the *discrete spectrum*, which decomposes discretely.

When $P\neq G$,

$$L^2_{P,\psi}\cong I^G_P(L^2_{M,\psi,dx})$$

where I^G_P is *parabolic induction* and M is a Levi subgroup. This was one of the first contributions of Langlands to this subject. The isomorphism is proved using the theory of Eisenstein series.

What is the L-function of a π, $L(s,\pi)$?

We can decompose $\pi=\bigotimes_v{}' \pi_v$ if π_v is an irreducible representation of $\mathrm{GL}_n(K_v)$. Then define

$$L(s,\pi)=\prod_v L(s,\pi_v).$$

Defining $L(s,\pi_v)$ uses representation theory and harmonic analysis of $\mathrm{GL}_n(K_v)$. We will see this definition only make sense when π is an automorphic representation, and we will see how the decomposition with respect to ψ is relevant.

There is a local version of the above theorem.

Theorem 6.6 (Harris–Taylor, Henniart)**.** *Let K be a non-archimedean local field. There is a bijection*

$$r\colon \mathrm{Irr}_n(W_K)\to \mathrm{Cusp}(\mathrm{GL}_n(K))$$

preserving L and ϵ factors (and satisfying extra properties).

Theorem 6.7. *Let $L_K=W_K\times \mathrm{SL}_2(\mathbb{C})$ be the* Langlands group. *There is a bijection*

$$r\colon \mathrm{Rep}_n(L_K)\to \mathrm{Irr}(\mathrm{GL}_n(K))$$

preserving L and ϵ factors (and satisfying extra properties).

Sometimes, a representation $\rho\colon G_k \to \mathrm{GL}(V)$ factors through another linear algebraic group $\hat{G}$. For example, ρ might preserve a symplectic or orthogonal pairing. Then Langlands associates to $\hat{G}$ a reductive group G/K.

Conjecture (Local Langlands Conjecture)**.** Therre exists a finite-to-one map:

$$\mathrm{Irr}(G(K)) \to \mathrm{Rep}(L_K, \hat{G})$$

with a description of the fibers (L-packets), satisfying some extra properties.

Conjecture (Global Langlands Conjecture)**.** Given $\pi = \bigotimes' \pi_v$, the multiplicity of π in L^2 is

$$\sum_{\substack{\rho\colon G_K \to \hat{G} \\ \rho_v \leftrightarrow \pi_v}} (*)$$

where $(*)$ depends on the L-packets in the Local Langlands Conjecture.

Remark 6.8. There is also a *Functoriality conjecture* which says that any L-homomorphism between L-groups induces a transfer map between automorphic forms.

7. Automorphic representations of $\mathrm{SL}_2(\mathbb{R})$

We are interested in representations of $\mathrm{GL}_2(\mathbb{A})$. We noted above (without proof) that such representations decompose

$$\pi = \bigotimes_{p \le \infty}' \pi_p$$

(if admissible) where π_p is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. We therefore start by discussing representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ for all p. In this chapter, we deal with $p = \infty$.

The Lie group $\mathrm{SL}_2(\mathbb{R})$.

Remark 7.1. Most representations of $\mathrm{SL}_2(\mathbb{R})$ are infinite-dimensional. The next example describes all the finite-dimensional ones.

Example 7.2. For each $N \in \mathbb{N}$, let $\mathcal{M}_N$ be the set of homogeneous degree N polynomials in 2 variables, i.e.

$$\left\{ \sum a_{i,j} X^i Y^j \;\middle|\; a_{i,j} \in \mathbb{C},\ i + j = N \right\}.$$

Then $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathcal{M}_N$. Moreover, $\mathcal{M}_N$ is a finite-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ of dimension $N+1$. It is unique such up to isomorphism.

Definition 7.3. A representation of G is a complex Hilbert space V, together with a group homomorphism

$$\pi\colon G \to \mathrm{Aut}_{\mathrm{cts}}(V)$$

such that the map

$$G \times V \to V$$

is continuous.

A representation (π, V) is *unitary* if π takes values in the unitary operators (i.e. it preserves the scalar product).

We have the following useful subgroups:

$$T = \begin{bmatrix} * & \\ & * \end{bmatrix}, \quad U = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = U \rtimes T, \quad K = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \cong S^1.$$

Definition 7.4 (Parabolic induction)**.** Let $\chi\colon T \to \mathbb{C}^\times$ be continuous character of T. Define

$$I_B^G \chi = \left\{ f\colon G \to \mathbb{C} \mid f(tug) = \delta_B(t)^{1/2}\chi(t)f(g) \right\}.$$

This is a representation of G by right translations. The functions f satisfy extra properties, soon to be defined.

Questions.

(1) What is δ_B and why is it there?
(2) Qhat kind of functions f are we actually taking?

Aside on Haar measures. Let H be locally compact topological group. Let H be a Haar measure dh (left-invariant[2]). For $g \in H$, consider the functionals

$$f \mapsto \int_H f(h)dh, \quad f \mapsto \int_H f(hg)dh.$$

These are two Haar functionals, so by uniqueness, there is $\delta_H(g) \in \mathbb{R}_{>0}$ such that

$$\int_H f(hg)dh = \delta_H(g)\int_H f(h)dh$$

for all f.

Exercise. The function $\delta_H\colon H \to \mathbb{R}_{>0}$ is a continuous group homomorphism.

Definition 7.5. The function δ_H is the *modulus character*. The group H is called *unimodular* if $\delta_H = 1$.

Examples 7.6. Unimodular groups include: abelian groups, compact groups, and reductive groups. In particular, T, U, K, G are all unimodular. However, B is not unimodular.

When H is a real Lie group,

$$\delta_H(h) = |\det(\mathrm{Ad}(h) \mid \mathrm{Lie}(H))|.$$

Here, $\underline{\mathrm{Ad}}(h)\colon H \to H$ and $\mathrm{Ad}(h) = d\underline{\mathrm{Ad}}(h)\colon \mathrm{Lie}(H) \to \mathrm{Lie}(H)$.

In the case of $H = B$, we have that

$$\underline{\mathrm{Ad}}\begin{bmatrix} t_1 & u \\ 0 & t_2 \end{bmatrix} = |t_1/t_2|.$$

This is the character which appeared in Definition 7.4. This (partially) answers question (1).

Consider $H \subseteq G$ locally compact. We have the G-homogeneous space G/H.

Question. Is there a G-invariant measure on G/H?

[2]In the first part of the course, we dealt with abelian groups, so we never had to specify if we're using right or left-invariant measures.

If G, H are real Lie groups, such a measure amounts to an H-invariant top form on $T_1(G/H) = \mathrm{Lie}(G)/\mathrm{Lie}(H)$. Note that H acts on G/H by conjugation. Check that

$$(\mathrm{Ad}(h) \mid T_1(G/H)) = (\mathrm{Ad}(h) \mid \mathrm{Lie}(G)/\mathrm{Lie}(H)).$$

Therefore

$$\frac{\det(\mathrm{Ad}(h) \mid \mathrm{Lie}(G))}{\det(\mathrm{Ad}(h) \mid \mathrm{Lie}(H))} = \frac{\delta_G(h)}{\delta_H(h)}.$$

Answer. A G-invariant measure on G/H exists if and only if $\delta_G|_H = \delta_H$.

But for $B \subseteq G$, this is false, since $\delta_G = 1$, $\delta_B \neq 1$. In particular, there will never be a G-invariant measure on G/H.

Lemma 7.7. *There exists a G-invariant Haar functional on the space of functions*

$$f \colon G \to \mathbb{C} \text{ such that } f(gh) = \frac{\delta_G(h)}{\delta_H(h)} f(g),$$

which are continuous and compactly supported mod H.

Proof. Let ω be a top form on $\mathrm{Lie}(G)/\mathrm{Lie}(H)$. Consider the differential form of degree $\dim(G) - \dim(H)$ on G given by

$$\omega(g) = L_g^* \omega$$

where L_G denotes left translation by g. It satisfies

$$\omega(gh) = \frac{\delta_H(h)}{\delta_G(h)} \cdot \omega(g).$$

Therefore, for f as above,

$$f \cdot \omega$$

is H-invariant, and hence a differential form of top degree on G/H. Integrating against it is the desired Haar functional. □

We now return to the discussion of parabolic induction (Definition 7.4), $I_B^G \chi$. Consider the case when χ is unitary. Then $f \in I_B^G \chi$ satisfies

$$(f \cdot \overline{f})(tug) = \delta_B(t)^{-1} f \overline{f}(g).$$

This belongs to the space of functions in Lemma 7.7, where we can integrate.

- We demand that $f \in L^2(B \backslash G)$.
- The representation $I_B^G \chi$ is unitary.

This completes the answer to questions (1) and (2) above in the case when χ is unitary. When χ is not unitary, we need the following lemma.

Lemma 7.8 (Iwasawa decomposition)**.** *There is a decomposition $G = B \cdot K$, i.e. $g \in G$ is the product $g = b \cdot k$ for $b \in B$, $k \in K$. Moreover, $B \cap K = \{\pm 1\} = Z_G$.*

Proof. **Exericse**. □

Caution. Here, neither of the two groups B and K normalize each other. Therefore, this is not just the usal product of groups from group theory.

By this Lemma, we have that

$$I_B^G \chi \cong \{f\colon K \to \mathbb{C} \mid f(\epsilon k) = \chi(\epsilon) f(k) \text{ for } \epsilon \in Z_G\}.$$

Which functions do we take? We simply require that $f \in L^2(K)$.

The two previous definitions agree by the integral formula:

$$\int_G f(g)dg = \int_K \int_B f(bk)dbdk.$$

Eventually, we will see that any irreducible representation is obtained by parabolic induction. For now, we need to do some more work.

Smooth and K-finite vectors.

Remark 7.9. Let V be a finite-dimensional representaion of G, $\pi\colon G \to \mathrm{GL}(V)$. Then

$$d\pi\colon \mathrm{Lie}(G) \to \mathrm{End}(V)$$

is a representation of $\mathrm{Lie}(G)$.

Why does this work?

(1) Continuity of $G \times V \to V$ implies continuity of $\pi\colon G \to \mathrm{GL}(V)$.
(2) The group $\mathrm{GL}(V)$ is a Lie group, and any continuous homomorphism of Lie groups is differentiable; in fact, real analytic.

However, this clear does not work for finite-dimensional representations.

Definition 7.10. A vector $v \in V$ is called

(1) *differentiable* if for each $X \in \mathfrak{g} = \mathrm{Lie}(G)$, the limit

$$X_V = \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tX))v = \lim_{t\to 0} \frac{\pi(\exp(tX))v - v}{t}$$

exists.
(2) *k-times differentiable* if it is differenitable and for all $X \in \mathfrak{g}$, Xv is $(k-1)$-differentiable,
(3) *smooth* if it is k-differentiable for all k.

We write $V^\infty \subseteq V$ for the subspace of smooth vector.

Fact 7.11. *The subspace V^∞ is G-invariant, but not closed, and is a representation of $\mathfrak{g}$.*

From now on, the real vector space $\mathfrak{g}$ will be replaced by its complexification $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. This still has a representation V^∞ by extending scalars. We do this to use the tools from representation theory of complex Lie algebras.

Universal enveloping algebra. Let L be a complex Lie algebra. Define

$$T(L) = \bigoplus_{n \geq 0} L^{\otimes n}$$

and let

$$I \text{ be the inhomogeneous ideal generated by } X \otimes Y - Y \otimes X - [X, Y].$$

The *universal enveloping algebra* is

$$\mathcal{U}(L) = \frac{T(L)}{I}.$$

Theorem 7.12 (Poincaré–Birkhoff–Witt (PBW))**.** *If $(x_1, \ldots, x_n)$ is an ordered basis of the $\mathbb{C}$-vector space L. Then*

$$(x_1^{k_1} \ldots x_n^{k_n})$$

is a basis of the $\mathbb{C}$-vector space $U(L)$. In particular, the map

$$L \hookrightarrow \mathcal{U}(L)$$

is injective.

Fact 7.13 (Universal property)**.** *Let A be an associative algebra. Define the Lie bracket on A by $[a, a'] = aa' - a'a$. Then any Lie algebra homomorphism $L \to A$ has a unique extension to an algebra homomorphism*

$$\mathcal{U}(L) \to A.$$

In particular, V^∞ is a $\mathcal{U}(G)$-representation.

The intuition is that the Lie algebra contains "level 1" differential operators, while the universal enveloping algebra contains all level operators.

Definition 7.14. A vector $v \in V$ is K*-finite* if $\operatorname{span}\{\pi(k)v \mid k \in K\}$ is finite-dimensional. Let V^{fin} be the space of K-finite vectors.

Then V^{fin} depends only on $\operatorname{Res}_K^G V$. This is a unitary representation (because we assumed V is a Hilbert space).

Theorem 7.15 (Peter–Weyl)**.** *Let K be a compact group.*

(0) *Any irreducible unitary representation of a compact group is finite-dimensional.*
(1) *The K-finite vectors in the space of continuous functions on K are dense with respect to the sup norm, and (if K is a Lie group) they are automatically smooth, because they are precisely the matrix coefficients of irreducible representations (which are finite-dimensional by (0)).*
(2) *If V is a unitary representation of K, then V is the Hilbert direct sum of irreducible representations. The space of K-finite vectors is the algebraic direct sum of these, hence dense.*

In particular, V^{fin} is dense in V.

How to produce a smooth K-finite vectors?

Given a continuous compactly supported function $f\colon G \to \mathbb{C}$, define

$$\begin{aligned} \pi(f)\colon V &\to V \\ v &\mapsto \int_G f(g)\pi(g)v dg. \end{aligned}$$

Fact 7.16.

(1) *If f is smooth, so is $\pi(f)v$ and $X\pi(f)v = \pi(Xf)v$.*
(2) *If f is K-finite on the left, then $\pi(f)v$ is K-finite.*

Lemma 7.17. *The space $V^{\text{fin}} \cap V^\infty$ is dense in V. In particular, V^∞ is dense.*

Proof. Recall the Iwasawa decomposition $G = BK = KB$, Lemma 7.8. Let $f_1\colon K \to \mathbb{C}$ be K-finite (and hence smooth), and $f_2\colon B \to \mathbb{C}$ smooth and compactly-supported. Then $f(kb) = f_1(k)f_2(b)$ is smooth and left K-finite. Given $v \in V$, $\varepsilon > 0$, consider the open subgroup

$$\{g \in G \mid |\pi(g)v - v| < \varepsilon\}.$$

Choose f_1, f_2 such that f is supported inside it. Then

$$|\pi(f)v - v| < \epsilon$$

and $\pi(f)v$ is smooth and K-finite by Fact 7.16. □

Admissible representations.

Theorem 7.18. *Let (π, V) be an irreducible unitary representation of G. The multiplicity of $\tau \in \operatorname{Irr}(K)$ in V^{fin} is $\leq \dim(\tau)$.*

Definition 7.19. A representation (π, V) of G is *admissible* if V^{fin} is a direct sum of irreducible representations of K with finite multiplicities.

Examples 7.20.

(1) Unitary representations are admissible by Theorem 7.18.
(2) Parabolically induced representations are admissible. In fact, Casselman proved that any irreducible admissible representation is a subrepresentation of a parabolically induced one. We will work towards proving this theorem, see Theorem 7.41.

Lemma 7.21. *If (π, V) is an admissible representation, then $V^{\text{fin}} \subseteq V^\infty$ and V^{fin} is $\mathfrak{g}$-invariant.*

Proof. For each $\tau \in \operatorname{Irr}(K)$, $(V^{\text{fin}} \cap V^\infty)[\tau]$ is dense in $V[\tau]$, but the latter is finite-dimensional, so $(V^{\text{fin}} \cap V^\infty)[\tau] = V[\tau]$. Therefore, $V[\tau]$ consists of smooth vectors, and since

$$V^{\text{fin}} = \bigoplus V[\tau],$$

we are done.

We now show that V^{fin} is stable under $\mathfrak{g}$. Taking $v \in V^{\text{fin}}$, let $W \subseteq V^{\text{fin}}$ be finite-dimensional and K-stable, with $v \in W$. Observe that $\mathfrak{g}W$ is finite-dimensional (but not necessarily a representation of $\mathfrak{g}$, since $\mathfrak{g}$ is not associative). For $w \in W$, $X \in \mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$, $Y \in \mathfrak{g}$, we have that

$$XYw = \underbrace{YXw}_{\in \mathfrak{g}W} + \underbrace{[X,Y]w}_{\in \mathfrak{g}W} \in \mathfrak{g}W.$$

In particular, this shows that the vector Yw is K-finite. □

By this lemma, we obtain a representation of $\mathfrak{g}$ and of K on V^{fin}. However, note that V^{fin} is not invariant under G.

Definition 7.22. A $(\mathfrak{g}, K)$-*module* is a $\mathbb{C}$-vector space which is a representation of $\mathfrak{g}$ and K, subject to the conditions:

(1) differentiating the K-action gives the same action as the restriction of the $\mathfrak{g}$-action to $\mathfrak{k}$,
(2) $\ell X v = (\mathrm{Ad}(\ell)X)\ell v$ for $\ell \in K$, $X \in \mathfrak{g}$.

Definition 7.23. A $(\mathfrak{g}, K)$-module W is *admissible* if W is the direct sum of irreducible representations of K with finite multiplicity. These are also known as *Harish-Chandra modules*.

Remark 7.24. If V is an admissible representation, then V^{fin} is an admissible $(\mathfrak{g}, K)$-module.

Definition 7.25.

(1) A G-representation V is *irreducible* if it has no non-trivial proper **closed** invariant subspaces.
(2) A $(\mathfrak{g}, K)$-module is *irreducible* if it has no non-trivial proper subspaces.

The natural question to ask is: if a representation of G is irreducible, is the associated $(\mathfrak{g}, K)$-module irreducible, and vice versa.

Definition 7.26. Two representations of G are called *infinitesimally equivalent* if their $(\mathfrak{g}, K)$-modules are isomorphic.

Remark 7.27. If two representations are isomorphic, then they are clearly infinitesimally equivalent. The converse is actually not true in general. However, a deep theorem due to Harish-Chandra states that for irreducible unitary representations, these two notions are equivalent.

Matrix coefficients. Let (π, V) be an admissible representation.

Definition 7.28. For $v \in V$, $\eta \in V^\vee$, define the *matrix coefficient* as

$$f_{v,\eta} \colon G \to \mathbb{C}$$
$$f_{v,\eta}(g) = \langle \pi(g)v, \eta \rangle.$$

If v, η are K-finite, we call $f_{v,\eta}$ *K-finite.*

Theorem 7.29. *A K-finite matrix coefficient is a real analytic function.*

Observe that for $D \in \mathcal{U}(\mathfrak{g})$,

$$Df_{v,\eta} = f_{\pi(D)v,\eta}. \tag{3}$$

Corollary 7.30. *If $U_0 \subseteq V^{\text{fin}}$ is $\mathfrak{g}$-invariant, then $U = \overline{U_0}$ (the closure in V) is G-invariant.*

Proof. Since G acts by continuous automorphisms, it is enough to check that $GU_0 \subseteq U$. Fix $v \in U_0$ and $\eta \in U^\perp \subseteq V^\vee$. For small $X \in \mathfrak{g}$,

$$\begin{aligned}
\langle \pi(\exp(X))v, \eta\rangle &= \sum_{n=0}^{\infty} X^n \langle \pi(x)v, \eta\rangle\Big|_{x=1} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\langle X^n v, \eta\rangle && \text{by equation (3)} \\
&= 0 && \text{as } X^n v \in U_0.
\end{aligned}$$

The matrix coefficient $f_{v,\eta}$ is zero on $\exp(\mathfrak{g}) \subseteq G$, an open neighborhood. By Theorem 7.29, this shows that $f_{v,\eta} = 0$, i.e. $\langle gv, \eta\rangle = 0$. This shows that $gv \in U$. ☐

Corollary 7.31. *Taking closures gives a bijection*

$$\left\{ \begin{matrix} \mathfrak{g}\text{-invariant subspaces} \\ \text{of } V^{\text{fin}} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} G\text{-invariant subspaces} \\ \text{of } V \end{matrix} \right\}$$

In particular, V^{fin} is irreducible if and only if V is irreducible.

Corollary 7.32. *Let (π_1, V_1) and (π_2, V_2) be an admissible representation of G.*

1. *If π_1 and π_2 are infinitesimally equivalent, then they have the same K-finite matrix coefficients.*
2. *If π_1 and π_2 are irreducible and share a K-finite matrix coefficient, then π_1 is infinitesimally equivalent to π_2.*

Proof. We first prove (1). Let $v \in V_1$ and $\eta \in V_1^\vee$ be K-finite. By Theorem 7.29, $f_{v,\eta}$ is uniquely characterized by its derivatives at 1, which by equation (3) are given by

$$\langle \pi_1(D)v, \eta\rangle.$$

Infinitesimal equivalence preserves these.

For (2), assume π_i are irreducible and

$$\langle \pi_1(g)v, \eta\rangle = \langle \pi_2(g)v_2, \eta_2\rangle.$$

Let $V_3 \subseteq \mathcal{C}^\infty(G)$ be the subspace spanned by $\mathcal{U}(\mathfrak{g})f_{v,\eta}$. Any $v \in V_1$ is of the form $\pi_1(D)v_1$. Define

$$\varphi\colon V_1 \to V_3$$

by

$$\varphi(\pi_1(D)v_1) = D(f_{v_1,\eta}).$$

This is bijective by irreducibility and one checks equivariance to complete the proof. ☐

This allows us to prove a version of Schur's Lemma. Note that Schur's Lemma normally only holds for finite-dimensional representations.

Corollary 7.33 (Schur's Lemma)**.** *Assume V is irreducible. Then any endomorphism of the $\mathfrak{g}$-module V^{fin} is a scalar. In particular, the center of the universal enveloping algebra $Z(\mathcal{U}(\mathfrak{g}))$ acts by a character.*

Remark 7.34. In general, the algebra $Z(\mathcal{U}(\mathfrak{g}))$ is a polynomial algebra over $\mathbb{C}$. This is a theorem due to Harish-Chandra, but we omit it here. In our case, it will be a 1-dimensional polynomial algebra and we will explicitly write the generator.

Proof. If φ commutes with $\mathfrak{g}$, it does so with $\mathfrak{k}$, so preserves K-types, which are finite-dimensional, so has an eigenvalue, and the eigenspace is $\mathfrak{g}$-invariant and non-trivial, so all of V^{fin}. □

The Casimir element. Recall the *Killing form* on a Lie algebra L over $\mathbb{C}$ is the map

$$\begin{aligned} \kappa\colon L \times L &\to \mathbb{C} \\ (X, Y) &\mapsto \operatorname{tr}(\operatorname{ad}(X), \operatorname{ad}(Y)), \end{aligned}$$

where $\operatorname{ad}(X)\colon L \to L$ is the map $Z \mapsto [X, Z]$. Then κ is L-invariant and symmetric.

Theorem 7.35. *The Lie algebra L is semisimple if and only if κ is non-degenerate.*

Therefore:

(1) $\kappa\colon L \to L^*$ is an isomorphism,
(2) $\kappa \in L^* \otimes L^* \xrightarrow{\cong} L \otimes L \to \mathcal{U}(L) \ni C$.

The element C is called the *Casimir operator*.

Fact 7.36. *The Casimir operator is in the center, $C \in Z(\mathcal{U}(\mathfrak{g}))$.*

The differential equation satisfied by the matrix coefficients. Let (π, V) be an irreducible admissible representation of G. Then C acts on $V^f in$ by a scalar $\lambda \in \mathbb{C}$ by Schur's Lemma 7.33. By equation (3), we have that

$$C f_{v,\eta} = \lambda \cdot f_{v,\eta} \tag{4}$$

for K-finite v and η.

Let us introduce the following coordinates:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ a basis for } \mathfrak{g}$$

$$Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$C = \frac{1}{4}(H^2 + V^2 - Y^2).$$

This computation is left as an exercise.

Fact 7.37 (Cartan decomposition)**.** *Let*

$$A_- = \left\{ h(t) = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \middle| 0 < t \leq 1 \right\}$$

and

$$A_-^{\leq 1} = \left\{ h(t) = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \middle| 0 < t < 1 \right\}.$$

The multiplication map

$$K \times A_- \times K \to G$$

is surjective, and

$$K \times A_-^{\leq 1} \times K \to G \setminus K$$

is a double cover (local diffeomorphism of degree 2).

In the coordinates on $K \cong S^1$ given by angles ϕ_1, ϕ_2 and on $A_-^{\leq 1}$ given by $t \in (0,1)$, we have that

$$\begin{aligned} C = {} & \left(t \frac{\partial}{\partial t} \right)^2 - 2 \frac{t^2 + t^{-2}}{t^2 - t^{-2}} \cdot t \frac{\partial}{\partial t} \\ & + \frac{4}{(t^2 - t^{-2})^2} \left(\left(\phi_1 \frac{\partial}{\partial \phi_1} \right)^2 + \left(\phi_2 \frac{\partial}{\partial \phi_2} \right)^2 \right) \\ & - \frac{4(t^2 + t^{-2})}{(t^2 - t^{-2})^2} \phi_1 \frac{\partial}{\partial \phi_1} \phi_2 \frac{\partial}{\partial \phi_2}. \end{aligned}$$

Assume v, η have pure K-type n (i.e. associated to the representation $z \mapsto z^n$ of S^1).

Substitute $z = t^2$. Then equation (4) gives the differential equation

$$\left(\left(\frac{\partial}{\partial z} \right)^2 - \left(\frac{(1+z^2)z}{1-z^2} - \frac{1}{z} \right) \frac{\partial}{\partial z} - \frac{n^2}{z(1+z)^2} - \frac{\lambda}{z^2} \right) f(h(t)) = 0. \tag{5}$$

Here $u(t)$ is the function on $A_-^{\leq 1}$. In general,

$$f_{v,\eta}(e^{i\phi_1} h(t) e^{i\phi_2}) = e^{in(\phi_1 + \phi_2)} \cdot f_{v,\eta}(h(t)).$$

The differential equation (5) makes sense for $z \in D \subseteq \mathbb{C}$, the unit disc. It is an ODE with regular singularities and there is a theory for solving such equations.

Lemma 7.38. *Let* $\mu_i = \frac{1}{2}(1 + (-1)^i \sqrt{4\lambda + 1})$ *for* $i = 1, 2$. *There are holomorphic functions* F_1, F_2 *on* D *with* $F_1(0) = F_2(0) = 1$ *such that the space of solutions of (5) on* $D \setminus (-1, 0)$ *is given by*

(1) $az^{\mu_1} F_1(z) + bz^{\mu_2} F_2(z)$ *if* $\mu_1 - \mu_2 \notin \mathbb{Z}$,
(2) $az^{\mu_1} F_1(z) + bz^{\mu_2} \log(z) F_2(z)$ *if* $\mu_1 - \mu_2 \in \mathbb{Z}$.

Thus

$$f_{v,\eta}(h(t)) = t^{2\mu_1}(a_0 + a_1 t^2 + a_2 t^4 + \cdots) + t^{2\mu_2}(b_0 + b_1 t^2 + \cdots)$$

where a_0 and b_0 are not both 0. For v, η which are not of pure type, we still have this asymptotic expansion, but it may be true that $a_0 = b_0 = 0$.

Casselman submodule theorem. Recall that for a character $\chi\colon T \to \mathbb{C}^\times$,

$$I_B^G\chi = \mathrm{Ind}_B^G(\chi \otimes \delta_B^{1/2}),$$

where we inflate the representation χ of T to a representation of $B = U \rtimes T$ on the right hand side.

Fact 7.39 (Frobenius reciprocity). *For any $(\mathfrak{g}, K)$-module $\mathcal{V}$, we have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\mathcal{V}, (I_B^G\chi)^{\mathrm{fin}}) = \mathrm{Hom}_{(\mathfrak{t},K_T)}(\mathcal{V}/\mathfrak{u}\cdot\mathcal{V}, \mathbb{C}_{\chi+\delta_B^{1/2}}),$$

where $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{u} = \mathrm{Lie}(U)$, and $K_T = K \cap T = \{\pm\} = Z_G$.

The proof of this is essentially the same as Frobenius reciprocity.

Theorem 7.40. *If $\mathcal{V} = V^{\mathrm{fin}}$ is a finitely-generated admissible non-zero $(\mathfrak{g}, K)$-module, then $\mathcal{V}/\mathfrak{u}\mathcal{V}$ is a finitely-generated non-zero $(\mathfrak{t}, K)$-module.*

The assumption that $\mathcal{V}$ comes from an admissible representation V is unnecessary. We make it only because we did not introduce matrix coefficients for general $(\mathfrak{g}, K)$-modules (which can be done).

Proof. Let $v_1, \dots, v_n$ generate $\mathcal{V}$. By admissibility, these are contained in a finite-dimensional K-invariant subspace. Add some more vectors to span this K-invariant $\mathbb{C}$-vector space.

Recall that

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is a basis of $\mathfrak{g}$. Recall that H spans $\mathfrak{t}$, E_+ spans $\mathfrak{u}$, Y spans $\mathfrak{k}$.

Then PBW 7.12 gives $E_+^\alpha H^\beta Y^\gamma$ as a basis of $\mathcal{U}(\mathfrak{g})$. Therefore,

$$\{E_+^\alpha H^\beta Y^\gamma v_i\}$$

are generators for the $\mathbb{C}$-vector space $\mathcal{V}$. If $\alpha > 0$, the $E_+^\alpha H^\beta Y^\gamma v_i = 0$ in $\mathcal{V}/\mathfrak{u}\mathcal{V}$. On the other hand, $Y^\gamma v_i$ is a linear combination of $v_1, \dots, v_n$. Thus $H^\beta v_i$ span the $\mathbb{C}$-vector space $\mathcal{V}/\mathfrak{u}\mathcal{V}$, so v_i span the $(\mathfrak{t}, K_T)$- module $\mathcal{V}/\mathfrak{u}\mathcal{V}$.

To show that $\mathfrak{u}\mathcal{V} \subsetneq \mathcal{V}$, let us assume that V is irreducible. Let $v \in \mathcal{V}$, $\eta \in \mathcal{V}^\vee$. We have the asymptotic expansion of $f_{v,\eta}$. We have

$$\mathrm{Ad}(h(t))(E_+)v = t^2 E_+ v.$$

Therefore,

$$\begin{aligned} f_{E_+v,\eta}(h(t)) &= \langle h(t)E + v, \eta\rangle \\ &= t^2\langle E_+ h(t)v, \eta\rangle \\ &= -t^2\langle h(t)v, E_+\eta\rangle \\ &= -t^2 f_{v,E_+\eta}(h(t)) \end{aligned}$$

So any matrix coefficient for vector in $\mathfrak{u}\mathcal{V}$ goes to zero by t^2 faster than any matrix coefficient for vectors in $\mathcal{V}$. Therefore, $\mathfrak{u}\mathcal{V} \subsetneq \mathcal{V}$. ∎

The proof in fact shows that E_+v is not of pure K-type even if v is. Otherwise, the matrix coefficient $f_{E_+v,\eta}$ would have the same asymptotic expansion as before.

Theorem 7.41 (Casselman submodule theorem)**.** *Any irreducible G-representation V is infinitesimally equivalent to a subrepresentation of $I_B^G\chi$.*

Proof. By Theorem 7.40, $\mathcal{V}/\mathfrak{u}\mathcal{V}$ is a non-zero and finitely-generated $(\mathfrak{t}, K_T)$-module. Sicne $K_T = \mathbb{Z}/2\mathbb{Z}$, this is a direct sum of two $\mathfrak{t}$-modules. Any $\mathfrak{t}$-module is a $\mathcal{U}(\mathfrak{t})$-module and $\mathcal{U}(\mathfrak{t}) = \mathbb{C}[X]$. This has an irreducible quotient, which is C with X acting by a complex number s, i.e.

$$\mathcal{V}/\mathfrak{u}\mathcal{V} \to \mathbb{C}_{(s,m)}$$

for $s \in \mathbb{C}$, $m \in \mathbb{Z}/2\mathbb{Z}$, where $\mathbb{C}[X]$ acts as $Xv = sv$ and $\mathbb{Z}/2\mathbb{Z}$ acts as $\operatorname{sgn}(\cdot)^m$.

Then Frobenius reciprocity5.9 implies that

$$V^{\mathrm{fin}} \to (I_B^G\chi)^{\mathrm{fin}},$$

where $\chi(t) = |t|^s \operatorname{sgn}(t)^m$. □

The structure of $I_B^G\chi$. Recall that

$$I_B^G\chi = \{f\colon G \to \mathbb{C} \mid f(tug) = \chi(t)\delta_B^{1/2}(t)f(g)\}$$

Suppose f is of pure K-type n.

Recall the Iwasawa decomposition 7.8: $G = B \cdot K$. Thus

$$f(g) = f(tuk(\varphi)) = \chi(t)\delta_B^{1/2}(t)e^{in\varphi}f(1).$$

Thus $f \in \mathbb{C} \cdot \psi_n$, where

$$\psi_n(tuk(\varphi)) = \chi(t)\delta_B^{1/2}(t)e^{in\varphi}.$$

Therefore,

$$(I_B^G\chi)^{\mathrm{fin}} = \bigoplus_{n \equiv m\ (2)} \mathbb{C} \cdot \psi_n,$$

where $\chi(t) = |t|^s \operatorname{sgn}(t)^m$.

Recall that $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Define $P_+ = H + iV$, $P_- = H - iV$. Then P_+, P_-, Y are a basis of $\mathfrak{g}$.

Lemma 7.42. *We have that*

$$P_+\psi_n = (1 + s + n)\psi_{n+2}$$
$$P_-\psi_n = (1 + s - n)\psi_{n-2}.$$

Proof. Check that

$$\mathrm{Ad}(k(\varphi))P_+ = e^{2i\varphi}P_+$$
$$\mathrm{Ad}(k(\varphi))P_- = e^{-2i\varphi}P_-.$$

Therefore, we know the K-type of $P_+\psi_n$ is $n + 2$, and hence

$$P_+\psi_n = \alpha\psi_{n+2},$$

and we can evaluate it at 1 to get

$$(P_+\psi_n)(1) = \alpha\psi_{n+2}(1) = \alpha.$$

Now, $P_+ = H + 2iE_+ - iY$. Compute:

$$\begin{aligned} H\psi_n(1) &= s+1, \\ E_+\psi_n(1) &= 0, \\ Y\psi_n(1) &= in. \end{aligned}$$

The same argument shows the statement for P_-. □

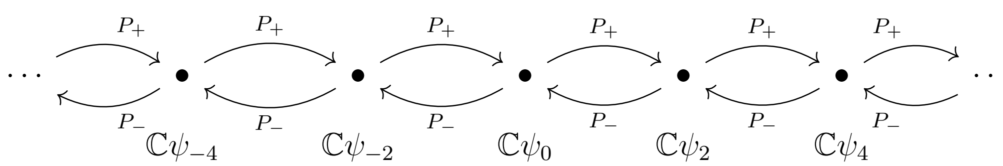
Corollary 7.43. *Casimir acts on $(I_B^G\chi)^{\mathrm{fin}}$ by the scalar $\frac{s^2-1}{4}$.*

Proof. We have that

$$C = -\frac{Y^2}{4} + \frac{P_+P_- + P_-P_+}{8}$$

and a computation gives the result. □

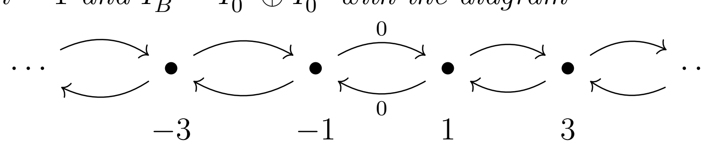
We have the following picture:



and Lemma 7.42 tells us when P_+ or P_- can be zero.

Corollary 7.44.

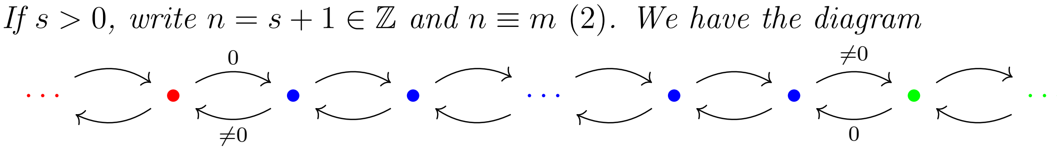
(1) *If $s \notin m+1+2\mathbb{Z}$, then $I_B^G\chi$ is irreducible.*
(2) *If $s \in m+1+2\mathbb{Z}$, then*
   (a) *$s=0$, then $m=1$ and $I_B^G = I_0^+ \oplus I_0^-$ with the diagram*



   and

$$I_0^\pm = \bigoplus_{k\in 1+2\mathbb{Z}_{\geq 0}} \psi_{\pm k}.$$

   (b) *If $s>0$, write $n = s+1 \in \mathbb{Z}$ and $n \equiv m$ (2). We have the diagram*



   where we have marked in red, a submodule $\mathcal{D}_n^-$, and in green, a submodule D_n^+. We hence have a non-split exact sequence

$$0 \longrightarrow \mathcal{D}_n^- \oplus \mathcal{D}_n^+ \longrightarrow I_B^G\chi \longrightarrow \mathcal{M}_{n-2} \longrightarrow 0.$$

   where the quotient $\mathcal{M}_{n-2}$ is marked above in blue.
   (c) *If $s<0$, write $n = -s+1 \in \mathbb{Z}$. we have the non-split exact sequence*

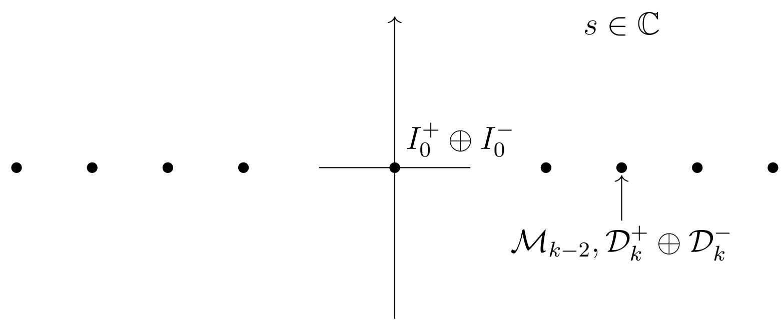
$$0 \longrightarrow \mathcal{M}_{n-2} \longrightarrow I_B^G\chi \longrightarrow \mathcal{D}_n^- \oplus \mathcal{D}_n^+ \longrightarrow 0.$$

Now, we have the irreducible modules

- $I_B^G\chi$, $\chi \leftrightarrow (s, m)$ and $s \notin m + 1 + 2\mathbb{Z}$,
- I_0^+, I_0^-,
- $\mathcal{D}_n^+, D_n^-$ for $n = 2, 3, 4, \ldots$,
- $\mathcal{M}_{n-2}$ for $n = 2, 3, \ldots$.

By consider K-types and Casimir eigenvalues, we see that the only possible isomorphisms among these are $I_B^G\chi \to I_B^G\chi^{-1}$ when these are irreducible. These are provided by the theory of intertwining operators.

We have the following diagram showing the representations parameterized by s:



All the representations we marked above ($(-1,1) \cup \mathbb{Z} \cup i\mathbb{R}$) are unitary. The representations on the imaginary axis are *principal series*, the representations on $(0,1)$ are called complementary series, and the representations $\mathcal{D}_k^{\pm}$ are the *discrete series*.

Intertwining operators. Consider $\chi(t) = |t|^s \operatorname{sgn}(t)^m$. Let

$$L_\infty \colon I_B^G\chi \to \mathbb{C}$$

be given by

$$L_\infty(f) = \int_U f(w \cdot u) dw$$

for

$$w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We write $I_B^G\chi = \mathcal{V}$.

Lemma 7.45. *The map L_∞ is a $(\mathfrak{t}, K_T)$-module homomorphism*

$$\mathcal{V}/\mathfrak{u}\mathcal{V} \to \mathbb{C}_{\chi^{-1}\cdot\delta_B^{1/2}}.$$

Proof. The map L_∞ is U-invariant by construction, so it factors through $\mathcal{V}/\mathfrak{u}\mathcal{V}$. For $t \in T$,

$$\begin{aligned} L_\infty(t \cdot f) &= \int_U f(w \cdot u \cdot t) du \\ &= \delta_B(t) \int_U f(w \cdot t \cdot u) du \\ &= \delta_B(t) \int_U f(t^{-1} w u) du \\ &= \chi(t)^{-1} \delta_B(t)^{-1/2} \delta_B(t) \int_U f(wu) du. \end{aligned}$$

This gives the result. □

By Frobenius reciprocity 7.39, we get a map

$$T\colon I_B^G \chi \to I_B^G \chi^{-1}.$$

By Iwasawa decomposition 7.8, restriction gives an isomorphism

$$I_B^G \chi \to \{f \in L^2(K) \mid f(\epsilon k) = \epsilon^m f(k),\ \epsilon \in \{\pm 1\} = Z_G\}.$$

On the left hand side, the action is independent of s but the vector space depends on s. On the right hand side, one can check that it is the opposite: the vector space visible does not depend on s, but the action of G depends on s.

Consider

$$L_\infty(s)\colon L^2_\epsilon(K) \to \mathbb{C}$$

as a function of $s \in \mathbb{C}$.

Proposition 7.46. *There exists a meromorphic function $\alpha\colon \mathbb{C} \to \mathbb{C}$ such that*

$$\widetilde{L_\infty}(s) = \alpha(s) L_\infty(s)$$

is always defined and never 0.

Proof. Take ψ_n and apply L_∞ to it. Note that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+u^2}} & \frac{-u}{\sqrt{1+u^2}} \\ 0 & \sqrt{1+u^2} \end{bmatrix} \cdot \begin{bmatrix} \frac{-u}{\#} & \frac{1}{\#} \\ \frac{-1}{\#} & \frac{-u}{\#} \end{bmatrix},$$

and hence

$$\psi_n\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \right) = (\sqrt{1+u^2})^{-s-1} \left(\frac{-u+i}{\sqrt{1+u^2}} \right)^n = (\sqrt{1+u^2})^{-s-1-|n|} (-u \pm i)^{|n|}.$$

We integrate this function over $u \in (-\infty, \infty)$. Applying binomial theorem to $(-u \pm i)^{|n|}$, we get

$$\int_{-\infty}^{\infty} (\sqrt{1+u^2})^{-s-1-|n|} \cdot u^k du = \begin{cases} 0 & k \text{ odd}, \\ \Gamma\left(\frac{s+|n|+k}{2}\right) \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{s+|n|+1}{2}\right)^{-1} & \text{otherwise.} \end{cases}$$

This completes the proof. □

Unitary and discrete series. Let (π, V) be an irreducible representation. We have the asymptotic expansion of its matrix coefficient, based on the eigenvalue $\lambda \in \mathbb{C}$ of the Casimir operator:

$$f_{v,\eta}(h(t)) \sim t^{2\mu_1}, t^{2\mu_2}$$

where $\mu_i = \frac{1}{2}(1 + \pm\sqrt{4\lambda + 1})$.

For $I_B^G(\chi)$ where $\chi(t) = |t|^s \operatorname{sgn}(t)^m$, we checked in Corollary 7.43 that

$$\lambda = \frac{s^2 - 1}{4}.$$

If π is unitary, then

- $|f_{v,\eta}(g)| \leq |v| \cdot |\eta|$ (so the matrix coefficients are uniformly bounded),
- $\lambda \in \mathbb{R}$.

Thus $s \in \mathbb{R}$ or $s \in i\mathbb{R}$.

(i) If $\lambda \leq -\frac{1}{4}$ (equivalently, $s \in i\mathbb{R}$), then $\mu_1, \mu_2 \in \frac{1}{2} + i\mathbb{R}$, so $f_{v,\eta}$ go to zero as fast as t. At the same time $I_B^G\chi$ is unitary, and the scalar product is:

$$((\cdot,\cdot)) \colon I_B^G\chi \times I_B^G\chi^{-1} \longrightarrow I_B^G\delta_B^{1/2} \xrightarrow{\text{Haar}} \mathbb{C}.$$

When χ is unitary, we have $\chi^{-1} = \overline{\chi}$.

(ii) If $-\frac{1}{4} \leq \lambda \leq 0$, then $s \in [-1, 1]$, and both $t^{2\mu_1}, t^{2\mu_2} \to 0$, but one of them is fast.

(iii) If $\lambda > 0$, then one of $t^{2\mu_i}$ goes to 0 fast, but the other does not go to 0 (actually, usually explodes).

In the previous section, we constructed an intertwining operator

$$T \colon I_B^G\chi \to I_B^G\chi^{-1}.$$

The approach is as follows: for $s \in \mathbb{R}$, define

$$I_B^G\chi \times I_B^G\chi \to \mathbb{C},$$
$$(f, g) \mapsto ((f, \overline{Tg})).$$

This is a G-invariant Hermitian pairing, so it is a scalar product if and only if it is positive definite.

For (ii), one can check that for $s = 0$, the map

$$T \colon I_B^G\chi \to I_B^G\chi$$

is the identity. (Note that $\chi = \chi^{-1}$ in this case). Thus

$$((f, g)) = ((f, Tg))$$

is a scalar product. Since $I_B^G\chi$ is irreducible for $s \in (0, 1)$, by continuity, we see that $((f, Tg))$ is still a scalar product.

Definition 7.47. The representations in case (i) are called *principal series* and the ones in case (ii) are called *complementary series*.

In case (iii), we see that $I_B^G\chi$ cannot be unitary, unless the offending exponent does not contribute.

For $s = k-1 \in \mathbb{Z}$, the representation is reducible, and we have

$$I_B^G\chi/\mathcal{M}_{k-2} \underset{\cong}{\xrightarrow{T}} \mathcal{D}_k^- \oplus \mathcal{D}_k^+,$$

or

$$I_B^G\chi/\mathcal{D}_k^- \oplus \mathcal{D}_k^+ \underset{\cong}{\xrightarrow{T}} \mathcal{M}_{k-2}.$$

This $((f, Tg))$ gives a scalar product on $\mathcal{D}_k^+$ and $\mathcal{D}_k^-$. One checks that

$$((\psi_n, \overline{T\psi_n})) = ((\psi_n, \overline{\widetilde{L_\infty}(\psi_n)\psi_n})) = \overline{\widetilde{L_\infty}(\psi_n)} \cdot \underbrace{((\psi_n, \overline{\psi_n}))}_{=1}.$$

A computation of $\widetilde{L_\infty}(\psi_n)$ using Proposition 7.46 shows that these numbers are positive.

Corollary 7.48. *The matrix coefficients of $\mathcal{D}_k^\pm$ are square-integrable. When $k > 2$, the matrix coefficients are even integrable.*

Proof. The dominant terms are $t^{2\mu_1}$ and $t^{2\mu_2}$ where $\mu_1 = \frac{1}{2}k$, $\mu_2 = 1 - \frac{1}{2}k$. Then, by boundedness, the term for μ_2 must vanish. (Some more work is required for $k = 2$, but we skip it here for the same of time.)

Thus $f_{v,\eta}(h(t)) \sim t^k$. We need the Haar measure in terms of the Cartan decomposition $G = KAK$:

$$\int_G f(g)dg = \int_0^{2\pi} \int_0^1 \int_0^{2\pi} f(k(\phi_1)h(t)k(\phi_2))d\phi_1 d\phi_2 (t^2 - t^{-2})\frac{dt}{t}.$$

From this, we see that if $f(h(t)) \sim t^a$, then f is integrable if and only if $a > 2$. This completes the proof. □

Fact 7.49. *The matrix coefficients for $I_B^G\chi$, χ unitary, lie in*

$$L^{2+\epsilon}(G)$$

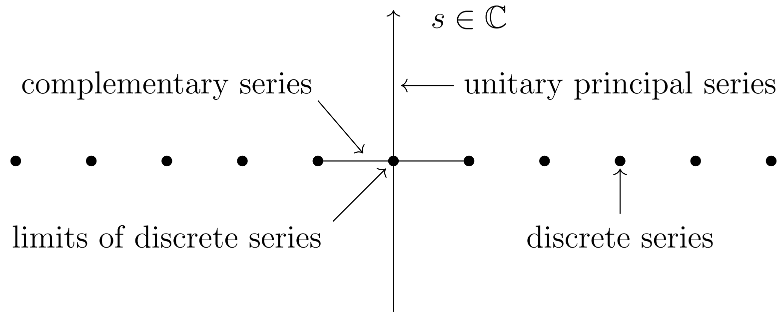
for any $\epsilon > 0$.

Proof. We know that $s \in i\mathbb{R}$. Then $f_{v,\eta}(h(t)) \sim t^1$, and we use the above formula to prove the statement. □

Definition 7.50. A representation of G is called

- *discrete series* if its matrix coefficients lie in $L^2(G)$,
- *tempered* if its matrix ceofficients lie in $L^{2+\epsilon}(G)$ for all $\epsilon > 0$.

In particular, $\mathcal{D}_k^-$ and $\mathcal{D}_k^+$ are the discrete series representations, and $I_B^G\chi$ for χ unitary are tempered. The representations I_0^+, I_0^- are sometimes called *limits of discrete series*, even tough they are not actually discrete series



This is one reason to call $\mathcal{D}_k^{\pm}$ *discrete series*. There is another reason to call them this. They are the only irreducible representations that occurs are discrete constituents of $L^2(G)$.

There is a verson of *Plancharet's Theorem* that says that

$$L^2(G) = \int^{\oplus} \pi d\pi$$

where the measure $d\pi$ is supported on tempered representations and is called *Plancharet measure*. This is an analog of the decomposition

$$L^2(\mathbb{R}) = \int^{\oplus} \mathbb{C}e^{ixy} dy$$

given by Fourier transforms and Fourier series.

The discrete series are the only representations which are given non-zero measure with respect to this measure, in fact:

$$\mathrm{vol}(\mathcal{D}_k^{\pm}, d\pi) = k - 1$$

for $\pm k = 2, 3, \ldots$.

Langlands correspondence. Let $\Gamma_{\mathbb{C}/\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We have the Weil group $W_{\mathbb{C}/\mathbb{R}}$ defined as an extension:

$$1 \longrightarrow \mathbb{C}^{\times} \longrightarrow W_{\mathbb{C}/\mathbb{R}} \longrightarrow \Gamma_{\mathbb{C}/\mathbb{R}} \longrightarrow 1$$

generated by $\mathbb{C}^{\times}$ and j (lifting complex conjugation) subject to the relations

$$jzj^{-1} = \overline{z}, \ j^2 = -1.$$

Definition 7.51.

(1) A *Langlands parameter* is a continuous semisimple group homomorphism
$$W_{\mathbb{C}/\mathbb{R}} \to \mathrm{PGL}_2(\mathbb{C}).$$
(2) Two such are *equivalent* if they are conjugate under $\mathrm{PGL}_2(\mathbb{C})$.
(3) A parameter φ is called *discrete* if any of the following equivalent conditions are satisfied:
- the image is not contained in B up to conjugation,
- it is an irreducible projective representation,
- $\mathrm{Centr}(\varphi, \mathrm{PGL}_2(\mathbb{C}))$ is finite.

(4) A parameter φ is tempered if its image is bounded.

Recall: the characters of $\mathbb{C}^\times$ are

$$\begin{aligned} \mathbb{C}^\times &\to \mathbb{C}^\times \\ z &\mapsto z^a \overline{z}^b \end{aligned}$$

for $a, b \in \mathbb{C}$, $a - b \in \mathbb{Z}$. If $z = re^{i\phi}$,

$$z^a \overline{z}^b = r^{a+b} e^{i\phi(a-b)}.$$

Lemma 7.52. *Any parameter is equivalent to one of the following:*

(1) $z \mapsto \begin{bmatrix} (z\overline{z})^s & 0 \\ 0 & 1 \end{bmatrix}$, $j \mapsto \begin{bmatrix} (-1)^m & 0 \\ 0 & 1 \end{bmatrix}$ *for* $s \in \mathbb{C}$, $m \in \mathbb{Z}/2\mathbb{Z}$,

(2) $z \mapsto \begin{bmatrix} (z/\overline{z})^a & 0 \\ 0 & 1 \end{bmatrix}$, $j \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *for* $a \in \frac{1}{2}\mathbb{Z}$.

Proof. Restrict φ to $\mathbb{C}^\times$. The image is a connected Lie subgroup consisting of commuting diagonalizable matrices.

(a) Suppose the image is trivial. Then j maps to an elements of order 2, and this is case 1 with $s = 0$.

(b) Suppose the image is non-trivial. After conjugating, the image is in $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \cong \mathbb{C}^\times$ where the isomorphism is given by renormalizing to make the right bottom entry 1. Thus

$$\begin{aligned} \varphi|_{\mathbb{C}^\times} : \mathbb{C}^\times &\to \mathbb{C}^\times \\ z &\mapsto \begin{bmatrix} z^a\overline{z}^b & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Now, the image of j normalizes $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and hence it is one of the following

$$\text{(b1): } \begin{bmatrix} d & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{(b2): } \begin{bmatrix} 0 & d \\ 1 & 0 \end{bmatrix}.$$

(b1) Then from $jzj^{-1} = \overline{z}$, we see that $a = b$, and hence

$$z \mapsto \begin{bmatrix} (z\overline{z})^a & 0 \\ 0 & 1 \end{bmatrix},$$

and from $j^2 = -1$, we get $d^2 = 1$. Thus this is case (1).

(b2) From $jzj^{-1} = \overline{z}$, we get $a = -b$. Therefore,

$$z \mapsto \begin{bmatrix} (z/\overline{z})^d & 0 \\ 0 & 1 \end{bmatrix}$$

and now $\begin{bmatrix} 0 & d \\ 1 & 0 \end{bmatrix}$ is conjugate under $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We are hence in case (2).

This completes the proof. □

Remark 7.53. In case (1), the only two equivalent representations correspond to s and $-s$. In case (2), a and $-a$ are the only equivalent representations. Between cases (1) and (2), the only equivalences are between $s = 0, m = 1$ and $a = 0$.

We compute the centralizers. In case (1) but not $s = 0$, $m = 1$, the centralizer is connected. In case (2), $a \neq 0$, centralizer is $\begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}$. In case (2) when $a = 0$, the centralizer is $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$, $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

Then discrete parameters are: case (2), $a \neq 0$. The tempered parameters are: discrete and case (1), $s \in i\mathbb{R}$.

We now describe the correspondence between irreducible admissible representations of G and representations of the Weil group.

Local Langlands correspondence for $\mathrm{SL}_2(\mathbb{R})$.

- In case (2), $a \neq 0$,
$$\varphi \leftrightarrow \{\mathcal{D}_k^+, \mathcal{D}_k^-\}$$
where $k = 2d + 1$. Note that $\{\mathcal{D}_k^+, \mathcal{D}_k^-\}$ is in bijection with $\mathrm{Irr}(\pi_0(\mathrm{Centr}(\varphi)))$.
- In case (2), $a = 0$,
$$\varphi \leftrightarrow \{I_0^+, I_0^-\}$$
and again $\{I_0^+, I_0^-\}$ is in bijection with $\mathrm{Irr}(\pi_0(\mathrm{Centr}(\varphi)))$.
- In case (1), not $(s = 0, m = 1)$, we have
$$\varphi \colon W_{\mathbb{C}/\mathbb{R}} \to \mathbb{C}^\times \hookrightarrow \mathrm{PGL}_2(\mathbb{C}).$$
Moreover, $W_{\mathbb{C}/\mathbb{R}}^{\mathrm{ab}} \cong \mathbb{R}^\times$ via the local Artin map
$$z \mapsto z\bar{z},$$
$$j \mapsto -1.$$
Therefore, φ corresponds to $\chi \colon \mathbb{R}^\times \to \mathbb{C}^\times$ with $\chi(t) = |t|^s \,\mathrm{sgn}(t)^m$. In this case,
$$\varphi \leftrightarrow I_B^G \chi$$
when the right hand side is irreducible, and
$$\varphi \leftrightarrow \mathcal{M}_{k-2}$$
otherwise.

Formal degree. Let $G = \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{Q}_p)$. Let (π, V) be a discrete series representation of G. We have not discussed what this means in the latter case, but once we do the following discussion will apply.

Lemma 7.54. *There is a unique real number* $\deg(\pi, dg)$ *such that for all* $v_1, v_2 \in V$, $v_1^\vee, v_2^\vee \in V^\vee$,
$$\int_G \langle gv_1, v_1^\vee \rangle \langle g^{-1} v_2, v_2^\vee \rangle dg = \langle v_1, v_2^\vee \rangle \langle v_2, v_1^\vee \rangle \deg(\pi, dg)^{-1}.$$

Definition 7.55. The constant $\deg(\pi, dg)$ in Lemma 7.54 is called the *formal degree*.

Exercise. Show that $\deg(\mathcal{D}_k^\pm, dg) = c_{dg}(k-1)$, where c_{dg} is a constant dependent only on the measure dg.

Proof of Lemma 7.54. The representation $V \otimes V^\vee$ of $G \times G$ is irreducible. We define two $G \times G$-equivariant pairings on $V \otimes V^\vee$:

$$\begin{aligned}(-,-)_1 &= \text{LHS},\\ (-,-)_2 &= \text{RHS without deg}.\end{aligned}$$

By irreducibility, these are scalar multiples of each other. We show that this scalar is nonzero and, in fact, belongs to $\mathbb{R}_{>0}$.

For this, recall that π is unitary (embed into $L^2(G)$ via a matrix coefficient). Let $(-,-)_h$ be an invariant scalar product on V. Choose $0 \neq v \in V$ and let $v^\vee = (-, v)_h \in V^\vee$. Then

$$\begin{aligned}(v \otimes v^\vee, v \otimes v^\vee)_1 &= \int_G |(gv, v)_h|^2 dg > 0,\\ (v \otimes v^\vee, v \otimes v^\vee)_2 &= |(v,v)_h|^2 > 0,\end{aligned}$$

showing that the scalar is a positive real number. □

8. Automorphic representations of $\mathrm{SL}_2(\mathbb{Q}_p)$

Basic structure of $\mathrm{SL}_2(\mathbb{Q}_p)$. Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$. We have groups $T \cong \mathbb{Q}_p^\times$, $U \cong \mathbb{Q}_p$ and $B = TU$, as before. However, in this case, a maximal compact is $K_0 = \mathrm{SL}_2(\mathbb{Z}_p)$, which is non-abelian.

Lemma 8.1 (Cartan decomposition)**.** *The group G decomposes as $G = \bigcup_{n\geq 0} K_0 \lambda_n(p) K_0$, a disjoint union, where*

$$\lambda_n(p) = \begin{bmatrix} p^n & 0 \\ 0 & p^{-n}\end{bmatrix}.$$

Furthermore, we have the integral formula

$$\int_G f(g)dg = \sum_{n\geq 0} p^{2n} \int_K \int_K f(k_1 \lambda_n(p) k_2) dk_1 dk_2$$

Proof. Elementary divisors. □

Note that this is the analog of $G = KAK$ where $A_- = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1}\end{bmatrix} \middle| 0 < t \leq 1 \right\}$, but we may get rid of the units here and the group A_- is now discrete.

Remark 8.2. The proof by elementary divisors works for $G = \mathrm{SL}_2(\mathbb{Q}_p)$. For other Gs, one needs to develop a general theory, *Bruhat–Tits theory.*

Lemma 8.3 (Iwasawa decomposition)**.** *We have a decomposition $G = TUK_0$ and we have the integral formula*

$$\int_G f(g)dg = \int_T \int_U \int_{K_0} f(tuk) dt du dk.$$

Proof. For $g \in G$, look where $g\langle e_1\rangle \subseteq \mathbb{Q}_p^2$ goes. We can return this line to $\langle e_1 \rangle$ via $k \in K_0$. But B is the stabilizer of $\langle e_1\rangle$. □

Lemma 8.4 (Bruhat decomposition)**.** *We have a decomposition $G = B \cup BwB$ for $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.*

Smooth and admissible representations.

Definition 8.5. Let (π, V) be a representation of G on a complex vector space.

(1) A vector $v \in V$ is called *smooth* if $\mathrm{Stab}(v) \subseteq G$ contains a compact open subgroup.
(2) The representation (π, V) is called *smooth* if all of its vectors are smooth.
(3) The representation (π, V) is called *admissible* if it is smooth and for each compact open subgroup (c.o.s.) $K \subseteq G$, $\dim_{\mathbb{C}} V^K < \infty$.

Remark 8.6. If (π, V) is any representation of G, then $V^\infty = \bigcup_{\substack{K \subseteq G \\ \text{c.o.s.}}} V^K$ is the space of smooth vectors. Unlike over $\mathbb{R}$ where V^∞ was only a representation of the Lie algebra $\mathfrak{g}$, V^∞ is still a G-representation. The Lie algebra in this case plays a different role.

Theorem 8.7 (Harish-Chandra)**.** *If (π, V) is an irreducible unitary representation of G, then V^∞ is dense in V and an admissible representation.*

This result is difficult and we will not attempt to prove it.

Remark 8.8. In fact, if π is smooth and irreducible, then π is admissible. We will almost prove this, but skip enough results that we will not present a complete prove.

Lemma 8.9. *The only irreducible smooth finite-dimensional representation of G is the trivial representation.*

Proof. The intersection of the stabilizers of a basis is the kernel of the representation and contains a compact open subgroup. Since the kernel is normal and bigger than the center, it equals all of G. □

Lemma 8.10.

(1) *Any finitely generated smooth representation has an irreducible quotient.*
(2) *Any smooth representation has an irreducible subquotient.*

Proof. Note that (2) follows from (1). For (1), consider the set of all proper subrepresentations , ordered by inclusion. By finite generation assumption, this set has the tower property. By Zorn's Lemma, this set has a maximal element. Quotienting by this maximal element gives an irreducible quotient. □

Example 8.11. A smooth finite-dimensional representation of $\mathbb{Q}_p^\times$ need not be reducible (i.e. a direct sum of irreducible representations).

Consider $V = \mathbb{C}^2$ with $\mathbb{Q}_p^\times$ acting via

$$\mathbb{Q}_p^\times \stackrel{\text{ord}}{\to} \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}),$$
$$1 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

This is clearly smooth, but is clearly not reducible.

Fact 8.12. *Let V be a smooth n-dimensional representation of $\mathbb{Q}_p^\times$ and let $(\chi_1, \dots, \chi_n)$ be the irreducible subquotients of a Jordan-Hölder series. If χ_i are pairwise distinct, then*

$$V = \bigoplus \mathbb{C}(\chi_i).$$

Proof. Exercise. □

Fact 8.13. *A smooth finite-dimensional representation of $\mathbb{Q}_p$ is the direct sum of irreducibles.*

Proof. The group $\mathbb{Q}_p$ is exhausted by $p^{-n}\mathbb{Z}_p$, compact opens. The restriction $V|_{p^{-n}\mathbb{Z}_p}$ decomposes as a sum of characters of $p^{-n}\mathbb{Z}_p$, which are compatible with n. This completes the proof. □

Induced representations. From now on, (π, V) is always assumed to be admissible.

For $\chi\colon T \to \mathbb{C}^\times$ smooth character, the *parabolic induction* of χ is

$$I_B^G\chi = \{f\colon G \to \mathbb{C} \mid f(tng) = \chi(t)\delta_B(t)^{1/2}f(g)\},$$

where f is right-invariant under some compact open subgroup K_f. One can show that in this case $\delta_B(t) = |t|^2$.

Clearly, $I_B^G\chi$ is a smooth representation by the restriction we made on f.

Lemma 8.14. *The parabolic induction $I_B^G\chi$ is finitely-generated and admissible.*

We skip the proof of this lemma for the sake of time.

Just like over $\mathbb{R}$, if χ is unitary, so is $I_B^G\chi$. The same argument as we presented for $\mathbb{R}$ works in this case. However, note that $I_B^G\chi$ is not complete.

Definition 8.15. Given a representation (π, V) of G, we define its *Jacquet module* $R_B^G\pi$ as the representation of T on $V_U = V/V(U)$, the coinvariance, where

$$V(U) = \langle v = \pi(u)v \mid v \in V, u \in U\rangle,$$

and the natural T-action is multiplied by $\delta_B^{-1/2}$.

We twist the action by $\delta_B^{-1/2}$ for convenience of notation. We did not do this for $\mathrm{SL}_2(\mathbb{R})$ and so a lot of the formulas involved $\delta_B^{-1/2}$.

Fact 8.16 (Frobenius reciprocity)**.** *For any representation (π, V) of G, we have*

$$\mathrm{Hom}_G(\pi, I_B^G\chi) \cong \mathrm{Hom}_T(R_B^G\pi, \chi).$$

Lemma 8.17. *We have*

$$V(U) = \left\{ v \in V \;\middle|\; \text{there exists a c.o.s. } U_0 \subseteq U \text{such that } \int_{U_0} \pi(u)v\,du = 0 \right\}.$$

Corollary 8.18. *The functor $V \mapsto V_U$ on the category of smooth B-representations is exact.*

Proof. Right-exactness is immediate (taking coinvariance is always right exact). Left exactness follows from Lemma 8.17. □

Corollary 8.19. *The functor R_B^G is exact.*

We first prove a lemma. We will use the notation:

$$\begin{aligned} e_K\colon V &\to V^K \\ v &\mapsto \frac{1}{\mathrm{vol}(K)}\int_K \pi(k)v dk. \end{aligned}$$

Lemma 8.20. *For a representation (π, V) of G, the following are equivalent:*

1. *$R^G_B\pi = 0$,*
2. *for any compact open subgroup $K \subseteq G$ and any vector $v \in V$, the function $g \mapsto e_K\pi(g)v$ is compactly supported,*
3. *any matrix coefficient of π is compactly supported.*

Proof. We first show that (2) is equivalent to (3). Assume (2) and show (3). Given $v \in V$, $\eta \in V^\vee$, choose $K \subseteq G$ c.o.s. such that $v \in V^K$, $\eta \in (V^\vee)^K$ (one exists by smoothness). Then

$$\begin{aligned} f_{v,\eta}(g) &= \langle gv, \eta\rangle \\ &= \langle gv, e_K\eta\rangle \\ &= \langle e_K gv, \eta\rangle \end{aligned}$$

which is compactly-supported since $e_K gv$ is. Conversely, given $v \in V$, choose a basis $\eta_1, \dots, \eta_n$ for $(V^\vee)^K$. Then $e_K gv = 0$ if and only if $\langle e_K gv, \eta_i\rangle = 0$ for all i, and hence

$$\mathrm{supp}(e_K gv) \subseteq \bigcup_i \mathrm{supp}(f_{v,\eta_i})$$

and the latter is compact. This shows that (2) is equivalent to (3).

Let us now show that (2) implies (1). Let $v \in V$ and K be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ such that $v \in V^K$. Let $t = \lambda_1(p) = \begin{bmatrix} p & 0 \\ 0 & p^{-1}\end{bmatrix}$. Then, by (2), $e_K t^m v = 0$ for sufficiently large m. Then

$$e_K t^m v = t^m e_{t^{-m}Kt^m} v = 0$$

for $m \gg 0$. Write $\overline{U}$ for the lower-triangular matrices with 1 on the diagonal. Then

$$K = \underbrace{(K\cap U)}_{K_u}\cdot\underbrace{(K\cap T)}_{K_T}\cdot\underbrace{(K\cap \overline{U})}_{K_{\overline{U}}},$$

see Remark 8.21. Now,

$$0 = e_{t^{-m}Kt^m}v = e_{t^{-m}K_Ut^m}e_{t^{-m}K_Tt^m}e_{t^{-m}K_{\overline{U}}t^m}v = e_{t^{-m}K_Ut^m}v.$$

By Lemma 8.17, $v \in V(U)$.

We just have to prove (1) implies (3), which is similar to the argument for (2) implies (1). Suppose $v \in V$, $\eta \in V^\vee$ are such that $f_{v,\eta}$ is not compactly supported. By Cartan decopmosition, the support of $f_{v,\eta}$ meets infinitely many K_0 double cosets. Choose $K \subseteq K_0$, compact open, such that $v \in V^K$, $\eta \in (V^\vee)^K$. Since K_0/K is finite, there exists $k_1, k_2 \in K_0$ such that the support of $f_{v,\eta}$ meets

$$k_i K t^m K k_2$$

for infinitely many m. Replace v by $k_2 v$ and η by $k_1^{-1}\eta$ to remove k_1, k_2 from the notation.

Now,

$$\begin{aligned} 0 &\neq f_{v,\eta}(t^m) \\ &= f_{v,e_{K\cap U}\eta}(t^m) \\ &= \langle t^m v, e_{K\cap U}\eta\rangle \\ &= \langle e_{K\cap U} t^m v, \eta\rangle \\ &= \langle t^m e_{t^{-m}K\cap U t^m} v, \eta\rangle. \end{aligned}$$

Therefore, $e_{t^{-m}K\cap U t^m} v \neq 0$ for infinitely many $m > 0$. Since t^{-m} acts expanding on $K \cap U$, we see that $e_{K_U} v \neq 0$ for any compact open $K_U \subseteq U$. By Lemma 8.17, this shows that $v \notin V(U)$, so the image of v in V_U is non-zero, so $V_U \neq 0$. □

Remark 8.21. We skipped a few details. First, we did not define the dual of an admissible representations: it is defined as $(V^\vee)^\infty$ (first, take the algebraic dual, and then take the smooth vectors). Second, we did not prove that the dual of an admissible representation is admissible. This is true and elementary, but we skipped it for the sake of time.

Remark 8.22. An compact open subgroup $K \subseteq G$ has a *Iwahori decomposition* if and only if the multiplication map

$$(K \cap U) \times (K \cap T) \times (K \cap \overline{U}) \to K$$

is bijective. The fact we used in the proof above is that congruence subgroups have Iwahori decomposition. Moreover, the element $t = \lambda_1(p)$, acting by conjugating, fixes $K \cap T$, is contracting on $K \cap U$, and is expanding on $K \cap \overline{U}$.

Definition 8.23. A representation (π, V) that satisfies the equivalent conditions of Lemma 8.20 is called *supercuspidal*.

Remark 8.24. Note that there are no *supercuspidal* representations of $\mathrm{GL}_2(\mathbb{R})$. Indeed, the matrix coefficients satisfy a differential equation and cannot be compactly supported unless they are 0.

Lemma 8.25. *An irreducible supercuspidal representation is both injective and projective in the category of smooth representations.*

Proof. The two statements are equivalent under duality.[3] So we prove projectivity.

Let (π, V) be an irreducible supercuspidal representation. Then, in particular, it is discrete series.[4] Choose $v_0 \in V$, $v_0^\vee \in V^\vee$ such that

$$\langle v_0, v_0^\vee\rangle = \deg(\pi, dg).$$

Let (σ, W) be a representation with a surjective map

$$f\colon (\sigma, W) \to (\pi, V).$$

Choose a preimage $w_0 \in W$ of v_0. Define

$$h\colon (\pi, V) \to (\sigma, W)$$

[3]The dual of a supercuspidal representation is supercuspidal, because their matrix coefficients are the same.

[4]Recall that discrete series are representations whose matrix coefficients are square integrable. This is definitely true if they are compactly supported.

by

$$h(v) = \int_G \langle \pi(g)^{-1}v, v_0^\vee \rangle \sigma(g) w_0 dg.$$

This integral clearly converges. For any $v^\vee \in V^\vee$,

$$\begin{aligned}
\langle f(h(v)), v^\vee \rangle &= \int_G \langle g^{-1}v, v_0^\vee \rangle \langle g v_0, v^\vee \rangle dg \\
&= \langle v_0, v_0^\vee \rangle \langle v, v^\vee \rangle \deg(\pi, dg)^{-1} && \text{by Lemma 7.54} \\
&= \langle v, v^\vee \rangle.
\end{aligned}$$

Therefore, $f(h(v)) = v$, and hence h is a splitting of f. □

Corollary 8.26. *No irreducible subquotient of $I_B^G\chi$ is supercuspidal.*

Proof. Suppose there is $V_1 \subsetneq V_2 \subseteq I_B^G\chi$ such that V_1/V_1 is irreducible and supercuspidal. By Lemma 8.25, the map $V_2 \to V_2/V_1$ splits as

$$V_2/V_1 \hookrightarrow V_@ \hookrightarrow I_B^G\chi.$$

By Frobenius reciprocity, we get a non-zero map

$$R_B^G(V_2/V_1) \to \mathbb{C}(\chi),$$

and hence $R_B^G(V_2/V_1) \neq 0$, which is a contradiction. □

Lemma 8.27. *There is a short exact sequence of T-modules*

$$0 \longrightarrow \chi^{-1} \longrightarrow R_B^G I_B^G \chi \longrightarrow \chi \longrightarrow 0$$

and it splits if and only if $\chi \neq 1$.

Proof. The representation $R_B^G I_B^G\chi$ only depends on the restriction of $I_B^G\chi$ to B. So we are looking at B-double costs of G. Recall that Bruhat decomposition

$$G = B \dot\cup BwB$$

where B is closed and BwB is open and dense. Let $V_0 \subseteq I_B^G\chi$ be the subspace of functions supported on BwB. It is a B-submodule. The functions in V_0 are precisely those f such that $f(1) = 0$. Thus we have a short exact sequence of B-modules:

$$0 \longrightarrow V_0 \longrightarrow I_B^G\chi \xrightarrow{f \mapsto f(1)} \mathbb{C}(\chi \cdot \delta_B^{1/2}) \longrightarrow 0.$$

Since the Jacquet module is exact, we get a short exact sequence

$$0 \longrightarrow V_0/V_0(U) \longrightarrow R_B^G I_B^G\chi \longrightarrow \mathbb{C}(\chi) \longrightarrow 0.$$

We claim that for any $f \in V_0$, there exists a compact open subgroup $U_1 \subseteq U$ such that f is supported on BwU_1. Since f is locally constant, there exists a compact open subgroup $\overline{U_1} \subseteq \overline{U}$ such that $f|_{\overline{U_1}} \equiv 0$.

Now, a matrix computation shows that:

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & 0 \\ 0 & -x \end{bmatrix} w \begin{bmatrix} 1 & x^{-1} \\ 0 & 1 \end{bmatrix}. \tag{6}$$

This shows that if $\overline{U_1}$ is described by $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ such that $\operatorname{ord}(x) \geq k$, then we may take U_1 to be $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ for $\operatorname{ord}(x) > -k$. This proves the claim.

Therefore, the integral

$$\int_U f(w \cdot u) du$$

converges for $f \in V_0$. Define

$$L_\infty \colon V_0 \to \mathbb{C}, \quad L_\infty(f) = \int_U f(w \cdot u) du.$$

Then we get a map of T-modules:

$$L_\infty \colon V_0/V_0(U) \to \mathbb{C}(\chi^{-1}\delta_B^{1/2}).$$

It is clearly surjective (for a good choice of f, we can show that the integral is non-zero). By Lemma 8.17, this map is an isomorphism. Therefore, the exact sequence above gives

$$0 \longrightarrow \mathbb{C}(\chi^{-1}) \longrightarrow R_B^G I_B^G \chi \longrightarrow \mathbb{C}(\chi) \longrightarrow 0$$

via the isomorphism $V_0/V_0(U) \cong \mathbb{C}(\chi^{-1}\delta_B^{1/2})$.

Now, we come to the splitting. By Fact 8.12, if $\chi \neq \chi^{-1}$, the sequence splits.

Now suppose $\chi \neq \chi^{-1}$, $\chi \neq 1$. We try to extend L_∞ to all of $I_B^G\chi$. There is a convergence issue, but we can use analytic continuation to overcome it. Let $\chi_s = \chi \cdot \delta_B^{s/2}$ for $s \in \mathbb{C}$. Consider

$$L_\infty(s) \colon I_B^G\chi_s \to \mathbb{C}(\chi_s^{-1}\delta_B^{1/2}).$$

Note that $I_B^G\chi_s$ is independent of s via restriction ot K. We are looking at

$$\int_{\mathbb{Q}_p} f\left(w \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}\right) du.$$

We split this as two integrals according to $\mathbb{Q}_p = \mathbb{Z}_p \cup \mathbb{Q}_p \setminus \mathbb{Z}_p$. The integral over $\mathbb{Z}_p$ converges so we only deal with the other part. Now, 6 shows that

$$w \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -u^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -u^{-1} & 0 \\ 0 & -u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u^{-1} & 1 \end{bmatrix}.$$

The integral over $\mathbb{Q}_p \setminus \mathbb{Z}_p$ via the substitution $u \mapsto u^{-1}$ becomes

$$\int\limits_{u \in p\mathbb{Z}_p} \chi_s(-u)\delta_B^{1/2}(-u)|u|^{-2} f\left(\begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}\right) du.$$

The convergence problem is not with f but with the characters in the integrand. There is N such that

$$f\Big|_{\begin{bmatrix} 1 & 0 \\ p^N\mathbb{Z}_p & 1 \end{bmatrix}}$$

is constant. The problematic part is (after substituting $u \mapsto -u$)

$$\int\limits_{p^N\mathbb{Z}_p} \chi_s(u) \underbrace{|u|^{-1}du}_{d^\times u} = \sum_{n \geq N} \chi_s(p^n) \int_{u \in \mathbb{Z}_p^\times} \chi(u) d^\times u.$$

If χ is ramified, this integral is 0. In this case, the analytic continuation comes for free.[5] When χ is unramified, we get that the sum is equal to

$$\mathrm{vol}(\mathbb{Z}_p^\times, d^\times u) = \frac{1}{1-\chi(p)p^{-s}}.$$

This has a pole at $s = 0$ if χ is trivial and no pole at $s = 0$ if $\chi \neq 1$. This shows the splitting when $\chi \neq 1$.

Dealing with the case $\chi = 1$ would take a whole other lecture, so we will skip it. We already know that T acts on $R_B^G\chi$ as $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ and one has to compute $*$ explicitly. □

Lemma 8.28. *Let σ be a subquotient of $I_B^G\chi$. Then σ is a submodule of either $I_B^G\chi$ or $I_B^G\chi^{-1}$.*

Proof. Take $V_1 \subseteq V_2 \subseteq I_B^G\chi$ such that σ_1 is V_2/V_1. Applying R_B^G, we get $R_B^G\sigma$ as a subquotient of $R_B^G I_B^G\chi$. By Corollary 8.26, $R_B^G\chi \neq 0$, so by Lemma 8.27, $R_B^G\sigma$ is one of χ, χ^{-1} or $R_B^G I_B^G\chi$. Applying Frobenius reciprocity completes the proof. □

Recall that (π, V) is supercuspidal if and only if $R_B^G\pi = 0$. Therefore, π is not supercuspidal, then $R_B^G\pi \neq 0$ and is finitely generated (as a representation of T), and hence has an irreducible quotient which is a character. Frobenius reciprocity then shows that $\pi \hookrightarrow I_B^G\chi$.

To describe all non-supercuspidal representations, we need to study the structure of $I_B^G\chi$.

Lemma 8.29. *The representation $I_B^G\chi$ has length* 1 *or* 2*.*

Proof. We claim that if $V_1 \subsetneq V_2 \subseteq I_B^G\chi$, then $R_B^G V_1 \subsetneq R_B^G V_2$.

The representation V_2/V_1 has an irreducible subquotient by Lemma 8.10 (2), i.e. we have

$$V_1 \subseteq V_3 \subsetneq V_4 \subseteq V_2$$

and V_4/V_3 is irreducible. By Corollary 8.26, is not supercuspidal. Thus, by exactness of Jacquet functor (Corollary 8.19),

$$R_B^G V_1 \subseteq R_B^G V_3 \subsetneq R_B^G V_4 \subseteq R_B^G V_2.$$

Assume by contradiction that

$$V_1 \subsetneq V_2 \subsetneq I_B^G\chi.$$

[5] Note that this was also the case in Tate's thesis.

By the claim we proved above,

$$R_B V_1 \subsetneq R_B^G V_2 \subsetneq R_B^G I_B^G \chi,$$

contradicting Lemma 8.27. ☐

Corollary 8.30. *If χ is unitary, then $I_B^G\chi$ is irreducible unless $\chi = \chi^{-1}$ and $\chi \neq 1$, in which case $I_B^G\chi$ is the direct sum of two inequivalent irreducible representations.*

Proof. Since χ is unitary, so is $I_B^G\chi$, so composition series splits, and hence it is enough to compute

$$\dim \operatorname{End}_G(I_B^G\chi).$$

But $\operatorname{End}_G(I_B^G\chi) = \operatorname{Hom}_T(R_B^G I_B^G\chi, \chi)$ had dimension 1 unless $\chi = \chi^{-1}$ and $\chi \neq 1$ (by Lemma 8.27). ☐

We now have to deal with the case where χ is not unitary.

Lemma 8.31. *The following are equivalent:*

(1) *the G-module $I_B^G\chi$ has a one-dimensional submodule,*
(2) *the U-module $I_B^G\chi$ has a one-dimensional submodule,*
(3) $\chi = \delta_B^{-1/2}$.

Proof. It is clear that (1) implies (2). To prove that (3) implies (1), note that when $\chi = \delta_B^{-1/2}$, then $I_B^G\chi$ is the set of smooth functions of $B\backslash G$, and we have a 1-dimensional submodule of constant functions.

We need to prove that (2) implies (3). Assume there is a 1-dimensional U-submodule spanned by $f \in I_B^G\chi$. Thus there is a character $\psi\colon U \to \mathbb{C}^\times$ such that

$$f(tugu') = \delta_B^{1/2}(t)\chi(t)f(g)\psi(u'). \tag{7}$$

In particular, the support of f is a union of (B,U)-double cosets. By Bruhat decomposition 8.4, the options are:

(i) B,
(ii) BwU,
(iii) G.

Clearly, (i) is not possible, because f is locally constant. Also, (ii) is no possible, because otherwise $f(1) = 0$, and by the proof of Lemma 8.27 we know that f is supported on BwU_1 for some compact open $U_1 \subseteq U$, and this contradicts 7.

Therefore, (iii) is true, and in particular $f(1) \neq 0$. Now,

$$f(1) = f(u) = f(1)\cdot\psi(u)$$

and hence $\psi \equiv 1$. By the matrix computation 6 in the proof of Lemma 8.27, we see that

$$f(1) = f\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right) = \chi(-x^{-1})\delta_B^{1/2}(-x^{-1})f(w)$$

for sufficiently small x. So the value $\chi(-x^{-1})\delta_B^{1/2}(-x^{-1})$ is constant for small x, hence $\chi = \delta_B^{-1/2}$. ☐

Corollary 8.32. *Assume χ is not unitary.*

(1) *If $\chi \neq \delta_B^{\pm 1/2}$, then $I_B^G\chi$ is irreducible.*
(2) *If $\chi = \delta_B^{-1/2}$, then we have the non-split exact sequence*

$$0 \longrightarrow \mathbb{C} \longrightarrow I_B^G\chi \longrightarrow \mathrm{St} \longrightarrow 0.$$

(3) *If $\chi = \delta_B^{1/2}$, we have the non-split exact sequence*

$$0 \longrightarrow \mathrm{St} \longrightarrow I_B^G\chi \longrightarrow \mathbb{C} \longrightarrow 0.$$

Definition 8.33. The representation St in Corollary 8.32 is the *Steinberg* representation.

Proof. Assume $I_B^G\chi$ is irreducible. Let $V_0 \subseteq I_B^G\chi$ be the $\mathbb{C}$-submodule $\{f \mid f(1) = 0\}$.

Fact. The module $V_0(U)$ is irreducible as a B-module.[6]

The composition series of $I_B^G\chi$ as a B-module has length 3 with two 1-dimensional subquotients and one infinite-dimensional. Assume $I_B^G\chi$ has a finite-dimensional B-submodule, so a finite-dimensional U-submodule, and hence a 1-dimensional one. By Lemma 8.31, $\chi = \delta_B^{-1/2}$. Since the length of $I_B^G\chi$ is 2 by Lemma 8.29, we are done.

If $I_B^G\chi$ does not have a finite-dimensional B-submodule, then $(I_B^G\chi)^\vee$ does, so we are almost done by dualizing. We just need to show that the two representations St are isomorphic. This follows from Lemma 8.34 below. □

Lemma 8.34. *Suppose $\chi \neq \chi^{-1}$. Then there is a non-trivial intertwining operator*

$$I_B^G\chi \to I_B^G\chi^{-1}.$$

In particular, if $I_B^G\chi$ is irreducible, then this is an isomorphism.

Proof. Apply Lemma 8.27 to get $R_B^G I_B^G\chi \to \chi^{-1}$ and Frobenius reciprocity to get the result. □

Remark 8.35. So far, we have all the non-supercuspidal irreducible representations:

(1) the trivial representation $\mathbb{C}$,
(2) the Steinberg St,
(3) the two distinct constituents of $I_B^G\chi$ when $\chi^2 = 1$, $\chi \neq 1$,
(4) the irreducible $I_B^G\chi \cong I_B^G\chi^{-1}$.

There are no isomorphisms between them by looking at the Jacquet modules.

Unitarity, temperedness, and discreteness. We list the properties of the representations listed above. Most of these things followed from what we discussed above, but we point out when something needs a proof.

- We know that $I_B^G\chi$ is unitary if χ is.

[6] This fact is proved using the theory of Kirillov models, which we will not discuss. It takes a while to set up and it is something that is only useful for $\mathrm{GL}(n)$, but does not generalize to other reductive groups. It uses the *mirabolic* subgroup $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ and the reason for the name is that such subgroups only exist for $\mathrm{GL}(n)$.

- We have *complementary series* for $\chi = \delta_B^{s/2}$ for $-1 < s < 1$, unitary. The proof is the same as in the real case.

Theorem 8.36 (Casselman's pairing)**.** *For (π, V) irreducible, admissible representation, there is a non-degenerate pairing*

$$\langle \mid \rangle \colon R_B^G \pi \otimes R_B^G \pi^\vee \to \mathbb{C}$$

satisfying the following condition: given $v \in V$, $v^\vee \in V^\vee$, there is $\epsilon > 0$ such that for $t \in T = \mathbb{Q}_p^\times$, $|t| < \epsilon$,

$$\langle \pi(t)v, v^\vee \rangle = \delta_B^{1/2}(t) \langle R_B^G \pi(t) v_u, v_u^\vee \rangle_B$$

where $v_u \in V_u$, $v_u^\vee \in V_U^\vee$ are the images.

The proof of this theorem would take a few lectures, so we will omit it here.

This theorem reduces the study of matrix coefficients to the Jacquet module, which has a simple description. We apply this together with Lemma 8.27 to see that:

- The matrix coefficient of $I_B^G \chi$ has the form
$$a \cdot \delta_B^{1/2} \chi(t) + b \delta_B^{1/2}(t) \chi^{-1}(t).$$
These are bounded as $t \to 0$ if and only if $|\chi(t)| = |t|^s$ with $-1 < \mathrm{Re}(s) < 1$.
- The representation $I_B^G \chi$ is tempered if and only if $s \in i\mathbb{R}$, i.e. χ is unitary.
- For Steinberg, $R_B^G \operatorname{St} = \mathbb{C} \delta_B^{1/2}$, so the matrix coefficients are $\delta_B(t) = |t|^2$ which are not integrable, but square-integrable. This is hence a discrete series representation.

Fact 8.37. *Steinberg is the unique discrete series representation of G that is not supercuspidal.*

Here is the analogy between the real and the p-adic case:

real case	p-adic case
non-unitary: $I_B^G \chi$ and χ not unitary	(same)
unitary non-tempered: composition series	(same)
tempered but not discrete $I_B^G \chi$ with χ unitary or its two pieces when $\chi^2 = 1$, $\chi \neq 1$	(same)
discrete: $\mathcal{D}_k^+ \oplus \mathcal{D}_k^-$ one for each finite-dimensional representation	discrete: Steinberg St (same)
supercupidal: none	supercuspidal: they exist!

We will see that supercuspidal representations exist in the p-adic case next time.

One can think of the discrete series together with supercuspidal case as one case, in which case the analogy does not break down. This can be seen using Harish–Chandra parameters, which we define below (Definition 8.41).

Hecke algebra. The Hecke algebra is the locally constant functions on G:

$$\mathcal{H} = \mathcal{H}(G) = \mathbb{C}_c^\infty(G),$$

which is an algebra under convolutions:

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy$$

where we normalize the measure so that $\mathrm{vol}(K_0, dy) = 1$.

Then $\mathcal{H}$ is an associative, non-commutative algebra with no unit. For a compact open subgroup $K \subseteq G$, consider

$$e_K = 1_K \cdot \mathrm{vol}(K, dy)^{-1}.$$

Then $e_K * e_K = e_K$ and we define

$$\mathbb{H}_K = e_K * \mathbb{H} * e_K,$$

which is an associative algebra, with unit e_K. In fact, it is the subspace of K-biinvariant functions in $\mathbb{H}$ and

$$\mathbb{H} = \bigcup_K \mathbb{H}_K.$$

Given a smooth G-representation (π, V), we promote it to an $\mathbb{H}$-module by defining the action as

$$\pi(f)v = \int_G f(g)\pi(g)vdg.$$

Moreover, V^K becomes a $\mathbb{H}_K$-module this way.

Proposition 8.38. *The functor $V \mapsto V^K$ gives a bijection between isomorphism classes of irreducible representations with $V^K \neq 0$ and isomorphism classes of irreducible $\mathbb{H}_K$-module.*

Definition 8.39. The distribution $\Theta_\pi(f) = \mathrm{tr}(\pi(f))$ is the *Harish–Chandra distribution* character of π.

Theorem 8.40. *There is a (unique) locally integrable function $\theta_\pi \colon G \to \mathbb{C}$ such that*

$$\Theta_\pi(f) = \int_G f(g)\theta_\pi(g)dg.$$

Definition 8.41. The *Harish-Chandra character* is θ_π.

These can be computed explicitly for all the representations we discussed.

Guest lecture by Stephen DeBacker (4/3/19): supercuspidal representations. We start with some motivation.

(1) We know that for finite groups, the set of characters of representations $\{\theta_\pi \mid \pi \in \hat{G}\}$ is supposed to separate conjugacy classes.

For p-adic groups, the Harish–Chandra character θ_π is a function on G^{reg} for any admissible π, so we kind of expect that in the p-adic setting $\{\theta_\pi : \pi \in \hat{G}\}$ should separate regular semisimple orbits.

(2) What is G^{reg}? In general,

$$G^{\mathrm{reg}} = \{\gamma \in G \mid C_G(\gamma)^0 \text{ is a torus}\} = \{\gamma \in G \mid D_G(\gamma) \neq 0\},$$

where $D_G(\gamma)$ is given by

$$\det_{\mathfrak{g}}(t - (\mathrm{Ad}(\gamma) - 1)) = t^{\cdots} + D_G(\gamma)t^{\ell}$$

where ℓ is the semisimple rank of G.

For GL_2 or SL_2, $\gamma \in G^{\mathrm{reg}}$ if and only if γ has distinct eigenvalues. In this case, $D_G(\gamma) = (\lambda - \lambda')^2$ where λ and λ' are the distinct eigenvalues of γ. Explicitly:

$$D_G\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+d)^2 - 4.$$

(3) Let

$$\mathcal{T} = \{S \subseteq G \mid S \text{ maximal } \mathbb{Q}_p\text{-torus}\}.$$

We write $k = \mathbb{Q}_p$. What do elements of $\mathcal{T}$ look like? They could be

- *Split* — tori which are G-conjugate to $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in k, ad = 1 \right\}$.
- *Elliptic* — tori of the form $T^{\theta,\eta} = \left\{ \begin{bmatrix} a & b\eta^{-1} \\ b\theta\eta & a \end{bmatrix} \;\middle|\; a^2 - b^2\theta = 1 \right\}$ for $\theta \in k^\times \setminus (k^\times)^2$, $\theta \in \{\epsilon, \omega, \omega\epsilon\}$ where $\epsilon \in (\mathbb{Z}_p^\times) \setminus ((\mathbb{Z}_p)^\times)^2$ and ω is the uniformizer of k. The element $\eta \in k^\times$ can be arbitrary for now, but we will next write down all the conjugacy classes.

  We have that

$$|N_G(A)/A| = 2$$

  and

$$|N_G(T^{\theta,\eta})/T^{\theta,\eta}| = \begin{cases} 1 & -1 \notin N(k(\sqrt{\theta})), \\ 2 & -1 \in N(k(\sqrt{\theta})). \end{cases}$$

  Elliptic tori in $\mathrm{SL}_2(k)$ up tp G-conjugation: for $\mathfrak{f} = \mathbb{F}_p$, the residue field of k,
  - if $-1 \in (\mathfrak{f}^\times)^2$: $T^{\epsilon,1}$, $T^{\epsilon,\omega}$, $T^{\omega,1}$, $T^{\omega,\epsilon}$, $T^{\epsilon\omega,1}$, $T^{\epsilon\omega,\epsilon}$,
  - if $-1 \notin (\mathfrak{f}^\times)^2$: $T^{\epsilon,1}$, $T^{\epsilon,\omega}$, $T^{\omega,1}$, $T^{\epsilon\omega,1}$.

Fact 8.42. *Fix a maximal k-torus T and $H \leq G$ open subgroup. The map*

$$\begin{aligned} H \times (T \cap G^{\mathrm{reg}}) &\to G^{\mathrm{reg}} \\ (h,t) &\mapsto {}^h t = hth^{-1} \end{aligned}$$

is a submersion. In particular, ${}^H(T \cap G^{\mathrm{reg}})$ is open.

Corollary 8.43. *We have that*

$$G^{\mathrm{reg}} = \bigcup_{T \in \mathcal{T}/G\text{-conj.}} {}^H(T \cap G^{\mathrm{reg}})$$

is a disjoint union of opens.

> Thus, in analogy with the situation for finite groups of Lie type[7], we expect irreducible representations of G to be, roughly, parameterized by pairs (T, ψ), where T is a maximal k-torus and $\psi \in \hat{T}$.

The goal will be to make this formal in order to write down supercupsidal repreentations.

We already saw that if $\chi \in \hat{A}$, then we can extend it to $\tilde{\chi} \in \hat{B}$ where

$$\widetilde{\chi}\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = \chi\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

We then have $I_B^G(\chi)$ and if $\chi^2 \neq 1$, this is irreducible, and if $\chi^2 = 1$, we need to do some work and pin down the irreducible components.

Basically, we have that

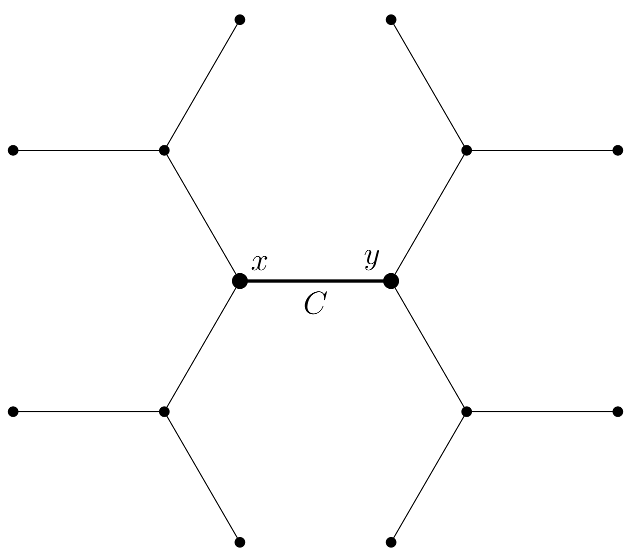
$$\theta_{I_B^G \chi}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = (\text{constants}) \cdot \left(\chi\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} + \chi\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\right).$$

What about the Harish–Chandra parameters for non-split groups?

Recall Mackey theory which says that for σ finite-dimensional, $\mathrm{Ind}_H^G \sigma$ is irreducible if and only if

(1) $\sigma \in \hat{H}$,
(2) if $g \in G$ intertwines σ with itself (i.e. $\langle \chi_\sigma, \chi_{g^*\sigma} \rangle_{g^{-1}H \cap H} \neq 0$), then $g \in H$.

The idea is that for an elliptic maximal torus T, we have a map $\mathcal{B}(T) \to \mathcal{B}(G)$. Here, $\mathcal{B}(-)$ is the *building* of a group. Since T is elliptic $\mathcal{B}(T)$ is a point. Since $\mathcal{B}(G)$ is an infinite tree, each vertex has valence $p + 1$. Although we assume $p \neq 2$, we can draw the tree of $\mathcal{B}(\mathrm{SL}_2(\mathbb{Q}_2))$ to illustrate what is looks like. It is an infinite tree with valency 3 so it looks as follows:



[7]For finite groups of Lie type Deligne–Lusztig theory is the correct approach. The relevant representations are $\pm R_{T,\psi}^G$, classified by a maximal torus T and a character χ.

The group G acts on the tree. For any part of the tree $P \subseteq \mathcal{B}(G)$, $\mathrm{Stab}_G(P)$ is a compact open subgroup of G. It is not too much of a lie to say that interesting compact open subgroups of G arise as $\mathrm{Stab}_G(P)$.

Pick any vertices x, y on the tree and C be the edge connecting them (as in the picture above). Then $\overline{C}$ is a fundamental domain for the action of G on $\mathcal{B}(G)$.

We set things up so that

$$\mathrm{Stab}_{\mathrm{SL}_2(k)}(x) = \mathrm{SL}_2(R) = K_0,$$

where $R = \mathbb{Z}_p$ is the ring of integers of k, and

$$\mathrm{Stab}_{\mathrm{SL}_2(k)}(y) = {}^{\pi}K_0 = Q_0,$$

where $\pi = \begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix}$, and

$$\mathrm{Stab}_{\mathrm{SL}_2(k)}(C) = K_0 \cap Q_0 = \mathcal{I},$$

the *Iwahori subgroup*. Then

$$\mathcal{B}(T^{\epsilon,1}) = x, \quad \mathcal{B}(T^{\epsilon,\omega}) = y$$

and all the other elliptic maximal tori give points in C.

We have the *Moy-Prasad filtration*

$$\begin{aligned} G_{x,0} &= K_0 = \mathrm{SL}_2(R) \\ G_{x,0^+} = G_{x,1} &= K_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in 1 + \mathfrak{p} \right\} \\ G_{x,1+} = G_{x,2} &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in 1 + \mathfrak{p}^2 \right\} \\ &\vdots \end{aligned}$$

In general, $G_{x,r} = G_{x,\lceil r \rceil}$ for $r > 0$. Roughly, this filtration is obtained by looking at $\mathrm{Stab}_{SL_2(k)}(C_r)$ where the region C_r of the graph is a *circle of radius r centered at x.*

To get the filtration for y, we conjugate by the matrix π. For any maximal elliptic torus $z \in C$, we have a similar filtration $G_{z,0} \supseteq G_{z,0^+} = G_{z,\frac{1}{2}} \supseteq G_{z,(\frac{1}{2})^+}$ and so on. One can write it down explicitly, but we omit this here.

Construction of supercuspidal representations. Let T be a maximal elliptic torus in $\mathrm{SL}_2(k)$. This gives a point $x \in \mathcal{B}(G)$. Let T_0 be the connected component of the identity, which lives between T_{0^+} and T. Define

$$T_r = T_0 \cap G_{x,r}.$$

Then

$$T_0 \cong \mathrm{Norm}^1(k(\sqrt{\theta}))^0$$

(the connected component of identity of norm 1 elements). The filtration T_r corresponds to the obvious filtration of $k(\sqrt{\theta})^\times$.

Take a character $\psi \in \hat{T}$. Suppose $\mathrm{Res}_{T_r} \psi \neq 1$ but $\mathrm{Res}_{T_{r^+}} \psi 1$. Then $\varrho(\psi) = r$ be the depth of ψ.

See Rabinoff's senior thesis for an exposition of the case $r = 0$. In this lecture, suppose $r > 0$ and fix $\Lambda\colon k^+ \to \mathbb{C}$ of conductor $\mathfrak{p}$.

By the magic of SL_2, there exists $X_\psi \in \mathfrak{t}_{-r} \setminus \mathfrak{t}_{-r^+}$ such that

$$\psi(\gamma) = \Lambda(\mathrm{Tr}(c^{-1}(\gamma)X_\psi))$$

for all $\gamma \in T_r$, where

$$\begin{aligned} c\colon \mathfrak{g}_{0^+} &\to G_{0^+} \\ X &\mapsto \frac{1+\frac{X}{2}}{1-\frac{X}{2}}. \end{aligned}$$

Fact. $\widehat{T_r/T_{r^+}} \cong \widehat{\mathfrak{t}_r/\mathfrak{t}_{r^+}} \cong \mathfrak{t}_{-r}/\mathfrak{t}_{-r^+}$.

We fatten ψ using the character Λ and the element X_Ψ. Note that $X_\psi \in \mathfrak{t}_{-r} \subseteq \mathfrak{g}_{x,-r}$. As above,

$$\mathfrak{g}_{x,-r}/\mathfrak{g}_{x,-r} \cong G_{x,\widehat{r/G_{x,r^+}}}.$$

In fact, $G_{x,\frac{r}{2}+}/G_{x,r^+}$ is abelian and

$$G_{x,\widehat{\frac{r}{2}+/G}_{x,r^+}} \cong \frac{\mathfrak{g}_{x,-r}}{\mathfrak{g}_{x,-\frac{r}{2}^+}}.$$

Take $\widetilde{\psi} \in T\hat{G}_{x,\frac{r}{2}+}$ given by

$$\widetilde{\psi}(tk) = \psi(t)\Lambda(\mathrm{Tr}(c^{-1}(k))X_\Psi).$$

Then

$$\mathrm{Stab}(\widetilde{\psi}, TG_{x,\frac{r}{2}+}) = TG_{x,\frac{r}{2}}.$$

If $G_{x,\frac{r}{2}} = G_{x,\frac{r}{2}+}$, then $\mathrm{Ind}_{TG_{x,\frac{r}{2}}}^{G} \widetilde{\psi}$ is irreducible and supercuspidal.

If $G_{x,\frac{r}{2}+} \neq G_{x,\frac{r}{2}}$, then let

$$N = \ker(\mathrm{Res}_{G_{x,\frac{r}{2}+}} \widetilde{\psi}) \leq G_{x,\frac{r}{2}+}.$$

It turns out that

$$TG_{x,\frac{r}{2}+}/N$$

is the center; the derived group of $TG_{x,\frac{r}{2}}/N$. This is hence a Heisenberg group, i.e. it fits in a short exact sequence of the form:

$$1 \longrightarrow A \longrightarrow H \longrightarrow A/H \longrightarrow 1.$$

By the Stone–von-Neumann Theorem, there is a unique irreducible representation $\sigma_{\widetilde{\psi}}$ of $TG_{x,\frac{r}{2}}$, whose restriction to $TG_{x,\frac{r}{2}+}$ contains $\widetilde{\psi}$. Then

$$\mathrm{Ind}_{TG_{x,\frac{r}{2}}}^{G} \sigma_{\widetilde{\psi}}$$

is an irreducible supercuspidal representation.

One can classify all irreducible supercuspidal representations this way by tracking through this argument.

The guest lecture ended here and Tasho Kaletha continues from next time.

A review of supercuspidal representations. Let F be any field.

Definition 8.44.

(1) A *maximal torus* of $\mathrm{GL}_2(F)$ is an embedded copy fo $A^\times$ where A/F is a quadratic étale algebra.
(2) A *maximal torus* of $\mathrm{SL}_2(F)$ is the intersection of one for $\mathrm{GL}_2(F)$.

There are two possibilities for A:

- $A = F \oplus F$,
- $A = E/F$, a separatble quadratic field extension.

To embed A into $\mathrm{Mat}(2,2,F)$, choose a basis of A/F and let A act on A by multiplication. Then

$$A \to \mathrm{Mat}(2,2,F) \overset{\det}{\to} F$$

is the norm $N\colon A \to F$. Thus for SL_2, we get a map $A^1 \to \mathrm{SL}_2(F)$, where A^1 denotes the norm 1 elements. Any A has a non-trivial Galois automorphism τ and $N(a) = a \cdot \tau(a)$. Note that for $A = F \oplus F$, τ permutes the two factors and hence:

$$A^1 = \{(x, x^{-1}) \mid x \in F^\times\}.$$

In the non-split case, choose $\eta \in E/F$, then $a + \eta b \in E$ acts on E in term of the basis $\{1, \eta^{-1}\}$ by the matrix $\begin{bmatrix} a & b \\ \eta^2 b & a \end{bmatrix}$. These are the non-split tori defined last time.

Fact 8.45. *All embeddings $A^\times \to \mathrm{GL}_2(F)$ are conjugate.*

This is not true for SL_2.

Fact 8.46. *The embeddings of A^1 into $\mathrm{SL}_2(F)$ are a torsor under $F^\times / N(A^\times)$.*

Examples 8.47.

- Let $F = \mathbb{R}$. The only non-split extension $E = \mathbb{C}$, i.e. the only non-split maximal torus in $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to S^1. We get two embeddings this way, which have the same image but differ by inversion.
- Now let $F = \mathbb{Q}_p$ for $p \neq 2$. There are 3 isomorphism classes of non-split maximal tori.

  When E/F is unramified, the two embeddings have different images and are distinguished by the two vertices in the Bruhat–Tits building.[8]

  For E/F ramified, two embeddings have the same image if and only if $-1 \in F^{\times,2}$ and then differ by inversion. The point in the Bruhat–Tits building associated to this torus is the center of an edge.
- When $F = \mathbb{Q}$, have infinitely many isomorphism classes of maximal toris, and in each infinitely many conjugacy classes.

[8] SL_2 does not act transitively on the vertices of the building, but it acts transitively on the edges. Two two embeddings correspond to the two endpoints, labeled x and y in the picture above.

Theorem 8.48. *Let $F = \mathbb{Q}_p$, $p \neq 2$. The irreducible supercupsidal representations are classified as follows.*

(1) *For each pair (S, θ) where $S \subseteq \mathrm{SL}_2$ is a maximal torus and $\theta \colon S \to \mathbb{C}^\times$ is a smooth character with $\theta^2 \neq 1$, get one. Two such representations are isomorphism if and only if the pairs are conjugate.*
(2) *Each of the two embeddings of $(\mathbb{Q}_p^1)^2$ together with an unramified sign character gives 2 supercuspidal irreducible representations, for a totall of 4.*

Representations in the first class are called *ordinary supercuspidals* (depth is equal to the depth of the character θ, discussed in the previous lecture), and in the second class are called *exceptional supercuspidals* (corresponding to depth 0, which has not discussed in the previous lecture).

Remark 8.49 (Discrete series of $\mathrm{SL}_2(\mathbb{R})$.)**.** The non-trivial characters of S^1 are of the form $z \mapsto z^k \in \mathbb{Z} \setminus \{0\}$. It is enough to fix one embedding and consider all k, or look at both and only $k > 0$. For $k > 0$, have $\mathcal{D}_k = D_k^+$ and $\mathcal{D}_{-k} = \mathcal{D}_k^-$.

Therefore, the discrete series are classified by (S, θ) as in the ordinary supercupsidal case. The *exceptional supercuspidals* have no analog in $\mathrm{SL}_2(\mathbb{R})$.

The Satake transform and unramified representations.

Definition 8.50. A smooth irreducible representation π is called *unramified* (K_0-spherical) if $\pi^{K_0} \neq 0$.

Note that

$$\{\text{Unramified irreducible representations}\} \leftrightarrow \{\text{unramified } \mathcal{H}_{K_0}\text{-modules}\},$$

where we recall that $\mathcal{H}_{K_0}$ is the Hecke algebra.

Given $t \in T$, $f \in \mathcal{H}_{K_0}$, define

$$f^B(t) = \delta_B(t)^{1/2} \int_U f(tu)\, du.$$

Then f^B is a compactly supported function on $T = \mathbb{Q}_p^\times$ and $T_0 = \mathbb{Z}_p^\times$-invariant. In other words, $f^B \in \mathcal{H}_{T_0}(T)$.

Theorem 8.51 (Satake isomorphism)**.** *The map*

$$S \colon \mathcal{H}_{K_0}(G) \to \mathcal{H}_{T_0}(T)^W, \qquad f \mapsto f^B$$

is an isomorphism of algebras, where $W = N(T)/T$.

Before we prove this theorem, we discuss some consequences.

Since $\mathcal{H}_{T_0}(T)$ is commutative, so is $\mathcal{H}_{K_0}(G)$. Moreover, unramified representations of $\mathbb{H}_{K_0}(G)$ correspond to characters $\eta \colon \mathcal{H}_{T_0}(T)^W \to \mathbb{C}$. We have a short exact sequence

$$1 \longrightarrow T_0 \longrightarrow T \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and so $\mathbb{H}_{T_0}(T) = \mathbb{C}[X]$. Therefore, unramified representations correspond to complex numbers.

Definition 8.52. A character of T is *unramified* if it is trivial on T_0, i.e. it is $t \mapsto \delta_B(t)^{s/2}$ for $s \in \mathbb{C}$.

Lemma 8.53. *Let χ be any character of T. Then $(I_B^G\chi)^{K_0}$ is 1-dimensional if χ is unramified and 0-dimensional otherwise.*

Proof. Iwasawa decomposition. □

Lemma 8.54. *For any unramified character χ and $f \in \mathbb{H}_{K_0}(G)$,*

$$\Theta_{I_B^G\chi}(f) = \Theta_\chi(f^B).$$

This is the *adjunction formula* between induction and taking the *constant term* $f \mapsto f^B$.

Proof. We have that $f = e_{K_0} * f * e_{K_0}$, so $\pi(f)\colon V_\pi \to V_\pi$ factors as

$$V_\pi \longrightarrow V_\pi^{K_0} \xrightarrow{\pi(f)} V_\pi \longrightarrow V_\pi^{K_0}.$$

To compute the trace of this operator, take any $0 \neq \phi \in V_\pi^{K_0}$ and note that

$$\operatorname{tr}(\pi(f)) = \pi(f) \cdot \phi.$$

Normalize ϕ such that $\phi(1) = 1$. Therefore:

$$\begin{aligned}
\operatorname{tr}(\pi(f)) &= (\pi(f)\phi)(1) \\
&= \int_G f(g)\phi(g)\,dg \\
&= \int_T\int_U\int_{K_0} f(tuk)\underbrace{\phi(tuk)}_{\delta_B^{1/2}\chi(t)}\,dt\,du\,dk \\
&= \int_T \chi(t)\underbrace{\delta_B^{1/2}(t)\int_U f(tu)\,du}_{f^B(t)}\,dt.
\end{aligned}$$

This gives the result. □

Corollary 8.55. *The character $\eta_\pi\colon \mathbb{C}[X] \to \mathbb{C}$ corresponding to the unramified subquotient of $I_B^G\chi$ sneds X to $\chi(p)$. In particular, $\chi = \delta_B^{s/2}$ and $\chi(p) = p^s$.*

Proof. Note that $X \in \mathbb{C}[X]$ corresponds to $\lambda_1(p) \in \mathcal{H}_{T_0}(T)$. Let f be its preimage in $\mathcal{H}_{K_0}(G)$. Then $\eta_\pi(x)$ be the action of $\pi(f)$ on $V_\pi^{K_0}$, which is equal to $\Theta_{I_B^G\chi}(f) = \theta_\chi(\lambda_1(p)) = \chi(p)$. □

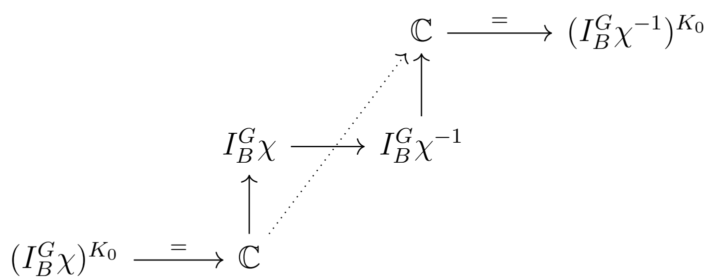
We now turn to the proof of the Satake isomorphism 8.51. We start with a lemma.

Lemma 8.56. *For χ unramified, the $\mathcal{H}_{K_0}$-modules $(I_B^G\chi)^{K_0}$ and $(I_B^G\chi^{-1})^{K_0}$ are isomorphic.*

Proof. If $\chi \neq \delta_B^{\pm 1/2}$, we know that $I_B^G \chi$ is irreducible and isomorphic to $I_B^G \chi^{-1}$. Let now $\chi = \delta_B^{-1/2}$. Then we have a short exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow I_B^G(\chi) \longrightarrow \mathrm{St} \longrightarrow 0.$$

Thus, $\mathbb{C} \cong (I_B^G \chi)^{K_0}$ and $\mathrm{St}^{K_0} = 0$. The intertwining operator now gives



and the diagonal dotted arrow is an isomorphism. □

Proof of the Satake isomorphism 8.51. We will use Lemma 8.54 throughout without specific references. We start by proving the W-invariance of the image:

$$\begin{aligned}\Theta_\chi((Sf)^w) &= \Theta_{{}^w\chi}(Sf) \\ &= \Theta_{\chi^{-1}}(Sf) \\ &= \Theta_{I_B^G\chi^{-1}}(f) \\ &= \Theta_{I_B^G\chi}(f) \\ &= \Theta_\chi(Sf).\end{aligned}$$

We then check that it is an algebra homomorphism. Linearity is obvious, so we just check multiplication:

$$\begin{aligned}\Theta_\chi((Sf_1) * (Sf_2)) &= \Theta_\chi(Sf_1) \circ \Theta_\chi(Sf_2) \\ &= \Theta_{I_B^G\chi}(f_1) \circ \Theta_{I_B^G\chi}(f_2) \\ &= \Theta_{I_B^G\chi}(f_1 * f_2) \\ &= \Theta_\chi(S(f_1 * f_2)).\end{aligned}$$

We now check bijectivity. Note that

- $\mathcal{H}_{T_0}(T)^W$ has basis $(\lambda_n(p) + \lambda_{-n}(p))T_0$ for $n \in \mathbb{Z}/\pm 1$,
- $\mathcal{H}_{K_0}(G)$ has basis $K_0\lambda_n(p)K_0$ for $n \in \mathbb{Z}/\pm 1$ (by the Cartan decomposition).

Therefore,

$$S(1_{K_0\lambda_n(p)K_0}) = \sum_{m \in \mathbb{Z}/\pm 1} a_{n,m} 1_{\lambda_m(p)+\lambda_{-m}(p)},$$

where

$$\begin{aligned} a_{n,m} &= S(1_{K_0\lambda_n(p)K_0})(\lambda_m(p)) \\ &= \delta_B(\lambda_n(p))^{1/2}\int_U 1_{K_0\lambda_n(p)K_0}(\lambda_m(p)u)\,du \\ &= p^u \operatorname{vol}(K_0\lambda_n(p)K_0 \cap K_0\lambda_m(p)U). \end{aligned}$$

We claim (and leave the proof as an exercise) that a matrix belongs to $K_0\lambda_n(p)K_0$ if and only if the minimal valuation of its entries is $-n$.

By Iwasawa decomposition, this implies that $a_{n,n} \neq 0$ and $a_{n,m} = 0$ if $m > n$. Therefore, the matrix is upper-triangular with non-zero entries on the diagonal, and hence invertible. $\square$

Local Langlands correspondence for tori. Next time, we will discuss the local Langlands correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$, but first, we need to discuss this correspondence for tori.

We will consider the cases $S = F^\times$ and $S = E^1$ where E/F is a quadratic eztension. Here, $F = \mathbb{R}$ or $F = \mathbb{Q}_p$.

Define $\hat{S} = \mathbb{C}^\times$ with Γ-action. This is trivial if $S = F^\times$ and the union non-trivial representation of $\Gamma_{E/F}$ if $S = E^1$. Define

$${}^L S = \hat{S} \rtimes W_F,$$

the *Langlands dual* of the torus S.

Theorem 8.57. *There is a group isomorphism between the group of continuous isomorphisms*

$$\begin{array}{ccc} W_F & \longrightarrow & {}^L S \\ & \underset{\text{id}}{\searrow} \quad \swarrow & \\ & W_F & \end{array}$$

up to conjugation by $\hat{S}$ and $\mathrm{Hom}_{\mathrm{cts}}(S, \mathbb{C}^\times)$.

Write $\varphi(w) = (\varphi_0(w), w)$. Then $\varphi_0 \in Z^1(W_f, \hat{S})$ and the $\hat{S}$-conjugacy class of φ corresponds to $[\varphi_0] \in H^1(W_F, \hat{S})$. Another way to state the theorem is that

$$H^1(W_F, \hat{S}) \cong \mathrm{Hom}_{\mathrm{cts}}(S, \mathbb{C}^\times).$$

Proof. In the split case,

$$H^1(W_F, \hat{S}) = \mathrm{Hom}(W_F, \mathbb{C}^\times) = \mathrm{Hom}(F^\times, \mathbb{C}^\times) = \mathrm{Hom}(S, \mathbb{C}^\times).$$

In the non-split case, we claim that corestriction gives an isomorphism

$$H^1(W_E, \hat{S})_{\Gamma_{E/F}} \underset{\cong}{\overset{\mathrm{Cor}}{\to}} H^1(W_F, \mathbb{C}^\times).$$

This follows from the following formula for $\varphi_0 = \mathrm{Cor}\varphi_0^E$. Fix $w \in W_F \setminus W_E$. For $x \in W_E$,

$$\varphi_0(x) = \varphi_0^E(x) \cdot \ {}^{w^{-1}}\varphi_0^E(wxw^{-1}),$$
$$\varphi_0(xw) = \varphi_0^E(x) \cdot \ {}^{w^{-1}}\varphi_0^E(wxw^{-1} \cdot w^2).$$

Now,

$$H^1(W_F, \hat{S}) \cong H^1(W_E, \mathrm{Ind}_{W_E}^{W_F} \mathbb{C}^\times) = \mathrm{Hom}(W_E, \mathbb{C}^\times) = \mathrm{Hom}(E^\times, \mathbb{C}^\times).$$

This completes the proof. □

Note that:

$$E^1 \hookrightarrow E^\times$$
$$\mathbb{C}\times \leftarrow \mathbb{C}^\times \times \mathbb{C}^\times$$
$$\frac{a}{b} \leftarrow\!\shortmid (a, b).$$

On the other hand,

$$E^1 \to E^1$$
$$x \mapsto \frac{x}{\tau x}$$
$$\mathbb{C}^\times \times \mathbb{C}^\times \leftarrow \mathbb{C}^\times$$
$$(a, a^{-1}) \leftarrow\!\shortmid a.$$

This is a manifestation of a general phenomenon, called *functoriality*. If S, T are tori over F and $S \to T$ is a homomorphism, there are dual homomorphism

$$\hat{T} \to \hat{S}$$
$${}^LT \mapsto {}^LS$$

and the local Langlands for tori is functorial.

Embedding the L-group of a maximal torus. Let $S = E^\times$, $S^1 = E^1$. We want to find embeddings:

$${}^LS \to \mathrm{GL}_2(\mathbb{C})$$
$${}^LS^1 \to \mathrm{PGL}_2(\mathbb{C}).$$

Turns out we need a choice: we can send

$$(\mathbb{C}^\times \times \mathbb{C}^\times) \times W_{E/F} \xrightarrow{{}^Lj} \mathrm{GL}_2(\mathbb{C})$$
$$(a, b) \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

but it is not clear where to sends elements of the Weil group. Recall that

$$1 \longrightarrow E^\times \longrightarrow W_{E.F} \longrightarrow \Gamma_{E/F} \longrightarrow 1$$

$$\dot{\tau} \longrightarrow \tau$$

where $\dot{\tau}$ is any lift. Take $e \in E^\times$. Then ${}^L j(1 \rtimes e)$ commutes with ${}^L j((a,b) \rtimes 1)$, so

$$ {}^L j(1 \rtimes e) = \begin{bmatrix} \chi_1(e) & 0 \\ 0 & \chi_2(e) \end{bmatrix} $$

for $\chi_1, \chi_2 \colon E^\times \to \mathbb{C}^\times$. Now, what remain is ${}^L j(\dot{\tau})$. It must be $\begin{bmatrix} 0 & a \\ -a^{-1} & 0 \end{bmatrix}$ where the $-$ is there to make determinant 1. There are more general reasons for putting the $-$ there, but we do not go into them here.

We know that $\dot{\tau} e \dot{\tau}^{-1} = {}^\tau e$. Thus

$$ \chi_2(e) = \chi_1(\bar{e}). $$

Thus,

$$ {}^L j(e) = \begin{bmatrix} \chi(e) & 0 \\ 0 & \chi(\bar{e}) \end{bmatrix} $$

for one $\chi \colon E^\times \to \mathbb{C}^\times$. We know that $\dot{\tau}^2 \in F^\times \setminus N_{E.F}(E^\times)$. Otherwise, $\dot{\tau}^2 = x \cdot {}^\tau x$ for $x \in E^\times$ and $(x^{-1}\dot{\tau})^2 = 1$, contradicting that the Weil group is non-split.

But we also know that

$$ {}^L j(\dot{\tau})^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. $$

Therefore, $\chi|_{F^\times}$ is the local class field theory sign character of E/F.

Fact 8.58. *Any $\chi \colon E^\times \to \mathbb{C}$ such that $\chi|_{F^\times}$ is the sign character for E/F given an L-embedding ${}^L S \to \mathrm{GL}_2(\mathbb{C})$, well-defined up to $\mathrm{GL}_2(\mathbb{C})$-conjugation.*

Warning. Different χ will in general lead to different conjugacy classes of embeddings.

Take χ and consider

$$ \begin{array}{ccc} {}^L S & \xrightarrow{{}^L j_\chi} & \mathrm{GL}_2(\mathbb{C}) \\ \downarrow & & \downarrow \\ {}^L S' & \xrightarrow{\overline{{}^L j_\chi}} & \mathrm{PGL}_2(\mathbb{C}), \end{array} $$

where we note that

$$ \overline{{}^L j_X}(e) = \begin{bmatrix} \chi(e/\bar{e}) & 0 \\ 0 & 1 \end{bmatrix}. $$

What are the choices of χ. In general, they are a torsor under $(E^\times/F^\times)^*$.

- When $F = \mathbb{R}$, $E = \mathbb{C}$, there is a preferred choice for χ, the complex argument.
- When $F = \mathbb{Q}_p$, E unramified, the preferred choice of χ is the unramified quadratic character.
- When $F = \mathbb{Q}_p$, E ramified (assuming again that $p \neq 2$ here), there are two tamely ramified χ and their quotient is the unramified quadratic character. For PGL_2, it turns out that this choice is irrelevant.

Altogether, we have shown the following fact.

Fact 8.59. *When $p \neq 2$, there is always a preferred choice of a map*
$$^{L}S^1 \to \mathrm{PGL}_2(\mathbb{C}).$$

Remark 8.60. Here is another embedding which may seem much more natural:
$$^{L}j((a,b) \rtimes 1) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$
$$^{L}j(1 \rtimes \tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$^{L}j(1 \rtimes e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
It leads to the *naïve Langlands correspondence*, which turns out not to be true. This is why we need to go through the more complicated process above.

The Local Langlands correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$. Let $\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ be a *Langlands parameter*. We describe below the L-packet associated to it.

(0) If $\varphi|_{\mathrm{SL}_2} \neq 1$, then $\varphi\colon\ \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is the natural projection. Then $\varphi|_{W_F} = 1$. We associate to it the Steinberg rerpresentation, $\Pi_\varphi = \{\mathrm{St}\}$.

(1) If $\varphi|_{\mathrm{SL}_2(\mathbb{C})} = 1$, note that $\varphi(P_F)$, where P_F is the wild inertia group, is a finite subgroup of p-power order—it has to be cyclic, generated by a semisimple element, so a subgroup of $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$. Therefore, $\varphi(W_F)$ is a subgroup of the normalizer of $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$, i.e.
$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \cup \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}.$$

   (a) Suppose $\varphi(W_F) \subseteq \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \cong \mathbb{C}^\times$. We get $\varphi\colon W_F \to \mathbb{C}^\times$ and by local Langlands correspondence for tori, this gives $\chi\colon F^\times \to \mathbb{C}^\times$. We then get $I_B^G\chi$, which is semisimple unless $\chi = \delta_B^{\pm 1/2}$, in which case the composition factors are $\mathbb{C}$ and St.
   If $\chi = \delta_B^{\pm 1/2}$, let $\Pi_\varphi = \{\mathbb{C}\}$.
   If $\chi \neq \delta_B^{\pm 1/2}$, let Π_φ be the constituents of $I_B^G\chi$. We have that $\varphi(w) = \begin{bmatrix} \chi(w) & 0 \\ 0 & 1 \end{bmatrix}$.
   We compute the centralizers $\mathrm{Cent}(\varphi(W_K), \hat{G})$:
      (i) $\varphi \equiv 1$, then it is $\mathrm{PGL}_2(\mathbb{C})$,
      (ii) $\varphi \not\equiv 1$, the centralizer is between $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \cup \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and it is equal to the latter if and only if $\chi^2 = 1$.
   In general, this verifies that
$$\Pi_\varphi \leftrightarrow \mathrm{Irr}(\pi_0(S_\varphi)).$$

   (b) Suppose $\varphi(W_F) \not\subseteq \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$. Then we have that
$$W_F \xrightarrow{\varphi} \mathrm{Norm}\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \to \mathbb{Z}/2\mathbb{Z}.$$

We get a quadratic extension E/F, so we get the torus $S = E^1$. We have a preferred map

$$\begin{array}{ccc} {}^L S & \longrightarrow & \mathrm{PGL}_2(\mathbb{C}) \\ & \nwarrow^{\varphi_S} \quad \nearrow_{\varphi} & \\ & W_F & \end{array}$$

so we get $\theta\colon S \to \mathbb{C}^\times$. We have 2 embeddings $S \hookrightarrow \mathrm{SL}_2(\mathbb{Q}_p)$, together with θ, so we get 2 supercuspidal representations (unless $S = E^1$ for E unramified and $\theta = \mathrm{sgn}$ where we get 4). Take them as Π_φ.
Let us now compute the centralizers. We have

$$\varphi(e) = \begin{bmatrix} \theta(e/\bar{e})\chi(e)^2 & 0 \\ 0 & 1 \end{bmatrix},\ \varphi(\dot{\tau}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Assume $\theta^2 \neq 1$. The the matrix is regular, so

$$S_\varphi = \begin{bmatrix} \pm & 0 \\ 0 & 1 \end{bmatrix}.$$

The remaining case, up to conjugation, has image

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 \\ \pm 1 & 0 \end{bmatrix} = (\mathbb{Z}/2\mathbb{Z})^2$$

and in self-centralizing in $\mathrm{PGL}_2(\mathbb{C})$.

Notice that in (1)(a), χ is unramified if and only if φ is unramified (trivial in I_F). Then φ is determined by $\varphi(\mathrm{Fr})$, a semisimple conjugacy class in $\mathrm{PGL}_2(\mathbb{C})$, i.e. $\begin{bmatrix} s & 0 \\ 1 & 0 \end{bmatrix} \equiv \begin{bmatrix} s^{-1} & 0 \\ 1 & 0 \end{bmatrix}$. Similarly, an unramified χ satisfies $\chi(p) = p^s$, $\chi(x) = |x|^t = e^{t \log |x|}$, where t corresponds to s via $e^t = s$. Moreover, s is also the Satake parameter (see Theorem 8.51).

The local Langlands correspondence for $\mathrm{GL}_2(F)$, $F = \mathbb{R}$, $F = \mathbb{Q}_p$.

Definition 8.61. An irreducible representation π of $\mathrm{GL}_2(F)$ is called

(1) *unitary* (same as before),
(2) *tempered* if unitary and matrix coefficients are in $L^{2+\epsilon}(G/Z)$,
(3) *discrete series* if unitary and matrix coefficients are in $L^2(G/Z)$,
(4) *essentially tempered/discrete series* if for some $n \in \mathbb{Z}$, $\pi \otimes |\det|^n$ is tempered/discrete series,
(5) *supercuspidal* if the matrix coefficients are compactly supported modulo the center.

The following diagram describes the relation between all these notions:

$$\begin{array}{ccccc} \text{supercupsidal} \hookrightarrow & \text{essentially discrete series} & \hookrightarrow & \text{essentially tempered} & \\ & \uparrow & & \uparrow & \\ & \text{discrete series} & \hookrightarrow & \text{tempered}\textit{book} & \hookrightarrow \text{unitary.} \end{array}$$

Remark 8.62. The representation theory for GL_2 becomes much simpler than that of SL_2. However, we presented the theory for SL_2 because a lot of the methods presented generalize to other reductive groups.

For $F = \mathbb{R}$, we consider $I_B^G\chi$ for $\chi\colon F^\times \times F^\times \to \mathbb{C}^\times$. This is irreducible, unless $\chi_1/\chi_2 = x^n \cdot \mathrm{sgn}(x)$ for $n \in \mathbb{Z} \setminus \{0\}$, un which case we get a finite-dimensional representation $\mathcal{M}_{n-2}$ and one discrete series $\mathcal{D}_n$.

For $F = \mathbb{Q}_p$, $I_B^G\chi$ is irreducible unless $\chi_1/\chi_2 = |x|^{\pm 1}$. This is equivalent to

$$\chi = (\phi \circ \det) \otimes \delta_B^{\pm 1/2}$$

for some $\phi\colon F^\times \to \mathbb{C}^\times$. Then we get two constituents:

$$\phi \circ \det \quad \text{and} \quad \phi \circ \det \otimes \mathrm{St}\,.$$

Moreover, for each maximal torus $E^\times \subseteq \mathrm{GL}_2(F)$ and non-trivial character $\theta\colon E^\times \to \mathbb{C}^\times$, we get one supercuspidal.

On the Galois side, we consider

$$\varphi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C}).$$

(0) If $\varphi\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ is the natural map, then $\varphi\colon W_F \to \mathbb{C}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ corresponds to $\phi\colon F^\times \to \mathbb{C}^\times$, and we associate to it $\mathrm{St} \otimes (\phi \circ \det)$.

(1) (a) $\varphi\colon W_F \to \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \subseteq \mathrm{GL}_2(\mathbb{C})$ corresponds to $\chi\colon F^\times \times F^\times \to \mathbb{C}^\times$ and we either take $I_B^G\chi$ if irreducible, or $(\phi \circ \det)$ if not.

(b) We must be careful about which ${}^L j\colon {}^L S \to \mathrm{GL}_2(\mathbb{C})$ to take. We do not go to these detials.

All centralizers are connected in this case.

In the unramified case, χ unramified if and only if φ is unramified. If χ is unramified it corresponds to an unordered pair $s_1, s_2 \in \mathbb{C}^\times$. Similarly, when φ is unramified, it corresponds to semisimple conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$, i.e. unordered pairs $s_1, s_2 \in \mathbb{C}^\times$. Again, $s_1, s_2\, in \mathbb{C}^\times$ are the *Satake parameters*.

9. Automorphic representations of $\mathrm{GL}(2, \mathbb{A})$

The blueprint for this theory is Tate's thesis.

Local factors. Let $M_2(F)$ be 2×2 matrices with F-coefficients. Let $\mathcal{S}(M_2(F))$ be the set of Schwartz–Bruhat functions on $M_2(F)$, defined in the same way as in Tate's thesis.

There is a Fourier transform

$$\mathcal{S}(M_2(F)) \to \mathcal{S}(M_2(F))$$

with respect to the bicharacter

$$\psi_p^0(\mathrm{tr}(X \cdot Y)),$$

where ψ_p^0 is the chosen character from Tate's thesis.

Definition 9.1. Let π be an irreducible admissible representation of $G(F)$, β a matrix coefficient of π and $\phi \in \mathcal{S}(M_2(F))$. Then the *zeta integral* is

$$Z(s,\phi,\beta) = \int_G \phi(g)\beta(g)|\det(g)|^{s+\frac{1}{2}}dg$$

where dg is the Haar measure on G, normalized so that $\mathrm{vol}(K_,dg) = 1$ if F is p-adic and $K_0 = \mathrm{GL}_2(\mathbb{Z}_p)$.

Fact 9.2. *The zeta integral converges in a right half-plane.*

Proof. The additive Haar measure on $M_2(F)$ is given by $dg^+ = |\det(g)|^{-2}dg$. Therefore:

$$Z(s,\phi,\beta) = \int_{M_2(F)} \phi(g)\beta(g)|\det(g)|^{s-3/2}d^+g.$$

Since ϕ is Schwartz, going to infinity in $M_2(F)$ (as a vector space) is not an issue. The other issue is the matrix coefficient β has a singular set, but making $\mathrm{Re}(s) \gg 0$, we can dominate the asymptotic expansion of β as g goes to the singular set. □

Remark 9.3. Since the leading exponents of β do not depend on the particular choice of β (but only on the representation), there is a right half plane such that $Z(s,\phi,\beta)$ converges there for all β.

Theorem 9.4.

(1) *The function $Z(s,\phi,\beta)$ has a meromorphic continuation to all of $\mathbb{C}$. When $p < \infty$, it is a rational function of p^{-s}.*
(2) *The family of functions*
$$\{Z(s,\phi,\beta) \mid \phi,\beta\}$$
has a gcd, that is, there is on number $L(s,\pi)$ such that
$$\frac{Z(s,\phi,\beta)}{L(s,\phi)}$$
is entire. When $p < \infty$, this family is an ideal in $C(p^{-s})$.
(3) *The local functional equation holds:*
$$Z(1-s,\hat{\phi},\beta^\vee) = \gamma(s,\pi)Z(s,\phi,\beta)$$
with a function $\gamma(s,\pi)$, rational in p^{-s} if $p < \infty$. Here, $\hat{\phi}$ is the Fourier transform of ϕ, and $\beta^\vee(g) = \beta(g^{-1})$, a matrix coefficient for $\pi^\vee$.

Definition 9.5. The *epsilon factor* is

$$\epsilon(s,\pi) = \gamma(s,\pi)\frac{L(s,\pi)}{L(1-s,\pi^\vee)}.$$

The goal is to prove the above theorem and give a more explicit description of all the functions involved.

The unramified case. Let $F=\mathbb{Q}_p$, $\chi=(\chi_1,\chi_2)$ be unramified, and $\pi=I_B^G\chi$. Then π^{K_0} is 1-dimensional, spanned by

$$f_\chi^0(tuk)=|a/b|^{1/2}\chi_1(a)\chi_2(b)$$

for $t=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

Take

$$\begin{aligned}\beta(g)&=\langle gf_\chi^0, f_{\chi^{-1}}^0\rangle\\&=\int_{K_0} gf_\chi^0(k)f_{\chi^{-1}}(k)dk\\&=\int_{K_0} f_\chi^0(kg)f_{\chi^{-1}}^0(k)dk\\&=f_\chi^0(g).\end{aligned}$$

Take $\phi=1_{M_2(\mathbb{Z}_p)}$. Then

$$\begin{aligned}Z(s,\phi,\beta)&=\int_G \phi(g)f_\chi^0(g)|\det(g)|^{s+\frac{1}{2}}dg\\&=\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p}\phi\left(\begin{bmatrix} a & b \\ 0 & d\end{bmatrix}\right)|a/d|^{1/2}\chi_1(a)\chi_2(d)|ad|^{s+\frac{1}{2}}\frac{d^\times a}{|a|}d^\times d db\\&=\int_{\mathbb{Z}_p\setminus\{0\}}\int_{\mathbb{Z}_p\setminus\{0\}}\chi_1(a)\chi_2(a)|a|^s|d|^s d^\times a d^\times d\\&=L(s,\chi_1)\cdot L(s,\chi_2).\end{aligned}$$

The principal series case.

Definition 9.6. Define the *L-function* as:

$$L(s,I_B^G\chi)=L(s,\chi_1)L(s,\chi_2)$$

and the *ϵ-factor* as:

$$\epsilon(s,I_B^G\chi)=\epsilon(s,\chi_1)\epsilon(s,\chi_2).$$

Lemma 9.7. *The function*

$$\frac{Z(s,\phi,\beta)}{L(s,\pi)}$$

is entire.

Proof. Let $f_\chi\in I_B^G\chi$ and $f_{\chi^{-1}}\in I_B^G\chi^{-1}$. Then

$$\beta(g)=\int_{K_0} f_\chi(kg)f_{\chi^{-1}}(k).$$

Writing

$$\xi(a,d)=\int_{\mathbb{Q}_p}\int_{K_0}\int_{K_0}\phi\left(k^{-1}\begin{bmatrix} a & b \\ 0 & d\end{bmatrix}k'\right)f_\chi(k')f_{\chi^{-1}}(k)dkdk'db,$$

we get that

$$Z(s,\phi,\beta) = \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^\times} \xi(a,d)\chi_1(a)\chi_2(d)|a|^s|d|^s d^\times a d^\times d.$$

One can check that ξ is a Schwartz–Bruhat function on $\mathbb{Q}_p \times \mathbb{Q}_p$. Then

$$\begin{aligned} Z(s,\phi,\beta) &= \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^\times} \xi(a,b)\chi_1(a)\chi_2(a)|a|^s|d|^s d^\times a d^\times d \\ &= P_\xi(p^s, p^{-s})L(s,\chi_1)L(s,\chi_2) && \text{by Tate's thesis.} \end{aligned}$$

This completes the proof. ☐

Lemma 9.8. *We have that* $\xi_{\hat{\psi},\beta^\vee} = \hat{\xi_{\phi,\beta}}$.

The supercuspidal case.

Lemma 9.9. *For any* ϕ, β, $Z(s,\phi,\beta)$ *is entire.*

Proof. Let $\omega\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the central character[9] of π, i.e. $\pi \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{z_a} = \omega(a)$. Then $\beta(z_a g) = \omega(a)\beta(g)$. We have that

$$Z(s,\phi,\beta) = \int_{G/Z} \left(\int_{\mathbb{Q}_p^\times} \phi(z_a g)\omega(a)|a|^{2s+1} sa \right) \beta(g)|\det(g)|^{s+1/2} dg.$$

The inner integral is a 1-dimensional ζ integral. The outer integral is entire (i.e. does not give any new poles), since β is compactly supported modulo Z.

There are now 2 possibilities. If ω is ramified, the inner integral is entire, so the whole integral is entire. If ω is unramified, then it has a pole at $s = -\frac{1}{2}$ (assuming ω is trivial) with residue $\phi(0)$, independent of g.

Thus the whole integral has a pole at $s = -\frac{1}{2}$ (ω is trivial) with residue

$$\phi(0) \int_{G/Z} \beta(g)|\det(g)|^{s+1/2} dg.$$

But

$$\int_{\mathbb{Q}_p} \beta\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g \right) dx = 0$$

(proving this is an exercise which involves considering the integral description of the Jacquet module).[10] Using the Iwasawa decomposition, this shows that the above integral is 0, and hence there is actually no pole. ☐

Definition 9.10. Define $L(s,\pi) = 1$ for π supercuspidal.

While the L-function of a supercupsidal representation is trivial, the ϵ-factor is actually interesting.

[9] We have not defined this, but this is done in the same way as for finite groups.

[10] We call a function β with this property a *supercuspidal function.*

Definition 9.11. For any admissible representation (π, V) and $\phi \in \mathcal{S}(M_2(F))$, define

$$\mathscr{Z}(s,\phi,\pi) = \int_G \phi(g)\pi(g)|\det(g)|^{s+\frac{1}{2}} dg \in \mathrm{End}_{\mathbb{C}}(V).$$

Remark 9.12. This does converge in a right half plane and

$$Z(s,\phi,\beta_{v,v^\vee}) = \langle \mathscr{Z}(s,\phi,\pi)v, v^\vee\rangle.$$

If $\mathrm{supp}(\phi) \subseteq G$, then $\phi \in \mathcal{C}_c^\infty(G)$ and then $Z(s,\phi,\pi)$ converges for all s and is entire. In fact, we can arrange that ϕ and $\hat{\phi}$ are supported on G.

Lemma 9.13. *Let π be discrete series. Then every smooth finite rank operator can be realized as $\mathscr{Z}(s,\phi,\pi)$.*

Proof. We may assume $s = -\frac{1}{2}$. A smooth finite rank operator is a linear combination of smooth rank 1 operators, i.e. $(v \otimes v^\vee)(w) = v \cdot \langle w, v^\vee\rangle$. Define

$$\phi(g) = \begin{cases} \deg(\pi, dg)\langle v, \pi^\vee(g)v^\vee\rangle & \text{if } |\det g| \in \{1, p\}, \\ 0 & \text{otherwise.} \end{cases}$$

One can check that $\hat{\phi}$ is supported on G. Then

$$\begin{aligned} \langle \mathscr{Z}(-1/2,\phi,\pi)w, w^\vee\rangle &= \int_{G/Z} \langle \pi(g)w, w^\vee\rangle\langle v, \pi^\vee(g)v^\vee\rangle \deg(\pi, dg) dg \\ &\langle v, w^\vee\rangle\langle w, v^\vee\rangle \qquad \text{Lemma 7.54} \\ &\langle\langle w, v^\vee\rangle v, w^\vee\rangle. \end{aligned}$$

This completes the proof. ∎

Lemma 9.14. *For $\phi, \psi \in \mathcal{S}(M_2(F))$ such that ψ, $\hat{\psi}$ are supported on G, we have*

$$\mathscr{Z}(1-s,\hat{\psi},\pi^\vee)^\vee \circ \mathscr{Z}(s,\phi,\pi) = \mathscr{Z}(s,\psi,\pi)\circ\mathscr{Z}(1-s,\hat{\phi},\pi^\vee)^\vee.$$

Proof. It is enough to check this after applying

$$\langle(-)v, v^\vee\rangle.$$

Then the two sides are:

$$\begin{aligned} \text{LHS} &= \int_G\int_G \phi(g)\hat{\psi}(h)\langle\pi(g)v, \pi(h)v^\vee|\det(g)|^{s+1/2}|\det(h)|^{3/2-s} dgdh, \\ \text{RHS} &= \int_G\int_G \hat{\phi}(g)\psi(h)\langle\pi(g^{-1})v, \pi(h^{-1})v^\vee|\det(g)|^{3/2-s}|\det(h)|^{s+1/2} dgdh. \end{aligned}$$

After a small massage, this is equivalent to the equation

$$\int_G \phi(g)\hat{\psi}(g)dg = \int_G \hat{\phi}(g)\psi(g)dg,$$

which is true. ∎

Proposition 9.15. *Let π be irreducible. There exists a unique scalar valued function $\gamma(s,\pi)$ such that*

$$\mathscr{Z}(1-s,\hat{\phi},\pi^\vee)^\vee = \gamma(s,\pi)\mathscr{Z}(s,\phi,\pi)$$

for all ϕ such that ϕ and $\hat{\phi}$ are supported on G.

Proof. We first show that there is a operator-valued $\gamma(s,\pi)$. Assume it existed and take $v \in V$. What is $\gamma(s,\pi)v$? By Lemma 9.13, there exists ϕ_v such that $Z(s,\phi_v,\pi) = v$. Then

$$\gamma(s,\pi)v = \mathscr{Z}(1-s,\hat{\phi}_v,\pi^v ee)v \tag{8}$$

by the relation γ should satisfy. This immediately shows uniqueness of $\gamma(s,\pi)$. We also see an approach to construct it.

We need to show (8) is independent of ϕ_v. Assume not an take ϕ_v, ϕ'_v such that

$$Z(1-s,\hat{\phi}_v-\hat{\phi}'_v,\pi^\vee)^\vee v = w \neq 0.$$

Choose ψ such that $Z(s,\psi,\pi)w = w \neq 0$. Then

$$\begin{aligned} 0 \neq w &= \mathscr{Z}(s,\psi,\pi)\mathscr{Z}(1-s,\hat{\phi}_v-\hat{\psi}'_v,\pi^\vee)^\vee v \\ &= \mathscr{Z}(1-s,\hat{\psi},\pi^\vee)^\vee \circ \mathscr{Z}(s,\phi_v-\phi'_v,\pi)v && \text{by Lemma 9.14} \\ &= 0 && \text{by assumption.} \end{aligned}$$

This is a contradiction, showign that equation (8) can be used to define $\gamma(s,\pi)$. A variation of this argument proves that $\gamma(s,\pi)$ is linear. Looking at how $\mathscr{Z}$ transforms under $\phi_h(x) = \phi(hx)$, we see that $\gamma(s,\pi)$ commutes with $\pi(h)$ for all $h \in G$, so $\gamma(s,\pi)$ is a scalar by Schur's Lemma. □

The computation of local factors for Steinberg is omitted here.

Preservation of local factors. Using the above computation and the description of the local Langlands correspondence, one can show the following theorem.

Theorem 9.16. *The Local Langlands correspondence preserves the local factors.*

Automorphic representations. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$, $G = \mathrm{GL}_2$, $K_\infty = O(2) \subseteq G(\mathbb{R})$ maximal compact, $K_p = \mathrm{GL}_2(\mathbb{Z}_p) \subseteq G(\mathbb{Q}_p)$ maximal compact open. Writing $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$,

$$K_F = \prod_{p<\infty} K_p \subseteq G(\mathbb{A}_F) \text{ maximal compact open,}$$

$$K = K_\infty \times K_f \subseteq G(\mathbb{A}) \text{ maximal compact.}$$

Finally, let

$$\mathscr{Z} = Z(\mathcal{U}(\mathfrak{g}_\infty))$$

be the center of the universal enveloping algebra.

Definition 9.17. A function $f\colon G(\mathbb{A}) \to \mathbb{C}$ is an *automorphic form* if

(1) f is $G(\mathbb{Q})$-invariant on the left,
(2) for some Hecke character $\psi\colon \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ such that

$$f(z_a g) = \psi(a)f(g) \quad \text{for } z_a = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

(3) for any $y \in G(\mathbb{A})$, the function $x_\infty \mapsto f(x_\infty y)$ is smooth and $\mathscr{Z}$-finite,

(4) f is *slowly increasing*: for any $c > 0$ and $\omega \subseteq G(\mathbb{A})$ compact, there are $C > 0$ and $N > 0$ such that
$$f\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right) \leq C|a|^N$$
for all $a \in \mathbb{A}^\times$, $|a| > c$, and $g \in \omega$.

Definition 9.18. A function $f\colon G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ is called *cuspidal* if
$$\int_{\mathbb{A}/\mathbb{Q}} f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) dx = 0$$
for almost all $g \in G(\mathbb{A})$.

Definition 9.19.

(1) Let $\mathcal{A}$ be the space of all automorphic forms.
(2) Let $\mathcal{A}^0$ be the space of cuspidal automorphic forms.
(3) Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$ be the set of all L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ satisfying (2) from Definition 9.17.
(4) Let $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$ be the subspace of all cuspidal functions.

Remark 9.20.

- The spaces L^2 and L^2_0 are (unitary) representations of $G(\mathbb{A})$ via right-translation.
- The space $\mathcal{A}$ is not a $G(\mathbb{A})$-representation, because $G(\mathbb{R})$ does not act. However, $G(\mathbb{A}_f)$ does act, so
$$\mathcal{H}(G(\mathbb{A}_f)) = \bigotimes_{p<\infty}{}' \mathcal{H}(G(\mathbb{Q}_p))$$
where the restricted tensor product is with respect to $\mathbb{1}_{K_p}$. In place of the $G(\mathbb{R})$-action, we have a $(\mathfrak{g}_\infty, K_\infty)$-module structure. In particular, $\mathcal{U}(\mathfrak{g}_\infty)$ acts and M_{K_∞} which is the convolution algebra[11] of finite measures on K_∞ also acts. We may hence define
$$\mathcal{H}(G(\mathbb{R})) = \mathcal{U}(\mathfrak{g}_\infty) \otimes M_{K_\infty}.$$
Overall, $\mathcal{A}$ is an $\mathcal{H}(G(\mathbb{A})) = \mathcal{H}(G(\mathbb{R})) \otimes \mathcal{H}(G(\mathbb{A}_f))$-module.

Fact 9.21. *The cuspdial automorphic forms $\mathcal{A}_0$ is a dense subspace of K-finite and $\mathscr{Z}$-finite vector in L^2_0.*

Definition 9.22. An *automorphic representation* is an $\mathcal{H}(G(\mathbb{A}))$-module which is an irreducible subquotient of $\mathcal{A}$. It is *cuspidal* if it is a subquotient of $\mathcal{A}_0$.

Theorem 9.23 (Gelfand–Graev–Piatetski-Shapiro)**.** *The space L^2_0 is a (completed) Hilbert direct sum of irreducible representations with finite multiplicities.*

Remark 9.24. The irreducible constituents of L^2_0 are precisely the cuspidal automorphic representations.

Problem. Describe the cuspidal automorphic representations and their multiplicities!

[11] Convolution is defined as $(\nu \circ \mu)(E) = \int 1_E(xy) d\nu(x) d\mu(y)$.

Proposition 9.25 (Flath's decomposition theorem). *An irreducible admissible representation π of $G(\mathbb{A})$ is*

$$\pi = \bigotimes_{p\leq\infty}{}' \pi_p$$

where π_p is an irreducible admissible representation of $G(\mathbb{Q}_p)$ and almost all π_p are unramified.

We have already classified all irreducible local representations. One can put any collection like that with almost all π_p unramified into a representation of $G(\mathbb{A})$. Such a representation may not be an automorphic representation and our task is to work out which such representations do occur in the L^2-spectrum.

Multiplicity one.

Theorem 9.26 (Multiplicity one). *An irreducible admissible representation of $G(\mathbb{A})$ occurs in L^2_0 with multiplicity ≤ 1.*

We give a sketch of the proof.

There are two components to prove:

- a global component called *global genericity of cusp forms*,
- a local component called *uniqueness of Whittaker model.*

Definition 9.27.

(1) An irreducible admissible representation π_p of $G(\mathbb{Q}_p)$ is *generic* if for one (hence any) character non-trivial $\xi_p\colon \mathbb{Q}_p \to \mathbb{C}^\times$,

$$\mathrm{Hom}_U(\pi_p, \xi_p) \neq 0.$$

(2) An automorphic representation π is *generic* if for one (hence any) non-trivial $\xi\colon \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$,

$$\mathrm{Hom}_U(\pi, \xi) \neq 0.$$

Remark 9.28. We justify the *one (hence any)* assertion in the definition. Note that

$$T \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$$

acts on $U \cong \mathbb{Q}_p$ by conjugation, i.e.

$$(a, b) \cdot u = \frac{a}{b} u.$$

If ξ_p^1, ξ_p^2 are two characters, then

$$\xi_p^2(x) = \xi_p^1(ax) \text{ for } a \in \mathbb{Q}_p.$$

Note that for SL_2, $T \cong \mathbb{Q}_p^\times$ acts on $U \cong \mathbb{Q}_p$ via $(a, u) \mapsto a^2 u$, so get $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times,2}$-many options for ξ_p.

The global case also follows.

Lemma 9.29. *If π is a cuspidal automorphic representation, it is generic.*

Proof. For any ξ and $f \in \pi$, we can take the ξ-Fourier coefficient:

$$f_\xi(g) = \int_{\mathbb{A}/\mathbb{Q}} f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} g\right) \overline{\xi}(x)dx.$$

Basic Fourier theory says that

$$f(g) = \sum_\xi f_\xi(g).$$

Since π is cuspidal, $f_1 = 0$. Therefore, $f_\xi \neq 0$ for some $\xi \neq 1$. Then $f \mapsto f_\xi(1)$ is an element of $\mathrm{Hom}_U(\pi, \xi) \neq 0$. □

Proposition 9.30. *For any irreducible admissible representation π_p of $G(\mathbb{Q}_p)$,*

$$\dim \mathrm{Hom}_U(\pi_p, \xi_p) \leq 1.$$

Instead of proving this proposition, we present a *toy proof*, when $F = \mathbb{F}_p$, where we do not have to worry about the analytic issues and we can just show the main idea.

Proof when $F = \mathbb{F}_p$. We want to prove that

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \xi)) \leq 1,$$

i.e. $\mathrm{Ind}_U^G \xi$ is multiplicity free. This is equivalent to showing that

$$\mathrm{End}_G(\mathrm{Ind}_U^G \xi) \text{ is commutative,}$$

where $\mathrm{End}_G(\mathrm{Ind}_U^G \xi)$ is a convolution algebra

$$\{f \colon G \to \mathbb{C} \mid f(ugu') = \xi(u)f(g)\xi(u')\}.$$

Bruhat decomposition gives representatives

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \quad \text{for } a, b \in \mathbb{Q}_p^\times$$

for $U\backslash G/U$. We can assign $f(g)$ for each such representative g almost freely. We have

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

if and only if $ax = by$. Thus, if $a \neq b$, we can find $x \neq y$, and get $f(g) = \xi(x-y)f(g)$, so $f(g) = 0$.

The following method of proof is called **Gelfand's trick**. We have the *anti-involution*

$$g \mapsto \mathrm{Ad} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} g^t.$$

This fixes all representatives g where $f(g) \neq 0$, hence gives an anti-involution of the Hecke algebra that is the identity. □

Remark 9.31. Gelfand's trick has later been generalized to other contexts. The general setting in which it works is related to so-called Gelfand pairs.

Definition 9.32.

(1) A non-zero element of $\mathrm{Hom}_U(\pi, \xi)$ is called a *Whittaker functional.*

(2) The image of a non-zero element of $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \xi)$ is called a *Whittaker model* for π.

What we proved above is the uniqueness of Whittaker models.

Proof of Multiplicity one Theorem 9.26. The global result tells us

$$0 \neq \mathrm{Hom}_U(\pi, \xi) = \bigotimes_p \mathrm{Hom}_U(\pi_p, \xi_p).$$

The local result says that $\mathrm{Hom}_U(\pi, \xi) \leq 1$. Therefore,

$$\mathrm{Hom}_U(\pi, \xi) = \mathbb{C}.$$

Therefore, we have the non-zero map

$$\ell_\xi \colon \mathbb{A}_0 \to \mathbb{C}$$

(taking the ξ-Fourier coefficient):

$$\ell_\xi(f) = \int_{\mathbb{A}/\mathbb{Q}} f\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) \xi(x) dx.$$

We obtain

$$\mathrm{Hom}_G(\pi, L_0^2) \hookrightarrow \mathrm{Hom}_U(\pi, \xi)$$
$$\alpha \mapsto \ell_\xi \circ \alpha.$$

Since the latter is 1-dimensional, this completes the proof. □

Remark 9.33. This fails for other groups. It works for GL_N and SL_2 (which was only proved in 2000 by Ramakrishnan), but not for SL_N when $N > 2$ (proved in 1994 by Blasius). In fact, the general Langlands conjectures predict the multiplicity that one should obtain.

Theorem 9.34 (Rigidity)**.** *If $\pi^1 = \bigotimes \pi_p^1$ and $\pi^2 = \bigotimes \pi_p^2$ are cuspidal automorphic representations and π_p^1 for almost all p, then $\pi^1 \cong \pi^2$.*

The Rigidity Theorem 9.34 together with Multiplicity one Theorem 9.26 gives *strong multiplicity one.*

Corollary 9.35 (Strong multiplicity one)**.** *If $\pi^1, \pi^2 \subseteq L_0^2$ are irreducible constituents and $\pi_p^1 \cong \pi_p^2$ for almost all p, then $\pi^1 = \pi^2$ (the same subspace of L_0^2).*

This theorem allows us to do the following. Consider the set

$$S = \left\{ \left\{ \{\alpha_p, \beta_p \mid \alpha_p, \beta_p \in \mathbb{C}^\times\}_p \right\}_{p \text{ primes}} \right\}$$

where we identify two collections

$$\{\{\alpha_p, \beta_p\}_p\} \sim \{\{\alpha_p', \beta_p'\}_p\}$$

if $\{\alpha_p, \beta_p\} = \{\alpha_p', \beta_p'\}$ for almost all p.

Strong multiplicity one gives an injection

$$\{\text{irreducible cuspidal representations}\} \hookrightarrow S/\sim$$

by assigning to π the collection of the Satake paremeters of its local representations.

Problem. Describe the image of this map, i.e. work out what collections complex numbers appear as Satake parameters of representations.

This is an extremely difficult thing to do.

Ramanujan conjecture.

Conjecture (Ramanujan)**.** If $\pi = \bigotimes \pi_p$ is a cuspidal automorphic representation, then each π_p is tempered.

Remark 9.36. This implies that the complementary series are never local components of cuspidal automorphic representations.

Remark 9.37. Already for Sp_4, there are non-tempered cuspidal automorphic representations. There are also non-generic cuspidal automorphic representations. However, all the examples of non-tempered cuspidal automorphic representations are non-generic. This leads to the generalized Ramanujan conjecture.

Conjecture (Generalized Ramanujan)**.** If π is a generic cuspidal automorphic representation of a quasisplit group G, then all π_p are tempered.

The Arthur conjecture extends a lot of the Langlands conjecture to non-tempered representation.

Classical modular forms. Recall that if $f\colon \mathbb{H} \to \mathbb{C}$ is a cusp form, then at the beginning of the class, we associated to it a function

$$\phi_f\colon \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$$

given by

$$\phi_f(g) = f(g \cdot i) j(g, i)^{-k}.$$

By strong approximation, we can think of it as a function

$$\phi_f\colon \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}.$$

Then one can show that $\phi_f \in L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover, ϕ_f spans an irreducible representation if f is a new eigenform.

Altogether, we get an injection

$$\text{new cuspidal eigenforms} \hookrightarrow \text{cuspidal automorphic representations}.$$

The cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ has

$$\pi_\infty \cong \mathcal{D}_k$$

where k is the weight of the modular form.

Cuspidal automorphic L-functions. Recall the space $\mathcal{S}(M_2(\mathbb{Q}_p))$ of Schwartz–Bruhat functions, for $p \leq \infty$. We formed the space

$$\mathcal{S}(M_2(\mathbb{A})) = \bigotimes_{p\leq\infty}{}' \mathcal{S}(M_2(\mathbb{Q}_p)),$$

restricted with respect to $\mathbb{1}_{M_2(\mathbb{Z}_p)}$.

Definition 9.38. Let π be a cuspidal automorphic representation. For $\phi \in \mathcal{S}(M_2(\mathbb{A}))$ and a matrix coefficient β of π, the *global zeta integral* is defined as

$$Z(s, \phi, \beta) = \int_{G(\mathbb{A})} \phi(g)\beta(g)|\det(g)|_{\mathbb{A}}^{s+1/2} dg.$$

Proposition 9.39. *This converges in a right half plane and equals to the product over all p of the local zeta integrals.*

Proof. Exercise. □

Recall that for $\mathbb{A}^1 = \{x \in \mathbb{A}^\times \mid |x|_{\mathbb{A}} = 1\}$ we have $\mathbb{A}^\times = \mathbb{A}^1 \times \mathbb{R}_{>0}$. We can similarly let

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) \mid \det(g) \in \mathbb{A}^1\}$$

and

$$A_\infty = Z(G)(\mathbb{R})^0 \cong \mathbb{R}_{>0}.$$

Once again, we have a decomposition

$$G(\mathbb{A}) = G(\mathbb{A})^1 \times \mathbb{R}_{>0}.$$

In the 1-dimensional case $\mathbb{A}^1\backslash\mathbb{Q}^\times$ is compact, but in the 2-dimensional case $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is no longer compact (but it at least has finite volume). This is related to the classical issue of cusps, which in the automorphic setting correspond to Borel subgroups. If one replaces G with the units in a definite quaternion algebra, there are no cusps (and no Borel subgroups) and this quotient is compact.

On the space $L^2_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$, we have a scalar product

$$\langle f_1, f_2\rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}^1)} f_1(x)\overline{f_2(x)}dx.$$

Therefore,

$$\beta(g) = \langle gf_1, f_2\rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} f_1(xg)\overline{f_2}(x)dx.$$

Proposition 9.40. *The function $Z(s, phi, \beta)$ has an analytic continuation to $s \in \mathbb{C}$ to a holomorphic function, and satisfies the functional equation*

$$Z(1-s, \hat{\phi}, \beta^\vee) = Z(s, \phi, \beta).$$

Proof. We have that:

$$\begin{aligned} Z(s,\phi,\beta) &= \int\limits_{G(\mathbb{A})} \phi(g) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} f_1(xg)\overline{f_2(x)}dx|\det(g)|^{s+1/2}dg \\ &= \int\limits_{G(\mathbb{A})} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(x^{-1}g)f_1(g)\overline{f_2(x)}|\det(g)|^{s+1/2}dx\,dg \\ &= \int\limits_{\mathbb{R}_{>0}} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{t\in G(\mathbb{Q})} \phi(x^{-1}tag)f_1(g)\overline{f_2}(x)a^{2s+1}dx\,dg\,d^\times a \end{aligned}$$

We split the integral over $\mathbb{R}_{>0}$ as a sum $\int_0^1 + \int_1^\infty$. The second integral is fine: it is absolutely convergent for all s to a holomorphic function. To compute the first integral, we want to apply the Poisson summation formula to

$$\sum_{t\in G(\mathbb{Q})} \phi(x^{-1}tag)f_1(g)\overline{f_2}(x)a^{2s+1}.$$

As in the GL(1) case, $\hat{\phi}$ is the **additive** Fourier transform, but the sum is over the multiplicative group. We hence add and subtract the terms from $M_2(\mathbb{Q}) \setminus G(\mathbb{Q})$, apply Poisson summation formula, and substitute $t \mapsto \frac{1}{t}$ to get

$$\begin{aligned} \int_1^\infty (\cdots) + \int_1^\infty \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{t\in G(\mathbb{Q})} \hat{\phi}(g^{-1}tax)f_1(g)\overline{f_2(x)}a^{3-2s}\,dx\,dg\,d^\times a \\ - \int_1^\infty \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi(0)f_1(g)\overline{f_2(x)}a^{2s+1}a^{2s+1}dgdxd^\times a \\ - \int_1^\infty \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{t\in M_2(\mathbb{Q})_1} \phi(x^{-1}ta^{-1}g)f_1(g)\overline{f_2(x)}a^{2s+1}\,dg\,dx\,d^\times a \\ + \int_1^\infty \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \hat{\phi}(0)f_1(g)\overline{f_2(x)}a^{2s+1}a^{2s+1}dgdxd^\times a \\ + \int_1^\infty \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{t\in M_2(\mathbb{Q})_1} \hat{\phi}(g^{-1}tax)f_1(g)\overline{f_2(x)}a^{2s+1}\,dg\,dx\,d^\times a. \end{aligned}$$

We can use *cuspidality* to get rid of the *boundary terms.*

The case $t = 0$. We have

$$\int\limits_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} f_1(g)dg = \langle f_1, 1\rangle = 0$$

because f_1 lies in π and 1 lies in the trivial representation $\mathbb{1}$ and

$$\pi, \mathbb{1} \subseteq L^2_d(G(\mathbb{Q})\backslash G(\mathbb{A}), \mathbb{1})$$

are orthogonal.

The case $t \in M_2(\mathbb{Q})_1$. We break $M_2(\mathbb{Q})$ into $G(\mathbb{Q})$-orbits under left multiplication. Each orbit has a representative of the form $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \gamma_2$ for $\gamma_2 \in G(\mathbb{Q})$. The stabilizer of such a representative in $G(\mathbb{Q})$ is $L(\mathbb{Q}) = \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$. Therefore, the contribution of that orbit to the boundary term is:

$$\int_1^\infty \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \sum_{\gamma_1 \in L(\mathbb{Q})\backslash G(\mathbb{Q})} \phi\left(x^{-1}\gamma_1^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \gamma_2 g\right) f_1(g)\overline{f_2}(x) a^{2s+1}\, dg\, dx\, d^\times a$$

$$= \int_1^\infty \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} \int_{L(\mathbb{Q})\backslash G(\mathbb{A})^1} \phi\left(y^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \gamma_2 g\right) f_1(g)\overline{f_2}(x) a^{2s+1}\, dg\, dy\, d^\times a \qquad \text{(where } y = \gamma_1 x).$$

Notice that, as a function of y, this is invariant not just under $L(\mathbb{Q})$ but under $L(\mathbb{A}) \supseteq U(\mathbb{A})$. Therefore we get an integral

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \overline{f_2(y)} dy = 0.$$

This completes the proof, since we showed vanishing of the boundary terms. □

Corollary 9.41. *When π is a cuspidal automorphic representation, $L(s, \pi)$ is holomorphic and satisfies the functional equation $L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi^\vee)$.*

Weil's converse theorem. Let π_p be an irreducible admissible **generic** representation of $G(\mathbb{Q}_p)$. We have a Whittaker model $W(\pi_p)$ of π_p. Given $W \in W(\pi_p)$, $\chi \colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$,

$$Z(s, g, \chi, W) = \int_{\mathbb{Q}_p^\times} W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right) \chi(a) |a|^{s-1/2} d^\times a.$$

This is almost like the 1-dimensional local zeta integral, but the function $W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right)$ is not locally constant, so it is not a Schwartz–Bruhat function.

Define $L(s, \pi_p \times \chi_p)$, $\epsilon(s, \pi_p \times \chi_p)$ as before. When $\chi = \mathbb{1}$. we get the old L and ϵ factors.

Theorem 9.42 (Converse theorem)**.** *Let $\pi = \bigotimes \pi_p$ be an irreducible admissible generic representation of $G(\mathbb{A})$. Then π is cuspidal automorphic if and only if $L(s, \pi \times \chi)$ is entire, bounded in vertical strips, and satisfies a functional equation, for all χ.*

Sketch of proof. Assume $L(s, \pi \times \chi)$ is nice (as above). We want an embedding $\pi \hookrightarrow L_0^2$ or, equivalently, of its Whittaker model $W(\pi) \hookrightarrow L_0^2$. A Whittaker function$W \colon G(\mathbb{A}) \to \mathbb{C}$ satisfies for $\xi \colon U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ and $u \in U(\mathbb{A})$ the invariance property $\omega(ug) = \xi(u) W(g)$. We want $G(\mathbb{Q})$-invariance, so we form

$$\phi_W(g) = \sum_{a \in \mathbb{Q}^\times} W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right).$$

This function is $B(\mathbb{Q})$-invariant. By Bruhat decomposition $G(\mathbb{Q}) = B(\mathbb{Q}) \cup B(\mathbb{Q}) w B(\mathbb{Q})$, we just need to check w-invariance:

$$\phi_W(g) = \phi_W(wg).$$

For a fixed g, consider the equation

$$\phi_W\left(\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right) = \phi_W\left(w\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} g\right).$$

To show these functions in $a \in \mathbb{A}$ are equal, we take Mellin transforms (as functions of χ) of both sides. On the left, we get

$$Z(s, g, \chi, W) = \prod_p Z_p(-) = L(s, \pi \times \chi) \prod_p \frac{Z_p(-)}{L_p(-)}.$$

On the right side, we get

$$L(1-s, \pi^\vee \times \chi^{-1}) \prod_p \frac{Z_p(1-s, wg, \chi_p^{-1}\xi_p^{-1}, W_p)}{L(1-s, \pi_p^\vee \times \chi_p^{-1})}.$$

The local and global functional equations combined give the equality of these function. □

Rigidity. The key input to the Strong Multiplicity One Theorem 9.35 was the Rigidity Theorem 9.34. We discuss the proof briefly.

Suppose $\pi_p^1 \cong \pi_p^2$ for $p \notin S$, S finite, where π^1 and π^2 are cuspidal automorphic representation. Apply the functional equation of L-functions to get that

$$\prod_{p \in S} \underbrace{\frac{L(1-s, (\pi_p^1)^\vee \times \chi_p^{-1})\epsilon(s, \pi_p^1 \times \chi_p)}{L(s, \pi_p^1 \times \chi_p)}}_{\gamma(1-s, (\pi_p^1)^\vee \times \chi_p^{-1})} = \prod_{p \in S} \frac{L(1-s, (\pi_p^2)^\vee \times \chi_p^{-1})\epsilon(s, \pi_p^2 \times \chi_p)}{L(s, \pi_p^2 \times \chi_p)}.$$

One uses this equality as follows.

(1) Stability of γ-factors: $L(s, \pi_p \times \chi_p) = 1$ if χ_p is very ramified and $\epsilon(s, \pi_p \times \chi_p) = \epsilon(s, \chi_p \cdot \omega_{\pi_p})\epsilon(s, \chi_p)$,
(2) Globalization of characters — find a global character χ which is highly ramified at some primes p and a specific χ_p at one place. This shows the local factors agree at each place $p \in S$, and they determine the local representations, so $\pi_p^1 \cong \pi_p^2$ for $p \in S$ also.

The global Langlands conjecture.

Conjecture (Langlands)**.** Let $\phi\colon G(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{C})$. There exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ such that for all v, $\phi|_{G(\overline{F_v}/F_v)}$ corresponds to ϕ_v via the Local Langlands Correspondence.

On the Galois side, we have Ind, Res. Conjecturally, on the automorphic side, induction corresponds to automorphic induction (AI), and restriction corresponds to base change (BC). Since we understand the Local Langlands Correspondence, we can describe the *candidates* for these representations.

Langlands showed existence of AI and BC in the case $G = \mathrm{GL}_2$ and E/F is cyclic. Later Arthur and Clozel generalized this to $G = \mathrm{GL}_n$ and E/F is cyclic.

Observation. The image of ϕ (in PGL_2) can be

(1) cyclic: Hecke theory,
(2) dihedral,
(3) tetrahedral (A_4),
(4) octahedral (S_4),
(5) icosahedral (A_5)

Cases (1)–(3) were handled by Langlands. Case (4) was handled by his student, Tunnel. Case (5) splits into two cases: odd (proved by Khare–Winterburger), even (this is an open problem).

We discuss the cases (2) and (3) briefly. Let ϕ be dihedral or tetrahedral. Then

$$\phi = \mathrm{Ind}_{W_E}^{W_F} \phi_E,$$

where E/F is cyclic of order 2 or 3. Now, ϕ_E is a character on $\mathbb{A}_E^\times$, so we may consider $I_B^G \phi_E$ on $\mathrm{GL}_2(\mathbb{A}_F)$. Using the trace formula, one can show that this corresponds to π, a cuspidal automorphic of $\mathrm{GL}_2(\mathbb{A}_F)$, corresponding to ϕ, as required.

One thing we have not discussed in this class is Eisenstein series. They describes fully the automorphic representations which are not cuspidal.