MATH 679: INTRODUCTION TO p-ADIC HODGE THEORY

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These are notes from Math 679 taught by Serin Hong in Winter 2020, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

Official notes for the class are available here:

http://www-personal.umich.edu/~serinh/Notes%20on%20p-adic%20Hodge%20theory.pdf

They contain many more details than these lecture notes.

This version is from July 28, 2020. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The class will consist of 4 chapters:

(1) Introduction, Section 1,
(2) Finite group schemes and p-divisible groups, Section 2,
(3) Fontaine's formalism,
(4) The Fargues–Fontaine curve.

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1. Introduction

1.1. A first glimpse of p-adic Hodge theory.

1.1.1. *The arithmetic perspective.* We start with an arithmetic perspective.

The goal is to study p-adic representations, i.e. continuous representations

$$\Gamma_K = \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_p)$$

where K is a p-adic field. This is quite different from studying ℓ-adic representations, i.e. continuous representations

$$\Gamma_K \to \mathrm{GL}_n(\mathbb{Q}_\ell) \quad \text{for } \ell \neq p.$$

Indeed, the topologies in this case are not quite compatible, so there are not as many representations as in the $\ell = p$ case.

We consider a motivating example. Let E be an elliptic curve over $\mathbb{Q}_p$ with good reduction. There is an elliptic scheme $\mathcal{E}$ over $\mathbb{Z}_p$ such that $\mathcal{E}_{\mathbb{Q}_p} = E$. For a prime ℓ (which may or may not be equal to p), we define the *Tate module*

$$T_\ell(E) = \varprojlim E[\ell^n](\overline{\mathbb{Q}_p}) \cong \mathbb{Z}_\ell^2$$

which has a continuous $\Gamma_{\mathbb{Q}_p}$-action. Tensoring with $\mathbb{Q}_\ell$, we get a continuous $\Gamma_{\mathbb{Q}_p}$-representation

$$V_\ell(E) = T_\ell(E) \otimes \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^2.$$

These representations see a lot of information about the elliptic curves. For example, we have the following fact.

Fact 1.1.1. *Given two elliptic curves E_1, E_2 over $\mathbb{Q}_p$, the natural maps*

$$\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(T_\ell(E_1), T_\ell(E_2))$$
$$\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(V_\ell(E_1), V_\ell(E_2))$$

are injective.

How to study $T_\ell(E)$? For $\ell \neq p$, we can consider the special fiber $\mathcal{E}_{\mathbb{F}_p}$, en elliptic curve over $\mathbb{F}_p$. The Tate module $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ is a continuous $\Gamma_{\mathbb{F}_p}$-representation. To describe the action, it is enough to describe the action of Frobenius (a topological generator for $\Gamma_{\mathbb{F}_p}$): it acts on $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ with characteristic polynomial $x^2 - ax + p$ where $a = p + 1 - \#(\mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p))$.

The punch line is that the reduction map

$$T_\ell(E) \to T_\ell(\mathcal{E}_{\mathbb{F}_p}) \tag{1}$$

is an isomorphism of $\Gamma_{\mathbb{Q}_p}$-representations, where the right hand side is a $\Gamma_{\mathbb{Q}_p}$-representation via the surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p} \cong \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Therefore:

(1) The action of $\Gamma_{\mathbb{Q}_p}$ factors through the map $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$.
(2) Frobenius of $\Gamma_{\mathbb{F}_p}$ acts with characteristic polynomial $x^2 - ax + p$.

The condition (1) is equivalent to the representation of $\Gamma_{\mathbb{Q}_p}$ being *unramified.*

Theorem 1.1.2 (Neron–Ogg–Shafarevich)**.** *An elliptic curve $E/\mathbb{Q}_p$ has good reduction if and only if $T_\ell(E)$ is unramified for all $\ell \neq p$.*

What about $\ell = p$? The key isomorphism (1) never holds. In fact,

$$T_p(\mathcal{E}_{\mathbb{F}_p}) \cong 0 \text{ or } \mathbb{Z}_p,$$

so it has the wrong rank. Let

$$I_{\mathbb{Q}_p} = \ker(\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p})$$

be the *intertia group.* Then there is a non-trivial contribution from $I_{\mathbb{Q}_p}$.

The solution to this problem was found by Grothendieck and Tate. We define

$$E[p^\infty] = \varinjlim E[p^n],$$

the p-divisible group of E. Note that this is a limit of schemes, not of the point of schemes.

Fact 1.1.3. *We can recover the action of $\Gamma_{\mathbb{Q}_p}$ on $T_p(E)$ from $E[p^\infty]$.*

The schemes $\mathcal{E}[p^\infty]$ and $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$ are also defined. We have maps

$$E[p^\infty] \xleftarrow{\otimes \mathbb{Q}_p} \mathcal{E}[p^\infty] \xrightarrow{\otimes \mathbb{F}_p} \mathcal{E}_{\mathbb{F}_p}[p^\infty]$$

Theorem 1.1.4 (Tate)**.** *The functor*

$$\left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \mathbb{Z}_p \end{matrix}\right\} \xrightarrow{\otimes \mathbb{Q}_p} \left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \mathbb{Q}_p \end{matrix}\right\}$$

is fully faithful.

Understanding the proof of the theorem and related results will be the goal of Chapter 2.

Theorem 1.1.5 (Dieudonné, Fontaine)**.** *There are equivalences of categories*

$$\left\{\begin{matrix} \textit{p-divisible groups} \\ \textit{over } \mathbb{F}_p \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{Dieduonné modules} \\ \textit{over } \mathbb{F}_p \end{matrix}\right\},$$

$$\left\{\begin{matrix} \textit{p-divisible groups} \\ \textit{over } \mathbb{Z}_p \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{Dieduonné modules} \\ \textit{over } \mathbb{F}_p \\ \textit{with an ``admissible filtration''} \end{matrix}\right\}.$$

Definition 1.1.6. A *Dieudonné module* over $\mathbb{F}_p$ is a free $\mathbb{Z}_p$-module M of finite rank with an endomorphism φ such that $\varphi(M) \supseteq pM$.

One should think of $\mathbb{Z}_p$ here are the ring of Witt vectors of $\mathbb{F}_p$, $\mathbb{Z}_p = W(\mathbb{F}_p)$.

The following summarizes the situation:

$$\begin{aligned} T_p(E) &\to E[p^\infty] \text{ } p\text{-divisible group} \\ &\to \left\{\begin{matrix} \text{Dieduonné module over } \mathbb{F}_p \\ + \text{ extra data} \end{matrix}\right\}. \end{aligned}$$

After inventing p, we also get

$$V_p(E) \to \left\{\begin{matrix} \text{``isocrystals'' over } \mathbb{F}_p \\ + \text{ extra data} \end{matrix}\right\}.$$

The general themes of p-adic Hodge theory are:

(1) To construct a dictionary between certain p-adic representations and certain semilinear algebraic objects.
(2) Change base field to $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$.

Since $\mathbb{Q}_p^{\mathrm{un}}$ is not p-adically complete any more, we need to work with $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ instead.

Many interesting properties of p-adic representations are encoded in the action of $I_{\mathbb{Q}_p}$. We note that:

$$I_{\mathbb{Q}_p} = I_{\mathbb{Q}_p^{\mathrm{un}}} = I_{\widehat{\mathbb{Q}_p^{\mathrm{un}}}}.$$

Usually, base changing to $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ simplifies things.

In the above correspondence, base changing to $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ roughly corresponds to replacing $\mathbb{F}_p$ by $\overline{\mathbb{F}_p}$.

Theorem 1.1.7 (Manin)**.** *The category of isocrystals over $\overline{\mathbb{F}_p}$ is semisimple.*

Question. Is there a *general framework* or *formalism* that provides all these general themes in more general scope?

To properly answer this question, we need to discuss the geometric side of the story.

1.1.2. *The geometric perspective.* The goal here is to use p-adic representations to study the geometry of algebraic varieties X over K. We look at the cohomology of X:

- $H_{\text{ét}}$: étale cohomology,
- H_{dR}: algebraic de Rham cohomology,
- H_{cris}: crystalline cohomology.

By definition, $H_{\text{ét}}$ is a p-adic Galois representation. The main goal is to find comparison theorems between the three cohomology theories.

In classical Hodge theory, there are many comparison theorems:

- between singular cohomology[1] and Hodge cohomology,
- between singular cohomology and de Rham cohomology

valid for proper smooth varieties over $\mathbb{C}$.

The reason for the name p-adic Hodge theory comes from the above motivation. The main issue in finding these comparison theorems is finding the correct *period ring*.

The obvious answer would be to work with $\widehat{\overline{K}}$, but we will soon see that this ring is not sufficient.

We first recall in more detail one of the comparison theorems from Hodge theory.

Theorem 1.1.8 (Hodge decomposition)**.** *Let Y be a proper smooth variety over $\mathbb{C}$. Then*

$$H^n(Y(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{i+j=n} H^i(Y, \Omega_Y^j).$$

Corollary 1.1.9. *The Hodge number of Y are topological invariants.*

Let $\mathbb{C}_K = \widehat{\overline{K}}$. It has a continuous Γ_K-action. The *p-adic cyclotomic character* is

$$\chi\colon \Gamma_K \to \mathbb{Z}_p^\times$$

such that for any p-power root of unity ζ,

$$\sigma(\zeta) = \zeta^{\chi(\sigma)}.$$

Definition 1.1.10. We define the *Tate twist* as a Γ_K-representation $\mathbb{C}_K(j)$ with the underlying vector space $\mathbb{C}_K$ and $\sigma \in \Gamma_K$ acting by $\chi^j(\sigma) \cdot \sigma$.

Theorem 1.1.11 (Hodge–Tate decomposition, Faltings)**.** *Let X be a proper smooth variety over K. Then*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_K(-j),$$

compatible with Γ_K-action, where

σ acts by $\sigma \otimes \sigma$ on the left hand side,
σ acts by $1 \otimes \sigma$ on the right hand side.

[1]One should think that singular cohomology over $\mathbb{C}$ corresponds to étale cohomology in the p-adic setting

Tate proved when X is an abelian variety with good reduction as a by product of the generic fiber functor theorem.

Define the *Hodge–Tate period ring*

$$B_{\mathrm{HT}} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}_K(j).$$

Then the Hodge–Tate decomposition 1.1.11 can be restated as

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}} \cong \left(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \right) \otimes B_{\mathrm{HT}}.$$

Theorem 1.1.12 (Tate–Sen)**.** *We have that* $B_{\mathrm{HT}}^{\Gamma_K} = K$*.*

As a consequence, we see that

$$\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes B_{\mathrm{HT}}\right)^{\Gamma_K} = \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}).$$

Here is another result from Hodge theory. There is an isomorphism

$$H^n(Y(\mathbb{C}), \mathbb{C}) \cong H^n_{\mathrm{dR}}(Y/\mathbb{C})$$

coming from the *period pairing*

$$\begin{aligned} H^n_{\mathrm{dR}}(Y/\mathbb{C}) \times H_{2d-n}(Y, \mathbb{C}) &\to \mathbb{C} \\ (\omega, \Gamma) &\mapsto \int_\Gamma \omega. \end{aligned}$$

Goal. Construct a *p-adic period ring.*

Fontaine constructed a p-adic period ring B_{dR} such that:

(1) B_{dR} carries Γ_K-action with $B_{\mathrm{dR}}^{\Gamma_K} = K$,
(2) B_{dR} carries a filtration with the accociated graded ring B_{HT}.

Theorem 1.1.13 (Faltings)**.** *We have that*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}$$

compatible with Γ_K*-actions and filtrations.*

By construction, $H^n_{\mathrm{dR}}(X/K)$ has a *Hodge filtration* such that the associated graded is

$$\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$$

The filtration on the right hand side of Faltings' Theorem 1.1.13 is given by the convolution filtration:

$$\mathrm{Fil}^m = \bigoplus_{a+b=m} \mathrm{Fil}^a \otimes \mathrm{Fil}^b.$$

Remarks 1.1.14.

(1) By passing to the associated graded in Faltings' Theorem 1.1.13, we recover the Hodge–Tate decomposition 1.1.11.
(2) We have that $\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes B_{dR}\right)^{\Gamma_K} \cong H^n_{\text{dR}}(X/K)$.
(3) We will not attempt to prove Faltings' Theorem 1.1.13, but we will use it as motivation.

Question. Is there a refinement of H_{dR} which recovers $H_{\text{ét}}$ itself?

Answer. Yes, cristalline cohomology H_{cris}.

Conjecture 1.1.15 (Grothendieck)**.** *Let $\mathcal{O}_K$ be the valuation ring of K and k be the residue field of $\mathcal{O}_K$. Let $W(k)$ be the ring of Witt vectors of k and $K_0 = \mathrm{Frac}(W(k))$. (If $K = \mathbb{Q}_p$ then $K_0 = \mathbb{Q}_p$, and if K is a finite extension of $\mathbb{Q}_p$, then K_0 is the maximal unramified subextension.)*

There should be a (purely algebraic) fully faithful functor D on a certain category of representations such that

$$D\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\right) = H^n_{\text{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K_0$$

for any proper smooth X with integral model $\mathcal{X}$ over $\mathcal{O}_K$.

Recall that for any elliptic curve E over $\mathbb{Q}_p$ with good reduction, we have seen that there is a fully faithful functor

$$V_p(E) \rightsquigarrow \{\text{filtered isocrystal}\}.$$

Now,

$$V_p(E) \cong \left(H^1_{\text{ét}}(E_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)\right)$$

and

$$\{\text{filtered isocrystal}\} \cong H^1_{\text{cris}}(\mathcal{E}_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Grothendieck's conjecture 1.1.15 is a generalization of this. By *purely algebraic* we mean that there should be a way to avoid going through p-divisible groups (which are geometric).

Fontaine constructed another period ring, called B_{cris} such that:

(1) B_{cris} carries an action of Γ_K such that $B^{\Gamma_K} = K_0$,
(2) B_{cris} carries a semi-linear endomorphsim φ called the *Frobenius action*,
(3) there is a natural map $B_{\text{cris}} \otimes_{K_0} K \hookrightarrow B_{\text{dR}}$, inducing a filtration on B_{cris}.

Theorem 1.1.16 (Faltings)**.** *Suppose X has good reduction with integral model $\mathcal{X}$. Then*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H^n_{\text{cris}}(\mathcal{X}_k/W(k)) \otimes B_{\text{cris}},$$

compatible with Γ_K-action, filtration, and Frobenius action.

Remark 1.1.17. By construction, $H^n_{\text{cris}}(\mathcal{X}_k/W(k))$ carries a *Frobenius action*. Frobenius acts only through B_{cris} on the left hand side and diagonally on the right hand side.

The isomorphism

$$H^n_{\text{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K \cong H^n_{\text{dR}}(X/K)$$

gives a filtration on H_{cris}. We use the convolution filtration on the right hand side.

Now, taking Γ_K-invariants of both sides gives:

$$\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{\Gamma_K} \cong H^n_{\text{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K_0.$$

There is an *inverse functor* so we get D, *Grothendieck's mysterios functor*, given by

$$D(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}.$$

This would prove Grothendieck's conjecture 1.1.15 if we define the domain of this functor and prove that it is fully faithful.

1.1.3. *Interplay via representation theory.* Fontaine built the formalism for functors that connect the geometric and arithmetic sides. This will be the focus of Chapter 3.

Let B be any period ring such as B_{HT}, B_{dR}, B_{cris}. Then define

$$\text{Rep}_{\mathbb{Q}_p}(\Gamma_K) = \text{category of } p\text{-adic representations of } \Gamma_K.$$

Define $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$. A representation $V \in \text{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if the natural maps

$$(V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \otimes B \to V \otimes B$$

is an isomorphism.

Now, D_B defines a functor on $\text{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, the category of B-admissible representations. The target category reflects the structure on B.

Examples 1.1.18. (1) If $B = B_{\text{HT}}$, the target category is the category of finite-dimensional graded vector spaces.
(2) If $B = B_{\text{dR}}$, the target category is the category of finite-dimensional filtered vector spaces.
(3) If $B = B_{\text{dR}}$, the target category is the category of finite-dimensional filtered vector spaces with Frobenius action.

Theorem 1.1.19 (Fontaine)**.** *The functors $D_{B_{\text{HT}}}$, $D_{B_{\text{dR}}}$, $D_{B_{\text{cris}}}$ are exact and faithful. Moreover, $D_{B_{\text{cris}}}$ is fully faithful.*

In particular, this proves Grothendieck's conjecture 1.1.15.

1.2. A first glimpse of the Fargues–Fontaine curve.

1.2.1. *Definition and key features.* There are two ways to describing the Fargues–Fontaine curve, the schematic curve and the adic curve. We will only describe the schematic curve, since we do not have the necessary language to talk about adic spaces. Fortunately, there is a GAGA type theorem, giving an equivalence between these two approaches.

For simplicity, we work with $K = \mathbb{Q}_p$. Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$. Let $F = \widehat{\overline{\mathbb{F}_p((u))}}$.

Recall Fontaine's ring B_{cris} with Frobenius action φ. There is a ring B^+_{cris} such that:

(1) B^+_{cris} is stable under φ, and $(B^+_{\text{cris}})^{\varphi=1} = \mathbb{Q}_p$,
(2) there exists $t \in B^+_{\text{cris}}$ such that $B^+_{\text{cris}}\left[\frac{1}{t}\right] = B_{\text{cris}}$ and $\varphi(t) = pt$.

Definition 1.2.1. The *Fargues–Fontaine curve* associated to $(\mathbb{Q}_p, F)$ is

$$X = \text{Proj}\left(\bigoplus_{n\geq 0}(B^+_{\text{cris}})^{\varphi=p^n}\right).$$

Remark 1.2.2. The Fargues–Fontaine curve X is

(1) a $\mathbb{Q}_p$-scheme,
(2) not of finite type over $\mathbb{Q}_p$, and hence not projective.

Slogan. The Fargues–Fontaine curve is the p-adic analogue of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$.

Theorem 1.2.3 (Fargues–Fontaine, Kedlaya)**.** *The curve X satisfies the following properties:*

(1) *it is Noetherian, connected, regular of dimension 1 over $\mathbb{Q}_p$,*
(2) *it is the union of two spectra of Dedekind domains,*
(3) *it is complete in the sense that for all $f \in K(X)$, $\text{div}(f)$ has degree 0,*
(4) $\text{Pic}(X) \cong \mathbb{Z}$*.*

In fact, X is an affine scheme of a PID together with a point at ∞. There exist closed points $x \in X$ such that

$$X \setminus \{x\} \cong \text{Spec}(B_e)$$
$$\widehat{\mathcal{O}_{X,x}} \cong B^+_{\text{dR}}$$

where

$$B_e = B^{\varphi=1}_{\text{cris}},$$
$$B^+_{\text{dR}} = \textit{valuation ring of } B_{\text{dR}}.$$

1.2.2. *Relation to the theory of perfectoid spaces.*

Definition 1.2.4. Let C be a field which is complete, non-archimedean, residue characteristic p.

(1) It is a *perfectoid field* if
   (a) the valuation is non-discrete,
   (b) the p-power map is surjective on $\mathcal{O}_C/p$.
(2) The *tilt* of C is defined as

$$C^\flat = \varprojlim_{x\mapsto x^p} C$$

with

$$(a\cdot b)_n = a_n \cdot b_n,$$
$$(a+b)_n = \lim_{n\to\infty}(a_{n+m}+b_{n+m})^{p^m}$$
$$|a|^\flat = |a_0|.$$

Remark 1.2.5. For any C, $C^\flat$ is a perfectoid field of characteristic p.

Examples 1.2.6. The field $\mathbb{C}_p$ is perfectoid of characteristic 0 with $\mathbb{C}_p^\flat \cong F$.

Remark 1.2.7. Scholze extended the de Rham Comparison Theorem 1.1.13 to rigid analytic varieties using the theory of perfectoid spaces.

Theorem 1.2.8 (Tilting equivalence)**.** *Suppose C is a perfectoid field.*

(1) *Every finite extension of C is a perfectoid field.*
(2) *There is a bijection*

$$\{\textit{finite extension of } C\} \leftrightarrow \{\textit{finite extension of } C^\flat\}$$
$$L \mapsto L^\flat,$$

(3) *The above bijection induces an isomorphism $\Gamma_C \cong \Gamma_{C^\flat}$.*

This allows to translate problems in characteristic 0 to problems in characteristic p.

Question. Can you parameterize a way of *untilting*?

Definition 1.2.9. An *untilt* of F is a pair (C, ι) where C is a perfectoid field of characteristic 0 and $\iota\colon C^\flat \cong F$.

Let φ_F be the Frobenius automorphism on F. It acts on the set of untilts of F by

$$\varphi_F \circ (C, \iota) = (C, \varphi_F \circ \iota).$$

Theorem 1.2.10 (Fargues–Fontaine)**.**

(1) *For any closed point $x \in X$, $k(x)$ is a perfectoid field of characteristic 0 with $k(x)^\flat \cong F$.*
(2) *There is a bijection*

$$\{\textit{closed points on } X\} \leftrightarrow \{\varphi_F\textit{-orbits of untilts}\}$$

induced by $x \mapsto k(x)$.

Remark 1.2.11. This theorem is one of the main motivations for the theory of *diamonds*. Just as

$$\text{Algebraic space} = \text{Scheme/étale equivalent relation},$$

one thing should that

$$\text{Diamond} = \text{Perfectoid space/pro-étale equivalence relation}.$$

1.3. Geometrization of p-adic representations.

Definition 1.3.1. Fix a closed point $\infty \in X$.

(1) A *vector bundle* on X is a locally free $\mathcal{O}_X$-module of finite rank.
(2) A *modification of vector bundles* on X is $(\mathcal{E}, \mathcal{F}, i)$ where
   - $\mathcal{E}$, $\mathcal{F}$ are vector bundles on X,
   - $i\colon \mathcal{E}|_{X\setminus\infty} \cong \mathcal{F}|_{X\setminus\infty}$.

Remark 1.3.2. There is a complete classification of vector bundles on X. We will see this later in the course. Roughly, it is analogous the fact that any vector bundle on $\mathbb{P}^1$ is isomorphic to $\bigoplus \mathcal{O}(\lambda)$.

Theorem 1.3.3 (Fargues–Fontaine). *There is a functorial commutative diagram:*

$$\begin{array}{ccc} \{\text{isocrystals over } \overline{\mathbb{F}_p}\} & \longleftrightarrow & \{\text{vector bundles on } X\} \\ \uparrow & & \uparrow \\ \{\text{filtered isocrystals over } \overline{\mathbb{F}_p}\} & \longrightarrow & \left\{\substack{\text{modifications of} \\ \text{vector bundles on } X}\right\} \end{array}$$

where the vertical arrows are forgetful functors. The top horizontal arrow is a bijection, but not an equivalence of categories.

Recall that there is a functor

$$D_{B_{\mathrm{cris}}} \colon \left\{\substack{B_{\mathrm{cris}}\text{-admissible} \\ \text{representations over } \mathbb{Q}_p}\right\} \to \left\{\substack{\text{filtered isocrystals} \\ \text{over } \mathbb{F}_p}\right\}$$

which is fully faithful.

Question. What is the essential image of this functor?

Theorem 1.3.4 (Colmez–Fontaine). *Given $N^0 = (N, \mathrm{Fil}^\bullet(N))$ over $\mathbb{F}_p$, define $\overline{N^0} = (\overline{N}, \mathrm{Fil}^\bullet(\overline{N}))$ over $\overline{\mathbb{F}_p}$. Via Theorem 1.3.3, we obtain a modification of vector bundles $(\mathcal{E}(\overline{N}), \mathcal{F}(\overline{N}), i(\overline{N}))$.*

Then N^0 is in the essential image of $D_{B_{\mathrm{cris}}}$ if and only if $\mathcal{F}(\overline{N})$ is trivial (i.e. $\mathcal{F}(\overline{N}) \cong \mathcal{O}_X^{\oplus n}$).

Remark 1.3.5. Let V_{cris} be the quasi-inverse of $D_{B_{\mathrm{cris}}}$. Then

$$V_{\mathrm{cris}}(\overline{N}) = H^0(X, \mathcal{F}(\overline{N})).$$

2. Foundations of p-adic Hodge theory

The goal of this chapter is to discuss:

(1) finite flat group schemes,
(2) p-divisible groups.

In particular, we will try to cover the main results of Tate's p-divisible groups [Tat67].

2.1. Finite flat group schemes.

The main reference for this chapter is Tate's finite flat group schemes [Tat97].

2.1.1. *Basic definition and properties.*

Definition 2.1.1. Let S be a base scheme. An S-scheme G is a *group scheme* if there are maps

- $m \colon G \times_S G \to G$ *multiplication*,
- $e \colon S \to G$ *unit section*,
- $i \colon G \to G$ *inverse*.

satisfying the following axioms:

(1) associativity:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{(\mathrm{id},m)} & G \times G \\ {\scriptstyle (m,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

(2) identity axiom:

$$\begin{array}{ccc} G \times_S S \cong G & \xrightarrow{\qquad \mathrm{id} \qquad} & G \\ & {\scriptstyle (\mathrm{id},e)}\searrow \quad \nearrow{\scriptstyle m} & \\ & G \times G & \end{array}$$

and similarly for $S \times_S G \cong G$,

(3) inverse:

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id},i)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow{\quad e \quad} & G \end{array}$$

Lemma 2.1.2. *Let G be an S-scheme. It is a group scheme if and only if $G(T)$ is a group functorial in T for all T/S.*

Definition 2.1.3. Let G, H be group schemes over S. A map $f\colon G \to H$ of S-schemes is a *homomorphism* if $G(T) \to H(T)$ is a group homomorphism for all T/S.

We define $\ker(f)$ to be an S-group scheme such that

$$\ker(f)(T) = \ker(G(T) \to H(T)).$$

Equivalently, $\ker(f)$ is the fiber of the unit section.

Example 2.1.4. The multiplication by n map $[n]_G\colon G \to G$ is defined by $g \mapsto g^n$.

Assume $S = \mathrm{Spec}(R)$.

Definition 2.1.5. Then $G = \mathrm{Spec}(A)$ is an R-group scheme if it has

- $\mu\colon A \to A \otimes_R A$ *comultiplcation,*
- $\epsilon\colon A \to R$ *counit,*
- $\iota\colon A \to A$ *coinverse.*

that correspond to multiplication, unit section, and inverse.

Examples 2.1.6.

(1) The *multiplcative group* over R is

$$\mathbb{G}_m = \mathrm{Spec}(R[t, t^{-1}]).$$

Then $\mathbb{G}_m(B) = B^\times$ with multiplication for any R-algebra B. Then

$$\mu(t) = t \otimes t, \quad \epsilon(t) = 1, \quad \iota(t) = t^{-1}.$$

(2) The *additive group* over R is

$$\mathbb{G}_a = \mathrm{Spec}(R[t])$$

Then $\mathbb{G}_a(B) = B$ with addition for any R-algebra B. Then

$$\mu(t) = 1 \otimes t + t \otimes 1, \quad \epsilon(t) = 0, \quad \iota(t) = -t.$$

(3) The *nth roots of unity* over R is

$$\mu_n = \mathrm{Spec}(R[t]/(t^n - 1)).$$

For any R-algebra B,

$$\mu_n(B) = \{b \in B \mid b^n = 1\}$$

under multiplication. The functions μ, ϵ, ι are all as in (1).

(4) If R has characteristic p, we can define

$$\alpha_p = \mathrm{Spec}(R[t]/t^p).$$

Then $\alpha_p(B) = \{b \in B \mid b^p = 0\}$ with addition for any R-algebra B. The functions μ, ϵ, ι are all as in (2).

(5) Let $\mathcal{A}$ be an abelian scheme over R. Then

$$\mathcal{A}[n] = \ker([n]_{\mathcal{A}})$$

is an affine group scheme over R. This is because $[n]_{\mathcal{A}}$ is a finite morphism.

(6) Let M be a finite abstract group. We can associate to it the constant group scheme $\underline{M}$ defined by

$$\underline{M} = \coprod_{m \in M} \mathrm{Spec}(R) \cong \mathrm{Spec}\left(\prod_{m \in M} R\right).$$

Writing $A = \prod_{m \in M} R$, note that

$$A \cong \{R\text{-valued functions on } M\}.$$

For any R-algebra B, we have that

$$\underline{M}(B) = \{\text{locally constant functions } \mathrm{Spec}(B) \to M\}$$

with the group structure induced by M. To describe μ, note that

$$A \otimes_R A = \{R\text{-valued functions on } M \times M\}.$$

We have that

$$\begin{aligned} \mu(f)(m, m') &= f(mm'), \\ \epsilon(f) &= f(1_M), \\ \iota(f)(m) &= f(m^{-1}). \end{aligned}$$

Assumption. From now on, R is a Noetherian local ring, $\mathfrak{m}$ is the maximal ideal of R, k is the residue field. The assumption R local is just for simplicity.

Definition 2.1.7. Let $G = \mathrm{Spec}(A)$ be an R-group scheme. It is a (commutative) *finite flat group scheme* of *order* n if:

(1) A is a locally free R-module of rank n,

(2) G is commutative, in the sense that:

$$\begin{array}{ccc} G \times G & \xrightarrow{(x,y)\mapsto(y,x)} & G \times G \\ & {\searrow}^{m} \quad {\swarrow}_{m} & \\ & G & \end{array}$$

Remark 2.1.8. (1) implies that $G \to \mathrm{Spec}(R)$ is finite and flat. (2) implies that $G(T)$ is commutative for all T over $S = \mathrm{Spec}(R)$. Note that $G(T)$ may not be of order n; for example, if $T = \mathrm{Spec}(B)$ if B is highly disconnected.

Example 2.1.9.

(1) The group scheme μ_n is finite flat of order n.
(2) If R has characteristic p, α_p is a finite flat R-group scheme of order p.
(3) Let $\mathcal{A}$ be an abelian scheme of dimension g over R. Then $\mathcal{A}[n]$ is a finite flat group scheme of order n^{2g}.
(4) If M is a finite abelian abstract group of order n, then $\underline{M}$ is a finite flat group scheme of order n.

We will assume two theorems in this section without proof.

Theorem 2.1.10 (Grothendieck)**.** *Suppose G is a finite flat R-group scheme of order m and $H \subseteq G$ is a closed finite flat R-subgroup scheme of order n. Then the quotient G/H exists as a finite flat R-group scheme of order m/n.*

As a result, we have a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

of R-group schemes.

Theorem 2.1.11 (Serre)**.** — *Let G be a finite flat R-group scheme of order n. Then $[n]_G$ kills G, i.e. $[n]_G$ factors through the unit section of G.*

Remark 2.1.12. This is unknown for noncommutative finite flat group schemes.

Lemma 2.1.13. *Suppose G is as above. Then $G_B = G \times_R B$ for any R-algebra B is a finite-flat B-group scheme.*

Proof. If $G = \mathrm{Spec}(A)$ with μ, ϵ, ι, then $G_B = \mathrm{Spec}(A_B)$ with $\mu \otimes 1$, $\epsilon \otimes 1$, $\iota \otimes 1$. □

2.1.2. *Cartier duality.*

Definition 2.1.14. Let G be as above. The *Cartier dual* $G^\vee$ of G is

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B)$$

with group structure induced by $(\mathbb{G}_m)_B$.

Using this definition, it is hard to see that $G^\vee$ is a finite flat group scheme. We will describe it differently soon which will make this apparent.

Remark 2.1.15. We could have defined $G^\vee = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$, where the sheaf Hom is on the big fppf site.

Lemma 2.1.16. *If $[n]_G$ kills G, then*

$$G^\vee(B) = \mathrm{Hom}_{B\text{-}grp}(G_B, (\mu_n)_B).$$

Proof. Recall that $\mu_n = \ker([n]_{\mathbb{G}_m})$. □

Theorem 2.1.17 (Cartier duality)**.** *Let $G = \mathrm{Spec}(A)$ be an R-group scheme of order n with μ, ϵ, ι as comultiplication, counit, coinverse. Define*

$$\begin{aligned} m_A \colon A \otimes_R A &\to A && \textit{ring multiplication,} \\ p \colon R &\to A && \textit{structure morphism,} \\ A^\vee &= \mathrm{Hom}_{R\text{-}mod}(A, R). \end{aligned}$$

Then:

(1) *the maps $\mu^\vee$ and $\epsilon^\vee$ given an R-algebra structure on $A^\vee$,*
(2) *$G^\vee \cong \mathrm{Spec}(A^\vee)$ with $m_A^\vee$, $p^\vee$, $\iota^\vee$ as comultiplication, counit, coinverse,*
(3) *$G^\vee$ is a finite flat R-group scheme of order n,*
(4) *$(G^\vee)^\vee \cong G$ canonically.*

Proof. Part (1) is straightforward. Parts (3) and (4) are consequences of (2). It suffices to prove (2) but we will do this next time. □

Examples 2.1.18.

(1) We have that $\mu_n^\vee \cong \underline{\mathbb{Z}/n\mathbb{Z}}$. Exercise: check this using Cartier duality 2.1.17.
(2) We have that $\alpha_p^\vee \cong \alpha_p$.

As a consequence, we have the following result.

Proposition 2.1.19. *Suppose $R = k$ is a field. Let $f \colon A \to B$ be an isogeny between abelian varieties over k. Then*

$$\ker(f)^\vee \cong \ker(f^\vee).$$

Proof. We have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow A \xrightarrow{f} B \longrightarrow 0.$$

Applying the $\underline{\mathrm{Hom}}$ functor, we get the long exact sequence sequence:

$$0 \longrightarrow \underline{\mathrm{Hom}}(B, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(A, \mathbb{G}_m) \longrightarrow \overbrace{\underline{\mathrm{Hom}}(\ker(f), \mathbb{G}_m)}^{\ker(f)^\vee}$$

$$\longrightarrow \underbrace{\underline{\mathrm{Ext}}^1(B, \mathbb{G}_m)}_{B^\vee} \xrightarrow{f^\vee} \underbrace{\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)}_{A^\vee}$$

We have that $\mathrm{Hom}(B, \mathbb{G}_m) = \mathrm{Hom}(A, \mathbb{G}_m) = 0$ since A, B are proper so any $A \to \mathbb{G}_m$ is constant. Hence the short exact sequence

$$0 \longrightarrow \ker(f)^\vee \longrightarrow B^\vee \xrightarrow{f^\vee} A^\vee,$$

completing the proof. □

Corollary 2.1.20. *Let $R = k$ be a field. Then $A[n]^\vee \cong A^\vee[n]$. This gives*

$$A[n] \times A^\vee[n] \to \mu_N,$$

called the Weil pariing.

Later, we will use a pairing

$$T_p(A) \times T_p(A^\vee) \to \mu_{p^\infty} \cong \mathbb{Z}_p(1)$$

obtained from the above corollary.

Proof of Cartier duality 2.1.17. Let $G = \mathrm{Spec}(A)$ and μ, ϵ, ι be the comultiplication, counit, and coinverse.

Let $p\colon R \to A$ be the structure morphism, $m_A\colon A \otimes_R A \to A$ be the ring multiplication. Consider

$$A^\vee = \mathrm{Hom}_R(A, R)$$

with R-algebra structure given by $\mu^\vee$ and $\epsilon^\vee$. Consider

$$G^\nabla = \mathrm{Spec}(A^\vee)$$

with $m_A^\vee$, $p^\vee$, $\iota^\vee$ as comultiplication, counit, and coinverse. We want to show that

$$G^\vee(B) \cong G^\nabla(B) \tag{2}$$

for all R-algebra B. We have that:

$$\begin{aligned} G^\vee(B) &\cong \mathrm{Hom}_{\mathrm{grp}}(G_B, (\mathbb{G}_m)_B) \\ &= \left\{ f \in \mathrm{Hom}_{B\text{-alg}}(B[t, t^{-1}], A_B) \;\middle|\; \substack{\mu_B(f(t)) = f(t) \otimes f(t), \\ \epsilon_B(f(t)) = 1, \\ \iota_B(f(t)) = f(t)^{-1}} \right\} \\ &= \left\{ u \in A_B^\times \;\middle|\; \substack{\mu(u) = u \otimes u, \\ \epsilon(u) = 1, \\ \iota(u) = u^{-1}} \right\} && \text{via } f \mapsto f(t) \\ &= \{ u \in A_B^\times \mid \mu(u) = u \otimes u \}, \end{aligned}$$

where the last equality follows from

$$\begin{aligned} (\mathrm{id}_B \otimes \epsilon_B) \circ \mu_B &= \mathrm{id}_B, \\ (\mathrm{id}_B \otimes \iota) \circ \mu_B &= p_B \circ \epsilon_B. \end{aligned}$$

Now, the right hand side of equation 2 is

$$\begin{aligned} G^{\nabla}(B) &= \mathrm{Hom}_{R\text{-alg}}(A^\vee, B) \\ &= \mathrm{Hom}_{B\text{-alg}}(A^\vee \otimes B, B) \\ &= \{f \in \mathrm{Hom}_{B\text{-mod}}(B, A_B) \mid \text{compatible with } m_B^\vee, p_B^\vee, \mu_B, \epsilon_B\} \\ &= \{u \in A_B^\times \mid \mu_B(u) = u \otimes u,\ \epsilon_B(u) = 1\} \\ &= \{u \in A_B^\times \mid \mu(u) = u \otimes u\}. \end{aligned}$$

This completes the proof if we check that the isomorphism respects the group structure. This is left as an exercise. □

Lemma 2.1.21. *Suppose $f\colon H \hookrightarrow G$ is a closed embedding of finite flat R-groups. Then*

$$\ker(f)^\vee \cong (G/H)^\vee.$$

Proof. We have that

$$\begin{aligned} \ker(f)^\vee(B) &= \ker(\mathrm{Hom}(G_B, \mathbb{G}_{m,B}) \xrightarrow{f} \mathrm{Hom}(H_B, \mathbb{G}_{m,B})) \\ &= \mathrm{Hom}((G/H)_B, \mathbb{G}_{m,B}) \\ &= (G/H)^\vee(B), \end{aligned}$$

as required. □

Proposition 2.1.22. *Taking the Cartier dual is an exact functor.*

Proof. We want to show that if

$$0 \longrightarrow G' \xrightarrow{\ f\ } G \xrightarrow{\ g\ } G'' \longrightarrow 0,$$

then

$$0 \longrightarrow (G'')^\vee \xrightarrow{\ g^\vee\ } G^\vee \xrightarrow{\ f^\vee\ } (G')^\vee \longrightarrow 0$$

is exact. Injectivity of $g^\vee$ is easy to check, since $\ker(f^\vee) \cong (G'')^\vee$.

To check that $f^\vee$ is surjective, note that $f^\vee\colon G^\vee \to (G')^\vee$ induces

$$G^\vee/(G'')^\vee \to (G')^\vee.$$

Its dual is

$$(G')^{\vee\vee} \to (G^\vee/(G'')^\vee)^\vee \cong \ker(g^{\vee\vee}) = \ker(g) = G',$$

which is an isomorphism. □

2.2. Finite étale group schemes.

Proposition 2.2.1. *For R Henselian, we have that:*

$$\{\textit{finite étale groups over } R\} \leftrightarrow \{\textit{finite abelian groups with a continuous } \Gamma_k\textit{-action}\}$$
$$G \mapsto G(\overline{k}).$$

Proof. Consider $\overline{m} \colon \operatorname{Spec}(\overline{k}) \to R$, a geometric point. Then

$$\pi_1(\operatorname{Spec}(R), \overline{m}) \cong \Gamma_k.$$

Hence

$$\{\text{finite étale schemes}/R\} \leftrightarrow \{\text{finite sets with a continuous } \Gamma_k\text{-action}\}.$$

Passing to group objects gives the result. □

Remark 2.2.2.

(1) This bijection is compatible with the order on each side.
(2) If $k = \overline{k}$, we have that $\Gamma_k = 1$.

Definition 2.2.3. Let $G = \operatorname{Spec}(A)$. The *augmentation ideal* is $I = \ker(\epsilon)$.

Lemma 2.2.4. *As R-modules, $A \cong R \oplus I$.*

Proof. The structure morphism $R \to A$ splits the short exact sequence:

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\epsilon} R \longrightarrow 0,$$

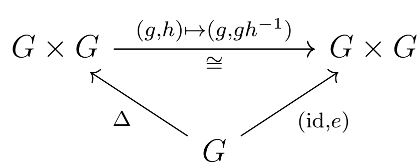
giving the desired isomorphism. □

Proposition 2.2.5. *Let $G = \operatorname{Spec}(A)$ and I be the augmentation ideal. Then*

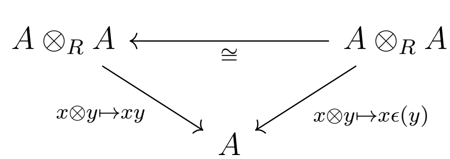
$$\Omega_{A/R} \cong I/I^2 \otimes_R A,$$
$$I/I^2 \cong \Omega_{A/R} \otimes_A A/I.$$

Remark 2.2.6. The multiplication on G defines an action on $\Omega_{A/R}$. The invariant forms under the G-action are determined by the values along the unit section. Any other form is an invariant form times a form on A.

Proof. We have the commutative diagram:



which corresponds to the commutative diagram



Let J be the kernel of the left map. Then $\Omega_{A/R} = J/J^2$ by definition.[2]

The kernel of the right hand side map is $J = A \otimes_R I$ since

$$A \otimes_R A \cong (A \otimes_R R) \oplus (A \otimes_R I)$$

and $I = \ker(\epsilon)$. Hence

$$J^2 = (A \otimes_R I)^2 = A \otimes_R I^2,$$

and so

$$J/J^2 = (A \otimes I)/(A \otimes I^2) \cong A \otimes_R I/I^2,$$

showing that

$$\Omega_{A/R} \otimes_A A/I = (I/I^2 \otimes_R A) \otimes A/I = (I/I^2) \otimes_R A/I \cong I/I^2.$$

This gives the result. □

Corollary 2.2.7. *Let $G = \operatorname{Spec}(A)$ be a finite flat R-group scheme. Then G is étale if and only if $I = I^2$.*

Proposition 2.2.8. *Every constant group scheme is étale.*

Proof. If $A = \prod_{m \in M} R$, then $I = \prod_{m \neq \mathrm{id}_M} R$, so $I = I^2$. □

Corollary 2.2.9. *Let $R = k = \overline{k}$ be a field of characteristic p. Then $\underline{\mathbb{Z}/p\mathbb{Z}}$ is the unique finite étale $\overline{k}$-group scheme of order p.*

In particular, $\underline{\mathbb{Z}/p\mathbb{Z}}$, μ_p, α_p are mutually non-isomorphic as finite flat groups of order p.

Proof. We know that $\underline{\mathbb{Z}/p\mathbb{Z}}$ is étale. Uniqueness follows from Proposition 2.2.1.

Since μ_p, α_p are not reduced, they are not isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$. Finally:

$$\mu_p = \operatorname{Spec}(k[t]/t^{p-1}) \quad \text{so } \mu_p^\vee \cong \underline{\mathbb{Z}/p\mathbb{Z}},$$

$$\alpha_p = \operatorname{Spec}(k[t]/t^p) \quad \text{so } \alpha_p^\vee \cong \alpha_p,$$

so they cannot be isomorphic. □

Proposition 2.2.10. *Let $G = \operatorname{Spec}(A)$ be a finite flat R-group scheme. Then G is étale if and only if the image of the unit section is open.*

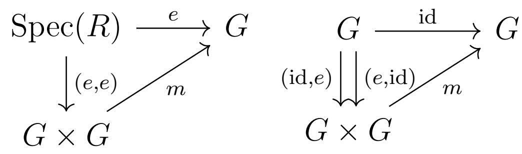
Proof. We have $\epsilon\colon \operatorname{Spec}(R) \to \operatorname{Spec}(A)$. The image of the unit section is $\operatorname{Spec}(A/I)$ which is open if and only if $I = I^2$. □

Proposition 2.2.11. *Let $G = \operatorname{Spec}(A)$ be a finite flat R-group scheme. If the order G is invertible in R, then G is étale.*

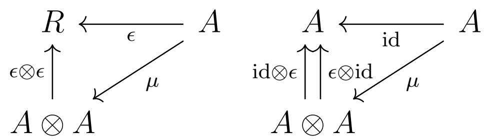
Corollary 2.2.12. *Every finite flat group scheme over a field of characteristic 0 is étale.*

Proof of Proposition 2.2.11. Let n be the order of G. We claim that $[n]_G$ induces multiplication by n on I/I^2. We have the diagrams

[2] This is an equivalent way to define $\Omega_{A/R}$. Indeed, $ds = 1 \otimes s - s \otimes 1$ is the universal derivation.



which correspond to



For all $x \in I$, $\epsilon \otimes \epsilon(\mu(x)) = 0$.

Since $A \cong R \oplus I$, we have that

$$A \otimes A \cong R \otimes R \oplus R \otimes I \oplus I \otimes R \oplus I \otimes I,$$

so

$$\mu(x) = a \otimes 1 + 1 \otimes b + I \otimes I$$

for $a, b \in I$. For $x = a = b$, we get

$$\mu(x) = 1 \otimes x + x \otimes 1 + I \otimes I$$

for all $x \in I$. Hence μ acts as $1 \otimes x + x \otimes 1$ on I/I^2. By induction, the assertion follows (indeed, $[n] = m \circ ([n-1], \mathrm{id})$ and we can run a similar argument).

We know that $[n]$ kills G by Serre's Theorem 2.1.11. Hence $[n]$ factors as:

$$[n]\colon G \to R \xrightarrow{e} G.$$

This gives

$$\Omega_{A/R} \to \underbrace{\Omega_{R/R}}_{=0} \to \Omega_{A/R},$$

so the induced map on $\Omega_{A/R}$ is 0. Thus $[n]_G$ induces the zero map on

$$\Omega_{A/R} \otimes_A A/I \cong I/I^2.$$

As n is invertible, multiplication by n on I/I^2 should be an isomorphism. □

2.3. **The connected étale sequence.** Let R be a Henselian local ring with residue field k.

Lemma 2.3.1. *An R-group G is étale if and only if G_k is étale.*

Proof. Étaleness is a fiberwise property. □

Lemma 2.3.2. *Let $T = \mathrm{Spec}(B)$ be a finite scheme over R. The following are equivalent:*

(1) *T is connected,*
(2) *B is a henselian local finite R-algebra,*
(3) *Γ_k acts transitively on $T(\overline{k})$.*

Proof. Clearly, (2) implies (1), because local implies connected. For (1) implies (2), suppose $B = \prod B_i$ for henselian local finite R-algebras. Then $\mathrm{Spec}(B_i)$ is a connected component of $\mathrm{Spec}(B)$. To show that (1) is equivalent to (3), let k_i be the residue field of B_i. Then

$$T(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(B, \overline{k}) = \coprod \mathrm{Hom}_k(k_i, \overline{k})$$

and $\mathrm{Hom}(k_i, \overline{k})$ is a Γ_k-orbit. □

Proposition 2.3.3. *Let $G = \mathrm{Spec}(A)$ and G^0 be a connected component of the unit section. Then $G^0(\overline{k}) = 0$.*

Proof. Let $G^0 = \mathrm{Spec}(A^0)$. Then A^0 is a henselian local finite R-algebra. We get a surjective homomorphism $A^0 \to R$. The residue field of A^0 is k. Then $G^0(\overline{k}) = \mathrm{Hom}_k(k, \overline{k}) = 0$. □

Theorem 2.3.4 (Connected–étale sequence)**.** *Let $G = \mathrm{Spec}(A)$ be a finite-flat R-group scheme. Then G^0 is a closed subgroup of G $G^{\text{ét}} = G/G^0$ is a finite étale group over R. We have a short exact sequence*

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0.$$

Proof. We have that $G^0 \times G^0$ is connected, since

$$(G^0 \times G^0)(\overline{k}) = G^0(\overline{k}) \times G^0(\overline{k}) = 0.$$

We hence have that $m(G^0 \times G^0) \subseteq G^0$ and $\iota(G^0) \subseteq G^0$, so G^0 is a closed subgroup. The unit section of $G^{\text{ét}}$ is G^0/G^0 which is open, since G^0 is open in G. □

Corollary 2.3.5. *A finite flat group scheme G is connected if and only $G(\overline{k}) = 0$.*

Corollary 2.3.6. *A finite flat group scheme G is étale if and only if $G^0 = 0$.*

Corollary 2.3.7. *If $f\colon G \to H$ is a group homomorphism with H is étale, then f uniquely factors through $G^{\text{ét}}$.*

Proof. We have that $f(G^0) \subseteq H^0 = 0$, so we get the result using the universal property of $G^{\text{ét}}$. □

Proposition 2.3.8. *Let $R = k = \overline{k}$ be a field. Then the connected–étale sequence splits. (This is also true if $R = k$ is a perfect field.)*

Proof. We want to show that there is a section of $G \twoheadrightarrow G^{\text{ét}}$. Consider

$$G^{\text{red}} = \mathrm{Spec}(A/\mathfrak{n})$$

where $\mathfrak{n}$ is the nilradical of A. We claim that G^{red} is a subgroup of G. Since a product of reduces schemes is reduced, $G^{\text{red}} \times G^{\text{red}}$ is reduced. Hence

$$m(G^{\text{red}} \times G^{\text{red}}) \subseteq G^{\text{red}}, \quad \iota(G^{\text{red}}) \subseteq G^{\text{red}}.$$

Moreover, G^{red} is étale because it is finite and reduced over k.

It suffices to show that the map $G \twoheadrightarrow G^{\text{ét}}$ induces $G^{\text{red}} \cong G^{\text{ét}}$. Since k is reduced, $G^{\text{red}}(\overline{k}) = G(\overline{k})$ and we also know that $G(\overline{k}) = G^{\text{ét}}(\overline{k})$. □

Example 2.3.9. Consider an elliptic curve E over $\overline{\mathbb{F}_p}$. We have a connected–étale sequence for the p-torsion:

$$0 \longrightarrow E[p]^0 \longrightarrow E[p] \longrightarrow E[p]^{\text{ét}} \longrightarrow 0.$$

We know that $E[p](\overline{\mathbb{F}_p})$ has order 1 or p. Hence $E[p]^{\text{ét}}(\mathbb{F}_p)$ has order p if E is ordinary of 1 if E is supersingular. Assume E is ordinary. Hence $E[p]^{\text{ét}}$ is étale of order p. By Corollary 2.2.9,

$$E[p]^{\text{ét}} \cong \underline{\mathbb{Z}/p\mathbb{Z}}.$$

Moreover,

$$(E[p]^{\text{ét}})^\vee \cong (\underline{\mathbb{Z}/p\mathbb{Z}})^\vee \cong \mu_p \hookrightarrow E[p]^\vee = E^\vee[p] \cong E[p].$$

Since μ_p is connected, $\mu_p \hookrightarrow E[p]^0$, so $\mu_p \cong E[p]^0$. Hence the connected-étale sequence is

$$0 \longrightarrow \mu_p \longrightarrow E[p] \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow 0.$$

By Proposition 2.3.8,

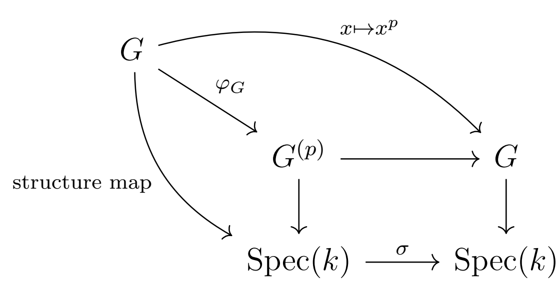
$$E[p] \cong \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}.$$

Remark 2.3.10. If E is supersingular, we know that $E[p]^{\text{ét}}$ is trivial. Then $E[p]$ is self-dual and we have a short exact sequence:

$$0 \longrightarrow \alpha_p \longrightarrow E[p] \longrightarrow \alpha_p \longrightarrow 0.$$

2.4. **The Frobenius morphism.** Let $R = k$ be a perfect field of characteristic p. Let σ be the Frobenius on k.

Definition 2.4.1. Let $G = \mathrm{Spec}(A)$ be a finite k-group. The *Frobenius twist* is $G^{(p)} = G \times_{k,\sigma} k$ and the (relative) *Frobenius* φ_G of G (over k) is defined by the diagram:



More generally,

$$G^{(p^r)} = (G^{(p^{r-1})})^{(p)},$$
$$\varphi_G^r = \varphi_{G^{(p^{r-1})}} \circ \varphi_G^{r-1}.$$

The *Verschiebung* of G is $\psi_G = \varphi_{G^\vee}^\vee$ where

$$\varphi_{G^\vee} : G^\vee \to (G^\vee)^{(p)}.$$

Remark 2.4.2. Verschiebung ψ_G is a map $G^{(p)} \cong ((G^\vee)^{(p)})^\vee \to G$.

Remark 2.4.3. We can check if a finite flat R-group scheme is connected or étale by passing to the special fiber. There are criteria for connected or étaleness for G_K in terms of Frobenius and the Verschiebung

Lemma 2.4.4.

(1) *The Frobenius φ_G induces a map*

$$\begin{aligned} A^{(p)} = A \otimes_{k,\sigma} k &\to A \\ a \otimes c &\mapsto c \cdot a^p. \end{aligned}$$

(2) *For any morphism $G \to H$ as schemes, we have induced maps*

$$\begin{array}{ccc} G & \xrightarrow{\varphi_G} & G^{(p)} \\ \downarrow & & \downarrow \\ H & \xrightarrow{\varphi_H} & H^{(p)} \end{array} \qquad \begin{array}{ccc} G^{(p)} & \xrightarrow{\psi_G} & G \\ \downarrow & & \downarrow \\ H^{(p)} & \xrightarrow{\psi_H} & H \end{array}$$

(3) *Both ψ_G and φ_G are group homomorphism.*

Example 2.4.5. We have that:

(1) $\varphi_{\alpha_p} = 0$, $\psi_{\alpha_p} = 0$,
(2) $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ is an isomorphism, $\psi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = 0$,
(3) $\varphi_{\mu_p} = 0$, ψ_{μ_p} is an isomorphism.

Proposition 2.4.6. *We have*

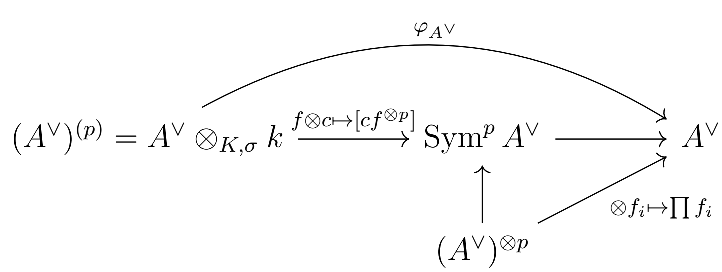
$$\psi_G \circ \varphi_G = [p]_G \quad \varphi_G \circ \psi_G = [p]_{G^{(p)}}.$$

Proof. This proof follows Richard Pink's notes [Pin04].

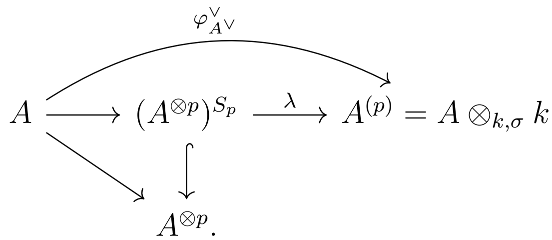
Since $\psi_G = (\varphi_{G^\vee})^\vee$, consider $\varphi_{A^\vee}^\vee$:

$$\varphi_{A^\vee} \colon A^\vee \otimes_{k,\sigma} k \to A^\vee.$$

Then:



The dual of this diagram is:

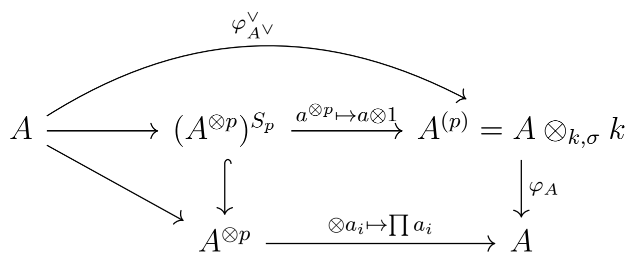


We compute the map λ explicitly. We have that

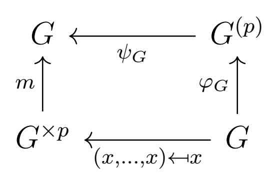
$$\begin{aligned}\lambda(a^{\otimes p})(f \otimes c) &= \epsilon_a^{\otimes p}([c \cdot f^{\otimes p}]) \\ &= c \cdot f(a)^p \\ &= f(a) \otimes c \\ &= (\epsilon_a \otimes 1)(f \otimes c),\end{aligned}$$

where ϵ_a is the identification $A \cong A^{\vee\vee}$, given by $\epsilon_a(f) = f(a)$.

Hence $\lambda(a^{\otimes p}) = a \otimes 1$ in the above diagram. The other elements of $(A^{\otimes p})^{S_p}$ will map to 0, because k has characteristic p. We hence have the diagram



On the level of groups, this yields:



Hence $\psi_G \circ \varphi_G = [p]_G$. The other equality follows immediately. □

Proposition 2.4.7. *Suppose G is a finite group scheme over k. Then G is connected if and only if $\varphi_G^r = 0$ for some r. Moreover, G is étale if and only if φ_G is an isomorphism.*

Proof. If G is connected, A is a local Artinian ring. It decomposes as $A = k \oplus I$ where $I = \ker(\epsilon)$. Since I is a maximal ideal, it is nilpotent, so there is $r > 0$ such that for all $x \in I$, $x^{p^r} = 0$. This shows that φ_G^r factors through the unit section.

Conversely, suppose $\varphi_G^r = 0$ for some r. Since φ_G^r induces an isomorphism $G(\overline{k}) \cong G^{(p^r)}(\overline{k})$, we have that $G(\overline{k}) = 0$, so G is connected.

If G is étale, $\ker(\varphi_G)$ is connected, so $\ker(\varphi_G) \subseteq G^0 = 0$. This shows that φ_G is injective. In fact, it is an injective homomorphism $\varphi_G\colon G \to G^{(p)}$ between groups of the same order, so it is an isomorphism.

Suppose now that φ_G is an isomorphism. It induces an isomorphism on G^0. Hence φ_{G^0} is an isomorphism, and hence $\varphi_{G^0}^r$ is an isomorphism. Since $\varphi_{G^0}^r = 0$ at some point (G^0 is connected), we see that $G^0 = 0$, and hence G is étale. ☐

Proposition 2.4.8. *Suppose G is a connected finite flat k-group. Then the order of G is a power of p.*

Proof. Let n be the order of G. We induct on n.

As usual, let $I = \ker(\epsilon)$ be the augmentation ideal. Choose $x_1, \dots, x_d \in I$ which lifts a basis of I/I^2. Since G is connected, $d > 0$.

Then A be a local ring with maximal ideal I.

Let $H = \ker(\varphi_G)$. We first claim that the order of H is p^d.

By Nakayama, $x_1, \dots, x_d$ generate I. Hence

$$H = \operatorname{Spec}(A/(x_1^p, \dots, x_d^p)).$$

We want to show that

$$\lambda\colon k[t_1, \dots, t_d]/(t_1^p, \dots, t_d^p) \xrightarrow{\cong} A/(x_1^p, \dots, x_d^p).$$

Surjectivity is clear. We have a natural map

$$\pi\colon A = k \oplus I \to I/I^2.$$

For each $j = 1, \dots, d$, define $D_j\colon A \to A$ as the composition

$$A \xrightarrow{\mu} A \otimes A \xrightarrow{(\mathrm{id},\pi)} A \otimes_k I/I^2 \xrightarrow{x_j \mapsto \delta_{ij}} A$$

We can check that $\lambda \frac{\partial}{\partial t_j} = D_j \lambda$ for all j by checking on the generators. Hence the kernel $\ker\lambda$ is stable under $\frac{\partial}{\partial t_j}$. Therefore, $\ker\lambda$ has to contain some constant, which shows that $\ker\lambda = 0$. This proves that λ is an isomorphism, and hence the claim that H has order p^d.

Since G is connected, $\varphi_G^r = 0$ for some r. Since φ_G^r on G/H is 0, G/H is connected. Finally, the order of G is the order of H times the order of G/H. Induction hence completes the proof. ☐

Recall that if the order of G is invertible in the base, then G is étale.

If R is a henselian local ring with perfect residue field, then there is another proof of the the proposition. Assume $R = k$ is a field. If k has characteristic p, the connected–étale sequence has $G^0 = 0$ if order is invertible in p. When k has characteristic 0, $G^0 \cong \operatorname{Spec}(k[t_1, \dots, t_d])$ when $d = \dim I/I^2$, so $d = 0$.

2.5. **p-divisible groups.** The references for this section are [Dem86] and [Tat67].

We assume throughout that the base ring R is a Henselian local noetherian ring.

2.5.1. *Basic definitions and properties.*

Definition 2.5.1. A *p-divisible group* of height h over R is an inductive system $G = \varinjlim G_v$ such that

(1) G_v is a finite flat R-groups of order p^{vh},
(2) there is an exact sequence

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1},$$

i.e. $G_v = G_{v+1}[p^v]$.

Examples 2.5.2.

(1) The *constant p-divisible group* is

$$\underline{\mathbb{Q}_p/\mathbb{Z}_p} = \varinjlim \underline{\mathbb{Z}/p^v\mathbb{Z}}$$

with the obvious transfer maps. It is a p-divisible group of height 1.
(2) The *p-power roots of unity* is

$$\mu_{p^\infty} = \varprojlim \mu_{p^v}$$

with the obvious transfer maps. It is a p-divisible group of height 1.
(3) If $\mathcal{A}$ is an abelian scheme over R,

$$\mathcal{A}[p^\infty] = \varinjlim \mathcal{A}[p^v]$$

with the obvious transfer maps is a p-divisible group of height $2g$, where $g = \dim \mathcal{A}$.

Definition 2.5.3. A map of p-divisible groups $f : G \to H$ is a *homomorphism* if $f = (f_i)$ is compatible system of R-group homomorphism:

$$\begin{array}{ccc} G_v & \xrightarrow{f_v} & H_V \\ \downarrow & & \downarrow \\ G_{v+1} & \xrightarrow{f_{v+1}} & H_{v+1} \end{array}$$

The *kernel* of f is $\ker(f) = \varinjlim \ker(f_v)$.

Remark 2.5.4. The kernel of f might not be a p-divisible group.

Example 2.5.5. The map $[n]_G = ([n]_{G_v})$ is a homomorphism, called *multiplication by n on G.*

We want to discuss Cartier duality for p-divisible groups. We first need a lemma.

Lemma 2.5.6. *Let $G = (G_v)$ be a p-divisible group over R. Then for any $v, t \in \mathbb{Z}_{\geq 0}$ there exist*

$$i_{v,t} : G_v \hookrightarrow G_{v+t},$$
$$j_{v,t} : G_{v+t} \to G_t$$

such that

(1) *$i_{v,t}$ induces $G_v = G_{v+t}[p^v]$,*

(2) *the diagram*

$$\begin{array}{ccc} G_{v+t} & \xrightarrow{[p^v]} & G_{v+t} \\ & {\scriptstyle j_{v,t}} \searrow \qquad \nearrow {\scriptstyle i_{v,t}} & \\ & G_t & \end{array}$$

commutes,

(3) *there is a short exact sequence:*

$$0 \longrightarrow G_v \xrightarrow{i_{v,t}} G_{v+t} \xrightarrow{j_{v,t}} G_r \longrightarrow 0.$$

Proof. We have that $i_{v,t} = i_{v+t-1} \circ i_{v+t-2} \circ \cdots \circ i_v : G_v \hookrightarrow G_{v+t}$. To check (1), we see that

$$\begin{aligned} G_{v+t}[p^v] &= G_{v+1}[p^{v+t-1}] \cap G_{v+t}[p^v] \\ &= G_{v+t-1} \cap G_{v+t}[p^v] \\ &= G_{v+t-1}[p^v]. \end{aligned}$$

To construct $j_{v,t}$, we first note that $[p^{v+t}]$ kills G_{v+t}. Hence $[p^v](G_{v+t})$ is killer by $[p^t]$. Hence

$$[p^v](G_{v+t}) \subseteq G_{v+t}[p^t] = G_t.$$

The composition defines a map $j_{v,t} : G_{v+t} \to G_t$ such that the diagram in (2) commputes.

Finally, it remains to check the surjectivity of $j_{t,v}$ to complete the proof of (3). We have that $\ker(j_{v,t}) = \ker[p^v] = G_v$. Hence $j_{v,t}$ induces a map

$$G_{v+t}/G_v \hookrightarrow G_t$$

between two groups of order $p^{v+t}/p^v = p^t$. It is hence an isomorphism, showing $j_{v,t}$ is surjective. □

Corollary 2.5.7. *The map $[p]$ on G is surjective as a map of fpqc schemes.*

Proposition 2.5.8 (Cartier duality for p-divisible groups)**.** *Let $G = \varinjlim G_v$ be a p-divisible group of height h over R.*

(1) *The sequence*

$$G_{v+1} \xrightarrow{[p^v]} G_{v+1} \xrightarrow{j_v = j_{1,v}} G_v \longrightarrow 0$$

is exact.

(2) *The injective limit $G^\vee = \varinjlim G_v^\vee$, the Cartier dual of G, is a p-divisible group of height h over R with transfer maps $j_v^\vee$.*

(3) *There is a canonical isomorphism $G^{\vee\vee} \cong G$.*

Proof. We start with (1). We have a commutative diagram with an exact row:

$$\begin{array}{ccccccccc} & & & & G_1 & & & & \\ & & & {\scriptstyle j_{v,1}} \nearrow & & \searrow {\scriptstyle i_{1,v}} & & & \\ 0 \longrightarrow & G_V & \xrightarrow{i_v = i_{v,1}} & G_{v+1} & \xrightarrow{\qquad [p^v] \qquad} & G_{v+1} & \xrightarrow{j_{1,v} = j_v} & G_v & \longrightarrow 0 \end{array}$$

We have that $\ker(j_{1,v}) = G_1 = \operatorname{im}([p^v]_{G_{v+1}})$. We hence get (1).

For (2), we dualize to get an exact sequence

$$0 \longrightarrow G_v^\vee \xrightarrow{j_v^\vee} G_{v+1}^\vee \xrightarrow{p^v} G_{v+1}^\vee$$

by Cartier duality 2.1.17. Hence $G_i^\vee = \varinjlim G_v^\vee$ is a p-divisible group.

Part (3) is obvious. □

Examples 2.5.9. We have that:

(1) $\left(\underline{\mathbb{Q}_p/\mathbb{Z}_p}\right)^\vee \cong \mu_{p^\infty}$,
(2) $\mathcal{A}[p^\infty]^\vee \cong \mathcal{A}^\vee[p^\infty]$.

Proposition 2.5.10 (Connected–étale sequence for p-adic groups)**.** *Let $G = \varinjlim G_v$ be a p-divisible group over R. Then there are p-divisible groups over R:*

$$G^0 = \varinjlim G_v^0,$$
$$G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$$

such that

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0.$$

Proof. We have a diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_v^0 & \longrightarrow & G_v & \longrightarrow & G_v^{\text{ét}} & \longrightarrow & 0 \\
 & & \vdots\, i_v^0 & & \downarrow i_v & & \vdots\, i_v^{\text{ét}} & & \\
0 & \longrightarrow & G_{v+1}^0 & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & 0 \\
 & & \downarrow [p^v] & & \downarrow [p^v] & & \downarrow [p^v] & & \\
0 & \longrightarrow & G_{v+1}^0 & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & 0
\end{array}$$

where the dotted maps are to be constructed. There is a unique $i_v^{\text{ét}}$ such that the top right square commutes. For exactness, we can pass to $\overline{k}$-points and see that it follows the middle column on $\overline{k}$-points.

There is also a unique closed embedding i_v^0 such that the left top square commutes.

We want to show that $G_v^0 = G_{v+1}^0[p^v]$. Obviously, $G_v^0 \subseteq G_{v+1}^0[p^v]$. Also, $G_{v+1}^0[p^v] \subseteq G_v^0$ and $G_{v+1}^0[p^v] \subseteq G_{v+1}[p^v] = G_v$. Finally, $G_{v+1}^0[p^v](\overline{k}) \subseteq G_{v+1}^0(\overline{k}) = 0$. □

Definition 2.5.11. Let $R = k$ be a perfect field of characteristic p. There is a *Frobenius twist*:

$$G^{(p)} = \varinjlim G_v^{(p)}.$$

There is a *Frobenius morphism* $\varphi_G = (\varphi_{G_v})$ and a *Verschiebung morphism* $\psi_G = (\psi_{G_v})$.

Proposition 2.5.12. *If G is a p-divisible group of height h,*

(1) *$G^{(p)}$ is a p-divisible group of height h,*
(2) *φ_G and ψ_G are homomorphisms,*
(3) *$\psi_G \circ \varphi_G = [p]_G$,*
(4) *$\varphi_G \circ \psi_G = [p]_{G^{(p)}}$.*

Proof. The proof is obivous by working on finite levels. □

Definition 2.5.13. Let $R = k$ be a field. The *Tate module of G* is

$$T_p(G) = \varprojlim G_v(\overline{k}),$$

where the transfer maps are given by $j_v \colon G_{v+1} \to G_v$.

Proposition 2.5.14. *Let $R = k$ be a field of characteristic not equal to p. Then there is an equivalence:*

$$\{p\text{-divisible groups over } k\} \leftrightarrow \left\{ \begin{matrix} \text{finite free } \mathbb{Z}_p\text{-modules} \\ \text{with continuous } \Gamma_K\text{-action} \end{matrix} \right\},$$
$$G \mapsto T_p(G).$$

Proof. Use the corresponding equivalence for finite flat k-groups (Proposition 2.2.1) and the fact that groups with invertible orders are étale (Proposition 2.2.11). □

2.6. **Serre–Tate equivalence for connected p-divisible groups.** A key correspondence for p-divisible groups is the Serre–Tate equivalence:

$$\left\{ \begin{matrix} \text{connected } p\text{-divisible} \\ \text{groups over } R \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{formal group laws} \\ \text{over } R \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} p\text{-divisible} \\ \text{formal Lie groups} \end{matrix} \right\}.$$

Let R be a complete local noetherian ring, with residue characteristic p.

Definition 2.6.1. Let $G = \varinjlim G_v$ be a p-divisible group over R. We say that G is:

- *connected* if each G_v is connected,
- *étale* if each G_v is étale.

Examples 2.6.2.

(1) The p-divisible group μ_{p^∞} is connected.
(2) The p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale.

Definition 2.6.3. Let $\mathcal{A} = R[\![t_1, \ldots, t_d]\!]$. Then define

$$\mathcal{A} \widehat{\otimes} \mathcal{A} = R[\![t_1, \ldots, t_d, u_1, \ldots, u_d]\!].$$

We will also write $T = (t_1, \ldots, t_d)$, $U = (u_1, \ldots, u_D)$ for the variables.

A *formal group law* of dimension d over R is a (continuous) map $\mu \colon \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $\Phi(T, U) = (\Phi_i(T, U))$ for each $\Phi_i(T, V)$ a power series of $2d$ variables and

$$\Phi_i(T, V) = \mu(t_i)$$

satisfying the following properties:

(1) *associativitiy*: $\Phi(T, \Phi(V, V)) = \Phi(\Phi(T, V), V)$,
(2) *unit section*: $\Phi(T, 0_d) = \Phi(0_d, T) = T$,
(3) *commutativity*: $\Phi(T, V) = \Phi(V, T)$.

Lemma 2.6.4. *If μ is a formal group law over R, then*

(1) *the diagrams*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mu} & \mathcal{A}\widehat{\otimes}\mathcal{A} \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle(\mu,\mathrm{id})} \\ \mathcal{A}\widehat{\otimes}\mathcal{A} & \xrightarrow{(\mathrm{id},\mu)} & \mathcal{A}\widehat{\otimes}\mathcal{A}\otimes\mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\widehat{\otimes}\mathcal{A} & \xrightarrow{x\otimes y\mapsto y\otimes x} & \mathcal{A}\widehat{\otimes}\mathcal{A} \\ {\scriptstyle\mu}\nwarrow & & \nearrow{\scriptstyle\mu} \\ & \mathcal{A} & \end{array}$$

commute,

(2) *the map $\epsilon\colon \mathcal{A} \to R$ given by $t_i \mapsto 0$ makes the diagram*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{A} & \xrightarrow{\cong} & \mathcal{A}\widehat{\otimes}R \\ & {\scriptstyle\mu}\searrow & & \nearrow{\scriptstyle(\mathrm{id},\epsilon)} & \\ & & \mathcal{A}\widehat{\otimes}\mathcal{A} & & \end{array}$$

and a symmetric diagram commute,

(3) *there is a continuous map $\iota\colon \mathcal{A} \to \mathcal{A}$ such that*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mu} & \mathcal{A}\otimes\mathcal{A} \\ \downarrow{\scriptstyle\epsilon} & & {\scriptstyle\iota\otimes\mathrm{id}}\downarrow\downarrow{\scriptstyle\mathrm{id}\otimes\iota} \\ R & \longrightarrow & \mathcal{A} \end{array}$$

commutes.

Proof. Parts (1) and (2) are clear. For (3), we need to define $I_i(T) = \iota(t_i)$, $I(T) = I_i(T)$ such that

$$\Phi(I(T), T) = 0 = \Phi(T, I(T)).$$

We want $P_j(T)$: a family of polynomials of degree j such that $I(T) = \lim P_j(T)$, i.e.

(i) $P_j(T) = P_{j-1}(T) \mod \text{degree } j$,
(ii) $\Phi(P_j(T), T) = 0 \mod \text{degree } j+1$.

Since $\Phi(T, U) = T + U \mod \text{degree } 2$, we may take $P_1(T) = -T$. We define $P_j(T)$ by recursion on j. We have that

$$\Phi(P_j(T), T) = \Delta_j(T) \mod \text{degree } j+2,$$

where $\Delta_j(T)$ is a homogeneous polynomial of degree $j+1$. Define

$$P_{j+1}(T) = P_j(T) + \Delta_j(T).$$

Then (i) is clearly satisfied. For (ii), we note that

$$\Phi(P_{j+1}(T), T) = \Phi(P_j(T) + \Delta_j(T), T) \equiv \Phi(P_j(T), T) + \Delta_j(T) \equiv 0 \mod \text{degree } j+2.$$

This proves (3). $\square$

Remark 2.6.5 (Formal schemes and groups)**.** A formal scheme is a scheme together with an infinitesimal neighborhood.

If A is a ring, we define $\mathrm{Spec}(A)$ as the set of prime ideals.

If A is a topological ring, we define $\mathrm{Spf}(A)$, the formal spectrum, as the set of open prime ideals of A.

Formal groups are group objects in the category of formal schemes. The lemma says that any formal group law over R defines a formal group structure on $\mathrm{Spf}(A)$, written G_μ.

Example 2.6.6. The multiplicative formal group law is

$$\begin{aligned} \mu_{\widehat{\mathbb{G}_m}} \colon R[\![t]\!] &\to R[\![t,u]\!], \\ t &\mapsto (1+t)(1+u)-1. \end{aligned}$$

Definition 2.6.7. Let μ, ν be formal group laws of dimension d over R. A continuous map $\gamma A \to A$ is a *homomorphism* from μ to ν if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\nu} & A\widehat{\otimes}A \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma\otimes\gamma} \\ A & \xrightarrow{\mu} & A\widehat{\otimes}A \end{array}$$

commutes.

Lemma 2.6.8. *A continuous map $\gamma\colon A \to A$ given by $\Xi(T) = (\Xi_i(T))$ where $\Xi_i(T) = \gamma(t_i)$ if a homomorphism if and only if, writing $\Phi(T,V)$ and $\Psi(T,V)$ for the functions associated to μ and ν, we have that*

$$\Psi(\Xi(T), \Xi(V)) = \Xi(\Phi(T,V)).$$

Example 2.6.9. The multiplication by n map $[n]_\mu$ on μ is a homomorphism.

Definition 2.6.10.

(1) The ideal $\mathcal{I} = (t_1, \ldots, t_d) = \ker \epsilon$ is the *augmentation ideal of μ.*
(2) A formal group law μ is p-divisible if $[p]_\mu$ is finite flat in the sense that $\mathcal{A}$ is a free module of finite rank over itself.

Remark 2.6.11. A formal group law μ is p-divisible if and only if $[p]$ on G_μ is surjective with finite kernel.

Proposition 2.6.12. *Let μ be a p-divisible formal group law of dimension d over R. Define*

$$\begin{aligned} A_v &= \mathcal{A}/([p^v]_\mu(\mathcal{I})), \\ A[p^v] &= \mathrm{Spec}(A_v). \end{aligned}$$

Then

(1) *each $\mu[p^\vee]$ is a connected finite flat R-group,*
(2) $\mu[p^\infty] = \varinjlim \mu[p^v]$ *is a connected p-divisible group over R.*

Proof. We may write

$$\begin{aligned} A_v &= \mathcal{A}/[p^v]_\mu(\mathcal{I}) \\ &= (\mathcal{A}/\mathcal{I}) \otimes_{\mathcal{A},[p^v]} \mathcal{A} \\ &= R \otimes_{\mathcal{A},[p^v]} \mathcal{A}. \end{aligned}$$

Then $1 \otimes \mu$, $1 \otimes \epsilon$, $1 \otimes \iota$ define comultiplication, counit, and coinverse on A_V.

Let r be the rank of $\mathcal{A}$ over $[p](\mathcal{A})$. Then r^v is the rank of A over $[p^v](\mathcal{A})$. Hence $\mathrm{Spec}(A_v)$ is a finite flat R-group scheme of order r^v.

Since R is complete, $\mathcal{A}$ is also a lcoal ring. Hence each A_v is a local ring, showing that $\mathrm{Spec}(A_v)$ is connected over R. Since $\mathrm{Spec}(A_1)$ has order $p^h = r$, and $\mathrm{Spec}(A_v)$ has order p^{hv}. This completes the proof of (1).

For (2), we need to check that $\mu[p^v]$ is the p^v-torsion of $\mu[p^{v+1}]$. The natural surjective map

$$A_v = \mathcal{A}/[p^v](\mathcal{I}) \twoheadrightarrow [p]\mathcal{A}/[p^{v+1}](\mathcal{I})$$

is an isomorphism as it is an R-linear map between R-modules of the same rank. We hence have a surjection

$$A_{v+1} = \mathcal{A}/[p^{v+1}](\mathcal{I}) \twoheadrightarrow [p]A/[p^{v+1}](\mathcal{I}) \cong A_v$$

induced by $[p]$, and hence $[p^v]$ will be 0. □

Remark 2.6.13. We have that $G_\mu[p^v] = \mathrm{Spec}(A_v)$.

Theorem 2.6.14 (Serre–Tate equivalence)**.** *There functor*

$$\begin{Bmatrix} p\text{-divisible formal group laws} \\ \text{over } R \end{Bmatrix} \to \begin{Bmatrix} \text{connected } p\text{-divisible} \\ \text{groups over } R \end{Bmatrix}$$
$$\mu \mapsto \mu[p^\infty]$$

is an equivalence of categories.

The map above is really the following. We have a formal group scheme G_μ associated to μ. Then the connected p-divisible group over R associated to μ is

$$\varinjlim G_v \cong \varinjlim G_\mu[p^v],$$

where we recall that

$$G_v = \mathrm{Spec}(\mathcal{A}/[p^v](\mathcal{I})).$$

Remark 2.6.15. Local class field theory can be stated in terms of Lubin–Tate formal group laws. Local Langlands for GL_1 is local class field theory. It can hence be stated in terms of certain p-divisible groups.

For GL_n, Harris and Taylor [HT01] proved the local Langlands correspondence via moduli spaces of p-divisible groups: Rapoport–Zink spaces and local Shimura varieties.

We now work towards the proof of the Serre–Tate equivalence 2.6.14.

The following proposition shows the essential surjectivity over k in the Serre–Tate equivalence 2.6.14.

Proposition 2.6.16. *Let* $G = \varinjlim G_v$ *be a connected* p*-divisible group over* R*, where* $G_v = \mathrm{Spec}(A_v)$*. Then*

$$\varprojlim A_v \otimes k \cong k[\![t_1, \ldots, t_d]\!].$$

Proof. Let $\overline{G} = G \times_R k$. Define $H_v = \ker(\varphi^v)$ and note that $H_v \subseteq \ker([p^v]) = \overline{G_v}$. Since

$$\varphi^v \circ \varphi^v = [p^v],$$

writing $H_v = \mathrm{Spec}(B_v)$ and we have $A_v \otimes k \twoheadrightarrow B_v$.

We have that $\overline{G_v}$ is a connected finite flat k-group. Hence $\varphi^w = 0$ on $\overline{G_v}$, so $G_v \subseteq H_w$ showing that $B_w \twoheadrightarrow A_v \otimes k$. Hence

$$\varprojlim A_v \otimes k \cong \varprojlim B_v.$$

Let J_v be the augmentation ideal of H_v and $J = \varprojlim J_v$. Then $B_v/J_v \cong k$. Let $y_1, \ldots, y_d \in J$ lift a basis of J_1/J_1^2. We have a commutative diagram:

$$\begin{array}{ccc} k = (B_v/J_v) \otimes_{k,\sigma} k & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ B_v^{(p)} = B_v \otimes_{k,\sigma} k & \xrightarrow{x \otimes c \mapsto cx^p} & B_v \end{array}$$

so

$$B_1 \cong B_v/J_v^{(p)}$$

where $J_v^{(p)}$ is the ideal generated by p-powers of elements in J.

Since $J_1/J_1^2 \cong J_v/J_v^2$, the images of $y_1, \ldots, y_d$ generate J_v/J_v^2. By Nakayma's Lemma, they generate J_v. We hence have a map

$$k[t_1, \ldots, t_d] \twoheadrightarrow B_v.$$

We hence have

$$k[t_1, \ldots, t_d]/(t_1^{p^v}, \ldots, t_d^{p^v}) \twoheadrightarrow B_v,$$

since $H_v = \ker(\varphi^v)$. We want to show this is an isomorphism.

We proceed by induction on v. When $v = 1$, we checked this in the proof of Proposition 2.4.8. For the induction step, we argue on ranks. We ant to show that p^{vd} is the order of H_v. For that, we observe that the sequence

$$0 \longrightarrow H_1 \longrightarrow H_{v+1} \xrightarrow{\varphi} H_v^{(p)} \longrightarrow 0$$

is exact. Since $H_1 = \ker(\varphi)$, we just need to check that φ is surjective. Recall that $[p]$ is surjective by Corollary 2.5.7. We know that $\varphi \circ \psi = [p]$, so φ is surjective. Recall that $H_{v+1} = \ker(\varphi^{v+1})$, so $\varphi(H_{v+1}) \subseteq \ker(\varphi^{\vee}_{\overline{G^{(p)}}})$, and the preimage of $H_v^{(p)}$ is $\ker(\varphi^{\vee}_{\overline{G^{(p)}}})$.

This shows that the order of H_{v+1} is $p^d \cdot p^{vd} = p^{d(v+1)}$, completing the proof. $\square$

Lemma 2.6.17. *Let μ be a p-divisible formal group law R. Letting*
$$A_v = \mathcal{A}/[p^v](\mathcal{I}),$$
we have that
$$\mathcal{A} \cong \varprojlim A_v.$$

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Then $\mathfrak{M} = \mathfrak{m}A + \mathcal{I}$ is a maximal ideal of $\mathcal{A}$. For each v, i, we have that
$$[p^v](I) + \mathfrak{m}^i A \supseteq \mathfrak{M}^w$$
for some w, since
$$\mathcal{A}/([p^v](\mathcal{I}) + \mathfrak{m}^i A) = A_v/\mathfrak{m}^i A_v,$$
which is local Artinian.

Moreover, $[p](\mathcal{I}) \subseteq p\mathcal{I}+\mathcal{I}^2$, because $[n]$ acts as multiplication by n on $\mathcal{I}/\mathcal{I}^2$ by the argument in the proof of Proposition 2.2.11. Alternatively, recall that $\Phi(T,U) = T+U+(\text{degree} \geq 2)$.

This shows that
$$[p^v](\mathcal{I}) + \mathfrak{m}^i \mathcal{A} \subseteq \mathfrak{M}^{w'}$$
for some w'.

Altogether, we see that:
$$\begin{aligned}
\mathcal{A} &\cong \varprojlim \mathcal{A}/\mathfrak{M}^w \\
&= \varprojlim_{v,i} \mathcal{A}/([p^v](\mathcal{I}) + \mathfrak{m}^i A) \\
&= \varprojlim_{v,i} A_v/\mathfrak{m}^i A_v \\
&\cong \varprojlim_{v} A_v && \text{since } A_v \text{ is } \mathfrak{m}\text{-adically complete.}
\end{aligned}$$
This completes the proof. □

Proof of Theorem 2.6.14. We first check that the functor is fully faithful. Let μ, ν be p-divisible formal group laws over R. Then for $B_v = \mathcal{A}/[p^v]_\nu(\mathcal{I})$:
$$\begin{aligned}
\mathrm{Hom}(\mu,\nu) &= \mathrm{Hom}_{\nu,\mu}(\mathcal{A},\mathcal{A}) \\
&= \mathrm{Hom}_{\nu,\mu}(\varprojlim B_v, \varprojlim A_v) && \text{by Lemma 2.6.17} \\
&= \varprojlim \mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v) \\
&= \varinjlim \mathrm{Hom}_{\mathrm{grp}}(\mu[p^v], \nu[p^v]) \\
&= \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty]).
\end{aligned}$$

For essential surjectivity, consider $G = \varinjlim G_v$ be a connected p-divisible group. Let
$$\overline{G} = G \times_R k,$$
and $G_v = \mathrm{Spec}(A_v)$. By Proposition 2.6.16,
$$k[\![t_1, \ldots, t_d]\!] \cong \varprojlim A_v \otimes k.$$
We want to lift to $f\colon \mathcal{A} \to \varprojlim A_v$. We hence need lifts $f_v\colon \mathcal{A} \to A_v$, which lifts the above isomorphism, such that

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{f_{v+1}} & A_{v+1} & \longrightarrow & A_{v+1} \otimes k \\ & \searrow^{f_v} & \downarrow\!\!\downarrow p_v & & \downarrow\!\!\downarrow \\ & & A_v & \longrightarrow & A_v \otimes k. \end{array}$$

Let f_1 be any lift over $k[\![t_1, \dots, t_d]\!] \to A_1 \otimes k$. We define f_v by recursion on v. Choose $y_1, \dots, y_d \in A_{v+1}$ which lift images of $t_1, \dots, t_d$ under

$$k[\![t_1, \dots, t_d]\!] \to A_{v+1} \otimes k.$$

Then $p_v(y_1), \dots, p_v(y_d)$ must lift the images of $t_1, \dots, t_d$ after the map

$$k[\![t_1, \dots, t_d]\!] \to A_v \otimes k.$$

We know that $f_v(t_1), \dots, f_v(t_d)$ also lift the images of $t_1, \dots, t_d$ under this map. Then $f_v(t_i) - p_v(y_i) \in \mathfrak{m}A_v$, so there exist $z_i \in \mathfrak{m}A_{v+1}$ such that

$$p_v(z_v) = f_v(t_i) - p_v(y_i).$$

Defining f_{v+1} by $f_{v+1}(t_i) = y_i + z_i$ gives the desired lift.

We want to show that the resulting map

$$f\colon \mathcal{A} \to \varprojlim A_v$$

is an isomorphism. Surjectivity is clear by Nakayama's Lemma. We want to show that $\ker(f) = 0$. We know that $\ker(f) \otimes_R k = 0$, i.e. $\mathfrak{m}\ker(f) = \ker(f)$. We now note that

$$\mathfrak{M}\ker(f) = (\mathfrak{m}\mathcal{A} + I)(\ker f) = \ker(f),$$

so f is injective by Nakayama's Lemma.

We have an isomorphism

$$f\colon \mathcal{A} \to \varprojlim A_v.$$

To prove essential surjectivity, We define $G = \varinjlim G_v$ for $G_v = \mathrm{Spec}(A_v)$. Then μ_v is a comultiplication on G_v, and $\mu = \varprojlim \mu_v$ defines a formal group law over R such that $\mu[p^v] = G_v$.

We just need to check that G is p-divisible. We omit the details of this; roughly, ones uses that the map $j_{v,t}\colon G_{v+t} \twoheadrightarrow G_t$ induces an injection $A_t \hookrightarrow A_{v+t}$. □

Definition 2.6.18. For a p-divisible group $G = \varinjlim G_v$ over R,

$$\dim(G) = \text{dimension of the formal group law associated to } G^0$$

(via the Serre–Tate equivalence 2.6.14).

In the course of the proof of Theorem 2.6.14, we showed the following result.

Corollary 2.6.19. *Let $\overline{G} = G \times_R k$. Then $\ker(\varphi_{\overline{G}})$ has order $p^{\dim(G)}$.*

Example 2.6.20. Recall that $\mu_{\widehat{\mathbb{G}_m}}(t,u) = (1+t)(1+u) - 1$. Then $[p^v](t) = (1+t)^{p^v} - 1$, so

$$\mu_{\widehat{\mathbb{G}_m}}[p^\infty] = \mu_{p^\infty}.$$

Theorem 2.6.21. *Let G be a p-divisible group over R. Then*

$$\mathrm{ht}(G) = \dim(G) + \dim(G^\vee).$$

Proof. By passing to the residue field, we may assume that $R = k$ is a perfect field of characteristic p. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\varphi) & \longrightarrow & G & \xrightarrow{\varphi} & G^{(p)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle [p]} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \xrightarrow{\mathrm{id}} & G & \longrightarrow & 0 \end{array}$$

is commutative with exact rows, since φ is surjective, because $\varphi \circ \psi = [p]_{G^{(p)}}$ and $\ker(\varphi)$ is killed by $[p]$ because $\psi \circ \varphi = [p]$.

Snake Lemma then gives a short exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow \ker([p]) \longrightarrow \ker(\psi) \longrightarrow 0.$$

Since $\ker(\varphi)$ has order $p^{\dim(G)}$ and $\ker([p]) = G_1$ has order $p^{\mathrm{ht}(G)}$, and $\psi = \varphi^\vee_{G^\vee}$ implies that $\ker(\psi)$ has order $p^{\dim(G^\vee)}$, we are done by multiplicativity of orders in short exact sequences. □

Corollary 2.6.22. *Let G be a p-divisible group over R with residue field $k = \overline{k}$ of height 1. Then G is isomorphic to μ_{p^∞} or $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$.*

Proof. By Theorem 2.6.21, we know that $\dim G = 0$ or $\dim G^\vee = 0$. If $\dim G = 0$, G is étale, so $G \cong \underline{\mathbb{Q}_p/\mathbb{Z}_p}$. Otherwise, $\dim G^\vee = 0$, so $G^\vee \cong \underline{\mathbb{Q}_p/\mathbb{Z}_p}$, so $G \cong \mu_{p^\infty}$. □

One can also prove this result using Dieudonné theory, which we will soon explain.

Example 2.6.23. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}_p}$. Then there is a short exact sequence

$$0 \longrightarrow E[p^\infty]^0 \longrightarrow E[p^\infty] \longrightarrow E[p^\infty]^{\text{ét}} \longrightarrow 0.$$

Since $E[p]^0$ and $E[p]^{\text{ét}}$ are both non-trivial, so are $E[p^\infty]$ and $E[p^\infty]^{\text{ét}}$. Finally, $E[p^\infty]$ is of height 2, so Corollary 2.6.22 shows that

$$E[p^\infty]^0 = \mu_{p^\infty}, \quad E[p^\infty]^{\text{ét}} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}.$$

The short exact sequence splits, because it splits at each finite level. Hence

$$E[p^\infty] = \mu_{p^\infty} \times \underline{\mathbb{Q}_p/\mathbb{Z}_p}.$$

Remark 2.6.24. We discuss Serre–Tate deformation theory for ordinary elliptic curves.

In general, Serre–Tate deformation theory says that the deformations of an abelian variety A/k are equivalent to the deformations of $A[p^\infty]$ (i.e. p-divisible groups G/R such that $G \times_R k \cong A[p^\infty]$).

Therefore, a deformation of an elliptic curve E over $k = \overline{k}$ corresponds to a deformation of $E[p^\infty]$. The deformation space of $E[p^\infty]$ is

$$\underline{\mathrm{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}),$$

since if G is a deformation over R, the connected–étale sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

and $G^0 = \mu_{p^\infty}$ and $G^{\text{ét}} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

We also have a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}_p} \longrightarrow \underline{\mathbb{Q}_p} \longrightarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow 0.$$

The long exact sequence after applying $\operatorname{Ext}(-, \mu_{p^\infty})$ gives

$$\underline{\operatorname{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \cong \underline{\operatorname{Hom}}(\underline{\mathbb{Z}_p}, \mu_{p^\infty}).$$

Therefore, the deformation space has the structure of a formal torus of dimension 1, given by $\mu_{\widehat{\mathbb{G}_m}}$.

2.7. **Dieudonné–Manin classification.** Let k be a perfect field of characteristic p. Let σ be the Frobenius automorphism over k.

Definition 2.7.1. We write $W(k)$ for the ring of *Witt vectors* over k. We write $K_0(k)$ for the fraction field of $W(k)$. The *Frobenius* $\sigma_{W(k)}$ on $W(k)$ is

$$\sigma\left(\sum_{n\geq 0} \tau(x_n)p^n\right) = \sum_{n\geq 0} \tau(x_n^p)p^n$$

where $\tau\colon k \to W(k)$ is the Teichmüller lift. Finally, $\sigma_{K_0(k)}$ is the unique field of automorphism on $K_0(k)$ extending $\sigma_{W(k)}$.

Example 2.7.2. Let $k = \mathbb{F}_q$ and ζ_{q-1} be a primitive $(q-1)$st root of unity. Then

$$W(k) = \mathbb{Z}_p[\zeta_{q-1}], \quad K_0(k) = \mathbb{Q}_p[\zeta_{q-1}]$$

and σ acts on $W(k)$ by

$$\sigma(\zeta_{q-1}) = \zeta_{q-1}^p,$$

and trivially on $\mathbb{Z}_p$.

Definition 2.7.3. A *Dieudonné module* over k is a pair (M, φ) where

- M is a finite free module over $W(k)$,
- $\varphi\colon M \to M$ is an additive map such that:
  (1) φ is σ-linear, i.e. $\varphi(am) = \sigma(a)\varphi(m)$ for all $a \in W(k)$, $m \in M$,
  (2) $\varphi(M) \supseteq pM$.

Theorem 2.7.4 (Dieudonné)**.** *There is an anti-equivalence:*

$$\mathbb{D}\colon \{p\text{-divisible groups over } k\} \to \{\text{Dieudonné modules over } k\}$$

such that

(1) $\operatorname{rk}(\mathbb{D}(G)) = \operatorname{ht}(G)$,
(2) *G is étale if and only if $\varphi_{\mathbb{D}(G)}$ is an isomorphism,*
(3) *G is connected if and only if $\varphi_{\mathbb{D}(G)}$ is topologically nilpotent,*
(4) *$[p]_G$ induces multiplication by p on $\mathbb{D}(G)$.*

For a proof, see [Dem86].

Remark 2.7.5. There is a notion of duality for Dieudonné modules, compatible with Cartier duality.

Examples 2.7.6. We have that

(1) $\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p)$ is $W(k)$ with $\varphi_{\mathbb{D}(\mathbb{Q}_p/\mathbb{Z}_p}$ given by $\sigma_{W(k)}$,
(2) $\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is $W(k)$ with $\varphi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}}$ given by $p\sigma_{W(k)}$,
(3) if E is an ordinary elliptic curve over $\overline{k}$, $\mathbb{D}(E[p^\infty]) = W(k)^{\oplus 2}$ with $\varphi = \sigma_{W(k)} \oplus p\sigma_{W(k)}$.

Definition 2.7.7. A map of p-divisible groups $f\colon G \to H$ is an isogeny if it is surjective and $\ker f$ is finite flat.

Proposition 2.7.8. *The following are equivalent:*

(1) $f\colon G \to H$ *is an isogeny,*
(2) $\mathbb{D}(f)\colon \mathbb{D}(H) \to \mathbb{D}(G)$ *is injective,*
(3) $\mathbb{D}(f)[1/p]\colon \mathbb{D}(H)[1/p] \xrightarrow{\cong} \mathbb{D}(G)[1/p]$.

This is easy to check using the properties in Theorem 2.7.4.

Definition 2.7.9. An *isocrystal* over k is a finite-dimensional $K_0(k)$-vector space N with a σ-linear bijection $\varphi\colon N \to N$.

Remark 2.7.10. If G is a p-divisible group over k, $\mathbb{D}(G)[1/p]$ is an isocrystal which determines the isogeny class of G.

Example 2.7.11. Let $\lambda \in \mathbb{Q}$ be $\lambda = \frac{d}{r}$ for $(d, r) = 1$, $r > 0$. The *simple isocrystal* $N(\lambda)$ *of slope* λ is $K_0(k)^{\oplus r}$ with

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = e_3, \quad \varphi(e_r) = p^d e_1.$$

Theorem 2.7.12 (Manin)**.** *Let $k = \overline{k}$. The category of isocrystals over k is semisimple with simple objects given by $N(\lambda)$.*

In other words, any N over k has a decomposition

$$N = \bigoplus N(\lambda_i)^{\oplus m_i}$$

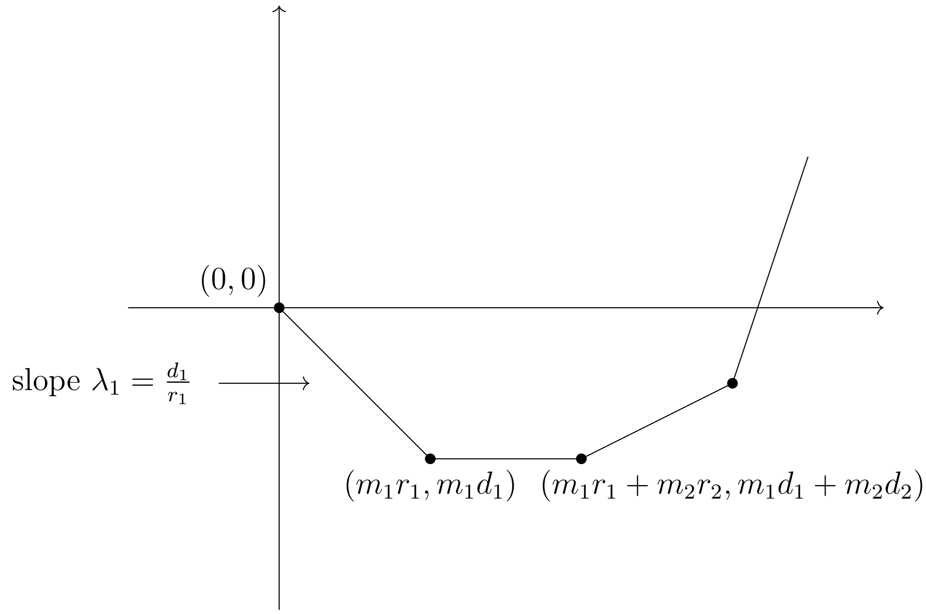
for $\lambda_1 < \cdots < \lambda_\ell$.

Definition 2.7.13. For $\lambda_i = \frac{d_i}{r_i}$, $(d_i, r_i) = 1$, $r_i > 0$.

(1) The Newton polygon of N is the lower convex hull of the points

$$(m_1 r_1 + \cdots + m_i r_i, m_1 d_1 + \cdots + m_i d_i), \quad i = 1, \ldots, \ell.$$

Here is a schematic diagram of N:



(2) The *dimension* of N is $\dim(N) = m_1d_1 + m_2d_2 + \cdots m_\ell d_\ell$.
(3) The *slope* of N is $\mu(N) = \frac{\dim(N)}{\operatorname{rank}(N)}$.

Proposition 2.7.14. *If G is a p-divisible group over $\overline{k}$, then $\mathbb{D}(G)[1/p]$ has rank $\operatorname{ht}(G)$ and dimension $\dim(G)$. Moreover, if*

$$\mathbb{D}(G)[1/p] = \bigoplus_{i=1}^{\ell} N(\lambda_i)^{\oplus m_i},$$

then

$$\mathbb{D}(G^\vee)[1/p] = \bigoplus N(1-\lambda_i)^{\oplus m_i}.$$

Theorem 2.7.15 (Serre, Honda–Tate, Oort)**.** *Let N be an isocrystal over $\overline{k}$. Then*

$$N \cong \mathbb{D}(A[p^\infty])[1/p]$$

for some abelian variety A over $\overline{k}$ if and only if

(1)

Example 2.7.16. Let A be a principally polarized abelian variety of dimension g over $\overline{k}$. Then A is ordinary if $\mathbb{D}(A[p^\infty])[1/p]$ has Newton polygon connecting $(0,0)$ to $(g,0)$ to $(2g,g)$. Hence $A[p^\infty]$ is isogenous to $\mu_{p^\infty}^g \times \underline{\mathbb{Q}_p/\mathbb{Z}_p}^g$.

We claim that $A[p^\infty] \cong \mu_{p^\infty}^g \times \underline{\mathbb{Q}_p/\mathbb{Z}_p}^g$. We have the short exact sequence

$$0 \to A[p^\infty]^0 \to A[p^\infty] \to A[p^\infty]^{\text{ét}} \to 0.$$

We have that

$$(A[p^\infty])^{\text{ét}} \cong (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g,$$

and

$$((\mathbb{Q}_p/\mathbb{Z}_p)^g)^\vee \cong \mu_{p^\infty}^g \hookrightarrow A[p^\infty]^\vee = A^\vee[p^\infty] = A[p^\infty]^0,$$

so

$$A[p^\infty]^0 \cong \mu_{p^\infty}.$$

Finally,

$$A[p^\infty] \cong A[p^\infty]^0 \times A[p^\infty]^{\text{ét}} = \mu_{p^\infty}^g \times (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g.$$

Remark 2.7.17. One can extend Serre–Tate deformation theory to show that the deformation space of A is a formal torus of dimension $g(g+1)/2$.

2.8. **Hodge–Tate decomposition.**

The goal is to prove he following two results.

(1) The Hodge–Tate decomposition for Tate modules.
(2) The full faithfulness of the generic fiber functor for p-divisible groups.

The reference is [Tat67].

2.8.1. *The completed algebraic closure of a p-adic field.*

Definition 2.8.1. A *p-adic field* K is a discrete valued complete non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field of characteristic p.

Example 2.8.2.

(1) Every finite extension of $\mathbb{Q}_p$.
(2) If k is a perfect field of characteristic p, $K_0(k)$, the fraction field of $W(k)$, is a p-adic field.

Remark 2.8.3. Let $k = \overline{\mathbb{F}_p}$. Then $K_0(\overline{\mathbb{F}_p})$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$.

Notation. If K is a p-adic field, we write

$$\begin{aligned} \Gamma_K &= \mathrm{Gal}(\overline{K}/K), \\ \mathcal{O}_K &= \text{valuation ring of } K, \\ \mathfrak{m} &= \text{maximal ideal of } \mathcal{O}_K, \\ k &= \mathcal{O}_K/\mathfrak{m}. \end{aligned}$$

Definition 2.8.4. The completed algebraic closure of K is $\mathbb{C}_K = \widehat{\overline{K}}$. We write $\mathcal{O}_{\mathbb{C}_K}$ for the valuation ring of $\mathbb{C}_K$.

Remark 2.8.5. The field $\mathbb{C}_K$ is **not** a p-adic field. We will study it nonetheless. It is our first example of a characteristic 0 perfectoid field.

We fix a valuation on $\mathbb{C}_K$ so that $v(p) = 1$.

Lemma 2.8.6. *The action of Γ_K on $\overline{K}$ uniquely extends to a continuous action on $\mathbb{C}_K$.*

Proof. Obvious by continuity. □

Proposition 2.8.7. *The field $\mathbb{C}_K$ is algebraically closed.*

Proof. Consider $P(t) \in \mathbb{C}_K[t]$. We want to show $P(t)$ has a root in $\mathbb{C}_K$.

Exercise. We can assume $P(t)$ is monic over $\mathcal{O}_{\mathbb{C}_K}$.

We want to show that there is a Cauchy sequence (α_n) such that $P(\alpha_n)$ converges to 0.

Write
$$P(t) = t^d + a_1 t^{d-1} + \cdots + a_d \quad \text{for } a_i \in \mathcal{O}_{\mathbb{C}_K}.$$
Consider
$$P_n(t) = t^d + a_{n,1} t^{d-1} + \cdots + a_{n,d}$$
with $v(a_{i,n} - a_i) \geq dn$ and $a_{n,i} \in \mathcal{O}_{\overline{K}}$.

We construct α_n recursively such that α_n is a root of $P_n(t)$. Let α_1 be any root of $P_1(t)$. Suppose α_n such that $P_n(\alpha_n) = 0$. Then
$$P_{n+1}(\alpha_n) = P_{n+1}(\alpha_n) - P_n(\alpha_n)$$
$$\sum_{i=1}^{d} (a_{n+1,i} - a_{n,i}) \alpha_n^i$$
and hence
$$v(P_{n+1}(\alpha_n)) \geq dn.$$
Let
$$P_{n+1}(t) = \prod (t - \beta_{n+1,i}).$$
Since $\mathcal{O}_{\overline{K}}$ is integrally closed, $\beta_{n+1,i} \in \mathcal{O}_{\overline{K}}$. Then
$$P_{n+1}(\alpha_n) = \prod (\alpha_n - \beta_{n+1,i}),$$
so there exists i such that $v(\alpha_n - \beta_{n+1,i}) \geq n$. We define $\alpha_{n+1} = \beta_{n+1,i}$.

We have a sequence α_n such that
$$v(\alpha_n - \alpha_{n+1}) \geq n$$
$$P_n(\alpha_n) = 0.$$
Then (α_n) is Cauchy, so $\alpha_n \to \alpha \in \mathbb{C}_K$. Now,
$$P_n(\alpha) - P_n(\alpha_n) = \sum_i (a_i - a_{n,i}) \alpha^i.$$
Hence $v(p_n(\alpha)) \to 0$ as $n \to \infty$, showing that $p(\alpha) = 0$. □

Definition 2.8.8. A *p-adic representation* of Γ_K is a finite-dimensional $\mathbb{Q}_p$-vector sapce V with a continuous homomorphism $\Gamma_K \to \mathrm{GL}(V)$.

Examples 2.8.9.

(1) Let G be a p-divisible group over K. Then $V_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a p-adic representation.
(2) Let X be an algebraic variety over K. Then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic representation.

Notation. We write $\mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$ for the category of p-adic representations of Γ_K.

Definition 2.8.10. Let M be a $\mathbb{Z}_p$-module with continuous Γ_K-action. The *nth Tate twist* of M is
$$M = \begin{cases} M \otimes T_p(\mu_{p^\infty})^{\otimes n} & \text{if } n > 0, \\ \mathrm{Hom}_{\Gamma_K}(T_p(\mu_{p^\infty})^{\otimes -n}, M) & \text{if } n < 0. \end{cases}$$

Example 2.8.11. Recall that $\mathbb{Z}_p(1) = T_p(\mu_{p^\infty})$. As a Galois representation, this corresponds to the *p-adic cyclotomic character* of K:

$$\chi_K \colon \Gamma_K \to \mathrm{Aut}(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p^\times.$$

We will usually simply write χ for χ_K.

Lemma 2.8.12. *Suppose M is a $\mathbb{Z}_p$-module with a continuous Γ_K-action. Then*

$$M(m+n) \cong M(m) \otimes \mathbb{Z}_p(n),$$
$$M(m)^\vee \cong M^\vee(-m).$$

Proof. These are simple consequences of the definition. □

Lemma 2.8.13. *Let M be a $\mathbb{Z}_p$-module with a continuous Γ_K-action $\varrho \colon \Gamma_K \to \mathrm{Aut}(M)$. Then $M(n)$ is identified with the $\mathbb{Z}_p$-module M with $\sigma \in \Gamma_K$ acting by $\chi(\sigma)^n \varrho(\sigma)$.*

Proof. We have that $M(n) = M \otimes \mathbb{Z}_p(n)$ with Γ_K-action $\varrho \otimes \chi^n$. □

We will assume the following theorem without proof.

Theorem 2.8.14 (Tate–Sen)**.** *The Galois cohomology of $\mathbb{C}_K(j)$ is given by*

$$H^i(K, \mathbb{C}_K(j)) = \begin{cases} K & \text{if } i = 0,1 \text{ and } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.8.15. The proof of this theorem requires the full power of the higher ramification theory and local class field theory. It would take several lectures to prove, which is why we omit it here.

If $i = j = 0$, the theorem says that $\mathbb{C}_K^{\Gamma_K} = K$. This has an elementary proof, c.f. [BC09, Prop. 2.1.2].

Lemma 2.8.16 (Serre–Tate)**.** *Let $V \in \mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$. Then the natural map*

$$\alpha_V \colon \bigoplus_{n \in \mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \otimes_K \mathbb{C}_K(-n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

is injective and Γ_K-equivariant.

Proof. For each $n \in \mathbb{Z}$, we have

$$\alpha_{V,n} \colon (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \otimes_K K(-n) \hookrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n) \otimes_K K(-n) = V \otimes_{\mathbb{Q}_p} \mathbb{C}_K.$$

This extends to a $\mathbb{C}_K$-linear map. Taking the direct sum of these maps give α_V which is now clearly Γ_K-equivariant.

We need to show α_V is injective. Suppose that $\ker(\alpha_v) \neq 0$. For each $n \in \mathbb{Z}$, choose a basis $(v_{m,n})$ of $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}$. Since the individual maps$\alpha_{V,n}$ are injective, we can identify $v_{m,n}$ as a vector $\alpha_{V,n}(v_{m,n})$ in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$.

Now, the vectors $(v_{m,n})_{m,n}$ span the source of α_V. Since we assume that $\ker(\alpha_V) \neq 0$, there is a non-trivial relation

$$\sum c_{m,n} v_{m,n} = 0.$$

Choose such a relation with minimal length and assume that $c_{m_0,n_0} = 1$ for some m_0, n_0. For $\sigma \in \Gamma_K$,

$$\begin{aligned} 0 &= \sigma\left(\sum c_{m,n} v_{m,n}\right) - \chi(\sigma)^{n_0}\left(\sum c_{m,n} v_{m,n}\right) \\ &= \sum \sigma(c_{m,n})\chi(\sigma)^{-n} v_{m,n} - \chi(\sigma)^{n_0}\left(\sum c_{m,n} v_{m,n}\right) \\ &= \sum \left(\sigma(c_{m,n}\chi(\sigma)^{-n} - \chi(\sigma)^{n_0} c_{m,n})\right) v_{m,n} \end{aligned}$$

If $m = m_0$, $n = -n_0$, the coefficient is 0. By minimality assumption, we see that

$$\sigma(c_{m,n})\chi(\sigma)^{-n} - \chi(\sigma)^{n_0} c_{m,n} = 0.$$

Hence

$$\sigma(c_{m,n})\chi(\sigma)^{-n-n_0} = c_{m,n}.$$

The left hand side is the Galois action on $\mathbb{C}(-n-n_0)$. If $n \neq -n_0$, $c_{m,n} = 0$. Hence $c_{m,n} \neq 0$ possibly only if $n = -n_0$.

If $n = -n_0$, $c_{m,n} \in \mathbb{C}_K^{\Gamma_K} = K$. Hence

$$\sum_m c_{m,n} v_{m,-n_0} = 0$$

is a K-linear relation, which is a contradiction. □

Definition 2.8.17. A representation $V \in \mathrm{Rep}_{\Gamma_K}(\mathbb{Q}_p)$ is *Hodge–Tate* if α_V is an isomorphism.

We now present the general idea of the proof of the Hodge–Tate decomposition for Tate modules. Recall that if G is a Lie group, $\log_G \colon G \to \mathrm{Lie}(G)$ is a local homeomorphism.

In our context, if G is a p-divisible group over $\mathcal{O}_K$, G^0 gives a formal group $\mathcal{G}$. We get a *p-adic Lie group* $\mathcal{G}(\mathcal{O}_C)$ and

$$\log \colon \mathcal{G}(\mathcal{O}_C) \to T_{\mathcal{G}}.$$

We will relate $T_p(G)$ to $t_{G^\vee}$.

2.8.2. *Formal points on p-divisible groups.* Fix $R = \mathcal{O}_K$. Let L be the p-adic completion of an algebraic extension of K (e.g. $L = \mathbb{C}_K$). Let $\mathcal{O}_L$ be the valuation ring of L and $\mathfrak{m}_L$ be its maximal ideal.

Definition 2.8.18. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. The group of *$\mathcal{O}_L$-valued formal points* on G is

$$G(\mathcal{O}_L) = \varprojlim G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) = \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Remark 2.8.19. This terminology is not standard. In [Tat67], $G(\mathcal{O}_L)$ is the *group of $\mathcal{O}_L$-points*, but it will soon become clear why these are just "formal points".

Example 2.8.20. We have that $\mu_{p^\infty}(\mathcal{O}_L) = 1 + \mathfrak{m}_L$ with the multiplicative structure. We check this:

$$\begin{aligned}\mu_{p^\infty}(\mathcal{O}_L) &= \varprojlim \mu_{p^\infty}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &= \{x \in \mathcal{O}_L^\times \mid v(x^{p^v} - 1) \text{ can be arbitrarily large}\} \\ &= 1 + \mathfrak{m}_L \qquad\qquad x^{p^v} - 1 \equiv (x-1)^{p^v} \mod \mathfrak{m}_L.\end{aligned}$$

Remark 2.8.21. The *ordinary* $\mathcal{O}_L$-valued points are

$$\mu_{p^\infty}(\mathcal{O}_L) = \varinjlim \mu_{p^v}(\mathcal{O}_L) = p\text{-power torsion points in } \mathcal{O}_L^\times.$$

Proposition 2.8.22. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.*

(1) *If $G_v = \mathrm{Spec}(A_v)$, $G(\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim A_v, \mathcal{O}_L)$.*
(2) *The $\mathcal{O}_L$-formal points on G, $G(\mathcal{O}_L)$ form a $\mathbb{Z}_p$-module with torsion:*

$$G(\mathcal{O}_L)_{\mathrm{tor}} = \varinjlim \varprojlim G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

(3) *If G is étale, then $G(\mathcal{O}_L) \cong G(k_L)$ is a torsion group (where k_L is the residue field of $\mathcal{O}_L$).*

Remark 2.8.23. The comultiplication on G_v defines a formal group on $G = \mathrm{Spf}(\varprojlim A_v)$. Then $G(\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim A_v, \mathcal{O}_L)$, which agree with our definition by Proposition 2.8.22 (1).

Proof of Proposition 2.8.22. We start with (1). Recall that $\mathcal{O}_L$ is complete, so

$$\mathcal{O}_L = \varprojlim \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L.$$

Since A_v is finite free over $\mathcal{O}_K$, A_v is $\mathfrak{m}$-adically complete, so

$$A_v = \varprojlim_i A_v/\mathfrak{m}^i A_v.$$

By definition,

$$G(\mathcal{O}_L) = \varprojlim \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Hence

$$\begin{aligned}G(\mathcal{O}_L) &= \varprojlim \varinjlim \mathrm{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &= \varprojlim \varinjlim \mathrm{Hom}_{\mathcal{O}_K}(A_v/\mathfrak{m}^iA_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &= \varprojlim_i \mathrm{Hom}_{\mathcal{O}_K}(\varprojlim_v A_v/\mathfrak{m}^iA_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &= \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim_{i,v} A_v/\mathfrak{m}^iA_v, \varprojlim_i \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\ &= \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\varprojlim A_v, \mathcal{O}_L).\end{aligned}$$

For (2), note that $G(\mathcal{O}_L)$ is obviously a $\mathbb{Z}_p$-module and

$$G(\mathcal{O}_L)_{\mathrm{tor}} = \text{set of } p\text{-power torsion}.$$

We have an exact sequence:

$$0 \longrightarrow G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \xrightarrow{[p^v]} G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$$

and taking $\varprojlim_i$,

$$0 \longrightarrow \varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \longrightarrow \underbrace{\varprojlim_i G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)}_{G(\mathcal{O}_L)} \xrightarrow{[p^v]} \underbrace{\varprojlim_i G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)}_{G(\mathcal{O}_L)}$$

Hence the p^v-torsion on $G(\mathcal{O}_L)$ is $\varprojlim_i G_v(\mathcal{O}_L, \mathfrak{m}^i\mathcal{O}_L)$. Hence

$$G(\mathcal{O}_L)_{\text{tor}} = \varinjlim \varprojlim G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

For (3), if G is étale, G_v is étale and hence formally étale, so

$$G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong G_v(\mathcal{O}_L/\mathfrak{m}^{i+1}\mathcal{O}_L).$$

Hence

$$G(\mathcal{O}_L) = \varprojlim \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) = \varprojlim \varinjlim G_v(k_L) = G(k_L),$$

completing the proof. ☐

Corollary 2.8.24. *If G is connected, take μ to be the formal group law under the Serre–Tate equivalence 2.6.14. Then*

$$G(\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_K\text{-cont}}(\mathcal{O}_K[\![t_1, \ldots, t_d]\!], \mathcal{O}_L)$$

where $d = \dim(G)$ and multipliciation by p is given by $[p]_\mu$.

Proposition 2.8.25. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. Then*

$$0 \longrightarrow G^0(\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \longrightarrow G^{\text{ét}}(\mathcal{O}_L) \longrightarrow 0$$

is exact.

Proof. Let $G_v = \mathrm{Spec}(A_v)$, $G_v^0 = \mathrm{Spec}(A_v^0)$, and $G_v^{\text{ét}} = \mathrm{Spec}(A_v^{\text{ét}})$. Let

$$\mathcal{A} = \varprojlim A_v, \quad \mathcal{A}^{\text{ét}} = \varprojlim A_v^{\text{ét}}.$$

This sequence is left exact since colimits and limits are both left exact. We need to show that $G(\mathcal{O}_L) \to G^{\text{'et}}(\mathcal{O}_L)$ is surjective, i.e. the map

$$\mathrm{Hom}_{\text{cont}}(\mathcal{A}, \mathcal{O}_L) \to \mathrm{Hom}_{\text{cont}}(\mathcal{A}^{\text{ét}}, \mathcal{O}_L)$$

is surjective. Recall that

$$G^0(\mathcal{O}_L) = \mathrm{Hom}_{\text{cont}}(\mathcal{O}_K[\![t_1, \ldots, t_d]\!], \mathcal{O}_L)$$

where $d = \dim(G)$. Moreover,

$$(\mathcal{A}^{\text{'et}} \otimes k)[\![t_1, \ldots, t_d]\!] \cong \mathcal{A} \otimes k$$

since over k the connected–étale sequence splits.

We get $f \colon \mathcal{A}^{\text{ét}}[\![t_1, \ldots, t_d]\!] \to \mathcal{A}$ (by the same argument as in Serre–Tate). We claim that this map is an isomorphism.

For surjectivity, assume $\operatorname{coker}(f) \neq 0$. Then there exists a maximal ideal $\mathfrak{M}$ of $\mathcal{A}$ such that $\operatorname{coker}(f)_{\mathfrak{M}} \neq 0$. Hence $\operatorname{coker}(f) \otimes_{\mathcal{O}_K} k = 0$, so $\mathfrak{m}\operatorname{coker}(f) = \operatorname{coker}(f)$, and hence

$$\operatorname{coker}(f)_{\mathfrak{M}} = \mathfrak{m}\operatorname{coker}(f)_{\mathfrak{M}} = \mathfrak{M}\operatorname{coker}(f)_{\mathfrak{M}}.$$

Since $\operatorname{coker}(f)_{\mathfrak{M}}$ is finitely-generated over $\mathcal{A}_{\mathfrak{M}}$, we are done by Nakayama's Lemma.

For injectivity, let $\mathcal{I} = (t_1, \ldots, t_d)$ and $\mathcal{I}$ be the image of I under f. We have a short exact sequence

$$0 \longrightarrow \ker(f)/\ker(f) \cap I^j \longrightarrow \mathcal{A}^{\text{ét}}[\![t_1, \ldots, t_d]\!]/I^j \longrightarrow \mathcal{A}/\widetilde{I}^j \longrightarrow 0,$$

so $\ker(f)/\ker(f) \cap I^j = 0$, showing that $\ker(f) \subseteq I^j$. Since $\bigcap I^j = 0$, this shows that $\ker(f) = 0$.

We have hence shown that f is an isomorphism. This gives a surjection $\mathcal{A} \to \mathcal{A}^{\text{'et}}$ which splits the embedding $\mathcal{A}^{\text{ét}} \to \mathcal{A}$. We hence get a splitting of

$$\operatorname{Hom}_{\text{cont}}(\mathcal{A}, \mathcal{O}_L) \to \operatorname{Hom}_{\text{cont}}(\mathcal{A}^{\text{ét}}, \mathcal{O}_L),$$

showing this map is surjective. □

Corollary 2.8.26. *For all $x \in G(\mathcal{O}_L)$, $p^n x \in G^0(\mathcal{O}_L)$ for some n.*

Proof. The group $G^{\text{ét}}$ is torsion. Hence for some some n, the image of $p^n x$ in $G^{\text{ét}}(\mathcal{O}_L)$ is trivial. We are hence done by the connected–étale sequence. □

Proposition 2.8.27. *If the field L is algebraically closed (e.g. $L = \mathbb{C}_K$), multiplication by p on $G(\mathcal{O}_L)$ is surjective.*

Proof. By the connected–étale sequence, can work on $G^0(\mathcal{O}_L)$ and $G^{\text{ét}}(\mathcal{O}_L)$ separately. Since $G^{\text{ét}}(\mathcal{O}_L) = G^{\text{ét}}(k_L)$, using equivalence to finite free $\mathbb{Z}_p$-modules, multiplication by p is surjective.

The group $G^0(\mathcal{O}_L)$ is p-divisible by the p-divisibility of the corresponding p-divisible formal group μ. Surjectivity on $G^0(\mathcal{O}_L)$ follows. □

Remark 2.8.28. These facts will imply that $\log\colon G(\mathcal{O}_{\mathbb{C}_K}) \to t_G(\mathbb{C}_K)$ is surjective.

2.8.3. *The logarithm for p-divisible groups.* Let L be the p-adic completion of an algebraic extension of K. Recall that

$$G(\mathcal{O}_L) = \text{group of formal } \mathcal{O}_L\text{-valued points.}$$

Then

$$G^0(\mathcal{O}_L) = \operatorname{Hom}_{\text{cont}}(\mathcal{O}_K[\![t_1, \ldots, t_d]\!], \mathcal{O}_L)$$

and

$$0 \longrightarrow G^0(\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \longrightarrow G^{\text{ét}}(\mathcal{O}_L) \longrightarrow 0$$

is exact.

Definition 2.8.29. Let G be a p-divisible group over $\mathcal{O}_K$ of dimension d. Let μ be the formal group law associated to G^0 and $I = (t_1, \dots, t_d) \subseteq \mathcal{O}_K[\![t_1, \dots, t_d]\!]$ be the augmentation ideal of μ.

(1) Let M be an $\mathcal{O}_K$-module. The *tangent space* of G with values in M is

$$t_G(M) = \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(I/I^2, M).$$

(2) The *cotangent space* of G with values in M is

$$t_G^*(M) = I/I^2 \otimes_{\mathcal{O}_K} M.$$

Remark 2.8.30.

(1) There is a formal group G associated to G. The tangent (cotangent space of G) agrees with the above notion.
(2) For any real $\lambda > 0$,

$$\mathrm{Fil}^\lambda G^0(\mathcal{O}_L) = \{f \in G^0(\mathcal{O}_L) \mid v(f(x)) \geq \lambda \text{ for all } x \in I\}$$

(this makes sense since $f \in G^0(\mathcal{O}_L) = \mathrm{Hom}(\mathcal{O}_K[\![t_1, \dots, t_d]\!], \mathcal{O}_L)$).

Definition 2.8.31. The *log map* for G is

$$\begin{aligned} \log_G \colon G(\mathcal{O}_L) &\to t_G(L) = \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(I/I^2, L) \\ f &\mapsto \left(x \mapsto f(x) = \lim_{n\to\infty} \frac{(p^n f)(x)}{p^n}\right) \end{aligned}$$

This definition only makes sense when we prove the limit exists.

Lemma 2.8.32. *Let* $f \in \mathrm{Fil}^\lambda G^0(\mathcal{O}_L)$. *Then*

$$pf \in \mathrm{Fil}^k(G^0(\mathcal{O}_L))$$

where $k = \min(1+\lambda, 2\lambda)$.

Proof. Recall that $[p]_\mu(x) = px + y$ for any $x \in I$, where $y \in I^2$. Hence

$$\begin{aligned} (pf)(x) &= f([p]_\mu(x)) \\ &= f(px+y) \\ &= f(px) + f(y) \\ &= pf(x) + f(y). \end{aligned}$$

Hence

$$v((pf)(x)) = v(pf(x) + f(y))$$

and

$$\begin{aligned} v(pf(x)) &= 1 + v(f(x)) \geq 1 + \lambda \\ v(f(y)) &\geq 2\lambda. \end{aligned}$$

Therefore, $v((pf)(x)) \geq 1+\lambda, 2\lambda$. $\square$

Lemma 2.8.33. *For every $x \in I$, $f \in G(\mathcal{O}_L)$,*

$$\lim_{n\to\infty} \frac{(p^n f)(x)}{p^n}$$

exists in L and equal zero if $x \in I^2$.

Proof. Recall that for any $f \in G(\mathcal{O}_L)$, $p^n f \in G^0(\mathcal{O}_L)$ for $n \gg 0$ by Corollary 2.8.26. Hence we can apply Lemma 2.8.32 to $p^n f \in G^0(\mathcal{O}_L)$.

By an easy induction, there exists c such that

$$p^n f \in \operatorname{Fil}^{n+c} G^0(\mathcal{O}_L) \text{ for } n \gg 0.$$

Indeed, if $\lambda \geq 1$, $\min(1+\lambda, 2\lambda) = 1+\lambda$ and if $\lambda < 1$, $\min(1+\lambda, 2\lambda) = 2\lambda$.

We now want to show that $\left(\frac{(p^n f)(x)}{p^n}\right)$ is Cauchy. We have that

$$\begin{aligned}
\frac{(p^{n+1}f)(x)}{p^{n+1}} - \frac{(p^n f)(x)}{p^n} &= \frac{(p^n f)([p]_\mu(x))}{p^{n+1}} - \frac{(p^n f)(px)}{p^{n+1}} \\
&= \frac{(p^n f)([p]_\mu(x) - px)}{p^{n+1}} \\
&= \frac{(p^n f)(y)}{p^{n+1}}
\end{aligned}$$

has valuation $\geq 2(n+c) - (n+1) = n + 2c - 1$. This shows that the limit exists.

We finally want to show that the limit is 0 if $x \in I^2$. By the same calculation as above,

$$v\left(\frac{(p^n f)(x)}{p^n}\right) \geq 2(n+c) - n \geq n + 2c,$$

so the sequence tends to 0. $\square$

Corollary 2.8.34. *Definition 2.8.31 of $\log_G$ makes sense.*

Remark 2.8.35. By the Serre–Tate equivalence, there is a smooth formal group $\mathcal{G}^0$ associated to G^0. One can then show that $\mathcal{G}^0(\mathcal{O}_L) = G^0(\mathcal{O}_L)$ has a structure of a p-adic analytic group.

One can hence define log on $G^0(\mathcal{O}_L)$. For all $f \in G(\mathcal{O}_L)$, $p^n f \in G^0(\mathcal{O}_L)$, so we define

$$\log(f) = \frac{\log(p^n f)}{p^n}.$$

Example 2.8.36. Suppose $G = \mu_{p^\infty}$. Then

$$\begin{aligned}
\mu_{p^\infty}(\mathcal{O}_L) &\cong \operatorname{Hom}_{\text{cont}}(\mathcal{O}_K[\![t]\!], \mathcal{O}_L) \\
&\cong \mathfrak{m}_L & f &\mapsto f(t) \\
&\cong 1 + \mathfrak{m}_L & f(t) &\mapsto 1 + f(t)
\end{aligned}$$

Moreover, $t_{\mu_{p^\infty}}(L) = \operatorname{Hom}_{\mathcal{O}_K}(I/I^2, L) = L$ and $I = (t)$.

We claim that the diagram

$$\begin{array}{ccccc} f & & \mu_{p^\infty}(\mathcal{O}_L) & \xrightarrow{\log_{\mu_{p^\infty}}} & t_G(L) & & g \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 1+f(t) & & 1+\mathfrak{m}_L & \xrightarrow{\log_p} & L & & g(t) \end{array}$$

commutes. We have that

$$\begin{aligned} \log(f)(t) &= \lim_{n\to\infty} \frac{(p^n f)(t)}{p^n} \\ &= \lim_{n\to\infty} \frac{f([p^n]_\mu(t))}{p^n} \\ &= \lim_{n\to\infty} \frac{f((1+t)^{p^n}-1)}{p^n} \\ &= \lim_{n\to\infty} \frac{(1+f(t))^{p^n}-1}{p^n}. \end{aligned}$$

Now,

$$\log_p(1+x) = \lim_{n\to\infty} \frac{(1+x)^{p^n}-1}{p^n} = \lim_{n\to\infty} \sum_{i=1}^{p^n} \frac{1}{p^n}\binom{p^n}{i} x^i.$$

We claim that

$$\frac{1}{p^n}\binom{p^n}{i}x^i - \frac{(-1)^{i-1}}{i}x^i \to 0.$$

This is equal to

$$\frac{(p^n-1)\dots(p^n-i+1)-(-1)^{i-1}(i-1)!}{i!}.$$

Hence

$$v\left(\frac{1}{p^n}\binom{p^n}{i}x^i - \frac{(-1)^{i-1}}{i}\right) \geq n + iv(x) - v(i!) \geq n + iv(x) - \frac{i}{p-1}.$$

This shows that

$$\lim_{n\to\infty} \frac{1}{p^n}\binom{p^n}{i}x^i = \frac{(-1)^{i-1}}{i}x^i.$$

Hence

$$\log_p(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}x^i$$

is the usual power series for log

Proposition 2.8.37.

(1) *The log map* $\log_G$ *is a group homomorphism.*
(2) *The log map* $\log_G$ *is a local isomorphism, in the sense that for all* $\lambda \geq 1$*:*

$$\mathrm{Fil}^\lambda G^0(\mathcal{O}_L) \xrightarrow{\cong} \mathrm{Fil}^\lambda t_G(L) = \{\tau \in t_G(L) \mid v(\tau(x)) \geq \lambda \text{ for all } x \in I/I^2\}.$$

The filtration on the left hand side is what defines the topology.
(3) *The kernel* $\ker(\log_G) = G(\mathcal{O}_L)_{tor}$.
(4) *The log map* $\log_G$ *induces an isomorphism* $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong t_G(L)$.

Proof. We first check (1). For all $f, g \in G(\mathcal{O}_L)$, we want to show that

$$\log_G(f+g) = \log_G(f) + \log_G(g).$$

We have that

$$\begin{aligned} \frac{p^n(f+g)(x)}{p^n} &= \frac{(p^n f \otimes p^n g)(\mu(x))}{p^n} \\ &= \frac{(p^n f)(x) + (p^n g)(x) + y}{p^n} \qquad \text{for } y \in (p^n f)I \otimes (p^n g)(I). \end{aligned}$$

Since the valuation of y gets really large as $n \to \infty$, this shows that

$$\frac{p^n(f+g)(x)}{p^n} - \frac{(p^n f)(x)}{p^n} - \frac{(p^n g)(x)}{p^n} \to 0.$$

For (2), note that for all $f \in \operatorname{Fil}^\lambda G^0(\mathcal{O}_L)$,

$$(p^n f) \in \operatorname{Fil}^{\lambda+n} G^0(\mathcal{O}_L),$$

so

$$v\left(\frac{(p^n f)(x)}{p^n}\right) \geq \lambda,$$

showing that

$$v(\log_G(f)(x)) \geq \lambda$$

for all $x \in I/I^2$. There is also an inverse:

$$\begin{aligned} \operatorname{Fil}^\lambda t_G(L) &\to \operatorname{Fil}^\lambda G^0(\mathcal{O}_L) \\ \tau &\mapsto \text{the unique element } f \text{ such that } \log f(t_i) = \tau(t_i). \end{aligned}$$

Exercise. *This is actually the exponential map in terms of p-adic Lie groups.* Find an elementary proof of this fact.

To show (3), we first note that $t_G(L)$ has no torsion. Then $G(\mathcal{O}_L)_{\text{tor}} \subseteq \ker(\log_G)$. We want to show that $\ker(\log_G) \subseteq G(\mathcal{O}_L)$. For $f \in \ker(\log_G)$, $p^n f \in G^0(\mathcal{O}_L)$ for $n \gg 0$, so $p^n f \in \operatorname{Fil}^1 G^0(\mathcal{O}_L)$, which shows that $p^n f = 0$.

This also shows injectivity in (4) and we just need to show surjectivity. For $\tau \in t_G(L)$, $p^n \tau \in \operatorname{Fil}^1 t_G(L)$, so there exists $f \in \operatorname{Fil}^1 G^0(\mathcal{O}_L)$ such that $\log_G(f) = p^n \tau$, so τ is in the image. □

2.8.4. *Proof of Hodge–Tate decomposition.*

Theorem 2.8.38 (Tate)**.** *Let G be a p-divisible group over $\mathcal{O}_K$. Then*

$$\operatorname{Hom}(T_p(G), \mathbb{C}_K) \cong t_{G^\vee}(\mathbb{C}_K) \oplus t_G^*(\mathbb{C}_K)(-1),$$

where $T_p(G) = T_p(G \times K)$.

Corollary 2.8.39. *We have that* $\dim(G) = \dim \operatorname{Hom}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K}$.

In fact, we will prove this Corollary along the way to proving Tate's theorem 2.8.38.

Lemma 2.8.40. *Let G be a p-divisible group over $\mathcal{O}_K$. Then $G_v(\overline{K}) \cong G_v(\mathbb{C}_K) \cong G_v(\mathcal{O}_{\mathbb{C}_K})$.*

Proof. Since K has characteristic 0, any finite flat K-group is étale. This gives the first isomorphism. The second isomorphism follows from the valuative criterion for properness. $\square$

Lemma 2.8.41. *We have that* $G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} = G(\mathcal{O}_K)$ *and* $t_G(\mathbb{C}_K)^{\Gamma_K} = t_G(K)$.

Proof. Since $\mathbb{C}_K^{\Gamma_K} = K$ and $\mathcal{O}_{\mathbb{C}_K}^{\Gamma} = \mathcal{O}_K$, this is immediate. $\square$

Lemma 2.8.42. *We have that*

$$\bigcap_{n=1}^{\infty} p^n G^0(\mathcal{O}_K) = 0.$$

Proof. Since the valuation on K is discrete, there exists $\delta > 0$, a minimal valuation. If $f \in \operatorname{Fil}^\lambda G^0(\mathcal{O}_K)$, $pf \in \operatorname{Fil}^\kappa G^0(\mathcal{O}_K)$ for $\kappa = \min(\lambda + 1, 2\lambda)$. Hence

$$p^n f \in \operatorname{Fil}^{n\delta} G^0(\mathcal{O}_K).$$

Since $\bigcap_{n=0}^{\infty} \operatorname{Fil}^{n\delta} G^0(\mathcal{O}_K) = 0$, this gives the result. $\square$

Corollary 2.8.43. *The group* $G^0(\mathcal{O}_K)$ *does not contain any element which is infinitely* p*-divisible, i.e.* $G^0(\mathcal{O}_K)$ *does not contain any* $\mathbb{Q}_p$*-space.*

Definition 2.8.44. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. Then

$$\begin{aligned} T_p(G) &= T_p(G \times_{\mathcal{O}_K} K) = \varprojlim G_v(\overline{K}) && \textit{Tate module,} \\ \Phi_p(G) &= \varinjlim G_v(\overline{K}) = G(\overline{K}) && \textit{Tate comodule.} \end{aligned}$$

Example 2.8.45. When $G = \mu_{p^\infty}$,

$$\begin{aligned} T_p(\mu_{p^\infty}) &= \mathbb{Z}_p(1) \\ \Phi_p(\mu_{p^\infty}) &= \varinjlim \mu_{p^v}(\overline{K}) = \mu_{p^\infty}(\overline{K}). \end{aligned}$$

Proposition 2.8.46. *We have the following duality isomorphisms:*

$$\begin{aligned} T_p(G) &= \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)), \\ \Phi_p(G) &= \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})). \end{aligned}$$

Proof. Note that

$$\begin{aligned} T_p(G) &= \varprojlim G_v(\overline{K}) \\ &= \varprojlim \operatorname{Hom}_{\overline{K}}((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}) && \text{Cartier duality} \\ &= \operatorname{Hom}(\varprojlim (G_v^\vee(\overline{K})), \varprojlim \mu_{p^v}(\overline{K})) && \text{both étale} \\ &= \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), T_p(\mu_p^\infty)). \end{aligned}$$

For the other isomorphism,

$$\begin{aligned}\Phi_p(G) &= \varprojlim G_v(\overline{K}) \\ &= \varprojlim_{\overline{K}}((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}) \\ &= \varprojlim(G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})),\end{aligned}$$

as stated. □

Proposition 2.8.47. *We have a short exact sequence*

$$0 \longrightarrow \Phi_p(G) \longrightarrow G(\mathcal{O}_{\mathbb{C}_K}) \xrightarrow{\log_G} t_G(\mathbb{C}_K) \longrightarrow 0.$$

Proof. We know that $\Phi_p(G) = G(\overline{K}) \subseteq G(\mathcal{O}_{\mathbb{C}_K})$. We need to check that $\log_G$ is surjective and its kernel is $\Phi_p(G)$.

Recall that $\log_G$ induces an isomorphism $G(\mathcal{O}_{\mathbb{C}_K}) \otimes \mathbb{Q}_p \cong t_G(\mathbb{C}_K)$, so $\log_G$ is surjective after inverting p.

Since $\mathbb{C}_K$ is algebraically closed, $G(\mathcal{O}_{\mathbb{C}_K})$ is p-divisible (i.e. multiplication by p on $G(\mathcal{O}_{\mathbb{C}_K})$ is surjective). Hence p is already invertible in $G(\mathcal{O}_{\mathbb{C}_K})$, showing that $\log_G$ is surjective.

We now want to show that $\ker(\log_G) = \Phi_p(G)$. Then

$$\begin{aligned}\ker(\log_G) &= G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}} \\ &= \varinjlim_v \varprojlim_i G_v(\mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}^i\mathcal{O}_{\mathbb{C}_K}) \\ &= \varinjlim_v G_v(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varinjlim_v G_v(\overline{K}) \\ &= \Phi_p(G),\end{aligned}$$

as required. □

Example 2.8.48. Let $G = \mu_{p^\infty}$. Then

$$\Phi_p(\mu_{p^\infty}) = \mu_{p^\infty}(\overline{K})$$

and $\mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) = 1 + \mathfrak{m}_{\mathbb{C}_K}$, $t_{\mu_{p^\infty}}(\mathbb{C}_K) = \mathbb{C}_K$. The short exact sequence is

$$0 \longrightarrow \mu_{p^\infty}(\overline{K}) \longrightarrow 1 + \mathfrak{m}_{\mathbb{C}_K} \xrightarrow{\log_p} \mathbb{C}_K \longrightarrow 0.$$

Proposition 2.8.49. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\alpha & & \downarrow d\alpha & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee),\mu_{p^\infty}(\overline{K})) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee),1+\mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee),\mathbb{C}_K) & \longrightarrow & 0 \end{array}$$

where α and $d\alpha$ are $\mathbb{Z}_p$-linear, Γ_K-equivariant, and injective.

Proof. Since $T_p(G^\vee)$ is a finite free $\mathbb{Z}_p$-module, the bottom row is exact. The left vertical map is an isomorphism by Proposition 2.8.46.

We construct the map α. We have that

$$\begin{aligned} T_p(G^\vee) &= \varprojlim G_v^\vee(\overline{K}) \\ &= \varprojlim G_v^\vee(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_K}\text{-grp}}((G_v)_{\mathcal{O}_{\mathbb{C}_K}},(\mu_{p^v})_{\mathbb{C}_{\mathbb{C}_K}}) \\ &= \mathrm{Hom}_{p\text{-div}}(G,\mu_{p^\infty}). \end{aligned}$$

For any $g \in G(\mathcal{O}_{\mathbb{C}_K})$, we define

$$\alpha(g)(u) = u_{\mathcal{O}_{\mathbb{C}_K}}(g)$$

where $u \in T_p(G^\vee)$ defines a map $u_{\mathcal{O}_{\mathbb{C}_K}} : G(\mathcal{O}_{\mathbb{C}_K}) \to \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K})$. Can similarly define $d\alpha$.

Exercise. Both α and $d\alpha$ are $\mathbb{Z}_p$-linear and Γ_K-equivariant.

The right square commutes by the functoriality of $\log_G$:

$$\begin{array}{ccc} G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) \\ \downarrow & & \downarrow \\ \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_p} & \mathbb{C}_K \end{array}$$

commutes.

The left square also commutes, because both vertical maps come from Cartier duality.

We want to show that α and d_α are injective. Snake Lemma gives and isomorphism $\ker(\alpha) \cong \ker(d\alpha)$. We just need to show that d_α is injective. Also, $d\alpha$ is $\mathbb{Q}_p$-linear, so $\ker(d\alpha)$ is a $\mathbb{Q}_p$-vectors space.

Step 1. The map α is injective on $G(\mathcal{O}_K)$. Otherwise, let $0 \neq g \in \ker\alpha \cap G(\mathcal{O}_K)$. Then $0 \neq p^n g \in G^0(\mathcal{O}_K) \cap \ker\alpha$. We may hence assume $g \in G^0(\mathcal{O}_K) \cap \ker\alpha$. Hence $G^0(\mathcal{O}_K)$ contains a $\mathbb{Q}_p$-vector space, contradicting Corollary 2.8.43.

Step 2. We show that $d\alpha$ is injective on $t_G(K)$. Since $\log_G$ induces $\log_G(G(\mathcal{O}_K)) \otimes \mathbb{Q}_p \cong t_G(K)$, it is enough to show injectivity on $\log_G(G(\mathcal{O}_K))$.

We want to show that if $h \in G(\mathcal{O}_K)$ and $d\alpha(\log_G(h)) = 0$, then $\log_G(h) = 0$.

Since $\ker(\alpha) = \ker(d\alpha)$ via $\log_G$ and $\log_G(h) \in \ker(d\alpha)$, we have that $\log_G(h) = \log_G(h')$ for $h' \in \ker(\alpha)$. This shows that

$$h - h' \in \ker(\log_G) = G(\mathcal{O}_K)_{\text{tors}}.$$

Therefore, $p^n(h - h') - 0$, so

$$p^n h = p^n h' \in \ker(\alpha) \cap G(\mathcal{O}_K) = 0.$$

Hence $p^n h = 0$, so $\log_G(h) = 0$.

Step 3. Finally, $d\alpha$ factors as

$$\begin{aligned} t_G(\mathbb{C}_K) = t_G(K) \otimes \mathbb{C}_K &\hookrightarrow \operatorname{Hom}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K \\ &\hookrightarrow \operatorname{Hom}(T_p(G^\vee, \mathbb{C}_K)). \end{aligned}$$

The first map is injective. The second is injective by the Serre–Tate Lemma 2.8.16. □

Note that Snake Lemma also shows that $\operatorname{coker} \alpha \cong \operatorname{coker} d\alpha$. Also, we note that $\ker \alpha = \ker d\alpha$ is a $\mathbb{Q}_p$-vector space. We will use these facts later.

Theorem 2.8.50. *The maps α, $d\alpha$ from Proposition 2.8.49 induce isomorphisms on G_K-invariants:*

$$\begin{aligned} \alpha_K \colon G(\mathcal{O}_K) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}), \\ d\alpha_K \colon t_G(K) &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K). \end{aligned}$$

Proof. By Proposition 2.8.49, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\alpha} & \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \operatorname{coker}(\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \log_G} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \operatorname{coker}(d\alpha) & \longrightarrow & 0 \end{array}$$

Applying $(\cdot)^{\Gamma_K}$, we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(\mathcal{O}_K) & \xrightarrow{\alpha_K} & \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \operatorname{coker}(\alpha)^{\Gamma_K} \\ & & \downarrow{\scriptstyle \log_G} & & \downarrow & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & t_G(K) & \xrightarrow{d\alpha} & \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \operatorname{coker}(d\alpha)^{\Gamma_K} \end{array}$$

By exactness, we have a commutative diagram:

$$\begin{array}{ccc} \operatorname{coker}(\alpha_K) & \hookrightarrow & \operatorname{coker}(\alpha)^{\Gamma_K} \\ \downarrow & & \downarrow \\ \operatorname{coker}(d\alpha_K) & \hookrightarrow & \operatorname{coker}(d\alpha)^{\Gamma_K} \end{array}$$

Since $\operatorname{coker}(\alpha_K) \hookrightarrow \operatorname{coker}(d\alpha_K)$, it is enough to show that $d\alpha_K$ is surjective.

Let

$$W = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K),$$
$$V = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K).$$

Then $d\alpha_K : t_G(K) \to V^{\Gamma_K}$, so $\dim_K(V^{\Gamma_K}) \geq \dim_K t_G(K) = \dim G = d$.

We want to show $\dim_K(V^{\Gamma_K}) = \dim_K(t_G(K))$. We also know that

$$\dim_K(W^{\Gamma_K}) \geq \dim_K(t_{G^\vee}(K)) = \dim(G^\vee) = d^\vee$$

and hence

$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \geq d + d^\vee = h.$$

It is enough to show that

$$\dim_K(V^{\Gamma_K}) + \dim(W^{\Gamma_K}) \leq h.$$

Note that $\dim_{\mathbb{C}_K}(V) = h = \dim_{\mathbb{C}_K}(W)$. Recall that

$$T_p(G) \cong \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1))$$

as a Γ_K-module, which induces a perfect Γ_K-equivariant pairing

$$T_p(G) \times T_p(G^\vee) \to \mathbb{Z}_p(1).$$

This gives a perfect Γ_K-equivariant pairing

$$V \times W \to \mathbb{C}_K(-1).$$

Taking Γ_K-invariant, we get

$$V^{\Gamma_K} \times W^{\Gamma_K} \to \mathbb{C}_K(-1)^{\Gamma_K} = 0.$$

This shows that $V^{\Gamma_K} \otimes \mathbb{C}_K$ and $W^{\Gamma_K} \otimes \mathbb{C}_K$ are orthogonal under this pairing. Hence

$$\dim_{\mathbb{C}_K}(V^{\Gamma_K} \otimes \mathbb{C}_K) + \dim_{\mathbb{C}_K}(W^{\Gamma_K} \otimes \mathbb{C}_K) \leq \dim_{\mathbb{C}_K}(V) = h,$$

completing the proof. □

Corollary 2.8.51. *We have that*

$$\dim(G) = \dim_K \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K) = \dim_k(T_p(G) \otimes \mathbb{C}_K(-1))^{\Gamma_K}.$$

In particular, the dimension of G is determined by $G \times_{\mathcal{O}_K} K$.

Proof. The first identity follows from Theorem 2.8.50. For the second identity, use

$$T_p(G) \otimes \mathbb{C}_K(-1) \cong \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

(e.g. by the pairing in the proof of Theorem 2.8.50). □

Proof of the Hodge–Tate decomposition 2.8.38. Let

$$W = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K),$$
$$V = \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K).$$

Then $V^{\Gamma_K} \cong t_G(K)$ and $W^{\Gamma_K} \cong t_{G^\vee}(K)$ by Theorem 2.8.50. We also had a perfect pairing

$$V \times W \to \mathbb{C}_K(-1),$$

inducing an isomorphism

$$W \cong \mathrm{Hom}(V, \mathbb{C}_K(-1)).$$

Under this isomorphism,

$$V^{\Gamma_K} \otimes \mathbb{C}_K \cong t_G(\mathbb{C}_K)$$
$$W^{\Gamma_K} \otimes \mathbb{C}_K \cong t_{G^\vee}(\mathbb{C}_K)$$

and they are orthogonal under this pairing. We now have that

$$t_G(\mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K) \subseteq W.$$

Moreover,

$$\underbrace{\dim t_G(\mathbb{C}_K)}_{d} + \underbrace{\dim t_{G^\vee}(\mathbb{C}_K)}_{d^\vee} = \dim(W) = h.$$

We hence get an exact sequence

$$0 \longrightarrow t_{G^\vee}(\mathbb{C}_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \longrightarrow \underbrace{\mathrm{Hom}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1))}_{\cong t_G^*(\mathbb{C}_K)(-1)} \longrightarrow 0.$$

To prove the theorem, we want to show that this sequence splits uniquely. Indeed,

$$\mathrm{Ext}^1(t_{G^*}(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \cong \mathrm{Ext}^1(\mathbb{C}_K(-1)^{\oplus d}, \mathbb{C}_K^{\oplus d^\vee}) \cong H^1(\Gamma_K, \mathbb{C}_K(-1))^{\oplus dd^\vee} = 0$$

by the Tate–Sen Theorem 2.8.14, and

$$\mathrm{Hom}(t_G(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \cong H^0(\Gamma_K, \mathbb{C}_K(-1))^{\oplus dd^\vee} = 0,$$

proving the theorem. □

Corollary 2.8.52. *The representation $V_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is Hodge–Tate.*

Proof. Recall that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge–Tate if

$$\alpha_V \colon \bigoplus (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \otimes_K \mathbb{C}_K(-n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_K.$$

We check that $V = V_p(G)$ is Hodge–Tate. By Hodge–Tate decomposition 2.8.38, we have that

$$(V \otimes \mathbb{C}_K(n))^{\Gamma_K} = \begin{cases} t_G(\mathbb{C}_K) & \text{if } n = 0 \\ t_G^*(\mathbb{C}_K) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since α_V is always injective, it must be an isomorphism for dimension reasons. □

Proposition 2.8.53. *Suppose A is an abelian variety over K with good reduction. Then*

$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j).$$

Proof. Since A has good reduction, there is an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ such that the generic fiber is $\mathcal{A} \times K \cong A$. Moreover, we know that

$$\mathcal{A}^\vee[p^\infty] \cong \mathcal{A}[p^\infty]^\vee.$$

We have the following facts:

(1) $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(\mathcal{A}[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,
(2) the formal completion of $\mathcal{A}$ at the unit element gives the formal group law corresponding to $\mathcal{A}[p^\infty]^0$ under the Serre–Tate equivalence 2.6.14,
(3) we have isomorphisms:

$$H^0(A, \Omega^1_{A/K}) \cong t^*_e(A),$$
$$H^1(A, \mathcal{O}_A) \cong t_e(A^\vee).$$

(4) we have isomorphisms:

$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \bigwedge^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p),$$
$$H^i(A, \Omega^j_{A/K}) \cong \bigwedge^i H^1(A, \mathcal{O}_A) \otimes \bigwedge^j H^0(A, \Omega^1_{A/K}).$$

By (4), it is enough to prove the result for $n = 1$. We have that

$$H_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{C}_K \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A[p^\infty]), \mathbb{C}_K).$$

By (2) and (3), we have that

$$H^0(A, \Omega^1 A/K) \cong t^*_{A[p^\infty]}(\mathbb{C}_K)$$
$$H^1(A, \mathcal{O}_A) \cong t_{A[p^\infty]^\vee}(\mathbb{C}_K).$$

Hence the result following from the Hodge–Tate decomposition 2.8.38 for $A[p^\infty]$. □

2.9. **Generic fibers of p-divisible groups.**

Theorem 2.9.1 (Tate)**.** *The generic fiber functor for the category of p-divisible groups over $\mathcal{O}_K$ is fully faithful.*

Corollary 2.9.2. *The functor $G \mapsto T_p(G)$ is fully faithful.*

Proposition 2.9.3. *Let $G = \varinjlim_v G_v$ is a p-divisible group over $\mathcal{O}_K$, where $G_v = \mathrm{Spec}(A_v)$. Then*

$$\mathrm{disc}(A_v/\mathcal{O}_K) = (p^{dvp^{hv}})$$

where $d = \dim(G)$, $h = \mathrm{ht}(G)$.

Sketch of proof. Recall that we have an exact sequence

$$0 \longrightarrow G_1 \xrightarrow{i_{v,1}} G_{v+1} \xrightarrow{j_{1,v}} G_v \longrightarrow 0.$$

We can then show that

$$\mathrm{disc}(A_{v+1}/\mathcal{O}_K) = \mathrm{disc}(A_v/\mathcal{O}_K)^{p^h} \cdot \mathrm{disc}(A_1)^{p^{hv}}.$$

By induction, we reduce to the case $v = 1$. The connected–étale sequence is

$$0 \longrightarrow G_1^0 \longrightarrow G_1 \longrightarrow G_1^{\text{ét}} \longrightarrow 0.$$

We can show that $\mathrm{disc}(A_1^{\text{ét}}/\mathcal{O}_K) = (1)$. It is hence enough to show that $\mathrm{disc}(A_1^0/\mathcal{O}_K) = (p^{d\cdot p^h})$. Using Serre–Tate correspondence2.6.14,

$$A_1 = \mathcal{O}_K \otimes_{A_1[p]_\mu} A$$

and

$$\mathrm{disc}(A_1/\mathcal{O}_K) = \mathrm{disc}(\mathcal{A}/[p]\mathcal{A}).$$

This is hard so we omit the details. □

Lemma 2.9.4. *Consider a homomorphism $f\colon G \to H$ between p-divisible groups. If $\widetilde{f}\colon G \times_{\mathcal{O}_K} K \to H \times_{\mathcal{O}_K} K$ is an isomorphism, f is an isomorphism.*

Proof. Let $G = \varinjlim_v G_v$, $H = \varinjlim_v H_v$, $G_v = \mathrm{Spec}(A_v)$, $H_v = \mathrm{Spec}(B_v)$. The map f consists of maps $\alpha_v\colon B_v \to A_v$ such that $\alpha_v \otimes 1\colon B_v \otimes K \xrightarrow{\cong} A_v \otimes K$.

Since both A_v, B_v are finite free over $\mathcal{O}_K$, $B_v \hookrightarrow A_v$. If $\mathrm{disc}(A_v/\mathcal{O}_K) = \mathrm{disc}(B_v/\mathcal{O}_K)$, then we are done. Recall that $\dim(G)$ is determined by $T_p(G)$. □

Remark 2.9.5. This statement is not true for finite flat $\mathcal{O}_K$-group schemes. However, if $K/\mathbb{Q}_p$ is finite with $e < p-1$, then Lemma 2.9.4 also holds (this is a Theorem of Raynaud).

Proposition 2.9.6. *Let G be a p-divisible group over $\mathcal{O}_K$. Let M be a $\mathbb{Z}_p$-direct summand of $T_p(G)$, stable under Γ_K-action. Then there exists a p-divisible group H over $\mathcal{O}_K$ with a homomorphism $H \to G$ (in fact, a closed embedding), which induces $T_p(H) \cong M$.*

Proof. There is a p-divisible group $\widetilde{H}$ over K with $\widetilde{H} \to G \times_{\mathcal{O}_K} K$ such that $T_p(\widetilde{H}) \cong H$, where $\widetilde{H} = \varinjlim \widetilde{H}_v$.

Consider the scheme closure $\underline{H_v}$ of $\widetilde{H_v}$ in G_v.

Remark. The injective limit $\varinjlim_v \underline{H_v}$ may not be a p-divisible group over $\mathcal{O}_K$.

We get maps $\underline{H_v} \hookrightarrow \underline{H_{v+1}}$ induced from $\widetilde{H}_v \hookrightarrow \widetilde{H}_{v+1}$.

We claim that there exists v_0 such that

$$H_v = \underline{H_{v+v_0}}/\underline{H_{v_0}}$$

such that $\varinjlim H_v$ is a p-divisible group.

On the generic fiber,

$$H_v \times K \cong \widetilde{H}_{v+v_0}/\widetilde{H}_{v_0} \cong \widetilde{H}_v.$$

The map $[p]$ on $\underline{H_{v+1}}$ factors through $\underline{H_v}$, since $\widetilde{H}_{v+1}/\widetilde{H}_v$ is killed by p, so $\underline{H_{v+1}}/\underline{H_v}$ is killed by p.

Hence $[p]$ induces:

$$\delta_v\colon \underline{H_{v+2}}/\underline{H_{v+1}} \to \underline{H_{v+1}}/\underline{H_v}.$$

On generic fibers, δ_v is an isomorphism. Writing $\underline{H_{v+1}}/\underline{H_v} = \mathrm{Spec}(B_v)$, δ_v induces a map

$$B_v \to B_{v+1}$$

which becomes an isomorphism after tensoring with K. Hence $B_v \hookrightarrow B_{v+1}$ and $\{B_v\}$ is an increasing order in $B_1 \otimes K$.

Fact. The integral closure of $\mathcal{O}_K$ in $B_1 \otimes K$ is Noetherian.

Hence there exists v_0 such that

$$B_v \cong B_{v+1} \quad \text{for all } v \geq v_0.$$

If $v \geq v_0$, we have that

$$\underline{H_{v+2}}/\underline{H_{v+1}} \cong \underline{H_{v+1}}/\underline{H_v}.$$

Now,

$$\begin{array}{ccc} H_{v+1} = \underline{H_{v+1+v_0}}/\underline{H_{v_0}} & \xrightarrow{[p^v]} & \underline{H_{v+1+v_0}}/\underline{H_{v_0}} \cong H_{v+1} \\ \big\downarrow & & \big\uparrow \\ \underline{H_{v+1+v_0}}/\underline{H_{v_0+v}} & \xrightarrow{\cong} & \underline{H_{v_0+1}}/\underline{H_{v_0}} \end{array}$$

Finally, $\ker([p^v]) = \underline{H_{v+v_0}}/\underline{H_{v_0}} = H_v$. □

Proposition 2.9.7. *There is a bijection:*

$$\operatorname{Hom}(G, H) \cong \operatorname{Hom}(G \times K, H \times K).$$

Proof. If you have a homomorphism $f \colon G \times K \to H \times K$. Then $\widetilde{f}$ uniquely extends to $f \colon G \to H$.

For uniqueness: if $G_v = \operatorname{Spec}(A_v)$, $H_v = \operatorname{Spec}(B_v)$, then $\widetilde{f_v} \colon B_v \otimes K \to A_v \otimes K$, so there is at most one extension to $B_v \to A_v$ (by choosing generators).

We need tos how existence. Consider the graph of $T = T_p f \colon T_p(G) \to T_p(H)$:

$$M \subseteq T_p(G) \oplus T_p(H).$$

We claim that M is a $\mathbb{Z}_p$-direct summand. Note that

$$\begin{aligned} T_p(G) \oplus T_p(H)/M &\xrightarrow{\cong} T_p(H) \\ (x, y) &\mapsto y - T(x), \end{aligned}$$

so $T_p(G) \oplus T_p(H)/M$ is torsion-free. Hence the short exact sequence

$$0 \longrightarrow M \longrightarrow T_p(G) \oplus T_p(H) \longrightarrow T_p(G) \oplus T_p(H)/M \longrightarrow 0$$

splits.

Since $T_p(G \times H) = T_p(G) \oplus T_p(H)$, Proposition 2.9.6, there exists a p-divisible group G' over $\mathcal{O}_K$ with a homomorphism $\iota \colon G' \to G \times H$ such that $T_p(G') \cong M$.

Consider the projection maps

$$\pi_1 \colon G \times H \to G,$$
$$\pi_2 \colon G \times H \to H.$$

Then $\pi_1 \circ \iota\colon G' \to G$ is an isomorphism by Lemma 2.9.4. Then $f = \pi_2 \circ \iota \circ (\pi_1 \circ \iota)^{-1}$ extends $\widetilde{f}$. □

Remark 2.9.8.

(1) Theorem 2.9.1 extends to any base ring R such that
  (a) R is integrally closed and notherian,
  (b) R is an integral domain with $\mathrm{Frac}(R)$ of characterictic 0.
  by Hartog's Lemma.
(2) The special fiber functor is faithful, i.e. $\mathrm{Hom}(G, H) \hookrightarrow \mathrm{Hom}(G \times k, H \times k)$.

3. Period rings and functors

The goal is to define and study:

- period rings $B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{cris}}$,
- de Rham and crystalline representations.

There is another important period ring, B_{st}, related to semistable representations. We will omit this here entirely.

3.1. **Fontain's formalism on period rings.**

The reference for this section is [BC09, Section 5].

Let K be a p-adic field and Γ_K be the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ and $I_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{un}})$ be the *inertia group of* K.

3.1.1. *Definitions and examples.*

Definition 3.1.1. Let B be a $\mathbb{Q}_p$-algebra with an action of Γ_K and let C be the fraction field of B with the natural Γ_K-action.

We say that B is $(\mathbb{Q}_p, \Gamma_K)$-*regular* if

(1) $B^{\Gamma_K} = C^{\Gamma_K}$,
(2) any $b \in B$ with $b \neq 0$ is a unit if $\mathbb{Q}_p \cdot b$ is stable under the Γ_K-action.

Example 3.1.2. Every field extension of $\mathbb{Q}_p$ under any Γ_K-action is $(\mathbb{Q}_p, \Gamma_K)$-regular.

Remark 3.1.3. If F is a field and G is a group, we can define (F, G)-regular rings by replacing $\mathbb{Q}_p$ with F and Γ_K with G in the above definition.

We can also extend our formalism to this setting.

Definition 3.1.4. Suppose B is a $(\mathbb{Q}_p, \Gamma_K)$-regular ring and $E = B^{\Gamma_K}$. Then

(1) for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, define
$$D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}.$$
(2) a representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if
$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

We denote by $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ the category of *B-admissible p-adic representations.*

Remark 3.1.5. Let R be a topological ring with a continuous Γ_K-action. Then

$$H^1(\Gamma_K, \mathrm{GL}_d(R)) = \{\text{continuous } d\text{-dimensional semilinear } \Gamma_K \text{ representations over } R\}/\cong .$$

Exercise. Check this.

For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, we can consider the class $[V] \in H^1(\Gamma_K, \mathrm{GL}_n(\mathbb{Q}_p))$. Let $[V]_B$ be its image in $H^1(\Gamma_K, \mathrm{GL}_n(B))$. Then V is B-admissible if and only if $[V]_B$ is trivial.

Examples 3.1.6.

(1) For any $(\mathbb{Q}_p, \Gamma_K)$-regular B, $V = \mathbb{Q}_p$ with trivial Γ_K-action is B-admissible. Indeed, $D_B(V) = B^{\Gamma_K} = E$.
(2) Consider $B = \overline{K}$. Then $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *$\overline{K}$-admissible* if and only if V is potentially trivial (i.e. the action of Γ_K on V factors through some finite quotient). This follows from the group cohomology interpretation and Hilbert 90.
(3) Consider $B = \mathbb{C}_K$. Then $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is $\mathbb{C}_K$-admissible if and only if V is potentially unramified, i.e. the action of the inertia group factors through a finite quotient. This fact is quite difficult; it follows from Sen theory and is almost as difficult as the Tate–Sen theorem 2.8.14.

Theorem 3.1.7. *Let B and E be as above and $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$.*

(1) *the natural map $\alpha_V \colon D_B(V) \otimes_E B \to V \otimes_{\mathbb{Q}_p} B$ is B-linear, Γ_K-equivariant, and injective,*
(2) $\dim_E D_B(V) \le \dim_{\mathbb{Q}_p}(V)$ *with equality if and only if α_V is an isomorphism*

Compare this to the Serre–Tate Lemma 2.8.16 and Definition 2.8.17.

Proof. In (1), α_V is defined as the composition

$$\begin{aligned} D_B(V) \otimes_E B &= (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \otimes_E B \\ &\to (V \otimes_{\mathbb{Q}_p} B) \otimes_E B \\ &= V \otimes_{\mathbb{Q}_p} (B \otimes_E B) \\ &\to V \otimes_{\mathbb{Q}_p} B \end{aligned}$$

so it is clearly Γ_K-equivariant and B-linear.

We want to show that α_V is injective. Let $C = \mathrm{Frac}(B)$, which is $(\mathbb{Q}_p, \Gamma_K)$-regular.

Then we get a map $\beta_V \colon D_C(V) \otimes C \to V \otimes_{\mathbb{Q}_p} C$ with

$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_{\mathbb{Q}_p} B \\ \big\downarrow & & \big\downarrow \\ D_C(V) \otimes_R C & \xrightarrow{\beta_V} & V \otimes_{\mathbb{Q}_p} C \end{array}$$

where we have used $E = B^{\Gamma_K} = C^{\Gamma_K}$ (Condition (1) in Definition 3.1.1). To show that $\ker \alpha_V = 0$, it is enough to show that $\ker \beta_V = 0$. We may hence assume that B is a field.

Let (e_i) be a basis of $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ over E. We regard each e_i as in $V \otimes_{\mathbb{Q}_p} B$.

Assume $\ker(\alpha_V) \neq 0$ and consider a non-trivial relation $\sum b_i e_i = 0$ for $b_i \in B$. We follow the proof of the Serre–Tate Lemma 2.8.16. Take such a relation of minimal length with $b_r = 1$ for some r.

For all $\gamma \in \Gamma_K$,

$$0 = \gamma(\sum b_i e_i) - \sum b_i e_i = \sum (\gamma(b_i) - b_i) e_i,$$

a shorter relation since $\gamma(b_r) - b_r = 1 - 1 = 0$. By minimality,

$$v(b_i) = b_i$$

for all i, so $b_i \in B^{\Gamma_K} = E$. This is a contradiction, proving (1).

For (2), $\alpha_V \colon D_B(V) \otimes_E B \hookrightarrow V \otimes_{\mathbb{Q}_p} B$ induces

$$\alpha_V \otimes 1 \colon D_B(V) \otimes_E C \hookrightarrow V \otimes_{\mathbb{Q}_p} C.$$

Taking C-dimensions, we obtain $\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p} V$.

If α_V is an isomorphism, so is $\alpha_V \otimes 1$, so $\dim_E D_B(V) = \dim_{\mathbb{Q}_p}(V)$. Then the map $\alpha_V \otimes 1$ is automatically an isomorphism.

Conversely, assume that $d = \dim_E d_B(V) = \dim_{\mathbb{Q}_p}(V)$. Let e_i be an E-basis of $D_B(V)$, (v_i) be a $\mathbb{Q}_p$-basis of V. In these bases, α_V is a $d \times d$ matrix M_V. Since $\alpha_V \otimes 1$ is an isomorphism, $\det(M_V) \neq 0$. We want to show that

$$\det(M_V) \in B^\times.$$

By definition of determinant:

$$\alpha_V(e_1 \wedge \cdots \wedge e_d) = \det(M_V)(v_1 \wedge \cdots \wedge v_d).$$

For any $\gamma \in \Gamma_K$,

$$\gamma(v_1 \wedge \cdots \wedge v_d) = c_\gamma(v_1 \wedge \cdots v_d) \quad \text{for some } c_\gamma \in \mathbb{Q}_p$$

and $e_1 \wedge \cdots \wedge e_d$ is Γ_K-invariant. This shows that

$$\gamma(\det(M_v)) = \frac{1}{c_\gamma} \det(M_V).$$

Condition (2) in Definition 3.1.1 implies that $\det(M_V) \in B^\times$. □

3.1.2. *Hodge–Tate representations.* We want to see how Hodge–Tate representations fit into this formalism.

Definition 3.1.8. The *Hodge–Tate period ring* is

$$B_{\mathrm{HT}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_K(n).$$

Then:

(1) B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular,

(2) $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge–Tate if and only if V is B_{HT}-admissible.

Let χ be the p-adic cyclotomic character of K, i.e. $\chi\colon \Gamma_K \to \mathrm{Aut}(T_p(\mu_{p^\infty})) = \mathrm{Aut}(\mathbb{Z}_p(1)) = \mathbb{Z}_p^\times$.

Lemma 3.1.9. *The image of inertia I_K under χ is infinite.*

We have the following extension of Tate–Sen Theorem 2.8.14. We will assume it without proof.

Theorem 3.1.10 (Tate)**.** *Let $\eta\colon \Gamma_K \to \mathbb{Z}_p^\times$ be a continuous character. Define*

$$\mathbb{C}_K(\eta) = \mathbb{C}_K \text{ with twisted } \Gamma_K\text{-action of } \gamma \text{ given by } \eta(\gamma)\cdot\gamma.$$

For $i = 0, 1$, we have that

$$H^i(\Gamma_K, \mathbb{C}_K(\eta)) = \begin{cases} 0 & \text{if } \eta(I_K) \text{ is infinite,} \\ K & \text{if } \eta(I_K) \text{ is finite.} \end{cases}$$

Note that $\mathbb{C}_K(\chi^n) \cong \mathbb{C}_K(n)$ and $\chi(I_K)$ is infinite, so we recover the Tate–Sen Theorem 2.8.14 for $i = 0, 1$.

Proof of Lemma 3.1.9. Recall that $\chi\colon \Gamma_K \to \mathbb{Z}_p^\times = \mathrm{Aut}(\mu_{p^\infty})$. For any $\gamma \in \Gamma_K$, $\zeta \in \mu_{p^\infty}(\overline{K})$, we have that

$$\gamma(\zeta) = \zeta^{\chi(\gamma)}$$

by definition of χ. It is enough to show that the field extension $K(\mu_{p^\infty}(\overline{K}))/K$ is infinitely ramified. Let e_n be the ramification degree of $K(\mu_{p^n}(\overline{K})/K)$ over K. We then know that the ramification of degree of $\mathbb{Q}_p(\mu_{p^n}(\overline{\mathbb{Q}_p}))/\mathbb{Q}_p$ is $p^{n-1}(p-1)$ and

$$e_n \geq p^{n-1}(p-1)/e \to \infty,$$

completing the proof. □

Proposition 3.1.11. *The ring B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular.*

Proof. Let $C_{\mathrm{HT}} = \mathrm{Frac}(B_{\mathrm{HT}})$. We first check condition (1) in Definition 3.1.1: $B_{\mathrm{HT}}^{\Gamma_K} = C_{\mathrm{HT}}^{\Gamma_K}$. By Tate–Sen 2.8.14, $B_{\mathrm{HT}}^{\Gamma_K} = K$. We want to show that $C_{\mathrm{HT}}^{\Gamma_K} = K$.

Observe that $B_{\mathrm{HT}} \cong \mathbb{C}_K[t, t^{-1}]$ with $\gamma \in \Gamma_K$ acting by

$$\gamma\left(\sum a_n t^n\right) = \gamma(a_n)\chi^n(\gamma)t^n.$$

Now, $C_{\mathrm{HT}} = \mathbb{C}_K(t) \subseteq \mathbb{C}_K(\!(t)\!)$. It is hence enough to show that $\mathbb{C}_K(\!(t)\!)^{\Gamma_K} = K$. If $\sum a_n t^n \in C_K(\!(t)\!)^{\Gamma_K}$, we have that

$$\gamma(a_n)\chi(\gamma)^n = a_n \quad \text{for all } \gamma \in \Gamma_K.$$

Hence

$$a_n \in \mathbb{C}_K(n)^{\Gamma_K} = \begin{cases} 0 & \text{if } n \neq 0 \\ K & \text{if } n = 0, \end{cases}$$

showing that $\sum a_n t^n = a_0 \in K$. This proves condition (1) in Definition 3.1.1.

We now check condition (2) in the definition: if $0 \neq b \in B_{\mathrm{HT}}$ satisfies $\mathbb{Q}_p b$ is stable under Γ_K, then $b \in B_{\mathrm{HT}}^\times$.

Let $b = \sum a_n t^n$. There is a character $\eta\colon \Gamma_K \to \mathbb{Q}_p^\times$ such that

$$\gamma(b) = \eta(\gamma)b$$

for all $\gamma \in \Gamma_K$, which is continuous. It hence factors through $\eta\colon \Gamma_K \to \mathbb{Z}_p^\times$. Then

$$\eta(\gamma)b = \gamma(b) = \sum \gamma(a_n)\chi^n(\gamma)t^n,$$

so

$$\gamma(a_n)\chi(\gamma)^n = \eta(\gamma)a_n,$$

i.e.

$$\gamma(a_n)(\eta^{-1}\chi^n)(\gamma) = a_n.$$

Hence $a_n \in \mathbb{C}_K(\eta^{-1}\chi^n)^{\Gamma_K}$.

If $a_n \neq 0$, $\chi^{-1}\chi^n(I_K)$, Theorem 3.1.10

We want to show that $b = a_n t^n \in B^\times$. If $a_n \neq 0$ and $a_m \neq 0$, then $\eta^{-1}\chi^n(I_K)$ and $\eta^{-1}\chi^m(I_K)$ are both finite. Hence $\chi^{n-m}(I_K)$ is finite, contradicting Lemma 3.1.9. □

Remark 3.1.12. *This remark was made in response to the question if the ring* $B_{\mathrm{HT}}^+ = \bigoplus_{n\geq 0} \mathbb{C}_K(n)$ *is* $(\mathbb{Q}_p, \Gamma_K)$*-regular.* The answer is no: the proof of Proposition 3.1.11 shows that this ring, isomorphic to $\mathbb{C}_K[t]$, is not $(\mathbb{Q}_p, \Gamma_K)$-regular, because t is not invertible.

Proposition 3.1.13. *A representation* $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ *is Hodge–Tate if and only if it is* B_{HT}*-admissible.*

Proof. We recall the definitions:

- V is Hodge–Tate if and only if
$$\widetilde{\alpha_V}\colon \bigoplus (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \otimes_K \mathbb{C}_K(-n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$
is an isomorphism.
- V is B_{HT}-admissible if and only if $\dim_E D_{B_{\mathrm{HT}}}(V) = \dim_{\mathbb{Q}_p}(V)$; we check that this is also equivalent to α_V being an isomorphism.

Since

$$D_{B_{\mathrm{HT}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = \bigoplus_{n\in\mathbb{Z}} (V \otimes \mathbb{C}_K(n))^{\Gamma_K},$$

the map $\widetilde{\alpha_V}$ is an isomorphism if and only if $\dim_E D_{B_{\mathrm{HT}}}(V) = \dim_{\mathbb{Q}_p}(V)$. □

Remark 3.1.14. One could also prove this by relating the maps α_V and $\widetilde{\alpha_V}$. It is important that they are not the same:

- α_V is a homomorphism between graded vector spaces,
- $\widetilde{\alpha_V}$ is a map between the graded 0-pieces.

Theorem 3.1.15. *Consider the functor* $D_B\colon \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K) \to \mathrm{Vec}_E$*, where* Vec_E *is the category of finite-dimensional* E*-vector spaces. We have that:*

(1) *D_B is exact and faithful,*
(2) *$\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is stable under taking subrepresentations and quotients,*
(3) *$\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ is stable under taking tensors, exterior and symmetric powers, duals,*
(4) *D_B commutes with the operations in (3):*

$$D_B(V \otimes W) \cong D_B(V) \otimes D_B(W),$$
$$D_B\left(\bigwedge^i V\right) \cong \bigwedge^i D_B(V),$$
$$D_B\left(\mathrm{Sym}^i V\right) \cong \mathrm{Sym}^i D_B(V),$$
$$D_B(V^\vee) \cong D_B(V)^\vee.$$

We prove this as a series of propositions.

Proposition 3.1.16. *The functor D_B is exact and faithful.*

Proof. To check that it is faithful, suppose $V, W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ and $f \in \mathrm{Hom}_{\mathbb{Q}_p[\Gamma_K]}(V, W)$ satisfies

$$0 = D_B(f) \colon D_B(V) \to D_B(W).$$

We want to show $f = 0$. Indeed:

$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{D_B(f)\otimes 1 = 0} & D_B(W) \otimes_E B \\ \downarrow \cong & & \downarrow \cong \\ V \otimes_{\mathbb{Q}_p} B & \xrightarrow{f \otimes 1} & W \otimes_{\mathbb{Q}_p} B, \end{array}$$

so $f = 0$.

To show exactness, suppose

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence in $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$.

Fact 3.1.17. *Every algebra over a field is faithfully flat.*

Hence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U \otimes_{\mathbb{Q}_p} B & \longrightarrow & V \otimes_{\mathbb{Q}_p} B & \longrightarrow & W \otimes_{\mathbb{Q}_p} B & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & D_B(U) \otimes_E B & \longrightarrow & D_B(V) \otimes_E B & \longrightarrow & D_B(U) \otimes_E B & \longrightarrow & 0 \end{array}$$

which shows that

$$0 \longrightarrow D_B(U) \longrightarrow D_B(V) \longrightarrow D_B(W) \longrightarrow 0$$

is exact. □

Remark 3.1.18. In practice, we enhance D_B to a functor into a category of E-spaces with some additional structures. We will need some work for exactness of this enhanced functor.

Proposition 3.1.19. *If $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, any subrepresentation or quotient of V is also B-admissible.*

Proof. Suppose

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is an exact sequence in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ with $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$. We want to show that U and W are also B-admissible. Recall that

$$D_B(V) = (V \otimes B)^{\Gamma_K}.$$

Since D_B is left-exact,

$$0 \longrightarrow D_B(V) \longrightarrow D_B(V) \longrightarrow D_B(W),$$

we have that

$$\begin{aligned} \dim_E D_B(V) &\leq \dim_E D_B(V) + \dim_E D_B(W) \\ &\leq \dim_{\mathbb{Q}_p} V + \dim_{\mathbb{Q}_p} W \\ &= \dim_{\mathbb{Q}_p} V. \end{aligned}$$

Since V is B-admissible, all the inequalities are equalities, showing that U and W are also B-admissible. □

Remark 3.1.20. *This remark is an answer to the question: Is the category $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ closed under extensions?*

The answer is no. In fact, there is an example which is Hodge–Tate but not de Rham given any non-split extension V:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow V \longrightarrow \mathbb{Q}_p(1) \longrightarrow 0.$$

Hence the category of B_{dR}-admissible representations is not closed under extensions. However, the proof of the existence of such a non-split extension is very hard.

Proposition 3.1.21. *If $V, W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, then $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with*

$$D_B(V \otimes_{\mathbb{Q}_p} W) \cong D_B(V) \otimes_E D_B(W).$$

Proof. We have a natural E-linear map:

$$\begin{aligned} D_B(V) \otimes_E D_B(W) &\to (V \otimes_{\mathbb{Q}_p} B) \otimes (W \otimes_{\mathbb{Q}_p} B) \\ &\to (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B \qquad (*) \end{aligned}$$

The image of the first map is $(V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \otimes (W \otimes B)^{\Gamma_K}$. The second map is Γ_K-equivariant, so we get a map

$$D_B(V) \otimes D_B(W) \to ((V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} = D_B(V \otimes_{\mathbb{Q}_p} W) \qquad (**).$$

The map $(*)$ is injective, since it extends to a B-linear map:

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \to ((V \otimes_{\mathbb{Q}_p} B) \otimes (W \otimes_{\mathbb{Q}_p} B)) \otimes_{\mathbb{Q}_p} B \to (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B.$$

The resulting map:

$$\begin{aligned} (D_B(V) \otimes_E B) \otimes_B (D_B(W) \otimes_E B) &\to (V \otimes_{\mathbb{Q}_p} B \otimes_E B) \otimes (W \otimes_{\mathbb{Q}_p} B \otimes_E B) \\ &\to (V \otimes_{\mathbb{Q}_p} B) \otimes_B (W \otimes_{\mathbb{Q}_p} B) \end{aligned}$$

is exactly $\alpha_V \otimes \alpha_W$. Since V and W are B-admissible, this map is an isomorphism.

Hence $(**)$ is injective. This show that

$$\begin{aligned} \dim D_B(V \otimes W) &\geq \dim_E D_B(V) \cdot \dim_E D_B(W) \\ &= \dim_{\mathbb{Q}_p}(V) \cdot \dim_{\mathbb{Q}_p}(W) \\ &= \dim_{\mathbb{Q}_p}(V \otimes_{\mathbb{Q}_p} W). \end{aligned}$$

Since the other inequality is clear, this completes the proof. $\square$

Proposition 3.1.22. *If $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, then $\bigwedge^n V$ and $\mathrm{Sym}^n V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\mathbb{Q}_p)$ with natural isomorphisms*

$$D_B\left(\bigwedge^n V\right) \cong \bigwedge^n D_B(V),$$
$$D_B\left(\mathrm{Sym}^n V\right) \cong \mathrm{Sym}^n D_B(V).$$

Proof. We only prove this for $\bigwedge^n V$, since $\mathrm{Sym}^n V$ can be treated similarly.

Since V is B-admissible, $V^{\otimes n}$ is B-admissible by Proposition 3.1.21 and hence $\bigwedge^n V$ is B-admissible by Proposition 3.1.19.

We get a commutative diagram:

$$\begin{array}{ccc} D_B(V)^{\otimes n} \cong D_B(V^{\otimes n}) & \xrightarrow{\qquad\qquad} & D_B(\bigwedge^n V) \\ & \searrow \qquad \nearrow{\scriptstyle (*)} & \\ & \bigwedge^n D_B(V) & \end{array}$$

by Propositions 3.1.16 and 3.1.21.

We want to show that $(*)$ is an isomorphism. We know that $(*)$ is surjective by the commutativity of the diagram. Moreover,

$$\begin{aligned}\dim_E D_B(\bigwedge^n V) &= \dim_{\mathbb{Q}_p}(\bigwedge^n V)\\ &= \binom{\dim_{\mathbb{Q}_p} V}{n}\\ &= \binom{\dim_E D_B(V)}{n}\\ &= \dim_E \bigwedge D_B(V),\end{aligned}$$

so $(*)$ must be an isomorphism. □

Proposition 3.1.23. *If $V \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$, $V^\vee \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with a perfect pairing:*

$$D_B(V) \otimes_E D_B(V^\vee) \xrightarrow{\cong} D_B(V \otimes_{\mathbb{Q}_p} V^\vee) \cong D_B(\mathbb{Q}_p) = E \quad (*).$$

Proof. **Case 1.** $\dim_{\mathbb{Q}_p} V = 1$.

We want to show $\dim_E D_B(V^\vee) = 1 = \dim_{\mathbb{Q}_p} V^\vee$. Choose a basis v of V over $\mathbb{Q}_p$. There exists a character $\eta\colon \Gamma_K \to \mathbb{Q}_p^\times$ such that

$$\gamma(v) = \eta(\gamma)v \quad \text{for all } \gamma \in \Gamma_K.$$

Since V is B-admissible, $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ is 1-dimensional. Hence, there exists $b \in B$ such that $v \otimes b$ is a Γ_K-invariant E-basis of $D_B(V)$.

Since V is B-admissible, Theorem 3.1.7 shows that the map

$$\alpha_V\colon D_B(V) \otimes_E B \xrightarrow{\cong} V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism, and hence it maps $v \otimes b$ to a basis of $V \otimes_{\mathbb{Q}_p} B$. Hence $b \in B^\times$.

Finally:

$$\begin{aligned}\gamma(v \otimes b) &= \gamma(v) \otimes \gamma(b)\\ &= \eta(v)v \otimes \gamma(b)\\ &\ v \otimes \eta(v)\gamma(b).\end{aligned}$$

Hence $b = \eta(\gamma)\gamma(b)$ for all $\gamma \in \Gamma_K$. This shows that

$$D_B(V^\vee) = (V^\vee \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$$

contains a non-zero $v^\vee \otimes b^{-1}$ where $v^\vee$ is a dual basis.

Hence $V^\vee$ is B-admissible and $D_B(V^\vee)$ is spanned by $v^\vee \otimes b^{-1}$. One easily checks that $(*)$ is perfect.

Case 2. General case.

Let $d = \dim_{\mathbb{Q}_p} V$. There is a natural Γ_K-equivariant isomorphism

$$\Phi\colon \underbrace{\det(V^\vee)}_{\bigwedge^d V^\vee} \otimes \bigwedge^{d-1} V \cong V^\vee$$

given by

$$(f_1 \wedge \cdots \wedge f_d) \otimes (w_2 \wedge \cdots \wedge w_d) \mapsto (w_1 \mapsto \det(f_i(w_j))).$$

Since V is B-admissible, $\det(V) = \bigwedge^d V$ is B-admissible, hence

$$\det(V^\vee) = \det(V)^\vee \text{ is } B\text{-admissible}$$

by Case 1.

Since $\bigwedge^{d-1} V$ is B-admissible by Proposition 3.1.22, this shows that $V^\vee$ is also B-admissible.

We want to show that $(*)$ is perfect.

Fact. *If W, W' are vector spaces with $d = \dim_E W = \dim_E W'$ then $W \times W' \to E$ is perfect if and only if*

$$\det(W) \times \det(W') \to E$$

is perfect.

Finally, $(*)$ induces the pairing:

$$\begin{array}{ccc} \det(D_B(V)) \otimes \det(D_B(V^\vee)) & \longrightarrow & E \\ \big\downarrow = & & \big\downarrow = \\ D_B(\det(V)) \otimes D_B(\det(V^\vee)) & \longrightarrow & E \end{array}$$

Since $\dim \det(V) = 1$, this completes the proof. □

3.2. **De Rham representations.** The goal is to define and study:

- the de Rham period ring B_{dR},
- de Rham representations.

The references for this section are [BC09, Sections 4, 6] and [Sch12].

Outline of the construction of B_{dR}.

The field $\mathbb{C}_K$ is perfectoid. Hence $F = \mathbb{C}_K^\flat$ is a perfectoid field of characteristic p. Let $\mathcal{O}_F$ be the valuation ring of F.

We get a surjective ring homomorphism:

$$\theta \colon W(\mathcal{O}_F) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$$

which gives

$$\theta \colon W(\mathcal{O}_F)[1/p] \twoheadrightarrow \mathbb{C}_K$$

and we may consider $\ker(\theta)$. Then

$$B_{\mathrm{dR}}^+ = \varprojlim_j W(\mathcal{O}_F)[1/p]/(\ker\theta)^j$$

$$B_{\mathrm{dR}} = \mathrm{Frac}(B_{\mathrm{dR}}^+).$$

3.2.1. *Perfectoid fields and tilting.*

Definition 3.2.1. Let C be a complete non-archimedean field of residue characteristic p with valuation ring $\mathcal{O}_C$. Then C is a perfectoid field if:

(1) the valuation on C is non-discrete,
(2) the pth power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.

Lemma 3.2.2. *Let C be a complete non-archimedean field of residue characteristic p with non-trivial valuation. Assume that the pth power map is surjective on C. Then C is perfectoid.*

Proof. We first check property (1). Let v be the valuation on C and suppose v is discrete. Then there exists $x \in C$ with minimal positive valuation. Also, $x = y^p$ for some $y \in C$ by the surjecitivity of the pth power map.

Then
$$0 < v(y) = \frac{1}{p}v(x) < v(x)$$
which is a contradiction.

For (2), it suffices to show surjectivity on $\mathcal{O}_C$. For all $x \in \mathcal{O}_C$, there exists $y \in C$ such that $x = y^p$. Then $v(y) = \frac{1}{p}v(x) > 0$, so $y \in \mathcal{O}_C$. □

Proposition 3.2.3. *The field $\mathbb{C}_K$ is perfectoid.*

Proof. This follows from Lemma 3.2.2, since $\mathbb{C}_K$ is algebraically closed. □

Proposition 3.2.4. *A non-archimedean field of characteristic p is perfectoid if and only if it is complete and perfect.*

Proof. The 'only if' direction is immediate. The 'if' direction follows from Lemma 3.2.2. □

Fix a perfectoid field C. Write $\mathcal{O}_C$ for the valuation ring of C and v for the valuation on C.

Definition 3.2.5. The *tilt* of C is
$$C^\flat = \varprojlim_{x \mapsto x^p} C$$
with the natural multiplication.

A priori, $C^\flat$ is a multiplicative monoid. We will later define a topology on it, which turns out to be equivalent to the inverse limit topology.

We want to show $C^\flat$ is a perfectoid field of characteristic p.

Lemma 3.2.6. *Fix $\varpi \in C^\times$ such that $0 < v(\varpi) \leq v(p)$. For all $x, y \in \mathcal{O}_C$ with $x - y \in \varpi\mathcal{O}_C$, then*
$$x^{p^n} - y^{p^n} \in \varpi^{n+1}\mathcal{O}_C.$$

Proof. By the inequality, ϖ divides p in $\mathcal{O}_C$. We have that
$$x^{p^n} - y^{p^n} = (y^{p^{n-1}} - (y^{p^{n-1}} - x^{p^{n-1}}))^p - y^{p^n}$$
which shows the result by induction. □

Remark 3.2.7. In practice, if C has characteristic 0, then we may choose $\varpi = p$.

If C has characteristic p, $C^\flat \cong C$, so in practice, we might as well assume C has characteristic 0.

Proposition 3.2.8. *Fix $\varpi \in C^\times$ such that $0 < v(\varpi) \leq v(p)$. Then we have a multiplicative bijection:*

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_C \to \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$$

induced by $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\varpi\mathcal{O}_C$.

Proof. The map is clearly multiplicative, so we only need to construct an inverse. Define

$$\ell\colon \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \to \varprojlim_{x\mapsto x^p} \mathcal{O}_C$$

by setting for $\overline{c} = (\overline{c}_n) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ for $\overline{c_n} \in \mathcal{O}_C/\varpi\mathcal{O}_C$:

$$\begin{aligned} \ell(\overline{c}) &= (\ell_n(\overline{c})) \\ \ell_n(\overline{c}) &= \lim_{m\to\infty} c_{n+m}^{p^m} \qquad \text{where } c_n \in \mathcal{O}_C \text{ lifts } \overline{c_n}. \end{aligned}$$

For $\ell, m, n \gg 0$,

$$c_{n+m+\ell}^{p^\ell} - c_{n+m} \in \varpi\mathcal{O}_C,$$

because

$$\begin{aligned} \overline{c_{n+m+\ell}^{p^\ell}} - \overline{c_{n+m}} &= \overline{c_{n+m}} - \overline{c_{n+m}} \\ &= 0. \end{aligned}$$

Hence Lemma 3.2.6 shows that

$$c_{n+m+\ell}^{p^{\ell+m}} - c_{n+m}^{p^m} \in \varpi^{m+1}\mathcal{O}_C.$$

Therefore, for all n, $(c_{n+m}^{p^m})$ is a Cauchy sequence in $\mathcal{O}_C$. Therefore,

$$\lim_{m\to\infty} c_{n+m}^{p^m} \text{ exists.}$$

To check ℓ is well-defined, choose another lift c_n' of $\overline{c_n}$. Then

$$c_n - c_n' \in \varpi\mathcal{O}_C,$$

so Lemma 3.2.6 implies that

$$c_{n+m}^{p^m} - c_{n+m}'^{p^m} \in \varpi^{m+1}\mathcal{O}_C.$$

Hence the limit does not depend on the choice.

Finally, we need to show that ℓ is inverse to the reduction map in the statement. We have that:

$$(c_n) \mapsto (\overline{c_n}) \mapsto \left(\lim_{m\to\infty} c_{n+m}^{p^m}\right) = \left(\lim_{n\to\infty} c_n\right) = (c_n),$$

$$(\overline{c_n}) \mapsto \left(\lim_{m\to\infty} c_{n+m}^{p^m}\right) \mapsto \left(\lim_{m\to\infty} \overline{c_{n+m}^{p^m}}\right) = \left(\lim_{n\to\infty} \overline{c_n}\right) = (\overline{c_n}),$$

showing that ℓ is the inverse. □

Remark 3.2.9. Since Proposition 3.2.8 gives a bijection

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_C \to \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$$

we may choose to work on either side of it. We will mostly work on the left hand side and only use the right hand side if needed. This gives a simple valuation (as we will see below) but makes the addition structure complicated.

Scholze [Sch12], on the other hand, chooses to work on the right hand side. Conversely, this makes the addition structure easy but the valuation is complicated.

Proposition 3.2.10. *The tilt $C^\flat$ is a perfectoid field of characteristic p with valuation ring $\mathcal{O}_{C^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_C$.*

Proof. **Step 1.** We show that $C^\flat$ is a perfect field of characteristic p. Fix $\varpi \in C^\times$ as before. Since ϖ divides p, $\mathcal{O}_C/\varpi\mathcal{O}_C$ is of characteristic p. Hence

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$$

has a ring structure with natural addition and multiplication. This induces a ring structure on $\mathcal{O}_{C^\flat}$ via Proposition 3.2.8. In fact, if $a = (a_n)$ and $b = (b_m)$ are in $\mathcal{O}_{C^\flat}$, then

$$(a+b)_n = \left(\lim_{m\to\infty} (a_{n+m} + b_{n+m})^{p^m}\right)$$

does not depend on the choice of ϖ. Recall that

$$C^\flat = \varprojlim_{x\mapsto x^p} C$$

so we may identify $C^\flat$ as the fraction field of $\mathcal{O}_{C^\flat}$. Hence $C^\flat$ is perfect of characteristic p.

Step 2. The field $C^\flat$ admits a valuation $v^\flat$ such that $v^\flat(c) = v(c_0)$ for all $c = (c_n) \in C^\flat$.

We have that $v^b(c) = \infty$ implies that $v(c_0) = \infty$, so $c_0 = 0$. Then $c_n = 0$ for all n, so $c = 0$. It is also clear that $v^\flat$ is multiplicative by definition.

We need to check the triangle inequality: for $a = (a_n)$, $b = (b_n)$ in $(C^\flat)^\times$, we have that

$$v^\flat(a+b) \geq \min(v^\flat(a), v^\flat(b)).$$

Without loss of generality, assume that $v^\flat(a) \geq v^\flat(b)$. For any n,

$$\nu(a_n) = \frac{1}{p^n} v(a_0) = v^\flat(a) \geq v^\flat(b) = v(b_0) = \frac{1}{p^n} v(b_n)$$

by multiplicativity. This shows that

$$\frac{a_n}{b_n} \in \mathcal{O}_C^\times \quad \text{for all } n.$$

Hence $a = b \cdot r$ for some $r \in \mathcal{O}_C^b$ and

$$v^\flat(a+b) = v^\flat(b(r+1)) = v^\flat(b) \cdot v^\flat(r+1) \geq v^\flat(b)$$

since $r + 1 \in \mathcal{O}_C^\flat$. We hence checked that $v^\flat$ defines a valuation on $C^\flat$.

Step 3. The valuation ring is $\mathcal{O}_{C^\flat}$.

For any $c = (c_n) \in C^\flat$,

$$v(c_n) = \frac{1}{p^n} v(c_0) = \frac{1}{p^n} v^\flat(c)$$

and $v^\flat(c) \geq 0$ if and only if $v(c_n) \geq 0$ for all n.

Step 4. The $v^\flat$-adic topology on $C^\flat$ is complete.

Given some $N > 0$, we have that $\nu(c_n) \geq \nu(\varpi)$ for all $n \geq N$ if and only if $n(c_0) \geq p^n v(\varpi)$. Hence $v(c_1) \geq p^{N-1} v(\varpi)$ etc.

Hence the $v^\flat$-adic topology on $\mathcal{O}_{C^\flat}$ is the same as the inverse topology on $\varprojlim \mathcal{O}_C/\varpi\mathcal{O}_C$. The latter topology is complete by definition.

By Proposition 3.2.4, this shows that $C^\flat$ is perfectoid. $\square$

3.2.2. *The de Rham period ring* B_{dR}. Let $F = \mathbb{C}_K^\flat$, a perfectoid field of characteristic p. Write $\mathcal{O}_F$ for its valuation ring and $v^\flat$ for its valuation.

Let $W(\mathcal{O}_F)$ be the Witt vectors over $\mathcal{O}_F$.

We want to construct a ring homomorphism

$$\theta \colon W(\mathcal{O}_F) \to \mathcal{O}_{\mathbb{C}_K}.$$

Lemma 3.2.11 (Universal property of Witt vectors)**.** *Let A be a perfect $\mathbb{F}_p$-algebra and R be p-adically complete. Given a ring homomorphism $\overline{\pi} \colon A \to R/pR$, $\overline{\pi}$ lifts uniquely to:*

- *a multiplicative map $\hat{\pi} \colon A \to R$,*
- *a ring homomorphism $\pi \colon W(A) \to R$.*

Also,

$$\pi\left(\sum [a_n] p^n\right) = \sum \widehat{\pi}(a_n) p^n.$$

Remark 3.2.12. There is another universal property [BC09, Proposition 4.3.4]: for

- A: perfect $\mathbb{F}_p$-algebra,
- B a p-ring (i.e. a ring for which B/p is perfect),

any $\overline{\pi} \colon A \to B/p$ uniquely lifts to a ring homomorphism $W(A) \to B$.

We cannot use this universal property, however, because $\mathcal{O}_{\mathbb{C}_K}/p$ is not perfect.

Proposition 3.2.13. *There is a ring homomorphism*

$$\theta \colon W(\mathcal{O}_F) \to \mathcal{O}_{\mathbb{C}_K}$$

such that

$$\theta\left(\sum [c_n] p^n\right) = \sum c_n^\# p^n$$

where $c_n = (c_{n,k}) \in \mathcal{O}_F = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}$ and $c_n^\# = c_{n,0} \in \mathcal{O}_{\mathbb{C}_K}$.

Proof. We have the ring homomorphism:

$$\overline{\pi}\colon \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_k}$$
$$c \mapsto \overline{c^{\#}},$$

where $\overline{c^{\#}}$ is the modulo p reduction of $C^{\#}$. Since the natural map

$$\hat{\pi}\colon \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}$$
$$c \mapsto c^{\#}$$

is multiplicative, Lemma 3.2.11 gives the required map $\theta = \pi$. □

Definition 3.2.14. The *infinitesimal period ring* is:

$$A_{\text{inf}} = W(\mathcal{O}_F),$$

where $F = \mathbb{C}_K^\flat$.

We constructed a homomorphism

$$\theta\colon A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_K}.$$

Proposition 3.2.15. *The map θ is surjective.*

Lemma 3.2.16. *For any $x \in \mathcal{O}_{\mathbb{C}_K}$, there exists $y \in \mathcal{O}_F$ such that $x - y^{\#} \in p\mathcal{O}_{\mathbb{C}_K}$.*

Proof. Let $\overline{x}$ be the image of x in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Since the pth power map is surjective on $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$, there exists

$$y' \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \quad \text{with } y'_0 = \overline{x}.$$

By Proposition 3.2.8, we have that:

$$\mathcal{O}_F = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K} \overset{\cong}{\to} \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$$
$$y \leftarrow\!\shortmid y'$$

and y works. □

Proof of Proposition 3.2.15. We have that

$$\theta\left(\sum [c_n]p^n\right) = \sum c_n^{\#}p^n.$$

For all $x \in \mathcal{O}_{\mathbb{C}_K}$, we have:

$$\begin{aligned} x &= c_0^{\#} + px_0 && c_0 \in \mathcal{O}_F, x_0 \in \mathcal{O}_{\mathbb{C}_K} \\ &= c_0^{\#} + p(c_1^{\#} + px_1) && c_0, c_1 \in \mathcal{O}_F, x_1 \in \mathcal{O}_{\mathbb{C}_K} \\ &\vdots \\ &= c_0^{\#} + pc_1^{\#} + p^2c_2^{\#} + \cdots. \end{aligned}$$

This gives the result by completeness of $\mathcal{O}_{\mathbb{C}_K}$. □

Remark 3.2.17. Where does this come from? Recall that B_{dR} is a refinement of B_{HT} for the de Rham comparison theorem. Observe that the de Rham cohomology has a Hodge filtration whose associated graded algebra equal to the Hodge cohomology. We want to construct B_{dR} as ring with graded algebra B_{HT}.

Fontaine's idea was to construct a complete DVR B_{dR}^+ such that

$$B_{\mathrm{dR}}^+/\mathfrak{m} \cong \mathbb{C}_K, \quad \mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{C}_K(1).$$

In characteristic p, the theory of Witt vectors provides a complete DVR with a specified residue field. Therefore, we want to build "$W(\mathbb{C}_K)$", but this does not work well, because $\mathbb{C}_K$ has characteristic 0. We should hence pass to characteristic p.

The ring $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ has characteristic 0, but is not perfect. Fontaine defined the *perfection* of $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ as

$$R_K = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}.$$

Then define $A_{\mathrm{inf}} = W(R_K)$.

Note that $R_K \cong \mathcal{O}_F$, so we have just been considering a more modern treatment of Fontaine's idea.

Fontaine finally realizes $\mathcal{O}_{\mathbb{C}_K}$ as a quotient of A_{inf}; indeed θ is surjective by Proposition 3.2.15. We have an induced map:

$$\theta_{\mathbb{Q}} \colon A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K,$$

so $\mathbb{C}_K$ is a quotient of $A_{\mathrm{inf}}[1/p]$.

Definition 3.2.18. Define

$$B_{\mathrm{dR}}^+ = \varprojlim_j A_{\mathrm{inf}}[1/p]/\ker(\theta_{\mathbb{Q}})^j.$$

However, B_{dR}^+ is not $(\mathbb{Q}_p, \Gamma_K)$-regular.

Definition 3.2.19. The *de Rham period ring* is:

$$B_{\mathrm{dR}} = \mathrm{Frac}(B_{\mathrm{dR}}^+).$$

Having laid out the strategy for constructing B_{dR}, we need to prove it has all the right properties.

To prove B_{dR}^+ is a complete DVR, we study $\ker(\theta)$.

Fix $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\# = p$. For example:

$$p^\flat = (p, p^{1/p}, p^{1/p^2}, \ldots).$$

Consider the associated element

$$\xi = [p^\flat] - p \in A_{\mathrm{inf}}.$$

We want to show that $\ker(\theta)$ is a principal ideal, generated by ξ.

Lemma 3.2.20. *We have that*

$$\ker(\theta) \cap p^n A_{\mathrm{inf}} = p^n \ker(\theta).$$

Proof. The '$\supseteq$' inclusion is obvious, so we only prove '$\subseteq$'. For $x \in \ker(\theta) \cap p^n A_{\text{inf}}$, we can write

$$x = p^n y \quad \text{for some } y \in A_{\text{inf}}.$$

Then

$$0 = \theta(x) = \theta(p^n y) = p^n \theta(y).$$

Since $\mathcal{O}_{\mathbb{C}_K}$ has no nonzero p-torsion, $\theta(y) = 0$, so $y \in \ker(\theta)$. □

Lemma 3.2.21. *We may write* $x \in \ker(\theta)$ *as*

$$x = c\xi + dp$$

for some $c, d \in A_{\text{inf}}$.

Proof. We want to show that

$$x \in (\xi, p) = ([p^\flat] - p, p) = ([p^\flat], p).$$

The element x has a Teichmuller expansion

$$x = \sum [c_n] p^n \quad \text{for } c_n \in \mathcal{O}_{\mathbb{C}_K}.$$

Hence

$$0 = \theta(x) = \sum c_n^\# p^n,$$

so

$$c_0^\# \equiv 0 \mod p.$$

Hence:

$$v^\flat(c_0) = v(c_0^\#) \geq v(p) = v((p^\flat)^\#) = v^\flat(p^\flat)$$

This shows that c_0 is divisible by $p^\flat$ in $\mathcal{O}_F$. Hence $[c_0]$ is divisible by $[p^\flat]$ in A_{inf}.

Finally,

$$x = [c_0] + \sum_{n \geq 1} [c_n] p^n \in ([p^\flat], p),$$

completing the proof. □

Proposition 3.2.22. *We have that* $\ker(\theta) = (\xi)$.

Proof. Note that $\xi \in \ker(\theta)$, because

$$\theta(\xi) = \theta([p^\flat]) - \theta(p) = (p^\flat)^\# - p = p - p = 0.$$

We want to show that $\ker(\theta) \subseteq (\xi)$.

For any $x \in \ker(\theta)$, Lemma 3.2.21 shows that we may write

$$x = c_0\xi + px_0 \quad \text{for some } c_0, x_0 \in A_{\text{inf}}.$$

We have that

$$px_0 = x - c_0\xi \in \ker(\theta),$$

so $x_0 \in \ker(\theta)$ by Lemma 3.2.20. Hence

$$x = c_0\xi + p(c_1\xi + px_1)$$

by Lemma 3.2.21 and we keep applying the two lemmas to write

$$x = c_0\xi + pc_1\xi + p^2c_2\xi + \cdots \in (\xi),$$

completing the proof. $\square$

Remark 3.2.23. We say that $x = \sum [c_n] p^n \in A_{\text{inf}}$ is *primitive of degree 1* if

$$v^\flat(c_0), \quad v^\flat(c_1) = 0.$$

We will see that x generates $\ker(\theta)$ if and only if x is primitive of degree 1.

In fact, we get a bijection:

$$\left\{ \substack{\text{primitive elements} \\ \text{of degree 1 in } A_{\text{inf}}} \right\} \leftrightarrow \left\{ \substack{\text{untilts} \\ \text{of } F} \right\}$$
$$\alpha \mapsto A_{\text{inf}}[1/p]/(\alpha).$$

Hence for any algebraically closed perfectoid field F of characteristic p, all untilts are algebraically isomorphic to $A_{\text{inf}}/\ker(\theta)$.

They are not (generally) topologically isomorphic: there is a counterexample. The intuition is that we define the topology dependent on the choice of generator α.

Proposition 3.2.24. *We have that*

$$\ker(\theta_{\mathbb{Q}})^j \cap A_{\text{inf}} = \ker(\theta)^j.$$

Proof. We proceed by induction on j. For $j = 1$, clearly $\ker(\theta_{\mathbb{Q}}) \cap A_{\text{inf}} \supseteq \ker(\theta)$, so we just have to show the other inclusion. If $x \in \ker(\theta_{\mathbb{Q}}) \cap A_{\text{inf}}$, since $\theta_{\mathbb{Q}} \colon \mathbb{A}_{\text{inf}}[1/p] \to \mathbb{C}_K$, there exists n such that $p^n x \in \ker(\theta) \cap A_{\text{inf}}$. This shows that $x \in \ker(\theta)$ by Lemma 3.2.20.

In the induction step, the inclusion

$$\ker(\theta_{\mathbb{Q}})^j \cap A_{\text{inf}} \supseteq \ker(\theta)^j$$

is again obvious and we prove the other inclusion.

For any $x \in \ker(\theta_{\mathbb{Q}})^j \cap A_{\text{inf}}$, there exists $n \geq 0$ such that $p^n x \in \ker(\theta)^j$, so

$$p^n x = r\xi^j$$

for some $r \in A_{\text{inf}}$. Hence

$$x \in \ker(\theta_{\mathbb{Q}})^j \cap A_{\text{inf}} \subseteq (\ker(\theta_{\mathbb{Q}}))^{j-1} \cap A_{\text{inf}} = \ker(\theta)^{j-1}$$

by the inductive hypothesis. Hence $x = \xi^{j-1} \cdot s$ for some $s \in A_{\text{inf}}$. We have that

$$r\xi^j = p^n s \xi^{j-1},$$

so

$$r\xi = p^n s.$$

This shows that

$$p^n s = r\xi \in \ker(\theta).$$

Using Lemma 3.2.20, $s \in \ker(\theta)$, so $s = \xi s'$ for some $s' \in A_{\text{inf}}$. Finally, this shows that

$$x = \xi^j \cdot s' \in \ker(\theta)^j,$$

as required. $\square$

Proposition 3.2.25. *We have that*

$$\bigcap_{j=1}^{\infty} \ker(\theta_{\mathbb{Q}})^j = \bigcap_{j=1}^{\infty} \ker(\theta)^j = 0.$$

Proof. By Proposition 3.2.24, we have that

$$\bigcap_{j=1}^{\infty} \ker(\theta_{\mathbb{Q}})^j = \bigcap_{j=1}^{\infty} \ker(\theta)[1/p]. \tag{3}$$

We just need to show that

$$\bigcap_{j=1}^{\infty} \ker(\theta)^j = 0.$$

For $x = \sum [c_n]p^n \in \bigcap_{j=1}^{\infty} \ker(\theta)^j$, x is infinitely divisible by ξ.

Hence c_0 is infinitely divisible by $p^\flat$. Since $v^\flat(p^b) = v(p) = 1 > 0$, we have that $c_0 = 0$. This shows that x is divisible by p and we write

$$x = p \cdot x_1 \quad \text{for } x_1 \in A_{\text{inf}}.$$

Then

$$x_1 \in \left(\bigcup_{j=1}^{\infty} \ker(\theta)^j \right) [1/p] \cap A_{\text{inf}}.$$

By equation 3,

$$x_1 \in \left(\bigcup_{j=1}^{\infty} \ker(\theta_{\mathbb{Q}})^j \right) \cap A_{\text{inf}} = \bigcap_{j=1}^{\infty} \ker(\theta)^j,$$

so x is infinitely divisible by p in $\bigcap_{j=1}^{\infty} \ker(\theta)^j$. This shows that $x = 0$. ∎

Lemma 3.2.26. *The natural map*

$$A_{\text{inf}}[1/p] \to \varprojlim A_{\text{inf}}[1/p]/\ker(\theta_{\mathbb{Q}})^j = B_{\text{dR}}^+$$

is injective. In particular, we can regard $A_{\text{inf}}[1/p]$ as a subring of B_{dR}^+.

Definition 3.2.27. The map θ induces a map

$$\theta_{\text{dR}}^+ \colon B_{\text{dR}}^+ \twoheadrightarrow A_{\text{inf}}[1/p]/\ker(\theta_{\mathbb{Q}}) \cong \mathbb{C}_K.$$

Theorem 3.2.28. *The ring B_{dR}^+ is a complete DVR with $\ker(\theta_{\text{dR}}^+)$ as maximal ideal, $\mathbb{C}_K$ as residue field, and ξ as uniformizer.*

Proof. **Step 1.** We show that B_{dR}^+ is a local ring.

By construction, $B_{\text{dR}}^+/\ker(\theta_{\text{dR}}^+) \cong \mathbb{C}_K$, so $\ker(\theta_{\text{dR}}^+)$ is a maximal ideal. We need to show that there are no other maximal ideals.

Fact. *If R is any ring and $I \subseteq R$ is an ideal such that $\bigcup_{n=1}^{\infty} I^n = 0$, and we write $\hat{R} = \varprojlim R/I^n$ for the completion of R with respect to I, then $x \in \hat{R}$ is a unit if and only if the image in R/I is a unit.*

Hence $x \in B_{\mathrm{dR}}^+$ is a unit if and only if $\theta_{\mathrm{dR}}^+(x)$ is a unit in $\mathbb{C}_K$ i.e. $x \notin \ker(\theta_{\mathrm{dR}}^+)$. This shows that B_{dR}^+ is local.

Step 2. We can show that any $x \in B_{\mathrm{dR}}^+$ has a unique expression $x = \xi^i u$ with $u \in (B_{\mathrm{dR}}^+)^\times$.

Exercise. Check this.

By construction, B_{dR}^+ is the ξ-adic completion of $A_{\mathrm{inf}}[1/p]$, so B_{dR}^+ is complete. □

Since de Rham cohomology has a filtration, we need a filtration on B_{dR}.

Corollary 3.2.29. *For any uniformizer ϖ of B_{dR}^+,*

$$\{\varpi^i B_{\mathrm{dR}}^+\}_{i\in\mathbb{Z}} = \{\ker(\theta_{\mathrm{dR}}^i)\}_{i\in\mathbb{Z}}$$

has the following properties:

(1) $\varpi^i B_{\mathrm{dR}}^+ \supseteq \varpi^{i+1} B_{\mathrm{dR}}^+$,
(2) $\bigcap_{i\in\mathbb{Z}} \varpi^i B_{\mathrm{dR}}^+ = 0$, $\bigcup_{i\in\mathbb{Z}} \varpi^i B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}$,
(3) $\varpi^i B_{\mathrm{dR}}^+ \cdot \varpi^j B_{\mathrm{dR}}^+ \subseteq \varpi^{i+j} B_{\mathrm{dR}}^+$.

Therefore, B_{dR} has a natural structure of a filtered ring.

Finally, we want to show that $B_{\mathrm{dR}}^{\Gamma_K} = K$ with graded algebra isomorphic to B_{HT}.

Proposition 3.2.30. *Let $K_0 = \mathrm{Frac}(W(k))$ where k is the residue field of K. Then:*

(1) *K is a finite totally ramified extension of K_0,*
(2) *there is a unique map $\overline{K} \to B_{\mathrm{dR}}^+$ making the triangle:*

$$\begin{array}{ccc} \overline{K} & \longrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \downarrow{\scriptstyle \theta_{\mathrm{dR}}^+} \\ & & \mathbb{C}_K \end{array}$$

commute.

Remark 3.2.31. The map $\overline{K} \to B_{\mathrm{dR}}^+$ is not continuous.

Proof. Recall that if A is a perfect $\mathbb{F}_p$-algebra and R is p-adically complete, then any ring homomorphism $A \to R/p$ uniquely lifts to a homomorphism $W(A) \to R$ by Lemma 3.2.11.

For (1), note that the quotient map $\mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/\mathfrak{m} = k$ has a canonical section $k \to \mathcal{O}_K/p\mathcal{O}_K$ (induces by $\overline{k} \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$), giving a map $W(k) \to \mathcal{O}_K$, and hence a map

$$K_0 \to K$$

of discrete valued fields. (Alternatively, one can use another universal property of Witt vectors to obtain the desired map $W(K) \to \mathcal{O}_K$.)

Then $e(K/K_0) = \frac{1}{v(\varpi)}$ where ϖ is a uniformizer of K and $f(K/K_0) = 1$. This shows (1).

For (2), consider the composition

$$k \to \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \to \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} = \mathcal{O}_F$$

which gives a map

$$W(k) \to A_{\mathrm{inf}}$$

and hence

$$K_0 \to A_{\mathrm{inf}}[1/p] \to B_{\mathrm{dR}}^+.$$

This lifts to a map $\overline{K} \to B_{\mathrm{dR}}^+$ by Hensel's lemma.

We only proved that the map $\overline{K} \to B_{\mathrm{dR}}^+$ exists, but we did not prove uniqueness. This can be found in [BC09], but we omit it here. □

Proposition 3.2.32. *There is a refinement of the DVR topology on B_{dR}^+ such that*

(1) $A_{\mathrm{inf}} \to B_{\mathrm{dR}}^+$ *is a closed embedding,*
(2) $\theta_{\mathbb{Q}} \colon A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K$ *is open and continuous,*
(3) *there is a continuous logarithm map*

$$\log \colon \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^+$$

given by

$$\log(x) = \sum (-1)^n \frac{([x]-1)^n}{n}$$

where we identify

$$\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \{c \in \mathcal{O}_F \mid c^\# = 1\},$$

(4) *multiplication by any uniformizer of B_{dR}^+ is a closed embedding,*
(5) B_{dR}^+ *is complete.*

Remark 3.2.33. A sketch of the proof is in [BC09, Exercise 4.5.3].

Remark 3.2.34. The DVR topology does not satisfy properties (1), (2), (3). The issue is that the DVR topology "ignores" the valuation topology on $\mathbb{C}_K$. In fact, the Γ_K-action of B_{dR}^+ is not continuous for the DVR topology.

Fix $\epsilon \in \mathbb{Z}_p(1)$ with $\epsilon \neq 1$, i.e. $\epsilon = (\zeta_{p^n})$ is a compatible system of p^nth roots of unity. Set

$$t = \log(\epsilon) \in B_{\mathrm{dR}}^+.$$

This will be a uniformizer, which is more convenient to work with than ξ, because of the simple Galois action.

Remark 3.2.35. We have the following (tentative) equalities:

$$\begin{aligned} \gamma(t) &= \gamma(\log(\epsilon)) \\ &= \log(\gamma(\epsilon)) && \text{if log is equivariant,} \\ &= \log(\epsilon^{\chi(\gamma)}) \\ &= \chi(\gamma)\log(\epsilon) && \text{if log is additive,} \\ &= \chi(\gamma)t \end{aligned}$$

Lemma 3.2.36. *We have that* $v^\flat(\epsilon - 1) = \frac{p}{p-1}$.

Proof. Indeed,

$$\begin{aligned} v^\flat(\epsilon - 1) &= v((\epsilon - 1)^\#) \\ &= v\left(\lim_{n\to\infty} (\zeta_{p^n} - 1)^{p^n}\right) \\ &= \lim_{n\to\infty} \left(p^n v(\zeta_{p^n} - 1)\right) \\ &= \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} \\ &= \frac{p}{p-1}, \end{aligned}$$

as required. □

Proposition 3.2.37. *The element* $t \in B_{\mathrm{dR}}^+$ *is a uniformizer.*

Proof. We have that $\theta([\epsilon] - 1) = \epsilon^\# - 1 = 1 - 1 = 0$. Hence

$$[\epsilon - 1] \in \ker(\theta) \subseteq \xi B_{\mathrm{dR}}^+.$$

Now,

$$t = \log(\epsilon) = \sum_{n\geq 1} (-1)^{n+1} \frac{([\epsilon] - 1)^n}{n} \in \xi B_{\mathrm{dR}}^+.$$

We want to show that t is not divisible by ξ^2. When $n \geq 2$, $\frac{([\epsilon]-1)^n}{n}$ is divisible by ξ^2. It is hence enough to check that $[\epsilon] - 1$ is no divisible by ξ^2.

We look at the first coefficient in the Teichmuller expansions:

$$[\epsilon - 1] \quad \text{and} \quad [(p^\flat)^2].$$

Considering valuations:

$$v^\flat(\epsilon - 1) = \frac{p}{p-1} < 2 = 2v(p)v((p^\flat)^2),$$

if $p > 2$. If $p = 2$, we look at the second coefficients:

$$[\epsilon - 1] \quad \text{and} \quad [(p^\flat)^4].$$

Again,

$$v^\flat(\epsilon - 1) = \frac{p}{p-1} = 2 < 4 = v^\flat((p^\flat)^4).$$

This completes the proof. □

Lemma 3.2.38. *For any* $m \in \mathbb{Z}_p$, $\log(\epsilon^m) = m \log(\epsilon)$.

Proof. **Case 1.** m is an integer.

We have that

$$\log((1 + x)^m) = m \log(1 + x)$$

as formal power series. Since

$$[\epsilon] - 1 \in \xi B_{\mathrm{dR}}^+,$$

the power series converges in B_{dR}^+ for $x = [\epsilon] - 1$.

Case 2. General case.

Choose a sequence $m_i \in \mathbb{Z}$ such that $\lim m_i = m$ in $\mathbb{Z}_p$. Then

$$\lim m_i \log(\epsilon) = (\lim m_i) \log(\epsilon) = m \log(\epsilon)$$

where the first equality follows since $t = \log(\epsilon)$ is a uniformizer in B_{dR}^+.

Also, $\lim \epsilon^{m_i} = \epsilon^m$ in the valuation topology on F. By continuity of log, we have that

$$\log(\epsilon^m) = \log(\lim \epsilon^{m_i}) = \lim \log(\epsilon^{m_i}) = \lim m_i \log(\epsilon) = m \log(\epsilon),$$

completing the proof. □

Theorem 3.2.39 (Fontaine)**.** *The natural Γ_K-action on B_{dR} has the following properties:*

(1) *any $\gamma \in \Gamma_K$ acts by $\gamma(t) = \chi(\gamma)t$,*
(2) *$t^i B_{\mathrm{dR}}^+$ is stable,*
(3) $\displaystyle\bigoplus_i m t^i B_{\mathrm{dR}}^+ / t^{i+1} B_{\mathrm{dR}}^+ \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_K(n) = B_{\mathrm{HT}}$,
(4) *B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{\mathrm{dR}}^{\Gamma_K} = K$.*

Proof. The natural Γ_K-action on $\mathbb{C}_K$ induces an action on $F = \mathbb{C}_K^\flat$ by

$$\gamma(x_n) = (\gamma(x_n))$$

for all $(x_n) \in F$. By functoriality, this gives a Γ_K-action on $A_{\mathrm{inf}} = W(\mathcal{O}_F)$; explicitly:

$$\gamma\left(\sum [c_n] p^n\right) = \sum [\gamma(c_n)] p^n.$$

It is clear that θ, $\theta_{\mathbb{Q}}$ are Γ_K-equivariant and hence $\ker(\theta)$, $\ker(\theta_{\mathbb{Q}})$ are Γ_K-stable. This gives a natural Γ_K-action on

$$B_{\mathrm{dR}}^+ = \varprojlim A_{\mathrm{inf}}[1/p]/(\ker(\theta_{\mathbb{Q}})^j)$$

which extends to B_{dR}.

We now check that this action satisfies the 4 properties. For (1), if $\epsilon \in \mathbb{Z}_p(1)$, we have that

$$\gamma\epsilon = \epsilon^{\chi(\gamma)}$$

for all $\gamma \in \Gamma_K$ by definition of χ, so

$$\gamma(t) = \gamma(\log(\epsilon)) = \log(\gamma(\epsilon)) = \log(\epsilon^{\chi(\gamma)}) = \chi(\gamma)\log(\epsilon) = \chi(\gamma)t$$

since log is Γ_K-equivariant and by Lemma 3.2.38.

Part (2) is immediate from (1). For (3), we have a natural map

$$B_{\mathrm{dR}}^+ / \ker(\theta_{\mathrm{dR}}^+) = B_{\mathrm{dR}}^+ / t B_{\mathrm{dR}}^+ \cong A_{\mathrm{inf}}[1/p]/\ker(\theta_{\mathbb{Q}}) \cong \mathbb{C}_K$$

which is Γ_K-equivariant. Hence, for any $n \in \mathbb{Z}$, we have that

$$t^n B_{\mathrm{dR}}^+ / t^{n+1} B_{\mathrm{dR}}^+ \cong \mathbb{C}_K(n)$$

which is canonical (since t is uniquely determined up to $\mathbb{Z}_p^\times$-multiple by Lemma 3.2.38). Taking the direct sum of these shows (3).

We just need to check (4). There is a natural injective homomorphism

$$\begin{array}{ccc} \overline{K} & \hookrightarrow & B_{\mathrm{dR}}^{+} \\ & \searrow & \downarrow \theta_{\mathrm{dR}}^{+} \\ & & \mathbb{C}_K \end{array}$$

which is Γ_K-equivariant.

We hence have
$$K = \overline{K}^{\Gamma_K} \hookrightarrow (B_{\mathrm{dR}}^{+})^{\Gamma_K} \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K}.$$
By (3), we get an injective K-algebra homomorphism
$$\bigoplus (B_{\mathrm{dR}}^{\Gamma_K} \cap t^n B_{\mathrm{dR}}^{+})/(B_{\mathrm{dR}}^{\Gamma_K} \cap t^{n+1} B_{\mathrm{dR}}^{+}) \hookrightarrow B_{\mathrm{HT}}^{\Gamma_K} = K$$
with the last equality following from Tate–Sen Theorem 2.8.14. Since the source has dimension ≤ 1 over K, we have that $\dim_K B_{\mathrm{dR}}^{\Gamma_K} \leq 1$, and hence $B_{\mathrm{dR}}^{\Gamma_K} = K$. □

3.3. **Properties of de Rham representations.**

Definition 3.3.1. A representation $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *de Rham* if it is B_{dR}-admissible, i.e. $\dim_K D_{\mathrm{dR}} = \dim_{\mathbb{Q}_p} V$ where $D_{\mathrm{dR}} = D_{B_{\mathrm{dR}}}$.

We write $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ for the category of de Rham representations.

Example 3.3.2.

(1) The representation $\mathbb{Q}_p(n)$ is de Rham for all $n \in \mathbb{Z}$. Indeed, we have that
$$D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes B_{\mathrm{dR}})^{\Gamma_K} \ni (1 \otimes t^{-n}),$$
so $D_{\mathrm{dR}}(\mathbb{Q}_p(n))$ is not trivial. Hence the inequality $\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p}(\mathbb{Q}_p(n)) = 1$ has to be an equality.

(2) By a result of Sen, every $\mathbb{C}_K$-admissible representation is de Rham. We will not prove this.

(3) If X is a proper smooth variety over K, the representation
$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$$
is a de Rham representation by a theorem of Faltings. We will not prove this.

By the general formalized of B-admissible representations (cf. Theorem 3.1.15)

(1) $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ is closed under taking subquotients, tensors, and duals,
(2) D_{dR} commutes with tensors, duals in Vec_K.

What we want to do next is to upgrade these properties to be compatible with the filtration coming from B_{dR}.

Definition 3.3.3. Define Fil_K to be the category of finite-dimensional filtered vector spaces over K:

(1) the objects are finite-dimensional vector spaces V over K, endowed with $\{\mathrm{Fil}^n(V)\}$ such that:
  (a) $\mathrm{Fil}^n(V) \supseteq \mathrm{Fil}^{n+1}(V)$,

(b) $\bigcap_{n\in\mathbb{Z}} \mathrm{Fil}^n = 0$,
(c) $\bigcup_{n\in\mathbb{Z}} \mathrm{Fil}(V) = V$,

(2) morphisms are K-linear maps $f\colon V \to W$ such that

$$f(\mathrm{Fil}^n(V)) \subseteq \mathrm{Fil}^n(W).$$

Example 3.3.4. If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, the vector space

$$D_{\mathrm{dR}}(V) = (V \otimes B_{\mathrm{dR}})^{\Gamma_K}$$

has a filtration defined by

$$\mathrm{Fil}^n(V) = (V \otimes t^n B_{\mathrm{dR}}^+)^{\Gamma_K}.$$

Hence D_{dR} is a functor into Fil_K.

Remark 3.3.5. Falting's de Rham comparison theorem gives a Γ_K-equivariant isomorphism:

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}), \mathbb{Q}_p) \cong H_{\mathrm{dR}}(X/K),$$

identifying the filtration on the left hand side with the Hodge filtration.

Definition 3.3.6. Let $V \in \mathrm{Fil}_K$. Then

$$\mathrm{gr}(V) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Fil}^n(V)/\mathrm{Fil}^{n+1}(V)$$

is the *associated graded* vector space of V.

Example 3.3.7. By Theorem 3.2.39 (3), we have that $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$

The idea is to study $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\Gamma_K)$ by passing to $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\Gamma_K)$ by taking $\mathrm{gr}(-)$.

Definition 3.3.8. For $V, W \in \mathrm{Fil}_K$, define the *convolution filtration* on $V \otimes_k W$ by

$$\mathrm{Fil}^n(V \otimes W) = \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes \mathrm{Fil}^j(W).$$

Example 3.3.9. The *unit object* is Fil_K is $K[0]$: the vector space K with

$$\mathrm{Fil}^n(K[0]) = \begin{cases} K & \text{if } n \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for all $V \in \mathrm{Fil}_K$,

$$V \otimes K[0] \cong K[0] \otimes V \cong V.$$

Lemma 3.3.10. *For $V, W \in \mathrm{Fil}_K$, a bijective morphism $f\colon V \to W$ is an isomorphism if and only if $\mathrm{gr}(f)\colon \mathrm{gr}(V) \to \mathrm{gr}(W)$ is an isomorphism.*

Proof. The 'only if' implication is obvious. We have to check the 'if' implication.

The map $\mathrm{gr}(f)\colon \mathrm{gr}(V) \to \mathrm{gr}(W)$ is an isomorphism of graded vector space, so

$$\mathrm{Fil}^n(V)/\mathrm{Fil}^{n+1}(V) \cong \mathrm{Fil}^n(W)/\mathrm{Fil}^{n+1}(W).$$

Since f is a bijection, we have that

$$\mathrm{Fil}^n(W) \hookrightarrow \mathrm{Fil}^n(W)$$

for all $n \in \mathbb{Z}$. Now,

$$\dim_k \operatorname{Fil}^n(V) = \sum_{i \leq n} \dim \operatorname{Fil}^i(V)/\operatorname{Fil}^{i+1}(V) = \sum_{i \leq n} \dim \operatorname{Fil}^i(W)/\operatorname{Fil}^{i+1}(W) = \dim_k \operatorname{Fil}^n W,$$

the the map $\operatorname{Fil}^n(W) \hookrightarrow \operatorname{Fil}^n(W)$ is an isomorphism. ☐

Example 3.3.11. Define $K[1]$ to be the filtered vector space whose underlying vector space is K and

$$\operatorname{Fil}^n K[1] = \begin{cases} K & n \leq 1, \\ 0 & n > 1. \end{cases}$$

The map $K[0] \to K[1]$ given by id_K is not an isomorphism. Indeed, the map

$$0 = \operatorname{Fil}^1 K[0] \to \operatorname{Fil}^1 K[1] = K$$

is cannot be an isomorphism. On graded vector spaces, we have that $\operatorname{gr} K[0] \to \operatorname{gr} K[1]$ is the 0 map.

This shows that a bijection on the underlying vector spaces may not be an isomorphism of graded vector spaces.

Lemma 3.3.12. *Let $V \in \operatorname{Fil}_K$. Then there is a basis $\{v_i\}$ for V such that for all $n \in \mathbb{Z}$,*

$$\{v_i\} \cap \operatorname{Fil}^n(V) \text{ is a basis for } \operatorname{Fil}^n(V).$$

Definition 3.3.13. A basis with the above property is called a *filtration oriented basis.*

Proof. Since

$$\operatorname{Fil}^n(V) = V \quad \text{for sufficiently small } n,$$
$$\operatorname{Fil}^n(V) = 0 \quad \text{for sufficiently large } n,$$

we may use induction to extend the basis of $\operatorname{Fil}^n(V)$ to $\operatorname{Fil}^{n-1}(V)$. ☐

Proposition 3.3.14. *For $V, W \in \operatorname{Fil}_K$,*

$$\operatorname{gr}(V \otimes W) \cong \operatorname{gr}(V) \otimes \operatorname{gr}(W).$$

Proof. Let $(v_{i,k})$ be a filtration oriented basis for V and $(w_{j,\ell})$ for a filtration oriented basis of W, where i and j denote the largest filtered pieces they belong to.

Let $(\overline{v_{i,k}})$ and $(\overline{w_{j,\ell}})$ denote their images under the maps $\operatorname{Fil}^i(V) \to \operatorname{gr}^i(V)$ and $\operatorname{Fil}^j(W) \to \operatorname{gr}^j(W)$. Recall that

$$\operatorname{Fil}^n(V \otimes W) = \sum_{i+j=n} \operatorname{Fil}^i(V) \otimes \operatorname{Fil}^j(W)$$

is spanned by

$$\{u_{i,k} \otimes v_{j,\ell} \mid i + j \leq n\},$$

so $\operatorname{gr}^n(V \otimes W)$ is spanned by

$$\{\overline{u_{i,k}} \otimes \overline{v_{j,\ell}} \mid i + j = n\}.$$

The vector space

$$\bigoplus_{i+j=n} \operatorname{gr}^i(V) \otimes \operatorname{gr}^j(W)$$

is also spanned by $\{\overline{u_{i,k}} \otimes \overline{v_{j,\ell}} \mid i + j = n\}$.

This gives a canonical isomorphism

$$\mathrm{gr}^n(V \otimes W) \cong \bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes \mathrm{gr}^j(W),$$

so taking $\bigoplus_{n\in\mathbb{Z}}$ gives the result. ☐

Definition 3.3.15. For $V \in \mathrm{Fil}^K$, the *dual filtration* for $V^\vee$ is defined by

$$\mathrm{Fil}^n(V^\vee) = \{f \in V^\vee \mid \mathrm{Fil}^{1-n}(V) \subseteq \ker(f)\}$$

We use Fil^{1-n}, not Fil^{-n}, to guarantee that $K[0]^\vee \cong K[0]$.

Facts. *We have that*

(1) $(V^\vee)^\vee \cong V$,
(2) $(V \otimes W)^\vee \cong (V^\vee \otimes W^\vee)$.

This finishes the general discussion of filtered representations.

Lemma 3.3.16. *Consider $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$. Then V is de Rham if and only if $V(n)$ is de Rham.*

Proof. Recall that $V(n) \cong V \otimes \mathbb{Q}_p(n)$, so $V \cong V(n) \otimes \mathbb{Q}_p(-n)$. Since $\mathbb{Q}_p(n)$ is de Rham for any $n \in \mathbb{Z}$ and $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\Gamma_K)$ is stable under $\otimes$ (Theorem 3.1.15), the result follows. ☐

Proposition 3.3.17. *If $V \in \mathrm{Rep}^{\Gamma_K}_{\mathbb{Q}_p}$, then V is Hodge–Tate and*

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V).$$

Proof. For any integer n, we have a short exact sequence:

$$0 \longrightarrow t^{n+1}B^+_{\mathrm{dR}} \longrightarrow t^n B^+_{\mathrm{dR}} \longrightarrow \mathbb{C}_K(n) \longrightarrow 0,$$

since $\mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$ (Theorem 3.2.39 (3)). Tensoring with V and taking Γ_K-invariants, we have a left exact sequence:

$$0 \longrightarrow (V \otimes t^{n+1}B^+_{\mathrm{dR}})^{\Gamma_K} \longrightarrow (V \otimes t^n B^+_{\mathrm{dR}})^{\Gamma_K} \longrightarrow (V \otimes \mathbb{C}_K(n))^{\Gamma_K} \longrightarrow 0.$$

This shows that

$$\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \hookrightarrow (V \otimes \mathbb{C}_K(n))^{\Gamma_K}.$$

Taking the sum over all $n \in \mathbb{Z}$, we have that

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow \bigoplus (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = D_{\mathrm{HT}}(V).$$

To check this is an isomorphism, we compute the dimensions:

$$\dim_k D_{\mathrm{dR}}(V) = \dim_k \mathrm{gr}\, D_{\mathrm{dR}}(V) \leq \dim_k D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p}(V).$$

Since V is de Rham, $\dim_k D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$, so all the above inequalities have to be equalities. This shows that V is Hodge–Tate and the injection above is an isomorphism. ☐

Example 3.3.18. Let V be a p-adic representation of Γ_K which fits into the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(\ell) \longrightarrow V \longrightarrow \mathbb{Q}_p(m) \longrightarrow 0$$

where $\ell \neq m$. We claim that V is automatically Hodge–Tate.

Tensoring with $\mathbb{C}_k(n)$, we get a short exact sequence

$$0 \longrightarrow \mathbb{C}_p(\ell+n) \longrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n) \longrightarrow \mathbb{C}_p(m+n) \longrightarrow 0.$$

Taking Γ_K-invariants gives a long exact sequence

$$0 \longrightarrow \mathbb{C}_p(\ell+n)^{\Gamma_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\Gamma_K} \longrightarrow \mathbb{C}_p(m+n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(\ell+n)).$$

By Tate–Sen 2.8.14, we have

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = \begin{cases} K & \text{if } n = -\ell, -m, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\dim_k D_{\mathrm{HT}}(V) = \sum \dim_k (V \otimes \mathbb{C}_K(n))^{\Gamma_K} = 2 = \dim_{\mathbb{Q}_p} V$.

Remark 3.3.19. If $\ell = m = 0$, then V may not be Hodge–Tate. There exists a 2-dimensional representation V over $\mathbb{Q}_p$ where $\gamma \in \Gamma_K$ acts by

$$\begin{pmatrix} 1 & \log_p(\chi(\gamma)) \\ 0 & 1 \end{pmatrix}.$$

In particular, the category $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{HT}}(\Gamma_K)$ is not closed under taking extensions.

Example 3.3.20. Let V be a p-adic representation of Γ_K which fits into the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \longrightarrow V \longrightarrow \mathbb{Q}_p(m) \longrightarrow 0$$

where $n > m$. We claim that V is de Rham.

We may assume that $m = 0$ and $n > 0$. Note that D_{dR} is left-exact by construction. We hence have a sequence:

$$0 \longrightarrow \underbrace{D_{\mathrm{dR}}(\mathbb{Q}_p(n))}_{\dim 1} \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow \underbrace{D_{\mathrm{dR}}(\mathbb{Q}_p)}_{\dim 1}$$

We need check that $\dim D_{\mathrm{dR}}(V) = 2$. This will follow if we show that the map

$$D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K$$

is surjective.

There is a long exact sequence:

$$0 \longrightarrow (\mathbb{Q}_p(n) \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow (V \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow (\mathbb{Q}_p \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mathbb{Q}_p \otimes B_{\mathrm{dR}}^+).$$

Note that:

$$(\mathbb{Q}_p(n) \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (t^n B_{\mathrm{dR}}^+)^{\Gamma_K}$$

since Γ_K acts on t by χ.

Moreover, $(t^n B_{\mathrm{dR}}^+)^{\Gamma_K} = 0$, because we have a commutative diagram

$$\begin{array}{ccc} \overline{K} & \hookrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \downarrow \\ & & \mathbb{C}_K \end{array}$$

and t is a uniformizer so the image of $\overline{K}$ is disjoint from $t^n B_{\mathrm{dR}}^+$.

Also,

$$(\mathbb{Q}_p \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} = (B_{\mathrm{dR}}^+)^{\Gamma_K} = K$$

by Theorem 3.2.39 (4).

Altogether, the long exact sequence above becomes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & (V \otimes B_{\mathrm{dR}}^+)^{\Gamma_K} & \longrightarrow & K & \longrightarrow & H^1(\Gamma_K, \mathbb{Q}_p \otimes B_{\mathrm{dR}}^+) \\ & & & & \big\downarrow & & \big\downarrow = & & \\ & & & & D_{\mathrm{dR}}(V) = (V \otimes B_{\mathrm{dR}})^{\Gamma_K} & \longrightarrow & K & & \end{array}$$

The proof will be complete if we show that $H^1(\Gamma_K, \mathbb{Q}_p \otimes B_{\mathrm{dR}}^+) = 0$.

We start with the short exact sequence

$$0 \longrightarrow t^{n+1} B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow \mathbb{C}_K(n) \longrightarrow 0.$$

The long exact sequence in cohomology gives

$$0 = \mathbb{C}_K(n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(n)) = 0$$

since $n > 0$ and using Tate–Sen 2.8.14.

By induction, this reduces to the case $n = 1$.

We handle this case directly. We consider $\alpha_1 \in H^1(\Gamma_K, B_{\mathrm{dR}}^+)$ and show that $\alpha_1 = 0$. Using the isomorphism above, we get sequences (α_m), y_m such that

(1) $\alpha_m \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+)$, $y_m \in t^m B_{\mathrm{dR}}^+$,
(2) $\alpha_{m+1}(\gamma) = \alpha_m(\gamma) + \gamma(y_m) - y_m$

Since t is a uniformizer in B_{dR}^+, $y = \sum y_m \in B_{\mathrm{dR}}^+$. Then

$$\alpha_1(y) + \gamma(y) - y \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+) \text{ for all } m \geq 1.$$

Therefore,

$$\alpha_1(y) + \gamma(y) - y = 0$$

for any γ, showing that α_1 is a coboundary. Hence $\alpha_1 = 0$, as required.

Example 3.3.21. Let V be a p-adic representation of Γ_K which fits into the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow V \longrightarrow \mathbb{Q}_p(1) \longrightarrow 0.$$

As we saw about, it is Hodge–Tate. However, if the short exact sequence above does not split, it is not de Rham. The proof is not easy, so we omit it here.

It is not hard to show that there exists such a non-split extension using Tate's local duality, but we do not discuss it here either.

Definition 3.3.22. If V is Hodge–Tate, $n \in \mathbb{Z}$ is a *Hodge–Tate weight* of V if

$$\dim_k (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\Gamma_K} \neq 0.$$

Proposition 3.3.23. *If V is de Rham, then n is a Hodge–Tate weight of V if and only if* $\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \neq 0$.

Proof. This is clear since by Proposition 3.3.17, there is an isomorphism of graded algebras $\mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V)$ so $\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \cong (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n))^{\Gamma_K}$. □

Remark 3.3.24. The Hodge–Tate weights are the positions of jumps in the filtration of $D_{\mathrm{dR}}(V)$.

Remark 3.3.25. If X is a smooth proper variety over K, then

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K).$$

Proposition 3.3.26. *If V is de Rham, there is an isomorphism*

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

in Fil_K.

Proof. We have a natural map

$$\begin{aligned} D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} &\to (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \otimes_K B_{\mathrm{dR}} \\ &\overset{\cong}{\to} V \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_K B_{\mathrm{dR}}) \\ &\to V \otimes B_{\mathrm{dR}} && \text{multiplication.} \end{aligned}$$

This is a morphism in Fil_K. To show that it is an isomorphism, we just need to show that it induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \to \mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})$$

by Lemma 3.3.10.

We have that

$$\begin{aligned} \mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) &\cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \otimes_K B_{\mathrm{dR}} && \text{by Proposition 3.3.14} \\ &\cong B_{\mathrm{HT}} \otimes B_{\mathrm{HT}} && \text{3.3.17.} \end{aligned}$$

Moreover,

$$\begin{aligned} \operatorname{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) &\cong V \otimes \operatorname{gr}(B_{\mathrm{dR}}) && \text{by Proposition 3.3.14} \\ &\cong V \otimes B_{\mathrm{HT}} && 3.3.17. \end{aligned}$$

We hence get an induced map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \to V \otimes_K B_{\mathrm{HT}},$$

which is an isomorphism because V is Hodge–Tate (by Proposition 3.3.17). □

Proposition 3.3.27. *The functor*

$$D_{\mathrm{dR}} \colon \operatorname{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\Gamma_K) \to \operatorname{Fil}_K$$

is faithful and exact.

Proof. Since D_{dR} is faithful with values in the category Vec_K and the forgetful functor $\operatorname{Fil}_K \to \operatorname{Vec}_K$ is faithful, the above functor is also faithful.

To show exactness, consider a short exact sequence of de Rham representations:

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

For any $n \in \mathbb{Z}$, consider the left exact sequence

$$0 \longrightarrow \operatorname{Fil}^n(D_{\mathrm{dR}}(U)) \longrightarrow \operatorname{Fil}^n(D_{\mathrm{dR}}(V)) \longrightarrow \operatorname{Fil}^n(D_{\mathrm{dR}}(W)).$$

We want to show that this sequence is also right exact.

Since U, V, W are de Rham, they are also Hodge–Tate (Proposition 3.3.17). We get an exact sequence

$$0 \longrightarrow D_{\mathrm{HT}}(U) \longrightarrow D_{\mathrm{HT}}(V) \longrightarrow D_{\mathrm{HT}}(W) \longrightarrow 0$$

of graded vector spaces (by the general formalism, cf. Theorem 3.1.15).

By Proposition 3.3.17, we get a short exact sequence:

$$0 \longrightarrow \operatorname{gr}^n(D_{\mathrm{dR}}(U)) \longrightarrow \operatorname{gr}^n(D_{\mathrm{dR}}(V)) \longrightarrow \operatorname{gr}^n(D_{\mathrm{dR}}(W)) \longrightarrow 0.$$

Finally,

$$\begin{aligned} \dim_K \operatorname{Fil}^n(V) &= \sum_{i \geq n} \dim_K \operatorname{gr}^i(D_{\mathrm{dR}}(V)) \\ &= \sum_{i \geq n} \dim_K \operatorname{gr}^i(D_{\mathrm{dR}}(U)) + \dim_K \operatorname{gr}^i(D_{\mathrm{dR}}(W)) \\ &= \dim_K \operatorname{Fil}^n(D_{\mathrm{dR}}(U)) + \dim_K \operatorname{Fil}^n(D_{\mathrm{dR}}(W)). \end{aligned}$$

Hence the left exact sequence

$$0 \longrightarrow D_{\mathrm{HT}}(U) \longrightarrow D_{\mathrm{HT}}(V) \longrightarrow D_{\mathrm{HT}}(W) \longrightarrow 0$$

must be exact. □

Corollary 3.3.28. *If V is a de Rham representation, every W subquotient of V is de Rham and $D_{\mathrm{dR}}(W)$ is naturally a subquotient of $D_{\mathrm{dR}}(V)$.*

Proof. Since W is de Rham by Theorem 3.1.15, we deduce the assertion from Proposition 3.3.27. □

Proposition 3.3.29. *For $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$, $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with a natural isomorphism of filtered vector spaces*

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W) \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W).$$

Proof. By Theorem 3.1.15, this assertion is true in the category of vector spaces. By construction, we can check that the natural map

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W) \to D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)$$

is a morphism in Fil^K. To check it is an isomorphism, we pass to the graded vector spaces; cf. Lemma 3.3.10. We want to show that

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)) \cong \mathrm{gr}(D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)).$$

We have that

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \otimes \mathrm{gr}(D_{\mathrm{dR}}(W)) \cong D_{\mathrm{HT}}(V) \otimes D_{\mathrm{HT}}(W)$$

by Propositions 3.3.14 and 3.3.17. Similarly,

$$\mathrm{gr}(D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)) \cong D_{\mathrm{HT}}(V \otimes_{\mathbb{Q}_p} W),$$

and we know that $D_{\mathrm{HT}}(V) \otimes D_{\mathrm{HT}}(W) \cong D_{\mathrm{HT}}(V \otimes_{\mathbb{Q}_p} W)$ by Theorem 3.1.15. □

Proposition 3.3.30. *If V is de Rham, then $\bigwedge^n V$, $\mathrm{Sym}^n V$ are both de Rham and*

$$\bigwedge^n (D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}\left(\bigwedge^n V\right)$$

$$\mathrm{Sym}^n(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}\left(\mathrm{Sym}^n V\right)$$

in Fil_K.

Proof. Once again, by Theorem 3.1.15, this assertion is true in the category of vector space. Since $\otimes$ and quotients commute with D_{dR} in Fil^k by the above results. □

Proposition 3.3.31. *If V is de Rham, then $V^\vee$ is de Rham with a natural perfect pairing*

$$D_{\mathrm{dR}}(V) \otimes D_{\mathrm{dR}}(V^\vee) \cong D_{\mathrm{dR}}(V \otimes V^\vee) \to D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K[0]$$

in Fil_K.

Proof. Once again, by Theorem 3.1.15, we get the above perfect pairing in the category of vector spaces. By the above results, each map is a morphism in Fil_K, and we can check it is an isomorphism by passing to associated gradeds (Lemma 3.3.10). We have that

$$\begin{aligned} \mathrm{gr}(D_{\mathrm{dR}}(V)^\vee) &\cong D_{\mathrm{HT}}(V^\vee) \\ &\cong D_{\mathrm{HT}}(V)^\vee \\ &\cong \mathrm{gr}(D_{\mathrm{dR}}(V))^\vee \end{aligned}$$

which completes the proof. □

We have hence showed that all the results of Theorem 3.1.15 hold for $B = \mathrm{dR}$ with D_{dR} valued in the category of filtered vector spaces.

We discuss some further properties of de Rham representations.

Proposition 3.3.32. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ and K'/K be a finite extension so that $\Gamma_{K'} \subseteq \Gamma_K$. Then:*

(1) $D_{\mathrm{dR},K} \otimes_K K' \cong D_{\mathrm{dR},K'}(V)$ *in* $\mathrm{Fil}_{K'}$,
(2) *V is de Rham if and only if V is de Rham as a representation of $\Gamma_{K'}$.*

Proof. We only have to check the first assertion. Note that B_{dR} only depends on $\mathbb{C}_K$ and $\mathbb{C}_K \cong \mathbb{C}_{K'}$, we have a natural map:

$$(V \otimes B_{\mathrm{dR}})^{\Gamma_K} = D_{\mathrm{dR},K} \otimes_K K' \to D_{\mathrm{dR},K'}(V) = (V \otimes_{B_{\mathrm{dR}}})^{\Gamma_{K'}}$$

in Fil_K. We need to check that

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) \otimes_K K' \xrightarrow{\cong} \mathrm{Fil}^n(D_{\mathrm{dR},K'}(V)).$$

By definition of the filtration:

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) \otimes_K K' = (V \otimes t^n B_{\mathrm{dR}}^+)^{\Gamma_K} \otimes_K K'$$
$$\mathrm{Fil}^n(D_{\mathrm{dR},K'}(V)) = (V \otimes t^n B_{\mathrm{dR}}^+)^{\Gamma_{K'}}$$

By passing to the Galois closure of K', we may assume that K'/K is Galois. Then:

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR},K'}(V))^{\mathrm{Gal}(K'/K)}.$$

We are hence done by Galois descent. □

Remark 3.3.33. We only prove this when K' is a finite extension of K. In fact, this holds for any complete discretely valued extension K'/K. The main example to keep in mind is $K' = \widehat{K^{\mathrm{un}}}$.

Corollary 3.3.34. *If V is 1-dimensional, then V is de Rham if and only if V is Hodge–Tate.*

Proposition 3.3.35. *The functor D_{dR} is not full.*

Proof. Consider any potentially trivial representation V, i.e. there exists a finite extension K'/K such that V is trivial as a representation of $\Gamma_{K'}$. Then V is de Rham by Proposition 3.3.32.

However,

$$D_{\mathrm{dR}}(V)_{K'} \cong D_{\mathrm{dR},K'}(V) \cong K'[0]$$

in $\mathrm{Fil}_{K'}$. Hence $D_{\mathrm{dR}}(V) \cong K[0]$, because at filtration level n the above isomorphisms give:

$$\mathrm{Fil}^n D_{\mathrm{dR}}(V)_{K'} \cong \mathrm{Fil}^n D_{\mathrm{dR},K'}(V) \cong \begin{cases} K' & \text{if } n < 0, \\ 0 & \text{if } n \geq 0. \end{cases}$$

Hence the functor is not full. □

We end the discussion of de Rham representations by discussing the Fontaine–Mazur conjecture.

By de Rham comparison theorem, for any proper smooth variety X over K, the representation $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham.

Question. Does every de Rham representation come from geometry? I.e. for any de Rham representation V, do there exist integers n, m and a proper smooth variety X over X such that V is a subquotient of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(m)$?

The answer is no in general. However, for a global number field, we have the following conjectural criterion for representations to be *geometric*.

Conjecture 3.3.36 (Fontaine–Mazur)**.** *Let E be a number field and $\mathcal{O}_E$ be the ring of integers of E. Consider a finite-dimensional representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}, E)$ over $\mathbb{Q}_p$ such that:*

(1) *V is unramified (i.e. $I_{E_{\mathfrak{p}}}$ acts trivially) at all but finitely many primes of $\mathcal{O}_E$*
(2) *for any prime $\mathfrak{p}$ over p in $\mathcal{O}_E$, the representation $V|_{\mathrm{Gal}(\mathbb{Q}_p/E_{\mathfrak{p}})}$ is de Rham.*

Then there exists a proper smooth variety X over E such that V is a subquotient of $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)(m)$.

Remark 3.3.37. Very little is known about this conjecture. We know:

- when $\dim V = 1$, it is true by class field theory,
- when $\dim V = 2$, it is known in many cases by the work of Kisin.

3.4. **Crystalline representations.** The goal is to study the period ring B_{cris} and crystalline representations. So far, the only result we assumed was the Tate–Sen Theorem 2.8.14 (and one smaller result about the new topology on B^+_{dR}). In this section, we will starting assuming more results without proof.

3.4.1. *Crystalline period ring.* Recall the following notation:

- $F = \mathbb{C}_K^\flat$,
- $A_{\mathrm{inf}} = W(\mathcal{O}_F)$,
- $p^\flat \in \mathcal{O}_F$ such that $(p^\flat)^\# = p$,
- $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$, a generator of $\ker(\theta)$ where $\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$,
- k is the residue field of $\mathcal{O}_K$,
- $W(k)$ is the ring of Witt vectors over k,
- $K_0 = \mathrm{Frac}(W(k))$.

Definition 3.4.1. Define

$$A_{\text{cris}} = \left\{ \sum_{n\geq 0} a_n \frac{\xi^n}{n!} \in B_{\text{dR}}^+ \mid a_n \in A_{\text{inf}},\ a_n \to 0 \right\} \subseteq B_{\text{dR}}^+$$
$$B_{\text{cris}}^+ = A_{\text{cris}}[1/p].$$

Remark 3.4.2.

(1) We will always work on the new topology on B_{dR}^+.
(2) The definition is different from the original one by Fontaine.

Proposition 3.4.3. *The element $t = \log([\epsilon])$ belongs to A_{cris} and $t^{p-1} \in pA_{\text{cris}}$.*

Proof. Recall that $t = \log([\epsilon])$ where $\epsilon \in \mathcal{O}_F$ satisfies $\epsilon^\# = 1$, $\epsilon \neq 1$. Hence $\epsilon = (\zeta_{p^n})$ for p^nth roots of unity ζ_{p^n}. We checked that $[\epsilon] - 1 \in \ker(\theta)$, so $[\epsilon] - 1 \in \xi A_{\text{inf}}$, i.e. $[\epsilon] - 1 = \zeta \cdot c$ for some $c \in A_{\text{inf}}$.

We want to show that $t \in A_{\text{cris}}$. We have that

$$\begin{aligned} t &= \log([\epsilon]) \\ &= \sum_{n\geq 1} (-1)^{n+1} \frac{([\epsilon]-1)^n}{n} \\ &= \sum_{n\geq 1} (-1)^{n+1} (n-1)! c^n \frac{\xi^n}{n!} \\ &\in \mathbb{A}_{\text{cris}} \end{aligned}$$

since $(n-1)!c^n \to 0$ as $n \to \infty$.

We now want to show that $t^{p-1} \in pA_{\text{cris}}$. Consider the truncation

$$\check{t} = \sum_{n=1}^{p} (-1)^{n+1} \frac{([\xi]-1)^n}{n}.$$

Note that $(n-1)!$ is divisible by p for all $n > p$, and hence

$$t = \check{t} + p \cdot a$$

for some $a \in A_{\text{cris}}$. We only need to check that

$$(\check{t})^{p-1} \in pA_{\text{cris}}.$$

For $1 \leq n \leq p-1$, $(-1)^{n+1} \dfrac{([\epsilon]-1)^n}{n}$ is divisible by $[\epsilon] - 1$ in A_{cris}. For $n = p$, we have that:

$$(-1)^{p+1} \frac{([\epsilon]-1)^p}{p} = (-1)^{p+1} \frac{([\epsilon]-1)^{p-1}}{p} \cdot ([\epsilon]-1).$$

Hence

$$\check{t} = ([\epsilon]-1) \left(a + (-1)^{p-1} \frac{([\epsilon]-1)^{p-1}}{p} \right)$$

for some a. We only need to show that:

$$([\epsilon]-1)^{p-1} \in pA_{\text{cris}}.$$

We know that $[\epsilon]-1=[\epsilon-1]$ in pA_{inf}. It is hence enough to show that

$$[(\epsilon-1)^{p-1}] \in pA_{\text{cris}}.$$

Recall that $v^\flat(\epsilon-1)=\frac{p}{p-1}$. Hence

$$v^\flat((\epsilon-1)^{p-1})=p=pv^\flat(p^\flat)=v^\flat((p^\flat)^p),$$

so

$$([\epsilon-1]^{p-1}) \text{ is divisible by } [(p^\flat)^p]=(\xi+p)^p.$$

Finally,

$$\xi^p=p(p-1)!\frac{\xi^p}{p!}.$$

Since we know that $\frac{\xi^p}{p!}\in A_{\text{cris}}$, this shows that $\xi^p\in pA_{\text{cris}}$. □

Corollary 3.4.4. *We have that $B_{\text{cris}}^+[1/t]\cong A_{\text{cris}}[1/t]$.*

Proof. Since $t^{p-1}\in pA_{\text{cris}}$, p is a unit in $A_{\text{cris}}[1/t]$. Hence $B_{\text{cris}}^+[1/t]=A_{\text{cris}}[1/p,1/t]=A_{\text{cris}}[1/t]$. □

Definition 3.4.5. The *crystalline period ring* B_{cris} is defined as $B_{\text{cris}}^+[1/t]=A_{\text{cris}}[1/t]$.

Remark 3.4.6. Where is this construction coming from? The motivation for B_{cris} is Grothendieck's mysterious functor conjecture. He conjectured there is a functor D such that

$$D(H^n_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p))=H^n_{\text{cris}}(\mathcal{X}/W(k))$$

where X is proper and smooth with good reduction.

The idea is to define $D=D_{B_{\text{cris}}}$ for a period ring which is a subring of B_{dR} with a natural Frobenius action in line with Frobenius action on crystalline cohomology.

They key observation is that we can get such a ring by adjoining to A_{inf} *divided powers* of ξ, i.e. elements $\frac{\xi}{n!}$. These are the elements we considered in the definition of B_{cris}.

The class continued remotely from here on, but I stopped typing the notes.

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