MATH 711: REPRESENTATION THEORY OF SYMMETRIC GROUPS

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These are notes from Math 711: Representation Theory of Symmetric Groups taught by Professor Andrew Snowden in Fall 2017, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from December 18, 2017. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The first part of the course will be devoted to the representation theory of symmetric groups and the main reference for this part of the course is [Jam78]. Another standard reference is [FH91], although it focuses on the characteristic zero theory. There is not general reference for the later part of the course, but specific citations have been provided where possible.

Contents

Introduction. Why study the representation theory of symmetric groups?

(1) We have a natural isomorphism

$$V \otimes W \xrightarrow{\sim} W \otimes V$$

$$v \otimes w \longrightarrow w \otimes v$$

We then get a representation of S_2 on $V \otimes V$ for any vector space V given by $\tau(v \otimes w) = w \otimes v$. More generally, we get a representation of S_n on $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$. If V is

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a representation of G, then $V^{\otimes n}$ is a representation of $G \otimes S_n$. Hence representations of S_n arise naturally when studying any representations, especially when considering tensor products.

(2) Suppose X is a topological space and let $\mathrm{Conf}_n(X)$ be the *configuration space of n points in X*, defined by

$$X^n \setminus \{\text{locus where two coordinates are equal}\}.$$

The symmetric group S_n acts natrually on this configuration space, giving a representation of S_n on $H^i(\mathrm{Conf}_n(X), \mathbb{C})$.

(3) We actually do not need a real reason, the subject is fundamental in itself.

1. Preliminaries

Let G be a group, k be a field, and V be a vector space over k.

Definition 1.1. A *representation* of G on V is any (and all) of the following

- A homomorphism $G \to \mathrm{GL}(V)$, the group of automorphisms of V,
- A linear action of G on V, i.e. a map $G \times V \to V$ such that
  - $(gh)v = g(hv)$,
  - $1 \cdot v = v$,
  - $g(\alpha v + \beta w) = \alpha(gv) + \beta(gw)$.
- A $k[G]$-module structure on V, where $k[G]$ is the *group algebra*:

$$k[G] = \left\{ \sum_{g \in G} a_g[g] : a_g \in k, a_g = 0 \text{ for all but finitely many } g \right\}$$

$$\left(\sum_g a_g[g]\right)\left(\sum_h b_h[h]\right) = \left(\sum_{g,h} a_g b_h [gh]\right).$$

If V, W are representations of G, we can form the following constructions

- $V \oplus W$ is a representation of G by $g(v \oplus w) = gv \oplus gw$,
- $V \otimes W$ is a representation of G by $g(v \otimes w) = gv \otimes gw$,
- V^* is a representation of G by $(g\lambda)(v) = \lambda(g^{-1}v)$,
- $\mathrm{Hom}(V, W) = \{\text{all linear maps } V \to W\}$ is a representation of G by
$$(gf)(v) = gf(g^{-1}v),$$
- $\mathrm{Hom}_G(V, W) = \{\text{all linear maps } f\colon V \to W \text{ s.t. } f(gv) = gf(v) \text{ for all } v \in V\}$,
- *invariant subspace*:
$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\},$$
- *coinvariant subspace*:
$$V_G = \frac{V}{\mathrm{span}(gv - v \mid g \in G, v \in V)}.$$

Fact 1.2.

- $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$.
- $(V^*)^G = (V_G)^*$.
- *Not true that* $(V^G)^* = (V^*)_G$ *in general, but true if* $\dim(V) < \infty$.

Definition 1.3.

- The *trivial representation* of G is $V = k$, $gv = v$ for all $g \in G$, $v \in V$.
- The *left regular representation* of G is $V = k[G]$ and G acts by left multiplication. For the *right regular representation*,
$$g \cdot \left(\sum_h a_h h \right) = \left(\sum_h a_h h \right) g^{-1}.$$
- If G acts on a set X, the *permutation representation* associated to X is $V = k[X]$ (vector space with basis X), and G acts by left multiplication
$$g \cdot \sum_{x \in X} a_x [x] = \sum_{x \in X} a_x [gx].$$

Definition 1.4.

- A representation V of G is *irreducible* if $V \neq 0$ and the only subrepresentations of V are 0 and V (or, equivalently, V is a simple $k[G]$-module).
- A representation V is *semi-simple* (or *completely reducible*) if it is a direct sum of simple representations.

Fact 1.5. *Any subrepresentation or quotient representation of a semi-simple representation is semi-simple.*

Proposition 1.6. *The following are equivalent:*

- *Every representation of* G *is semi-simple.*
- *If* V *is a representation of* G, $W \subseteq V$ *subrepresentation, then there exists a subrepresentation* $W' \subseteq V$ *such that* $V = W \oplus W'$.
- *Every extension of representations splits, i.e. if*
$$0 \longrightarrow V_1 \xrightarrow{\ i\ } V_2 \xrightarrow{\ p\ } V_3 \longrightarrow 0$$
is an exact sequence of representations then there exists a map $s \colon V_3 \to V_2$ *such that* $ps = \mathrm{id}_{V_3}$ *(called a splitting).*
- *Every representation of* G *is projective (injective).*

Definition 1.7. An R-module M is *projective* if every exact sequence
$$0 \longrightarrow N \longrightarrow N' \longrightarrow M \longrightarrow 0$$
splits.

Proposition 1.8. *The following are equivalent:*

(1) Every representation of G *is semi-simple.*
(2) The trivial representation is projective.

Proof. By Proposition 1.6, (1) implies (2). Conversely, assume (2) and let V be a representation of G with subrepresentation $W \subseteq V$. We then have a surjective map

$$\mathrm{Hom}(V, W) \twoheadrightarrow \mathrm{Hom}(W, W).$$

Taking G-invariants, we get a map

$$\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_G(W, W).$$

As the trivial representation is projective, we get a map

$$\begin{array}{ccc} & & k \cdot \mathrm{id}_W \\ & \swarrow & \downarrow \\ \mathrm{Hom}(V, W) & \twoheadrightarrow & \mathrm{Hom}(W, W) \end{array}$$

Thus the map $\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_G(W, W)$ is surjective and the identity id_W comes from a map $s \in \mathrm{Hom}_G(V, W)$, which provides a splitting for $W \subseteq V$. □

Theorem 1.9. *Suppose G is finite and $|G| \neq 0$ in k. Then every representation of G is semi-simple.*

Proof. By Proposition 1.8, we need to show the trivial representation is projective, which is equivalent to showing that if $f \colon V \to W$ is a surjection of representations, then $V^G \to W^G$ is surjective.

Given $w \in W^G$, pick $v_0 \in V$ such that $f(v_0) = w$. Now, define

$$v = \frac{1}{|G|} \sum_{g \in G} g v_0.$$

For all $h \in G$ we then have

$$\begin{aligned} hv &= \frac{1}{|G|} \sum_{g \in G} (hg) v_0 \\ &= \frac{1}{|G|} \sum_{g \in G} g v_0 \\ &= v \end{aligned}$$

and so $v \in V^G$. Finally,

$$\begin{aligned} f(v) &= \frac{1}{|G|} \sum_{g \in G} g f(v_0) \\ &= \frac{1}{|G|} \sum_{g \in G} w \\ &= w, \end{aligned}$$

completing the proof. □

Remark 1.10. Theorem 1.9 is **not** true if $|G| = 0$ in k. For example, let $G = \mathbb{Z}/p$, $k = \mathbb{F}_p$, $V = \mathbb{F}_q^2 = \mathbb{F}_p e_1 \oplus \mathbb{F}_p e_2$ with action of $a \in G$ given by the matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Then there exists a short exact sequence

$$0 \longrightarrow \mathrm{triv} \longrightarrow V \longrightarrow \mathrm{triv} \longrightarrow 0$$

but $V \ncong \mathrm{triv} \oplus \mathrm{triv}$.

Remark 1.11. In general, if $p = \mathrm{char}(k) \big| |G|$, then $k[G]$ is not semi-simple. We show that $k[G]^G \cong k$. If $x = \sum_{h \in G} a_h h$, then

$$gx = \sum_{h \in G} a_{g^{-1}h} h = \sum_{h \in H} a_h h,$$

so x is G-invariant if and only if $a_{g^{-1}h} = a_h$ for all g. Thus x is a scalar multiple of $y = \sum_{g \in G} g$. Hence indeed

$$k[G]^G = ky.$$

We define the *augmentation map* by

$$\epsilon \colon k[G] \to k, \ \epsilon(g) = 1.$$

Then $\epsilon(y) = |G| = 0$ in this case, and hence ϵ does not split.

Suppose for now k and G are arbitrary. Let $\{L_i\}_{i \in I}$ be the representatives of isomorphism classes of irreducible representations of G. Suppose V is a semi-simple representation. Then there exists an isomorphism

$$f \colon \bigoplus_{i \in I} L_i^{\oplus n_i} \to V$$

for some n_i's. This isomorphism is **not** canonical.

Fact 1.12. *The embedding* $f(L_i^{\oplus n_i}) \subseteq V$ **is** *canonical.*

It is the sum of images of all maps $L_i \to V$, called the L_i*-isotypic piece* of V. Thus there exists a canonical decomposition

$$V = \bigoplus_{i \in I} (L_i\text{-isotypic piece of } V),$$

and each L_i-isotypic piece of V admits a non-canonical decomposition as $L_i^{\oplus n_i}$ for some n_i.

Proposition 1.13 (Schur's Lemma)**.** *Suppose V and W are irreducible representations.*

1. *Any G-map $V \to W$ is either 0 or an isomorphism.*
2. $\mathrm{End}_G(V)$ *is a division algebra.*
3. *If V is finite dimensional and k is algebraically closed then* $\mathrm{End}_G(V) = k$.

Proof. For (1), say $f \neq 0$. Then $\ker(f) \subsetneq V$, so $\ker(f) = 0$ as V is irreducible, and $0 \neq \mathrm{im}(f) \subseteq W$ so $\mathrm{im}(f) = W$. Then (2) follows from (1). For (3), let $f \colon V \to V$ be a G-endomorphism. Let λ be an eigenvalue of f. Then

$$\ker(f - \lambda \mathrm{id}) \neq 0,$$

so $f = \lambda \mathrm{id}$. □

Example 1.14 ((3) does not hold in general)**.** Let $G = \mathbb{C}^*$ and consider $\mathbb{C}$ as a 2-dimensional real representation. Then V is irreducible. But $\mathrm{End}_G(V) = \mathbb{C}$. The same example works for $G = \{1, i, -1, -i\} \cong \mathbb{Z}/4$.

Remark 1.15. There are examples where $\text{End}_G(V)$ is genuinely not a field. For instance, try to think of a representation whose endomorphism ring is the quaternions.

Assume $k = \bar{k}$ and let V be a finite-dimensional isotypic representation of G (so there is only 1 non-zero isotypic piece). Say L_i is the irreducible in V.

Definition 1.16. The L_i *multiplicity space* in V is $\text{Hom}_G(L_i, V)$.

Fact 1.17. *If V isotypic, then the canonical map*

$$\text{Hom}_G(L_i, V) \otimes L_i \to V$$

is an isomorphism.

To show this, reduce to the case where V is a direct sum of L_i's and then to V being just L_i, in which case, $\text{Hom}_G(L_i, V) = k$ by Schur's Lemma 1.13 (3), so the isomorphism is $k \otimes L_i \to V$.

Let V be a finite-dimensional semi-simple representation and M_i be the multiplicity space of L_i, i.e. $M_i = \text{Hom}_G(L_i, V)$. Then the canonical map

$$\bigoplus_{i \in I} M_i \otimes L_i \to V$$

is an isomorphism.

Remark 1.18. If V is not semisimple, the map is still injective and its image is the *socle* of V, the maximal semi-simple subrepresentation of V.

Proposition 1.19. *Let k be algebraically closed and G, H be groups. Any finite-dimensional irreducible representation of $G \times H$ has the form $V \otimes W$ with V an irreducible representation of G and W an irreducible representation of H. Conversely, any representation of this form is irreducible.*

Proof. We first show that $V \otimes W$ is irreducible. Let $U \subseteq V \otimes W$ be a subrepresentation. As a representation of H, $V \otimes W$ is isotypic and its W-multiplicity space is V, so

$$V \cong \text{Hom}_H(W, V) \otimes W.$$

Now, $\text{Hom}_H(W, V)$ is a G-subrepresentation of V, so it is 0 or V, and hence V is 0 or $V \otimes W$.

Suppose U is a finite-dimensional irreducible representation of $G \times H$. Let W be an irreducible representation of H contained in U. We then have an injective map

$$\underbrace{\text{Hom}_H(W, V)}_{\text{a rep. of } G} \otimes W \to U$$

and it is actually surjective as U is irreducible and the map is non-zero. □

We now assume G is finite, $k = \bar{k}$, $|G| \neq 0$ in k. The goal is to understand $k[G]$, the left regular representation of G.

We first think of $k[G]$ as a representation of $G \times G$, with action defined by

$$(g, h) \cdot x = gxh^{-1}$$

(so both the left and the right regular representations).

We know that $k[G]$ is a semi-simple representation of $G \times G$.

Lemma 1.20. *Let V be a representation of $G \times G$. We have that*

$$\mathrm{Hom}_{G\times G}(k[G], V) = V^G,$$

where $G \subseteq G \times G$ is the diagonal copy of G.

Proof. The image of 1 under any map in $\mathrm{Hom}_{G\times G}(k[G], V)$ gives an element of V^G. The map defined this way is the isomorphism. □

Therefore, we have that

$$\mathrm{Hom}_{G\times G}(k[G], L_i \otimes L_j) = (L_i \otimes L_j)^G = \mathrm{Hom}_G(L_i^*, L_j) = \begin{cases} k & \text{if } L_j \cong L_i^*, \\ 0 & \text{otherwise,} \end{cases}$$

where the last equality follows from Schur's Lemma 1.13. Therefore, as $G\times G$-representations:

$$k[G] \cong \bigoplus_{i\in I} L_i \otimes L_i^*.$$

In fact, we get a canonical isomorphism

$$k[G] \to \bigoplus_{i\in I} \mathrm{End}(L_i).$$

By considering these vector spaces as representations of G, we get the following theorem.

Theorem 1.21. *The left regular representation $k[G]$ of G decomposes as $\bigoplus_{i\in I} L_i^{\oplus \dim L_i}$.*

Corollary 1.22. *We have that*

$$\sum_{i\in I} \dim(L_i)^2 = |G|.$$

Remark 1.23. These results are sometimes proven using character theory.

Induction. Let $R \to S$ be a ring homomorphism. By restriction of modules, we get a functor:

$$\mathrm{Mod}_S \to \mathrm{Mod}_R.$$

It has a left and a right adjoint:

- The left adjoint is *extension of scalars* $S \otimes_R -$:
$$\mathrm{Hom}_S(S \otimes_R M, N) = \mathrm{Hom}_R(M, N).$$
- The right adjoint is *co-extension of scalars* $\mathrm{Hom}_R(S, -)$:
$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M).$$

Now, suppose $H \subseteq G$ is a subgroup, and left $R = k[H]$, $S = k[G]$. Then we get a functor

$$\mathrm{Res}^G_H\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H).$$

Its left adjoint is

$$\mathrm{Ind}^G_H(V) = k[G] \otimes_{k[H]} V,$$

i.e. we have

$$\mathrm{Hom}_G(\mathrm{Ind}^G_H(V), W) = \mathrm{Hom}_H(V, \mathrm{Res}^G_H(W)).$$

Its right adjoint is

$$\mathrm{CoInd}_H^G(V) = \mathrm{Hom}_{k[H]}(k[G], V),$$

i.e. we have

$$\mathrm{Hom}_G(W, \mathrm{CoInd}_H^G(V)) = \mathrm{Hom}_H(\mathrm{Res}_H^G(W), V).$$

The two adjunction statements are called *Frobenius reciprocity.*

Fact 1.24.

- *If* $[H : G] < \infty$ *then* $\mathrm{Ind}_H^G \cong \mathrm{CoInd}_H^G$.
- *Both* Ind_H^G *and* CoInd_H^G *are exact.*
- *Induction is transitive: if* $K \subseteq H \subseteq G$*, then*

$$\mathrm{Ind}_H^G \mathrm{Ind}_K^H = \mathrm{Ind}_K^G .$$

The proof is left as an exercise.

Examples 1.25.

- $\mathrm{Ind}_{\{1\}}^G(k) = k[G]$, where k is the trivial representation of $H = \{1\}$
- $\mathrm{Ind}_H^G(k) = k[G/H]$, the permutation representation of G acting on G/H
- $\mathrm{Ind}_H^G(k[H]) = k[G]$

Character theory. Let V be a finite-dimensional representation of G. The *character* of V is the function $\chi_V : G \to k$ given by $\chi_V(g) = \mathrm{tr}(g|V)$. This is a *class function*, i.e. $\xi_V(hgh^{-1}) = \xi_V(g)$.

Remark 1.26. These characters are only interesting in characteristic zero. In positive characteristic, one has to use *Brauer characters.*

Now assume $k = \mathbb{C}$ and G is finite.

Fact 1.27.

- $\chi_{V \oplus W} = \chi_V + \chi_W$
- $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
- $\chi_{V^*} = \overline{\chi_V}$
- $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$

Proof. Over $k = \mathbb{C}$, the trace is the sum of the eigenvalues, and the first three formulas follow immediately. The final formula follows from the previous ones by noting $\mathrm{Hom}(V, W) \cong V^* \otimes W$. □

For $\varphi, \psi : G \to \mathbb{C}$, put

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g).$$

Proposition 1.28. *Let V, W be finite-dimensional representations of G. Then*

$$\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_G(V, W).$$

Proof. First assume V is the trivial representation. Define $p\colon W \to W$ by

$$p(w) = \frac{1}{|G|} \sum_{g \in G} gw,$$

which give a projection of W onto W^G. Hence

$$\dim \operatorname{Hom}_G(\mathrm{triv}, W) = \dim W^G = \operatorname{tr}(p) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g|W) = \langle \chi_{\mathrm{triv}}, \chi_W \rangle.$$

In general, we have that

$$\operatorname{Hom}_G(V, W) = \operatorname{Hom}_G(\mathrm{triv}, \operatorname{Hom}(V, W))$$

and hence using the previous part, we obtain

$$\dim \operatorname{Hom}_G(V, W) = \langle \chi_{\mathrm{triv}}, \chi_{\operatorname{Hom}(V,W)} \rangle = \langle \chi_{\mathrm{triv}}, \overline{\chi_V} \chi_W \rangle = \langle \chi_V, \chi_W \rangle,$$

completing the proof. □

Corollary 1.29. *If V, W are irreducible, then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

Proof. This follows from Proposition 1.28 and Schur's Lemma 1.13. □

Corollary 1.30. *The set $\{\chi_V \mid V \text{ irreducible}\}$ is a basis for the space of class functions on G.*

Thus, the data about representations of a group can be arranged into a *character table* of G:

	— conjugacy classes of G —
| irreducible representations of G |	

2. Representations of symmetric groups

Every element σ of S_n admist a decomposition $\sigma = C_1 \dots C_r$, where the C_i's are disjoint cycles and every element of $\{1, \dots, n\}$ appears once.

A *partition* of n is an unordered collection of positive integers summing to n:

$$n = m_1 + \cdots + m_r,$$

where we typically take $m_1 \geq m_2 \geq \cdots \geq m_r$.

We associate to σ the partition

$$p(\sigma) = \#C_1 + \cdots + \#C_r$$

of n. (The notation $p(\sigma)$ is not standard but we assume it temporarily.) Note that

$$\tau\sigma\tau^{-1} = C_1^\tau \cdots C_r^\tau$$

and hence $p(\tau\sigma\tau^{-1}) = p(\sigma)$. We therefore get a map

$$p\colon \{\text{conjugacy classes in } S_n\} \to \{\text{partitions of } n\}.$$

Fact 2.1. *This is a bijection.*

Corollary 2.2. *The number of conjugacy classes in S_n is the number of partitions of n.*

Corollary 2.3. *The number of complex irreducible representations of S_n is the number of partitions of n.*

Proof. This follows from Corollary 2.2 and Corollary 1.30. □

Examples 2.4.

- The trivial representation.
- We have the sign homomorphism
  $$\text{sgn}\colon S_n \to \{\pm 1\},$$
  defining a 1-dimensional *sign (or alternating) representation.*
- Since S_n naturally permutes the set $\{1, \ldots, n\}$, we get a permutation representation of S_n on $\mathbb{C}^n$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, then $\sigma(e_i) = e_{\sigma(i)}$. Then $\mathbb{C}^n$ is the *permutation representation* of S_n.
- $\displaystyle\sum_{i=1}^{n} e_n$ is S_n-invariant, and hence $\mathbb{C}^n$ is not irreducible unless $n = 1$.
- Define $\epsilon\colon \mathbb{C}^n \to \mathbb{C}$ by $\epsilon(e_i) = 1$ (the *augmentation map*). Then $\ker(\epsilon)$ is an S_n-subrepresentation. Hence
  $$\mathbb{C}^n = \ker(\epsilon) \oplus \mathbb{C}.$$

Proposition 2.5. *For $n > 1$, $\ker(\epsilon)$ is irreducible.*

Proof. Let $V \subseteq \ker(\epsilon)$ be a non-zero subrepresentation. Let

$$v = \sum_{i=1}^{r} a_i e_i$$

be a nonzero element of V with minimal r (so no a_i is 0 after permuting the e_i's).

Trivially, $r \neq 1$ or otherwise $V = 0$. If $r = 2$, v is a scalar multiple of $e_1 - e_2$, which generates $\ker(\epsilon)$, so $V = \ker(\epsilon)$.

Suppose $r > 2$. There exist $1 \leq i, j \leq r$ such that $a_i \neq a_j$. We may assume that $i = r - 1$, $j = r$. We then note that

$$a_{r-1}v - a_r(r\ r-1)v \in V$$

but it is equal to

$$\underbrace{(a_{r-1}-a_r)\sum_{i=1}^{r-2} a_i e_i}_{\neq 0} + (a_{r-1}^2 - a_r^2)e_{r-1}.$$

This contradicts the minimality of r. □

Definition 2.6. The representation $\ker(\epsilon)$ is called the *standard representation* of S_n.

We list the irreducible representation of S_n for small n.

n	number	list	notes
1	1	triv	
2	2	triv, sgn = std	
3	3	triv, sgn, std	In particular, we can conclude that $\mathrm{sgn}\otimes\mathrm{std}\cong\mathrm{std}$, because our list is complete.
4	5	triv, sgn, std, $\mathrm{sgn}\otimes\mathrm{std}$, ?	To see that $\mathrm{sgn}\otimes\mathrm{std}\ncong\mathrm{std}$, we compute the characters: $\chi_{\mathrm{std}}((12))=1,\ \chi_{\mathrm{std}\otimes\mathrm{sgn}}((12))=-1$. We note that there is one irreducible representation still missing.

Remark 2.7. Note that if G acts on X, the character of the permutation representation at $g\in G$ is the number of fixed points of g on X. We can use that to compute the character of the standard representation.

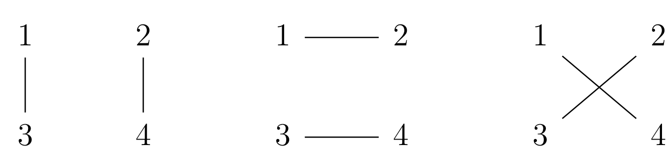
We write down the character table of S_4 for the representations that we know.

partition	1^4	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
permutation	1	(12)	(12)(34)	(123)	(1234)
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
std	3	1	-1	0	-1
$\mathrm{std}\otimes\mathrm{sgn}$	3	-1	-1	0	1
?					

Definition 2.8. A *perfect matching* on a set S is a graph (undirected with no loops) in which every vertex belongs to exactly 1 edge.

Let $\mathcal{M}_n = \{\text{matchings on } \{1,2,\dots,n\}\}$. Clearly, S_n acts on $\mathcal{M}_n$, so we get the permutation representation $\mathbb{C}[\mathcal{M}_n]$.

Let us think about the $n=4$ case. The set $\mathcal{M}_4$ has 3 elements:



Let $\epsilon\colon C[\mathcal{M}_4] \to \mathbb{C}$ be the augmentation map. Is $V = \ker \epsilon$ irreducible? To check, we compute the character by looking at how many matchings are fixed by the consecutive permutations and subtracting 1:

- 1 fixes all 3 matchings,
- (12) fixes only the second matching,
- (12)(34) fixes all 3 matchings,
- (123) does not fix any matchings,
- (1234) fixes only the first matching.

Hence the character χ_V of V is

partition	1^4	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
permutation	1	(12)	(12)(34)	(123)	(1234)
χ_V	2	0	2	-1	0

Looking at the value on (123), we see that χ_V is not a positive linear combination of triv and sgn, and it is 2-dimensional, so it is irreducible.

Thus the complete character table of S_4 is

partition	1^4	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
permutation	1	(12)	(12)(34)	(123)	(1234)
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
std	3	1	-1	0	-1
std $\otimes$ sgn	3	-1	-1	0	1
χ_V	2	0	2	-1	0

We will find more irreducible representations of S_n inside of permutation representations. For example, for a subgroup $H \subseteq S_n$, we can look at the permutation representation associated to the action of S_n on the cosets S_n/H. Hence we want a good source of subgroups of S_n.

Definition 2.9. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n, the *Young subgroup* S_λ of S_n is

$$S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r} \subseteq S_n.$$

We can now consider the permutation representation $\mathbb{C}[S_n/S_\lambda]$. Recall that the number of irreducible representations of S_n is the number of partitions of S_n, so we get the correct number of representations; however, they will not be irreducible.

A λ-*partition* of $\{1, 2, \dots, n\}$ is a decomposition

$$\{1, 2, \dots, n\} = \coprod_{i=1}^{r} A_i \text{ with } \#A_i = \lambda_i.$$

The set of λ-partitions has a natural S_n-action, and if

$$P = \coprod_{i=1}^{r} \left\{ \sum_{j=1}^{i} \lambda_j + 1, \dots, \sum_{j=1}^{i} \lambda_j + \lambda_{i+1} \right\}$$

then S_λ is the stabilizer of P. Thus

$$\mathbb{C}[S_n/S_\lambda] = \mathbb{C}[\text{set of all } \lambda\text{-partitions}].$$

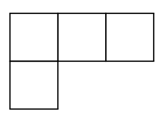
(This is sometimes called the generalized orbit-stablizer theorem.)

Examples 2.10.

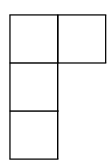
- For $\lambda = (n)$, $S_\lambda = S_n$, and $\mathbb{C}[S_n/S_\lambda] = \text{triv}$.
- For $\lambda = (1^n)$, $S_\lambda = \{1\}$, and $\mathbb{C}[S_n/S_\lambda]$ is the regular representation.
- For $\lambda = (n-1, 1)$, $S_\lambda = S_{n-1}$, and $\mathbb{C}[S_n/S_\lambda] = \mathbb{C}^n = \text{triv} \oplus \text{std}$.
- For $\lambda = (2, 2)$, $n = 4$, $S_\lambda = S_2 \times S_2$, and $\mathbb{C}[S_n/S_\lambda] = \text{triv} \oplus \text{std} \oplus V$, where $V = \ker \epsilon$ for the augmentation map $\epsilon \colon \mathbb{C}[\mathcal{M}_4] \to \mathbb{C}$.
  Note that $\mathbb{C}[S_n/S_\lambda]$ has a representation of $S_4 \times S_2$. The decomposition as a representation of $S_4 \times S_2$,
  $$[(\text{triv} \oplus V) \boxtimes \text{triv}] \oplus [(\text{std} \boxtimes \text{sgn})],$$
  where by $V \boxtimes W$ we mean the representation of $S_4 \times S_2$ obtained from tensoring V as a representation of S_4 with W as a representation of S_2. (We use $\boxtimes$ to distinguish this from tensoring two representations of $S_4 \times S_2$.)

Definition 2.11. The *Young diagram* of a partition λ is a diagram with λ_1 boxes in the first row, λ_2 in the second and so on.

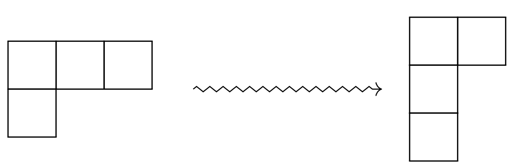
Examples 2.12. The Young diagram of $\lambda = (3, 1)$ is



The Young diagram of $\lambda = (2, 1, 1)$ is



Note that Young diagrams have a symmetry: we can flip along the diagonal line:



This gives an involution on the set of partitions , called the *conjugation* or *transpose*, $\lambda \mapsto \lambda^\dagger$. Explicitly,

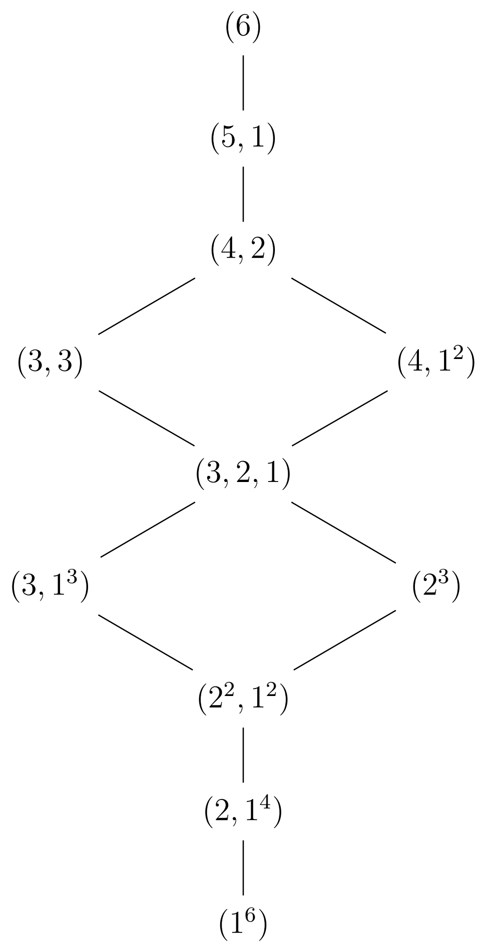
$$(\lambda^\dagger)_i = \max j \text{ such that } \lambda_j \geq i.$$

Definition 2.13. If λ, μ are two partitions, we say that λ *dominates* μ and write $\lambda \rhd \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k \text{ for all } k.$$

This defined a partial ordering on the set of partitions.

Example 2.14. The partitions of 6 form the following graph where λ is above μ if and only if $\lambda \rhd \mu$:



We make the following observation:

$$\mathbb{C}[S_n/S_\lambda] = (\text{some irreducible}) \oplus \left(\begin{array}{c} \text{sum of irreducibles,} \\ \text{each of which appears} \\ \text{in some } \mathbb{C}[S_n/S_\mu] \\ \text{with } \mu \rhd \lambda \end{array} \right)$$

The evidence for this is given by Examples 2.10.

Examples 2.15.

- $\mathbb{C}[S_n/S_{n-1}] = \text{std} \oplus \text{triv}$, where triv appears in $\mathbb{C}[S_n/S_n]$, and std is new.
- $\mathbb{C}[S_4/S_{(2,2)}] = \underbrace{V}_{\text{new}} \oplus \underbrace{\text{std}}_{\mathbb{C}[S_4/S_{(3,1)}]} \oplus \underbrace{\text{triv}}_{\mathbb{C}[S_4/S_{(4)}]}.$

We will see later that this observation is actually true.

Definition 2.16. A λ-*tableau* is a way of filling the Young diagram of λ with numbers $1, \dots, n$ such that each number appears once.

Examples 2.17. The following are two λ-tableux obtained from the same Young diagram

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Definition 2.18. A λ-*tabloid* is the same, but the order within the rows is irrelevant.

A λ-tabloid is drawn with only horizontal lines.

Example 2.19. The first two λ-tabloids are the same, while the last one is different:

$$\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{ccc} \hline 1 & 3 & 2 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{ccc} \hline 4 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Note that S_n permutes the λ-tableaux and λ-tabloids, and the associated representations are:

$$\mathbb{C}[\lambda\text{-tableaux}] = \text{regular representation},$$
$$M^\lambda := \mathbb{C}[\lambda\text{-tabloids}] = \mathbb{C}[S_n/S_\lambda].$$

Recall that we expect M^λ to contain some simple L^λ, and all others appear in M^μ for some $\mu \rhd \lambda$, $\mu \neq \lambda$.

Example 2.20 ($n = 4$)**.** For the different partitions λ of 4, we get the following simple modules L^λ:

- $M^{(4)} = \text{triv}$, so $L^{(4)} = \text{triv}$,
- $M^{(3,1)} = \text{triv} \oplus \text{std}$, so $L^{(3,1)} = \text{std}$,
- $M^{(2,2)} = \text{triv} \oplus \text{std} \oplus V$, so $L^{(2,2)} = V$,
- $M^{(2,1,1)} = \text{triv} \oplus \text{std}^{\oplus 2} \oplus V \oplus (\text{std} \otimes \text{sgn})$, so $L^{(2,1,1)} = \text{std} \otimes \text{sgn}$,
- $M^{(1^4)} = \mathbb{C}[S_4]$, so $L^{(1^4)} = \text{sgn}$.

Recall that by Corollary 2.3 there exists a bijection between irreducibles and partitions—this example shows a specific bijection between them.

Note that
$$M^\lambda = \text{Ind}_{S_\lambda}^{S_n}(\text{triv}).$$
We also define
$$N^\lambda = \text{Ind}_{S_{\lambda^\dagger}}^{S_n}(\text{sgn}) = M^{\lambda^\dagger} \otimes \text{sgn}\,.$$

Example 2.21 ($n = 4$)**.** We have:

- $N^{(1^4)} = \text{sgn} = L^{(1^4)}$
- $N^{(2,1,1)} = \text{sgn} \oplus \text{std} \oplus \text{sgn} = L^{(1^4)} \oplus L^{(2,1,1)}$

It appears that $N^\lambda = L^\lambda \oplus (\bigoplus L^\mu)$ with $\mu \lhd \lambda$, whereas $M^\lambda = L^\lambda \oplus (\bigoplus L^\mu)$ with $\mu \rhd \lambda$.

Based on this observation, we expect there to be a unique map (up to scaling) $N^\lambda \to M^\lambda$, whose image will be L^λ.

Think of elements of N^λ as tableaux of shape λ such that permuting a column changes by sign of that permutation ("*signed tabloids*"):

$$\left(\begin{array}{c|c|c} 1 & 3 & 5 \\ 2 & 4 & \end{array} \right) = - \left(\begin{array}{c|c|c} 2 & 3 & 5 \\ 1 & 4 & \end{array} \right)$$

We will define an averaging map to get the following diagram

$$N^\lambda \xrightarrow{\text{averaging map}} \mathbb{C}[\text{tableaux}] \longrightarrow M^\lambda$$

Let t be a tableau. Define

- $\{t\} \in M^\lambda$ the tabloid defined by t,
- $\{t\}' \in N^\lambda$ the signed tabloid defined by t,
- C_t the column stabilizer of t, i.e. the subgroup of S_n fixing columns of t,
- R_t the row stablizer of t,
- $\kappa_t = \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\sigma \in \mathbb{C}[S_n]$.

Note that

$$M^\lambda = \mathbb{C}[\lambda\text{-tabloids}] = \mathbb{C}[\lambda\text{-tableaux}]/\langle \sigma t - \sigma \mid \sigma \in R_t,\ t\ \lambda\text{-tableau}\rangle,$$

$$N^\lambda = \mathbb{C}[\lambda\text{-tableuax}]/\langle \sigma t - \operatorname{sgn}(\sigma) t \mid \sigma \in C_t,\ t\ \lambda\text{-tableau}\rangle.$$

We define the averaging map by

$$\{t\}' \mapsto \kappa_t t$$

for any t. In other words, a tableau t goes $e_t = \kappa_t\{t\} \in M^\lambda$, which is a *polytabloid*.

The image of $N^\lambda \to M^\lambda$ is the span of the e_t's. It is called the *Specht module* and denoted S^λ.

Example 2.22. Let t be

2	4	1
3	5	

Then

$$C_t = \langle (23), (54) \rangle \subseteq S_5$$

and so

$$\kappa_t = 1 - (23) - (54) + (23)(54).$$

Thus e_t is

$$\begin{array}{l} \hline 2\ \ 4\ \ 1 \\ \hline 3\ \ 5 \\ \hline \end{array} - \begin{array}{l} \hline 3\ \ 4\ \ 1 \\ \hline 2\ \ 5 \\ \hline \end{array} - \begin{array}{l} \hline 2\ \ 5\ \ 1 \\ \hline 3\ \ 4 \\ \hline \end{array} + \begin{array}{l} \hline 3\ \ 5\ \ 1 \\ \hline 2\ \ 4 \\ \hline \end{array}$$

Lemma 2.23. *Let λ, μ be partitions of n, t a λ-tableau, t' a μ-tableau. Suppose that, for all i, all numbers in the ith row of t' appear in different columns in t. Then*

$$\lambda \trianglerighteq \mu$$

Proof. We rearrange the numbers step by step:

(1) We can use C_t to move all numbers in the first row of t' into the first row of t. Thus $\lambda_1 \geq \mu_1$.
(2) We can use C_t to move all numbers in the second row of t' to the first or second row of t. Thus $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$.

Continuing this way, we get the result. □

Lemma 2.24. *Let λ and μ be partitions of n, t a λ-tableau, t' a μ-tableau. Suppose*

$$\kappa_t\{t'\} \neq 0.$$

Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then $\kappa_t\{t'\} = \pm e_t$.

Proof. Let a and b be two numbers in the same row of t'. Then $(a\ b)\{t'\} = \{t'\}$. Hence a, b are in different columns of t, or otherwise $(a\ b) \in C_t$ and we have

$$\kappa_t\{t'\} = (a\ b)\kappa_t\{t'\} = \operatorname{sgn}((a\ b))\kappa_t\{t'\} = -\kappa_t\{t'\},$$

contradicting $\kappa_t\{t'\} \neq 0$. By Lemma 2.23, $\lambda \trianglerighteq \mu$.

If $\lambda = \mu$, then $t' \in C_t t$ by the above argument, and so $\kappa_t\{t'\} = \pm\kappa_t\{t\} = \pm e_t$. □

Corollary 2.25. *Given $m \in M^\lambda$ and a λ-tableau t, $\kappa_t m \in \mathbb{C}e_t$.*

Proof. By definition, m is a linear combination of $\{t'\}$ for λ-tableaux t'. Moreover, we know that $\kappa_t\{t'\} = 0$ or $\pm e_t$ by Lemma 2.24, which completes the proof. □

Definition 2.26. Define a symmetric bilinear inner product $\langle -, - \rangle$ on M^λ by

$$\langle \{t\}, \{t'\} \rangle = \delta_{\{t\},\{t'\}} = \begin{cases} 1 & \text{if } \{t\} = \{t'\} \\ 0 & \text{otherwise.} \end{cases}$$

Note that this product is invariant under S_n.

Proposition 2.27. *Suppose $V \subseteq M^\lambda$ is a subrepresentation. Then $V \supseteq S^\lambda$ or $V \subseteq (S^\lambda)^\perp$.*

Proof. Let $v \in V$. If $\kappa_t v \neq 0$ then $e_t \in V$ by Corollary 2.25, and so $e_{t'} \in V$ for all t', since e_t's form a single orbit under S_n. Thus $S^\lambda \subseteq V$.

Otherwise, $\kappa_t v = 0$ for any $v \in V$ and t, and we obtain

$$0 = \langle \kappa_t v, \{t\} \rangle = \langle v, \underbrace{\kappa_t\{t\}}_{e_t} \rangle = \langle v, e_t \rangle.$$

Thus $v \in (S^\lambda)^\perp$, showing that $V \supseteq (S^\lambda)^\perp$. □

Corollary 2.28. *The Specht module S^λ is irreducible.*

Proof. Suppose $V \subsetneq S^\lambda$ is a subrepresentation. By Proposition 2.27, $V \subseteq (S^\lambda)^\perp$. Then

$$V \subseteq S^\lambda \cap (S^\lambda)^\perp = 0$$

because everything is defined over $\mathbb{R}$ and $\langle -, - \rangle$ is positive definite over $\mathbb{R}$. (Alternatively, we can define the inner product above to be Hermitian instead and use this.) $\square$

We therefore have

$$M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$$

and the first summand is irreducible.

Lemma 2.29. *Suppose $f \colon M^\lambda \to M^\mu$ is a map of representations and $S^\lambda \not\subset \ker(f)$. Then $\lambda \trianglerighteq \mu$ and if $\lambda = \mu$, then f_{S_λ} is a scalar.*

Proof. Let t be a λ-tableau. If $e_t \notin \ker(f)$, then

$$0 \neq f(e_t) = f(\kappa_t\{t\}) = \kappa_t f(\{t\}).$$

Since $f\{t\}$ is a linear combination of μ-tabloids, we can use Lemma 2.24 to conclude that $\lambda \trianglerighteq \mu$.

If $\lambda = \mu$ then $\kappa_t f(\{t\}) \in \mathbb{C}e_t$ by Lemma 2.24. Thus $f(e_t)$ is a scalar multiple of e_t for any t, and since S_n acts transitively on e_t, it must be the same scalar α, so $f(v) = \alpha v$ for any $v \in S^\lambda$. $\square$

Corollary 2.30. *We have that $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$ and*

$$(S^\lambda)^\perp = \text{sum of } (S^\mu)'s \text{ for } \mu \triangleright \lambda,\ \mu \neq \lambda.$$

Furthermore, every irreducible of S_n is isomorphic to some S^λ.

Proof. We claim that if $S^\lambda \cong S^\mu$, then $\lambda = \mu$. Indeed, choose $f \colon M^\lambda \to M^\mu$ by extending this isomorphism (this is possible because the characteristic is 0):

$$\begin{array}{ccc} M^\lambda & \xrightarrow{f} & N^\lambda \\ \uparrow & & \uparrow \\ S^\lambda & \xrightarrow{\cong} & S^\mu \end{array}$$

By Lemma 2.29, $\lambda \trianglerighteq \mu$. By symmetry, $\mu \trianglerighteq \lambda$, and hence $\lambda = \mu$.

Thus every irreducible of S_n is S^λ, since there are the correct number of them (Corollary 2.3).

Finally, suppose S^μ appears in $(S^\lambda)^\perp$. Then choose an f extending the inclusion, i.e. so that

$$\begin{array}{ccc} M^\mu & \xrightarrow{f} & M^\lambda \\ \uparrow & & \uparrow \\ S^\mu & \hookrightarrow & (S^\lambda)^\perp \end{array}$$

commutes. By Lemma 2.29, $\mu \trianglerighteq \lambda$ and we cannot have $\mu = \lambda$ since $S^\lambda \cap (S^\lambda)^\perp = 0$. $\square$

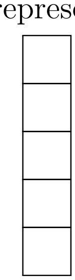
Remark 2.31. We will use the following fact in the next propositon. Recall that
$$M^\lambda = \mathbb{C}[S_n/S_\lambda] = \mathrm{Ind}_{S_\lambda}^{S_n}(\mathrm{triv}),$$
and so by Frobenius reciprocity
$$\mathrm{Hom}_{S_n}(M^\lambda, V) = \mathrm{Hom}_{S_\lambda}(\mathrm{triv}, V) = V^{S_\lambda}$$
via the map
$$(f\colon M^\lambda \to V) \mapsto f(\{t\}) \in V^{S_\lambda}$$
(noting that $S_\lambda = R_t$, the row stabilizer).

Proposition 2.32. *We have that* $S^\lambda \otimes \mathrm{sgn} = S^{\lambda^\dagger}$.

Proof. Fix a λ-tableau and let $t^\dagger$ be the transpose tableau. Then there is a unique S_n-equivariant map
$$f\colon M^{\lambda^\dagger} \to S^\lambda \otimes \mathrm{sgn}$$
$$f(\{t^\dagger\}) = e_t \otimes 1.$$
because $e_t \otimes 1$ is invariant under $R_{t^\dagger} = C_t$ (note: e_t is skew-invariant under C_t, so $e_t \otimes 1$ is invariant). Then
$$f(e_{t^\dagger}) = f(\kappa_{t^\dagger}\{t^\dagger\}) = \kappa_{t^\dagger}(e_t \otimes 1) = (\rho_t e_t) \otimes 1.$$
We show that $\rho_t e_t \neq 0$:
$$\langle \rho_t e_t, \{t\}\rangle = \langle e_t, \rho_t\{t\}\rangle = |R_t|\langle e_t, \{t\}\rangle = |R_t| \neq 0.$$
Hence we have shown the map $f_{S^{\lambda^\dagger}}\colon S^{\lambda^\dagger} \to S^\lambda \otimes \mathrm{sgn}$ is non-zero, and since both sides are irreducible, it gives an isomorphism by Schur's Lemma 1.13. ☐

Example 2.33.

- $S^{(n)} = \mathrm{triv}$, since $M^{(n)} = \mathrm{triv}$.
- $S^{(1^n)} = \mathrm{sgn}$, since $M^{(1^n)}$ is the regular representation and its elements have the shape



  with n boxes, so $C_t = S_n$, and e_t projects $\{t\} \in \mathbb{C}[S_n]$ to sgn component.
- We claim that $S^{(n-1,1)} = \mathrm{std}$. First, $M^{(n-1,1)}$ is naturally the permutation representation $\mathbb{C}^n$ via
$$\begin{array}{l}\hline \cdots\ \ \cdots\ \ \cdots \\ \hline i \\ \hline\end{array} \mapsto v_i, \text{ the } i\text{th basis vector of } \mathbb{C}^n.$$
  If t is the tableau
$$\begin{array}{|c|c|c|c|}\hline i & & \cdots & \\ \hline j \\ \cline{1-1}\end{array}$$
  Then

$$e_t = \begin{array}{|c|c|c|c|}\hline i & & \cdots & \\ \hline j \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|c|}\hline j & & \cdots & \\ \hline i \\ \cline{1-1}\end{array}$$

and hence e_t corresponds to $v_j - v_i \in \ker \epsilon$.

We know $\sigma e_t = \operatorname{sgn}(\sigma) e_t$ for $\sigma \in C_t$. But note that $\sigma e_t \neq e_t$ for $\sigma \in R_t$, in general. We now want to find more linear relations between the e_t's.

Example 2.34. For the regular representation, we have the relation

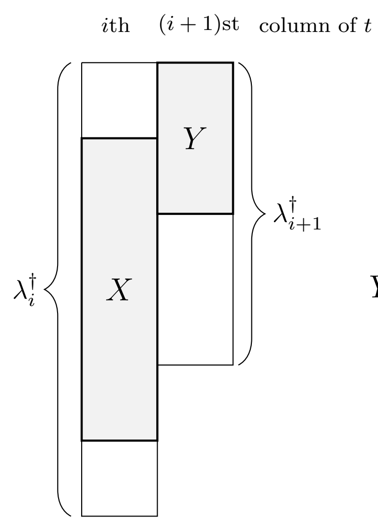
$$(v_k - v_j) + (v_j - v_i) = (v_k - v_i).$$

and we get the following linear relation in $S^{(n-1,1)}$:

$$\begin{array}{|c|c|c|}\hline j & i & \cdots \\ \hline k \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline i & i & \cdots \\ \hline j \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline i & j & \cdots \\ \hline k \\ \cline{1-1}\end{array} = 0.$$

after applying e.

Let us generalize this to arbitrary S^λ. Let t be a tableau of shape λ. Pick sets X and Y as follows:



$$X \subseteq \{\text{numbers in } i\text{th column}\}$$

$$Y \subseteq \{\text{numbers in } (i+1)\text{st column}\}$$

Define

$$G_{X,Y} = \sum_{\sigma \in S_{X \cup Y}} (\operatorname{sgn} \sigma)\sigma,$$

called a *Garnir element*.

Proposition 2.35. *Assume* $|X \cup Y| \geq \lambda_i^\dagger$. *Then* $G_{X,Y} e_t = 0$.

This is called a *Garnir relation*.

Proof. Let $\sigma \in C_t$. There exist $a, b \in X \cup Y$ such that a and b are in the same row of σt (by the pigeon hole principle). Then

$$(a\ b)\{\sigma t\} = \{\sigma t\}$$

and hence

$$G_{X,Y}\{\sigma t\} = 0$$

because $(a\ b) \in S_{X\cup Y}$. Since e_t is a linear combination of $\{\sigma t\}$'s with $\sigma \in C_t$, we get $G_{X,Y} e_t = 0$. □

Remark 2.36. Note that $S_X \times S_Y \subseteq C_t$, so it acts on e_t through sgn. Thus $G_{X,Y}$ acts by

$$|S_X \times S_Y| \cdot \underbrace{\sum_{\sigma \in S_{X\cup Y}/S_X \times S_Y} (\operatorname{sgn}\sigma)\sigma}_{\text{Garnir element}},$$

where the sum is over a set of coset representatives.

Example 2.37.

- Let t be the tableau

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

  and $X = \{1, 2\}$, $Y = \{3\}$. Then

$$S_{X\cup Y}/S_X \times S_Y = \{1, (23), (13)\}$$

  and the Garnir relation is

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} = 0$$

  (after applying e).
- Let t be the tableau

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array}$$

  and $X = \{1, 2, 3\}$, $Y = \{4\}$. Then

$$S_{X\cup Y}/S_X \times S_Y = \{1, (14), (24), (34)\}$$

  and the Garnir relation is

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & 6 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}$$

  (after applying e).
- For $X = \{2, 3\}$, $Y = \{4, 5\}$, we get a relation with 6 terms.

Definition 2.38. A tableau t is *standard* if rows and columns of t are increasing.

Example 2.39. The tableau

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

is standard, while

1	4	6
2	3	
5		

is not standard.

Theorem 2.40. *The set* $\{e_t\}_{t \text{ standard}}$ *is a basis of* S^λ*.*

Proof. We define a total order $<$ on tabloids. Let $r_i(\{t\})$ be the row in which i appears in t. Define $\{t\} < \{t'\}$ if

$$(r_n(\{t\}), r_{n-1}(\{t\}), \dots) < (r_n(\{t'\}), r_{n-1}(\{t'\}), \dots)$$

in the lexicographical order, i.e., explicitly, $\{t\} < \{t'\}$ if

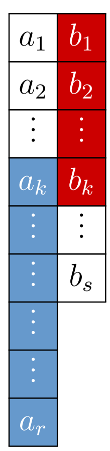
- $r_n(\{t\}) < r_n(\{t'\})$,
- **or** $r_n(\{t\}) = r_n(\{t'\})$ **and** $r_{n-1}(\{t\}) < r_{n-1}(\{t'\})$,
- **or** ...

We observe that if t is standard, then $\{t\}$ is the biggest tabloid (with respect to $<$) appearing in $C_t\{t\}$. Therefore, $\{e_t\}_{t \text{ standard}}$ are linearly independent in S^λ, since

$$e_t = \{t\} + (\text{smaller things under } <).$$

We need to show that for any tableau t, e_t is a linear combination of $e_{t'}$ with t' standard. We do this by induction on t, using the *transpose* of the total order $<$.

Fix t. We may assume that the columns of t are increasing and that t is not standard. We can find adjact columns where a row decreases: suppose $a_k > b_k$ in the following adjacent columns



Let X be the blue set above and Y be the red set above. We apply the Garnir relation on X, Y. This writes e_t as a linear combination of $e_{t'}$s, where t' is bigger than t in our order (because as get moved into b slots). This completes the proof by induction. $\square$

Corollary 2.41. *For any partition* λ*,* $\dim S^\lambda = \#\{$*standard* λ*-tableaux*$\}$*.*

Moreover, by the proof of Theorem 2.40 we also get the following result.

Corollary 2.42. *Every e_t is a $\mathbb{Z}$-linear combination of $e_{t'}$s with t' standard. Thus the matrix of any $\sigma \in S_n$ in S^λ with respect to the standard basis has integer entries.*

Example 2.43. Let $\lambda = (3,2)$. The standard tableaux are

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Thus $\dim S^{(3,2)} = 5$. By taking transpositions, we note that $S^{(2,2,1)} = S^{(3,2)} \otimes \text{sgn}$, so $\dim(2,2,1) = 5$.

Example 2.44. Let $\lambda = (3,1,1)$. The picture is

$$\begin{array}{|c|c|c|}\hline 1&x_1&x_2\\\hline y_1\\\cline{1-1} y_2\\\cline{1-1}\end{array}$$

and choosing any x_1 and x_2 determines y_1 and y_2. Thus $\dim S^{(3,1,1)} = \binom{4}{2} = 6$.

We now know what the irreducible representations of S_n are and how to compute their dimensions. However, we have no relation between the representations of S_n and $S_k \subseteq S_n$ for $k < n$.

Important question. Let λ be a partition of n. How does $S^\lambda|_{S_{n-1}}$ decompose?

Answer. We will show $S^\lambda|_{S_{n-1}} = \bigoplus_{\mu \subset \lambda} S^\mu$, where $\mu \subset \lambda$ is containment of Young diagrams.

Let $r_1 < \cdots < r_k$ be the rows of a tableau obtained from removing a box from λ. Define a filtration

$$F^i S^\lambda = \text{span of all } e_t\text{'s where the number } n \text{ appears in row } \leq r_i \text{ in } t.$$

Clearly, $F^i S^\lambda$ is S_{n-1}-stable.

Lemma 2.45. *The space $F^i S^\lambda$ has a basis $\{e_t\}$ where t is standard and n appears in row $\leq i$.*

Proof. Linear independence follows from Theorem 2.40. Suppose t is not standard and n is in row $\leq r_i$. Applying a Garnir relation, similarly as in the proof of Theorem 2.40, writes e_t as a linear combination of $e_{t'}$s where t' still has the property that n is in row $\leq r_i$. □

Lemma 2.46. *As S_{n-1}-representations, we have that*

$$F^i S^\lambda / F^{i-1} S^\lambda \cong S^{\lambda(i)},$$

where $\lambda(i)$ is λ with a box in row r_i removed.

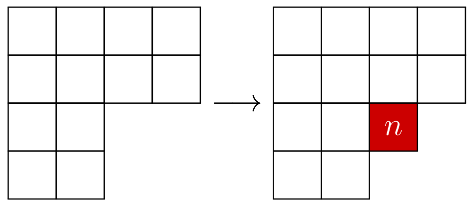
Proof. Define a linear map $f\colon S^{\lambda(i)} \to F^i S^\lambda / F^{i-1} S^\lambda$ by

$$f(e_t) = e_{t'}$$

where t is a standard $\lambda(i)$-tableau and t' is t with n placed in missing box. This is clearly an isomorphism of vector spaces. We have to show it is S_{n-1}-equivariant. We only present a sketch of this proof.

We want to show $f(e_{\sigma t}) = \sigma f(e_t) = \sigma e_{t'} = e_{\sigma t'}$ for $\sigma \in S_{n-1}$ and t standard.

The map does the following



To see where it maps $e_{\sigma t}$, we have to use Garnir relations to express it in terms of standard $\lambda(i)$-tableau. The point is that the Garnir relations are the same on either side when working modulo $F^{i-1}S_\lambda$, which gives the desired equivariance. □

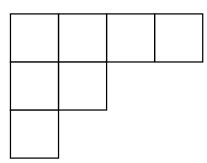
Since we are in characteristic 0, we have a splitting

$$F^i S^\lambda \cong (F^{i-1} S^\lambda) \oplus (F^i S^\lambda / F^{i-1} S^\lambda),$$

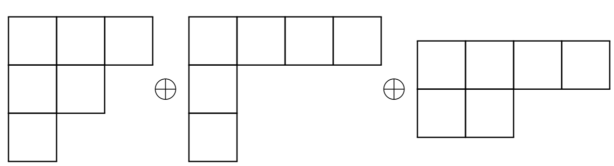
and hence

$$S^\lambda|_{S_{n-1}} = \bigoplus_{\substack{\mu \subset \lambda \\ |\mu| = n-1}} S^\mu.$$

Example 2.47. The Specht module associated to the tableau



restricted to S_6 splits as



Remark 2.48. We could have attempted to prove the above using the map f mapping e_t to $e_{t'} \in F^i S^\lambda$.

$$\begin{array}{ccc} S^{\lambda(i)} & \xrightarrow{\sim} & F^i S^\lambda / F^{i-1} S^\lambda \\ & \searrow^{f} & \uparrow\!\uparrow \\ & & F^i S^\lambda \end{array}$$

However, the map f is not S_{n-1}-equivariant.

Indeed, consider $f\colon S^{(2)} \to S^{(2,1)}$ mapping e_t to $e_{t'}$. In the discussion below, to simplify notation, we will identify t with its image e_t in S^λ. The map f is defined by

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

But (12) applied to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ gives $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ which splits as $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ in $F^1 S^\lambda$.

Proposition 2.49 (Pieri's rule)**.** *Let μ be a partition of $n-1$. Then*

$$\operatorname{Ind}_{S_{n-1}}^{S_n} S^\mu = \bigoplus_{\substack{\mu\subset\lambda \\ |\lambda|=n}} S^\lambda .$$

Proof. By Frobenius reciprocity, the multiplicity of S^λ in $\operatorname{Ind}_{S_{n-1}}^{S_n}(S^\mu)$ is the multiplicity of S^μ in $S^\lambda|_{S_{n-1}}$, which is

$$\begin{cases} 1 & \text{if } \mu \subset \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

completing the proof. ☐

2.1. **The Grothendieck group.** Let G be a finite group and k be a field.

Definition 2.50. The *representation ring* of G over k, denoted $R_k(G)$, is

$$\frac{\text{free } \mathbb{Z}\text{-module on symbols } [V] \text{ for } V \text{ finite-dimensional representation of } G}{\begin{array}{c}[V_2] = [V_1] + [V_3] \text{ whenever we have a short exact sequence} \\ 0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0\end{array}}$$

We define multiplication on this ring by $[V][W] = [V \otimes_k W]$.

Exercise. Verify this works: $R_k(G)$ is a well-defined, commutative, associative, unital ring.

Remark 2.51. To motivate the definition, we make the following observations.

- If $\operatorname{char}(k) = 0$, then $V_2 = V_1 \oplus V_3$ if we have a short exact sequence above, so $+$ is simply $\oplus$.

- A *generalized Euler characteristic* for representations of G is a way to assign a number $\chi(V)$ to a representation V of G such that $\chi(V_2) = \chi(V_1) + \chi(V_3)$ if we have a short exact sequence above. Hence giving an Euler characteristic χ is the same as giving a homomorphism $\chi\colon R_k(G) \to \mathbb{C}$.

More generally, we can make the following construction for any (essentially small) abelian category $\mathcal{A}$.

Definition 2.52. The *Grothendieck group* of $\mathcal{A}$, denoted $\mathcal{K}(\mathcal{A})$ is

$$\frac{\text{free } \mathbb{Z}\text{-module on symbols } [M] \text{ for } M \in \mathcal{A}}{\begin{array}{c}[M_2] = [M_1] + [M_3] \text{ whenever we have a short exact sequence} \\ 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0\end{array}}$$

In general, $\mathcal{K}(\mathcal{A})$ is just an abelian group.

If $\mathcal{A}$ has a symmetric $\otimes$-product that is exact, then $\otimes$ induces a ring structure on $\mathcal{K}(\mathcal{A})$ via $[M][N] = [M \otimes N]$.

Remark 2.53. More generally, if $\otimes$ is not exact, but it is right exact, and derived functors Tor_* exist, and $\mathrm{Tor}_n = 0$ for $n \gg 0$, then we can given $\mathcal{K}(\mathcal{A})$ a ring structure by

$$[M][N] = \sum_{p \geq 0} (-1)^p [\mathrm{Tor}_p(M, N)].$$

Remark 2.54. When $\mathcal{A} = \mathrm{Rep}_k^{\mathrm{fd}}(G)$, we recover the representation ring of G:

$$R_k(G) = \mathcal{K}(\mathrm{Rep}_k^{\mathrm{fd}}(G)).$$

Let G be a finite group and let $L_1, \ldots, L_r$ be the irreducible complex representations. Then

$$R_{\mathbb{C}}(G) = \bigoplus_{i=1}^{r} \mathbb{Z}[L_i]$$

and hence $R_{\mathbb{C}}(G)$ is free of rank r. It is clear that $[L_i]$ generate $R_k(G)$ under $+$. To show that there are no linear relations, we present two similar arguments

(1) We have an exact functor
$$\mathrm{Hom}(L_i, -)\colon \mathrm{Rep}_{\mathbb{C}}^{\mathrm{fd}}(G) \to \mathrm{Vec}$$
and this induces
$$R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(\mathrm{triv}) = \mathbb{Z}$$
since $L_i \mapsto 1$ and $L_j \mapsto 0$ if $i \neq j$. Thus there are no linear relations.

(2) Taking characters gives a map
$$R_{\mathbb{C}}(G) \to \{\text{class functions } G \to \mathbb{C}\}.$$
Since χ_{L_i} are linearly independent, there cannot be a linear relation between $[L_i]$.

Remark 2.55. If $\mathcal{A}$ is an abelian category and all its objects have finite length, then

$$\mathcal{K}(\mathcal{A}) = \bigoplus_{i \in I} \mathbb{Z}[L_i]$$

where $\{L_i\}_{i \in I}$ are the simple objects of $\mathcal{A}$. This follows from the Jordan-Hölder Theorem.

Example 2.56. We note that $R_{\mathbb{C}}(S_3) \cong \mathbb{Z}^3$ and it has a basis $s_{(3)}$, $s_{(1^3)}$, $s_{(2,1)}$, where we write $s_\lambda = [S^\lambda]$. We have that

$$s_{(3)} = [\mathrm{triv}] = \text{ identity in the ring,}$$
$$s_{(1^3)}^2 = 1,\ s_{(1^3)} s_{(2,1)} = s_{(2,1)},$$
$$s_{(2,1)}^2 = s_{(3)} + s_{(1^3)} + s_{(2,1)}.$$

Exercise. Try to realize the last decomposition explicitly.

Hence

$$R_{\mathbb{C}}(S_3) = \frac{\mathbb{Z}[a,s]}{(a^2=1,\ as=s,\ s^2=1+a+s)}.$$

Remark 2.57. More generally, if V is self-dual, irreducible representation, then $(V^{\otimes 2})_G \neq 0$.

In general,

$$R_{\mathbb{C}}(S_n) = \bigoplus_{|\lambda|=n} \mathbb{Z} s_\lambda$$

and

$$s_\lambda s_\mu = \sum_{|\nu|=n} g_{\lambda\mu\nu} s_\nu,$$

where $g_{\lambda\mu\nu}$ are the *Kronecker coefficients* and

$$g_{\lambda\mu\nu} = \dim(S^\lambda \otimes S^\mu \otimes S^\nu)^{S_n}$$

because representations are self-dual.

Remark 2.58. The Kronecker coefficients are poorly understood in general, so it is hard to understand multiplication in a representation ring of S_n.

A different ring is better understood and more important. Define

$$\Lambda = \bigoplus R_{\mathbb{C}}(S_n) = \bigoplus_{\lambda} \mathbb{Z} s_\lambda.$$

For $|\lambda| = n$, $|\mu| = m$, we define

$$s_\lambda s_\mu = \left[\operatorname{Ind}_{S_n \times S_m}^{S_{n+m}} (S^\lambda \otimes S^\mu)\right].$$

Exercise. This makes Λ a commutative, associative, unital ring with $s_{(0)} =$ [trivial representation of S_0] as the unit.

Remark 2.59. The ring Λ is graded: s_μ has degree $|\mu|$.

Proposition 2.60. *The ring Λ is a polynomial ring in the elements $\{s_{(n)}\}_{n\geq 1}$, where $s_{(n)} =$ [trivial representation of S_n].*

Proof. Suppose $\lambda = (\lambda_1, \dots, \lambda_r)$, $|\lambda| = n$. We have that

$$s_{(\lambda_1)} s_{(\lambda_2)} \dots s_{\lambda_r} = \left[\operatorname{Ind}_{S_\lambda}^{S_n}(\text{triv})\right] = \left[M^\lambda\right] = s_\lambda + (\text{sum of } s_\mu\text{'s with } \mu \lhd \lambda).$$

Thus we have upper triangular relations between the s_λ's and monomials in $s_{(n)}$'s. □

Proposition 2.61. *We have that* $s_{(1)} s_\lambda = \sum_{\substack{\lambda \subset \mu \\ |\mu| = |\lambda|+1}} s_\mu$.

Proof. This follows from $\operatorname{Ind}_{S_{n-1}}^{S_n} S^\lambda = \bigoplus_{\substack{\lambda \subset \mu \\ |\mu| = |\lambda|+1}} S^\lambda$, Pieri's rule 2.49. □

Remark 2.62.

- General Pieri's rule describes $s_{(n)}s_\lambda$.
- Littlewood-Richardson rule describes $s_\lambda s_\mu$.

Question. Is Λ naturally the Grothendieck group of something?

Definition 2.63. A *representation of* S_* is a sequence $(M_n)_{n\geq 0}$, where M_n is a representation of S_n.

The category $\mathrm{Rep}_{\mathbb{C}}(S_*)$ is an abelian category. Every finite-dimensional representation of S_* is a direct sum of S^λ's, a representations of S_* concentrated in degree $|\lambda|$. We then have that

$$\mathcal{K}(\mathrm{Rep}^{\mathrm{fd}}_{\mathbb{C}}) = \Lambda,$$

which answers the above questions.

For $M_*, N_* \in \mathrm{Rep}(S_*)$, define

$$(M_* \otimes N_*)_n = \bigoplus_{i+j=n} \mathrm{Ind}^{S_n}_{S_i\times S_j}(M_i \otimes N_j).$$

Fact 2.64. *This defines a* $\otimes$*-product on* $\mathrm{Rep}(S_*)$*, which gives* $\mathcal{K}(\mathrm{Rep}^{fd}_{\mathbb{C}})$ *a ring structure, with which* $\mathcal{K}(\mathrm{Rep}^{fd}_{\mathbb{C}}) = \Lambda$ *is a ring isomorphism.*

Theorem 2.65. *The category* Rep^{fd}_* *is the universal* $\mathbb{C}$*-linear* $\otimes$*-category, i.e. given a* $\mathbb{C}$*-linear* $\otimes$*-category* $\mathcal{A}$*, the functor*

$$\{\otimes\text{-functors } \mathrm{Rep}^{fd}_{\mathbb{C}}(S_*) \to \mathcal{A}\} \to \mathcal{A}$$
$$\mathcal{F} \mapsto \mathcal{F}(S^1)$$

is an equivalence of categories.

This is analogous to the statement

$$\{\text{ring homomorphisms } \mathbb{Z}[x] \to R\} \cong R.$$

3. Young's Rule, Littlewood–Richardson Rule, Pieri Rule

We start with considerations that are purely combinatorial and we later apply them to representation theory of symmetric groups.

Let $\mu = (\mu_1, \mu_2 \ldots)$ be a sequence of non-negative integers. A sequence of positive integers has type μ if it has exactly μ_i i's.

Definition 3.1. Let $a = (a_1, a_2 \ldots)$ be a sequence of positive integers. We define each term of a to be *good* as follows:

- All 1s in a are good.
- An $(i+1)$ is good if and only if $\#\{\text{previous good } i\text{'s}\} \geq \#\{\text{previous good } (i+1)\text{'s}\}$.

If a term of a is *bad* if it is not good.

Example 3.2. The sequence $a = (1, 2, 1, 1, 2)$ has type $(3, 2)$ and all terms are good.

One way to think of good terms which will be useful later is to replace all the good i's be left parenthesis and all the $i+1$'s as right parenthesis. Then an $i+1$ is bad if it does not complete a correct expression.

Example 3.3. Consider the sequence $a = (1, 2, 3, 1, 1, 3, 2)$ of type (3,2,2). To check which 2s are good, we consider the parentheses

$$()3((3).$$

Thus all 2s are good in a. To check which 3s are good, we consider the parentheses

$$1()11)(.$$

Thus the first 3 is good but the second 3 is bad.

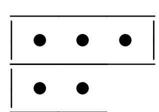
Definition 3.4. A *pair of partitions* is a pair (μ, μ') where $\mu = (\mu_1, \mu_2, \ldots)$ is a sequence of non-negative integers and μ' is a non-increasing sequence of non-negative integers such that $\mu'_i \leq \mu_i$. We write $s(\mu, \mu')$ for the set of type μ sequences such that $\#(\text{good } i\text{'s}) \geq \mu'_i$.

Note that:

- $s(\mu, 0)$ is the set of all type μ sequences,
- $s(\mu, \mu') = s(\mu, (\mu_1, \mu'_2, \mu'_3, \ldots))$ because all 1's are good, so we may (and will) always assume $\mu'_1 = \mu_1$.

We represent (μ, μ') graphically as follows: we draw μ like a tabloid with vertical lines indicating μ'_is, and add black circles to represent the empty spaces.

Example 3.5. The pair (μ, μ') with $\mu = (3, 2)$ and $\mu' = (2, 1)$ can be represented as follows[1]

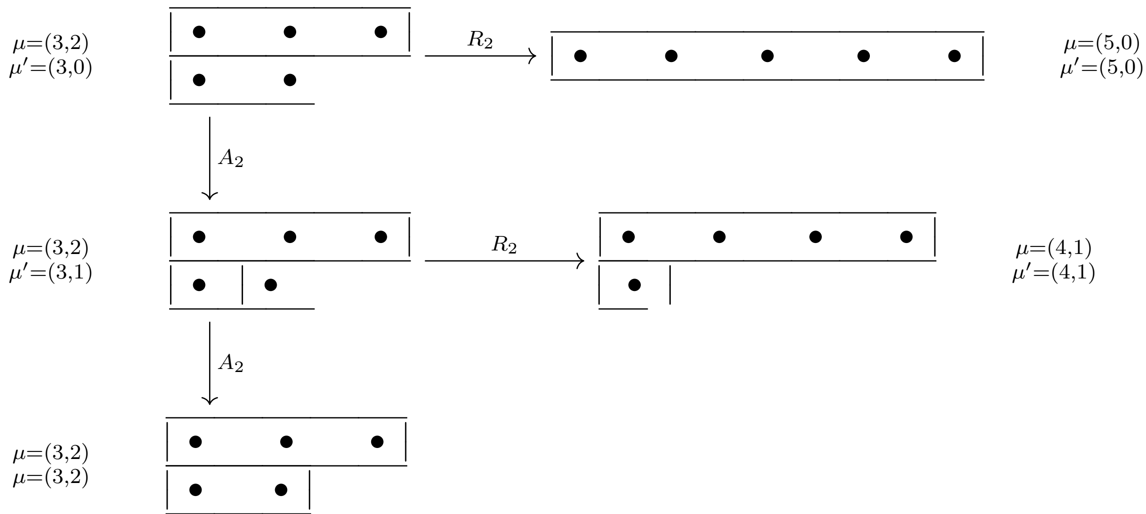


Definition 3.6. Let (μ, μ') be a pair of partitions. Let $c > 1$ be such that $\mu_{c-1} = \mu'_{c-1}$ and $\mu'_c < \mu_c$.

- Suppose $\mu'_c < \mu'_{c-1}$. Define $A_c\mu'$ to be μ' but with cth entry changed to $\mu'_c + 1$. (We think of A_c as **A**dding a box to μ' in the cth row.)
  This gives a new pair $(\mu, A_c\mu')$. By convention $(\mu, A_c\mu') = (0, 0)$ if $\mu'_c = \mu'_{c-1}$.
- Similarly, $R_c\mu$ is obtained from μ by changing μ_c to μ'_c and μ_{c-1} to $\mu_{c-1} + (\mu_c - \mu'_c)$. (We think of R_c as **R**aising boxes in row c to the right of μ' to row $c-1$.)
  This gives a new pair $(R_c\mu, \mu')$.

Example 3.7. The following diagram shows what the operations A_c and R_c do to some pairs (μ, μ').

[1]The author apologizes for the quality of the pictures of pairs (μ, μ'). Making these proved more difficult than excepted. If you a better way to produce these, please let me know!



A priori, the images of applying R_2 would have $\mu' = (3, 0)$ and $\mu' = (3, 1)$, but we always normalize to assume that $\mu_1 = \mu'_1$

Remark 3.8.

- Starting with any (μ, μ') and applying a sufficiently long sequence of A's and R's, we eventually get a pair (λ, λ).
  This is because applying both of the operators gets you closer to having all the boxes in the μ' part.

- Given any (μ, μ'), there exists a sequence ν and a sequence of A's and R's that takes $(\nu, (\nu_1, 0, \dots, 0))$ to (μ, μ').
  Take $\nu = (\mu'_1, \dots, \mu'_r, \mu_1 - \mu'_1, \dots, \mu_n - \mu'_n)$. First apply all the As to get the prime piece correct and the R's to get extra boxes in the correct positions.

Theorem 3.9. *Let (μ, μ') be a set of partitions such that A_c and R_c are defined. We have a bijection*

$$s(\mu, \mu') \setminus s(\mu, A_c\mu') \to s(R_c\mu, \mu')$$

given by changing all the bad c's to $(c - 1)$'s.

Proof. Let $a \in s(\mu, \mu') \setminus s(\mu, A_c\mu')$. Suppose A_c is non-trivial, so $\mu_{c-1} = \mu'_{c-1}$. Then the number of good $c - 1$'s in a is at least $\mu'_{c-1} = \mu_{c-1}$, so it is exactly μ_{c-1}. Moreover, note that $a' \in s(\mu, A_c\mu')$ if the number of good i's in a' is at least μ'_i for $i \neq 0$, and the number of good c's in a' is at least $\mu'_c + 1$. Hence an element $a \in s(\mu, \mu') \setminus s(\mu, A_c\mu')$ has to contain exactly μ'_c good c's. To sum up, we have that:

- a contains μ_{c-1} $(c - 1)$'s, all of which are good,
- a contains μ'_c good c's,
- a contains $\mu_c - \mu'_c$ bad c's.

Let b be obtained from a by changing the bad c's to $(c - 1)$'s. Then b has type $R_c\mu$, since a contained $\mu_c - \mu'_c$ bad c's, and hence the number of $(c - 1)$'s in b is $\mu_{c-1} + \mu_c - \mu'_c$, and the number of c's in b is μ'_c.

To prove that the map is well-defined, we still have to show b has time (R_c, μ'). We will first prove the following statement

$$(*) \qquad \text{For all } j, \#(\text{good } (c-1)\text{'s before } j\text{th term of } b) \geq \#(\text{good } (c-1)\text{'s before } j\text{th term of } a).$$

by induction on j. It is clear for $j = 1$. Assume $(*)$ holds for $j = i$. The inequality is obviously true for $j+1$ unless jth term is $(c-1)$ which is good in a and bad in b. But in this case, the inequality at j is strict because the number of $(c-2)$'s before j is the same in a and b. Hence $(*)$ holds for $j+1$, completing the induction.

Applying $(*)$, b has at least μ'_{c-1} good $(c-1)$s, because a does, and all c's in b are good. If $i \neq c, c-1$, then an i in b is good if and only if it is good in a. This shows that $b \in s(R_c\mu, \mu')$. Hence the map is well-defined.

Now, we will show that it is a bijection. Let $a \in s(\mu, \mu') \setminus s(\mu, A_c\mu')$ and b be its image. We think about replacing the $(c-1)$'s in b with left parenthesis and c's with right parenthesis. Since all c's in b are good, all the right parentheses are matched. The sequence b now looks as follows

$$b_0(b_1(\ldots(b_r$$

where each b_i is a correctly parenthesized expression, and "(" are the unmatched left parentheses.

We claim that the first $\mu_c - \mu'_c$ unmatched left parenthesis were flipped from a. Otherwise, we would create a new good c or destroy an existing good c. Indeed, in a we know that all the left parenthesis are good (since all $(c-1)$'s are good). If a looked like $a_0(a_1)a_2$, then we would not flip the right parenthesis around b_1, because it corresponded to a good c.

This shows that the map is injective, because it determines b from a.

If $b \in s(R_c\mu, \mu')$ is given, we say that a $(c-1)$ is red if it is an unmatched and green otherwise. Let a' be the sequence obtained by changing the first $\mu_c - \mu'_c$ red $(c-1)$'s to c's. This is the candidate for the inverse image of b. We need to check that $a' \in s(\mu, \mu') \setminus s(\mu, A_c\mu')$. We must show

$$(**) \qquad \text{Every } c-1 \text{ in } a' \text{ is good.}$$

To do that, we proceed in the following steps.

(1) For any j, $\#(\text{green } (c-1)\text{'s before } j) \leq \#(\text{good } (c-1)\text{'s before } j)$.
 Suppose the jth term is red $(c-1)$. Then
$$\#(\text{green } (c-1)\text{'s before } j) = \#(c\text{'s before } j) \leq \#(\text{good } (c-1)\text{'s before } j).$$
 The same argument applies to the entire sequence, so the inequality holds if j is maximal. Now, we complete the proof by argueing by descending induction on j, using the step above.

(2) (Assume $c > 2$.) For every $c-1$ in a',
$$\#(\text{previous good } (c-2)\text{'s}) > \#(\text{previous good } (c-1)\text{'s}).$$
 As $b \in s(R_c\mu, \mu')$, b contains at most $\mu_c - \mu'_c$ bad $(c-1)$'s. For any $c-1$ in b,

$$\#(\text{previous good } (c-2)\text{'s}) > \#(\text{previous } (c-1)\text{'s}) - (\mu_c - \mu'_c).$$

Therefore, the inequality in the claim holds holds after the first $\mu_c - \mu'_c$ red $(c-1)$'s. Suppose the jth term of a' is $c-1$ that is before the $(\mu_c - \mu'_c)$th red $c-1$. Then

$$\begin{aligned} \#((c-1)\text{'s before } j \text{ in } a') &= \#(\text{green } (c-1)\text{'s before } j \text{ in } b) \\ &\leq \#(\text{good } (c-1)\text{'s before } j \text{ in } b) \\ &\leq \#(\text{good } (c-2)\text{'s before } j \text{ in } b) \end{aligned}$$

and the last two inequalities are strict if j is a good $(c-1)$ in b.

Then (2) proves $(**)$, and hence completes the proof. □

Recall Λ is the free $\mathbb{Z}$-module on $\{s_\lambda\}_{\lambda \text{ partition}}$, where s_λ is the class of S^λ in the Grothendieck ring. Given a pair (μ, μ'), define a linear endomorphism $E_{\mu,\mu'}$ of Λ by

$$E_{\mu,\mu'}(s_\lambda) = \sum_{\lambda \subset \nu} a_\nu s_\nu,$$

where a_ν is the number of ways to fill the boxes of $\nu \setminus \lambda$ with numbers such that

(a) rows are non-decreasing,
(b) columns are increasing,
(c) by reading rows top to bottom and right to left, we get a sequence in $s(\mu, \mu')$.

In particular, (c) shows that $|\nu| = |\lambda| + |\mu|$.

Lemma 3.10. *Let μ be a partition. Then $E_{\mu,\mu}(s_0) = s_\mu$.*

Proof. We have that

$$E_{\mu,\mu}(s_0) = \sum_{\nu} a_\nu s_\nu.$$

We will show that $a_\nu = 0$ for $\nu \neq \mu$ and $a_\mu = 1$. Consider a filling of ν that satisfies (a)–(c). We show that the only possible filling has to have 1's in the first row, 2's in the second row, etc.

Suppose some i appears in row j of ν and $j < i$, and take a minimal such i. Then no $(i-1)$'s appear above i by minimality, and none appear to to the right by (a). Therefore, this i is bad. Hence the sequence is not in $s(\mu, \mu')$, contradicting (c).

We also know that i does not appear in row j with $j > i$ by (b). Therefore, every i appears in the ith row. This implies that $\mu = \nu$ and $a_\nu = 1$. □

Remark 3.11. Suppose $\mu = (\mu_1, \ldots, \mu_r)$. Then

$$E_{\mu,0}(s_0) = \underbrace{s_{\mu_1} \ldots s_{\mu_r}}_{[M^\mu]}.$$

This gives us the decomposition of M^μ into irreducibles. We have that $E_{\mu,0} = \sum a_\nu s_\nu$, and a_ν is the number of fillings of diagram ν such that (a) and (b) holds, and the number of $i's$ is μ_i. This is the number of semi-standard (i.e. satisfying (a) and (b)) tableaux of shape ν and type μ. Thus this result is equivalent to the following statement.

Theorem 3.12 (Young's Rule)**.** *The representation M^μ decomposes into irreducibles as*

$$M^\mu = \bigoplus (S^\nu)^{\oplus a_\nu},$$

where a_ν is the number of semistandard tableaux of shape ν and type μ.

We will only prove this theorem towards the end of this chapter. We first use Theorem 3.9 to prove that $E_{\mu,\mu'} = E_{\mu,A_c\mu'} + E_{R_c\mu,\mu'}$. This will allow us to prove the Littlewood-Richardson rule 3.14 using Young's rule 3.12.

Proposition 3.13. *Assume A_c and R_c are defined on (μ, μ'). Then*

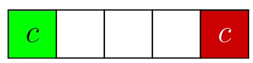
$$E_{\mu,\mu'} = E_{\mu,A_c\mu'} + E_{R_c\mu,\mu'}.$$

Proof. Let $|\mu| = r$ and let λ, ν be partitions of $n - r$ and n with $\lambda \subset \nu$. Fill $\nu \setminus \lambda$ such that we get a sequence in $s(\mu, \mu') \setminus s(\mu, A_c\mu')$.

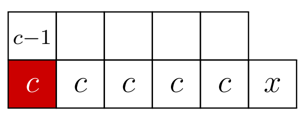
We claim that changing all the bad c's to $(c-1)$'s gives a filling satisfying (a)–(c) if and only if the original filling satisfies (a)–(c). This will prove the proposition by Theorem 3.9. Throughout the proof, we mark the bad c's with red and the good c's with green.

Suppose first that (a)–(c) holds for the initial filling. There could be two possible problems for the modified filling.

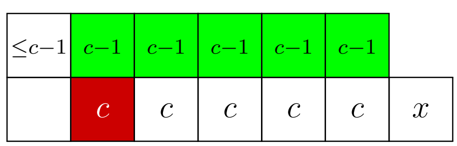
- There could be a bad c to the right of a good c in the same row.



  This is impossible because a c to the left of a bad c cannot be good.
- There could be a bad c in box (i, j) and $c - 1$ in box $(i - 1, j)$. Let $m - 1$ be number of c's following the c at (i, j).



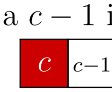
  Above each of these c's is a $c - 1$ by (a) and (b). All of the $(c-1)$'s are good by assumption.



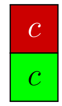
  When we hit box $(i, j + m - 1)$, we have built up m good $(c-1)$'s, so the next m c's are good. Hence at (i, j) the c is actually good.

Now, suppose after changing bad c's to $(c-1)$'s, properties (a)–(c) hold. Let us show the original filling satisfied (a)–(c). Again, there could be two possible problems with the original filling.

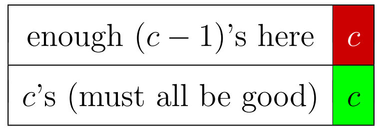
- There could be a bad c to the left of a $c - 1$ in the same row.



  But this cannot happen: the $c - 1$ is good, so the c is also good.
- There could be a bad c directly above a good c, say $(i - 1, j)$ and (i, j):



We have that $\#((c-1)\text{'s to the left of } (i-1,j) \text{ in row } i-1) \geq \#(\text{good } c\text{'s in } i\text{th row})$. Below every $c-1$ in $(i-1)$st row, we must have a c by (a) and (b), and they must be good because of the conditions (a) and (b) after the change.



This completes the proof. ∎

Theorem 3.14 (Littlewood-Richardson rule)**.** *We have that*

$$s_\lambda s_\mu = E_{\mu,\mu}(s_\lambda) = \sum_{\lambda \subset \nu} a_\nu s_\nu,$$

where a_ν is the number of semi-standard fillings of $\nu \setminus \lambda$ such that the resulting sequence has type μ and all terms in it are good.

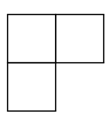
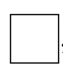
The coefficients a_ν are sometimes written $c^\nu_{\lambda,\mu}$ and called the *Littlewood-Richardson coefficients*. Before giving the proof, we present a few special cases, one of which we will have to prove before we prove the theorem.

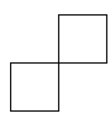
Corollary 3.15 (Pieri rule)**.** *We have that*

$$s_{(n)} s_\lambda = \sum_{\lambda \subset \nu} s_\nu$$

where $|\nu| = |\lambda| + n$ and $\nu \setminus \lambda$ has no two boxes in the same column.

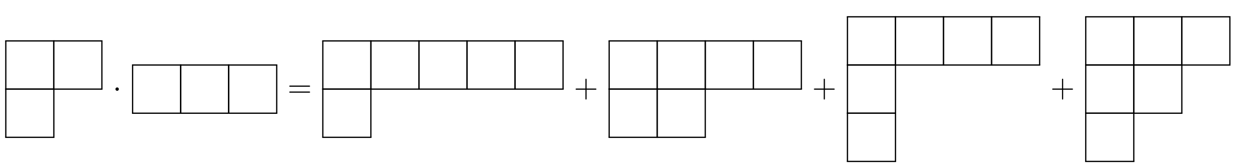
Note that Pieri rule is multiplicity free.

Example 3.16. If $\nu =$  and $\lambda =$ , then

$\nu \setminus \lambda =$ 

This is what we mean by $\nu \setminus \lambda$ having no two boxes in the same column.

Example 3.17. We have that



Remark 3.18. Pieri rule 3.15 is much easier to use in practice than the Littlewood-Richardson rule 3.14, because the statement does not involve as much counting. It is easy to make mistakes when using the Littlewood-Richardson rule 3.14.

Remark 3.19 (Alternative Pieri rule)**.** We have that

$$s_{(1^n)}s_\lambda = \sum_{\lambda \subset s_\nu} s_\nu$$

for $|\nu| = |\lambda| + n$ and $\nu \setminus \lambda$ has no two boxes in the same row.

Corollary 3.20 (Pieri rule II)**.** *We have that*

$$[M^\mu]s_\lambda = \sum_{\lambda \setminus \mu} b(\nu \setminus \lambda, \mu)s_\nu,$$

where $b(\nu \setminus \lambda, \mu)$ is the number of semi-standard fillings of $\nu \setminus \lambda$ of type μ.

Proof. This follows from iterating the normal Pieri rule 3.15. □

We now restate Pieri's rule II as a lemma to Theorem 3.14. We will prove it using Young's rule 3.12:

$$[M^\mu] = \sum b(\nu, \mu)s_\nu,$$

which we assume for now and prove later.

Lemma 3.21 (Pieri rule II)**.** *We have that*

$$[M^\mu]s_\lambda = \sum_{\lambda \setminus \mu} b(\nu \setminus \lambda, \mu)s_\nu,$$

where $b(\nu \setminus \lambda, \mu)$ is the number of semi-standard fillings of $\nu \setminus \lambda$ of type μ.

Proof. We prove this by induction on λ (using the dominance order). We have that

$$\sum_\nu b(\nu, \mu)[M^\lambda]s_\nu = [M^\mu][M^\lambda] = [M^{\mu \cup \lambda}] = \sum_\omega b(\omega, \lambda \cup \mu)s_\omega.$$

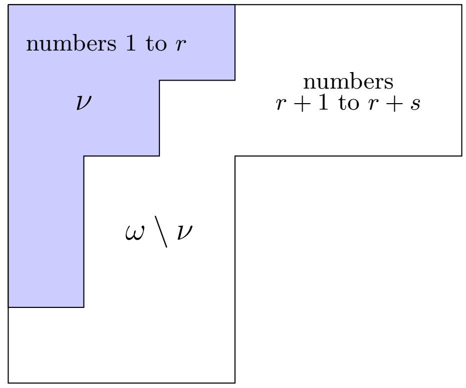
To show that this is equal to

$$\sum_{\nu, \omega} b(\nu, \mu)b(\omega, \nu, \lambda)s_\omega,$$

we just have to show that

$$b(\omega, \mu \cup \lambda) = \sum_\nu b(\nu, \mu)b(\omega \setminus \nu, \lambda). \tag{1}$$

Filling ω with type $\mu \cup \lambda$ is the same as choosing a subset $\nu \subseteq \omega$ and first filling ω with type μ and then $\omega \setminus \nu$ with type λ. If $\mu = (\mu_1, \dots, \mu_r)$, $\lambda = (\lambda_1, \dots, \lambda_s)$, this is summarized by the following diagram



This gives a bijection between the element on the left hand size and right hand side of equation (1), which shows that it holds. ☐

Proof of Littlewood-Richardson rule 3.14. Let ν be a partition. We apply a long sequence of A's and R's to $(\nu, 0)$ to end up with (ω, ω)'s. By Proposition 3.13,

$$E_{\nu,0} = \sum_{\omega \trianglelefteq \nu} a_\omega E_{\omega,\omega}.$$

Therefore, we have an upper-triangular change of basis. Write

$$E_{\mu,\mu} = \sum_\alpha b_\alpha E_{\alpha,0} \text{ for } b_\alpha \in \mathbb{Z}.$$

We have that

$$\begin{aligned} E_{\mu,\mu}(s_\lambda) &= \sum b_\alpha E_{\alpha,0}(s_\lambda) \\ &= \left(\sum b_\alpha E_{\alpha,0}(s_0)\right) s_\lambda \\ &= E_{\mu,\mu}(s_0) s_\lambda \\ &= s_\mu s_\lambda && \text{by Lemma 3.10} \end{aligned}$$

This gives the result. ☐

We finally prove Young's rule 3.12, i.e. the multiplicity of S^λ in M^μ, which is $\dim \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$, is the number of semistandard tableau of shape λ, type μ.

Let

$$\mathcal{T}(\lambda, \mu) = \{\text{tableaux of shape } \lambda \text{ and type } \mu\}.$$

Fix a tableau t of shape λ and type 1^n (which labels the boxes of λ). This makes S_n permute the *boxes* of λ, and hence we get an S_n action on $\mathcal{T}(\lambda, \mu)$.

The action is transitive and the stabilizer of an element on S_μ is

$$M^\mu \cong \mathbb{C}[\mathcal{T}(\lambda, \mu)].$$

We say that $T, T' \in \mathcal{T}(\lambda, \mu)$ are *row equivalent* if $T = \sigma T'$ for some $\sigma \in R^t$ (and similarly for *column equivalent*).

Definition 3.22. Let $T \in \mathcal{T}(\lambda, \mu)$ and define $\theta_T \colon M^\lambda \to M^\mu$ to be the unique S_n-equivariant map taking

$$\{t\} \mapsto \sum_{\substack{T' \text{ row} \\ \text{equiv. to } T}} T',$$

and $\widehat{\theta}_T = (\theta_T)|_{S^\lambda}$.

Theorem 3.23. *The set*

$$\left\{\widehat{\theta}_T \mid T \text{ is semistandard}\right\}$$

forms a basis for

$$\operatorname{Hom}_{S_n}(S^\lambda, M^\mu).$$

Remark 3.24. This theorem implies Young's rule 3.12.

Write $[T]$ for the column equivalence class of T. Define an order on these equivalence classes generated by letting $[T_1] \lhd [T_2]$ if $[T_2]$ is obtained from $[T_1]$ by switching a and b where $a < b$ and the columns of a is to the right of the column of b.

Remark 3.25. If T is semistandard and T' is row equivalent to T, then $[T'] \unlhd [T]$.

Lemma 3.26. *The set*

$$\left\{\widehat{\theta}_T \mid T \text{ is semistandard}\right\}$$

is linearly independent.

Proof. Suppose $\sum_{T \text{ sstd}} a_T \widehat{\theta}_T = 0$ is a nontrivial linear relation.[2] Let $[T']$ be maximal with $a_{T'} \neq 0$. We have that

$$\left(\sum_{T \text{ sstd}} a_T \widehat{\theta}_T\right)(\{t\}) = a_{T'} T' + (\text{linear combination of } T'' \text{ s.t. } [T'] \ntrianglelefteq [T'']),$$

and hence applying κ_t

$$\left(\sum_{T \text{ sstd}} a_T \widehat{\theta}_T\right)(e_t) = a_{T'} \underbrace{\kappa_t T'}_{\neq 0} + (\text{linear combination of } \kappa_t(T'') \text{ s.t. } [T'] \ntrianglelefteq [T'']) \neq 0,$$

which is a contradiction. □

Now, we have an injective map

$$\bigoplus_\lambda S^\lambda \otimes (\text{span of } \widehat{\theta}_T \text{ with } T \text{ semistandard}) \hookrightarrow M^\mu.$$

Remark 3.27. To show that the $\widehat{\theta}_T$'s with T semistandard span, it suffices to show that the dimensions above are equal. We have that

[2] We write sstd for semistandard in the summation.

$$\text{LHS has a basis} \quad \bigcup_{\lambda} \left(\begin{matrix} \text{standard tableau of} \\ \text{shape } \lambda \end{matrix} \quad \begin{matrix} \text{semistandard tableau of} \\ \text{shape } \lambda \text{ type } \mu \end{matrix} \right)$$

$$\text{RHS has a basis} \qquad \text{tabloids of shape } \mu$$

The RSK (Robinson–Schensted–Knuth) correspondence gives an explicit bijection. This argument will not be presented here.

Lemma 3.28. *Let* $\widehat{\theta} \in \mathrm{Hom}_{S_n}(S^\lambda, M^\lambda) \neq 0$. *Put*

$$\widehat{\theta}(e_t) = \sum_{T \in \mathcal{T}(\lambda,\mu)} c_T T.$$

Then

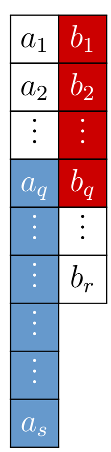
(1) $c_{T'} = 0$ *if* T' *has a repeat in some column,*
(2) *there exists a semistandard* T' *with* $c_{T'} \neq 0$.

Proof. For (1), suppose that T' has a repeat in the boxes labeled by i and j in t in the same column. Then

$$\sum c_T (i\ j) T = (i\ j)\widehat{\theta}(e_t) = \widehat{\theta}((i\ j)e_t) = -\widehat{\theta}(e_t) = -\sum c_T T.$$

Since $(i\ j)T' = T'$, $c_{T'} = -c_{T'}$.

To show (2), we note that, more generally, if $\sigma \in C_t$, then $c_{\sigma T} = \pm c_T$ for any T. Let $[T_1]$ be maximal with $c_{T_1} \neq 0$. By the above, we can assume that the columns of T_1 are increasing. We want to show that the rows of T_1 are semistandard. Assume not, so say there exists a decrease in row q, going from column j to column $j+1$. We have the following situation



with $a_q > b_q$ and $a_1 < \ldots, < a_s$ and $b_1 < \ldots < b_r$. Let X be the blue set and Y be the red set of T_1. Consider the Garnir relation 2.35 for $X \cup Y$:

$$\sum_{\sigma \in S_{X\cup Y}/S_X \times S_Y} (\operatorname{sgn}\sigma)\sigma e_t = 0.$$

Apply $\widehat{\theta}$ to get

$$\sum_{T\in\mathcal{T}(\lambda,\mu)} C_T \sum_{\sigma\in S_{X\cup Y}/S_X\times S_Y} (\operatorname{sgn}\sigma)\sigma T = 0.$$

Consider the coefficient of T_1 in this sum. We know it is 0. Hence there exists a nonzero term canceling c_1T_1, and hence there is a $\sigma \in S_{X\cup Y}/S_X \times S_Y$ such that $c_{\sigma T_1} \neq 0$. But $[T_1] \lhd [\sigma T_1]$, a contradiction. □

Lemma 3.29. *The set*

$$\left\{\widehat{\theta}_T \mid T \textit{ is semistandard}\right\}$$

spans $\operatorname{Hom}(S^\lambda, M^\mu)$.

Proof. Consider a nonzero $\widehat{\theta} \in \operatorname{Hom}(S^\lambda, M^\mu)$, and write

$$\widehat{\theta}(e_t) = \sum_{T\in\mathcal{T}(\lambda,\mu)} c_T T.$$

Let T_1 be semistandard and $[T_1]$ be maximal with $c_{T_1} \neq 0$. Then

$$\widehat{\theta}_{T_1}(e_t) = \underbrace{\kappa_t T_1}_{\substack{\text{coeff. of} \\ T_1 \text{ is } 1}} + (\text{smaller things}),$$

and hence in

$$(\widehat{\theta} - c_{T_1}\widehat{\theta}_{T_1})(e_t)$$

the coefficient of T_1 is 0, and we have modified by elements $\lhd T_1$. We can now continue inductively to prove the assertion. □

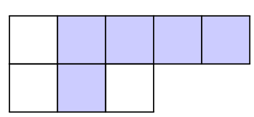
Altogether, this completes the proof of Young's rule 3.12.

3.1. **Hook length formula.**

Definition 3.30. The (i, j) *hook* of a partition λ consists of the box at (i, j) and all boxes directly below or directly right of it.

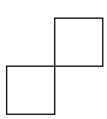
The *hook length* at (i, j) is the number of boxes in the hook at (i, j).

Example 3.31. For $\lambda = (5, 3)$, $(i, j) = (1, 2)$ the hook is marked in blue below

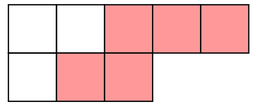


and the hook length is 5.

Definition 3.32. A *boder strip* (or *skew hook*) of λ is a shape of the form $\lambda \setminus \mu$ such that it is connected and contains no 2×2 box.

Not that  is not connected.

Example 3.33. A border strip of $\lambda = (5, 3)$ is marked in red below



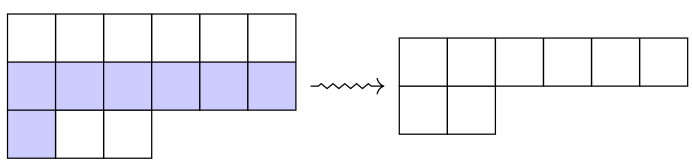
Note that there is a bijection between the hooks and the border strips, and this preserves the number of boxes.

We know that
$$[M^\lambda] = s_\lambda + (\text{sum of } s_\mu' s \text{ with } \mu \lhd \lambda),$$
where the sum is determined by Young's rule 3.12. Hence there is an upper-triangular change of basis between $[M^\lambda] = s_{(\lambda_1)} \dots s_{(\lambda_r)}$ and s_λ if $\lambda = (\lambda_1, \dots, \lambda_r)$.

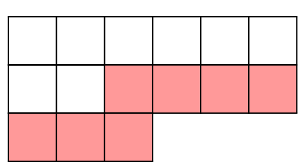
Now we want to invert this and express s_λ in terms of the $[M^\lambda]$'s.

Theorem 3.34. *We have that*
$$s_\lambda = \det\left((s_{\lambda_i - i + j})_{1 \le i, j \le r}\right)$$
where $s_{(k)} = 0$ *if* $k < 0$, $s_{(0)} = 1$.

Proof. Let $A(\lambda)$ be the matrix $(s_{\lambda_i - i+j})_{1 \le i,j \le r}$. We will compute $\det(A(\lambda))$ by Laplace expression in final column. Consider the matrix obtained from $A(\lambda)$ by deleting the final column and kth row. The (i, j) entry of this matrix is
$$s_{(\lambda_i - i + j} \text{if } i < k$$
$$s_{(\lambda_{i+1} - 1 - i + j)} \text{if } i \ge k$$
and hence this is the matrix
$$A(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1} - 1, \dots, \lambda_r - 1).$$
This partition is obtained from λ by deleting the kth row and subtracting 1 from all the following rows. Thus this corresponds to deleting the hook of λ at $(k, 1)$, and shifting the boxes up to get a Young diagram.



This is the same as removing the corresponding border strip β_k.

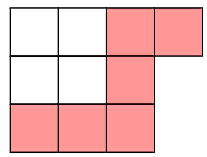


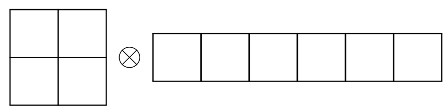
Hence this matrix is $\det(\lambda \setminus \beta_k)$. We have that
$$\begin{aligned}\det(A(\lambda)) &= \det(A(\lambda \setminus \beta_r)) s_{(\lambda_r)} - \det(A(\lambda \setminus \beta_{r-1})) s_{\lambda_{r-1}+1} + \dots \pm \det(A(\lambda \setminus \beta_1)) s_{\lambda_1 + r - 1} \\ &= s_{\lambda \setminus \beta_r} s_{h_{r,1}} - s_{\lambda \setminus \beta_{r-1}} s_{h_{r-1,1}} + \dots \pm s_{\lambda \setminus \beta_1} s_{h_{r,1}},\end{aligned}$$
by induction on the size of the partition, and writing $|\beta_k| = h_{k,1}$.

We compute the products using Pieri rule 3.15. We show that s_λ appears once, everything else cancels.

First, s_λ appears in $s_{\lambda\backslash\beta_r} s_{h_r,1}$ with multiplicity 1 and does not appear in any other term because β_k with $k < r$ has boxes in the same column.

Let X_k be the diagrams in $s_{\lambda\backslash\beta_k} s_{(h_{k,1})}$ that contain final box in row k of λ, and let Y_k be the complement of X_k. Then $X_r = \{\lambda\}$ and $Y_1 = \emptyset$.





One can now show that $X_{k-1} = Y_k$. This shows that everything but X_r cancels, we we get s_λ. ☐

Corollary 3.35. *For a partition* $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\dim S^\lambda = n! \det \left(\frac{1}{(\lambda_i - i + j)!} \right)_{1 \le i,j \le r},$$

where $\frac{1}{m!} = 0$ *if* $m < 0$.

Proof. Consider the additive function

$$\varphi \colon \Lambda \to \mathbb{Q}$$
$$V \mapsto \frac{\dim V}{n!}$$

for any representation V of S_n.

We show that this is a ring homomorphism. If V is a representation of S_n and W is a representation of S_m, then

$$[V][W] = \left[\operatorname{Ind}_{S_n \times S_m}^{S_{n+m}} (V \otimes W) \right],$$

and the dimension is

$$(\dim V)(\dim W) \frac{(n+m)!}{n!m!}.$$

Hence

$$\varphi([V][W]) = \frac{(\dim V)(\dim W)\frac{(n+m)!}{n!m!}}{(n+m)!} = \varphi([V])\varphi([W]).$$

To get the result, apply φ to $s_\lambda = \det(s_{\lambda_i - i + j})$ and note that $s_{(m)}$ is the trivial representation of S_m, so $\varphi(s_{(m)}) = \frac{1}{m!}$. ☐

Example 3.36. Let $\lambda = (3, 2)$. Then

$$s_{(3,2)} = \det \begin{pmatrix} s_{(3)} & s_{(4)} \\ s_{(1)} & s_{(2)} \end{pmatrix} = s_{(2)}s_{(2)} - s_{(4)}s_{(1)} = s_{(3,2)} + s_{(4,1)} + s_{(5)} - s_{(5)} - s_{(4,1)},$$

which agree with Theorem 3.34. By Corollary 3.35, the dimension is

$$\dim S^{(3,2)} = 5! \det \begin{pmatrix} \frac{1}{3!} & \frac{1}{4!} \\ 1 & \frac{1}{2!} \end{pmatrix} = 5! \left(\frac{1}{12} - \frac{1}{24} \right) = \frac{5!}{24} = 5.$$

Theorem 3.37 (Hook length formula)**.** *For a partition λ,*

$$\dim S^\lambda = \frac{n!}{\prod(\textit{all hook lengths})}.$$

Example 3.38. We have the following hook lengths at every box of $\lambda = (3, 2)$

4	3	1
2	1	

and hence the dimension is

$$\dim S^\lambda = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5.$$

This agrees with Example 3.36.

Proof of Theorem 3.37. We take $r = 3$ for exposition, the general case is treated the same way. By Corollary 3.35, it is enough to show that

$$\det \left(\frac{1}{(\lambda_i - i + j)!} \right) = \frac{1}{\prod(\text{hook lengths})}.$$

Note that $\lambda_i - i + j = h_{i,1} + j - r$. We have that

$$\det \begin{pmatrix} \frac{1}{(h_{11}-2)!} & \frac{1}{(h_{11}-1)!} & \frac{1}{(h_{11})!} \\ \frac{1}{(h_{21}-2)!} & \frac{1}{(h_{21}-1)!} & \frac{1}{(h_{21})!} \\ \frac{1}{(h_{31}-2)!} & \frac{1}{(h_{31}-1)!} & \frac{1}{(h_{31})!} \end{pmatrix} = \frac{1}{h_{11}!h_{21}!h_{31}!} \det \begin{pmatrix} h_{11}(h_{11}-1) & h_{11} & 1 \\ h_{21}(h_{21}-1) & h_{21} & 1 \\ h_{31}(h_{31}-1) & h_{31} & 1 \end{pmatrix}$$
$$= \frac{(h_{11} - h_{21})(h_{11} - h_{31})(h_{21} - h_{31})}{h_{11}!h_{21}!h_{31}!}$$

which shows the result. ∎

4. Representations of $\mathrm{GL}_n(\mathbb{C})$

Everything in this chapter is over $\mathbb{C}$.

Definition 4.1. An *m-dimensional representation* of $\mathrm{GL}_n(\mathbb{C})$ is *algebraic* if the homomorphism $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ induced by a map of varieties $\mathrm{GL}_n \to \mathrm{GL}_m$.

An infinite dimensional representation is *algebraic* if it is a union of finite-dimensional representations.

Remark 4.2. Equivalently, an algebraic representation of GL_n is a comodule over $\mathbb{C}[\mathrm{GL}_n]$, the coordinate ring of GL_n.

To define a comodule, first note that an algebra A comes with a map $A \otimes A \to A$, and a coalgebra will be equipped with a map $A \to A \otimes A$ satisfying similar axioms. A module over an algebra comes with a map $A \otimes M \to M$, whereas a comodule comes with a map $M \to A \otimes M$. The reason the arrows are flipped is that $\mathcal{O}(-)$ is a contravariant functor.

Example 4.3. The following representations are algebraic:

- $\mathbb{C}^n$, the standard representation of GL_n,
- $(\mathbb{C}^n)^*$, det,
- any $\otimes$-construction on algebraic representations; e.g. $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$, $\bigwedge^k(\mathbb{C}^n)$, ...

A non-algebraic representation of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^* = S^1 \otimes \mathbb{R}$ is

$$(x, y) \mapsto \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Theorem 4.4. *Algebraic representation of* GL_n *are semisimple.*

Proof. Weyl's unitary trick. Let $U_n \subseteq \mathrm{GL}_n(\mathbb{C})$ be the unitary group. Then U_n is compact, so there is a Haar measure dg on U_n.

Let V be an algebraic representation of GL_n and let $W \subseteq V$ be a subrepresentation. Let $\langle , \rangle_0$ be a hermitian form on V. We average it out over U_n: define

$$\langle v, w \rangle = \int_{U_n} \langle gv, gw \rangle_0 dg,$$

a non-degenerate hermitian form on V which is U_n-invariant.

If $W^\perp$ is the orthogonal space to W with respect to $\langle \ , \ \rangle$. Then

$$V = W \oplus W^\perp$$

because $\langle \ \rangle$ is U_n-invariant, $W^\perp$ is U_n-stable.

Since U_n is Zariski-dense in GL_n, any U_n-stable subgroup of V is GL_n-stable, so $W^\perp$ is a GL_n-subrepresentation. □

Remark 4.5. Weyl's trick can be summarized as follows: semisimplicity can be transferred from a Zariski-dense subgroup to the whole group.

Now, we just need to classify the simple algebraic representations of GL_n. The first step is the following proposition.

Proposition 4.6. *Suppose V is a simple algebraic representation of* GL_n. *Then there exists $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ and a simple subrepresentation of* $(\mathbb{C}^n)^{\otimes k}$ *such that* $V \cong W \otimes \det^m$.

Proof. Let $v \in V$. Then we get an algebraic function $\mathrm{GL}_n \to V$ mapping $g \mapsto gv$. Then

$$V^* \to \mathbb{C}[\mathrm{GL}_n]$$
$$\lambda \mapsto (g \mapsto \lambda(gv))$$

is GL_n-equivariant. Hence algebraic irreducible is contained in $\mathbb{C}[\mathrm{GL}_n]$ with left regular action. Since

$$\mathrm{GL}_n = \{m \in M_n \mid \det(m) \neq 0\},$$

we $\mathbb{C}[\mathrm{GL}_n]$ is the localization at det:

$$\mathbb{C}[\mathrm{GL}_n] = \mathbb{C}[M_n]\left[\frac{1}{\det}\right].$$

Thus if V is an irreducible algebraic representation of GL_n, it is finite-dimensional, and hence $\det^m \otimes V \subseteq \mathbb{C}[M_n]$ for $m \gg 0$. Since

$$M_n = \mathbb{C}^n \otimes (\mathbb{C}^n)^*,$$

we have that

$$\mathbb{C}[M_n] = \mathrm{Sym}(\mathbb{C}^n \otimes (\mathbb{C}^n)^*) = \bigoplus_{k \geq 0} \mathrm{Sym}^k(\mathbb{C}^n \otimes (\mathbb{C}^n)^*).$$

Finally,

$$\mathrm{Sym}^k(\mathbb{C}^n \otimes (\mathbb{C}^n)^*) \subseteq (\mathbb{C}^n \otimes (\mathbb{C}^n))^{\otimes k} = \bigoplus (\mathbb{C}^n)^{\otimes k},$$

and hence $\det^m \otimes V$ is contained in one of these piecies, $(\mathbb{C}^n)^{\otimes k}$ for some k. □

It is hence enough to understand the structure of $(\mathbb{C}^n)^{\otimes k}$. This is a representation of $\mathrm{GL}_n \times S_k$, where S_k permutes the $\otimes$-factors.

Definition 4.7. Given a partition λ of k, then we define an algebraic GL_n-representation $S_\lambda(\mathbb{C}^n)$ to be the multiplicity space of S^λ in $(\mathbb{C}^n)^{\otimes k}$, i.e.

$$S_\lambda(\mathbb{C}^n) = \mathrm{Hom}_{S_k}(S^\lambda, (\mathbb{C}^n)^{\otimes k}).$$

Let $\ell(\lambda)$ be the number of parts of λ, i.e. the number of rows in the Young diagram.

Theorem 4.8.

- *The representation $S_\lambda(\mathbb{C}^n)$ is 0 if $\ell(\lambda) > n$, and irreducible if $\ell(\lambda) \leq n$*
- *If $\lambda \neq \mu$ and both have $\ell \leq n$, then $S_\lambda(\mathbb{C}^n) \not\cong S_\mu(\mathbb{C}^n)$.*

Corollary 4.9. *Every irreducible algberaic representation of GL_n has the form $S_\lambda(\mathbb{C}^n) \otimes \det^m$ for some $\lambda, m \in \mathbb{Z}$.*

Proof. This is because $(\mathbb{C}^n)^{\otimes k} = \bigoplus_\lambda S^\lambda \otimes S_\lambda(\mathbb{C}^n)$ (because $S_\lambda(\mathbb{C}^n)$ is defined as a multiplicity space), and it follows from Theorem 4.8. □

Remark 4.10. Note that λ and m are no quite unique: for example, $S_{1^n}(\mathbb{C}^n) = \det = S_0(\mathbb{C}^n) \otimes \det$.

Lemma 4.11. *For any λ, $S_\lambda(\mathbb{C}^n)$ is 0 or irreducible.*

Proof. We have that

$$(\mathbb{C}^n)^{\otimes k} = \bigoplus_\lambda S^\lambda \otimes S_\lambda(\mathbb{C}^n).$$

Hence

$$\operatorname{End}_{S_k}((\mathbb{C}^n)^{\otimes k}) = \bigoplus_{\lambda} \operatorname{End}(S_\lambda(\mathbb{C}^n)),$$

and

$$\operatorname{End}_{S_k}((\mathbb{C}^n)^{\otimes k}) = \operatorname{End}((\mathbb{C}^n)^{\otimes})^{S_k} = (\operatorname{End}(\mathbb{C}^n)^{\otimes k})^{S_k}.$$

This is spanned as a $\mathbb{C}$-vector space by elements of the form $a \otimes \cdots \otimes a$ for $a \in \operatorname{End}(\mathbb{C}^n)$.

For $1 \le i \le k$, let $x_i(a) = 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$ with a in the ith position. Then note that

$$a \otimes \cdots \otimes a = x_1 \otimes \cdots \otimes x_k.$$

The ring of symmetric polynomials $\mathbb{C}[x_1, \ldots, x_k]^{S_k}$ is generated as a $\mathbb{C}$-algebra by elements of the form $x_1^r + \cdots + x_k^r$ for all $r \ge 0$. Hence power sums in $x_i(a)$'s generate $(\operatorname{End}(\mathbb{C}^n)^{\otimes k})^{S_k}$ as a $\mathbb{C}$-algebra. Since $x_i(a)^r = x_i(a^r)$, we only need the first powers. Hence the elements $x_1(a) + x_2(a) + \cdots + x_n(a)$ for $a \in \operatorname{End}(\mathbb{C}^n) = \mathfrak{gl}_n$ (Lie algebra) generated the invariant algebra. Note that the element $x_1(a) + \cdots + x_k(a)$ is the action of $a \in \mathfrak{gl}_n$ on $(\mathbb{C}^n)^{\otimes k}$.

Therefore, we have shown that the map

$$\mathcal{U}(\mathfrak{gl}_n) \to \operatorname{End}_{S_k}((\mathbb{C}^n)^{\otimes k})$$

is surjective, where $\mathcal{U}(\mathfrak{g})$ is the *universal enveloping algebra*:

$$\mathcal{U}(\mathfrak{g}) = \frac{T(\mathfrak{g})}{(X \otimes Y - Y \otimes X)},$$

where $T(\mathfrak{g})$ is the tensor algebra. Thus

$$\mathcal{U}(\mathfrak{gl}_n) \twoheadrightarrow \operatorname{End}(S_\lambda(\mathbb{C}^n)).$$

Hence $S_\lambda(\mathbb{C}^n)$ is either 0 or irreducible as a representation of $\mathfrak{gl}_n$. Thus $S_\lambda(\mathbb{C}^n)$ is either 0 or an irreducible GL_n-representation. ☐

Lemma 4.12. *If $n \le \ell(\lambda)$ then $S_\lambda(\mathbb{C}^n) = 0$.*

Proof. We need to show that

$$\operatorname{Hom}_{S_k}(S^\lambda, (\mathbb{C}^n)^{\otimes k}) = 0,$$

so consider a map $f \colon S^\lambda \to (\mathbb{C}^n)^{\otimes k}$. It suffices to show that $f(e_t) = 0$ for any tableau t. Recall that $e_t = \kappa_t\{t\}$ and $\kappa_t e_t = |C_t| e_t$. Hence

$$f(e_t) = \frac{1}{|C_t|} f(\kappa_t e_t) = \frac{\kappa_t}{|c_t|} f(e_t).$$

Thus it suffices to show that $\kappa_t(\mathbb{C}^n)^{\otimes k} = 0$. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{C}^n$. Then $e_i \otimes e_j$ is a basis for $\mathbb{C}^n$, so $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$ is a basis for $(\mathbb{C}^n)^{\otimes k}$. Consider some basis vector. Since $\ell(\lambda) > n$, there exist distinct numbers j and j' in the first column of t such that $i_j = i_{j'}$. Then the basis vector is fixed by $(j\ j')$, and hence it is killed by κ_t. ☐

The algebraic irreducibles of $\mathbb{G}_m = \mathrm{GL}_1$ are 1-dimensional and given by $z \mapsto z^n$ for $n \in \mathbb{Z}$. Similarly, the irreducibles of $\mathbb{G}_m^n$ are 1-dimensional and given by

$$(z_1, \ldots, z_n) \mapsto z_1^{a_1} \ldots z_n^{a_n}$$

for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$.

Definition 4.13. The group $\mathbb{G}_m^n$ is called a *torus* and its 1-dimensional algebraic representations are called *weights*.

Let V be an algebraic representation of $\mathrm{GL}_n \supseteq \mathbb{G}_m^n$ (diagonal matrices). Then

$$V = \bigoplus_{a \in \mathbb{Z}^n} V_a,$$

where V_a is the isotypic piece for $V_{|\mathbb{G}_m^n}$ corresponding to the weight a. Then V_a is called the *weight space* for a.

Lemma 4.14. *Let λ be a partition of k. Suppose $\ell(\lambda) \leq n$. The 1^k-weight space of $S_\lambda(\mathbb{C}^n)$ is naturally isomorphic to S^λ as a representation of S_k.*

Remark 4.15. Recall that $S_n \subseteq \mathrm{GL}_n$ as permutation matrices and S_n normalizes $\mathbb{G}_m^n$, so S_n acts on the weights. In this case, 1^k is fixes by $S_k \subseteq S_n$, so S_k acts on the weight spaces.

Proof of Lemma 4.14. The 1^k-weight space of $S_\lambda(\mathbb{C}^n)$ is

$$\mathrm{Hom}_{S_k}(S^\lambda, (1^k\text{-weight space of } (\mathbb{C}^n)^{\otimes k})).$$

A basis element $e_{i_1} \otimes \cdots \otimes e_{i_k}$ of $(\mathbb{C}^n)^{\otimes k}$ has weight $a = (a_1, \ldots, a_n)$ where a_j is the number of is equal to j. Thus the 1^k-weights space of $(\mathbb{C}^n)^{\otimes k}$ has a basis $e_{i_1} \otimes \cdots \otimes e_{i_k}$ where

$$\{i_1, \ldots, i_k\} = \{1, \ldots, k\}.$$

This has an action of $S_k \times S_k$, where the first S_k is the one we Hom over and it acts by permuting the $\otimes$-factors, and the second S_k is the subgroup of GL_n and it acts by relabeling the indices. As an $S_k \times S_k$-set, the set of basis vectors looks like S_k with two S_k's acts by right and left multiplication. Hence the 1^k-weight space of $(\mathbb{C}^n)^{\otimes k}$ is isomorphic to $\mathbb{C}[S_k]$ with the usual action of $S_k \times S_k$. Thus

$$\mathrm{Hom}_{S_k}(S^\lambda, (1^k\text{-weight space of } (\mathbb{C}^n)^{\otimes k})) \cong S^\lambda.$$

This completes the proof. □

Corollary 4.16. *If $\ell(\lambda) \leq n$, then $S_\lambda(\mathbb{C}^n)$ is irreducible.*

Proof. By Lemma 4.14, $S_\lambda(\mathbb{C}^n) \neq 0$, and hence it is irreducible by Lemma 4.11. □

Corollary 4.17. *If $\lambda \neq \mu$ and $\ell \leq n$, then $S_\lambda(\mathbb{C}^n) \ncong S_\mu(\mathbb{C}^n)$.*

This completes the proof of Theorem 4.8.

Proposition 4.18. *For $|\lambda| = k$, $|\mu| = \ell$,*

$$S_\lambda(\mathbb{C}^n) \otimes S_\mu(\mathbb{C}^n) \cong \bigoplus_\nu S_\nu(\mathbb{C}^n)^{c^\nu_{\lambda,\mu}},$$

where $c^\nu_{\lambda,\mu}$ are the Littlewood-Richardson coefficients (see the Littlewood-Richardson rule 3.14).

Proof. We have that

$$\begin{aligned}
\mathrm{Hom}_{S_k}(S^\lambda, (\mathbb{C}^n)^{\otimes k}) \otimes \mathrm{Hom}_{S_k}(S^\mu, (\mathbb{C}^n)^{\ell}) &\cong \mathrm{Hom}_{S_k \times S_\ell}(S^\lambda \otimes S^\mu, (\mathbb{C}^n)^{\otimes(k+\ell)}) \\
&\cong \mathrm{Hom}_{S_{k+\ell}}(\mathrm{Ind}_{S_k \times S_\ell}^{S_{k+\ell}}(S^\lambda \otimes S^\mu), (\mathbb{C}^n)^{\otimes(k+\ell)}) \\
&\qquad \text{(by Frobenius reciprocity)} \\
&\cong \mathrm{Hom}_{S_{k+\ell}}(\bigoplus_\nu (S^\nu)^{\oplus c^\nu_{\lambda,\mu}}, (\mathbb{C}^n)^{\otimes k+\ell}) \\
&\qquad \text{(by Littlewood-Richardson rule 3.14)} \\
&= \bigoplus_\nu S_\nu(\mathbb{C}^n)^{\oplus c^\nu_{\lambda,\mu}}
\end{aligned}$$

which completes the proof. □

Corollary 4.19. *We have that* $S_\lambda(\mathbb{C}^n) \otimes \det = S_{\lambda+1^n}(\mathbb{C}^n)$.

Proof. Recall that $S_{1^n}(\mathbb{C}^n) = \det$ and $S_\lambda(\mathbb{C}^n) \otimes S_{1^n}(\mathbb{C}^n)$ decomposes according to the Pieri rule 3.15. Since $S_\nu(\mathbb{C}^n) = 0$ if $\ell(\nu) > n$, we must put the n boxes in the first n rows. □

Corollary 4.20. *Every algebraic irreducible representation of* GL_n *has the form* $S_\lambda(\mathbb{C}^n) \otimes \det^m$ *where* $\lambda_n = 0$ *and* $m \in \mathbb{Z}$ *and* λ, m *are unique.*

Therefore, we have established a bijection

$$\left\{ \begin{matrix} \text{irreducibles of } \mathrm{GL}_n \\ \text{appearing in } (\mathbb{C}^n)^{\otimes k} \\ \text{for some } k \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducibles } S^\lambda \\ \text{of } S^k \\ \text{with } \ell(\lambda) \leq n \end{matrix} \right\}$$

This motivates the following definition.

Definition 4.21. A representation of GL_n is *polynomial* if it can be realized as a subquotient of a direct product of tensor powers of the standard representation.

The condition "*appearing in* $(\mathbb{C}^n)^{\otimes k}$ *for some* k" in the bijection above reduces to saying they are *polynomial*.

Examples 4.22.

- The representations $S_\lambda(\mathbb{C}^n)$ are polynomial.
- The representation $(\mathbb{C}^n)^*$ is **not** polynomial.

Remark 4.23. The following are equivalent for m-dimensional representation of GL_n:

(1) V is polynomial,
(2) V is algebraic and all weights in V are non-negative,
(3) in the homomorphism $\mathrm{GL}_n \to \mathrm{GL}_m$ the entries of GL_m are polynomials in the entries of GL_m

Definition 4.24. A *polynomial representation* of $\mathrm{GL}_\infty = \bigoplus_{n \geq 1} \mathrm{GL}_n$ is one appearing as a subquotient of $\oplus$ of $\otimes$-powers of $\mathbb{C}^\infty = \bigcup_{n\geq 1} \mathbb{C}^n$.

We write $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ for the category of polynomial representations.

Theorem 4.25. *The category* $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ *is semi-simple and its simple objects are* $S_\lambda(\mathbb{C}^\infty) = \mathrm{Hom}_{S_k}(S^\lambda, (\mathbb{C}^\infty)^{\otimes k})$ *for all partitions* λ*.*

Proof. We have that

$$S_\lambda(\mathbb{C}^\infty) = \bigcup_{n \geq 1} S_\lambda(\mathbb{C}^n).$$

Suppose $V \subseteq S_\lambda(\mathbb{C}^\infty)$ is non-zero and GL_∞-stable. For $n \gg 0$, $V \cap S_\lambda(\mathbb{C}^n)$ is non-zero and GL_n-stable, so $S_\lambda(\mathbb{C}^n) \subseteq V$, whence $V = S_\lambda(\mathbb{C}^\infty)$. Hence $S_\lambda(C\infty)$ is simple.

Moreover,

$$(\mathbb{C}^\infty)^{\otimes k} = \bigoplus_\lambda S^\lambda \otimes S_\lambda(\mathbb{C}^\infty),$$

so $(\mathbb{C}^\infty)^{\otimes k}$ is a semisimple representation of GL_∞. Any direct sum, subrepresentation or quotient of semisimple objects is semisimple, hence every object of $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ is semisimple. □

Remark 4.26. We have that

$$\mathrm{Hom}_{S_k}(S^\lambda, -) - ((S^\lambda)^* \otimes -)^{S_k} = (S^\lambda \otimes -)^{S_k},$$

and sometimes it is preferable to use the latter form rather than the former, because it is covariant.

Theorem 4.27 (Schur–Weyl duality)**.** *We have an equivalence of categories*

$$\mathrm{Rep}(S_*) \longleftrightarrow \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$$

$$(M_k)_{k\geq 0} \longrightarrow \bigoplus_{k \geq 0} (M_k \otimes (\mathbb{C}^\infty)^{\otimes k})^{S_k}$$

$$(1^n \textit{ weight space of } V)_{n \geq 0} \longleftarrow V$$

which is compatible with $\otimes$*-products (using induction* $\otimes$*-product on the left hand side).*

Proof. Both sides are semisimple abelian categories, so we just need to check that the bijective functor between the two sides sends simple objects to simple objects, which we showed above. □

We define a third category, which will also be equivalent to the above two. For any vector space V, put $S_\lambda = (V^{\otimes n} \otimes S^\lambda)^{S_n}$. This defines a functor

$$S_\lambda \colon \mathrm{Vec}^{\mathrm{f}} \to \mathrm{Vec},$$

called a *Schur functor.*

We will define a class of functors from Vec to Vec which are the most natural consider in this context, called *polynomial functors.* Note first that the set of functors Vec → Vec forms an abelian category. Let

$$\begin{aligned} T_n\colon \mathrm{Vec}^{\mathrm{f}} &\to \mathrm{Vec} \\ V &\mapsto V^{\otimes n}. \end{aligned}$$

Definition 4.28. A functor $F\colon \mathrm{Vec}^{\mathrm{f}} \to \mathrm{Vec}$ is *polynomial* if it ccurs as a subquotient (in the functor category) of direct sum of T_n's. Let $\mathcal{P}$ be the abelian category of polynomial functors.

Exercise. The functors $T_n S_\lambda$ are polynomial.

Remark 4.29. Consider a functor $F\colon \mathrm{Vec}^{\mathrm{f}} \to \mathrm{Vec}^{\mathrm{f}}$. Then the following are equivalent:

(1) F is a subquotient of $\oplus$ of $(T_n)|_{\mathrm{Vec}^{\mathrm{f}}}$,
(2) for any $V, W \in \mathrm{Vec}^{\mathrm{f}}$,

$$\mathcal{F}\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}(\mathcal{F}V, \mathcal{F}W)$$

is a polynomial function.

Theorem 4.30. *The category $\mathcal{P}$ is semisimple and the Schur functors S_λ are simple.*

Proof. We first prove that S_λ is simple. Suppose $F \subseteq S_\lambda$ is a nonzero subobject. There exists an n such that $F(\mathbb{C}^n) \neq 0$. Then $F(\mathbb{C}^n) = S_\lambda(\mathbb{C}^n)$, because $S_\lambda(\mathbb{C}^n)$ is irreducible as a GL_n-representation. Suppose $m > n$. We have the standard inclusion $i\colon \mathbb{C}^n \to \mathbb{C}^m$ and the standard projection $p\colon \mathbb{C}^m \to \mathbb{C}^n$, with $p \circ i = \mathrm{id}$. Then the diagram

$$\begin{array}{ccc} F(\mathbb{C}^n) & \xrightarrow{=} & S_\lambda(\mathbb{C}^n) \\ \big\downarrow{\scriptstyle F(i)} & & \big\downarrow{\scriptstyle S_\lambda(i)} \\ F(\mathbb{C}^m) & \hookrightarrow & S_\lambda(\mathbb{C}^m) \end{array}$$

commutes. Clearly, $F(i)$ is injective, because $F(p) \circ F(i) = F(p \circ i) = \mathrm{id}$. Hence $F(\mathbb{C}^m) \neq 0$ and so $F(\mathbb{C}^m) = S_\lambda(\mathbb{C}^m)$.

Now suppose $m < n$. If $m < \ell(\lambda)$, then $S_\lambda(\mathbb{C}^m) = 0$, so $F(\mathbb{C}^m) = S_\lambda(\mathbb{C}^m)$. So suppose $\ell(\lambda) \leq m < n$. Then the diagram

$$\begin{array}{ccc} F(\mathbb{C}^n) & \xrightarrow{=} & S_\lambda(\mathbb{C}^n) \\ \big\downarrow{\scriptstyle F(p)} & & \big\downarrow{\scriptstyle S_\lambda(p)} \\ F(\mathbb{C}^m) & \longrightarrow & S_\lambda(\mathbb{C}^m) \end{array}$$

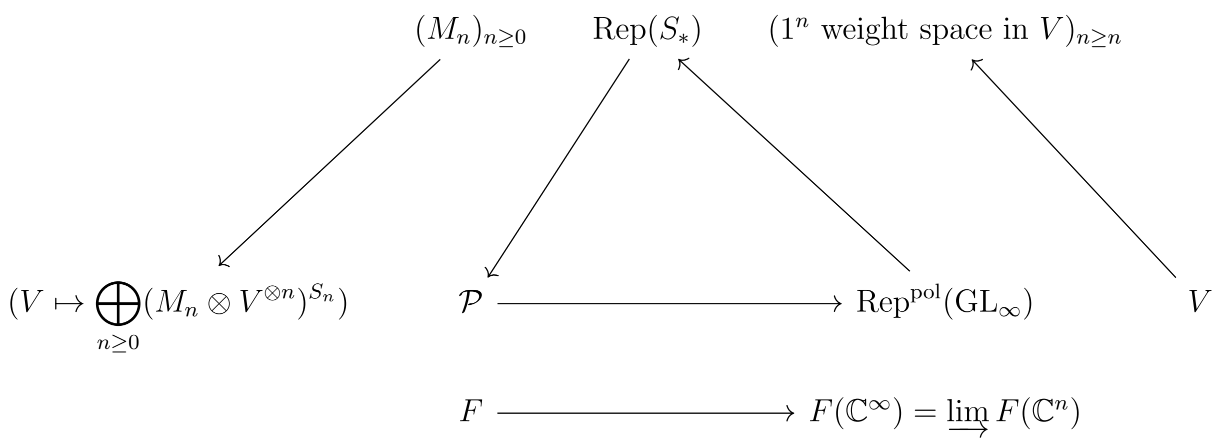
commutes. Since $S_\lambda(p) \circ S_\lambda(i) = \mathrm{id}$, $S_\lambda(p) \neq 0$. Thus $F(p) \neq 0$, and hence $F(\mathbb{C}^m) \neq 0$, and hence $F(\mathbb{C}^m) = S_\lambda(\mathbb{C}^n)$. This proves S_λ is simple.

Note that S_n acts on T_n in the category $\mathcal{P}$. Then

$$T_n = \bigoplus_{|\lambda|=n} S^\lambda \otimes S_\lambda$$

holds functorially, and hence T_n is a direct sum of simples, and hence it is semisimple. Any $\oplus$ or subquotient of semisimple objects is semisimple, so $\mathcal{P}$ is semisimple. $\square$

To summarize, we have the following equivalences of categories



We have an analogy between $\mathrm{Vec}^{\mathrm{f}}$ and $\mathbb{A}^1$, summarized by the following table

$\mathrm{Vec}^{\mathrm{f}}$ is an abelian $\otimes$-category	$\mathbb{A}^1$ is a ring variety
$\mathcal{P}$ are its automorphisms	$\mathbb{C}[t]$ and its automorphisms
(1) Given $V \in \mathrm{Vec}$, $\mathrm{ev}_v \colon F \mapsto F(V)$	(1) Given $a \in \mathbb{C}$, $\mathrm{ev}_a \colon \mathbb{C}[t] \to \mathbb{C}$
(2) $F, G \in \mathcal{P}$, $F \circ G \in \mathcal{P}$	(2) $f, g \in \mathbb{C}[t]$, $f \circ g \in \mathbb{C}[t]$
(3) $\mathcal{P}$ is a symmetric abelian $\otimes$-category with $(F \otimes G)(V) = F(V) \otimes G(V)$	(3) $\mathbb{C}[t]$ is a ring
(4) have coaddition $\mathcal{P} \to \mathcal{P}^2$ given by $F \mapsto ((V,W) \mapsto (V \oplus W))$ and comultiplication $\mathcal{P} \to \mathcal{P}^2$ given by $F \mapsto ((V,W) \mapsto (V \otimes W))$	(4) $\mathbb{C}[t]$ has coaddition $t \mapsto t \otimes 1 + 1 \otimes t$ and comultiplication $t \mapsto t \otimes t$

In (4) above, we define $\mathcal{P}^n$ as the category of polynomial functors $(\mathrm{Vec}^{\mathrm{f}})^n \to \mathrm{Vec}$ similarly to $\mathcal{P}$, with

$$T_{k_1,\dots,k_n}(V_1, \dots, V_n) = V_1^{\otimes k_1} \otimes \cdots \otimes V_n^{\otimes k_n}.$$

Fact 4.31. *The category $\mathbb{P}^n$ is semisimple category which objects are $S_{\lambda_1} \otimes \cdots \otimes S_{\lambda_n}$.*

Two interesting questions to ask are

(1) Describe how the extra structure described above works on S_λ's.
(2) Describe how the extra structure works on $\mathrm{Rep}(S_*)$ or $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$.

Let us start with describing coaddition on S_λ with $|\lambda| = n$. In other words, consider the functor $(V, W) \mapsto S_\lambda(V \oplus W)$. We have that

$$\begin{aligned} S_\lambda(V \oplus W) &= \mathrm{Hom}_{S_n}(S^\lambda, (V \oplus W)^{\otimes n}) \\ &= \mathrm{Hom}_{S_n}\left(S^\lambda, \bigoplus_{i+j=n} (\otimes\text{-strings with } i\ V\text{'s and } j\ W\text{'s})\right) \\ &= \mathrm{Hom}_{S_n}\left(S^\lambda, \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n}(V^{\otimes i} \otimes W^{\otimes j})\right) \\ &= \bigoplus_{i+j=n} \mathrm{Hom}_{S_i \times S_j}(\mathrm{Res}_{S_i \times S_j}^{S_n}(S^\lambda), V^{\otimes i} \otimes W^{\otimes j}) \\ &= \bigoplus_{i+j=n} \bigoplus_{\mu,\nu} C_{\mu,\nu}^\lambda \otimes \mathrm{Hom}_{S_i \times S_j}(S^\mu \otimes S^\nu, V^{\otimes i} \otimes W^{\otimes j}) \end{aligned}$$

and hence

$$S_\lambda(V \otimes W) = \bigoplus_{\substack{\mu,\nu \\ |\mu|=|\nu|=|\lambda|}} (S_\mu(V) \otimes S_\nu(W))^{\oplus g_{\lambda,\mu,\nu}},$$

where $g_{\lambda,\mu,\nu}$ are the *Kronecker coefficients*.

Exercise. Show that $\mathrm{Sym}^n(V \otimes W) = \bigoplus_{|\lambda|=n} S_\lambda(V) \otimes S_\lambda(W)$, which is known as the *Cauchy rule*.

Decomposing $S_\lambda \circ S_\nu$ is called the *plethysm problem*, and it is more or less impossible to do in general.

Exercise. $\mathrm{Sym}^n \circ \mathrm{Sym}^2 = \bigoplus_{\substack{|\lambda|=2n \\ \text{all points of} \\ \lambda \text{ even}}} S_\lambda$

If $\mathcal{A}$ is an abelian symmetric $\otimes$-category, we can do commutative algebra within $\mathcal{A}$. For example, a commutative algebra in $\mathcal{A}$ is an object A of $\mathcal{A}$ with multiplication $m\colon A \otimes A \to A$ satisfying the usual rules. For example, it is commutative

$$\begin{array}{ccc} A \otimes A & & \\ \Big\downarrow {\scriptstyle \text{symmetry of } \otimes} & \searrow^{m} & \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

Similarly, we can talk about A-modules in $\mathcal{A}$.

Of course, general abelian symmetric $\otimes$-categories are difficult to study, so we specialize to $\mathcal{A} = \mathrm{Rep}(S_*) \cong \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$.

Definition 4.32. A *twisted commutative algebra* (tca) is a commutative algebra in $\mathrm{Rep}(S_*)$.

A tca A is a graded vector space $A = \bigoplus_{n \geq 0} A_n$ with an action of S_n on A_n. We have multiplication

$$A \otimes A \to A$$

given by

$$\bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n}(A_i \otimes A_j) \to A_n,$$

which is the same as giving $S_i \times S_j$-equivariant maps

$$A_i \otimes A_j \to A_{i+j}$$

for any i and j.

These maps make A into an associative, unital graded $\mathbb{C}$-algebra. The commutativity law is

$$\begin{array}{ccccc} x \otimes y & & A_i \otimes A_j & \xrightarrow{m} & A_{i+j} \\ \downarrow & & \downarrow & & \downarrow \sigma \\ y \otimes x & & A_j \otimes A_i & \xrightarrow{m} & A_{i+j} \end{array}$$

where σ is a *twist* given by $\sigma(1) = j+1, \ldots, \sigma(i) = i+j, \sigma(i+1) = 1, \ldots, \sigma(i+j) = j$. This justifies the name twisted commutative algebra.

Under the equivalence $\mathrm{Rep}(S_*) \cong \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ tca's correspond to commutative associative unital $\mathbb{C}$-algebras with polynomial action of GL_∞.

Example 4.33. Let U be a vector space and $A = T(U)$ be the tensor algebra, i.e. $A_n = U^{\otimes n}$ with the permutation action of S_n. Then A is a tca. In fact,

$$A = \mathrm{Sym}(U \otimes S^{(1)}).$$

Under the Schur-Weyl duality 4.27, A corresponds to $\mathrm{Sym}(U \otimes \mathbb{C}^\infty)$.

Definition 4.34. A tca A is *noetherian* if Mod_A is locally noetherian, i.e. every object is $\varinjlim$ of noetherian objects.

Proposition 4.35. *If* $\dim(U) < \infty$*, then* $A = T(U)$ *is noetherian as a tca.*

Sketch of a proof. Let M be a finitely generated A-module. Then M is a quotient of $A \otimes V$ for some finite length object V in $\mathrm{Rep}(S_*)$. (The objects $A \otimes V$ are the projective objects in this category.)

We have that

$$A = \bigoplus_{\ell(\lambda) \leq d} S_\lambda(U) \otimes S_\lambda(\mathbb{C}^\infty)$$

by Cauchy rule. Hence all partitions in A have $\leq d$ rows. Hence all partitions in M have $\leq N$ rows for some N.

Now, think of Schur functors. Note that if $L \subseteq L' \subseteq M$ and $L(\mathbb{C}^N) = L'(\mathbb{C}^N)$, then $L = L'$. This is because we can decompose these as multiplicity spaces and they all have to agree.

Therefore, there exists an order-preserving injection

$$\{\text{submodules of } M\} \to \{A(\mathbb{C}^N)\text{-submodules of } M(\mathbb{C}^N)\}.$$

Since $A(\mathbb{C}^N)$ is a finitely-generated $\mathbb{C}$-algebra, $M(\mathbb{C}^N)$ is a finitely-generated module over $M(\mathbb{C}^N)$, and hence the right hand side satisfies the ascending chain condition. Hence the left hand side satisfies the ascending chain condition. □

Let us specialize to $\dim U = 1$. Then $A = \mathbb{C}[t]$ and S_n acts trivially on A_n. An A-module is a sequence (M_n) in $\mathrm{Rep}(S_*)$ with transition maps $M_n \to M_{n+1}$ which are S_n-equivariant.[3] These sequences have to satisfy a few extra conditions which we omit here.

We present an alternative way to view A-modules in this case.

Definition 4.36. Define FI to be the category whose objects are finite sets and maps are injections between the sets. An FI-module is a functor $FI \to \mathrm{Vec}$.

(The notion of FI-modules was introduced by Church–Ellenberg–Farb around 2012.)

If M is an FI-module, then

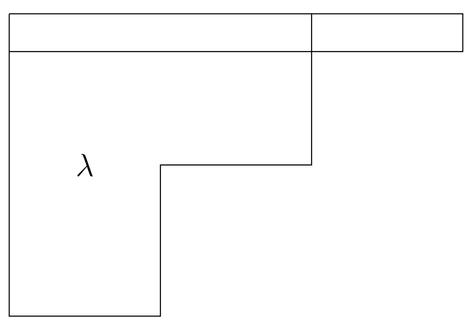
$$M_n = M([n]) \text{ if an } S_n\text{-representation},$$

where $[n] = \{1, \ldots, n\}$ and the inclusion $[n] \to [n+1]$ induces a transition map $M_n \to M_{n+1}$.

Fact 4.37. *We have an equivalence of categories* $\mathrm{Mod}_{\mathbb{C}[t]} \cong \mathrm{Mod}_{FI}$.

Every A-module is a quotient of direct sums of $A \otimes S^\lambda$. We can decompose this using the Pieri rule 3.15.

The following shapes appear in Pieri rule 3.15 for λ:



Let L_λ be the sum of all of these representations in $A \otimes S6\lambda$.

Fact 4.38. *The representation L_λ is an A-quotient of $A \otimes S^\lambda$.*

Fact 4.39. *If M is a finitely-generated A-module, then there exists a filtration $0 = F_0 \subset \cdots \subset F_n = M$ such that $F_i/F_{i-1} \cong L_\lambda$ (up to finite error, i.e. truncating to $\geq n$ on either side).*

Definition 4.40. Let M, N be A-modules. Let $M \boxtimes N$ be defined by $(M \boxtimes N)_n = M_n \otimes N_n$.

Note that $A \boxtimes A = A$, so if M, N are A-modules, so is $M \boxtimes N$.

Fact 4.41. *If M, N are finitely-generated, so is $M \boxtimes N$.*

[3] These correspond to multiplication by $t \in \mathbb{C}[t]$, and it is enough to say what the generator does.

The idea is to reduce to the case $M = A \otimes \mathbb{C}[S_m]$ and $N = A \otimes \mathbb{C}[S_n]$. For these modules, $\mathrm{Hom}_A(M, -) = (-)_m$, so these are analogs of free modules. In this case, we explicitly compute.

Recall the Kronecker coefficient $g_{\lambda,\mu,\nu}$ is the multiplicity of S^ν in $S^\lambda \otimes S^\mu$, where $|\lambda| = |\mu| = |\nu|$. We define $\lambda[n] = (\lambda_1 + n, \lambda_2 + n, \ldots)$.

We mentioned before that these coefficents are really hard to understand. However, there are partial results towards understanding them, the first of which is the following theorem.

Theorem 4.42 (Murnaghan). *Given any λ, μ, ν, $g_{\lambda[n],\mu[n],\nu[n]}$ is constant for $n \gg 0$.*

Remark 4.43. The eventual value is called the *stable Kronecker coefficient* $G_{\lambda,\mu,\nu}$.

Instead of presenting the original proof by Murnaghan, we show an easier proof by Church–Ellenberg–Farb which uses the introduced theory.

Proof. Since $L_\lambda \boxtimes L_\mu$ is a finitely-generated A-module, we have that

$$[L_\lambda \boxtimes L_\mu] = [L_{\nu_1}] + \cdots + [L_{\nu_k}]$$

in the Grothendieck group $\mathcal{K}(\mathcal{A})$ (up to finite error). For n large,

$$g_{\lambda[n],\mu[n],\nu[n]} = \#\{i \mid \nu_i = \nu\},$$

completing the proof. □

5. Representation theory in positive characteristic

We finally move on to representation theory in characteristic $p > 0$. We first review a few basic notions.

Let k be an algebraically closed field of characteristic $p > 0$ and G be a finite group.

$p \nmid \lvert G\rvert$	$p \mid \lvert G\rvert$
Representations of G over k are semisimple.	Representations of G over k are not semisimple in general.
The number of simples is the number of conjugacy classes of G.	The number of simples is the number of p-regular[4] conjugacy classes.

Example 5.1. If G is a p-group, then the number of p-regular conjugacy classes is 1, so there is only one simple, the trivial representations.

Example 5.2. Let $G = S_3$

If $p = 2$, then the 2-regular conjugacy classes are 1, (123). Then 2 simples are triv and std.

Indeed, std $= \ker \epsilon$, the kernel of the augmentation map $\epsilon \colon k^3 \to k$, and the augmentation map splits as $e_1 + e_2 + e_3 \in (k^3)^{S_3}$ has non-zero image under ϵ as $3 \neq 0$ in characterstic 2. Thus $k^3 = k \oplus \mathrm{std}$, and it is not hard to show that std is simple.

If $p = 3$, then the 3-regular conjugacy classes are 1, (12), so there are 2 simples: triv and sgn.

Therefore, std is an extension of the two. We have $\mathrm{std} = \ker \epsilon$ but

$$\epsilon(e_1 + e_2 + e_3) = 3 = 0$$

in characteristic 3, so

$$\mathrm{triv} = k(e_1 + e_2 + e_3) \subseteq \mathrm{std}\,.$$

The quotient is 1-dimensional, spanned by the image of $e_1 - e_2$, so it must be either triv or sgn. Since $(12)(e_1 - e_2) = -(e_1 - e_2)$, the quotient must be sgn.

First goal. Understand the simples of S_n in characteristic $p > 0$.

The number of p-regular conjugacy classes in S_n is the number of partitions of n into parts prime to p. However, we will actually parametrize the irreducibles by a different class of partitions.

Definition 5.3. A partition is *p-singular* if some part occurs at least p times (i.e. $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$) and *$p$-regular* otherwise.

We will index the irreducibles of S_n in characteristic p by p-regular partitions. Let us first see that their number is correct. Note that

$$\prod_{n\geq 1} \frac{1}{1-x^n} = \prod_{n\geq 1}(1 + x^n + x^{2n} + \cdots),$$

and hence the coefficient of x^n is the number of partitions of n. In other words, this is the generating function for the number of partitions of n.

Lemma 5.4. *The number of p-regular partitions of n is the number of p-regular conjugacy classes in S_n.*

Proof. Consider the product

$$P = \prod_{n\geq 1} \frac{1-x^{np}}{1-x^n}$$

in two ways to get the result.

(1) By canceling the fractions, we obtain
$$P = \prod_{p\,\nmid\, n} \frac{1}{1-x^n} = \prod_{p\,\nmid\, n}(1 + x^n + x^{2n} + \cdots),$$
so the coefficient of x^n is the number of p-regular conjugacy classes in S_n.

(2) By expanding $1 - x^{np}$, we obtain
$$P = \prod_{n\geq 1}(1 + x^n + x^{2n} + \cdots + x^{(p-1)n})$$
so the coefficient of x^n is the number of p-regular partitions of n.

This shows the desired equality. □

We define the *space of tabloids* M^λ and the *Specht module* $S^\lambda \subset M^\lambda$ as in characteristic 0.

Remark 5.5. Some of the results/proofs in characteristic 0 are still valid in characteristic $p > 0$:

- Garnir relations 2.35,
- Standard basis 2.40 (so $\dim S^\lambda$ is the same in all characteristics),
- First version of the Pieri rule 3.15: if $|\lambda| = n$, then $S^\lambda|_{S_{n-1}}$ has a filtration where graded pieces are S^μ where μ are what appears in the Pieri rule.

We have an invariant form $\langle\ ,\ \rangle$ on M^λ given by

$$\langle \{t\}, \{t'\} \rangle = \begin{cases} 1 & \text{if } \{t\} = \{t'\} \\ 0 & \text{otherwise} \end{cases}$$

which induces a form on S^λ.

Proposition 5.6. *If $V \subseteq M^\lambda$ is a subrepresentation, then $V \supseteq S^\lambda$ or $V \subseteq (S^\lambda)^\perp$.*

Proof. (Same as in characteristic 0.) Pick $v \in V$, tableau t. Then $\kappa_t v$ is a scalar multiple of e_t. If $\kappa_t v \neq 0$ for some v, t, then $e_t \in V$, so $S^\lambda \subset V$. If $\kappa_t v = 0$ for all v, t, then $0 = \langle \kappa_t v, \{t\} \rangle = \langle v, e_t \rangle = 0$ for all v, t, so $V \subseteq (S^\lambda)^\perp$. □

Proposition 5.7. *The quotient*

$$S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$$

is either 0 *or irreducible.*

Proof. Let $V \subseteq S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ be subrepresentations, and

$$S^\lambda \cap (S^\lambda) \subseteq \widetilde{V} \subseteq S^\lambda$$

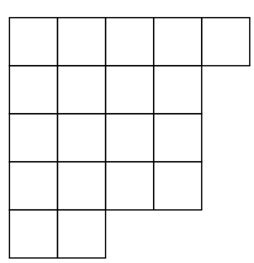
be the inverse image under the quotient map. By Proposition 5.6, either $\widetilde{V} \supseteq S^\lambda$, and then $\widetilde{V} = S^\lambda$, and V is the whole space, or $\widetilde{V} \subseteq (S^\lambda)^\perp$, and then $\widetilde{V} = S^\lambda \cap (S^\lambda)^\perp$, so $V = 0$. □

The difference in positive characteristic is that the quotient could actually be 0. We will try to understand when this happens.

Proposition 5.8. *The inner product $\langle\ ,\ \rangle$ is 0 on S^λ if and only if λ is p-singular.*

Proof. Suppose λ is p-singular. We consider $\langle e_t, e_{t'} \rangle$. If a part of λ appears with multiplicity k, we get an action of $\mathbb{Z}/k\mathbb{Z}$ on the set of tabloids in both e_t and $e_{t'}$ by cycling the relevant rows.

For example, in



we have an action of $\mathbb{Z}/3\mathbb{Z}$ on the three middle rows.

Then $\langle e_t, e_{t'} \rangle$ is a multiple of k. Taking $k = p$ (this is possible, since λ is p-singular), we get $\langle e_t, e_{t'} \rangle = 0$.

Suppose λ is p-regular. Let t be a tableau and t' the tableau obtained from t by reversing each row:

$$t = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 & 10 \\ \cline{1-3}\end{array} \qquad t' = \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline 7 & 6 & 5 \\ \cline{1-3} 10 & 9 & 8 \\ \cline{1-3}\end{array}$$

Consider $\langle e_t, e_{t'}\rangle$. Suppose $\sigma \in C_t$ and satisfies i and $\sigma(i)$ below to the rows of same length for all i. Then $\sigma\{t\}$ appears both in e_t and in $e_{t'}$. In the example above, $\sigma = (58)$ is allowed but $\sigma = (15)$ is not allowed.

One can see that these $\sigma\{t\}$ account for all tabloids in common between t and t'. Then $\langle e_t, e_{t'}\rangle$ is the number of such σ. This is equal to

$$\prod_{n\geq 1}(\#(\text{rows of length } n)!)^n,$$

and $\#(\text{rows of length } n) < p$, because λ is p-regular, which shows that $\langle e_t, e_{t'}\rangle \neq 0$. □

Definition 5.9. For λ p-regular, put $D^\lambda = S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$.

By Propositions 5.7 and 5.8, D^λ is an irreducible representation of S_n (in particular, it is nonzero).

The next step will be to show that all D^λ's are distinct. Note that there are the correct number of D^λ's, so if we show they are all distinct, this will show that they are all the irreducibles.

Lemma 5.10. *Let λ, μ be partitions of n with λ p-regular. Consider a nonzero homomorphism*

$$\theta\colon S^\lambda \to M^\mu/U$$

for some $U \subset M^\mu$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then $\operatorname{im}\theta = \frac{S^\mu+U}{U}$.

Recall the following Lemma we proved in Chapter 2.

Lemma (Lemma 2.24)**.** *Let λ and μ be partitions of n, t a λ-tableau, t' a μ-tableau. Suppose*

$$\kappa_t\{t'\} \neq 0.$$

Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then $\kappa_t\{t'\} = \pm e_t$.

The proof still works in any characteristic, so we may use it here.

Proof of Lemma 5.10. Let t be a λ-tableau, t' be the row revesal of t, as in th proof of Proposition 5.7. Then $\langle e_t, e_{t'}\rangle \neq 0$.

We have that

$$0 \neq \langle e_{t'}, \kappa_t\{t\}\rangle = \langle \kappa_t e_t', \{t\}\rangle.$$

Hence $\kappa_t e_t' \neq 0$, so $\kappa_t e_{t'} = h e_t$ for some nonzero $h \in k$. We then have that

$$h\theta(e_t) = \theta(\kappa_t e_{t'}) = \kappa_t\theta(e_{t'}) \neq 0,$$

WIndeed, we have $\theta(e_t) \neq 0$ because $\theta \neq 0$ and $e_{t'}$ generates S^λ, and hence $h \neq 0$. Thus $\theta(e_{t'}) \in M^\mu/U$ is not killed by κ_t. Hence $\kappa_t M^\mu \neq 0$, and by Lemma 2.24, $\lambda \trianglerighteq \mu$.

If $\lambda = \mu$, then $\theta(e_t) = h^{-1}\kappa_t\theta(e_{t'})$ and by Lemma 2.24, this is a scalar multiple of $e_t \pmod{U}$, so $\operatorname{im}\theta = \frac{S^\mu + U}{U}$. □

Corollary 5.11. *Let λ, μ, U be as in Lemma 5.10. Suppose $\theta\colon D^\lambda \to M^\mu/U$ is nonzero. Then $\lambda \trianglerighteq \mu$ and if $U \supset S^\mu$ then $\mu \neq \lambda$.*

Proof. Consider $S^\lambda \twoheadrightarrow D^\lambda \xrightarrow{\theta} M^\mu/U$. By Lemma 5.10, we get $\mu \trianglerighteq \mu$, and the other assertion is clear. □

Theorem 5.12.

(1) Every irreducible of S_n is isomorphic to D^λ for some λ p-regular.
(2) No two distinct D^λ's are isomorphic.
(3) Each D^λ is self-dual, absolutely irreducible, defined over $\mathbb{F}_p$.

Proof. For (2), suppose $D^\lambda \cong D^\mu$. This gives a nonzero map

$$D^\lambda \to D^\mu \subset \frac{M^\mu}{S^\mu \cap (S^\mu)^\perp},$$

so Corollary 5.11 gives $\lambda \trianglerighteq \mu$, and we also get $\mu \trianglerighteq \lambda$ by symmetry so $\lambda = \mu$. Then (1) follows by counting the number of irreducibles.

Finally for (3), it is obvious that D^λ is defined over $\mathbb{F}_p$, and it is absolutely irreducible (because we never used properties of k, just that is has characteristic p). Finally, $\langle\ \rangle$ is a perfect pairing on M^λ, which induces a perfect pairing on $S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$, so D^λ is self-dual. □

Theorem 5.13. *All the Jordan-Hölder constituents of M^μ have the form D^λ with $\lambda \trianglerighteq \mu$. The module D^μ occurs if and only if μ is p-regular, in which case it has multiplicity 1.*

Proof. What we just proved shows that all constituents of M^μ/S^μ are D^λ with $\lambda \triangleright \mu$. Indeed, if D^λ is a constituent, we get an injection $D^\lambda \to M^\mu/U$ for some $U \supset S^\mu$, and $\lambda \triangleright \mu$ follows from Corollary 5.11.

We have an isomorphism

$$(S^\mu)^\perp \cong (M^\mu/S^\mu)^*$$

coming from the inner product $\langle\ ,\ \rangle$. Hence all the constituents of $(S^\mu)^\perp$ have the form

$$(\text{constituent of } M^\mu/S^\mu)^* = (D^\lambda)^* = D^\lambda$$

with $\lambda \triangleright \mu$. The only left over part of M^μ is

$$S^\mu/(S^\mu \cap (S^\mu)^\perp) = \begin{cases} 0 & \mu \text{ is } p\text{-singular,} \\ D^\mu & \text{otherwise.} \end{cases}$$

Indeed, we have the short exact sequences

$$0 \longrightarrow S^\mu \longrightarrow M \longrightarrow M/S^\mu \longrightarrow 0$$

$$0 \longrightarrow S^\mu \cap (S^\mu)^\perp \longrightarrow S^\mu \longrightarrow S^\mu/(S^\mu \cap (S^\mu)^\perp) \longrightarrow 0$$

This completes the proof. □

Remark 5.14. It is in general very hard to know the multiplicity of D^μ in S^λ, as we did in characteristic 0.

Remark 5.15. For any finite group G, we have a well-defined map

$$\mathcal{K}(\mathrm{Rep}_{\mathbb{Q}}(G)) \longrightarrow \mathcal{K}(\mathrm{Rep}_{\mathbb{F}_p}(G))$$

$$[V] \longrightarrow [M/pM]$$

where $M \subseteq V$ is a G-stable lattice. (We can always get such a lattice by taking any lattice in V and averaging over the group G.)

In $\mathcal{K}(\mathrm{Rep}_{\mathbb{Q}}(S_n))$, we have Young's rule

$$[M^\mu] = [S^\mu] + (\text{sum of } [S^\mu]\text{'s with } \lambda \rhd \mu)\,.$$

Therefore, the same holds in $\mathcal{K}(\mathrm{Rep}_{\mathbb{F}_p})(S_n)$, because M^μ and S^μ are defined integrally, so they are preserved by the functor.

We can use this to prove that the D^λ's are all the irreducibles without counting. We will prove by induction on μ that all constituents of M^μ are D^λ's. We have that

$$[M^\mu] = [S^\mu] + \underbrace{(\text{sum of } [S^\mu]\text{'s with } \lambda \rhd \mu)}_{\text{sum of } [D^\nu]\text{'s by induction}},$$

and

$$[S^\mu] = [S^\mu \cap (S^\mu)^\perp] + \underbrace{[S^\mu/S^\mu \cap (S^\mu)^\perp]}_{=0 \text{ or } D^\mu},$$

and $S^\mu \cap (S^\mu)^\perp \subseteq (S^\mu)^\perp \cong (M^\mu/S^\mu)^*$, which is a sum of $[D^\nu]$'s by induction.

Note that $M^{1^n} = k[S_n]$ and all the constituents of D^λ's, so D^λ's are all irreducible.

Question. When is S^λ irreducible?

Let us give an example where we can answer this question.

Lemma 5.16. *The standard representation $S^{(n-1,1)}$ of S_n is irreducible if and only if $p \nmid n$.*

Proof. Suppose

$$0 \neq x = \sum_{i=1}^n a_i e_i \in \overbrace{\ker(\epsilon \colon k^n \to k)}^{\text{std}}.$$

Because $p \nmid n$, there exist $i \neq j$ such that $a_i \neq a_j$. Then

$$(i\ j)x - x = (a_i - a_j)(e_j - e_i).$$

Hence if $V \subseteq \mathrm{std}$ is a nonzero subrepresentation, then V contains $e_i - e_j$ for some $i \neq j$. Then V contains $e_i - e_j$ for all i, j by using S_n, so $V = \mathrm{std}$.

If $p|n$, then $k(e_1 + \cdots + e_n) \subset \mathrm{std}$ is S_n-stable. □

Next, we will give a necessary and sufficient condition for a general S^λ for p-regular λ to be irreducible.

Lemma 5.17. *Suppose* $\mathrm{End}_{S_n}(S^\mu) = k$.[5] *Then S^μ is irreducible if and only if it is self dual.*

Proof. Any irreducible is self-dual by Theorem 5.12. Suppose S^μ self-dual. Let $U \subseteq S^\mu$ be an irreducible submodule. Then

$$\underbrace{S^\mu \cong (S^\mu)^*}_{\text{assumption}} \to \underbrace{U^\mu \cong U}_{U \text{ irreducible}} \subseteq S^\mu.$$

Hence the composition $S^\mu \to S^\mu$ must be a scalar multiple of id because $\mathrm{End}(S^\mu) = k$. This shows $U = S^\mu$. □

Let $g^\mu = \gcd(\langle e_t, e_{t'} \rangle \mid t, t' \text{ any } \mu\text{-tableaux})$, computed in S^μ over $\mathbb{Q}$. We showed that $p|g^\mu$ if and only if μ is p-singular. In fact, our proof showed more: if $z_j = \#(\text{parts of } \mu = j)$, then

$$\prod z_j! \;\Big|\; g^\mu \;\Big|\; \prod (z_j!)^j$$

where we got the first one by counting the common tabloids in e_t and $e_{t'}$, and the second one by looking at $\langle e_t, e_{t'} \rangle$ where t' was the row reversal of t (see the proof of Proposition 5.8).

Recall that

$$\kappa_t = \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\sigma,$$

$$\rho_t = \sum_{\sigma \in R_t} \sigma.$$

Lemma 5.18.

(1) *The gcd of the coefficients of the tabloids in $\rho_t \kappa_t \{t\}$ is $g^{\mu^\dagger}$.*
(2) *We have that $\kappa_t \rho_t \kappa_t = \prod (\text{hook lengths of } \mu)\, \kappa_t \{t\}$.*

Proof. Suppose t is a μ-tableau.

[5] This assumption holds always in characteristic $p \neq 2$ and often in characteristic 2. This follows from our results on semistandard homomorphisms.

For (1), note that $g^{\mu^\dagger} = \gcd_{\pi \in S_n} \langle e_{t^\dagger}, \pi e_{t^\dagger} \rangle$. Then

$$\begin{aligned}
\operatorname{sgn}(\pi)\langle e_{t^\dagger}, \pi e_{t^\dagger}\rangle &= \operatorname{sgn}(\pi)\langle \{t^\dagger\}, \kappa_{t^\dagger}\pi\kappa_{t^\dagger}\{t^\dagger\}\rangle \\
&= \sum_{\substack{\sigma,\tau \in C_{t^\dagger} \\ \sigma\pi\tau \in R_{t^\dagger}}} \operatorname{sgn}(\sigma\pi\tau) \\
&= \sum_{\substack{\omega \in R_{t^\dagger} \\ \tau \in C_{t^\dagger} \\ \omega\tau^{-1}\pi^{-1} \in C_{t^\dagger}}} \operatorname{sgn}(\omega) \\
&= \sum_{\substack{\omega \in C_t \\ \tau \in R_t \\ \omega\pi^{-1}\tau^{-1}}} \operatorname{sgn}(\omega) \\
&= \langle \{t\}, \pi^{-1}\rho_t\kappa_t\{t\}\rangle \\
&= \langle \{\pi\{t\}, \rho_t\kappa_t\{t\}\}
\end{aligned}$$

Hence $g^{\mu^\dagger} = \gcd_{\pi \in S_n}\langle \pi\{t\}, \rho_t\kappa_t\{t\}\rangle =$ gcd of coefficients in $\rho_t\kappa_t\{t\}$.

For (2), we know that $\kappa_t\rho_t\kappa_t\{t\} = c\kappa_t\{t\}$ for some c. We have a map

$$\begin{aligned}
M^\mu &\to \mathbb{Q}[S_n] \\
\sigma\{t\} &\mapsto \sigma\rho_t
\end{aligned}$$

and hence $S^\mu = \mathbb{Q}[S_n]\kappa_t\rho_t$. We know that $(\kappa_t\rho_t)^2 = c\kappa_t\rho_t$. Let U be the complement in $\mathbb{Q}[S_n]$ to $\mathbb{Q}[S_n]\kappa_t\rho_t$. Consider right multiplication by $\kappa_t\rho_t$ on $\mathbb{Q}[S_n]$, and compute its trace in two ways.

First, $\mathbb{Q}[S_n] = \mathbb{Q}[S_n]\kappa_t\rho_t \oplus U$. Since $\kappa_t\rho_t$ is a projector onto its image, it acts by 0 on U. The matrix for this operator is hence

$$\left(\begin{array}{c|c} c & 0 \\ \hline 0 & 0 \end{array}\right)$$

Hence the trace of right multiplication by $\kappa_t\rho_t$ is $c \cdot \dim S^\mu$.

We compute the trace in another way. The coefficient of 1 in $\kappa_t\rho_t$ is 1 because $R_t \cap C_t = \{1\}$. Hence the coefficient of σ in $\sigma\kappa_t\rho_t$ is 1 for all $\sigma \in S_n$. Hence in the basis $\{\sigma\}_{\sigma \in S_n}$ of $\mathbb{Q}[S_n]$, the matrix for $\kappa_t\rho_t$ has 1s on the diagonals, which shows that its trace is $n!$.

This shows that $c \cdot \dim S^\mu = n!$ which, by the Hook Length Formula 3.37, implies the result $c = \prod(\text{hook lengths in } \mu)$. ∎

Proposition 5.19. *Let $\theta\colon M_k^\mu \to S_k^\mu$ be the map*

$$\{t\} \mapsto \underbrace{\left(\frac{1}{g^{\mu^\dagger}}\rho_t\kappa_t\{t\}\right)}_{\textit{computed in } M_{\mathbb{Z}}^\mu} \mod p.$$

Theorem 5.20. *We have that* $\ker\theta \supseteq (S^\mu_k)^\perp$ *and if* $\ker\theta = (S^\mu_k)^\perp$ *and* $\mathrm{End}(S^\mu) = k$ *then* $\operatorname{im}\theta = S^\mu$ *and* S^μ *is irreducible.*

Proof. Let $\theta_{\mathbb{Q}} \colon M^\mu_{\mathbb{Q}} \to S^\mu_{\mathbb{Q}}$ be given by the same formula. Then

$$\theta_{\mathbb{Q}}(\kappa_t\{t\}) = (\text{nonzero rational number}) \cdot e_t,$$

so it is nonzero. Hence $\theta_{\mathbb{Q}}$ is a nonzero multiplication of the standard projection, which shows that $\ker(\theta_{\mathbb{Q}}) = (S^\mu_{\mathbb{Q}})^\perp$.

Consider $(S^\mu_{\mathbb{Q}})^\perp \cap M^\mu_{\mathbb{Z}}$, a free $\mathbb{Z}$-module of rank $(\dim S^\mu_{\mathbb{Q}})^\perp$. Consider a $\mathbb{Z}$-basis. Its reduction mod p is linearly independent in $(S^\mu_k)^\perp$, and hence it spans a subspace of $(S^\mu_k)^\perp$ of dimension equal to $\dim(S^\mu_{\mathbb{Q}})^\perp = (S^\mu_k)^\perp$, since they both have basis given by semistandard tableaux. Hence element of $(S^\mu_k)^\perp$ lifts to an element of $(S^\mu_{\mathbb{Q}})^\perp$. Thus $\ker(\theta) \supseteq (S^\mu_k)^\perp$.

If $\ker\theta = (S^\mu_k)^\perp$, then θ induces an isomorphism

$$\underbrace{M^\mu/(S^\mu)^\perp}_{\cong (S^\mu)^*} \to S^\mu,$$

which shows that S^μ is self-dual, and hence it is irreducible by Lemma 5.17. $\square$

Theorem 5.21. *Suppose* μ *is* p*-regular. Then* S^μ *is reducible if and only if* p *divides*

$$\underbrace{\frac{\prod(\textit{hook lengths in } \mu)}{g^{\mu^\dagger}}}_{\textit{this is an integer by Lemma 5.18}}.$$

Proof. By Theorem 5.20, S^μ is reducible if and only if $\ker\theta \supsetneq (S^\mu)^\perp$. Note that $\frac{S^\mu + (S^\mu)^\perp}{(S^\mu)^\perp}$ is the unique minimal submodule of $M^\mu/(S^\mu)^\perp$. Hence S^μ is irreducible if and only if $\ker\theta \supset S^\mu$.

Finally, we have that

$$\theta(\kappa_t\{t\}) = \frac{1}{g^{\mu^\dagger}} \kappa_t \rho_t \kappa_t \{t\} = \frac{\prod(\text{hook lengths})}{g^{\mu^\dagger}} \{t\} = 0$$

if and only if p divides $\frac{\prod(\text{hook lengths})}{g^{\mu^\dagger}}$. $\square$

Theorem 5.22 (James and Murphy, conjectured by Carter)**.** *Consider the diagram obtained from* λ *by placing in each box the* p*-adic valuation of the hook length. Then the following are equivalent:*

(1) the numbers are constant along the columns,
(2) λ *is* p*-regular and* S^λ *is irreducible.*

5.1. **Polynomial representations of** GL_m**.** Recall that in characteristic 0 we had an equivalence between $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ and $\mathrm{Rep}(S_*)$. We will see what happens in characteristic $p > 0$.

Fix a field k of characteristic $p > 0$.

Definition 5.23. An *algebraic representation* of GL_m is a comodule of the coordinate ring $k[\mathrm{GL}_m]$. If k is infinite, a finite-dimensional algebraic representation is the same as a representation o f$\mathrm{GL}_n(k)$ such that the matrix entries are rational functions. (This is not true is k is finite.)

Example 5.24. Let $m = 1$, i.e. consider the algebraic representations of $\mathrm{GL}_1 = \mathbb{G}_m$, $k[\mathbb{G}_m] = k[t, t^{-1}]$. The comultiplication is

$$\begin{aligned} \Delta \colon k[\mathbb{G}_m] &\to k[\mathbb{G}_m] \otimes k[\mathbb{G}_m] \\ t &\mapsto t \otimes t \end{aligned}$$

and the counit is

$$\begin{aligned} \eta \colon k[\mathbb{G}_m] &\to k \\ t &\mapsto 1. \end{aligned}$$

Suppose M is a comodule for $k[\mathbb{G}_m]$ with the comodule structure given by $\Delta \colon M \to k[\mathbb{G}_m] \otimes M$ satisfying the approporiate axioms. Let

$$M_n = \{m \in M \mid \Delta(m) = t^n \otimes m\}.$$

We claim that $M = \bigoplus_{n \in \mathbb{Z}} M_n$. For $m \in M$,

$$\Delta(m) = \sum_{n \in \mathbb{Z}} t^n \otimes m_n \quad \text{for some } m_n \in M.$$

By one of the comodule axioms $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, which gives

$$\sum_{n \in \mathbb{Z}} t^n \otimes t^n \otimes m_n = \sum_{n \in \mathbb{Z}} t^n \otimes \Delta m_n,$$

which shows that $\Delta m_n = t^n \otimes m_n$, i.e. $m_n \in M_n$. By the counit axiom, we obtain

$$m = (\eta \otimes 1)\Delta m = \sum_{n \in \mathbb{Z}} m_n,$$

which proves the claim.

This shows that

$$\mathrm{Rep}^{\mathrm{alg}}(\mathbb{G}_m) \cong (\text{graded vector spaces})$$

so it is semisimple, and for each $n \in \mathbb{Z}$ there is a simple which has a 1-dimensional graded vector space concentrated in degree n.

We present two applications of this example.

(1) If we consider $\mathbb{G}_m \subseteq \mathrm{GL}_m$ as scalar matrices. Every algebraic GL_m-representation of V breaks up as $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $\mathbb{G}_m$ acts by $t \mapsto t^n$ in V_n.

(2) The same analysis applies to:

$$\mathrm{Rep}(\mathbb{G}_m^m) = (\mathbb{Z}^m\text{-graded vector spaces})$$

and $(\mathbb{G}_m)^m \subseteq \mathrm{GL}_m$ as diagonal matrices. Hence if V is an algebraic representation of GL_m, we get a decomposition of vector spaces

$$V = \bigoplus_{\lambda \in \mathbb{Z}^m} V_\lambda,$$

where G_m^m acts on V_λ via $(t_1, \ldots, t_n) \mapsto t^n$. This is the *weight decomposition* of V.

Remark 5.25. Note that $t \mapsto 1$ and $t \mapsto t^{p-1}$ are two non-isomorphic representation of $\mathbb{G}_m$, but they are isomorphic representations of $\mathbb{G}_m(\mathbb{F}_p)$.

Examples 5.26.

- The usual symmetric and exterior powers are representations of GL_m.
- Symmetric powers are **not** irreducible in general. Let $V = k^m$ be the standard representation of GL_m and consider

  $$\mathrm{Sym}^p(V) = \text{span of degree } p \text{ polynomials in variables } x_1, \ldots, x_m.$$

  Note that the span of the x_i^p is GL_m-stable when k is infinite. The reason is that if $g \in \mathrm{GL}_m(k)$ and

  $$g = \begin{pmatrix} a_{11} & \cdots \\ a_{21} & \cdots \\ \vdots & \\ a_{n1} & \cdots \end{pmatrix}$$

  then

  $$gx_1^p = (gx_1)^p = (a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m)^p = a_{11}^p x_1^p + a_{21}^p x_2^p + \cdots + a_{m1}^p x_m^p.$$

  Thus we have an exact sequence

  $$0 \longrightarrow (\text{span of } p\text{th powers}) \longrightarrow \mathrm{Sym}^p(V) \longrightarrow (\text{quotient}) \longrightarrow 0.$$

  One can see that this sequence is not split. Indeed, it is enough to show that any non-zero subrepresentation of $\mathrm{Sym}^p(V)$ contains x_i^p. This can be done by considering the operators $E_{i,j} = x_j \frac{d}{dx_i}$.

  Hence $\mathrm{Rep}^{\mathrm{alg}}(\mathrm{GL}_m)$ is not semisimple for $m > 1$.
- We define

  $$\mathrm{Sym}^n(V) = (V^{\otimes n})_{S_n} \qquad \text{(the coninvariants)}$$

  $$D^n(V) = (V^{\otimes n})^{S_n} \qquad \text{(the invariants)}$$

  where the latter is called the *nth divided power*. Then we get a diagram

  $$D^n(V) \hookrightarrow V^{\otimes n} \twoheadrightarrow \mathrm{Sym}^n(V)$$

  where the composition is an isomorphism in characteristic 0 but not in characteristic $p > 0$ (in general).

  We also get the short exact sequence

$$0 \longrightarrow (\text{kernal}) \longrightarrow D^p(V) \longrightarrow (\text{span of } p\text{th powers}) \longrightarrow 0$$

$$x_i \otimes \cdots \otimes x_i \longrightarrow x_i^p$$

kills other basis vectors

This shows that $D^p(V)$ is not isomorphic to $\mathrm{Sym}^p(V)$ even by a different isomorphism.

- If V is any algebraic representation of GL_m, we can precompose with the Frobenius map $F\colon \mathrm{GL}_m \to \mathrm{GL}_m$, taking $g = (g_{ij}) = (g_{ij}^p)$, to get a new representation $F(V)$.

  For example, when $m = 2$, $F(\mathrm{std}) = ke_1 \oplus ke_2$, and
  $$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\text{in } F(\mathrm{std})} \cdot e_1 = \underbrace{\begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}}_{\text{in std}} \cdot e_1 = a^p e_1 + c^p e_2.$$
  Hence $F(\mathrm{std}) = (\text{sum of } p\text{th powers}) \subset \mathrm{Sym}^p$. This is what appeared in two of the exact sequences above.
- For any algebraic representation V, V^* is naturally an algebraic representation. Note that if $V = ke_1 \oplus \cdots \oplus ke_m$ is the standard representation then
  $$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} e_i = a_i e_i,$$
  and the weight is $\lambda = (0, \dots, 1, \dots, 0)$ with 1 in the ith spot. Hence the weights in V have one entry 1 and the rest 0s. A similar argument shows that the weights in V^* have one entry -1 and the rest 0s. Therefore, dualizing can take representations with positive weights to ones that have negative weights. When we restrict our attention to polynomial representations later, we will want the representations which have positive weights, and hence we define a different dual.

  Define $V^\vee$ to be V^* precomposed with the automorphism
  $$^t(-)^{-1}\colon \mathrm{GL}_m \to \mathrm{GL}_m.$$
  Then $V^\vee$ is V^* as a vector space with the action
  $$(g \cdot \lambda)(v) = \lambda(^t g v).$$
  We then have that
  $$\mathrm{Sym}^n(V)^\vee = D^n(V^\vee)$$
  for all vector spaces V, canonically. If V is the standard representation of GL_m, then $V \cong V^\vee$ as GL_m-representations.

Definition 5.27. A representation of GL_m is *polynomial* if it occurs as a subquotient of direct sums of $\otimes$-powers of std. Equivalently, a *polynomial representation* of GL_m is a comodule for $k[M_m]$ (where the comomdule structure comes from matrix multiplication).

Note that if W is a polynomial representation, so is $W^\vee$, since $\mathrm{std}^\vee \cong \mathrm{std}$ and
$$(W_1 \otimes W_2)^\vee \cong W_1^\vee \otimes W_2^\vee.$$

Proposition 5.28. *The category* $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ *(the degree n piece) is equivalent to the category of modules over* $D^n(\mathrm{End}(V)) = (\mathrm{End}(V)^{\otimes n})^{S_n}$, $V = k^m$.

Proof. The objects in $\mathrm{Rep}^{\mathrm{pol}(\mathrm{GL}_m)}$ are comodules over

$$k[S_m] = \mathrm{Sym}(\mathrm{End}\, V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathrm{End}(V)),$$

a sum of coalgebras. Hence objects in $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ are comodules over $\mathrm{Sym}^n(\mathrm{End}(V))$, which are modules over

$$\mathrm{Sym}^n(\mathrm{End}(V))^* = D^n((\mathrm{End}\, V)^*) = D^n(\mathrm{End}\, V).$$

This completes the proof. □

For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, put

$$D^\lambda(V) = D^{\lambda_1}(V) \otimes \cdots \otimes D^{\lambda_r}(V).$$

For $\lambda = (n)$, we see that $D^\lambda(V) = D^n(V)$; for $\lambda = (1, \dots, 1)$, $D^\lambda = V^{\otimes n}$.

Proposition 5.29. *Let* $|\lambda| \leq m = \dim(V)$. *Then* $D^\lambda(V)$ *is projective in* $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)$. *Hence every degree n object is the quotient of direct sums of* D^λ*'s with* $|\lambda| = n$.

Proof. Trivially, $D^n(\mathrm{End}(V))$ is projective as a $D^n(\mathrm{End}(V))$-module. The multiplication on $D^n(\mathrm{End}(V))$ is

$$\underbrace{D^n(\mathrm{End}(V))}_{D^n(V\otimes V^*)} \otimes \underbrace{D^n(\mathrm{End}(V))}_{D^n(V\otimes V^*)} \to D^n(\mathrm{End}(V))$$

is induced by pairing the V with the V^*. As a module,

$$D^n(\mathrm{End}(V)) = D^n(V^{\oplus m}) = \bigoplus_{|\lambda|=n//\ell(\lambda)\leq m} D^\lambda(V).$$

This shows that D^λ is projective if $\ell(\lambda) \leq m$ as direct summand of a projective module. □

Corollary 5.30. *The modules* $\mathrm{Sym}^\lambda(V) = \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V)$ *are injective for* $|\lambda| \leq m = \dim(V)$. *Every degree n representation with* $n \leq m$ *injects into a direct sum of these.*

Warning. The modules D^λ and Sym^λ are **not** indecomposable in general.

Remark 5.31. For $m' \geq m$, we have an exact functor

$$\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_{m'})_n \to \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$$
$$V \mapsto \begin{pmatrix} \text{sum of weight spaces} \\ W_\lambda \text{ where } \lambda_0 = 0 \text{ for } i > m \end{pmatrix}$$

One can show that this is an equivalence for $m', m \gg 0$.

We have functors for $m \geq n$

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n & \xrightarrow{\Phi} & \mathrm{Rep}(S_n) \\ W & \longrightarrow & (1^n \text{ weight space in } W) \end{array}$$

$$\begin{array}{ccc} Rep(S_n) & \xrightarrow{\Psi} & \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n \\ M & \longrightarrow & ((k^m)^{\otimes n} \otimes M)^{S_n} \end{array}$$

We note that Φ is exact and Φ is left exact.

Note that if V is any polynomial representation the weights in $F(V)$ are divisible by p, where F is the Frobenius twist. Hence $\Phi(F(V)) = 0$, so Φ is not an equivalence.

Proposition 5.32.

- *The functors* (Φ, Ψ) *are an adjoint pair.*
- *The counit* $\Phi\Psi \to \mathrm{id}$ *is an isomorphism.*

Proof. We have that

$$\begin{aligned} \Phi\Psi M &= (1^n \text{ weight space in } (V^{\otimes n} \otimes M)^{S_n}) \\ &= (\underbrace{(1^n \text{ weight space in } V^{\otimes n})}_{k[S_n]} \otimes M)^{S_n} \qquad (\text{ it is important that } m \geq n \text{ here}) \\ &\cong M \end{aligned}$$

Given a GL_m-equivariant map $W \to \Psi M$, apply Φ to get $\Phi W \to \Phi\Psi M \cong M$. This gives a map

$$(*) \qquad \mathrm{Hom}_{\mathrm{GL}_m}(W, \Psi M) \to \mathrm{Hom}_{S_n}(\Phi W, M).$$

We want to show $(*)$ is an isomorphism. Given W, pick a presentation

$$A \to B \to W \to 0,$$

where A and B are sums of D^λ's (cf. Proposition 5.29). To given an exact sequence

$$\Phi A \to \Phi B \to \Phi W \to 0$$

by exactness of Φ. We then obtain

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}(W, \Psi M) & \longrightarrow & \mathrm{Hom}(\Psi W, M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(B, \Psi M) & \longrightarrow & \mathrm{Hom}(\Psi B, M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A, \Psi M) & \longrightarrow & \mathrm{Hom}(\Psi A, M) \end{array}$$

with exact columns, and hence, by the Five Lemma, it is enough to show that the two bottom horizontal maps are isomorphisms. Therefore, it is enough to show that $(*)$ is an isomorphism when $W = D^\lambda(V)$. Note that

$$\Phi D^n = \operatorname{triv}.$$

Note. If V and W are polynomial representations of GL_m of degrees n_1, n_2 summing to at most m, then

$$\Phi(V \otimes W) = \underbrace{\Phi(V) \otimes \Phi(W)}_{\text{in } \operatorname{Rep}(S_*)}.$$

This is because

$$\begin{aligned}(1^n \text{ weight space in } V \otimes W) &= \bigoplus_{[n]=A \amalg B} (1^A \text{ weight space in } V) \otimes (1^B \text{ weight space in } W) \\ &= \operatorname{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}}((1^{n_1} \text{ weight space in } V) \otimes (1^{n_2} \text{ weight space in } W)) \\ &= \operatorname{Ind}_{S_{n_1} \times S_{n_2}}^{S_{n_1+n_2}} \Phi(V) \otimes \Phi(W).\end{aligned}$$

Therefore,

$$\Phi D^\lambda = M^\lambda = \operatorname{Ind}_{S_\lambda}^{S_n}(\operatorname{triv}),$$

which shows that

$$\Psi \Phi D^\lambda = (V^{\otimes n} \otimes \operatorname{Ind}_{S_\lambda}^{S_n}(\operatorname{triv}))^{S_n} = (V^{\otimes n} \otimes \operatorname{triv})^{S_\lambda} = D^\lambda(V).$$

Exercise. This gives the inverse to $(*)$. □

5.2. **Serre quotient categories.** We will now see that if we are in this situation, we can recover the target category from the original category via Serre quotients. In other words, our next goal is to prove that $\operatorname{Rep}(S_n)$ is a quotient of $\operatorname{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ for $m \geq n$.

The reference for this section is [Gab62]. We omit a lot of the proofs here and the reader is referred to this paper for details.

Suppose that $\Phi \colon \mathcal{A} \to \mathcal{B}$ is an exact functor of abelian categories. Define

$$\ker \Phi = \{M \in \mathcal{A} \mid \Phi(M) = 0\}.$$

Then $\ker \Phi$ is closed under sub/quotients and extensions:

(1) If $M \in \ker(\Phi)$ and N is a sub or quotient of M, then $N \in \ker \Phi$.
(2) Suppose

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is exact with $M_1, M_3 \in \ker \Phi$. Then $M_2 \in \ker \Phi$.

Both of these properties follow from exactness.

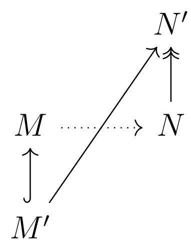
Definition 5.33. Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a full subcategory $\mathcal{K}$ satisfying (1) and (2).

Serre quotient construction. Let $\mathcal{K}$ be a Serre subcategory of $\mathcal{A}$. We define a new category $\mathcal{A}/\mathcal{K}$, called the *Serre quotient*, by

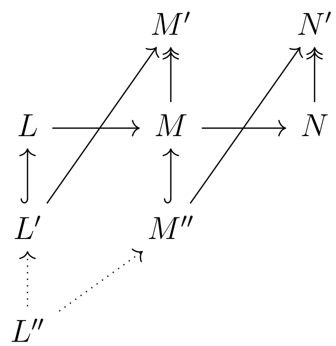
Objects: same as objects of $\mathcal{A}$,
Morphisms: $\operatorname{Hom}_{\mathcal{A}/\mathcal{K}}(M,N) = \varinjlim \operatorname{Hom}_{\mathcal{A}}(M',N')$, where the colimit is over all subobjects M' of M with $M/M' \in \mathcal{K}$ and all quotients N' of N with

$$\ker(N \twoheadrightarrow N') \in \mathcal{K},$$

where one can think of the following picture



Composition: induced by the following pull back L''

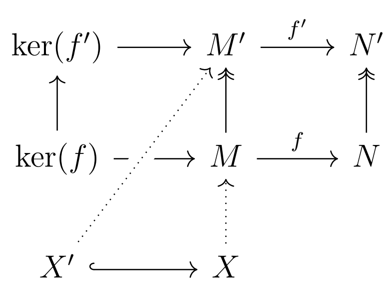


Fact. The Serre quotient $\mathcal{A}/\mathcal{K}$ is an abelian category.

We have a functor $T\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ which is the identity on objects and the natural map on morphisms.

Proposition 5.34. *The functor T is exact.*

Proof. Let us show $T(\ker(f)) = \ker T(f)$, where $f\colon M \to N$ is a morphism. We have the commutative diagram



Using this, we get a map $X \to \ker(f)$ in $\mathcal{A}/\mathcal{K}$. $\square$

Proposition 5.35. *We have that* $\ker(T) = \mathcal{K}$ *and if* $f\colon M \to N$ *is a morphism, then* $T(f)$ *is an isomorphism if and only if* $\ker(f)$ *and* $\operatorname{coker}(f)$ *are in* $\mathcal{K}$.

Proposition 5.36 (Mapping property)**.** *Let* $\mathcal{K} \subseteq \mathcal{A}$ *and* $\Phi\colon \mathcal{A} \to \mathcal{B}$ *such that* Φ *is exact and* $\ker \Phi \supseteq \mathcal{K}$*. Then there is a unique functor* $\Phi'\colon \mathcal{A}/\mathcal{K} \to \mathcal{B}$ *such that* $\Phi = \Phi' \circ T$*.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M, N) & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{B}}(\Phi M, \Phi N) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(M', N') & \xrightarrow{\Phi} & \mathrm{Hom}_{\mathcal{B}}(\Phi M', \Phi N') \end{array}$$

and the right vertical arrow is a bijection because the maps $\Phi N \to \Phi N'$ and $\Phi M' \to \Phi M$ are isomorphisms. Then Φ induces a map

$$\varinjlim \mathrm{Hom}_{\mathcal{A}}(M', N') \to \mathrm{Hom}_{\mathcal{A}/\mathcal{K}}(\Phi M, \Phi N),$$

which gives the desired Φ'. □

Definition 5.37. We say that $\mathcal{K}$ is a *localizing subcategory* if it is a Serre subcategory and T has a right adjoint $S\colon \mathcal{A}/\mathcal{K} \to \mathcal{A}$, called the *section functor.*

Remark 5.38. The section functor S be left exact and takes injectives to injectives.

Example 5.39. Let R be an integral domain and $K = \mathrm{Frac}(R)$. Let

$$\mathcal{A} = \mathrm{Mod}_R,$$

$$\mathcal{K} = \text{torsion modules}.$$

We have an exact functor

$$\begin{aligned} \mathrm{Mod}_r &\to \mathrm{Vec}_K \\ M &\mapsto K \otimes_R M \end{aligned}$$

with $\ker \Phi \supseteq \mathcal{K}$ (actually, they are equal). Therefore, we get a functor

$$\Phi'\colon \mathcal{A}/\mathcal{K} \to \mathrm{Vec}_K\,.$$

Exercise. The functor Φ' is actually an equivalence of categories.

(More generally, we can treat localization with respect to the multiplicative subset R.)

Finally, the restriction functor $S\colon\ \mathrm{Vec}_K \to \mathrm{Mod}_R$ is the section functor.

Example 5.40. If X is a topological space and $i\colon U \hookrightarrow X$ is the inclusion of an open subset, then we have the adjoint functors

$$\mathrm{Sh}(X) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Sh}(X)$$

It is (typically) true that

$$\mathrm{Sh}(U) \cong \frac{\mathrm{Sh}(X)}{\ker(i^*)}.$$

We recall that $\ker(i^*)$ consists of sheaves $\mathcal{F}$ on X such that $\mathcal{F}_x = 0$ for all $x \in U$.

The analogous statement is true for quasi-coherent sheaves on a scheme.

Proposition 5.41. *Let $M \in \mathcal{A}$. The following are equivalent:*

(1) *the unit $M \to STM$ is an isomorphism.*
(2) $\mathrm{Ext}^i_{\mathcal{A}}(K, M) = 0$ *for all $K \in \mathcal{K}$ and $i = 0, 1$,*
(3) *for any $f\colon A \to B$ in $\mathcal{A}$ such that $T(f)$ is an isomorphism, the induced map*

$$\mathrm{Hom}_{\mathcal{A}}(B, M) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(A, M),$$

(4) $\mathrm{Hom}_{\mathcal{A}}(N, M) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{K}}(TN, TM)$ *for all $N \in \mathcal{A}$.*

Definition 5.42. An object $M \in \mathcal{A}$ is *saturated* (or $\mathcal{K}$-saturated) if it satisfies conditions (1)–(4).

Proposition 5.43. *For any $M \in \mathcal{A}/\mathcal{K}$, SM is saturated.*

Proposition 5.44.

(1) *The counit $TS \to \mathrm{id}_{\mathcal{A}/\mathcal{K}}$ is an isomorphism.*
(2) *The kernel and cokernel of a unit $\mathrm{id}_{\mathcal{A}} \to ST$ always belongs to $\mathcal{K}$.*

Proposition 5.45. *Suppose $\Phi\colon \mathcal{A} \to \mathcal{B}$ is exact with right adjoint $\Psi\colon \mathcal{B} \to \mathcal{A}$ such that the counit $\Phi\Psi \to \mathrm{id}$ is an isomorphism. Then the canonical functor*

$$\mathcal{A}/\ker\Phi \to \mathcal{B}$$

is an equivalence and $\ker\Phi$ is a localizing subcategory.

Back to our case. We had functors

$$\Phi\colon\ \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty) \to \mathrm{Rep}(S_*),$$

$$\Psi\colon\ \mathrm{Rep}(S_*) \to \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty),$$

where Φ is exact, (Φ, Ψ) is an adjoint pair, and the counit $\Phi\Psi \to \mathrm{id}$ is an isomorphism.

By Proposition 5.45, $\mathrm{Rep}(S_*)$ is the Serre quotient of $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ by

$$\ker\Phi = \{V \mid (1^n\text{-weight space in } V) = 0 \text{ for all } n\}.$$

5.3. **Highest weight structure. Motivation.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, Borel subalgebra $\mathfrak{b}$ with radical $\mathfrak{u}$, and Cartan subalgebra $\mathfrak{h}$.

Example. Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $\mathfrak{b}$ consists of the upper triangular matrices, $\mathfrak{u}$ consists of the strictly upper triangular matrice, and $\mathfrak{h}$ are the traceless diagonal matrices.

Definition 5.46. The *category $\mathcal{O}$* is the category of $\mathfrak{g}$-modules satisfying:

(a) finitely generated over $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra,
(b) have weight decomposition, i.e. semisimple as $\mathfrak{h}$-modules,
(c) action of $\mathfrak{u}$ is locally nilpotent, i.e. every vector is killed by a power of $\mathfrak{u}$.

Definition 5.47. An element $M \in \mathcal{O}$ is a *highest weight module* of weight λ if $M^{\mathfrak{u}}$ is one-dimensional of weight λ and generates M.

Let $\mathfrak{u}^-$ be the opposite of $\mathfrak{u}$ (for example, for $\mathfrak{sl}_n$ take the strictly lower triangular matrices). Then

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{u}^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{u}).$$

Suppose M is a highest weight module of weight λ if N and N' are submodule of M without λ as a weight, then $N+N'$ satisfies the same condition, so there exists a maximal submodule N of M without λ as a weight. If K is any submodule of M then either $K \subset N$ or $K = M$, hence M/N is simple with λ as the highest weight.

If λ is a weight, we can regard λ as a representation of $\mathfrak{b}$ via the homomorphism $\mathfrak{b} \to \mathfrak{h} \cong \mathfrak{b}/\mathfrak{u}$.

Definition 5.48. The *Verma module* $V(\lambda)$ is $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}\lambda$.

This is a highest weight module of weight λ. Moreover, for any module M. we have

$$\mathrm{Hom}_{\mathfrak{g}}(V(\lambda), M) = \{\lambda\text{-weight vectors in } M^{\mathfrak{u}}\}$$

We get a simple quotient $L(\lambda)$ of $V(\lambda)$ with highest weight λ, and the $L(\lambda)$ are the simple modules.

The weights appearing in $\ker(V(\lambda) \to L(\lambda))$ are smaller than λ, so the simple constituents of $V(\lambda)$ other than $L(\lambda)$ have the form $L(\mu)$ with $\mu < \lambda$.

Fact. The category $\mathcal{O}$ has enough projectives.

However, $V(\lambda)$ are typically not projective (or injective). Let $P(\lambda)$ be a projective cover of $L(\lambda)$ of finite length. This gives a surjection

$$P(\lambda) \to V(\lambda).$$

We actually have a filtration

$$\cdots \supseteq F^1 \supseteq F^0 = P(\lambda)$$

such that

$$F^0/F^1 = V(\lambda) \text{ and } F^i/F^{i+1} = V(\mu) \text{ with } \mu > \lambda \text{ for } i \geq 1.$$

Fact. The category $\mathcal{O}$ has a duality. Therefore, we it has enough injectives, and co-Verma modules, giving a similar picture to the above.

This finishes the motivation and we return to the general case. The following definition, given by Cline–Parshall-Scott, allows to generalize the notion of highest weights.

Definition 5.49. A *highest weight category* is a k-linear (where k is a field) locally artinian abelian category $\mathcal{A}$ satisfying (AB5), i.e. the flitered colimits are exact and it has enough injectives, with a partially ordered set Λ (of *weights*) such that:

(1) The simples are indexed by Λ: $\{L(\lambda)\}_{\lambda\in\Lambda}$,
(2) For each $\lambda \in \Lambda$, there exists an object $A(\lambda)$ with an embedding $L(\lambda) \subset A(\lambda)$ such that all composition factors of $A(\lambda)/L(\lambda)$ have the form $L(\mu)$ with $\mu < \lambda$. We also require that

$$[A(\lambda) : L(\mu)] < \infty \text{ and } \dim \mathrm{Hom}(A(\lambda), A(\mu)) < \infty.$$

(3) There is an injective envelope of $I(\lambda)$ with a filtration
$$0 = F_0 \subset F_1 \subset \cdots$$
such that
(a) $F_1 = A(\lambda)$,
(b) $F_i/F_{i-1} = A(\mu)$ with $\mu > \lambda$ for $i \geq 2$,
(c) $[I(\lambda) : A(\mu)] = (\text{the number of times } F_i/F_{i-1} = A(\mu)) < \infty$,
(d) $I(\lambda) = \bigcup_{i\geq 1} F_i$.

Motivating example. The category $\mathcal{O}$ is highest weight category.

Our motivation. The category $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)$ is a highest weight category.

Proposition 5.50. *Suppose $\mathcal{A}$ is a highest weight category with $\#\Lambda < \infty$. Then*
$$\mathrm{gldim}\mathcal{A} < \infty.$$

Proof. We first show that $A(\lambda)$ has finite injective dimension by *descending* induction on λ. We have an exact sequence
$$0 \longrightarrow A(\lambda) \longrightarrow I(\lambda) \longrightarrow Q \longrightarrow 0,$$
where Q is the quotient, and we know that Q has pieces $A(\mu)$ with $\mu > \lambda$. Hence Q has finite injective dimension by inductive hypothesis, and so
$$\mathrm{injdim}A(\lambda) \leq \mathrm{injdim}Q + 1 < \infty.$$
We next show that $L(\lambda)$ has finite injective dimension by *ascending* induction on λ:
$$0 \longrightarrow L(\lambda) \longrightarrow A(\lambda) \longrightarrow Q \longrightarrow 0,$$
where Q has pieces $L(\mu)$ with $\mu < \lambda$. By inductive hypothesis Q has finite injective dimension, and hence
$$\mathrm{injdim}L(\lambda) \leq \max(\mathrm{injdim}A(\lambda), 1 + \mathrm{injdim}Q(\lambda)) < \infty.$$
This completes the proof. □

Definition 5.51. A ring R is *(left) hereditary* if any submodule of a (left) projective module is again projective. Equivalently, $\mathrm{lgldim}R \leq 1$.

Suppose R is a hereditary finite-dimensional k-algebra. We claim that Mod_R is a highest weight category.

Let J be the *socle* of R, i.e. the maximal semisimple subobject of R.

Fact 5.52.

- *The socle J is a 2-sided ideal.*
- *The ring R/J is hereditary.*
- *The modules J and R/J have no common composition factors.*

Define

$$0 = J_0 \subset J_1 = J \subset \cdots \subset J_r = R$$

by

$$J_n/J_{n-1} = \mathrm{soc}(R/J_{n-1}).$$

For a simple L, let let $n(L)$ be the n such that L is a summand of J_n/J_{n-1}. Partially order the simples by $L < L'$ if $n(L) > n(L')$.

If $n(L) = n$, then L is a quotient of J_n, so $P(L)$ surjects onto L, $\ker \subseteq J_{n-1}$, so it only has larger simples. Take linear duals to get a highest weight category, with $A(L) = I(L)$.

Definition 5.53. A 2-sided ideal J in a ring R is *hereditary* if

(1) J is projective as a left R-module,
(2) $\mathrm{Hom}_R(J, R/J) = 0$,
(3) $J\mathrm{rad}(R)J = 0$.

Definition 5.54. A ring R is *quasi-hereditary* if there is a chain of 2-sided ideals in R

$$0 = J_0 \subset J_1 \subset \cdots \subset J_r = R$$

such that $J_n/J_{n-1} \subset R/J_{n-1}$ is a hereditary ideal.

Example 5.55. If R is hereditary, we can take J_n to be the inverse image of $\mathrm{soc}\left(\frac{R}{J_{n-1}}\right)$. This gives a chain as above, so R is quasi-hereditary.

Theorem 5.56. *Let R be a finite-dimensional k-algebra. Then Mod_R is a highest weight category if and only if R is quasi-hereditary.*

5.4. **Highest weight structure on $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$.** We omit most of the proofs in this section. Parts of it follow [Wey03], but precise references are not provided.

Let λ be a partition of n with r rows and s columns, and $V = k^m$. The *Weyl module* K_λ is the image of

$$\begin{array}{ccc} D^{\lambda_1}(V) \otimes \cdots \otimes D^{\lambda_r}(V) & & \\ \downarrow & \searrow & \\ & & V^{\otimes n} \\ & \swarrow & \\ \bigwedge^{\lambda_1^\dagger} \otimes \cdots \otimes \bigwedge^{\lambda_s^\dagger}(V) & & \end{array}$$

The *Schur module* L_λ is the image of

$$\bigwedge^{\lambda_1^\dagger} \otimes \cdots \otimes \bigwedge^{\lambda_s^\dagger}(V) \to \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V).$$

It is easy to see that L_λ and K_λ both have λ as a weight with multiplicity 1, and all other weights ae smaller.

Fact 5.57. *The λ-weight vector generates K_λ and cogenerates L_λ.*

Hence K_λ has a simple quotient with highest weight λ and L_λ has a simple submodule with highest weight λ.

Remark 5.58. The Schur module L_λ has a basis indexed by semistandard tableaux of shape λ filled with numbers $1, \dots, m = \dim V$. In particular, the weight decomposition of L_λ, K_λ is independent of characteristic.

Examples 5.59. The Weyl module $K_{(n)}$ is $D^n(V)$ and similarly $K_{(1^n)} = \bigwedge^n(V)$. For Schur modules: $L_{(n)} = \mathrm{Sym}^n(V)$ and $L_{(1^n)} = \Lambda^n(V)$.

In characteristic 0, $L_\lambda \cong K_\lambda \cong S_\lambda(V)$.

We have shown that $D^\lambda = D^{\lambda_1} \otimes \cdots \otimes D^{\lambda_r}(V)$ is a projective object of $\mathrm{Rep}^{\mathrm{pol}}$.

Version of Pieri rule. The module $K_\lambda \otimes D^n$ has a filtration where the graded pieces are K_μ's where μ's are as in the usual Pieri rule 3.15, the top piece (quotient) is least dominant (boxes go as low as possible).

Corollary 5.60. *The module D^λ has a filtration with graded pieces equal to K_λ. We can deduce a similar filtration for summands of D^λ.*

Theorem 5.61. *The category $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ is a highest weight category. Its standard objects are K_λ's (filter projectives, analogous to Verma). Its costandard objects are L_λ's (filter injectives, $A(\lambda)$'s in definition of highest weight category).*

Corollary 5.62. *The category $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ has finite global dimension.*

Proof. This following by combining Proposition 5.50 with Theorem 5.61. □

Remark 5.63. Totaro computed the global dimension in the 90s. Let $\alpha_p(n)$ be the sum of the digits in base p expansion of n. Then

$$\mathrm{gldim} = 2(n - \alpha_p(n)).$$

Note that if $n < p$ then $\alpha_p(n) \geq n$, so the global dimension is 0. Hence $\mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_m)_n$ is semi-simple.

Corollary 5.64 (to Theorem 5.61)**.** *The Schur algebra $D^n(\mathrm{End}\, V)$ is quasi-hereditary.*

Remark 5.65. Let λ be a partition of n and $m \geq n$. Then

$$H^0\,(\text{flag variety for } \mathrm{GL}_m, \mathcal{L}(\lambda)) = K_\lambda$$

and higher cohomology groups vanish, where $\mathcal{L}(\lambda)$ is a G-equivariant line bundle.

6. Deligne interpolation categories and recent work

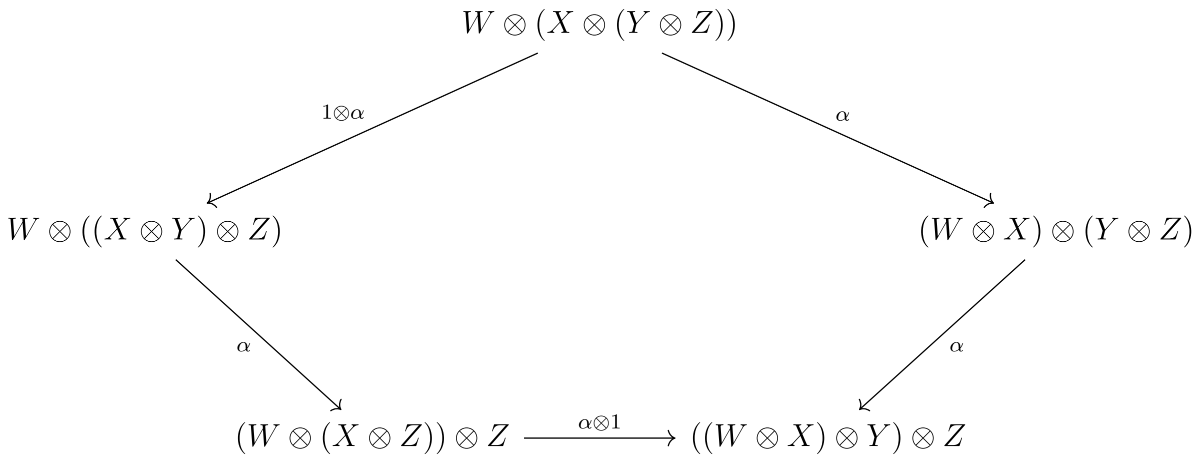
Let $\mathcal{C}$ be a category. We will define a monoidal structure on $\mathcal{C}$. Consider a functor

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

We want associativity, but equality $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ is too much to ask for. Instead, we want an isomorphism

$$\alpha \colon X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

of functors (considering both sides are functors $\mathcal{C}^3 \to \mathcal{C}$). We also require the following diagram to commute

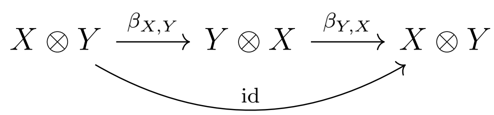


This is called them *pentagon axiom* (for obvious resons).

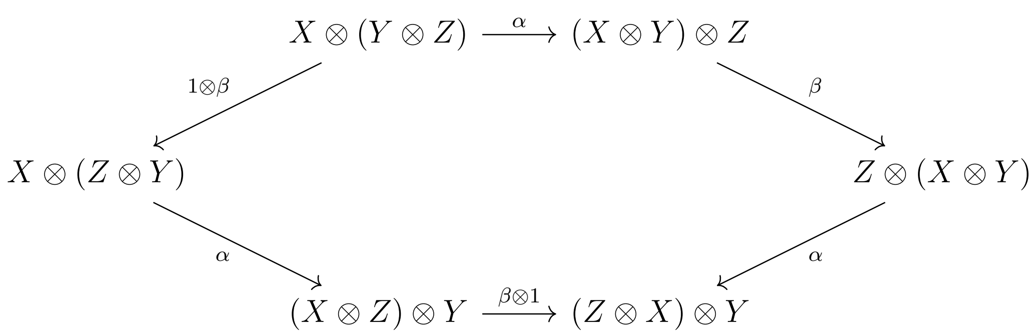
We also want commutativity, which corresponds to a functorial isomorphism

$$\beta\colon X \otimes Y \to Y \otimes X$$

such that



and the following diagram commutes



This is called the *hexagon axiom*.

Given $\mathcal{C}$ with $\otimes, \alpha, \beta$, a *unit object* of $\mathcal{C}$ is an object $\mathbb{1}$ with an isomorphism $\mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ such that $X \mapsto X \otimes \mathbb{1}$ is an equivalence of categories $\mathcal{C} \to \mathcal{C}$.

Definition 6.1. A *symmetric monoidal category* is a category $\mathcal{C}$ with functor $\otimes$ and α, β, $\mathbb{1}$ as above.

Remark 6.2. If $\mathcal{C}$ is a symmetric monoidal category, I is a finite set, then there exists a functor

$$\mathcal{C}^I \to \mathcal{C}$$
$$(X_i)_{i\in I} \mapsto \bigotimes_{i\in I} X_i.$$

Definition 6.3. The *internal Hom*, denoted $\underline{\mathrm{Hom}}(X,Y)$ is the right adjoint to $-\otimes X$:

$$\mathrm{Hom}(T, \underline{\mathrm{Hom}}(X,Y)) \cong \mathrm{Hom}(T\otimes X, Y).$$

Taking $T = \underline{\mathrm{Hom}}(X,Y)$, id_T corresponds to a map

$$\mathrm{ev}\colon \underline{\mathrm{Hom}}(X,Y)\otimes X \to Y$$

called the *evaluation map*.

Remark 6.4. We have that:

- $\underline{\mathrm{Hom}}(\mathbb{1}, X) \cong X$,
- $\mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(X,Y)) \cong \mathrm{Hom}(X,Y)$.

Definition 6.5. The *dual* of X, denoted $X^\vee$, is $\underline{\mathrm{Hom}}(X,\mathbb{1})$. We have an *evaluation map*

$$\mathrm{ev}\colon X\otimes X^\vee \to \mathbb{1}.$$

There is a canonical map $X \to X^{\vee\vee}$ and we say that X is *reflective* if this is an isomorphism.

Definition 6.6. A symmetrical monoidal category is called *rigid* if

(a) $\underline{\mathrm{Hom}}(X,Y)$ exists for all X, Y,
(b) all objects are reflective,
(c) $\underline{\mathrm{Hom}}(X,Y)\otimes\underline{\mathrm{Hom}}(X',Y') \to \underline{\mathrm{Hom}}(X\otimes X', Y\otimes Y')$ is an isomorphism

In particular, (c) for $Y=\mathbb{1}$, $X'=\mathbb{1}$ implies that

$$X^\vee \otimes Y' \cong \underline{\mathrm{Hom}}(X, Y').$$

Assume $\mathcal{C}$ is a rigid symmetric monoidal category. Then we have the evaluation map

$$\underline{\mathrm{Hom}}(X,X) \cong X\otimes X^\vee \xrightarrow{\mathrm{ev}} \mathbb{1}.$$

Apply $\mathrm{Hom}(\mathbb{1}, -)$ to get a map

$$\mathrm{tr}_X\colon\ \mathrm{End}(X) \to \mathrm{End}(\mathbb{1})$$

called the *trace*. The *rank* of X is

$$\mathrm{rank}(X) = \mathrm{tr}_X(\mathrm{id}_X) \in \mathrm{End}(\mathbb{1}).$$

Definition 6.7. A *tensor category* (or $\otimes$-category) is an additive symmetric monoidal category such that $\otimes$ is bi-additive.

Remark 6.8. There is no consistent definition of a tensor category in the literature. Some papers require it to be abelian, some require it to be rigid.

Examples 6.9.

- For a field k, $\mathrm{Vec}_k^{\mathrm{fd}}$ is a rigid abelian $\otimes$-category.

- For a commutative ring R, $\mathrm{Mod}_R^{\mathrm{fg}}$ is an $\otimes$-category and it is abelian if and only if R is Noetherian.
- For a commutative ring R, the category of projective, finitely-generated R-modules, $\mathrm{Proj}_R^{\mathrm{fg}}$, is a rigid $\otimes$-category (although it is typically not abelian).
- If k is a field and G is a group scheme over k, $\mathrm{Rep}_k^{\mathrm{fd}}(G)$ is a rigid abelian $\otimes$-category.
- The category $\mathrm{SVec}_k^{\mathrm{fd}}$ of finite-dimensional super vector sapces over k. A *super vector space* is a $\mathbb{Z}/2$-graded vector space:
$$V = V_0 \oplus V_1.$$
Then $\mathrm{SVec}_k^{\mathrm{fd}}$ is a symmetric monoidal category, using usual $\otimes$ product of graded vector spaces with usual α and $\mathbb{1}$ but with modified β:
$$\begin{aligned}\beta\colon X \otimes Y &\to Y \otimes X\\ x \otimes y &\mapsto (-1)^{\deg x \deg y} y \otimes x.\end{aligned}$$
Then $\mathrm{rank}(V) = \dim(V_0) - \dim(V_1) \in \mathbb{Z}$, but it can be negative.

Remark 6.10. Suppose $\mathcal{C}$ is a monoidal category, graded by a group G: every object has a degree in G, and if we tensor two objects their degree multiply in G.

We can try to modify α to get a new monoidal category:
$$\alpha'(X, Y, Z) = \phi(\deg X, \deg Y, \deg Z)\alpha(X, Y, Z),$$
where $\phi\colon G^3 \to \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$. This satisfies the hexagon axiom if ϕ is a 3-cocycle, and ones that differ by a coboundary give the same monoidal structure. Therefore, get new monoidal structures parameterized by
$$H^3(G, \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})).$$
For symmetric monoidal categories graded by abelian groups G, we can try to modify both α and β:
$$\beta'(X, Y) = \psi(\deg X, \deg Y)\beta(X, Y).$$
The hexagon axiom translates to an identity on ψ. The new structures are parameterized by the *abelian cohomology* $H^3_{\mathrm{ab}}(G, \mathrm{Aut}(\mathrm{id}_{\mathcal{C}}))$, which is defined appropriately for the axioms to hold.

Definition 6.11. A *fiber functor* on a k-linear $\otimes$-category is an exact faithful k-linear $\otimes$-functor
$$\omega\colon \mathcal{C} \to \mathrm{Vec}_k^{\mathrm{fd}}.$$

Theorem 6.12 (Main Theorem of Tanaka Duality)**.** *If $\mathcal{C}$ is a rigid, abelian k-linear $\otimes$-category such that $\mathrm{End}(\mathbb{1}) = k$ with fiber functor ω, then $\mathcal{C} \cong \mathrm{Rep}_k(G)$ for some group scheme G (where $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, defined in an appropriate way).*

Remark 6.13. We can recover G from $\mathrm{Rep}_k(G)$, the tensor structure, and ω.

If $k = \mathbb{C}$, we do not need ω—just the symmetric monoidal structure on $\mathrm{Rep}_k(G)$. However, we really do need the **symmetric** monoidal structure. There are examples of finite groups G, H such that
$$\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathrm{Rep}_{\mathbb{C}}(H)$$
as monoidal categories but $G \ncong H$. (The first example is due to Etingof and Gelaki).

It is important to know when $\mathcal{C}$ has a fiber functor. A necessary condition is that

$$\text{rank}(\text{any object}) = \text{non-negative integer} \in \text{End}(\mathbb{1}).$$

Theorem 6.14 (Deligne, [Del90])**.** *If $\mathcal{C}$ is a rigid abelian k-linear, $\otimes$-category,* $\text{char}(k) = 0$, $\text{End}(\mathbb{1}) = k$*, k is algebraically closed, then the following are equivalent:*

(1) $\mathcal{C}$ has a fiber functor,
(2) for all objects M, $\text{rank}(M)$ *is a non-negative integer,*
(3) for all objects M, there exists $n \gg 0$ such that $\bigwedge^n(M) = 0$.

Remark 6.15 (Rough idea of the proof)**.** In the proof, Deligne mimics algebraic geometry. If you have an object, you can pass to an *affine variety*, and then your fiber functor maps points to fibers over that points.

In the setting of Deligne's Theorem 6.14, is condition (2) automatic? No, because of the example of super vector spaces (cf. Examples 6.9).

If V is a super vector space, write $V[1]$ for the super vector space with $V[1]_i = V_{i+1}$. We can show

$$S_\lambda(V[1])) = S_{\lambda^\dagger}(V)[|\lambda|],$$

so for example

$$\bigwedge^n(V[1]) = \text{Sym}^n(V)[n].$$

If V is a finite-dimensional super vector space such that $V_1 \neq 0$ and $V_0 \neq 0$, then

$$\bigwedge^n(V) \neq 0, \quad \text{Sym}^n(V) \neq 0$$

for all n. This shows that condition (3) in Deligne's Theorem 6.14 does not hold. From this, one easily sees that there is no fiber functor on super vector spaces.

Question. Is the rank always an integers in the setting of Deligne's Theorem 6.14?

Answer. No. Deligne's interpolation categories give a counterexample.

From now on, we work over $k = \mathbb{C}$.

Question. How can we construct the category $\text{Rep}(S_n)$ without talking about S_n?

Answer. We will start with representations $\mathbb{C}^n$ of S_n and its $\otimes$-powers.

Lemma 6.16. *Let G be a finite group (or, more generally, a reductive algebraic group) and let V be a faithful representation of G. Then every irreducible representation of G is a quotient of a* $\text{Sym}^n(V)$ *for some n.*

Proof. Pick $v \in V$ with trivial stabilizer in G. We get a map $G \to V$ given by $g \mapsto gv$. This is a closed embedding (thinking of these as varieties), so we get a surjection of coordinate rings

$$\text{Sym}(V^*)) \to \text{Fun}(G).$$

Since any representation appears in $\text{Fun}(G)$, this completes the proof. $\square$

Hence all irreducible representations of S_n appear in some $T^r = (\mathbb{C}^n)^{\otimes r}$.

We now want to understand maps between these representations:

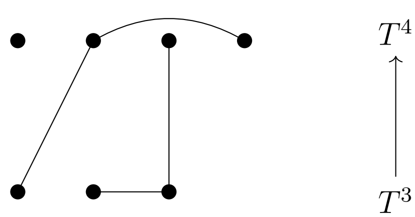
- S_r acts on T^r gives maps $T^r \to T^r$,
- $T^1 \to T^0 = \mathbb{C}$, the augmentation map $e_i \mapsto 1$,
- $T^0 \to T^1$, $1 \mapsto \sum_{i=1}^{n} e_i$,
- $\alpha_r \colon T^0 \to T^r$, $1 \mapsto \sum_{i=1}^{r} e_i \otimes \cdots \otimes e_i$,
- $\beta_r \colon T^r \to T^0$, $e_{i_1} \otimes \cdots \otimes e_{i_r} \mapsto \begin{cases} 1 & \text{if } i_1 = \cdots = i_r, \\ 0 & \text{otherwise,} \end{cases}$
- $T^1 \to T^2$, $e_i \mapsto e_i \otimes e_i$,
- $T^2 \to T^1$, $e_i \otimes e_j \mapsto \delta_{i,j} e_i$,
- if $r, s > 0$, define

$$\gamma_{r,s} \colon T^r \to T^s$$
$$e_{i_1} \otimes \cdots \otimes e_{i_r} \mapsto \delta_{i_1,\dots,i_r} e_{i_1} \otimes \cdots \otimes e_{i_r}$$

We write $[n] = \{1, \dots, n\}$. Given a partition of the set $[r] \amalg [s]$, we define a map $T^r \to T^s$ as follows:

- for a part of size k contained in $[r]$, use β_k,
- for a part of size k contained in $[s]$, use α_k,
- for a part with $k > 0$ vertices in $[r]$, and $\ell > 0$ vertices in $[s]$, use $\gamma_{k,l}$.

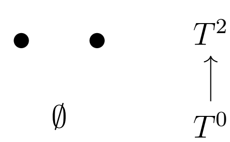
Example 6.17. Consider the partition of $[4] \amalg [3]$ given by



Then the associated map $T^3 \to T^4$ is given by

$$e_i \otimes e_j \otimes e_k \mapsto \delta_{j,k} \sum_{\ell=1}^{n} e_i \otimes e_j \otimes e_i$$

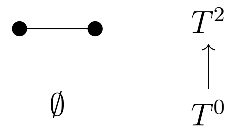
Example 6.18. Note that



gives the map

$$1 \mapsto \sum_{\ell=1}^{n} e_\ell \otimes \sum_{k=1}^{n} e_k$$

whereas



gives the map

$$1 \mapsto \sum_{\ell=1}^{n} e_\ell \otimes e_\ell.$$

Let $H_{r,s}$ be the vector space with basis given by these diagrams. We have now constructed a map

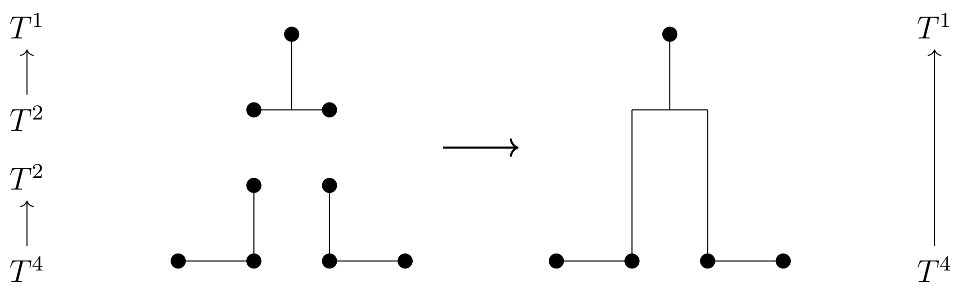
$$H_{r,s} \to \mathrm{Hom}_{S_n}(T^r, T^s).$$

Proposition 6.19. *This map is always surjective, and an isomorphism if $n \geq r + s$.*

Proof. Moving T^s over with a dual, we can reduce to $r = 0$. Note that S_n acts on basis vectors of T^s, so $(T^s)^{S_n}$ has a basis consisting of orbits of S_n acting on $[n]^s$. These orbits correspond to partitions of the set $[s]$. The associated invariants are related to the partitions in $H_{0,s}$ by an upper triangular change of variables. □

Question. How does composition $H_{r,s} \times H_{s,t} \to H_{r,t}$ work in terms of diagrams?

Example 6.20. We have the following example:



and the composition is

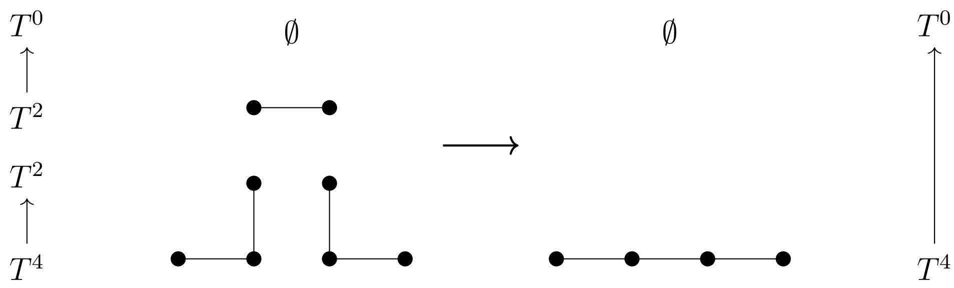
$$T^4 \to T^2 \to T^1$$

$$e_i \otimes e_j \otimes e_k \otimes e_\ell \mapsto \delta_{i,j}\delta_{k,l}$$

which does correspond to the diagram on the right.

Generalization: merge two partitions together, delete middle row vertices. If only one partition remains, and touches top or bottom row, that is the answer.

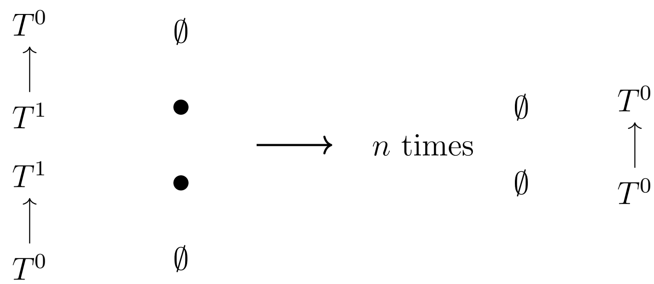
Example 6.21. Following this rule, we have the diagrams



and the composition is indeed

$$\begin{array}{rcccl} T^4 & \to & T^2 & \to & T^0 \\ e_i \otimes e_j \otimes e_k \otimes e_\ell & \mapsto & \delta_{ij}\delta_{k\ell} e_i e_\ell & \mapsto & \delta_{ij}\delta_{kl}\delta_{il}. \end{array}$$

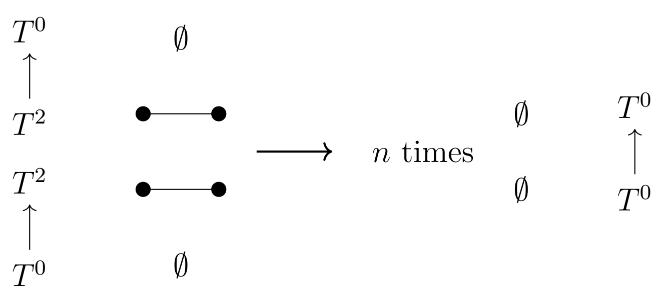
Examples 6.22. We give a few more examples to understand the general rule of composition. We have that



since this is given by

$$\begin{array}{ccccc} T^0 & \to & T^1 & \to & T^0 \\ 1 & \mapsto & \sum_{i=1}^n e_i & \mapsto & n. \end{array}$$

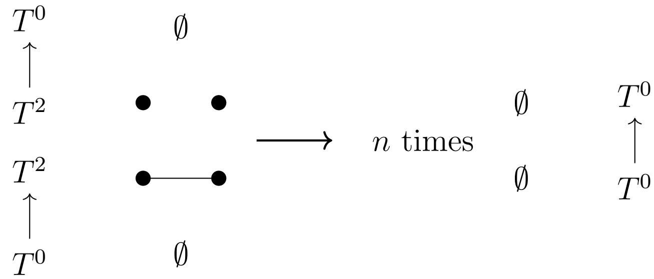
We have that



since this is given by

$$\begin{array}{ccccc} T^0 & \to & T^2 & \to & T^0 \\ 1 & \mapsto & \sum_{i=1}^n e_i \otimes e_i & \mapsto & n. \end{array}$$

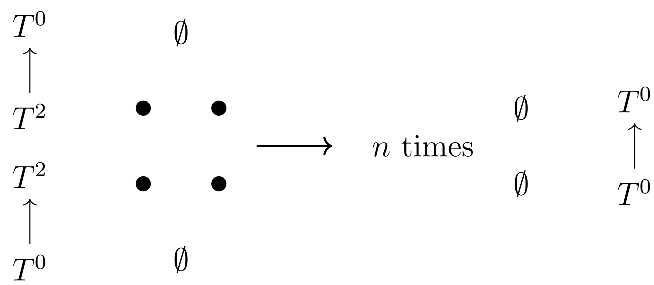
We have that



since this is given by

$$\begin{array}{ccccc} T^0 & \to & T^2 & \to & T^0 \\ 1 & \mapsto & \sum_{i=1}^{n} e_i \otimes e_i & \mapsto & n. \end{array}$$

We have that

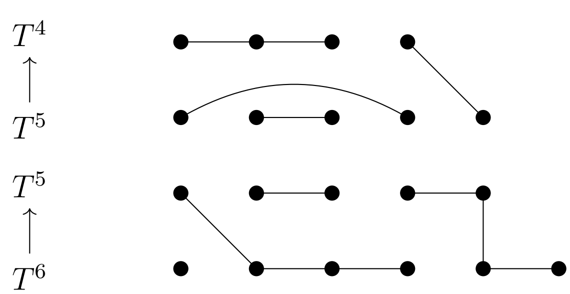


since this is given by

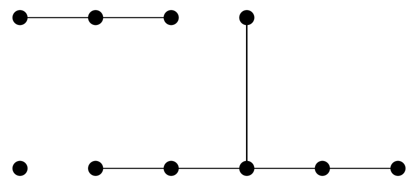
$$\begin{array}{ccccc} T^0 & \to & T^2 & \to & T^0 \\ 1 & \mapsto & \sum_{i,j=1}^{n} e_i \otimes e_j & \mapsto & n^2. \end{array}$$

General rule of composition: Overlay the two diagrams and (merge partitions). For each partition entirely in the middle row contributes to a factor of n. Discard these and what is left is the answer.

Example 6.23. The two diagrams



give

n times 

We work over $\mathbb{C}$ and fix $t \in \mathbb{C}$.

Define the category $\underline{\mathrm{Rep}}(S_t)_0$ as follows

- objects: T^r for $r \geq 0$,
- morphisms: $\mathrm{Hom}(T^r, T^s) = H_{r,s}$,
- composition: the rule above with n changed to t.

This category is $\mathbb{C}$-linear but not additive. We have a $\otimes$-product on $\underline{\mathrm{Rep}}(S_t)_0$:

- $T^r \otimes T^s = T^{rs}$,
- $\otimes\colon H_{r,s} \times H_{r',s'} \to H_{r+r',s+s'}$ is defined by putting the two diagrams next to each other.

This satisfies the conditions to be rigid. Moreover,

$$(T^r)^\vee = T^r$$

and

$$(-)^\vee\colon H_{r,s} \to H_{s,r}$$

flips the diagram upside down.

If $t \in \mathbb{N}$, we have a functor

$$\Phi\colon \underline{\mathrm{Rep}}(S_t)_0 \to \mathrm{Rep}(S_t)$$
$$T^r \mapsto (\mathbb{C}^t)^{\otimes r}$$

This functor is full by Proposition 6.19 (but not faithful).

We will now try to make an abelian category from $\underline{\mathrm{Rep}}(S_t)_0$.

Suppose $\mathcal{A}$ is an Ab-category (enriched over Ab). The *additive envelope* of $\mathcal{A}$ is a new category:

- objects: formal direct sums $M_1 \oplus \cdots \oplus M_r$, $M_i \in \mathcal{A}$,
- morphisms:

$$\mathrm{Hom}(M_1 \oplus \cdots \oplus M_r, N_1 \oplus \cdots \oplus N_s) = s \times r \text{ matrices with entries in } \mathrm{Hom}_{\mathcal{A}}(M_i, N_j),$$

- composition: defined by matrix multiplication.

This is an additive category.

Say $\mathcal{A}$ is an additive category. We define the *Karoubian envelope* of $\mathcal{A}$ as follows:

- objects: pairs (M, e) for $M \in \mathcal{A}$, $e \in \mathrm{End}(M)$ is an idempotent [think of these as representing eM],
- morphisms: $\mathrm{Hom}((M,e),(N,e')) = e\,\mathrm{Hom}_{\mathcal{A}}(M,N)e'$.

The Karoubian envelope is additive and all idempotent endomorphisms have images.

Definition 6.24. We define the *Deligne category* $\underline{\mathrm{Rep}}(S_t)$ as the Karoubian envelope of the additive envelope of $\underline{\mathrm{Rep}}(S_t)_0$.

This category has a structure of a rigid $\otimes$-category.

6.1. Relation to $\mathrm{Rep}(S_n)$.

Remark 6.25. For $t \in \mathbb{N}$, we still have an $\otimes$-functor

$$\Phi\colon \underline{\mathrm{Rep}}(S_t) \to \mathrm{Rep}(S_t)$$

which is still full. But now, it is also surjective on objects (which is why we took the Karoubian envelope).

Let $P_r(t) = \mathrm{End}(T^r)$ in $\underline{\mathrm{Rep}}(S_t)$. This is called the *partition algebra*.

For $t \in \mathbb{N}$, we have a surjection

$$P_r(T) \to \mathrm{End}_{S_t}((\mathbb{C}^t)^{\otimes r}).$$

An elementary argument to justify this is that any idempotent of the target lifts to and idempotent of the source.

Corollary 6.26. *The functor $\underline{\mathrm{Rep}}(S_t) \to \mathrm{Rep}(S_t)$ is essentially surjective.*

Proof. Consider simple object S^λ of $\mathrm{Rep}(S_t)$. We know that S^λ is a summand of some $(\mathbb{C}^t)^{\otimes r}$, so $S^\lambda = \epsilon(\mathbb{C}^t)^{\otimes r}$ for some idempotent $\epsilon \in \mathrm{End}_{S_t}((\mathbb{C}^t)^{\otimes r})$. Let $\tilde{\epsilon} \in P_r(t)$ be a lift of ϵ. Then

$$\Phi((T^r, \tilde{\epsilon})) \cong S^\lambda,$$

so this functor is essentially surjective on objects. We already saw that it is surjective on morphisms in Remark 6.25. □

Definition 6.27. A morphism $f\colon X \to Y$ (in some rigid $\otimes$-category) is *negligible* if $\mathrm{tr}(gf) = 0$ for all $g\colon Y \to X$.

Example 6.28. In $\underline{\mathrm{Rep}}(S_t)$, consider $\mathrm{End}(T^1)$. It has a basis

$$\alpha$$

$$\beta = \mathrm{id}_{T^1}$$

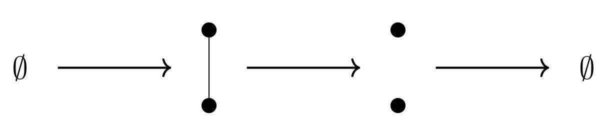
and $\alpha^2 = t\alpha$. We can compute

$$\mathrm{tr}(\alpha) = \mathrm{tr}(\beta) = t.$$

We have

$$T^0 \longrightarrow T^1 \otimes (T^1)^\vee \xrightarrow{\alpha,\beta\otimes\mathrm{id}} T^1 \otimes (T^1)^\vee \longrightarrow T^0$$

corresponding to the diagram



whose composition is t.

Consider $f = \alpha - \beta$. Then f is negligible if and only if $\mathrm{tr}(\alpha f) = \mathrm{tr}(\beta f) = 0$ and

$$\mathrm{tr}(\beta f) = \mathrm{tr}(f) = \mathrm{tr}(\alpha) - \mathrm{tr}(\beta) = 0$$
$$\mathrm{tr}(\alpha f) = \mathrm{tr}(\alpha^2 - \alpha)\,\mathrm{tr}((t-1)\alpha) = (t-1)t.$$

Hence f is negligible if and only if $t = 0$ or $t = 1$.

Remark 6.29. Note that $\mathrm{tr}(\beta) = t$ says $\mathrm{rank}(T^1) = t$.

Proposition 6.30. *Let $\mathcal{A}$ and $\mathcal{B}$ be rigid $\otimes$-categories with $\mathrm{End}_{\mathcal{A}}(\mathbb{1}) = \mathrm{End}(\mathcal{B})(\mathbb{1})$, and $\Phi\colon \mathcal{A} \to \mathcal{B}$ is a full $\otimes$-functor. Then for $f \in \mathrm{Mor}(\mathcal{A})$, f is negligible if and only if $\Phi(f)$ is negligible.*

Proof. Because Φ is an $\otimes$-functor, it preserves traces. Then

$$\mathrm{tr}(gf) = \mathrm{tr}(\Phi(g)\Phi(f))$$

for $f\colon X \to Y$, so $\mathrm{tr}(gf) = 0$ for all $g\colon Y \to X$ if and only if $\mathrm{tr}(\Phi(g)\Phi(f)) = 0$ for all $\Phi(g)\colon \Phi(Y) \to \Phi(X)$. Since Φ is full, $\Phi(g)$ are all morphisms $\Phi(Y) \to \Phi(X)$. □

Let $t \in \mathbb{N}$ and $\mathcal{N}$ be the class of negligible morphisms in $\underline{\mathrm{Rep}}(S_t)$.

Theorem 6.31. *There is an equivalence of categories $\underline{\mathrm{Rep}}(S_t)/\mathcal{N} \to \mathrm{Rep}(S_t)$.*

Proof. Let $\Phi\colon \underline{\mathrm{Rep}}(S_t) \to \mathrm{Rep}(S_t)$ be the usual functor. It suffices to show $\Phi(f) = 0$ if and only if f is negligible.

If $\Phi(f) = 0$ then $\Phi(f)$ is negligible, and so f is negligible by the previous proposition.

If f is negligible then $\Phi(f)$ is negligible, and so $\Phi(f) = 0$. (Note: if $\mathcal{C}$ is a semisimple $\mathbb{C}$-linear abelian rigid $\otimes$-category, then all negligible morphisms are 0. Indeed, if $f\colon X \to Y$ is non-zero, there exists $g\colon Y \to X$ such that gf is a nonzero projector so it has nonzero trace, so f is not negligible). □

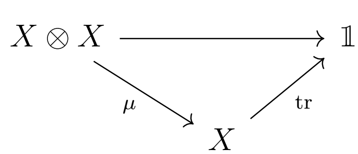
Note that this theorem is not true in positive characteristic.

6.2. **Universal property of Deligne Categories.** Let $\mathcal{C}$ be a rigid $\otimes$-category. Suppose X is a commutative unital associative algebra in $\mathcal{C}$ (so we have $\mu\colon X \otimes X \to X$ and $i\colon \mathbb{1} \to X$ satisfying the necessary properties).

Let $\mathrm{tr}\colon X \to \mathbb{1}$ be the composite

$$\mathbb{1} \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} X \otimes X \otimes X^\vee \xrightarrow{\mu\otimes\mathrm{id}} X \otimes X^\vee \xrightarrow{\mathrm{ev}} \mathbb{1}$$

This gives a pairing



which gives a map $X \to X^\vee$.

Definition 6.32. The algebra X is a *Frobenius algebra* if $X \to X^\vee$ is an isomorphism. We denote by $\mathcal{F}_t(\mathcal{C})$ the category of Frobenius algebras in $\mathcal{C}$ of rank t.

Remark 6.33. If X is a Frobenius algebra, $X \cong X^\vee$, so we get a coalgebra structure on X by dualizing the algebra structure. (Frobenius algebras are often defined in this way.)

Examples 6.34.

(1) The permutation representation $\mathbb{C}^n \in \mathrm{Rep}(S_n)$ is a Frobenius algebra and $\mathbb{C}^n = \prod_{i=1}^{n} \mathbb{C}$ as algebras.
(2) The object $T^1 \in \underline{\mathrm{Rep}}(S_t)$ is a Frobenius algebra with $\mu\colon T^1 \otimes T^1 \to T^1$ given by the diagram

It is clear that if $\Phi\colon \mathcal{A} \to \mathcal{B}$ is an $\otimes$-functor of rigid $\otimes$-categories then Φ carries a Frobenius algebra to a Frobenius algebra.

Hence if $\mathcal{A} = \underline{\mathrm{Rep}}(S_t)$, we get a functor

$$\mathrm{Fun}^{\otimes}(\underline{\mathrm{Rep}}(S_t), \mathcal{C}) \to \mathcal{F}_t(\mathcal{C}) \qquad (*)$$
$$\Phi \mapsto \Phi(T^1).$$

Theorem 6.35. *If $\mathcal{C}$ is an additive Karoubian category then $(*)$ is an equivalence.*

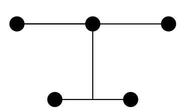
Remark 6.36. This is the universal property of $\underline{\mathrm{Rep}}(S_t)$. In words, giving a $\otimes$-functor $\underline{\mathit{Rep}}(S_t) \to \mathcal{C}$ is the same as giving a Frobenius algebra in $\mathcal{C}$ of rank t.

Proof of Theorem 6.35 (sketch). Let $X \in \mathcal{F}_t(\mathcal{C})$. We get a map

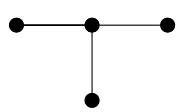
$$H_{r,s} \to \mathrm{Hom}_{\mathcal{C}}(X^{\otimes r}, X^{\otimes s}).$$

Idea: basic diagrams correspond to basic operations on X (multiplication, comultiplication, unit, counit).

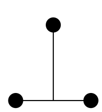
For example, when $r = 2$, $s = 3$, the diagram



is the composition of the diagrams



iterated comultiplication
$X \to X \otimes X \to X \otimes X \otimes X$



multiplication
$X \otimes X \to X$

This defines a functor $\underline{\mathrm{Rep}}(S_t)_0 \to \mathcal{C}$ and it is easy to see it is a $\otimes$-functor. This extends to $\underline{\mathrm{Rep}}(S_t)$ because $\mathcal{C}$ is additive and Karoubian.

Therefore, we have now constructed a functor

$$\mathcal{F}_t(\mathcal{C}) \to \mathrm{Fun}^{\otimes}(\underline{\mathrm{Rep}}(S_t), \mathcal{C})$$

and now one shows that it is equivalent to $(*)$. $\square$

6.3. **Representations of S_∞.** We give an overview of the topic, without proofs or details. See [SS15] for a more in depth treatment of the subject.

Let $T^r = (\mathbb{C}^\infty)^{\otimes r}$ with the natural action of S_∞. Recall that $\mathbb{C}^\infty = \bigcup_{n\geq 1} \mathbb{C}^n$, $S_\infty = \bigcup_{n\geq 1} S_n$.

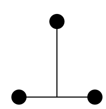
Definition 6.37. A representation of S_∞ is *algebraic* if it is a subquotient of a direct sum of T^r's. Let $\mathrm{Rep}(S_\infty)$ be the category of all algebraic representations of S_∞.

This is an abelian $\otimes$-category, but it is not rigid.

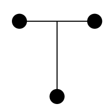
Remark 6.38. The category $\mathrm{Rep}(S_\infty)$ is not semisimple. For example, the augmentation map $\epsilon\colon T^1 \to T^0$ has no section, because $(T^1)^{S_\infty} = 0$.

Can describe maps between T^r's using the partition diagrams:

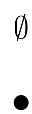
$$T^2 \to T^1, \qquad e_i \otimes e_j \mapsto \delta_{ij} e_i$$



$$T^1 \to T^2, \qquad e_i \mapsto e_i \otimes e_i$$



$$T^1 \to T^0, \qquad e_i \mapsto 1$$



However, we do not have the opposite map

$$T^0 \to T^1$$

Definition 6.39. The *downwards partition category* (dp) is the following

- objects: finite sets,
- a map $S \to T$ is a partition of $S \amalg T$ such that all parts meet S,
- composition is defined as before using the diagrams.

Remark 6.40. Because all partitions meet the bottom row, no parameter appears.

We have a functor

$$\begin{aligned} \kappa\colon (\mathrm{dp}) &\to \mathrm{Rep}(S_\infty) \\ \{1,\dots,r\} = [r] &\mapsto T^r \end{aligned}$$

Definition 6.41. A (dp)-*module* is a functor (dp) $\to$ Vec. Let $\mathrm{Mod}_{(\mathrm{dp})}$ be the category of (dp)-modules.

We then get a functor

$$\begin{aligned} \mathrm{Mod}_{(\mathrm{dp})}^{\mathrm{op}} &\to \mathrm{Rep}(S_\infty) \\ M &\mapsto \mathrm{Hom}_{(\mathrm{dp})}(M, \kappa). \end{aligned}$$

Theorem 6.42 (Sam–Snowden)**.** *This functor is an equivalence of categories of finite length objects.*

Define (up) like (dp) but with the opposite condition, i.e. (up) $\cong$ $(\mathrm{dp})^{\mathrm{op}}$. Then

$$\begin{aligned} \mathrm{Mod}_{(\mathrm{up})}^{f} &\cong (\mathrm{Mod}_{\mathrm{dp}}^{\mathrm{op}})^{f} \\ M &\mapsto (S \mapsto M(S)^*), \end{aligned}$$

where by f we mean the finite length objects. Therefore:

$$\mathrm{Rep}(S_\infty)^f \cong \mathrm{Mod}_{(\mathrm{up})}^{f}$$

(with no op).

Remark 6.43. This equivalence is of $\otimes$-categories, once one describes an $\otimes$-structure on the right hand side.

General construction. If $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is a functor of (essentially) small categories, we get a pullback functor

$$F^*\colon\ \mathrm{Mod}_{\mathcal{D}} \to \mathrm{Mod}_{\mathcal{C}}$$

which has a left adjoint $F_!$ and a right adjoint F_* (for example, by theory of Grothendieck abelian categories).

Therefore, if $\mathcal{C}$ is a monoidal category with monoidal structure $\amalg$, then we get two $\otimes$-structures on $\mathrm{Mod}_{\mathcal{C}}$ by using $\amalg_*$ or $\amalg_!$. If M, N are $\mathcal{C}$-modules,

$$M \otimes N = \amalg_*(M \boxtimes N) \text{ or } \amalg_!\,(M \boxtimes N)$$

where $M \boxtimes N\colon \mathcal{C} \times \mathcal{C} \to \mathrm{Vec}$ is given by $(x, y) \mapsto M(x) \otimes N(y)$.

We can apply this to (up) with $\amalg$ given by disjoint union to get an $\otimes$-structure on $\mathrm{Mod}_{(\mathrm{up})}$ (use $\amalg_*$). This corresponds to the usual $\otimes$ on $\mathrm{Rep}(S_*)$ under the equivalence.

We have the following universal property of $\mathrm{Rep}(S_\infty)$.

Theorem 6.44. *Giving a left-exact $\otimes$-functor* $\mathrm{Rep}(S_\infty) \to \mathcal{C}$*, where $\mathcal{C}$ is a $\mathbb{C}$-linear symmetric $\otimes$-category is the same as giving an object $A \in \mathcal{C}$ equipped with maps $m\colon A \otimes A \to A$, $\eta\colon A \to \mathbb{1}$, $\Delta\colon A \to A \otimes A$ such that … (the obvious properties hold: m is associative, Δ is co-associative, etc).*

This can be proved easily using the modules over (up) perspective, but it is much harder when working with $\mathrm{Rep}(S_\infty)$ directly.

Note that A is like a Frobenius algebra, but without a unit.

We can apply the universal property with $\mathcal{C} = \mathrm{Rep}(S_n)$ and $A = \mathbb{C}^n$. This gives a left exact $\otimes$-functor

$$\begin{aligned} \Gamma_n \colon \mathrm{Rep}(S_\infty) &\to \mathrm{Rep}(S_n), \\ T^r &\mapsto (\mathbb{C}^n)^{\otimes r} \end{aligned}$$

called the *specialization functor*.

One can give an elementary description of this functor. Let $H_n \subset S_\infty$ be the subgroup fixing each of $1, \dots, n$ (so $H_n \cong S_\infty$) and $G_n \subset S_\infty$ be the subgroup fixing each of $n+1, n+2, \dots$ (so $G_n \cong S_n$). We can then consider $G_n \times H_n \subseteq S_\infty$, which is the correct analog of a Young subgroup.

Hence if M is an S_∞-representation, M^{H_n} is a G_n-representation, and hence an S_n-representation.

Theorem 6.45. *For any S_∞-representation, $\Gamma_n(M) = M^{H_n}$.*

Remark 6.46. This implies $(M \otimes N)^{H_n} \cong M^{H_n} \otimes N^{H_n}$ for $M, N \in \mathrm{Rep}(S_\infty)$.

The simple objects of $\mathrm{Rep}(S_\infty)$ are parameterized by partitions, called them L_λ.

Theorem 6.47. *One can compute $R^i\Gamma_n(L_\lambda)$ using a Borel–Weil–Bott-style formula.*

We have seen that $\mathrm{Rep}(S_\infty)^f \cong \mathrm{Mod}^f_{(\mathrm{up})}$, which is useful for understanding the category $\mathrm{Rep}(S_\infty)^f$ and its monoidal structure by analogy to Deligne interpolation categories. However, it is not useful in understanding the abelian structure.

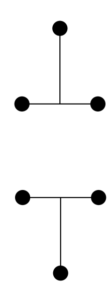
Note that $\mathrm{Mod}^f_{\mathcal{C}}$ is easy to understand if all morphisms in $\mathcal{C}$ go in one direction. If $x, y \in \mathcal{C}$, define $x \le y$ if $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$. Assume that this is a partial order. In this case, we call $\mathcal{C}$ directed. For fixed x, if M is a $\mathcal{C}$-module, we can define a submodule M' by

$$\begin{array}{ll} M_y & \text{if } y \ge x, \\ 0 & \text{otherwise.} \end{array}$$

Therefore, if M is simple, it is concentrated in one degree. Under this assumption, simple $\mathcal{C}$-modules correspond to pairs (x, V), where x is an object of $\mathcal{C}$ and V is an irreducible representation of $\mathrm{Aut}(x)$.

However, (up) is **not** directed. For example, we have the maps

$$T^1 \leftarrow T^2 \qquad \delta_{ij} e_i \leftarrow e_i \otimes e_j$$



$$T^2 \leftarrow T^1 \qquad e_i \otimes e_i \leftarrow e_i$$

so $[1] \leq [2] \leq [1]$, but $[2] \neq [1]$ in (up).

Recall that FI is the category whose objects and finite sets and who morphisms are injections. Note that FI **is** directed.

Define a functor

$$\begin{aligned} \Phi\colon \mathrm{Mod}_{FI} &\to \mathrm{Rep}(S_\infty) \\ M &\mapsto \varinjlim_{n\to\infty} M_n. \end{aligned}$$

This functor is exact.

Define $P_n \in \mathrm{Mod}_{FI}$ by $(P_n)_k = \mathbb{C}[\mathrm{Hom}_{FI}([n],[k])]$. Then $\mathrm{Hom}(P_n, M) = M_n$ for all FI-modules M. Therefore, these objects are projective.

Recall that $x \in M_n$ is *torsion* if x maps to $0 \in M_n$ for some $m \geq n$, and M is *torsion* if all its elements are.

This functor is exact and kills the torsion subcategory. By the mapping property of Serre quotients, we have the induced functor

$$\overline{\Phi}\colon \underbrace{\frac{\mathrm{Mod}_{FI}}{\mathrm{Mod}_{FI}^{\mathrm{tors}}}}_{\mathrm{Mod}_{FI}^{\mathrm{gen}}} \to \mathrm{Rep}(S_\infty).$$

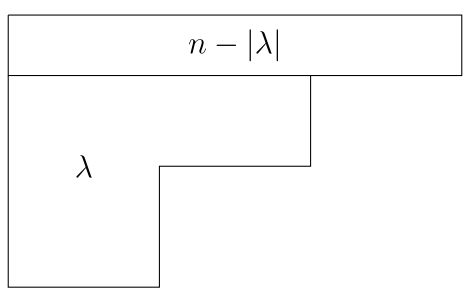
Theorem 6.48. *The functor $\overline{\Phi}$ is an equivalent.*

Idea of proof. We define

$$\begin{aligned} \Psi\colon \mathrm{Rep}(S_\infty) &\to \mathrm{Mod}_{FI} \\ V &\mapsto (\Gamma_n(V)), \end{aligned}$$

where Γ_n is the specialization functor defined before. It is easy to see that $\Phi\Psi \cong \mathrm{id}$ and Φ and Ψ are adjoint to each other. The theorem now following from general properties of Serre quotients. $\square$

Recall for a partition λ, we have an FI-module L_λ given by setting $(L_\lambda)_n$ to be



for $n \geq |\lambda| + \lambda_1$, and 0 otherwise. Recall that this is the sort of picture that appeared in the Pieri rule.

We showed that every finitely generated FI-module has a finite filtration with graded pieces L_λ, up to ignoring finitely many degrees. Therefore, the L_λ's are the simple objects of $\mathrm{Mod}_{FI}^{\mathrm{gen}}$.

Recall that $\mathrm{Mod}_{FI} \cong \mathrm{Mod}_A$, where $A = \mathrm{Sym}(\mathbb{C}^\infty) = \mathbb{C}[x_1, x_2, \ldots]$ and Mod_A is the category of A-modules with compatible polynomial representations of GL_∞. Under this identification:

$$\text{torsion } FI\text{-modules} \leftrightarrow \text{torsion } A\text{-modules},$$

where the torsion A-modules are ones where every element is annihilated by a power of $\mathfrak{m} = (x_1, x_2, \ldots)$. Roughly:

$$\begin{aligned} \mathrm{Mod}_A \leftrightarrow &\ \mathrm{GL}_\infty\text{-equivariant quasicoherent sheaves on } \mathbb{A}^\infty \ (\approx \mathrm{Spec}(A)) \\ \text{torsion} \leftrightarrow &\ \text{supported at } 0. \end{aligned}$$

Therefore, we expend $\mathrm{Mod}_A^{\mathrm{gen}}$ to correspond to GL_∞-equivariant quasicoherent sheaves on $\mathbb{A}^\infty \setminus \{0\}$.

This suggests that $\mathrm{Mod}_A^{\mathrm{gen}}$ is equivalent to representations of stabilizers of GL_∞ on some point in $\mathbb{A}^\infty \setminus \{0\}$. Consider the first basis vector. Then $\mathrm{Stab}(e_1)$ in GL_∞ is a generally affine group GA

$$\left(\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right)$$

and $GA \cong \mathrm{GL}_\infty \ltimes \mathbb{C}^\infty$. The actual theorem is the following.

Theorem 6.49. *We have that* $\mathrm{Mod}_A^{\mathrm{gen}} \cong \mathrm{Rep}^{\mathrm{pol}}(GA)$.

Giving a representation of GA is the same as giving a representation of $\mathbb{C}^\infty$ without a GL_∞-equivariance. Therefore, representations of $\mathbb{C}^\infty$ correspond to $\mathrm{Sym}(\mathbb{C}^\infty)$-modules.

Theorem 6.50. *We have that* $\mathrm{Rep}^{\mathrm{pol}}(GA) \cong \mathrm{Mod}_A^{\mathrm{tors}}$.

Corollary 6.51. *We have that* $\mathrm{Mod}_A^{\mathrm{gen}} \cong \mathrm{Mod}_A^{\mathrm{tors}}$.

It is easy to see that every finite length A-module has finite injective dimension.

Recall we have the specialization functor $\Gamma_n \colon \mathrm{Rep}(S_\infty) \to \mathrm{Rep}(S_n)$, which is left exact.

Problem: compute the derived functors of Γ_n.

Recall that we have

$$\begin{array}{ccccccc} V & & \mathrm{Rep}(S_\infty) & \xrightarrow{\cong} & \mathrm{Mod}_{FI}^{\mathrm{gen}} & \xrightarrow{\cong} & \mathrm{Mod}_A^{\mathrm{gen}} \\ & \searrow & & \searrow & \downarrow S & & \downarrow S \\ & & (\Gamma_n(V)) & & \mathrm{Mod}_{FI} & & \mathrm{Mod}_A \end{array}$$

Therefore, to compute $R^i\Gamma_n$, it is enough to compute R^iS.

We have

$$\mathrm{Mod}_A \underset{T=j^*}{\overset{S=j_*}{\rightleftarrows}} \mathrm{Mod}_A^{\mathrm{gen}}$$

where $j\colon \mathbb{A}^\infty \setminus \{0\} \hookrightarrow \mathbb{A}$.

Therefore, $R^i S$ corresponds to $R^i j_*$, which is just H^i on $\mathbb{A}^\infty \setminus \{0\}$. We have

$$\underline{\mathrm{Spec}}\left(\underbrace{\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)}_{=B}\right) = \mathbb{A}^\infty \setminus \{0\} \to \mathbb{P}^\infty$$

and one can show that the simples L_λ in $\mathrm{Mod}_A^{\mathrm{gen}}$ correspond to

$$B \otimes S_\lambda(R).$$

Then short exact sequence

$$0 \longrightarrow R \longrightarrow \mathbb{C}^\infty \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

then shows that

$$\bigoplus_{n\geq 0} R^i\Gamma_n L_\lambda \cong H^i(\mathbb{P}^\infty, B \otimes S_\lambda(R)),$$

which is computed by BWB.

Theorem 6.52 (Deligne)**.** *If $t \notin \mathbb{N}$ then $\underline{\mathrm{Rep}}(S_t)$ is semisimple and abelian.*

Basic idea: inductively decompose T^n into direct sums of objects L_λ and show $\mathrm{Hom}(L_\lambda, L_\mu) = 0$.

If $t \in \mathbb{N}$, then $\underline{\mathrm{Rep}}(S_t)$ is **not** abelian. For example,

$$T^1 \underset{g}{\overset{f}{\rightleftarrows}} T^0$$

and $fg = t$. If $t = 0$, $\ker(f)$ does not exist. For other t, there are other examples that show it is not abelian.

Recall that for an Ab-category, we constructed an additive envelope and then a Karoubian envelope, but it is more difficult to construct an "*abelian envelope*" in general.

In this case, Deligne constructed a rigid abelian $\otimes$-category $\underline{Rep}(S_t)^{\mathrm{ab}}$ with a fully faithful $\otimes$-functor

$$\underline{\mathrm{Rep}}(S_t) \to \underline{\mathrm{Rep}}(S_t)^{\mathrm{ab}}.$$

Basic idea. Work in $\underline{\mathrm{Rep}}(S_{-1})$, which is abelian. Consider $A = T^1 \oplus (T^0)^{\oplus(t+1)}$. This has rank t because T^1 has rank -1. We can give this the structure of a Frobenius algebra, which gives a functor

$$\underline{\mathrm{Rep}}(S_t) \to \mathrm{Mod}_A,$$

where Mod_A are the A-modules in $\underline{\mathrm{Rep}}(S_{-1})$. The category Mod_A is close to $\underline{\mathrm{Rep}}(S_t)^{\mathrm{ab}}$, but one still has to modify it slightly to make it rigid.

Deligne also conjectured a universal property, which was later proved by Comes-Ostrik.

Theorem 6.53. *Let $\mathcal{C}$ be a rigid abelian $\otimes$-category and $A \in \mathcal{C}$ be a Frobenius algebra of rank t. Let $F\colon \underline{\mathrm{Rep}}(S_t) \to \mathcal{C}$ be the $\otimes$-functor corresponding to A. Then F factors uniquely through either $\underline{\mathrm{Rep}}(S_t)^{\mathrm{ab}}$ or $\mathrm{Rep}(S_t)$.*

6.4. **Some generalities on embedding into abelian categories.** Suppose $\mathcal{A}$ is an additive category. Let us try to realize $\mathcal{A}$ as a full subcategory of a Grothendieck abelian category $\mathcal{B}$ such that every object of $\mathcal{B}$ is a quotient of a directs sum of objects of $\mathcal{A}$.

Given such an embedding, we can consider the class $\mathcal{E}$ of morphisms in $\mathcal{A}$ that are epimorphisms in $\mathcal{B}$.

Remark 6.54. Note that $\mathcal{E} \subseteq \{\text{class of all epimorphisms in } \mathcal{A}\}$, but it can be a proper subclass. For example, if $\mathcal{A}$ is the category of free $\mathbb{Z}$-modules, $\mathcal{B}$ is the category of all $\mathbb{Z}$-modules. The map $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 is an epimorphism in $\mathcal{A}$, but not in $\mathcal{B}$.

The class $\mathcal{E}$ is obtained from an embedding endow $\mathcal{A}$ is a subcanonical[6] Grothendieck topology[7] We take the morphisms in $\mathcal{E}$ to be the covering maps.

Now suppose $\mathcal{A}$, $\mathcal{E}$ are given and $\mathcal{E}$ gives a subcanonical Grothendieck topology on $\mathcal{A}$. We can take $\mathcal{B}$ to be the category of additive sheaves on abelian groups, i.e. its objects are $\mathcal{F}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$, where $\mathcal{F}$ is an additive functor and a sheaf, and the morphisms are morphisms of sheaves.

For example, we can always take $\mathcal{E}$ to be the split epimorphisms, whence $\mathcal{B} = \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab})$.

We cannot always take $\mathcal{E}$ to be all epimorphisms. However, for $\mathcal{A} = \underline{\mathrm{Rep}}(S_t)$, you can, and then $\mathcal{B}^f = \underline{\mathrm{Rep}}(S_t)^{\mathrm{ab}}$.

Additional remarks on $\underline{\mathrm{Rep}}(S_t)$:

- Indecomposable objects are X^λ, parameterized by partitions λ.
- If $t \in \mathbb{N}$, the image of X^λ in $\mathrm{Rep}(S_t)$ is $S^{\lambda[t]}$, where
$$\lambda[t] = (t - |\lambda_1|, \lambda_1, \dots, \lambda_r),$$
if this is a partition, and it is 0 otherwise.
- The dimension $\dim S^{\lambda[t]}$ is given by the Hook length formula 3.37. It is clearly a polynomial in t, say $P_\lambda(t)$. For example, if $\lambda = (2, 1)$, we have

[6] This means that the representable presheaves are sheaves.

[7] The notion of a Grothendieck topology can be translated into a few simple axioms. For example, if $X \to Y$ is a morphism and $U \to Y$ is in $\mathcal{E}$, there exists V such that the square
$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$
commutes. Note that this is not quite a fiber product, since we only assert the existence of a such a square. The other axioms also translate to similar properties.

$t-1$	$t-3$	$t-5$	$\dots$	2	1
2	1				
1					

and the Hook Length Formula shows that

$$\dim S^{\lambda[t]} = \frac{t!}{2 \cdot (t-5)! \cdot (t-3) \cdot (t-1)} = \frac{t(t-2)(t-4)}{2}.$$

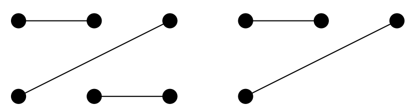
- The rank $\mathrm{rank}(X^\lambda)$ is $P_\lambda(t)$ for any $t \in \mathbb{C}$.

There is a connection between $\mathrm{Rep}(S_\infty)$ and Deligne categories for $t \in \mathbb{N}$. We have an $\otimes$-functor

$$\mathcal{F}\colon \mathrm{Rep}(S_\infty) \to \underline{\mathrm{Rep}}(S_t)^{\mathrm{ab}}.$$

Theorem 6.55 (Barter, Entoru-Eizenbud, Heidersdorf)**.** *The functor $\mathcal{F}$ is exact.*

Variant: orthogonal case. Note that $O(n)$, the orthogonal group, acts on $\mathbb{C}^n$ and let $T^r = (\mathbb{C}^n)^{\otimes r}$. Question: what is $\mathrm{Hom}(T^r, T^s)$? An example of such a map is $s = 6$ an $r = 4$, we have



and more generally these maps should be matchings. Similarly as before, we can define compositions using these diagrams.

6.5. **Harman's Theorem.** Define $\mathrm{Rep}_{\mathbb{C}}(S_n)^{\leq r}$ to be the category of representations that are direct sums of $S^{\lambda[n]}$ where $|\lambda| \leq r$.

Theorem 6.56. *We have an equivalence of categories*

$$\mathrm{Rep}_{\mathbb{C}}(S_n)^{\leq r} \to \mathrm{Rep}_{\mathbb{C}}(S_{n+1})^{\leq r}$$
$$S^{\lambda[n]} \mapsto S^{\lambda[n+1]}$$

for $n \geq 2r$.

Goal. Work towards a theorem of Nate Harman [Har15] that extends this to positive characteristic. In this case, we do not get stability but instead periodicity.

Application. If M is a finitely generated FI-module in characteristic 0, M_{n+1} is obtained from M_n for $n \gg 0$ via this functor. In characteristic $p > 0$, for $n \equiv n \mod p^k$, get M_m from M_n via Nate's equivalence.

We start by treating permutation modules. Fix a field k of characteristic $p > 0$. Define $\mathrm{Perm}_n^{\leq r}$ to be the full subcategory of $\mathrm{Rep}_k(S_n)$ on objects that are direct sums of $M^{\lambda[n]}$'s with $|\lambda| \leq r$.

Theorem 6.57. *For $2r < m < n$, suppose $p^{\lceil \log_p(r) \rceil} | n - m$. Then there is an equivalence*

$$\mathrm{Perm}_m^{\leq r} \to \mathrm{Perm}_n^{\leq r}$$
$$M^{\lambda[m]} \mapsto M^{\lambda[n]}.$$

We will take M^λ to be a vector space with basis $\langle A_1, \dots, A_r \rangle$ where $A_1 \amalg \cdots \amalg A_r = [n]$ and $|A_i| = \lambda_i$. (So A_i is the ith row of the tabloid).

Let τ be a tabloid of shape μ type λ. Define

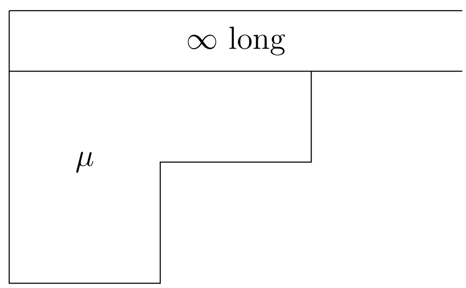
$$\begin{aligned} h^\tau \colon M^\mu &\to M^\lambda \\ \langle A_1, \dots, A_s \rangle &\mapsto \sum_{|A_i \cap B_j| = \tau_{ij}} \langle B_1, \dots, B_r \rangle. \end{aligned}$$

Theorem 6.58. *The maps h^τ form a basis of* $\operatorname{Hom}_{S_n}(M^\mu, M^\lambda)$.

The proof of this theorem is deferred for now, but it is similar to what we have done before.

The set of tabloids of shape $\mu[n]$ and type $\lambda[n]$ stabilizes once $n \geq 2\max(|\mu|, |\lambda|)$. We call these the *stable tabloids* of shape μ, type λ.

We can think of these as tabloids of shape $\mu[\infty]$, type $\lambda[\infty]$:



filled in with numbers so that:

- number of 1s is $\lambda[\infty]_1 = \infty$,
- number of 2s is $\lambda[\infty]_2 = \lambda_1$,
- and so on.

For a stable tabloid τ, we write $\tau[n]$ for the truncation of τ to size n.

Lemma 6.59. *The composition*

$$M^\nu \xrightarrow{h^\sigma} M^\mu \xrightarrow{h^\tau} M^\lambda$$

with $\ell(\nu) = t$, $\ell(\mu) = s$, $\ell(\lambda) = r$ is equal to

$$\sum_\rho c^\rho_{\sigma,\tau} h^\rho$$

where

$$c^\rho_{\sigma,\tau} = \prod_{k=1}^{s} \left(\sum_{\substack{\alpha = (\alpha_{i,j}) \\ \sum_i \alpha_{i,j} = \tau_{k,j} \\ \sum_j \alpha_{i,j} = \sigma_{i,k}}} \prod_{i,j} \binom{\rho_{i,j}}{\alpha_{i,j}} \right)$$

Proof. We have the following

$$\begin{aligned}
h^\tau(h^\sigma(\langle A_1,\dots,A_t\rangle)) &= h^\tau\left(\sum_B \left\{\begin{array}{ll} 1 & \text{if } A_i\cap B_j=\sigma_{i,j} \\ 0 & \text{otherwise}\end{array}\right\}\langle B_1,\dots,B_s\rangle\right)\\
&= \sum_{B,C}\left\{\begin{array}{ll} 1 & \text{if } A_i\cap B_j=\sigma_{i,j} \\ 0 & \text{otherwise}\end{array}\right\}\left\{\begin{array}{ll} 1 & \text{if } B_i\cap C_j=\tau_{i,j} \\ 0 & \text{otherwise}\end{array}\right\}\langle C_1,\dots,C_r,\rangle\\
&= \sum_C \underbrace{\#\left\{B \,\middle|\, \begin{array}{c} A_i\cap B_j=\sigma_{i,j} \\ B_i\cap C_j=\tau_{i,j}\end{array}\right\}}_{\text{need to count this}}\langle C_1,\dots,C_r\rangle.
\end{aligned}$$

Fix A, C and put $\rho_{i,j}=|A_i\cap C_j|$. Possible choices for B_1:

$$\begin{aligned}
B_1 = (\underbrace{A_1\cap C_1}_{\rho_{1,1}}\cap B_1)\amalg\cdots\amalg(\underbrace{A_1\cap C_r}_{\rho_{1,r}}\cap B_1)\\
\vdots\\
\amalg\,(A_t\cap C_1\cap B_1)\amalg\cdots\amalg(A_t\cap C_r\cap B_1)
\end{aligned}$$

Let $\alpha_{i,j}=|A_i\cap C_j\cap B_1|$. The result then follows. □

An *integer-valued polynomial* is a polynomial $F\in\mathbb{Q}[t]$ such that $F(n)\in\mathbb{Z}$ for all $n\in\mathbb{Z}$. Let R be the ring of integer-valued polynomials.

Lemma 6.60. *Let ρ,σ,τ be stable tabloids. Then $c^{\rho[n]}_{\sigma[n],\tau[n]}$ is an integer valued polynomial.*

Proof. By Lemma 6.59:

$$C^{\rho[n]}_{\sigma[n],\tau[n]} = \prod_{k=1}^{s}\left[\sum_{\substack{\alpha=(\alpha_{i,j})\\ \sum_i \alpha_{i,j}=\tau[n]_{k,j}\\ \sum_j \alpha_{i,j}=\sigma[n]_{i,k}}}\prod_{i,j}\binom{\rho[n]_{i,j}}{\alpha_{i,j}}\right].$$

It is clearly integer valued, and we just need to show it is a polynomial. Suppose $k>1$. Then

$$\sum_i \alpha_{i,j}=\tau[n]_{k,j}$$

is independent of n. Hence $\alpha_{i,j}$ is bounded and independent of n. Moreover,

$$\rho[n]_{i,j}=\begin{cases}\text{constant} & \text{if } i\neq 1 \text{ or } j\neq 1\\ \text{constant}+n & \text{if } i=j=1\end{cases}$$

Therefore

$$\binom{\rho[n]_{i,j}}{\alpha_{i,j}}=\begin{cases}\binom{\text{constant}}{\text{constant}} & \text{if } i\neq 1 \text{ or } j\neq 1\\ \binom{\text{constant}+n}{\text{constant}} & \text{if } i=j=1\end{cases}$$

This is clearly a polynomial in n.

Now suppose $k = 1$. For $j > 1$, $\sum_i \alpha_{i,j} = \tau[n]_{i,j}$ is constant, so $\alpha_{i,j}$ is bounded, independent of n. For $i > 1$, $\sum_j \sigma[n]_{i,k}$ is constant, so $\alpha_{i,j}$ is bounded independent of n.

Hence all $\alpha_{i,j}$'s except $\alpha_{1,1}$ are bounded independent of n. Then

$$(\alpha_{11} - n) + \sum_{i \neq 1} \alpha_{i,1} = \tau[n]_{1,1} - n = \text{constant}.$$

Hence $\alpha_{1,1} = \text{constant} + n$.

One then proceeds with a similar argument to the above to conclude the result. $\square$

Theorem 6.61 (Polya)**.** *The ring R has a $\mathbb{Z}$-basis of $x \mapsto \binom{x}{k}$ for $k \geq 0$.*

Proposition 6.62. *For $f \in R$ of degree d, $f(n) \mod p$ are periodic with period dividing $p^{\lceil \log_p(d) \rceil}$.*

Proof. Lucas congruence: if

$$n = n_0 + n_1 p + \cdots + n_r p^r,$$
$$m = m_0 + m_1 p + \cdots + m_r p^r,$$

then

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_r}{m_r} \mod p,$$

so $\binom{n}{m} \mod p$ only depends on the class on n modulo $p^{\lceil \log_p(d) \rceil}$. $\square$

Theorem 6.57 now follows for permutation modules. We will now show how to go from the permutation modules to the general case.

Let $\mathcal{C}$ be a highest weight category with poset Λ, $\Delta(\lambda)$ be the standard object and $\nabla(\lambda)$ be the costandard object. Then let

$$\mathcal{C}^\Delta = \text{category of objects with filtration having } \Delta(\lambda) \text{ as graded pieces},$$
$$\mathcal{C}^\Delta = \text{category of objects with filtration having } \nabla(\lambda) \text{ as graded pieces}.$$

Proposition 6.63. *For any $\lambda \in \Lambda$, there is a unique $T(\lambda) \in \mathcal{C}$ (up to isomorphism) such that*

1. *$T(\lambda) \in \mathcal{C}^\Delta \cap \mathcal{C}^\nabla$,*
2. *simple constituents of $T(\lambda)$ are $L(\lambda)$ with multiplicity 1 and $L(\mu)$ with $\mu < \lambda$,*
3. *$T(\lambda)$ is indecomposable.*

Definition 6.64. Modules of the form $T(\lambda)$, or direct sums of $T(\lambda)$'s, are called *tilting modules*. A *full tilting module* is one that contains all of $T(\lambda)$'s as summands.

Theorem 6.65 (Ringel duality)**.** *Let T be a full tilting module, $A = \mathrm{End}_{\mathcal{C}}(T)$. Then*

1. Mod_A *is a highest weight category with poset* Λ^{op}*, whose standard objects are* $\mathrm{Hom}_{\mathcal{C}}(\Delta(\lambda), T)$,
2. Mod_A *is independent of the choice of T, up to equivalence; this category is denoted $\mathcal{C}^\vee$ and called the Ringel dual,*
3. *we have an equivalence $\mathcal{C} \cong (\mathcal{C}^\vee)^\vee$.*

Let $\mathrm{Tilt}(\mathcal{C})$ be the full subcategory on tilting modules.

Corollary 6.66. *Let $\mathcal{C}$ and $\mathcal{C}'$ be highest weight categories with poset Λ. Then any equivalence $\mathrm{Tilt}(\mathcal{C}) \cong \mathrm{Tilt}(\mathcal{C}')$ taking $T(\lambda)$ to $T'(\lambda)$ extends to an equivalence $\mathcal{C} \cong \mathcal{C}'$ respecting the highest weight structure.*

Suppose $\mathcal{C}$ is a highest weight category with poset Λ. Let $\Lambda' \subset \Lambda$ be downwards closed (i.e. if $\lambda \in \Lambda'$ and $\mu \leq \lambda$ then $\mu \in \Lambda'$). Let $\mathcal{C}_{\Lambda'}$ be a Serre subcategory of $\mathcal{C}$ on $L(\lambda)$ with $\lambda \in \Lambda'$.

Proposition 6.67. *The category $\mathcal{C}_{\Lambda'}$ is a highest weight category with poset Λ'.*

Schur-Weyl duality.

Let $\mathcal{C}_n = \mathrm{Rep}^{\mathrm{pol}}(\mathrm{GL}_\infty)_n$. We have

$$\begin{aligned} T\colon \mathcal{C}_n &\to \mathrm{Rep}(S_n) \\ V &\mapsto 1^n\text{-weight space of } V \\ S\colon \mathrm{Rep}(S_n) &\to \mathcal{C}_n \\ V &\mapsto ((k^\infty)^{\otimes n} \otimes V)^{S_n} \end{aligned}$$

Then $\mathcal{C}_n$ is a highest weight category with Λ consisting of partitions of n where $\Delta(\lambda)$ is the Weyl module, $S(S^{\lambda^\dagger})$ and M^λ is the permutation module in $\mathrm{Rep}(S_n)$.

Fact 6.68. *There are unique indecomposable summands Y^λ (the Young submodules) of M^λ containing S^λ.*

Let $T' = T \otimes \mathrm{sgn}$, $S' = S(- \otimes \mathrm{sgn})$.

Proposition 6.69. *In characteristic at least 5, we have*

(1) $T'(\Delta(\lambda)) = S^{\lambda^\dagger}$, $T'(P(\lambda)) = Y(\lambda) \otimes \mathrm{sgn}$, $T'(L(\lambda)) = \begin{cases} D^{\lambda^\dagger} & \text{if } \lambda^\dagger \text{ isp-regular} \\ 0 & \text{otherwise,} \end{cases}$ $T'(T(\lambda)) = Y(\lambda^\dagger)$,
(2) $S'(S^{\lambda^\dagger}) = \Delta(\lambda)$, $S'(Y(\lambda^\dagger)) = T$,
(3) S', T' are mutually inverse equivalences on the subcategory $\mathcal{C}_n^\Delta$ and the subcategory of $\mathrm{Rep}(S_n)$ on modules admitting a filtration by Specht modules.

We previously saw that

$$\begin{aligned} \mathrm{Perm}_n^{\leq r} &\cong \mathrm{Perm}_m^{\leq r} \\ M^{\lambda[n]} &\mapsto M^{\lambda[m]} \end{aligned}$$

if $p^{\lceil \log_p(r) \rceil} | n - m$.

Let $\mathrm{Young}_n^{\leq r}$ be the subcategory of $\mathrm{Rep}(S_n)$ on all direct sums of $Y^{\lambda[n]}$ with $|\lambda| \leq r$.

Corollary 6.70. *We have that $\mathrm{Young}_n^{\leq r} \cong \mathrm{Young}_m^{\leq r}$ if $p^{\lceil \log_p(r) \rceil} | n - m$.*

Proof. This follows from the above since $\mathrm{Young}_n^{\leq r}$ is the Karoubian envelope of $\mathrm{Perm}_n^{\leq r}$. □

Let $\mathcal{C}_n^{\leq r}$ be the Serre subcategory of $\mathcal{C}_n$ on $L(\lambda[n]^\dagger)$ with $|\lambda| \leq r$.

Theorem 6.71. *In characteristic at least* 5*, we have*

$$\mathcal{C}_n^{\leq r} \cong \mathcal{C}_m^{\leq r}$$

as highest weight categories, if $p^{\lceil \log_p(r) \rceil} | n - m$*.*

Proof. Note that $\mathcal{C}_n^{\leq r}$ is a highest weight category with poset $\{\lambda[n]^\dagger \mid |\lambda| \leq r\}$, independent of n for $n \gg 0$. Tilting modules for $\mathcal{C}_n^{\leq r}$ are $T(\lambda[n])^\dagger$ with $|\lambda| \leq r$. We have that

$$\mathrm{Tilt}(C_n^{\leq}) \cong \mathrm{Young}_n^{\leq r} \cong \mathrm{Young}_m^{\leq r} \cong \mathrm{Tilt}(\mathcal{C}_m^{\leq r})$$

and the equivalence sends

$$T(\lambda[n]^\dagger) \mapsto T(\lambda[m]^\dagger).$$

By Corollary 6.66, we obtain $\mathcal{C}_n^{\leq r} \cong \mathcal{C}_m^{\leq r}$ □

Let $\mathrm{Rep}(S_n)_{\mathrm{im}}^{\leq r}$ be the essential image of $\mathcal{C}_n^{\leq r}$.

Theorem 6.72. *We have* $\mathrm{Rep}(S_n)_{\mathrm{im}}^{\leq r} \cong \mathrm{Rep}(S_m)_{\mathrm{im}}^{\leq r}$ *extending the equivalence on permutation categories if* $p^{\lceil \log_p(r) \rceil} | n - m$ *and characteristic is at least 5.*

Proof. Define $(\mathcal{C}_n^{\leq r})^{\mathrm{sing}}$ to be the Serre subcategory of $\mathcal{C}_n^{\leq r}$ on $L(\lambda[n]^\dagger)$ with λ p-singular. (These simples are in $\ker(T')$.) We have an equivalence

$$\mathrm{Rep}(S_n)_{\mathrm{im}}^{\leq r} \cong \mathcal{C}_n^{\leq r} / (\mathcal{C}_n^{\leq r})^{\mathrm{sing}}.$$

The reason is that $T' \colon \mathcal{C}_n^{\leq r} \to \mathrm{Rep}(S_n)_{\mathrm{im}}^{\leq r}$ is an exact functor which kills $(\mathcal{C}_n^{\leq r})^{\mathrm{sing}}$, and $S' \colon \mathrm{Rep}(S_n)_m^{\leq r} \to \mathcal{C}_n^{\leq r}$ is adjoint to T'. The equivalence then follows from general facts about Serre quotients.

Under the equivalence $\mathcal{C}_n^{\leq r} \cong \mathcal{C}_m^{\leq r}$, the singular subcategories coincide, because the highest weight structure is respected. The result now follows. □

Let $\mathrm{Rep}(S_n)^{\leq r}$ be the smallest abelian subcategory of $\mathrm{Rep}(S_n)$ containing $\mathrm{Perm}_n^{\leq r}$.

Theorem 6.73. *We have an equivalence* $\mathrm{Rep}(S_n)^{\leq r} \cong \mathrm{Rep}(S_n)^{\leq r}$ *extending the equivalence on the permutation categories if* $p^{\lceil \log_p(r) \rceil} | n - m$ *and the characteristic is at least 5.*

Proof. We know from the previous theorem that

$$\begin{array}{ccc} \mathrm{Rep}(S_n)_{\mathrm{im}}^{\leq r} & \xrightarrow{\cong} & \mathrm{Rep}(S_m)_{\mathrm{im}}^{\leq r} \\ \uparrow & & \uparrow \\ \mathrm{Perm}_n^{\leq r} & \xrightarrow{\cong} & \mathrm{Perm}_m^{\leq r} \end{array}$$

and this proves the result. □

One can then deduce the following theorem.

Theorem 6.74. *The following sequences are periodic with period equal to a power of* p*:*

(1) $\left[S^{\lambda[n]} : D^{\lambda[n]}\right]$,
(2) $\left[M^{\lambda[n]} : Y^{\mu[n]}\right]$,

(3) modular kronecker coefficients $\left[D^{\lambda[n]} \otimes D^{\mu[n]} : D^{\nu[n]}\right]$,
(4) modular versions of Littlewood-Richardson coefficients:

$$\left[\mathrm{Ind}_{S_k \times S_{n-k}}^{S_n}(D^{\nu} \otimes D^{\mu[n-k]}) : D^{\lambda[n]}\right].$$

Define $X_i \colon S_n \to \mathbb{Z}$ by $X_i(\sigma) = \#(i\text{-cycles in } \sigma)$. This is a class function.

Fact 6.75. *In characteristic 0, there is a polynomial* $P_\lambda \in \mathbb{Q}[X_1, X_2, \ldots]$ *such that*

$$\chi_{S^{\lambda[n]}}(\sigma) = P_\lambda(X_1(\sigma), X_2(\sigma), \ldots)$$

for $n \gg 0$.

Theorem 6.76 (Nate). *Given* λ*, there exists* $\ell \in \mathcal{N}$ *and* $P_{\lambda_{i_j}} \in \mathbb{Q}[X_1, X_2, \ldots]$ *for* $0 \le j \le p^\ell$ *such that*

$$\widehat{\chi}_{D^{\lambda}[n]}(\sigma) = P_{\lambda_{i_j}}(X_1(\sigma), X_2(\sigma), \ldots)$$

for $n \gg 0$, $j \equiv n \mod p^\ell$*, where by* $\widehat{\chi}_{D^{\lambda}[n]}$ *is a Brauer character.*

Proof. We can express $[D^{\lambda[n]}]$ in $K(S_n)$ in terms of $[S^{\mu[n]}]$'s, and this expression is periodic in p. We get a similar relation between $\hat{\chi}_{D^{\lambda[n]}}$ and $\hat{\chi}_{S^{\mu[n]}} = \chi_{S^{\mu[n]}}$, where the latter is a characteristic 0 character. Then Theorem 6.76 follows from Fact 6.75. □

Applications to FI-modules. If M is a finitely-generated FI-module in characteristic 0, there exist $\lambda_1, \ldots, \lambda_r$ such that

$$M_n \cong S^{\lambda_1[n]} \oplus \cdots \oplus S^{\lambda_r[n]}$$

for $n \gg 0$. It is not obvious how to carry this over to characteristic $p > 0$.

Example 6.77. Let $M_n = k^n$, a finitely-generated FI-module (where $\mathrm{char}(k) = p > 0$).

(1) Decomposition into irreducibles:

$$[M_n] = \begin{cases} 2[D^n] + [D^{(n-1,1)}] & \text{if } p|n \\ [D^n] + [D^{(n-1,1)}] & \text{otherwise} \end{cases}$$

which is not stable but it is periodic mod p.

(2) Decomposition into Specht modules:

$$[M_n] = [S^n] + [S^{(n-1,1)}]$$

independently of n.

Both of these properties hold for general finitely-generated FI-modules.

We start by explaining (1). We actually give a more general result of which (1) is a corollary.

Theorem 6.78. *Suppose* M *is a finitely-generated* FI*-module. Then there is an* r *such that* M_n *and* M_m *correspond under* $\mathrm{Rep}(S_n)^{\le r} \cong \mathrm{Rep}(S_m)^{\le r}$ *for all* $n, m \gg 0$ *with* $n \equiv m$ mod $p^{\lceil \log_p r \rceil}$.

Proof. Let $\mathcal{I}(q)$ be a free FI-module in degree q with

$$\mathcal{I}(q)_n = k[\mathrm{Hom}_{FI}([q],[n])] \in \mathrm{Perm}_n^{\leq q}.$$

Since $\mathrm{Rep}_k(FI)$ is locally noetherian, we have a presentation

$$P_1 \to P_0 \to M \to 0$$

where each P_i is a finite direct sum of $\mathcal{I}(q)$'s. The result now follows. □

Corollary 6.79. *Let M be a finitely generated FI-modules.*

1. *The expression for $[M_n]$ in terms of $[D^{\mu[n]}]$'s is periodic. (This gives the analog of (1) above.)*
2. *Decomposing M_n into indecomposables is also periodic.*
3. *For fixed i, $\dim H^i(S_n, M_n)$ is periodic in n.*

It is not clear how to deduce (3) from the above, but Harman claims it does follows. It is however definitely true and it was proved independently by Nagpal [Nag15].

For (2), we need a Theorem of Nagpal [Nag15]. If V is a representation of S_q, define an FI-module $\mathcal{I}(V)$, an induced FI-module, by

$$\mathcal{I}(V)_q = \mathrm{Ind}_{S_q\times S_{n-q}}^{S_n}(V \otimes \mathrm{triv}).$$

For example, if $V = k[S_q]$, $\mathcal{I}(V) = \mathcal{I}(q)$.

We say that an FI-module M is *semi-induced* if it has a finite filtration such that the graded pieces are induced.

Theorem 6.80 (Nagpal)**.** *Let M be a finitely-generated FI-module. Then there exists a complex*

$$0 \to M \to P_0 \to \cdots \to P_t \to 0$$

with P_i semi-induced and a complex is exact in degree $n \gg 0$.

In characteristic 0, this is a theorem by Snowden and Sam and one can even take P_i to be induced.

Theorem 6.81 (Harman)**.** *Let M be a finitely-generated FI-module. Then there exist partitions $\lambda_1, \dots, \lambda_r$ and integers $c_1, \dots, c_r$ such that*

$$[M_n] = c_1[S^{\lambda[n]}] + \cdots + c_r[S^{\lambda_r[n]}]$$

for all $n \gg 0$. In fact, we may assume that λ_i are p-regular (and hence the c's are unique).

Proof. By Nagpal's Theorem 6.80, we can reduce to the can $M = \mathcal{I}(V)$ with $V = S^\lambda$. Then

$$M_n = \mathrm{Ind}_{S_q\times S_{n-q}}^{S_n}(S^\lambda \otimes \mathrm{triv})$$

and the result follows from Pieri's rule. □

One can prove the periodicity of cohomology claimed in Corollary 6.79 (3) in another way. We present the sketch of an argument following [NS17].

Suppose M is an FI-module. Then

$$\bigoplus_{n\geq 0} H^i(S_n, M_n) = \Gamma^i(M)$$

by definition.

Question. What structure does $\Gamma^i(M)$ have that implies periodicity of dimensions?

Recall that $A = k[t]$ is a twisted commutative algebra. Note that $\Gamma^0\colon \operatorname{Rep}(S_*) \to \mathrm{Vec}^{\mathbb{N}}$ is a tensor functor. Hence

$$\Gamma^i(M) \text{ is a } \Gamma^0(A)\text{-module.}$$

Note that $\mathbb{D} = \Gamma^0(A)$ is a divided power algebra.

Let k be a commutative ring. A divided power algebra $\mathbb{D}$ has a k-basis $x^{[n]}$ with

$$x^{[n]} \cdot x^{[m]} = \binom{n+m}{m} x^{[n+m]}.$$

If k is a field of characteristic p,

$$\begin{aligned} k[y_0, \ldots]/(y_0^p, \ldots) &\cong \mathbb{D} \\ y_i &\mapsto x^{[p^i]}. \end{aligned}$$

Theorem 6.82. *The divided power algebra $\mathbb{D}$ is coherent (i.e. finitely generated ideals are finitely presented).*

Theorem 6.83. *If M is a finitely generated FI-module over a noetherian ring k, then*

$$\Gamma^i(M)$$

is (almost) a finitely-presented $\mathbb{D}$-module.

Main problems for twisted commutative algebras.

(1) Noetherian property.
(2) Structure of modules.

Conjecture 6.84. *All finitely generated twisted commutative algebras are noetherian.*

Status.

- Any finitely generated bounded twisted commutative algebra is noetherian. For example, $\operatorname{Sym}(\mathbb{C}^\infty)$, $\operatorname{Sym}(U \otimes \mathbb{C}^\infty)$ for $\dim U < \infty$ are noetherian.
- We have that

$$\begin{matrix} \operatorname{Sym}(\operatorname{Sym}^2 \mathbb{C}^\infty) \\ \operatorname{Sym}(\Lambda^2 \mathbb{C}^\infty) \end{matrix} = \bigoplus_{\substack{\lambda \\ \text{all parts even}}} S_\lambda(\mathbb{C}^\infty)$$

  are both noetherian. This is a theorem due to Nagpal–Sam–Snowden [NSS15].
- In Summer 2017, Draisma proved [Dra17] that finitely-generated twisted commutative algebras are topologically noetherian.

Structure of Mod_A for $A = \operatorname{Sym}(\mathbb{C}^\infty)$.

Theorem 6.85 (Structure theorem)**.** *Let $M \in D^b_{\mathrm{fg}}(A)$, the bounded, finitely generated derived category of A. Then there exists an exact triangle*

$$T \longrightarrow M \longrightarrow F \longrightarrow T[-1]$$

with $T, F \in D^b_{\mathrm{fg}}(A)$ and T a complex of torsion modules, F a complex of projective modules.

Corollary 6.86. *The Λ-module $\mathcal{K}_0(\mathrm{Mod}_A^{\mathrm{fg}})$ is free of rank 2 with basis $[\mathbb{C}]$ and $[A]$.*

As an application, for an A-module M, define the *Hilbert series* as

$$H_M(t) = \sum_{n \geq 0} \dim M_n \frac{t^n}{n!}$$

and note that $H_{M \otimes N} = H_M \cdot H_N$.

Theorem 6.87. *If M is finitely-generated, then $H_M(t) = p(t)e^t + q(t)$ for $p, q \in \mathbb{Q}[t]$.*

Define the *generalized Hilbert series* as

$$\widetilde{H}_M(t_1, t_2, \ldots) = \sum_{\lambda \text{ partition}} \mathrm{tr}(C_\lambda|_M) \frac{t^\lambda}{\lambda!}$$

where

$$C_\lambda \text{ is a conjugacy class in } S_{|\lambda|} \text{ corresponding to } \lambda,$$
$$t^\lambda = t_1^{m_1(\lambda)} t_2^{m_2(\lambda)} \ldots,$$
$$\lambda! = m_1(\lambda)! m_2(\lambda)! \ldots,$$
$$m_i = \#(i\text{s in } \lambda).$$

Theorem 6.88. *If M is finitely-generated, then*

$$\widetilde{H}_M(t) = p(t)e^{T_0}q(t)$$

where $p, q \in \mathbb{Q}[t_1, t_2, \ldots]$ and $T_0 = \sum_{i \geq 1} t_i$. Moreover, $p(t)$ is essentially a characteristic polynomial (up to conjugation of variables).

Local cohomology. Let $H^0_m(M)$ be a torsion submodule of M and H^i_m be the right derived functor of H^0_m.

Theorem 6.89. *If M is finitely-generated, $H^i_m(M)$ is finite length, vanishes for $i \gg 0$.*

Remark 6.90. We can recover q's in the Hilbert series from local cohomology.

General remarks. Let $\mathcal{A}$ be an abelian $\otimes$-category with some extra properties. Let A be a unit object. Then we can think of $\mathcal{A}$ as Mod_A. An *ideal* of A is a subobject of A. For ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$, we get a map

$$\mathfrak{a} \otimes \mathfrak{b} \to A \otimes A = A$$

and we can define $\mathfrak{ab}$ to be the image of this map.

Definition 6.91.

- The object A is a *domain* if $\mathfrak{ab} = 0$ implies $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$.

- An ideal $\mathfrak{p} \subset A$ is *prime* if $A/\mathfrak{p}$ is a domain.
- We let $\mathrm{Spec}(A)$ be the set of all prime ideals with the usual Zariski topology.
- For a domain A, we can define $\mathrm{Frac}(A)$ by
$$\mathrm{Mod}_{\mathrm{Frac}(A)} = \frac{\mathrm{Mod}_A}{\mathrm{Mod}_A^{\mathrm{tors}}}.$$
- For a prime $\mathfrak{p}$, we get a *residue "field"* given by $\mathrm{Frac}(A/\mathfrak{p})$.

Let $A = \mathrm{Sym}(U \otimes \mathbb{C}^\infty)$ for U finite dimensional. Then
$$\mathrm{Spec}(A) = \mathrm{Gr}(U) = \bigcup_{r=0}^{\dim U} \mathrm{Gr}_r(U),$$
the *total Grassmanian*. We can define the Serre subquotient $\mathrm{Mod}_{A,r}$ corresponding to Gr_r.

Proposition 6.92. *We have that*
$$\mathrm{Mod}_{A,r} \cong \mathrm{Mod}_B^0$$
where $B = \mathrm{Sym}(Q \otimes \mathbb{C}^\infty)$ on $\mathrm{Gr}_r(U)$ and the 0 means supported at 0.

Corollary 6.93. *We have that $\mathcal{K}_0(\mathrm{Mod}_{A,r}) = \Lambda \otimes \mathcal{K}_0(\mathrm{Gr}_r(U))$, where Λ is the ring of symmetric functions.*

Corollary 6.94. *We have that $\mathcal{K}_0(\mathrm{Mod}_A) = \bigoplus_{r=0}^{\dim U}$ is free of rank $2^{\dim U}$ as a Λ-module.*

For details of this, see [SS17].

Consider $A = \mathrm{Sym}(\mathrm{Sym}^2)$. Then
$$\mathrm{Spec}(A) = \mathbb{N}^2 \cup \{\infty\}.$$
The "residue field" at (r, s) is closely related to $\mathrm{Rep}(\mathrm{OSp}(r|s))$, the ortho-symplectic group. The residue field at ∞ is $\mathrm{Rep}^{\mathrm{alg}}(O_\infty)$.

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