# MATH 715: BERKOVICH SPACES

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These are notes from Math 715 taught by Professor Mattias Jonsson in Winter 2020, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from July 28, 2020. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The website for the class is

http://www.math.lsa.umich.edu/~mattiasj/715/

You can find the official syllabus, notes, and some homework sheets there.

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*Date*: July 28, 2020.

## 1. Introduction

Berkovich spaces are analytic spaces over (non-archimedean) valued fields.

**Definition 1.1.** A *valued field* is a pair $(k, |\cdot|)$ where $k$ is a field with a norm $|\cdot|\colon k \to \mathbb{R}_+ = \mathbb{R}_{\geq 0} = [0, \infty)$ such that

$$|a+b| \leq |a| + |b|$$
$$|a| = 0 \text{ if and only if } a = 0$$
$$|ab| = |a| \cdot |b|.$$

**Example 1.2.** The complex numbers $\mathbb{C}$ with the norm $|a+ib|_\infty = \sqrt{a^2+b^2}$ are a valued field $(\mathbb{C}, |\cdot|_\infty)$.

**Example 1.3.** Any field $k$ with the trivial absolute value $|a|_0 = 1$ for $a \neq 0$ is a valued field $(k, |\cdot|_0)$.

**Example 1.4.** The $p$-adic norm on $\mathbb{Q}$ is defined by writing $a = p^n \frac{b}{c}$ for $(p, bc) = 1$ and setting $|a| = r^n$ for $r \in (0,1)$. This makes $(\mathbb{Q}, |\cdot|)$ a valued field.

**Example 1.5.** For any $k$, $k((t))$ with the norm trivial on $k$ and $|t| = r \in (0,1)$ makes $(k((t)), |\cdot|)$ into a valued field.

**Definition 1.6.** A norm $|\cdot|$ is *non-archimedean* (NA) if $|a+b| \leq \max\{|a|, |b|\}$.

**Lemma 1.7.** *A norm $|\cdot|$ is non-archimedean if and only if $|n| \leq 1$ for all $n \in \mathbb{Z}$.*

*Proof.* The 'only if' implication is trivial. For the other implication, we see that

$$\begin{aligned}
|a+b| &= |(a+b)^n|^{1/n} \\
&= \left| \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \right|^{1/n} \\
&\leq \left( \sum_{j=0}^{n} |a|^j |b|^{n-j} \right)^{1/n} \\
&\leq \big( (n+1) \max\{|a|, |b|\}^n \big)^{1/n} \\
&= (n+1)^{1/n} \max\{|a|, |b|\}.
\end{aligned}$$

Letting $n \to \infty$, $(n+1)^{1/n} \to 1$, which gives the result. $\square$

**Exercise.** For a NA norm $|\cdot|$, $|a| \neq |b|$ implies that $|a+b| = \max\{|a|, |b|\}$.

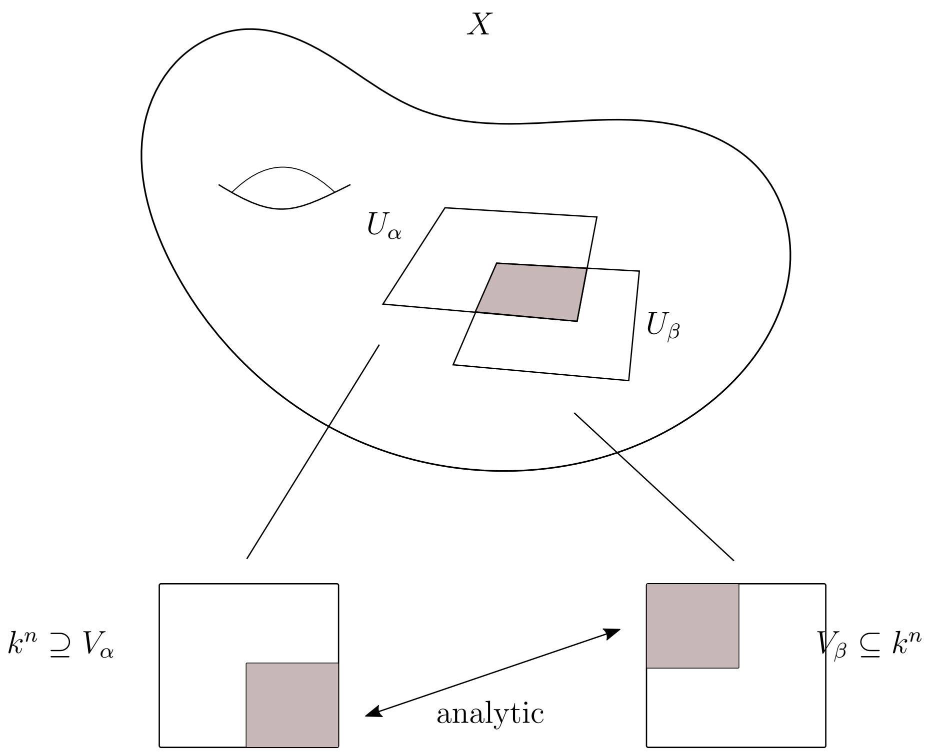
A norm $|\cdot|$ gives a distance on $k$ (i.e. $d(x,y) = |x-y|$). More generally, it defines a distance on $k^n$.

**Exercise.** If $|\cdot|$ is NA, then

- balls in $k$ and $k^n$ are both open and closed,
- $k$ and $k^n$ are totally disconnected.

**Naïve (?) definitions.**

- Analytic functions on $U \subseteq k^n$ are functions locally given by converging power series.
- A $k$-manifold of dimension $n$ is a topological space $X$ with an open covering $\{U_\alpha\}$ such that each $U_\alpha$ is homeomorphic to an open subset $V_\alpha$ in $k^n$ and for $U_\alpha \cap U_\beta$, the transition maps are analytic. See picture below.




**Problems.**

(1) The space $k^n$ is locally disconnected.
(2) Too many analytic functions, so we have a problem with analytic continuation. Indeed, in complex analysis, if an analytic function is 0 on $\mathbb{D}_{1/2}$ then it is zero on $\mathbb{D} = \mathbb{D}_1$. This is not the case with our definition of analytic functions for NA fields, because of the total disconnectedness.

**Approaches to NA geometry.**

(1) Tate (60s): rigid spaces.
(2) Raynaud (70s): formal schemes over $k^0$.
(3) Berkovich (80s): $k$-analytic spaces (Berkovich spaces).
(4) Huber(80s): adic spaces.

We will focus on on the third approach. For general introduction and a comparison between these approaches, see Brian Conrad's notes [Con07].

1.1. **The Berkovich affine space (as a topological space).** Eventually, this will be a topological space with a sheaf of analytic functions. For now, we focus on the definition of the topological space itself.

Let $k$ be a valued field (possibly archimedean). Consider

$$R = k[T_1, \dots, T_n].$$

We can associate to $R$ two algebro-geometric spaces:

- $\mathrm{Max}(R) = \{\text{maximal ideals of } R\} \supseteq k^n$,
- $\mathrm{Spec}(R) = \{\text{prime ideals of } R\}$ with the *Zariski topology*; this is usually denoted $\mathbb{A}^n_k$.

The *Berkovich affine space* is

$$\begin{aligned} \mathbb{A}^{n,\mathrm{an}}_k &= \{\text{semivaluations on } R \text{ extending the valuation on } k\} \\ &= \left\{ |\cdot|_x \colon R \to \mathbb{R}_+ \;\middle|\; \begin{smallmatrix} |f+g|_x \le |f|_x + |g|_x & f,g\in R, \\ |fg|_x = |f|_x|g|_x| & f,g\in R \\ |a|_x = |a| & a\in k \end{smallmatrix} \right\}. \end{aligned}$$

The topology is the weakest one such that the map

$$\begin{aligned} \mathbb{A}^{n,\mathrm{an}}_k &\to \mathbb{R}_+ \\ x &\mapsto |f|_x \end{aligned}$$

is continuous for all $f \in R$. Hence, a prebasis for the topology consists of sets

$$\{x \mid |f|_x < t\}$$
$$\{x \mid |f|_x > s\}$$

for $f \in R$ and $s, t \in \mathbb{R}$.

We can check that $\mathbb{A}^{n,\mathrm{an}}_k$ is:

- Hausdorff (this is easy),
- locally compact (this is an application of Tychanoff's Theorem),
- path connected (this is non-trivial).

**Lemma 1.8.** *We have that* $\mathrm{Max}(R) \subseteq \mathbb{A}^{n,\mathrm{an}}_k$.

*Sketch.* Assume $k$ is algebraically closed so that $\mathrm{Max}(R) = k^n$. The valuation associated to $a \in k^n$ is $|\cdot|_a \in \mathbb{A}^{n,\mathrm{an}}_k$ given by

$$|f|_a = |f(a)|.$$

The resulting map $\mathrm{Max}(R) \to \mathbb{A}^{n,\mathrm{an}}_k$ is injective. $\square$

**Fact 1.9.** *If* $(k, |\cdot|) = (\mathbb{C}, |\cdot|_\infty)$*, then* $\mathbb{A}^{n,\mathrm{an}}_k = \mathbb{C}^n$*, i.e. the map* $\mathrm{Max}(R) \to \mathbb{A}^{n,\mathrm{an}}_k$ *is surjective.*

This follows from the Gelfand–Mazur Theorem. We will talk about it later in the class.

**Fact 1.10.** *If $k$ is NA (and $n \ge 1$), then* $\mathrm{Max}(R) \to \mathbb{A}^{n,\mathrm{an}}_k$ *is* ***not*** *surjective.*

**Example 1.11.** Let us work out a special case for now:

- $n = 1$,
- $|\cdot| = |\cdot|_0$ is the trivial norm,

- $k$ is algebraically closed.

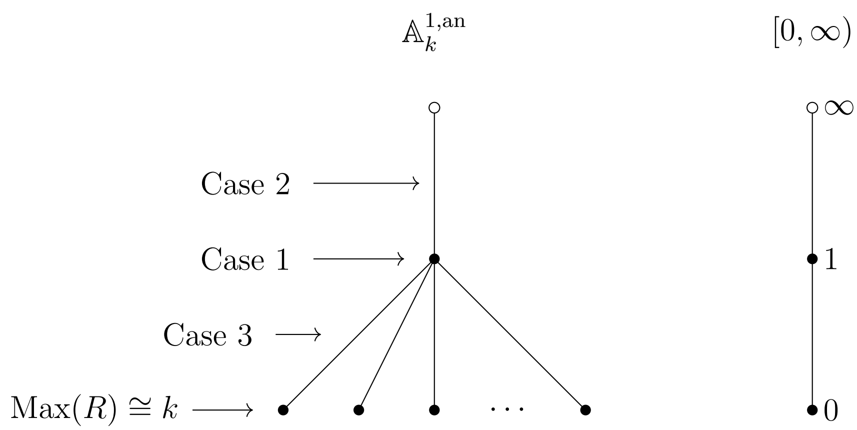
Then $x \in \mathbb{A}_k^{1,\mathrm{an}}$ corresponds to a semivaluation $|\cdot|_x$ on $k[T]$ which are trivial on $k$. It is determined by $|T-a|_x$ for $a \in k$.

Case 1: $|\cdot|_x$ is the trivial norm on $k[T]$.
Case 2: $|T|_x > 1$. Then $|\cdot|_x$ is uniquely determined by $r = |T|_x \in (1, \infty)$, because $|T-a|_x = r$ for all $a \in k$.
Case 3: there exists $a \in k$ such that $|T-a|_x < 1$. Then $a$ is unique, because

$$|T-b|_x = |T-a+(a-b)|_x = 1$$

for $b \neq a$. Moreover, $|\cdot|_x$ is uniquely determined by $a$ and $r = |T-a|_x \in [0,1)$.

This is the picture of the Berkovich affine line:




Recall that an affine scheme is defined as $X = \mathrm{Spec}(A)$ where $A$ is a ring. An important case is when $A$ is a finitely-generated $k$-algebra.

The Berkovich analogue of this is $X = M(A)$ where $A$ is a seminormal ring. We call $M(A)$ the *Berkovich spectrum*. If $X$ is a space, $A$ denotes the set of functions on $X$. An important case is when $A$ is a $k$-affinoid algebra, where $k$ is an NA field.

## 2. Seminormed commutative algebra

The main reference for this topic is [BGR84]. It is a little dry so we will not go into full details, but detailed notes are available here:

http://www.math.lsa.umich.edu/~mattiasj/715/BookParts/Seminormed%20Groups%20Jan%2012%202020.pdf

There are plenty of exercises and the reader is encouraged to read through them.

**Definition 2.1.** A *seminormed group* is an abelian group $G$ with a *seminorm* $\|\cdot\|\colon G \to \mathbb{R}_+$, i.e.

$$\begin{aligned} \|0\| &= 0, && \\ \|-f\| &= \|f\| && \text{for } f \in G \\ \|f+g\| &\le \|f\| + \|g\| && \text{for } f, g \in G \end{aligned}$$

It is *non-archimedean* (NA) if $\|f+g\| \le \max\{\|f\|, \|g\|\}$. It is a *normed group* if $\|f\| = 0$ if and only if $f = 0$.

There are several ways to make seminormed groups into a category.

**Definition 2.2.** A homomorphism of groups $\varphi\colon G \to H$ is

- *bounded* if there is $C > 0$, $\|\varphi(g)\| \le C\|g\|$ for all $g$; we then write $\|\varphi\|$ for the smallest such $C$,
- *contractive* if $\|\varphi\| \le 1$,
- *isometry* if $\|\varphi(g)\| = \|g \mid$ for all $g \in G$.

**Definition 2.3.** If $H \subseteq G$ is a subgroup, the *quotient seminorm* (or *residue seminorm*) on the coset space $G/H$ is given by

$$\|f\| = \inf_{\pi(g)=f} \|g\| \quad \text{for } f \in G/H$$

where $\pi\colon G \to G/H$.

**Definition 2.4.** A morphism $\varphi\colon G \to H$ is *admissible* if $G/\ker\varphi \cong \operatorname{im}\varphi$, i.e. there exists $C \ge 1$ such that

$$C^{-1} \inf_{k \in \ker\varphi} \|g+k\| \le \|\varphi(g)\| \le C\|g\|$$

for all $g \in G$. It is *contractively admissible* if $C = 1$.

**Definition 2.5.** A seminormed group $G$ is *complete* if every Cauchy sequence converges. (Note that limits are not necessarily unique in a seminormed group.)

A *Banach group* is a complete normed group.

A *separated completion* of a seminormed group $G$ is an isometry with dense image

$$G \to \widehat{G}$$

where $\widehat{G}$ is a Banach group. It satisfies an obvious universal property.

**Definition 2.6.** A *seminormed ring* is a ring $A$ with a seminorm $\|\cdot\|$ on $(A, +)$, as an abelian group, such that

$$\begin{aligned} \|ab\| &\le \|a\|\|b\| && \text{for } a, b \in A \\ \|1\| &\le 1. && \end{aligned}$$

One can then define a *normed ring* and a *Banach ring* in the obvious way.

**Remark 2.7.** If $\|1\| < 1$, then $\|a\| = 0$ for all $a \in A$. Nonetheless, we do not want to require that $\|1\| = 1$ in general so that we don't exclude examples like the zero ring.

**Definition 2.8.** A seminorm $\|\cdot\|$ on a ring $R$ is

- *power multiplicative* if $\|a^n\| = \|a\|^n$ for $a \in A$ and $n \geq 1$,
- *multiplicative* if $\|ab\| = \|a\|\|b\|$ for $a, b \in A$ and $\|1\| = 1$.

A *valuation* is a multiplicative norm. A *semivaluation* is a multiplicative seminorm.

**Examples 2.9.**

(1) $(A, |\cdot|_0)$ where $A$ is a ring, $|\cdot|_0$ is the trivial norm given by $|a|_0 = 1$ for $a\ neq 0$. This is a NA norm.
(2) $(\mathbb{Z}, |\cdot|_\infty)$ or $(\mathbb{C}, |\cdot|_\infty)$ are Banach ring.
(3) The zero ring $A$ satisfies all the above properties.
(4) Valued field $(k, |\cdot|)$ are examples of normed rings. For more details, see Homework 0.
(5) $(\mathbb{C}, \max\{|\cdot|_\infty, |\cdot|_0\})$ is the *hybrid norm* on $\mathbb{C}$.

**Definition 2.10.** A *non-archimedean field* (NA field) is a field which is complete with respect to a NA valuation.

**Lemma 2.11.** *If $A$ is a Banach ring, then the set of invertible elements of $A$ is open (and non-empty if $A \neq 0$).*

*Proof.* Assume $a \in A$ is invertible. Assume $b \in A$ satisfies

$$\|b - a\| < \|a^{-1}\|^{-1}.$$

We claim that $b$ is invertible. We have that

$$\left\| 1 - \frac{b}{a} \right\| = \|a^{-1}(a - b)\| \leq \|a^{-1}\|\|a - b\| < 1.$$

One can then check that $\frac{b}{a} = 1 - \left(1 - \frac{b}{a}\right)$ is invertible, with inverse $\sum_{n=0}^{\infty} \left(1 - \frac{b}{a}\right)^n$. Then $b = a \cdot \frac{b}{a}$ is invertible. □

**Corollary 2.12.** *If $A$ is a Banach ring and $\mathfrak{a} \subset A$ is a proper ideal, then*

- *$\overline{\mathfrak{a}}$ is a proper ideal of $A$,*
- *if $\mathfrak{a}$ is a maximal ideal, then $\mathfrak{a}$ is closed; then $A/\mathfrak{a}$ is a Banach field.*

2.1. **Relative discriminant algebras.** These are the Berkovich analogues of polynomial rings. Let $A$ be a seminormed ring and fix $r > 0$. We define

$$A\langle r^{-1}R\rangle = \left\{ f = \sum_{i=0}^{\infty} a_i T^i \;\middle|\; \|f\| = \sum_i \|a_i\| r^i < \infty \right\}.$$

This is again a seminormed ring. If $A$ is a Banach ring, so is $A\langle r^{-1}T\rangle$. In fact, this is the *completion* of $A[T]$ with respect to the above norm.

**Lemma 2.13.** *Assume $A$ is a Banach ring. If $a \in A$, then $1 - aT$ is invertible if and only if $\sum_{i=0}^{\infty} \|a^i\| r^i < \infty$.*

*Proof.* This is left as an exercise. □

2.2. **Seminormed rings and algebras.** Let $A$ be a seminormed ring.

**Definition 2.14.** A *seminormed module* $(M, \|\cdot\|)$ is

- an $A$-module $M$ with a seminorm $\|\cdot\|$ (as a group),
- $\|a \cdot m\| \leq \|a\| \cdot \|m\|$.

It is *faithfully seminormed* if $\|am\| = \|a\|\|m\|$.

**Definition 2.15.** A *seminormed algebra* is a seminormed ring $(B, \|\cdot\|)$ with a contractive ring homomorphism $\varphi\colon A \to B$ such that $a \circ \varphi = \varphi(a)b$.

2.3. **Seminormed tensor products.** Let $A$ be a seminormed ring and $M$, $N$ be seminormed $A$-modules. We consider the tensor product $M \otimes_A N$. There are two possible seminorms on $M \otimes_A N$:

(1) [General case] Define
$$\|v\| = \inf \left\{ \sum_i \|m_i\|\|n_i\| \;\middle|\; v = \sum m_i \otimes n_i \right\}.$$
(2) [NA case] For $A, M, N$ NA, define
$$\|v\| = \inf \left\{ \max_i \|m_i\|\|n_i\| \;\middle|\; v = \sum m_i \otimes n_i \right\}.$$

In each case, the universal property in this case is

$$\begin{tikzcd}
M \times N \arrow[rr] \arrow[dr, "\substack{\text{bounded}\\\text{bilinear}}"'] & & M \otimes_A N \arrow[dl, dotted, "\substack{\text{bounded}\\\text{bilinear}}"] \\
& P &
\end{tikzcd}$$

Suppose $A$ is a Banach ring and $M, N$ are Banach $A$-modules, then we define
$$M \widehat{\otimes}_A N = \text{separated completion of } M \otimes_A N.$$
This is again a Banach $A$-module, called the *complete tensor product*. The universal property is the same.

## 3. The Berkovich spectrum

**Definition 3.1.** The *Berkovich spectrum* of a seminormed ring $(A, \|\cdot\|)$ is
$$\begin{aligned} M(A) &= \{\text{bounded semivaluations on } A\} \\ &= \{|\cdot|_x\colon A \to \mathbb{R}_+ \text{ semivaluation such that } |\cdot|_x \leq \|\cdot\| \text{ on } A\}. \end{aligned}$$

**Notation.** If $x \in M(A)$ and $f \in A$, then we write $|f(x)|$ for $|f|_x$.

**Definition 3.2.** The topology on $M(A)$ is the weakest one such that the map
$$M(A) \ni x \mapsto |f(x)| \in \mathbb{R}_+$$
is continuous for all $f \in A$.

**Example 3.3.** For $A = 0$, $M(A) = \emptyset$. Indeed, if $x \in M(A)$, then $|1|_x = 1$, $|0|_x = 0$. If $0 = 1$, this is impossible.

**Example 3.4.** If $\|\cdot\| = 0$, then $M(A) = \emptyset$ for the same reason.

**Example 3.5.** If $k$ is a valued field, then $M(k) = \{\|\cdot\|\}$. Checking this is a simple exercise.

**Theorem 3.6.**

(1) *For any $A$, $M(A)$ is a compact (Hausdorff) topological space.*
(2) *The Berkovich spectrum $M(A)$ is empty if and only if $\|\cdot\| \equiv 0$.*

*Proof of (1).* We have a natural injective map

$$M(A) \xrightarrow{i} P = \prod_{f \in A} [0, \|f\|] = \{(t_f)_{f \in A}\}.$$

Note that $P$ is compact (and nonempty) by Tychanoff's Theorem.

By the definition of topologies on $M(A)$ and $P$, $i$ is an embedding. It suffices to show that $i(M(A))$ is closed. The image $i(M(A)) \subseteq P$ is defined by:

$$\begin{aligned} t_0 &= 0, \\ t_1 &= 1, \\ t_{f+g} &\leq t_f + t_g && \text{for all } f, g \in A, \\ t_{fg} &= t_f \cdot t_g && \text{for all } f, g \in A. \end{aligned}$$

These are clearly closed conditions. □

To prove the Theorem 3.6, it suffices to show that $M(A) \neq \emptyset$. We first talk mention the functoriality of $M$.

**Definition 3.7.** A morphism (i.e. a bounded ring homomorphism) $\varphi \colon A \to B$ of seminormed rings induces

$$\begin{aligned} M(B) &\xrightarrow{\varphi^*} M(A) \\ y &\mapsto (f \mapsto |\varphi(f)(y)|). \end{aligned}$$

**Exercise.** It is easy to check that this is well-defined and continuous. Therefore, $M(B) \neq \emptyset$ implies that $M(A) \neq \emptyset$.

We can now complete the proof of Theorem 3.6. We will need the following two lemmas.

**Lemma 3.8.** *If $(A, \|\cdot\|)$ is a norm field with $\|\cdot\|$ minimal, then the (separated) completion $(\hat{A}, \|\cdot\|)$ is a Banach field, and the induced norm $\|\cdot\|$ is also minimal.*

*Proof.* See the notes. □

**Lemma 3.9.** *If $(A, \|\cdot\|)$ is a Banach field, the following are equivalent:*

(1) *$\|\cdot\|$ is minimal,*
(2) *$\|f^{-1}\| = \|f\|^{-1}$ for all $f \in A^\times$,*
(3) *$\|\cdot\|$ is a valuation.*

*Proof.* **Exercise.** (2) is equivalent to (3).

It is also easy to see that (2) implies (1). Suppose $\|\cdot\|' \le \|\cdot\|$. Then for all $f \in A^\times$,

$$1 = \|1\|' = \|ff^{-1}\|' \le \|f\|'\|f^{-1}\|' \le \|f\|\|f\|^{-1} = 1$$

by (2). We hence have equalities throughout, so $\|f\|' = \|f\|$.

It remains to show that (1) implies (2). Since $\|\cdot\|$ is minimal, $\|\cdot\|$ is power multiplicative. Indeed $\varrho \le \|\cdot\|$ where

$$\varrho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}$$

is the *spectral seminorm*, so $\|\cdot\| = \varrho$ which is power-multiplicative.

Suppose there exists $f \in A^\times$ such that $r = \|f^{-1}\|^{-1} < \|f\|$. Set

$$B = A\langle r^{-1}T\rangle = \left\{ g = \sum_{i=0}^{\infty} f_i T^i \;\middle|\; \|g\| = \sum_i \|f_i\| r^i < \infty \right\}$$

(relative discriminant algebra[1]). This is a Banach $A$-algebra.

Then $1 - f^{-1}T$ is not invertible in $B$ by Lemma 2.13, because

$$\sum_{i=0}^{\infty} \underbrace{\|f^{-1}\| r^i}_{=1} = \infty.$$

Hence also $f - T$ is not invertible in $B$. Consider $\mathfrak{b} = \overline{(b - T)} \subset B$, which is a proper ideal. Define

$$(A', \|\cdot\|') = (B/\mathfrak{b}, \text{quotient norm}),$$

a non-zero Banach ring. There are maps $A \to B \to A'$ and the composition $\varphi$ is contractive.

Finally, $\|\cdot\|' \circ \varphi$ is a norm on $A$, $\|\cdot\|' \circ \varphi \le \|\cdot\|$, and

$$(\|\cdot\|' \circ \varphi)(f) = \|\varphi(f)\|' \le \|T\|' \le r < \|f\|.$$

The norm $\|\cdot\|'$ is hence strictly smaller than $\|\cdot\|$, contradicting minimality. $\square$

**Remark 3.10.** In general, if $(A, \|\cdot\|)$ is a normed field, then $\hat{A}$ is not necessarily a field. Therefore, the minimality assumption is crucial.

*Proof of Theorem 3.6 (2).* Assume $\|\cdot\| \not\equiv 0$. Consider the separated completion map $A \to \widehat{A}$. Since $\widehat{A}$, it is enough to prove the statement for $\widehat{A}$ by Exercise 3. We may hence assume without loss of generality that $A$ is a non-zero Banach ring.

Let $\underline{\mathfrak{m}}$ be a maximal ideal and consider the map $A \to A/\underline{\mathfrak{m}}$. By Exercise 3, we may assume without loss of generality that $A$ is a Banach field with a nonzero norm.

Now, consider

$$S = \{\text{norms } \|\cdot\|' \text{ on } A \text{ such that } \|\cdot\|' \le \|\cdot\| \text{ on } A\}.$$

Then $S$ is a non-empty poset.

**Exercise.** Any chain in $S$ has a lower bound in $S$. Hint: use $\|1\|' = 1$.

[1]Think of this as the analog of a polynomial ring.

By Zorn's Lemma, there is a minimal element $\|\cdot\|'$ in $S$. By passing to the completion and using Lemma 3.8, we may assume that $(A, \|\cdot\|')$ is a Banach field and $\|\cdot\|'$ is minimal. By Lemma 3.9, $\|\cdot\|'$ is a valuation. Hence $\|\cdot\|' \in M(A)$. ☐

3.1. **Points and complete residue fields.** Let $A$ be a seminormed ring. The points $x \in M(A)$ are semivaluations $|\cdot|_x : A \to \mathbb{R}_+$. Consider the prime ideal

$$\mathfrak{p}_x = \{|\cdot|_x = 0\} \subsetneq A.$$

Then $A/\mathfrak{p}_x$ is a valued integral domain. This induces a valuation on

$$\mathrm{Frac}(A/\mathfrak{p}_x).$$

**Definition 3.11.** The *complete residue field* of $x$ is the completion $\mathcal{H}(x)$ of $\mathrm{Frac}(A/\mathfrak{p}_x)$.

This plays the role of the residue field from algebraic geometry. We have maps

$$\begin{array}{ccccccc} A & \twoheadrightarrow & A/\mathfrak{p}_x & \hookrightarrow & \mathrm{Frac}(A/\mathfrak{p}_x) & \hookrightarrow & \mathcal{H}(x) \\ f & & \longrightarrow & & \longrightarrow & & f(x). \end{array}$$

**Remark 3.12.** Note that $|f(x)| = |f|_x$. This justifies the previously introduced notation.

**Definition 3.13.** The *Berkovich–Gelfand* transform is the map

$$A \to \prod_{x \in M(A)} \mathcal{H}(x)$$
$$f \mapsto (f(x))_{x \in M(A)}.$$

This map is contractive, because $|f(x)| = |f|_x \le \|f\|$.

Theorem 3.6 has the following consequence.

**Corollary 3.14.** *If $A$ is a Banach ring, then $f \in A$ is invertible if and only if $f(x) \neq 0$ for all $x \in M(A)$.*

*Proof.* For the 'only if' implication, if $fg = 1$, then for all $x \in M(A)$, $f(x)g(x) = 1$ in $\mathcal{H}(x)$, so $f(x) \neq 0$.

For the 'if' implication, assume that $f$ is not invertible. Then $f$ is contained in a maximal ideal $\mathfrak{m}$. Since $A$ is a Banach ring, $A/\mathfrak{m}$ is a Banach field (and hence a nonzero Banach ring). By Theorem 3.6, $M(A/\mathfrak{m}) \neq \emptyset$. If $x \in M(A)$ lies in the image of $M(A/\mathfrak{m}) \to M(A)$, then $f(x) = 0$. ☐

3.2. **Complex Banach algebras.** This is due to Gelfand and others. It served as a motivation for Berkovich when developing the theory of Berkovich spaces.

Let $A$ be a commutative complex Banach algebra, i.e. a Banach ring containing $(\mathbb{C}, \|\cdot\|_\infty)$. In the general theory, it is important to consider non-commutative algebras, without a unit, but we will focus on this simpler case.

**Example 3.15.** The canonical example is $A = C^\infty(X, \mathbb{C})$ for a compact topological space $X$ with the sup (or *uniform*) norm

$$\|f\ | = \sup_x |f(x)|.$$

The motivating question is: can we reconstruct $X$ from $A$?

For $f \in A$, consider the *spectrum of* $f$:

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \text{ is not invertible in } A\}.$$

Note that if $A$ is a Banach algebra of some operators acting on a vector space, this recovered the spectrum (set of eigenvalues) of a linear operator.

**Theorem 3.16.** *We have that $\sigma(f)$ is non-empty and compact.*

*Proof.* Using Lemma 2.13, $\|g\| < 1$ implies that $1 - g$ is invertible. Hence $\sigma(f)$ is closed and $\lambda \in \sigma(f)$ implies that $|\lambda| \leq \|f\|$. This shows that $\sigma(f)$ is compact.

We need to show that $\sigma(f)$ is non-empty. Suppose $\sigma(f) = \emptyset$. Then the *resolvent*

$$\mathbb{C} \ni \lambda \mapsto (\lambda - f)^{-1} \in A$$

is a bounded non-constant entire function. This contradicts Liouville's theorem. □

**Corollary 3.17** (Gelfand–Mazur)**.** *Any complex Banach field is isomorphic to* $\mathbb{C}$.

*Proof.* We have the structure map $\mathbb{C} \to A$. We want to show it is surjective.

For $f \in A$, $\sigma(f) \neq \emptyset$ by Theorem 3.16, so there exists $\lambda \in \mathbb{C}$ such that $\lambda - f$ is not invertible. Hence $\lambda - f = 0$, so $\lambda = f$. □

**Corollary 3.18.** *If $\mathfrak{m}$ is a maximal ideal in a complex Banach algebra $A$, then $\mathbb{C} \xrightarrow{\cong} A/\mathfrak{m}$.*

We have a map

$$\begin{aligned} \operatorname{Max}(A) &\to M(A) \\ \mathfrak{m} &\mapsto \text{seminorm on } A \\ &\qquad\text{induced by the Euclidean norm on } A/\mathfrak{m} \cong \mathbb{C}. \end{aligned}$$

**Corollary 3.19.** *The map* $\operatorname{Max}(A) \to M(A)$ *is bijective.*

*Proof.* We first check surjectivity. Pick $x \in M(A)$ corresponding to a bounded semivaluation $|\cdot|_x$ on $A$. We get the prime ideal $\mathfrak{p}_x = \{|\cdot|_x = 0\} \subseteq A$. Then

$$\mathcal{H}(x) = \text{completion of } \operatorname{Frac}(A/\mathfrak{p}_x)$$

is a valued complex Banach field. Hence $\mathcal{H}(x) \cong \mathbb{C}$. This shows that $\mathfrak{p}_x$ is a maximal ideal.

For injectivity, note that $\mathfrak{m} = \{|\cdot| = 0\}$, so we can recover $\mathfrak{m}$ from $|\cdot|$. □

**Remark 3.20.** If $A = C^0(X, \mathbb{C})$, then $M(A) = \operatorname{Max}(A) = X$. This answers the question posed in Example 3.15.

3.3. **The spectral radius.** The name spectral radius comes from the theory of complex Banach algebras. We discuss it in more generality.

Let $A$ be a Banach ring ($A \neq 0$). Let $\varrho\colon A \to \mathbb{R}_+$ be the *spectral radius* defined by

$$\varrho(f) = \lim_{n\to\infty} \|f^n\|^{1/n} = \inf_n \|f^n\|^{1/n}.$$

**Theorem 3.21** (Berkovich Maximum Modulus Principle)**.** *For $f \in A$,*

$$\varrho(f) = \max_{x\in M(A)} |f(x)|.$$

Note that the supremum is attained by compactness on $M(A)$ (Theorem 3.6).

*Proof.* For any $x \in M(A)$ and $f \in A$, $|f(x)| \le \|f\|$. Hence

$$|f(x)| = |f^n(x)|^{1/n} \le \|f^n\|^{1/n} \to \varrho(f)$$

as $f \to \infty$. This shows the inequality $\varrho(f) \ge \max_{x\in M(A)} |f(x)|$.

For the other inequality, pick $r > \max_{x\in M(A)} |f(x)|$. We must show that $r > \varrho(f)$. Consider the relative discriminant algebra:

$$B = A\langle rT\rangle = \left\{ g = \sum_{i=0}^{\infty} f_i T^i \;\middle|\; f_i \in A, \|g\| = \sum_{i=0}^{\infty} \|f_i\| r^{-i} < \infty \right\}.$$

Since $\|T\| = r^{-1}$, we see that $|T(y)| \le r^{-1}$ for all $y \in M(B)$. Hence

$$|(fT)(y)| < 1$$

for all $y \in M(B)$. Hence $(1 - fT)(y) \neq 0$ for all $y \in M(B)$. By Corollary 3.14, this is equivalent to $1 - fT$ being invertible in $B$. Hence Lemma 2.13 shows that

$$\sum_{i=0}^{\infty} \|f^i\| r^{-i} < \infty.$$

Hence $r > \varrho(f) = \lim_{n\to\infty} \|f^i\|^{1/i}$. $\square$

**Corollary 3.22.** *We have that $\varrho(f) = 0$ ($f$ is quasinilpotent) if and only if $f(x) = 0$ in $\mathcal{H}(x)$ for all $x \in M(A)$.*

**Remark 3.23.** If $A^u$ is the uniformization of $A$, i.e. the separated completion of $(A, \varrho)$, then there is a contractive map $A \to A^u$. Then the induced map $M(A^u) \to M(A)$ is an isomorphism.

**Example 3.24.** Consider $A = (\mathbb{Z}, |\cdot|_\infty)$. We will describe $M(A)$.

Given $x \in M(A)$, set $\mathfrak{p}_x = \{|\cdot|_x = 0\} \subseteq \mathbb{Z}$, which is a prime ideal. There are several cases.

(1) Suppose $\mathfrak{p}_x = p\mathbb{Z}$ for a prime $p$. (This means that $|p| = 0$.) Then $|\cdot|_x$ induces a valuation on $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.
    **Exercise.** The only valuation on $\mathbb{F}_p$ is the trivial one.

Hence the induced valuation on $\mathbb{Z}/p\mathbb{Z}$ is trivial. Thus $|\cdot|_x = |\cdot|_{p,0}$ is given by:

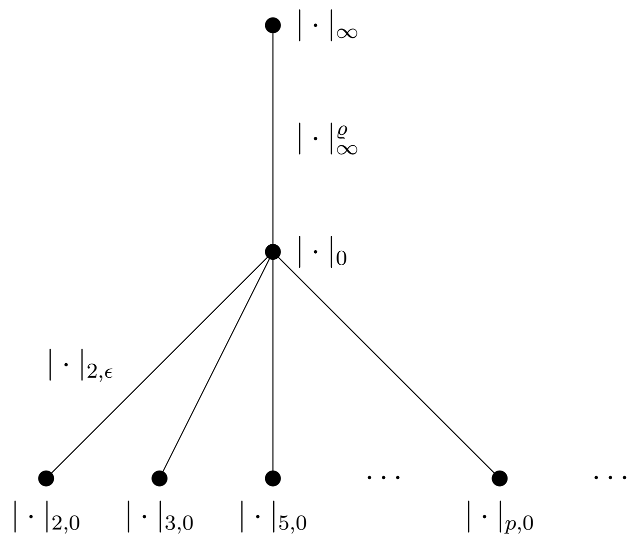
$$|n|_{p,0} = \begin{cases} 0 & \text{if } p \text{ divides } n, \\ 1 & \text{otherwise.} \end{cases}$$

(2) Suppose $\mathfrak{p}_x = 0$, i.e. $|\cdot|_x$ is a valuation on $\mathbb{Z}$ and hence extends to a valuation on $\mathbb{Q}$. Now, we can use Ostrowski's Theorem. There are three subcases.
   (a) $|\cdot|_x = |\cdot|_0$ is the trivial norm

$$|n|_0 = \begin{cases} 1 & n \neq 0, \\ 0 & n = 0, \end{cases}$$

   (b) $|\cdot|_x = |\cdot|_\infty^\varrho$ for $0 < \varrho \leq 1$,
   (c) $|\cdot|_x = |\cdot|_{p,\epsilon}$ for $0 < \epsilon < 1$ where for $n = p^k a$ with $(p, a) = 1$, we define $|n|_{p,\epsilon} = \epsilon^k$.

We have the following picture of $M(\mathbb{Z}, |\cdot|_\infty)$:




What about the topology? A prebasis is for $n \in \mathbb{Z}$ and $r \in \mathbb{R}_+^*$ by

$$\{x \in M(\mathbb{Z}) \mid |n|_x < r\},$$

$$\{x \in M(\mathbb{Z}) \mid |n|_x > r\}.$$

**Exercise.** Draw $\{|6| > \frac{1}{6}\} \subseteq M(\mathbb{Z})$ and $\{|6| \leq \frac{1}{6}\}$.

Let $A$ be a Banach ring.

**Definition 3.25.** A *character* on $A$ is a bounded (equivalently, contractive) ring homomorphism $\chi\colon A \to K$ for a complete valued field $K$.

**Example 3.26.** For $x \in M(A)$, the map $A \to \mathcal{H}(x) = \mathrm{Frac}(A/\mathfrak{p}_x)^\wedge$ is a character. Conversely, any character $\chi\colon A \to K$ induces a point in $M(A)$ given by $\{*\} = M(K) \to M(A)$.

One can prove the following.

**Proposition 3.27.** *For any Banach ring $A$,*

$$M(A) \cong \{\textit{equivalence classes of characters}\}.$$

**Exercise.** Define the equivalence relation and prove the proposition.

**Examples 3.28.**

(1) If $k$ is a valued field, then $M(k)$ is a point.
(2) We described $M(\mathbb{Z}, |\cdot|_\infty)$ in Example 3.24.
(3) Let $A$ be a trivially normed ring. We always have the *kernel map*:

$$\begin{aligned} \ker\colon M(A) &\to \operatorname{Spec}(A) \\ x &\mapsto \mathfrak{p}_x = \{|\cdot|_x = 0\}. \end{aligned}$$

When $A$ is trivially normed, this has a canonical section:

$$\begin{aligned} \operatorname{triv}\colon \operatorname{Spec}(A) &\to M(A) \\ \mathfrak{p} &\mapsto |\cdot|_{\mathfrak{p},0} \end{aligned}$$

where

$$|f|_{\mathfrak{p},0} = \begin{cases} 0 & \text{if } f \in \mathfrak{p}, \\ 1 & \text{if } f \notin \mathfrak{p}. \end{cases}$$

We also have a *reduction map* (or *center map*)

$$\begin{aligned} \operatorname{red}\colon M(A) &\to \operatorname{Spec}(A), \\ x &\mapsto \{|\cdot|_x < 1\}. \end{aligned}$$

**Exercise.**
- The map ker is continuous.
- The map red is anticontinuous (i.e. preimage of any open is closed), if $A$ is Noetherian.
- The map triv is neither.
- For any $x$, $\ker(x)$ is a specialization of $\operatorname{red}(x)$, i.e. $\ker(x) \subseteq \operatorname{red}(x)$.

(3a) Let $A$ be a DVR, e.g. $\mathbb{Z}_p$ or $k[\![T]\!]$. We consider $A$ with the trivial valuation, $(A, |\cdot|_0)$.
Let $\mathfrak{m} = (f)$ be the maximal ideal of $(f)$. Pick $x \in M(A)$. Then for all $g \in A$,

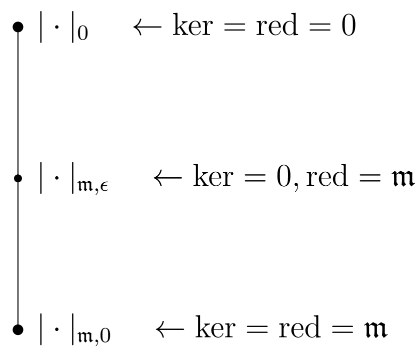
$$|g|_x \leq |g|_0 \in \{0, 1\}.$$

If $g \notin \mathfrak{m}$, it is invertible, so $|g|_x = 1$. The semivaluation $|\cdot|_x$ will hence be determined by $|f|_x$.

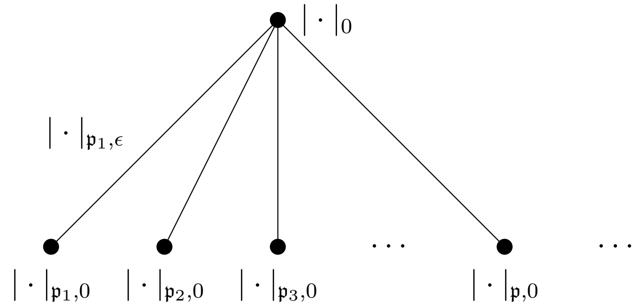
There are 3 cases:
(a) If $|f|_x = 1$, then $|\cdot|_x = |\cdot|_0$.
(b) If $|f|_x = 0$, then $|\cdot|_x = |\cdot|_{\mathfrak{m},0}$.
(c) If $|f|_x = \epsilon \in (0,1)$, then

$$|g|_x = \epsilon^{\max\{\ell \mid g \in \mathfrak{m}^\ell\}}.$$

Hence $M(A) \cong [0,1]$ in this case.




(3b) Let $A$ be a Dedekind ring, e.g. $\mathbb{Z}$ or $k[T]$. A similar analysis to Example 3.24 shows that $M(A)$ looks like:




where the points at the bottom are $\mathrm{Max}(A) = \mathrm{Spec}(A) \setminus \{0\}$.

## 4. Berkovich discs

Some references for this section include [BR10] and [Jon12].

Let $k$ be a non-archimedean field. By assumption, it is complete.

Let $r > 0$ be a radius. Define

$$\begin{aligned} \mathbb{E}(r) &= \{a \in k \mid |a| \le r\} && \text{the \textit{closed disc}}, \\ \mathbb{D}(r) &= \{a \in k \mid |a| < r\} && \text{the \textit{open disc}}. \end{aligned}$$

Any polynomial $f \in k[T]$ defines a continuous function

$$\begin{aligned} f \colon \mathbb{E}(r) &\to k \\ a &\mapsto f(a). \end{aligned}$$

A tentative norm on $k[T]$ is given by $\sup\limits_{\mathbb{E}(r)} |f|$.

**Exercise.** If the value group $|k^\times| \subseteq \mathbb{R}_+^\times$ is dense, then for any $f \in k[T]$, $f = \sum\limits_i a_i T^i$,

$$\sup_{\mathbb{E}(r)} |f| = \max_i |a_i| r^i.$$

**Exercise.** This is false if $k$ is discretely or trivially valued.

We ignore the tentative norm and use the following for any $k$.

**Definition 4.1.** For any $r > 0$, we define $\|f\| = \|f\|_r = \max_i |a_i| r^i$ if $f \in \sum_i a_i T^i$.

**Lemma 4.2** (Gauss)**.** *The norm* $\|\cdot\|_r$ *is a valuation on* $k[T]$ *(i.e. it is also multiplicative).*

*Proof.* Exercise. □

**Definition 4.3.** Define $k\{r^{-1}T\}$ to be the separated completion of $(k[T], \|\cdot\|_r)$:

$$k\{r^{-1}T\} = \left\{ f \in \sum_0^\infty a_i T^i \;\middle|\; \lim_i |a_i| r^i = 0 \right\} \subseteq k[\![T]\!]$$

with the norm $\|f\| = \max_i |a_i| r^i$.

To check that this is indeed the separated completion, one should check that it is a valued Banach $k$-algebra. We leave that as an exercise.

**Remark 4.4.** If $r = 1$, $k\{T\}$ is called the *Tate algebra* in 1 variable, also denoted $k\langle T\rangle$.

**Definition 4.5.** The *Berkovich closed disc of radius* $r$ is $E(r) = E_k(r) = M(k\{r^{-1}T\})$.

Note that $E(r) \subseteq \mathbb{A}_k^{1,\mathrm{an}}$ is a closed subset, where $\mathbb{A}_k^{1,\mathrm{an}}$ was defined in Example 1.11.

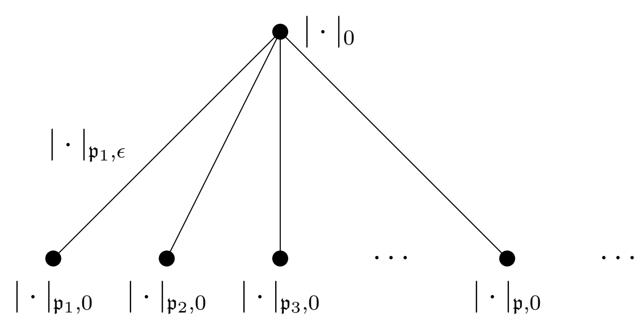
**Remark 4.6.** If $0 < r < s$, we have a map $k\{s^{-1}T\} \to k\{r^{-1}T\}$ which is bounded (not admissible), with dense image, which induces an embedding (homeomorphism onto the image)

$$E(r) \to E(s).$$

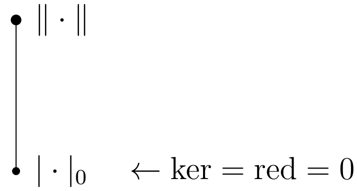
The next goal is to understand the structure of $E(r)$. This is naturally divided into three cases:

(a) $k$ is trivially valued,
(b) $k$ non-trivially valued, algebraically closed,
(c) $k$ is non-trivially valued, not algebraically closed.

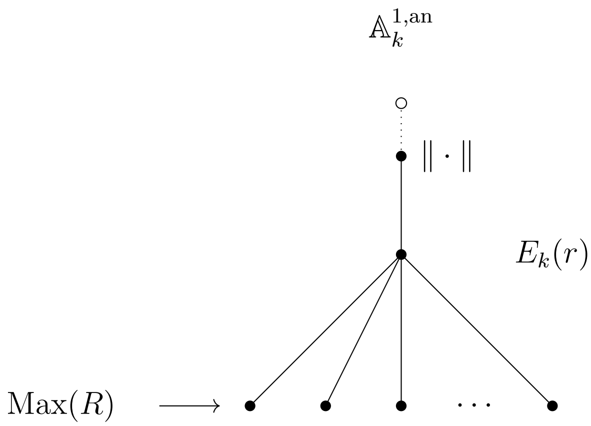
4.1. **Case (a): $k$ is trivially valued.** For $r = 1$, $k\{r^{-1}T\} = k[T]$ with the trivial norm $|\cdot|_0$. This is a Dedekind ring and we saw a picture of the Berkovich spectrum in Example 3.28 (3b):




For $0 < r < 1$, we have that $k\{r^{-1}T\} = k[\![T]\!]$. The picture in this case is:




For $r > 1$, $k\{r^{-1}T\} = k[T]$ and $E_k(r) \subseteq \mathbb{A}_k^{1,\mathrm{an}}$:




4.2. **Case (b): $k$ is algebraically closed but non-trivially valued.** Some examples of these fields include $\widehat{k((T))^a}$ (where the superscript $a$ is the algebraic closure) or $\mathbb{C}_p$.

**Remark 4.7.**

- In this case, $|k^\times| \subseteq \mathbb{R}_+^\times$ is dense.
- If $x \in E(r)$, $|\cdot|_x$ is uniquely determined by $|T-a|_x = |(T-a)(x)|$ for $a \in \mathbb{E}(r)$.

There are a few *types* of points.

**Type 1 points.** These are also called *classical* or *rigid points*. We have a map

$$\begin{aligned} \mathbb{E}(r) &\to E(r) \\ a &\mapsto \big(f \mapsto |f(a)|\big)\,. \end{aligned}$$

This defines a semivaluation with kernel $(T-a)$.

**Exercise.** The map $\mathbb{E}(r) \to E(r)$ is a homeomorphism onto its image.

The remaining points are valuations on $k[T] \subseteq k\{r^{-1}T\}$. We need to first talk about discs. Given $a \in \mathbb{E}(r) \subseteq k$ and $\varrho \in (0, r]$, set

$$\begin{aligned} \mathbb{E}(a,\varrho) &= \{b \in k \mid |b-a| \le \varrho\} \subseteq \mathbb{E}(r), \\ E(a,\varrho) &= \{x \in E(r) \mid |(T-a)(x)| \le \varrho\} \subseteq E(r). \end{aligned}$$

**Exercise.** Check that:

(1) $E(a, \varrho \cap k = \mathbb{E}(a,\varrho)$ via the map $\mathbb{E}(r) \subseteq E(r)$,
(2) $\mathbb{E}(a_1,\varrho_1) \subseteq \mathbb{E}(a_2,\varrho_2)$ if and only if $\varrho_1 \le \varrho_2$ and $|a_1 - a_2| \le \varrho_2$,

(2') $E(a_1, \varrho_1) \subseteq E(a_2, \varrho_2)$ if and only if $\varrho_1 \leq \varrho_2$ and $|a_1 - a_2| \leq \varrho_2$,
(3) if $a \in \mathbb{E}(r)$, then $T \mapsto T + a$ gives an automorphism of $k\{r^{-1}T\}$ and sends $E(b, \varrho)$ to $E(a + b, \varrho)$.

**Type 2 and 3 points.** Let $(a, \varrho)$ be as above. Define a norm $|\cdot|$ on $k[T]$ by

$$|f| = \max_i |c_i|\varrho^i \quad \text{for } f = \sum c_i(T-a)^i.$$

This norm is determined by $|T - b|_E = \max\{\varrho, |a - b|\}$. Gauss' Lemma 4.2 shows that $|\cdot|$ is a valuation.

**Exercise.**

- The norm $|\cdot|$ only depends on $E = E(a, \varrho)$; we denote it by $|\cdot|_E$.
- We have that $|\cdot|_E \leq \|\cdot\|$ on $k[T]$. Hence $|\cdot|_E$ extends to a valuation on $k\{r^{-1}T\}$. Therefore, $|\cdot|_E$ defined a point $p(E) \in E(r)$.

**Definition 4.8.** The point $p(E)$ is of *Type 2* if $\varrho(E) \in |k^\times|$. Otherwise, $p(E)$ is of *Type 3*.

The norm $|\cdot|_E$ is often called the *Gauss norm* associated to $E$ (because we check that it is a valuation using Gauss' Lemma 4.2).

**Exercise.** The point $p(E)$ is the maximal point of $E$, i.e. $|f(x)| \leq |f|_E$ for all $x \in E$, $f \in k\{r^{-1}T\}$.

**Exercise.** For two discs $E, E'$, $E \subseteq E'$ if and only if $|\cdot|_E \leq |\cdot|_{E'}$ on $k\{r^{-1}T\}$.

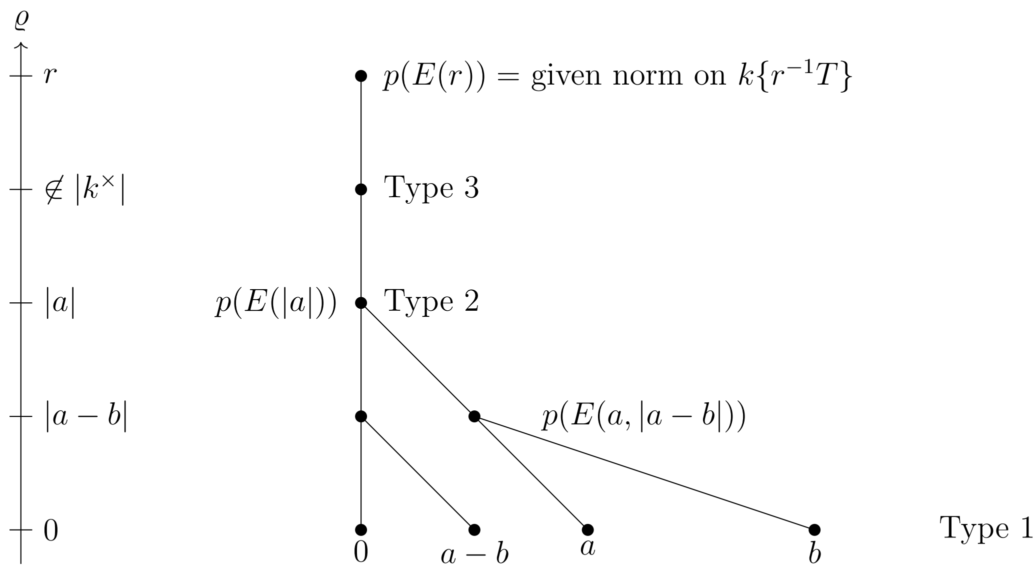
**Exercise.** If $f \in k\{r^{-1}T\}$,

- $|f|_E = \sup_{\mathbb{E}=E\cap k} |f|$ (the supremum over classical points),
- if $p(E)$ is of Type 2, the supremum is attained.

**Exercise.** Given $a \in \mathbb{E}(r)$, the map $[0, r] \ni \varrho \mapsto p(E(a, \varrho)) \in E(r)$ is a homeomorphism onto its image.

By convention, $p(E(a, 0))$ is a Type 1 point associated to $a \in \mathbb{E}(r)$. We get a path inside the Berkovich space starting at points of Type 1.

Here is a rough picture of what this space looks like.




Overall, we have the following points:

| | | |
|---|---|---|
| Type 1 points | $\leftrightarrow$ | points in $\mathbb{E}$, |
| Type 2, 3 points | $\leftrightarrow$ | discs in $\mathbb{E}$, |
| Type 4 points | $\leftrightarrow$ | decreasing families of discs in $\mathbb{E}$ with empty intersection. |

**Definition 4.9.** A field $k$ is *spherically complete* if the intersection of any chain of discs in nonempty.

**Exercise.**

(1) Any disc $E = E(a, \varrho)$ has a unique maximal point $p(E) \in E$.
(2) Under the partial order on $E(r)$, where $x \le y$ if and only if $|f(x)| \le |f(y)|$ for all $f \in k\{r^{-1}T\}$, $p(E)$, check that
$$E = \{y \in E(r) \mid y \le p(E)\}.$$
(3) Moreover, $p(E)$ is given by
$$|f_E| = \max_i |c_i|\varrho^i \quad \text{for } f = \sum_0^\infty c_i (T-a)^i.$$

We can clarify Definition 4.8 now.

**Definition 4.10.** We say that $x \in E(R)$ is of:

- *Type 1* if $x = p(E)$, $\varrho(E) = 0$,
- *Type 2* if $x = p(E)$, $\varrho(E) \in |k^\times|$,
- *Type 3* if $x = p(E)$, $\varrho(E) \in \mathbb{R}_+^\times \setminus |k^\times|$.

**Type 4 points.** A family $\mathcal{E} = \{E_i\}_{i \in I}$ of discs in $E(r)$ is a *chain* if for all $i, j \in I$, $E_i \subseteq E_j$ or $E_j \subseteq E_j$. This implies that $\bigcap_{i \in I} E_i \neq \emptyset$, since $E(r)$ is compact.

Set $\mathbb{E}_i = E_i \cap \mathbb{E}(r)$. Then $\mathbb{E}_i \neq \emptyset$ but it is possible that $\bigcap_{i \in I} \mathbb{E}_i = \emptyset$. Define

$$\varrho(\mathcal{E}) = \inf_i \varrho(E_i) \in [0, r]$$

and $p(\mathcal{E}) \in E(r)$ given by

$$|f|_{\mathcal{E}} = |f|_{p(\mathcal{E})} = \inf_i |f|_{E_i}.$$

There are 3 cases.

(1) If $\varrho(\mathcal{E}) = 0$, one can check that $\bigcap E_i = \bigcap \mathbb{E}_i = \{p(\mathcal{E})\}$. Hence $p(\mathcal{E})$ is of Type 1.

**Exercise.** Check this.

(2) If $\varrho(\mathcal{E}) > 0$ and $\bigcap_i \mathbb{E}_i \neq \emptyset$. Then $\bigcap E_i = E$ is a disc in $E(r)$ and $p(\mathcal{E})$ is of Type 2 or 3.

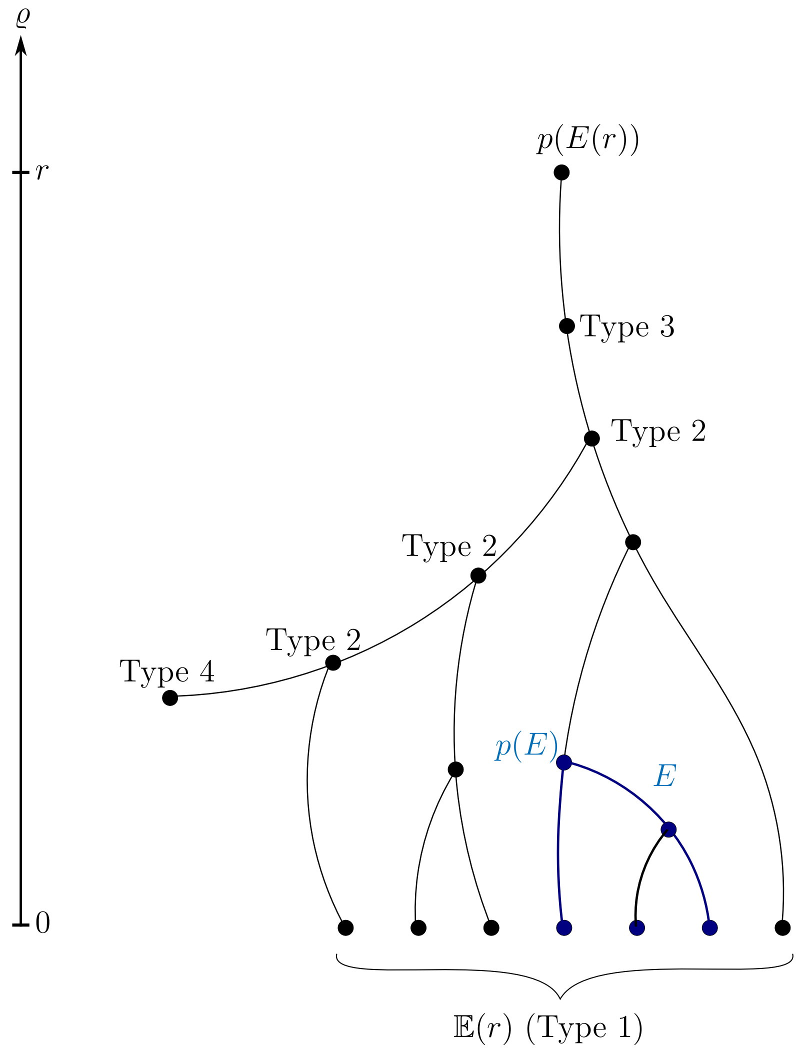
**Exercise.** Check this,

(3) If $\varrho(\mathcal{E}) > 0$ and $\bigcap_i \mathbb{E}_i = \emptyset$, we say that $p(\mathcal{E})$ is of *Type 4*.

**Remark 4.11.**

(1) Type 4 points exist if and only if $k$ is not spherically complete.
(2) Type 1 and Type 4 points are minimal with respect to the partial order.

Here is a schematic diagram of $E(r)$.




We marked an embedded disc $E$ in blue, it's maximal element is labelled $p(E)$.

**Theorem 4.12.** *For any algebraically closed non-archimedean field $k$ and any $r > 0$, every point of $E(r)$ is of Type 1–4.*

*Proof.* Given $x \in E(r)$ and $a \in \mathbb{E}(r)$, set

$$E_a = E(a, \underbrace{|(T-a)(x)|}_{\varrho_a}).$$

**Exercise.** The family $\mathcal{E} = \{E_a\}_{a \in \mathbb{E}(r)}$ is a chain.

**Exercise.** Check that $p(\mathcal{E}) = x$.

This completes the proof. $\square$

4.3. **Case (c): $k$ is not algebraically closed, non-trivially valued.** We first talk about fibers of morphisms. Let $A \xrightarrow{\varphi} B$ be a morphism of seminormed rings. This gives a continuous map

$$f = \varphi^* \colon M(B) \to M(A).$$

For $x \in M(A)$, we get a character $A \to \mathcal{H}(x)$, where $\mathcal{H}(X)$ is the complete residue field.

**Proposition 4.13.** *There is a homeomorphism*

$$f^{-1}(x) \cong M(B \otimes_A \mathcal{H}(x)).$$

*Proof.* See the notes. □

It is possible that the seminorm on $B \otimes_A \mathcal{H}(x)$ is 0. In that case $f^{-1}(x) = \emptyset$.

We now consider ground field extensions. Suppose $k$ is an NA field and $A$ is an NA Banach $k$-algebra. Let $k'/k$ is an NA field extension. Define

$$A' = A \widehat{\otimes}_k k',$$

where the completion is with respect to the NA seminorm on the tensor product. This is a Banach $k'$-algebra.

**Exercise.** If $A = k\{r^{-1}T\}$, $A' = k'\{r^{-1}T\}$.

The map $k \hookrightarrow k'$ gives an isometry $A \hookrightarrow A'$, which induces

$$M(A') \xrightarrow{f} M(A).$$

For $x \in M(A)$, Proposition 4.13 shows that

$$f^{-1}(x) \cong M(A' \widehat{\otimes}_A \mathcal{H}(x)) = M(k' \otimes_k \mathcal{H}(x)).$$

**Theorem 4.14** (Grusan). *If $V$ and $W$ are nonzero $k$-Banach spaces (and $k$ is NA), then*

$$0 \neq V \otimes_k W \hookrightarrow V \widehat{\otimes}_k W,$$

*i.e. the seminorm on $V \otimes_k W$ is a norm.*

**Corollary 4.15.** *The map $M(A') \to M(A)$ is surjective.*

We now discuss an important example of field extensions. Let $k$ be an NA field and $k^a$ be an algebraic closure of $k$. We will state a few facts without proof.

**Fact 4.16.** *The valuation on $k$ extends uniquely to a valuation on $k^a$.*

**Fact 4.17.** *The completion $k' = \widehat{h^a}$ is also algebraically closed.*

Let $G = \mathrm{Aut}(k^a/k)$.

**Fact 4.18.** *The group $G$ acts on $k^a$ by isometries. Hence the action extends to $k'$.*

If $A$ is a Banach $k$-algebra, $G$ acts on $A' = A \otimes_k k'$. Hence $G$ also acts on $M(A')$.

**Theorem 4.19.** *The map*

$$M(A')/G \to M(A)$$

*is a homeomorphism.*

We will soon use this to describe $E(r)$ for non-algebraically closed $k$. First, we summarize the structure of $E(r)$ for $k = k^a$.

Recall that we have a partial ordering on $E(r)$ (or even any $M(A)$), defined by

$$x \leq y \quad \text{if and only if} \quad |f(x)| \leq |f(y)| \text{ for all } f \in k\{r^{-1}T\}.$$

Here is a summary of the classification of points when $k$ is algebraically closed. For $a \in \mathbb{E}(r)$, $0 \leq \varrho \leq r$, we defined

$$E = E(a, \varrho) = \{x \in E(r) \mid |(T - a)(x)| \leq \varrho\}.$$

It has a maximal point $p(E) \in E \subseteq E(r)$. These points have the following types:

(Type 1) $\varrho(E) = 0$,
(Type 2) $\varrho(E) \in |k^\times|$,
(Type 3) $\varrho(E) \in \mathbb{R}_+^\times \setminus |k^\times|$.

Finally, there are Type 4 points, associated to chains $\mathcal{E} = \{E_i\}_{i \in I}$ of discs $E_i \subseteq E(r)$ such that

$$\bigcap_i E_i \cap \mathbb{E}(r) = \emptyset.$$

Then $\bigcap E_i = \{p(\mathcal{E})\}$ and $p(\mathcal{E})$ is a point of Type 4. Recall the picture of the Berkovich disc above. There is a *tree structure* of $(E(r), \leq, \varrho)$.

- There is a unique maximal element $x_0 = p(E(r))$.
- Every point dominates some minimal point.
- For any $x \in E(r)$, the set

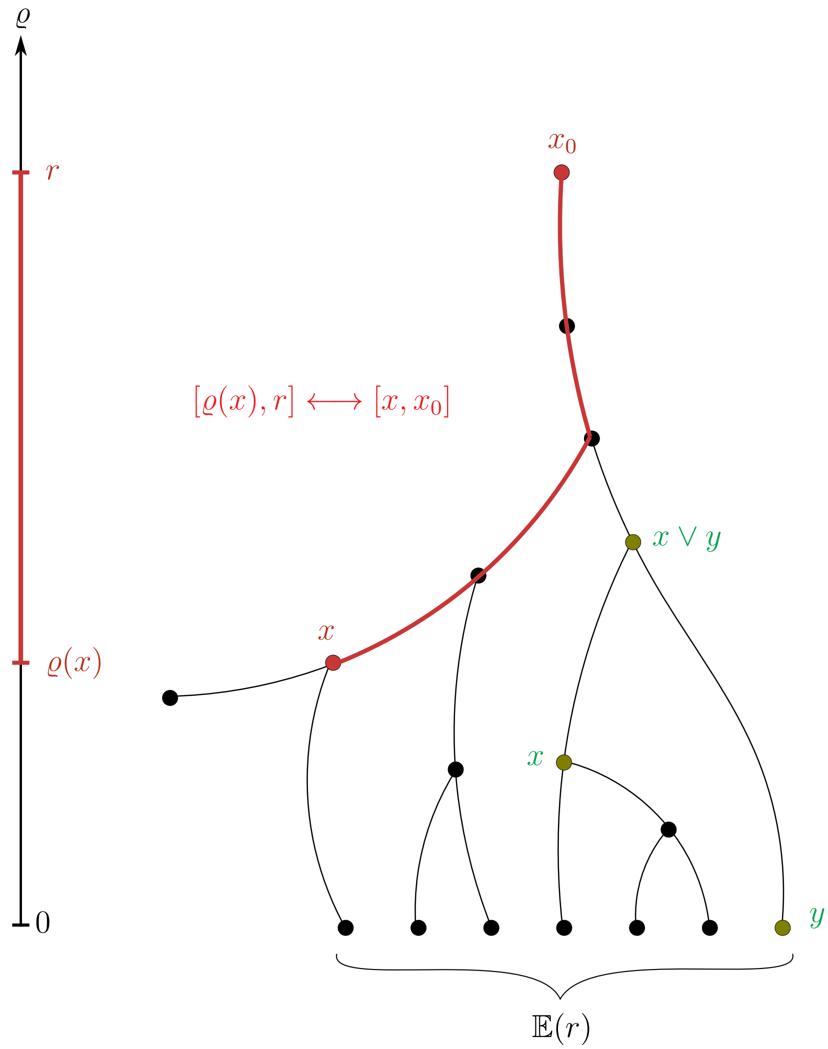
  $$[x, x_0] = \{y \in E(r) \mid x \leq y \leq x_0\}$$

  is totally ordered, and there is an order preserving bijection:

  $$\varrho\colon [x, x_0] \to [\varrho(x), r].$$

- Any two points $x, y \in E(r)$ have a supremum $x \vee y$.

Below is a sketch indicating an interval $[x, x_0]$ and a supremum $x \vee y$ of two points $x$, $y$.




We finally discuss Berkovich discs for general $k$. Consider $k' = \widehat{k^a}$ and $G = \mathrm{Aut}(k^a/k)$ acts on $k'$ by isometries. Hence $G$ acts on $E_{k'}(r)$ by homeomorphisms.

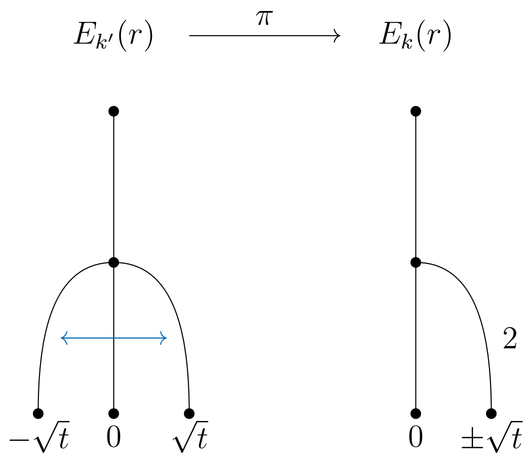
Theorem 4.19 in this case shows implies the following.

**Fact 4.20.** *The map $k \hookrightarrow k'$ induces a homeomorphism*

$$E_{k'}(r)/G \xrightarrow{\cong} E_k(r).$$

*Moreover, $G$ preserves the partial ordering $\leq$ on $E_{k'}(r)$ and the radius $\varrho$. Hence $G$ preserves the Type of a point.*

**Example 4.21.** Consider $k = \mathbb{C}[\![t]\!]$, formal Laurent series in $t$. Here is a picture for $k' = k(\sqrt{t})$.




**Exercise.**

(1) Points of Type 2, 3 have finite $G$-orbits.
(2) Points of Type 1 have finite $G$-orbits if and only if they come from $k^a$.
(3) Points of Type 4 may or may not have finite orbits, and both possibilities occur.
(4) The Berkovich disc $(E(r), \leq, \text{"}\pi_*\varrho\text{"})$ has a tree structure as before.

4.4. **Residue fields.** Let $A$ be a Banach ring and $x \in M(A)$. Recall that $\mathcal{H}(x)$ is the complete residue field, i.e. $\widehat{\mathrm{Frac}(A/\mathfrak{p}_x)}$. If $A$ is NA, $\mathcal{H}(x)$ is also NA.

Here are some invariants of NA fields. Let $k$ be an NA field. We have:

$$
\begin{aligned}
|k^\times| &\subseteq \mathbb{R}_+^\times && \text{the value group,}\\
\widetilde{k} &= \{|\cdot| \leq 1\}/\{|\cdot| < 1\} && \text{the classical residue field,}\\
\bigoplus_{r\in\mathbb{R}_+^\times} &\{|\cdot| \leq r\}/\{|\cdot| < r\} && \text{the graded residue field.}
\end{aligned}
$$

We will call $\widetilde{\mathcal{H}(x)}$ the *double residue field.*

**Example 4.22.** Let $A = k\{r^{-1}T\}$ for an algebraically closed NA field. Then $M(A) = E_k(r) = E(r)$. We want to compute $\mathcal{H}(x)$ for $x$ of Type 1,2,3,4.

**Type 1 points.** Let $x = p(E(a, 0))$ for $a \in k$. Then $\mathfrak{p}_x = (T_a)$, so

$$\mathcal{H}(x) = k.$$

**Type 3 points.** Let $x = p(E(a, \varrho))$ for $a \in \mathbb{E}(r)$, $\varrho \in (0, r] \setminus |k^\times|$. Without loss of generality, assume $a = 0$. For $0 \neq f \in k\{r^{-1}T\}$, we have

$$|f(x)| = \max_i |a_i|\varrho^i \quad \text{for } f = \sum_0^\infty a_i T^i.$$

Since $\varrho \notin |k^\times|$, the max is attained for a unique $i$. Otherwise, if $|a_i|\varrho^i = |a_j|\varrho^j$ for $i < j$, we have $\varrho^{j-i} = \frac{|a_j|}{|a_i|} \in |k^\times|$, so $\varrho \in |k^\times|$ because $|k^\times|$ is divisible. Then

$$f = a_i T^i \left( 1 + \sum_{j \neq i} \frac{|a_j|}{|a_i|} T^{j-i} \right),$$

where $\frac{|a_j|}{|a_i|}\varrho^{j-i} < 1$, so $\to 0$ as $j \to \infty$. We have a similar description for any $f \in \mathrm{Frac}(A)$ (i.e. $i \in \mathbb{Z}$).

From this, one can deduce that:

$$\mathcal{H}(x) \cong k\{\varrho^{-1}T\varrho T^{-1}\} = \left\{ f = \sum_{-\infty}^{\infty} T^i \;\middle|\; |a_i|\varrho^i \to 0 \text{ for } i \to \pm\infty \right\}$$

with $|f(x)| = \max |a_i|\varrho^i$.

The value group $|\mathcal{H}(x)^\times|$ is generated by $|k^\times|$ and $\varrho$. The double residue field is:

$$\widetilde{\mathcal{H}(x)} \cong \widetilde{k}.$$

## 5. Strictly $k$-affoinoid spaces

The reference for this section is [BGR84] and [Ber90, Section 2.1].

Let $k$ be an NA field. We will define an $k$-affinoid space as the Berkovich spectrum of a *$k$-affinoid algebra*:

$$\begin{aligned} A &= k\{r_1^{-1}T, \ldots, r_n^{-1}T\}/I, \\ X &= M(A). \end{aligned}$$

We will note that if $X$ is a $k$-affinoid space,

$$X \hookrightarrow E(r),$$

where $E(r)$ is a *polydisc*.

**Analogies.**

(1) *Complex analytic sets* are (locally) $X = \{f_1 = \cdots = f_m = 0\} \subseteq B \subseteq \mathbb{C}^n$.
(2) *Affine $k$-schemes* (of finite type) are $X = \mathrm{Spec}(A)$ where $k$ is a field and $A = k[T_1, \ldots, T_n]/I$ is a finitely generated $k$-algebra. Moreover:

$$X \hookrightarrow \mathbb{A}^n_k.$$

We now make the definitions formal.

For $r = (r_1, \dots, r_n) \in \mathbb{R}_+^{\times n}$, we define

$$\begin{aligned} k\{r^{-1}T\} &= k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \\ &= \left\{ f = \sum_{\nu \in \mathbb{Z}_+^n} a_\nu T^\nu \;\middle|\; \lim_{|\nu|\to\infty} |a_\nu| r^\nu = 0 \right\} \qquad \text{where } |\nu| = \nu_1 + \cdots \nu_n \\ &\subseteq k[\![T_1, \dots, T_n]\!]. \end{aligned}$$

We define

$$\|f\| = \max_\nu |a_\nu| r^\nu.$$

Gauss' Lemma 4.2 implies that $k\{r^{-1}T\}$ is a valued Banach $k$-algebra.

**Definition 5.1.** The *Berkovich polydisc* is $E(r) = M(k\{r^{-1}T\})$.

### 5.1. **The algebra and Banach structure of the Tate algebra.**

**Definition 5.2.** The *Tate algebra* is

$$\mathrm{Tate}_n = k\{T_1, \dots, T_n\} \subseteq k[\![T_1, \dots, T_n]\!],$$

i.e. we take $r = (1, \dots, 1)$, with the norm

$$\|f\| = \max_\nu |a_\nu|.$$

**Exercise.** If $k$ is trivially valued, then $\mathrm{Tate}_n = k[T_1, \dots, T_n]$ with the trivial norm.

**Reduction.**

$$\begin{aligned} \mathrm{Tate}_n &= k\{T_1, \dots, T_n\}, && \\ \mathrm{Tate}_n^\circ &= k^\circ\{R_1, \dots, T_n\} && \{\|\cdot\| \le 1\}, \\ \mathrm{Tate}_n^{\circ\circ} &= k^{\circ\circ}\{T_1, \dots T_n\} && \{\|\cdot\| < 1\}, \\ \widetilde{\mathrm{Tate}_n} &\cong \widetilde{k}[T_1, \dots, T_n] && \{\|\cdot\| \le 1\}/\{\|\cdot\| < 1\}. \end{aligned}$$

**Elementary properties.**

(1) An element $f \in \mathrm{Tate}_n$ is invertible if and only if $\|f - f(0)\| < \|f\| = \|f(0)\|$, where we note that $f(0) = a_0$.

*Sketch of proof.* Without loss of generality, $f(0) = 1$, i.e. $f = 1 + f_0$ with $f_0(0) = 0$. We need to then show that

$$1 + f_0 \text{ is invertible if and only if } \|f_0\| < 1.$$

Recall that the 'if' follows from Lemma 2.13. Conversely, suppose $fg = 1$. Since $f(0) = 1$, $\|f\| \ge 1$. Also, $g(0) = 1$, so $\|g\| \ge 1$. This shows that $\|f\| = \|g\| = 1$, since $\|\cdot\|$ is a valuation.

If $f, g \in \mathrm{Tate}_n$, $fg = 1$, then $\widetilde{f}\widetilde{g} = 1$ in $\widetilde{k}[T_1, \dots, T_n]$. The only invertible elements in $\widetilde{k}[T_1, \dots, T_n]$ are constants, and hence $\widetilde{f} = 1$. Therefore

$$1 = \widetilde{f} = 1 + \widetilde{f}_0,$$

showing that $\widetilde{f_0} = 0$, i.e.

$$f_0 \in \{\|\cdot\| < 1\}.$$

This shows that $\|f_0\| < 1$. ☐

(2) For any $f \in \mathrm{Tate}_n$, there exists $a \in k$ such that $|a| = \|f\|$ and $f - a$ is non-invertible.

*Proof.* If $|f(0)| = \|f\|$, pick $a = f(0)$. If $|f(0)| < \|f\|$, pick any $a \in k$ with $|a| = \|f\|$ (this is always possible). Then

$$|(f-a)(0)| = \|f - a\| = \|f\| = \|(f-a) - (f-a)(0)\|,$$

so $f - a$ is not invertible by (1). ☐

(3) The Jacobson radical of $\mathrm{Tate}_n$ is zero (so there are many maximal ideals).

*Proof.* Suppose $f \neq 0$ belongs to every maximal ideal. Then (2) implies that there exists $a \in k^\times$ such that $f - a$ is non-invertible. Then $f - a \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, so $a = f - (f - a) \in \mathfrak{m}$. This contradicts the maximality of $\mathfrak{m}$. ☐

(4) Every $k$-algebra homomorphism $\varphi\colon \mathrm{Tate}_n \to \mathrm{Tate}_m$ is contractive.

*Proof.* Suppose $\|\varphi(f)\| = \|f\|$ for $f \in \mathrm{Tate}_n$. Pick $a \in k$ such that $|a| = \|\varphi(f)\|$ and

$$\varphi(f) + a = \varphi(f + a) \text{ is not invertible.}$$

Hence $f + a$ is not invertible, contradicting (1). ☐

Hence any $\varphi\colon k\{T_1, \ldots, T_n\} \to k\{T_1, \ldots, T_m\}$ induces

$$\widetilde{\varphi}\colon \widetilde{k}[T_1, \ldots, T_n] \to \widetilde{k}[T_1, \ldots, T_m].$$

(5) If $\varphi$ as above is an isomorphism of $k$-algebras, then $m = n$ and $\varphi$ is an isometric isomorphism.

*Proof.* This follows immediately from (4). ☐

5.1.1. *The maximum modulus principle.* Consider

$$\mathbb{E}_k^n = \{x \in k^n \mid |x_i| \leq 1 \text{ for all } i\} = (k^\circ)^n.$$

Any $f \in \mathrm{Tate}_n$ defines a continuous function $f\colon \mathbb{E}_k^n \to k$ such that $|f(x)| \leq \|f\|$ for all $x$.

**Theorem 5.3** (Maximum modulus principle)**.** *If $\widetilde{k}$ is infinite, then for all $f \in \mathrm{Tate}_n$, there exists $x \in \mathbb{E}_k^n$ such that $|f(x)| = \|f\|$.*

*Proof.* Without loss of generality, suppose $\|f\| = 1$. If $\widetilde{k}$ is infinite, there exists $\widetilde{x} \in \widetilde{k}^n$ such that $\widetilde{f}(\widetilde{x}) \neq 0$.

Lift $\widetilde{x}$ to $x \in \mathbb{E}_k^n$. Then $\widetilde{f(x)} = \widetilde{f}(\widetilde{x}) \neq 0$, so $|f(x)| = 1$. ☐

5.1.2. *Algebraic properties.* We state the following results as a *black box*; the details can be found in the notes.

**Theorem 5.4.** *The Tate algebra* $\mathrm{Tate}_n$ *is Noetherian, Jacobson (every ideal is an intersection of maximal ideals), factorial (UFD) of Krull dimension* $n$.

The proof is omitted. It uses *Weierstrass polynomials*: $W = T^n + a_1 T^{n-1} + \cdots a_0 \in \mathrm{Tate}_{n-1}[T] \subseteq \mathrm{Tate}_n$ such that $\|a_i\| \leq 1$ for all $i$. One proves a *Weierstrass preparation theorem*, a division theorem, and so on.

5.1.3. *Properties as a Banach ring.*

**Definition 5.5.** Let $A$ be a Banach ring and $M$ be a Banach $A$-module.

(1) $M$ is *finite* (as a Banach $A$-module) if there exists an admissible surjective morphism $A^n \twoheadrightarrow M$ for some $n \geq 0$,
(2) $M$ is *Noetherian* (as a Banach $A$-module) if every submodule $M'$ of $M$ is a finite Banach $A$-module (in particular, any such $M'$ is closed),
(3) $A$ is *Noetherian* (as a Banach ring) if it is Noetherian as a Banach $A$-module (in particular, every ideal of $A$ is closed).

**Question.** Are these good definitions? It is one way to formulate this theory, but it is unclear if this is the most canonical way to do it.

**Theorem 5.6.** *The Banach ring* $\mathrm{Tate}_n$ *is Noetherian. In particular, every ideal of* $\mathrm{Tate}_n$ *is closed.*

We will work towards the proof of this theorem. It uses the *Banach open mapping theorem.*

**Theorem 5.7** (Open Mapping Theorem)**.** *Let $k$ be a complete non-trivially valued field, and $\varphi\colon A \to B$ be a bounded surjective morphism of $k$-Banach spaces. Then $\varphi$ is admissible (in particular, it is open).*

*Proof.* Copy the usual proof over $\mathbb{C}$. Use $|k^\times| \neq \{1\}$ to scale various balls. ☐

**Remark 5.8.** This is false if $|k^\times| = \{1\}$. Consider

$$\underbrace{k\{s^{-1}T\}}_{k[T]} \hookrightarrow \underbrace{k\{r^{-1}T\}}_{k[T]} \qquad \text{for } 1 \leq r < s.$$

This map is bounded and bijective (it is the identity) It is not admissible (or open), because $T^n \mapsto T^n$ but $s^n \gg r^n$ for $n \gg 0$.

**Lemma 5.9.** *Let $k$ be a non-trivially valued NA field, $A$ be a NA Banach $k$-algebra, and $M$ be a NA normed $A$-module such that the completion $\widehat{M}$ is a finite Banach $A$-module. Then $\widehat{M} = M$.*

We defer the proof until later and first show how the lemma is used.

**Corollary 5.10.** *If $k$ is NA, non-trivially valued, then any Banach $k$-algebra $A$ that is Noetherian as a ring is Noetherian as a Banach ring (in particular, every ideal is closed).*

*Proof.* Pick any ideal $\mathfrak{a} \subseteq A$. We want to apply Lemma 5.9 with $M = \mathfrak{a}$ and $\widehat{M} = \overline{\mathfrak{a}}$.

Since $A$ is Noetherian, there is an $A$-linear surjective map $\varphi\colon A^n \to \overline{\mathfrak{a}}$ for some $n \geq 0$.

**Exercise.** The map $\varphi$ is automatically bounded.

Using the Open Mapping Theorem 5.7, $\varphi$ is admissible. Therefore, $\overline{\mathfrak{a}}$ is a Banach $A$-module. Then Lemma 5.9 implies that $\overline{\mathfrak{a}} = \mathfrak{a}$. □

*Proof of Theorem 5.6.* We know that $\mathrm{Tate}_n$ is Noetherian as a ring by Theorem 5.4. If $|k^\times| \neq \{1\}$, we are done by Corollary 5.10.

Otherwise, $|k^\times| = \{1\}$, $\mathrm{Tate}_n = k[T_1, \ldots, T_n]$ with the trivial norm. □

The main takeaway of Theorem 5.6 is that every ideal of $\mathrm{Tate}_n$ is automatically closed.

We still need to prove Lemma 5.9 to complete the proof.

*Proof of Lemma 5.9.* There exist $m_1', \ldots, m_n' \in \widehat{M}$ such that $\pi\colon A^n \to \widehat{M}$ given by

$$\pi(a_1, \ldots, a_n) = \sum_1^n a_i m_i'$$

is surjective and admissible. Then

$$\pi((A^{\circ\circ})^n) = \sum_1^n A^{\circ\circ} m_i'$$

is an open neighborhood of $0 \in \widehat{M}$. Since $M \subseteq \widehat{M}$ is dense, we have that

$$m_i' \in M + \sum_1^n A^{\circ\circ} m_j' \quad \text{for } 1 \leq i \leq n,$$

so

$$m_i' = m_i + \sum_1^n a_{i,j} m_j' \quad \text{for } a_{ij} \in A^{\circ\circ}, m_i \in M.$$

We write this equation as

$$m = (I_n - B)m'$$

where $B = (a_{i,j})_{i,j=1}^n$.

**Exercise.** Use the invertibility criterion 2.13 to show that $I_n - B$ is invertible.

Then $m' \in M^n$, showing that $\widehat{M} = M$. □

5.2. **Strictly $k$-affinoid algebras.** Everything in this section is NA.

**Definition 5.11.** A Banach $k$-algebra is *strictly $k$-affinoid* if there is an admissible surjective morphism

$$\mathrm{Tate}_n \twoheadrightarrow A$$

for some $n \geq 0$.

**Remark 5.12.** In Tate's theory of rigid analytic spaces, one also assumes that $k$ is non-trivially valued. If $k$ is trivially valued, $A$ is a finitely-generated $k$-algebra with norm equivalent to the trivial norm.

**Example 5.13.** Consider $A = k\{T, T^{-1}\}$, the *unit circle algebra.* It is defined as

$$k\{T, T^{-1}\} = \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \;\middle|\; \lim_{i \to \pm\infty} |a_i| = 0 \right\}$$

with norm $\|f\| = \max_i |a_i|$.

The map

$$\begin{aligned} k\{S, T\} &\twoheadrightarrow k\{T, T^{-1}\} \\ (S, T) &\mapsto (T^{-1}, T) \end{aligned}$$

is admissible and surjective. Hence $A$ is strictly $k$-affonoid.

**Remark 5.14.** Note that the natural map $k\{T\} \hookrightarrow k\{T, T^{-1}\}$ induces and injective map

$$M(\{T, T^{-1}\}) \hookrightarrow M(k\{T\}) = E(1) \text{ the unit disc}$$

and its image is

$$\{x \in E(1) \mid |T(x)| = 1\},$$

the *unit circle.*

This is an analog of $\mathrm{Spec}(A) \supseteq \mathrm{Spec}(A_f)$ from algebraic geometry, which is an *open affine subset.* It is hence deceptively large for a *unit circle* inside a *unit disc.*

The category of strictly $k$-affinoid algebras is closed under various operators (ground field extensions, products, tensor products, ...).

5.2.1. *Algebraic and Banach ring properties.*

**Theorem 5.15.** *Any strictly $k$-affinoid algebra is Noetherian and Jacobson.*

*Proof.* Since $A$ is a quotient of a Tate algebra, this follows from Theorem 5.4. □

One can also prove a generalization of Theorem 5.6.

**Theorem 5.16.** *Any strictly $k$-affinoid algebra is Noetherian as a Banach ring.*

**Corollary 5.17.** *Every ideal of $A$ is closed.*

**Theorem 5.18** (Noether normalization)**.** *If $A \neq 0$ is strictly $k$-affinoid, then there is a finite injective morphism* $\mathrm{Tate}_d \hookrightarrow A$ *for some $d \geq 0$.*

The proof, once again, uses induction and Weierstrass polynomials. We omit it here.

Geometrically, we have a closed immersion $M(A) \hookrightarrow E^n$ by definition. Noether Normalization Theorem 5.18 says that there's also a finite map $M(A) \to E^d$:

$$\begin{tikzcd}
M(A) \arrow[r, hook, "\text{closed imm.}"] \arrow[d, "\text{finite}"] & E^n \\
E^d
\end{tikzcd}$$

**Corollary 5.19** (Nullstellensatz)**.** *If* $\mathfrak{m} \subseteq A$ *is a maximal ideal, then* $\dim_k(A/\mathfrak{m}) < \infty$*.*

*Proof.* Since $\mathfrak{m}$ is closed, $A/\mathfrak{m}$ is strictly $k$-affinoid. Noether Normalization Theorem 5.18 implies that there is a finite map

$$\mathrm{Tate}_d \hookrightarrow A/\mathfrak{m}.$$

This shows that $\mathrm{Tate}_d$ is a field which is only possible if $d = 0$ and $\mathrm{Tate}_d = k$. □

**Corollary 5.20.** *If* $\varphi\colon A \to B$ *is a morphism of strictly* $k$*-affinoid algebras and* $\mathfrak{n} \subseteq B$ *is a maximal ideal, then* $\varphi^{-1}(\mathfrak{n}) \subseteq A$ *is a maximal ideal.*

*Proof.* We have maps

$$k \hookrightarrow A/f^{-1}(\mathfrak{n}) \hookrightarrow B/\mathfrak{n}$$

and $\dim(B/\mathfrak{n}) < \infty$ by Corollary 5.19. Hence $A/f^{-1}(\mathfrak{n})$ is an integral domain which is finite over $k$, so it is a field. □

**Corollary 5.21.** *If* $A$ *is strictly* $k$*-affinoid, and* $\mathfrak{m} \subseteq A$ *is a maximal ideal, then the valuation on* $k$ *extends uniquely to* $A/\mathfrak{m}$*. In particular, there is a map*

$$\begin{aligned} \mathrm{Max}(A) &\hookrightarrow M(A), \\ \mathfrak{m} &\mapsto (f \mapsto \text{norm of image in } A/\mathfrak{m}). \end{aligned}$$

**Proposition 5.22.** *If* $k$ *is non-trivially valued, then* $\mathrm{Max}(A) \subseteq M(A)$ *is dense.*

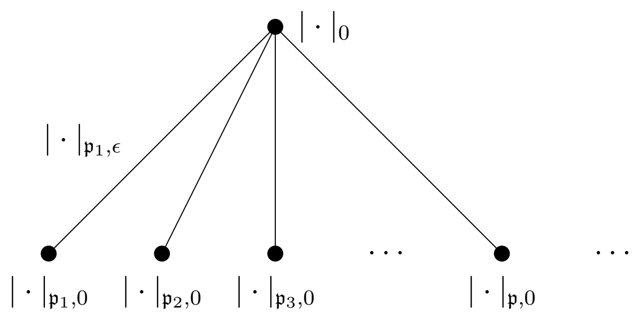
**Remark 5.23.** If $k$ is trivially valued, the closure of $\mathrm{Max}(A)$ in $M(A)$ is the image of the canonical section

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[rr, bend left, "\mathrm{triv}"] & & M(A) \arrow[ll, "\ker"]
\end{tikzcd}$$

$$\mathfrak{p} \longrightarrow \text{trivial valuation of } A/\mathfrak{p}.$$

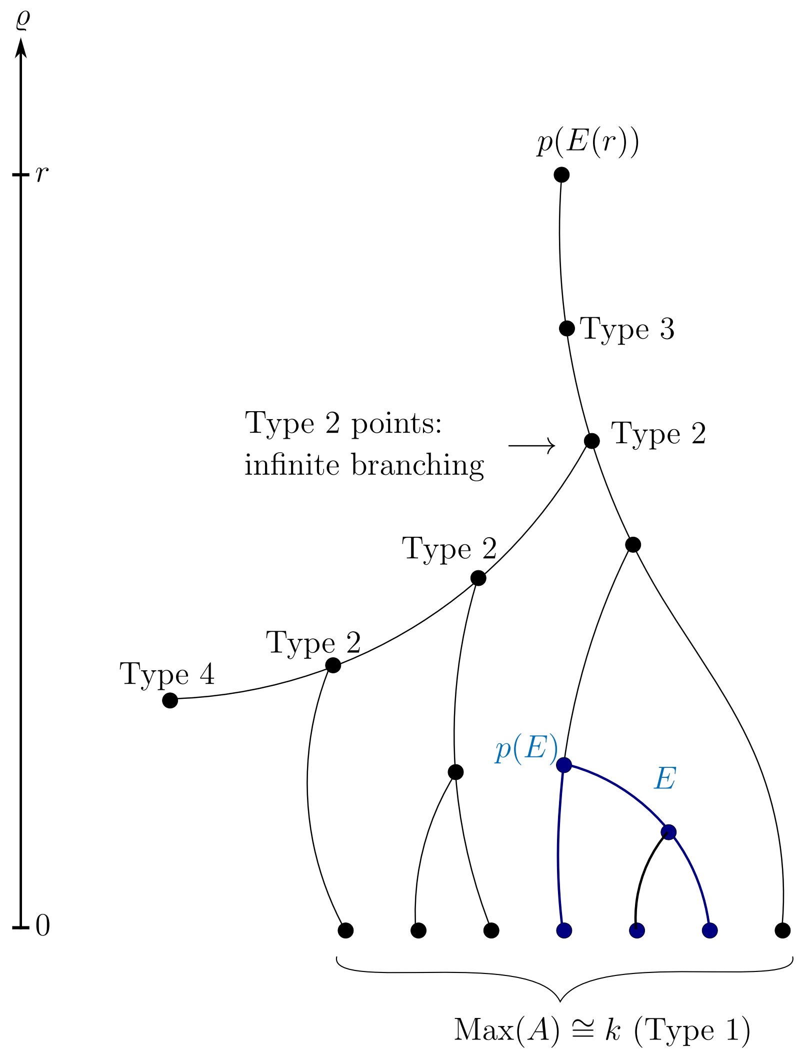
**Example 5.24.** Let $A = k\{T\} = \mathrm{Tate}_1$ so that $M(A) = E(1)$.

When $|k^\times| = \{1\}$,



and $|\cdot|_0 = \mathrm{triv}((0))$. The closure of $\mathrm{Max}(A)$ in $M(A)$ is exactly $\mathrm{Max}(A) \cup \{\mathrm{triv}((0))\}$, i.e. all the marked points.

When $|k^\times| \neq \{1\}$ and $k$ is algebraically closed:




Since Type 2 points have infinite branching, they will always be in the closure of $\mathrm{Max}(A) \cong k$.

*Sketch of proof of Proposition 5.22.* Pick $x_0 \in M(A)$ and $U \ni x_0$ is an open neighborhood. We must prove that $\mathrm{Max}(A) \cap U \neq \emptyset$.

Without loss of generality, assume that

$$U = \{x \in M(A) \mid |f_i(x)| < a_i,\ |g_j(x)| > b_i \quad 1 \leq i \leq m, 1 \leq j \leq n\}$$

for $a_i, b_j > 0$.

Since $|k^\times| \neq \{1\}$, there exist $p_i, q_i \in \underbrace{|k^\times|^{\mathbb{Q}}}_{\text{dense in } \mathbb{R}_+^\times}$ such that

$$\begin{cases} |f_i(x_0)| < p_i < a_i, \\ |g_j(x(0)| > q_j > b_j. \end{cases}$$

Replace $f_i, g_j$ with powers and then scale to assume without loss of generality that $p_i = q_j = 1$ for all $i, j$. Set

$$B = \underbrace{A\{S_1, \ldots, S_m, T_1, \ldots, T_n\}}_{\text{strictly } k\text{-affinoid}} / \underbrace{(f_i - S_i, g_j T_j - 1)}_{\text{closed}},$$

a strictly $k$–affoinoid algebra. The map $A \to B$ is bounded and hence induces a continuous map $M(B) \xrightarrow{h} M(A)$.

One can show that ($h$ is a homeomorphism onto its image and) the image of $h$ is

$$\underbrace{\{x \in M(A) \mid |f_i(x)| \leq 1, |g_j(x)| \geq 1\}}_{\text{Laurent domain in } M(A)} \subseteq U.$$

Then $h(\mathrm{Max}(B)) \subseteq \mathrm{Max}(A) \cap U$. Since the image of $M(B)$ contains $x_0$, $B \neq 0$, so $\mathrm{Max}(B) \neq$. Hence $\mathrm{Max}(A) \cap U \neq \emptyset$. □

Recall that the spectral radius associated to a seminorm $\|\cdot\|$ on $A$ is $\varrho(f) = \lim \|f^n\|^{1/n}$. Recall also the Berkovich maximum modulus principle 3.21

$$\varrho(f) = \max_{x \in M(A)} |f(x)| \tag{1}$$

for all $f \in A$, where $A$ is a Banach ring.

**Theorem 5.25** (Maximum modulus principle)**.** *If $A$ is a strictly $k$-affinoid algebra, then the max in equation (1) is attained on* $\mathrm{Max}(A) \subseteq M(A)$.

*Sketch of proof.* We did this for $A = \mathrm{Tate}_n$. In general, we reduce to this case using Noether Normalization Theorem 5.18. □

**Lemma 5.26.** *If* $\mathrm{Tate}_d \hookrightarrow A$ *is finite, then for any $f \in A$, there is a polynomial*

$$P(T) = T^m + a_1 T^{m-1} + \cdots + a_m \in \mathrm{Tate}_d[T]$$

*such that $P(f) = 0$ and $\varrho(f) = \max_i \varrho(a_i)^{1/i}$.*

*Idea of proof.* Use Berkovich MMP 3.21 to reduce to the case $\mathrm{Tate}_d = k$ $(d = 0)$. Indeed, the map $M(A) \to E^d$ is surjective with finite fibers an $\varrho(f) = \max_{x \in M(A)} |f(x)|$. Then take $P$ to be the minimal polynomial of $f$. □

**Corollary 5.27.** *The spectral radius $\varrho(f)$ is in the divisible value group $|k|^{\mathbb{Q}}$.*

*Proof.* Since $\varrho(a_i) \in |k|$, this follows from Lemma 5.26. □

We can also prove the MMP (Theorem 5.25)
$$\varrho(f) = \max_{x \in \mathrm{Max}(A)} |f(x)|$$
in the same way. Then the corollary also follows from MMP 5.25/

*Another proof of Corollary 5.27.* If $x \in \mathrm{Max}(A)$, $\mathcal{H}(x)/k$ is finite, so
$$|f(x)| \in \{0\} \cup |\mathcal{H}(x)^\times| \subseteq \{0\}\{0\} \cup |k^\times|^{\mathbb{Q}}$$
for every $f \in A$.

By Theorem 5.25, $\varrho(f) = \max_{x \in \mathrm{Max}(A)} |f(x)|$, which completes the proof. □

**Proposition 5.28.** *Let $f \in A$ be as above. The following are equivalent:*

1. *$f$ is power-bounded: $\sup_n \|f^n\| < \infty$,*
2. *$\varrho(f) \leq 1$.*

*Sketch of proof.* That (1) implies (2) is trivial. The converse implication is true when $A = \mathrm{Tate}_n$ (since it is valued). The general case follows by reducing to this case using Noether Normalization Theorem 5.18. □

**Proposition 5.29.** *For $A$, $f$ as above:*

1. *$f$ is nilpotent if and only if $\varrho(f) = 0$,*
2. *if $f$ is not nilpotent, then there exists $C = C_f > 0$ such that $\varrho(f)^n \leq \|f^n\| \leq C_F \varrho(f)^n$ for all $n \geq 1$.*

*Proof.* For (1), we know that $\varrho(f) = 0$ if and only if $f$ belongs to all maximal ideals of $A$ by MMP 5.25. Since $A$ is Jacobson (cf. Theorem 5.15), this is equivalent to $f$ being nilpotent.

In (2), the first inequality holds in general. We sketch the proof of the other inequality. By Noether Normalization Theorem 5.18, there exists a finite map $\varphi\colon \mathrm{Tate}_d \hookrightarrow A$. By Lemma 5.26, there exists $P(T) \in \mathrm{Tate}_d[T]$ such that $P(f) = 0$ and $\varrho(f) = \max_i \varphi(a_i)^{1/i}$.

**Case 1.** If $\varrho(f) = 0$, $a_i = 0$ for all $i$, so $f^m = 0$.

**Case 2.** If $\varrho(f) > 0$, we reduce to $\varrho(f) = 1$ by power-multiplicativity and scaling. Then
$$P(T) \in \mathrm{Tate}_d^\circ[T],$$
so
$$f^m \in \sum_{i=0}^{m-1} \varphi(\mathrm{Tate}_d^\circ) \cdot f^i \quad \text{since } P(f) = 0.$$
Hence
$$f^n \in \sum_{i=0}^{m-1} \varphi(\mathrm{Tate}_d^\circ) \cdot f^i \quad \text{for all } n \geq m.$$
This shows that $\sup_n \|f^n\| < \infty$. □

**Corollary 5.30.** *If $A$ is reduced, then $\varrho$ is a **norm**.*

A stronger result is true.

**Theorem 5.31.** *If $A$ is reduced, then $\varrho$ is a **complete** norm, equivalent to the given norm.*

**Remark 5.32.**

(1) Every reduced strictly $k$-affinoid algebra has a canonical norm.
(2) Suffices to prove that $\varrho$ is complete. The map

$$(A, \|\cdot\|) \xrightarrow{\text{id}} (A, \varrho)$$

is admissible by the Open Mapping Theorem 5.7.

*Idea of proof.* We use Noether Normalization Theorem 5.18 in the standard way. The new ingredient is that the valued field $K = \mathrm{Frac}(\mathrm{Tate}_d)$ is *weakly stable*. For any finite field extension $L \supseteq K$, we equip $L$ with the spectral norm[2]: for $f \in L$ with minimal polynomial $P(T) = T^m + a_1 T^{m-1} + \cdots + a_m$, we define

$$\|f\| = \max_i \|a_i\|^{1/i}.$$

Weak stability means that the normed $K$-vector space $(L, \|\cdot\|)$ is isomorphic to $K^n$, i.e. the natural map $K^n \to L$ is admissible. □

**Definition 5.33.** If $A$ is a strictly $k$-affinoid algebra, we call $M(A)$ a strictly *$k$-affinoid space*. A *morphism* $M(B) \to M(A)$ of strictly $k$-affinoid space is induced by a (bounded) map $A \to B$.

**Proposition 5.34.** *If $\varphi \colon A \to B$ is a finite morphism of strictly $k$-affinoid algebras, then $f \colon M(B) \to M(A)$ has finite fibers.*

*Proof.* There exists an admissible $A$-linear epimorphism $A^n \to B$ for some $n$. Pick any $x \in M(A)$. Take complete tensor product with $\mathcal{H}(x)$ to get an admissible epimorphism:

$$\mathcal{H}(x)^n \cong A^n \widehat{\otimes}_A \mathcal{H}(x) \to B \widehat{\otimes}_A \mathcal{H}(x).$$

This implies that $f^{-1}(x) \cong M(B \widehat{\otimes}_A \mathcal{H}(x))$ is finite, because $B \widehat{\otimes}_A \mathcal{H}(x)$ is a finite extension of $\mathcal{H}(x)$. □

**Proposition 5.35.** *Assume $\varphi \colon A \hookrightarrow B$ is a finite injective morphism. Then $f \colon M(B) \to M(A)$ is surjective (with finite fibers).*

*Proof.* We already know that $f$ has finite fibers by Proposition 5.34. Pick $x \in M(A)$. Then

$$f^{-1}(x) \cong M(B \otimes_A \mathcal{H}(x)).$$

We have to show that $B \otimes_A \mathcal{H}(x)$ is non-zero and has non-zero seminorm.

We use the fact that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective (a standard fact in algebraic geometry). Set $\mathfrak{p} = \ker(x) \subseteq A$ and let $L = \mathrm{Frac}(A/\mathfrak{p})$. Then $L$ is a valued field whose completion is $\mathcal{H}(x)$. The fiber of $\mathfrak{p}$ is $\mathrm{Spec}(B \otimes_A L)$. Hence $V = B \otimes_A L$, a non-zero $L$-vector space.

[2] If $K$ was complete, there would be a unique extension of the norm to $L$. However, $K = \mathrm{Frac}(\mathrm{Tate}_d)$ is not complete.

As above, have an admissible epimorphism $\mathcal{H}(x)^n \to B \otimes_A \mathcal{H}(x)$ of $A$-modules. But $B \otimes_A \mathcal{H}(x) \cong V \otimes_L \mathcal{H}(x)$ is non-zero vector space over $\mathcal{H}(x)$.

To summarize, we have an admissible epimorphism

$$\pi \colon \mathcal{H}(x)^n \to V \otimes_L \mathcal{H}(x)$$

of $\mathcal{H}(x)$-modules. Hence

$$V \otimes_L \mathcal{H}(x) \cong \frac{\mathcal{H}(x)^n}{W}$$

for $W = \ker \pi$. We must show that the seminorm on $\mathcal{H}(x)^n/W$ is a norm (or at least $\not\equiv 0$). This is equivalent to showing that $W \subseteq \mathcal{H}(x)^n$ is closed.

**Fact.** *This is automatic: any finite-dimensional vector space over a complete valued field is complete.*

Therefore, $B \otimes_A \mathcal{H}(x)$ is a seminormed ring with non-zero seminorm. This implies that $f^{-1}(x) = M(B \otimes_A \mathcal{H}(x)) \neq \emptyset$. □

Recall that we defined the kernel map

$$\ker \colon M(A) \to \mathrm{Spec}(A).$$

**Theorem 5.36.** *If $A$ is strictly $k$-affinoid, then* $\ker$ *is surjective.*

We first prove a lemma.

**Lemma 5.37.** *Suppose $A$ is an integral domain. Then there exists a bounded valuation on $A$. Equivalently, there exists $x \in M(A)$ such that* $\ker x = 0 \in \mathrm{Spec}(A)$*.*

*Proof.* By Noether Normalization Theorem 5.18, there is a finite injective map

$$\mathrm{Tate}_d \hookrightarrow A.$$

This induces commutative diagram:

$$\begin{tikzcd} M(A) \arrow[r, two heads] \arrow[d, "\ker"] & M(\mathrm{Tate}_d) = E^d \arrow[d, "\ker"] \\ \mathrm{Spec}(A) \arrow[r, two heads] & \mathrm{Spec}(\mathrm{Tate}_d) \end{tikzcd}$$

The rows are surjective by Proposition 5.35. Pick $x \in M(A)$ mapping to the Gauss point of $E^d$. We hence have elements

$$\begin{tikzcd} x \arrow[d] & M(A) \arrow[r, two heads] \arrow[d, "\ker"] & M(\mathrm{Tate}_d) = E^d \arrow[d, "\ker"] & \text{Gauss} \arrow[d] \\ \xi & \mathrm{Spec}(A) \arrow[r, two heads] & \mathrm{Spec}(\mathrm{Tate}_d) & \text{generic point} \end{tikzcd}$$

Then $\xi = \ker(x) \in \mathrm{Spec}(A)$ maps to the generic point of $\mathrm{Spec}(\mathrm{Tate}_d)$. By the corresponding algebraic geometry statement, we conclude that $\xi$ is a generic point of $\mathrm{Spec}(A)$. □

*Proof of Theorem 5.36.* Without loss of generality, $A$ is reduced. Indeed, $A \mapsto A/\text{nilradical}(A)$ does not change $M(A)$.

Consider $\xi \in \operatorname{Spec}(A)$ with associated prime ideal $\mathfrak{p} \subseteq A$. Then $A/\mathfrak{p}$ is a strictly Ind$k$-affinoid integral domain. We have a commutative diagram:

$$\begin{tikzcd} M(A/\mathfrak{p}) \arrow[r] \arrow[d, "\ker"] & M(A) \arrow[d, "\ker"] \\ \operatorname{Spec}(A/\mathfrak{p}) \arrow[r] & \operatorname{Spec}(A) \end{tikzcd}$$

By Lemma 5.37, there exists $y \in M(A/\mathfrak{p})$ such that $\ker(y) = \eta$, the generic point of $\operatorname{Spec}(A/\mathfrak{p})$. Then the image $x \in M(A)$ of $y \in M(A/\mathfrak{p})$ maps to $\xi$ under ker. □

5.3. **The Zariski topology.** Let $A$ be strictly affinoid. For an ideal $I \subseteq A$,

$$V(I) = \{x \in M(A) \mid f(x) = 0 \text{ for all } f \in I\}.$$

**Definition 5.38.** The *Zariski topology* on $M(A)$ is the weakest topology such that $V(I)$ are closed.

Since $V(I)$ are closed in the Berkovich topology, the Berkovich topology is stronger than the Zariski topology.

For $Z \subseteq M(A)$, we define

$$I_Z = \{f \in A \mid f|_Z \equiv 0\} \subseteq A,$$

which is an ideal.

**Proposition 5.39.**

(1) *The set $V(I_Z)$ is the Zariski closure of $Z$.*
(2) *For any ideal $I \subseteq A$, $I_{V(I)} = \sqrt{I}$, the radical of $I$.*

*Proof.* (1) is clear. For (2), $f \in \sqrt{I}$ implies that $f^n \in I$ for some $n \geq 1$, so $f^n|_{V(I)} \equiv 0$, and hence $f|_{V(I)} \equiv 0$, so $f \in I_{V(I)}$. Conversely, if $f \in I_{V(I)}$, then $f \equiv 0$ on $\operatorname{Max}(A) \cap V(I)$. This is equivalent to $f \in \mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal and $\mathfrak{m} \supseteq I$. Since $A$ is Jacobson, this is equivalent to $f \in \sqrt{I}$. □

5.4. **Reduction.** Let $(A, \|\cdot\|)$ be a NA Banach ring. Then the *graded reduction ring* is

$$\bigoplus_{r \in \mathbb{R}_+^\times} \underbrace{\{\|\cdot\| \leq r\}/\{\|\cdot\| < r\}}_{\widetilde{A}_r}.$$

The *classical reduction ring* is just $\widetilde{A}_1$.

**Definition 5.40.** If $A$ is a strictly $k$-affinoid algebra, then the *reduction* of $A$ is

$$\widetilde{A} = A^\circ / A^{\circ\circ}$$

where

$$A^\circ = \{\varrho \le 1\},$$
$$A^{\circ\circ} = \{\varrho < 1\}.$$

Then $\widetilde{A}$ is a $\widetilde{k}$-algebra, which only depends on $A/\mathrm{nil}(A)$, justifying the name *reduction.*

Note that there is no need to work with the *graded reduction ring* (for $\varrho$): by Corollary 5.27, we can always scale $\widetilde{A}_r$ to $\widetilde{A}_r$.

Any bounded map $\varphi\colon A \to B$ induces $\widetilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$, since $\varrho_B \circ f \le \varrho_A$.

**Lemma 5.41.** *The spectral norm $\varrho$ is multiplicative if and only if $\widetilde{A}$ is an integral domain.*

*Proof.* If $\varrho$ is multiplicative, and $\widetilde{a}, \widetilde{b} \neq 0$, then $\varrho(a) = \varrho(b) = 1$, so $\varrho(ab) = 1$, so $\widetilde{a}\widetilde{b} \neq 0$.

Conversely, suppose $f, g \in A$ and $\varrho(fg) < \varrho(f)\varrho(g)$. Since $\varrho(f), \varrho(g) \in |k^\times|^{\mathbb{Q}}$ by Corollary 5.27 and $\varrho$ is power-multiplicative, we may assume that

$$\varrho(f), \varrho(g) \in |k^\times|.$$

Replace $f$, $g$ by $af$, $bg$ for $a, b \in k^\times$ to assume that $\varrho(f), \varrho(g) = 1$. Then $\varrho(fg) < 1$. This is equivalent to $\widetilde{f}\widetilde{g} = \widetilde{fg} = 0$, but $\widetilde{f}, \widetilde{g} \neq 0$. □

**Proposition 5.42.** *If $\varphi\colon A \to B$ is a morphism of strictly $k$-affinoid algebras with $A$ reduced, the following are equivalent:*

1. *$\varphi$ is injective and admissible ($C^{-1}\|f\| \le \|\varphi(f)\| \le C\|f\|$ for some $C$)*
2. *$\varphi$ is an isometry for the spectral radius,*
3. *$\widetilde{\varphi}\colon \widetilde{A} \to \widetilde{B}$ is injective.*

*Proof.* Use:

- Corollary 5.27,
- since $A$ is reduced, without loss of generality, $\|\cdot\| = \varrho$ on $A$.

The details are left as an exercise. □

**Remark 5.43.** In general, it is not easy to related surjectivity of $\varphi$ and $\widetilde{\varphi}$.

We leave the following theorem as a *black box.*

**Theorem 5.44.** *If $\varphi\colon A \to B$ is a morphism of strictly $k$–affinoid algebras, then*

$$\varphi \text{ is finite if and only if } \widetilde{\varphi} \text{ is finite.}$$

We will not use the techniques used to prove it later in the class, which is why we will not discuss them in detail.

*Ingredients of the proof.*

- We reduce to $A = \mathrm{Tate}_n$ using Noether Normalization Theorem 5.18 and the definition of strictly $k$-affinoid algebras.

- Use $\underbrace{\text{Weierstrass}}$ polynomials to show the 'if' implication.
- Use $\widetilde{\mathrm{Tate}_n} = \widetilde{k}[T_1, \ldots, T_n]$ is a *Japanese* ring.

**Corollary 5.45.** *If $A$ is strictly $k$-affinoid, then $\widetilde{A}$ is a finitely-generated $\widetilde{k}$-algebra.*

*Proof.* Using Noether Normalization Theorem 5.18 to get a finite ma $\mathrm{Tate}_d \hookrightarrow A$ and Theorem 5.44 to get a finite map $\widetilde{\mathrm{Tate}_d} \to \widetilde{A}$. □

## 6. Affinoid spaces and the structure sheaf

Let $k$ be a NA field and $A$ a Banach $k$-algebra.

Recall that $A$ is *strictly $k$-affinoid* if there is an admissible epimorphism

$$k\{T\} = k\{T_1, \ldots, T_n\} \twoheadrightarrow A.$$

**Example 6.1.** The algebra $k\{r^{-1}T\}$ is strictly $k$-affinoid if and only if $r_i \in |k^\times|^{\mathbb{Q}}$ for $i = 1, \ldots, n$.

**Definition 6.2.** A *$k$-affinoid algebra* is a Banach $k$-algebra $A$ admitting an epimorphism $k\{r^{-1}T\} \twoheadrightarrow A$ for some $r \in (\mathbb{R}_+^\times)^n$ for $n \geq 0$. Then $M(A)$ is a $k$-affinoid space.

**Remark 6.3.** In Tate's theory of analytic spaces [Tat71], he only uses *strictly $k$-affinoid spaces* and refers to them as *$k$-affinoid spaces*.

### 6.1. **Ground field extension.**

Let $k'/k$ be a NA extension. Then $A\widehat{\otimes}_k k'$ is a Banach $k'$-algebra.

**Proposition 6.4.** *Let $A$ be $k$-affinoid. Then:*

1. *$A\widehat{\otimes}_k k'$ is $k'$-affinoid,*
2. *there exists an extension $k'/k$ such that $A\widehat{\otimes}_k k'$ is strictly $k'$-affinoid.*

*Proof.* For (1), note that $k\{r^{-1}T\} \twoheadrightarrow A$ tensored with $k'/k$ gives the desired map

$$k'\{r^{-1}T\} \twoheadrightarrow A\widehat{\otimes}_k k'.$$

In (2), we pick $k'$ of a spacial form. We say $r \in (\mathbb{R}_+^\times)^n$ is *$|k^\times|$-free* if the images of $r_1, \ldots, r_n$ in the $\mathbb{Q}$-vector space $\mathbb{R}_+^\times/|k^\times|^{\mathbb{Q}}$ are $\mathbb{Q}$-linearly independent. (This is equivalent to $r^\nu \in |k^\times|$ if and only if $\nu = 0$.)

If $r$ is $|k^\times|$-free, define

$$k_r = \left\{ f = \sum_{\nu \in \mathbb{Z}^n} a_\nu T^\nu \;\middle|\; \lim_{|\nu| \to \infty} |a_\nu| r^\nu = 0 \right\} \subseteq k[\![T_1^{\pm 1}, \ldots, T_n^{\pm 1}]\!]$$

with the norm $\|f\| = \max_\nu |a_\nu| r^\nu$. This is a field because the maximum is attained for a unique $\nu$. Then we can pick $k' = k_r$ in (2).

The general case follows easily, so we omit this here. □

**Remark 6.5.** We can describe $A\widehat{\otimes}_k k_r$ concretely:

$$A\widehat{\otimes}_k k_r = \left\{ f = \sum_{\nu\in\mathbb{Z}^n} a_\nu T^\nu \;\middle|\; \lim_{|\nu|\to\infty} \|a_\nu\| r^\nu = 0, \quad a_\nu \in A \right\}$$

with the seminorm and spectral norm:

$$\|f\| = \max_\nu \|a_\nu\| r^\nu,$$
$$\varrho(f) = \max_\nu \varrho(a_\nu) r^\nu.$$

In particular, we have a diagram:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}"] & A\widehat{\otimes}_k k_r \arrow[r, two heads, "\sum a_\nu T^\nu \mapsto a_0"'] & A \end{tikzcd}$$

How do ground field extensions look for spaces? For $X = M(A)$, define $X_{k'} = M(A\widehat{\otimes}_k k')$. Then the map $k \hookrightarrow k'$ induces a continuous surjective map

$$X_{k'} \to X.$$

**Example 6.6.** Let $A = k\{r^{-1}T\}$ for $r \in \mathbb{R}_+^\times$. Then $X = E_k(r)$ is a disc. We consider $r \notin |k^\times|^{\mathbb{Q}}$, $k' = k_r$ and $X_{k'} = E_{k'}(r)$.

Then $p(E_k(r))$ is a Type 3 point, with a unique branch coming down. However, $p(E_{k'}(r))$ is a Type 2 point with infinitely many branches coming down.

Let $\pi\colon E_{k'}(r) \to E_k(r)$ be the map above. The fibers of $\pi$ are

$$\pi^{-1}(x) \cong M(\mathcal{H}(x)\widehat{\otimes}_k k')$$

As above,

$$\mathcal{H}(x)\widehat{\otimes}_k k_r = \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \;\middle|\; a_i \in \mathcal{H}(x),\ \lim_i |a_i| r^i = 0 \right\}.$$

Therefore, $\pi^{-1}(x)$ is a *circle of radius $r$ over $\mathcal{H}(x)$*.

There is always a section

$$\sigma\colon E_k(r) \to E_{k'}(r)$$

given by

$$\sigma(x) = \text{Gauss point of } \mathcal{H}(x)\widehat{\otimes}_k k_r.$$

### 6.2. Properties of $k$-affinoid algebras.

**Proposition 6.7.** *Let $A$ be a $k$-affinoid algebra. Then:*

(1) *$A$ is a Noetherian Banach ring (i.e. $A$ is Noetherian and every ideal is closed),*
(2) *if $\mathfrak{a} \subseteq A$ is an ideal, then $A/\mathfrak{a}$ is $k$-affinoid.*

*Sketch of proof.* We will prove that $A$ is Noetherian and every ideal $\mathfrak{a} \subseteq A$ is closed.

As in the proof of Proposition 6.4, pick $r \in (\mathbb{R}_+^\times)^n$ which is $|k^\times|$-free such that $A' = A\widehat{\otimes}_k k_r$ is strictly $k$-affinoid.

The key property is that

$$\mathfrak{a} = A \cap (A' \cdot \mathfrak{a}) \tag{2}$$

where $A' \cdot \mathfrak{a}$ is an ideal in $A'$. We use

$$\begin{tikzcd} A \arrow[r, hook] \arrow[rr, bend left, "\mathrm{id}"] & A\widehat{\otimes}_k k_r \arrow[r, two heads] & A. \end{tikzcd}$$

To prove this, pick generators $f_1, \ldots, f_n \in \mathfrak{a}$ of $A' \cdot \mathfrak{a}$. Pick $f \in \mathfrak{a}$. Then

$$f = \sum_1^n f_j g_j$$

for

$$g_j \in \sum_{\nu \in \mathbb{Z}^n} g_{j\nu} T^\nu, \quad g_{j\nu} \in A.$$

Hence

$$f = \sum_1^n f_j g_{j0}$$

and $g_{j0} \in A$. This shows that $\mathfrak{a}$ is generated by $f_1, \ldots, f_n$ and equation (2) holds. □

Similarly, one proves the following.

**Proposition 6.8.** *If $A$ is $k$-affinoid and $f \in A$,*

(1) $\varrho(f) = 0$ *if and only if $f$ is nilpotent,*
(2) $\varrho(f) \leq 1$ *if and only if* $\sup_n \|f^n\| < \infty$.

**Proposition 6.9.** *If $A$ is a reduced $k$-affinoid algebra, then $\varrho$ is a complete norm, equivalent to the given norm.*

**Remark 6.10.** The classical reduction $A \mapsto \widetilde{A}$ is not well-behaved for $k$-affinoid algebras. Instead, use the graded reduction:

$$\widetilde{A} = \bigoplus_{r \in \mathbb{R}_+^\times} \{\varrho \leq r\}/\{\varrho < r\}.$$

6.3. **Rational domains and the structure sheaf.** Let $k$ be a NA field, $A$ a $k$-affinoid algebra.

Then $X = M(A)$ is an affinoid space and $A$ are the analytic functions on $X$.

The goal is to define a structure sheaf $\mathcal{O}_X$ such that

$$\Gamma(V, \mathcal{O}_V) = \{\text{analytic functions on } V\}$$

for $V \subseteq X$.

It is easier to first work with certain **closed** subset $V \subseteq X$, called *affinoid domains*.

To define $\mathcal{O}_X$, we want:

- $\Gamma(V, \mathcal{O}_X) = A_V$ is a $k$-affinoid algebra,
- restriction maps $A \to A_V$,
- sheaf axiom.

The idea is to define $V$ using inequalities.

We will have the following analogy between algebraic and NA geometry:

| Algebraic geometry | NA geometry |
|---|---|
| $X = \mathrm{Spec}(A)$ | $X = M(A)$ |
| $V = \{f \neq 0\}$ $= \mathrm{Spec}(A[f^{-1}])$ | $V = \{\|f\| \geq 1\}$ $= M(A\{f^{-1}\})$ (Laurent domain) |
| $V \subseteq X$ affine open | $V \subseteq X$ affinoid |

We will have the following classes of affinoid domains:

$$\text{Weierstrass} \Rightarrow \text{Laurent} \Rightarrow \text{Rational} \Rightarrow \text{Affinoid}.$$

6.3.1. *Weierstrass domains of $X = M(A)$.* Given $f_1, \dots, f_n \in A$, $p_1, \dots, p_n \in \mathbb{R}_+^\times$, set

$$V = X(p^{-1}f) = \{x \in X \mid |f_i(x)| \leq p_i \quad 1 \leq i \leq n\}.$$

Then

$$A_V = A\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}/(f_i - T_i)$$

with the quotient norm is a $k$-affinoid algebra. The map $A \to A_V$ is contractive with dense image, so

$$M(A_V) \xrightarrow{\tau} M(A) = X$$

is a homeomorphism onto its image.

**Lemma 6.11.** *We have that $\tau(M(A_V)) = V$.*

*Sketch of proof.* For the '$\subseteq$' inclusion, note that for $y \in M(A_V)$, $|f_i(y)| = |T_i(y)| \leq p_i$ for $1 \leq i \leq n$, so if $x = \tau(y) \in X$, then $|f_i(x)| \leq p_i$.

For the '$\supseteq$' inclusion, given $x \in V$ i.e. a bounded semivaluation on $A$ with $|f_i| \leq p_i$ for all $i$, we can extend this to $A_V$ by

$$|g| = \inf \left\{ \max_\nu |a_\nu| \cdot |f|^\nu \;\middle|\; g = \sum a_\nu T^\nu \right\}.$$

One has to check that this is actually a norm, but we omit this here. ☐

6.3.2. *Laurent domains of $X = M(A)$.* Given $f_i, g_j \in A$, $p_i, q_j \in \mathbb{R}_+^\times$, $1 \le i \le m$, $1 \le j \le n$, define

$$V = X(p^{-1}f, qg^{-1}) = \{x \in X \mid |f_i(x)| \le p_i,\ |g_j(x)| \ge q_j\} \subseteq X.$$

Then

$$A_V = A\{p_1^{-1}T_1, \ldots, p_m^{-1}T_m, q_1S_1, \ldots, q_nS_n\}/(T_i - f_i, g_jS_j - 1)$$

with the quotient norm is a $k$-affinoid algebra.

The analog of Lemma 6.11 still holds for Laurent domains.

**Lemma 6.12.** *The map $A \to A_V$ induces $M(A_V) \overset{\tau}{\hookrightarrow} M(A)$ with image $V$.*

The proof is similar. The reason why $M(A_V) \to M(A)$ is injective follows from a general lemma.

Suppose $\varphi\colon A \to B$ is a morphism of Banach rings. Define $\varphi^+(A) \subseteq B$ to be the *augmented image*:

$$\{\varphi(a)/\varphi(a') \mid a, a' \in A,\ \varphi(a') \text{ is invertible in } B\}.$$

**Lemma 6.13.** *If $\varphi^+(A) \subseteq B$ is dense, then $M(B) \to M(A)$ is injective.*

**Exercise.** For a Laurent domain $V$, the augmented image of $A \to A_V$ is dense.

**Remark 6.14.** The Laurent domains define the topology on $X = M(A)$, i.e.

$$\{\mathrm{int}(V) \mid V \subseteq X \text{ is a Laurent domain }\}$$

is a basis for the topology. Recall that the topology was originally defined by requiring that $x \mapsto f(x)$ is continuous for all $f$. This will be given by the kind of (strict) inequalities as in the definition of Laurent domains. Clearly, Weierstrass domains are not enough.

6.3.3. *Rational domains of $X = M(A)$.* Given $\begin{cases} g, f_1, \ldots, f_n \in A \text{ without common zero on } X, \\ p_1, \ldots, p_n \in \mathbb{R}_+^\times \end{cases}$

set

$$V = X\left(p^{-1}\frac{f}{g}\right) = \{x \in X \mid |f_i(x)| \le p_i|g(x)| \text{ for all } i\} \subseteq X.$$

Then

$$A_V = A\{p_1^{-1}T_1, \ldots, p_n^{-1}T_n\}/(gT_i - f_i)$$

and $A \to A_V$ induces $M(A_V) \hookrightarrow M(A)$ with image $V$ (the augmented image of $A \to A_V$ is dense).

**Remark 6.15.** If $p_i, q_j \in |k^\times|^{\mathbb{Q}}$ above, we call $V$ *strictly Weierstrass/Laurent/Rational.* We can then always pick $p_i = q_j = 1$.

We discuss some elementary properties of the above domains. We abbreviate them by their first letters below.

(1) **Pullbacks:** if $f\colon Y \to X$ is a morphism and $V \subseteq X$ is W/L/R, then $f^{-1}(V) \subseteq Y$ is W/L/R.
(2) **Finite intersections:** if $V_1, \ldots, V_N \subseteq X$ are W/L/R, then $V_1 \cap \cdots \cap V_N \subseteq X$ is W/L/R.

**Exercise.** Check that the intersection of rational domains is rational.

(3) For domains $V \subseteq X$, Weierstrass implies Laurent implies Rational. (Indeed, a general Laurent domain is the intersection of Laurent domains with 1 inequality which are clearly Rational.)

We now discuss transitivity. Suppose $U \subseteq V \subseteq X = M(A)$.

**Proposition 6.16.** *If $V \subseteq X$ is W/R and $U \subseteq V$ is W/R, then $U \subseteq X$ is W/R.*

**Remark 6.17.** This is false for Laurent domains.

*Sketch of proof in the W case.* Suppose $V = X(p^{-1}f)$ where $|f_i| \leq p_i$, $f_i \in A$, and $U = V(r^{-1}g)$ for $g_j \in A_V$. Since $A$ is dense in $A_V$, we can find $h_j \in A$ such that $|h_j - g_j| \leq r_j$ on $V$. Hence

$$U = V(r^{-1}h) = X(r^{-1}h, p^{-1}f)$$

where $|f_i| \leq p_i$, $|h_j| \leq r_j$ on $X$. $\square$

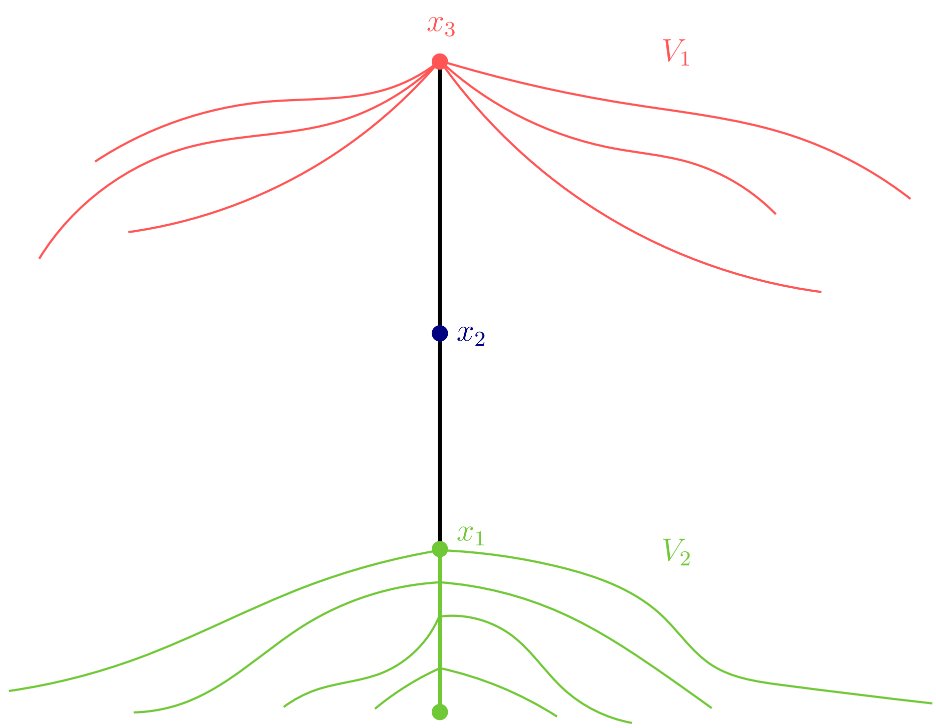
**Remark 6.18.** One finally defines affinoid domains using a universal property. One then show that they are finite unions of rational domains. We delay this until Section 6.5.

**Example 6.19.** Let $X = E = M(k\{T\})$ be the unit disc. For $0 < |a| < 1$, $a \in k$, consider

$$V = \{|T| \leq |T - a|^2\}$$

is (strictly[3]) rational, but not Laurent. We note that $V = V_1 \cup V_2$ is a disjoint union with

$$V_1 = \{|T| = 1\}, \quad V_2 = \{|T| \leq |a|^2\}.$$




[3]I.e. all $p_i$ are in the divisible value group. One has similar notions of strictly W and L domains.

One can check that this domain is not Laurent, i.e.

$$V \neq \{|f_i| \leq p_i,\ |g_j| \geq q_j\}$$

for any $f_i$, $g_j$, $p_i$, $q_j$. We can test this at points marked $x_1$, $x_2$, $x_3$ in the figure above.

Consider a rational domain $V = \{|f_i| \leq p_i|G|\} = X(p^{-1}f/g) \subseteq X$. Then

$$A_V = A\{p^{-1}T\}/(gT_i - f_i)$$

be a $k$-affinoid algebra.

**Question.** Is $A_V$ canonically associated to $V$ (and $A$)? (I.e. independent of $p_i$, $f_i$, $g$.)

**Answer.** Yes, by a universal property.

Recall that the map $\varphi\colon A \to A_V$ induces a map $M(A) \xrightarrow{\tau} M(A) = X$.

**Theorem 6.20.**

(1) *The map $\tau$ is a homeomorphism onto its image.*
(2) *If $\sigma\colon Z \to X$ is a morphism of $k$-affinoid spaces such that $\sigma(Z) \subseteq V$, then $\sigma$ factors through $\tau$:*

$$\begin{tikzcd} Z \arrow[rr, dotted] \arrow[dr, "\sigma"'] & & M(A_V) \arrow[dl, "\tau"] \\ & X & \end{tikzcd}$$

(3) *If $y \in M(A_v)$, then $A \xrightarrow{\varphi} A_v$ induces an isomorphism $\mathcal{H}(y) \xrightarrow{\cong} \mathcal{H}(\tau(y))$*

*Proof of (1) and (3).* For any $x \in X$, $\tau^{-1}(x) = M(B_x)$ where

$$B_x = \mathcal{H}(x)\widehat{\otimes}_A A_V \cong \mathcal{H}(x)\{p^{-1}T\}/(g(x)T_i - f_i(x)).$$

If $x \in V$, then

- $g(x) \neq 0$ (or else $f_i(x) = 0$, a contradiction),
- $\left(T_i - \frac{f_i(x)}{g(x)}\right)$ is a maximal ideal in $\mathcal{H}(x)\{p^{-1}T\}$,
- $\mathcal{H}(x) \xrightarrow{\cong} B_x$ and $\#\tau^{-1}(x) = 1$.

If $x \notin V$, then

- $\tau^{-1}(x) = \emptyset$ since if $y \in M(A_V)$, then
$$|f_i(y)| = |T_i(y)| \cdot |g(y)| \leq p_i|g(y)|,$$
which shows that $\tau(y) \in V$.

This proves (1) and (3). $\square$

It remains to prove (2) in Theorem 6.20. We will need the following lemma.

**Lemma 6.21.** *Let $\psi\colon A \to C$ be a morphism of $k$-affinoid algebras, $h_1, \dots, h_n \in C$, $p_1, \dots, p_n \in \mathbb{R}_+^\times$. Assume $\varrho(h_i) \leq p_i$ for $1 \leq i \leq n$. Then there is a unique map $\Psi\colon A\{p^{-1}T\} \to C$ extending $\psi$ such that $\Psi(T_i) = h_i$.*

*Proof.* The uniqueness is clear: we must set

$$\Psi\left(\sum_\nu a_\nu T^\nu\right) = \sum_\nu \psi(a_\nu) h^\nu.$$

The question is: is this power series convergent?

It suffices to prove that $\|\psi(a_\nu)\| \cdot \|h^\nu\| \to 0$ as $|\nu| \to \infty$. Since $C$ is $k$-affinoid, there exists $D_h, N_h > 0$ such that

$$\|h_i^{\nu_i}\| < D_h \varrho(h_i)^{\nu_i} \leq D_h p_i^{\nu_i}$$

for $\nu_i \geq N_h$, $i = 1, \dots, n$. This implies the converges since $\|\psi(a_\nu)\| \leq \|a_\nu\|$ and $\|a_\nu\| p^\nu \to 0$ as $|\nu| \to \infty$. □

*Proof of (2) in Theorem 5.7.* Let $\sigma\colon Z \to X$ be a map induces by $\psi\colon A \to C$. Then

$$A_V = A\{p^{-1}T\}/(gT_i - f_i).$$

Note that $\psi(g) \in C$ is invertible since $|\psi(g)(z)| = |g(\sigma(z))| \neq 0$ for all $z \in Z$. We can use Lemma 6.21 to lift $\psi$ to

$$\Psi\colon A\{p^{-1}T\} \to C \quad \text{with } \Psi(T_i) = \underbrace{\frac{\psi(f_i)}{\psi(g)}}_{h_i}.$$

Note that the lemma does apply to $h_i$ because

$$\begin{aligned} \varrho(h_i) &= \max_{z \in Z} |h_i(z)| && \text{by BMMP 3.21} \\ &= \max_{z \in Z} \frac{|f_i(\sigma(z))|}{g(\sigma(z))} \\ &\leq p_i && \text{since } \sigma(z) \in V. \end{aligned}$$

It is clear that $\Psi(gT_i - f_i) = 0$, so $\Psi$ induces $\psi_V\colon A_V \to C$. (We also get uniqueness.) □

6.4. **Tate acyclicity.** Let $X = M(A)$ be a $k$-affinoid space. The idea is to prove that

$$\{(V, A_V) \mid V \subseteq X \text{ rational domain}\}$$

is a sheaf.

A rough is example is the following. Suppose $X = V_1 \cup V_2$ where $V_i$ are rational. Let $f_i = A_{V_i}$ such that $f_1 = f_2$ on $V_1 \cap V_2$. Then there exists $f \in A$ such that $f|_{V_i} = f_i$.

In general, given a finite covering $X = V_1 \cup \cdots \cup V_n$ by rational domains, we get the following complex:

$$0 \longrightarrow A \xrightarrow{(\varphi_i)_i} \prod_i A_{V_i} \xrightarrow{(\varphi_i - \varphi_j)_{i,j}} \prod_{i,j} A_{V_i \cap V_j} \longrightarrow \cdots \qquad (*)$$

where $\varphi_i\colon A \to A_{V_i}$, $\varphi_{i,j}\colon A_{V_i} \to A_{V_i \cap V_j}$ are the natural maps

**Theorem 6.22** (Tate's Acyclicity)**.** *The above complex* $(*)$ *is exact (as a complex of Banach* $A$*-modules).*

*Sketch of proof.* **Step 1.** Use $\widehat{\otimes}_k k_r$ to reduce to the *strict case*.

**Step 2.** Reduce to the special covering

$$X = \{|f| \leq 1\} \cup \{|f| \geq 1\} \quad \text{for } f \in A.$$

**Step 3.** Treat this case by hand: show that

$$0 \longrightarrow A \longrightarrow A\{f\} \times A\{f^{-1}\} \longrightarrow A\{f, f^{-1}\} \longrightarrow 0$$

is exact. □

## 6.5. **Affinoid domains.**

6.5.1. *Monomorphisms and immersions.* We review different nortions of embeddings: monomorphisms, closed immersions, *Weierstrass embeddings*, and *Runge embeddings*.

**Definition 6.23.** A map $Y \xrightarrow{\varphi} X$ be a *monomorphism* if whenever

$$\begin{tikzcd} Z \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] & Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

we have that $\varphi \circ \psi_1 = \varphi \circ \psi_2$ if and only if $\psi_1 = \psi_2$.

This is equivalent to the map $Y \to Y \times_X Y$ being an isomorphism, and hence to the map

$$B \widehat{\otimes}_A B \to B$$

being an isomorphism.

**Definition 6.24.** A map $Y \to X$ is a *closed immersion* if the corresponding map $A \to B$ is an admissible epimorphism.

**Example 6.25.** For any $k$-affinoid domain $A$ we have $k\{r^{-1}T\} \twoheadrightarrow A$, so $X \hookrightarrow E(r)$ is a closed immersion.

**Definition 6.26.** A map $Y \to X$ is a *Weierstrass embedding* if $B = A_V$ where $V \subseteq X$ is a Weierstrass domain.

**Example 6.27.** For $0 < r < 1$, $E(r) \subset E$ is a Weierstrass embedding.

**Definition 6.28.** A map $Y \to X$ is a *Runge immersion* if it factors:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, "\text{closed immersion}"'] & & X \\ & X' \arrow[ur, "\text{Weierstrass}"'] & \end{tikzcd}$$

**Example 6.29.** The composition $E(r) \hookrightarrow E^2(r) \to E^2(1)$ for $0 < r < 1$ is a Runge immersion.

**Lemma 6.30.** *A map $Y \to X$ is a Runge immersion if and only if the associated map $A \xrightarrow{\varphi} B$ has dense image.*

*Proof.* The 'only' is implication is clear, since this is true both for Weierstrass embeddings and for closed immersions.

For the 'if' implication, suppose $A \overset{\varphi}{\to} B$ has dense image. Since $B$ is a $k$-affinoid algebra, there exists an admissible epimorphism $k\{r^{-1}T\} \twoheadrightarrow B$. Applying $\widehat{\otimes}_k A$, we get

$$A\{r^{-1}T\} \twoheadrightarrow B.$$

Without loss of generality, (by perturbing the map a little bit) we may suppose that $T_i \mapsto \varphi(f_i)$ for $f_i \in A$. We get a commutative triangle:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & A\{r^{-1}T\}/(T_i - f_i) & \end{tikzcd}$$

showing that the associated map $Y \to X$ is a Runge immersion. ∎

**Corollary 6.31.** *The composition of two Runge immersions is a Runge immersion.*

**Corollary 6.32.** *Any Runge immersion $Y \to X$ is a monomorphism.*

**Theorem 6.33** (Temkin)**.** *If $\varphi\colon Y \to X$ is a monomorphism of $k$-affinoid spaces, then:*

(1) *$\varphi$ is injective,*
(2) *for any $y \in Y$, $\mathcal{H}(x) \to \mathcal{H}(y)$ is an isomorphism, where $x = \varphi(y)$,*
(3) *there exists a finite covering $X = X_1 \cup \ldots \cup X_n$ by rational domains such that*

$$\varphi^{-1}(X_i) \to X_i \text{ is a Runge immersion.}$$

**Remark 6.34.** This is also true in the *strict case*, when $|k^\times| \neq \{1\}$.

*Proof of Theorem 6.33, parts (1) and (2).* Pick $y \in Y$ and set $x = \varphi(y) \in X$. Then

$$\varphi^{-1}(x) \cong M(B_x) \quad \text{where } B_x = \mathcal{H}(x)\widehat{\otimes}_A B.$$

We know that

$$B\widehat{\otimes}_A B \to B \text{ is an isomorphism}$$

because $\varphi$ is a monomorphism. Base changing this with $A \to \mathcal{H}(x)$, we get that

$$B_x\widehat{\otimes}_{\mathcal{H}(x)} B_x \to B_x$$

is an isomorphism.

**Theorem** (Grusan)**.** *The seminorm on $B_x \otimes_{\mathcal{H}(x)} B_x$ is a norm.*

Therefore, $B_x \otimes_{\mathcal{H}(x)} B_x \to B_x\widehat{\otimes}_{\mathcal{H}(x)} B_x$ is an isometric injection. Hence the composition

$$B_x \otimes_{\mathcal{H}(x)} B_x \to B_x$$

is injective, showing that

$$(\dim_{\mathcal{H}(x)} B_x)^2 \leq \dim_{\mathcal{H}(x)} B_x,$$

which is only possible if

$$\dim_{\mathcal{H}(x)} B_x = 1.$$

This shows that the map $\mathcal{H}(x) \to B_x$ is an isomorphism. This shows both (1) and (2). ∎

We will skip the details of the proof of part (3) of Theorem 6.33, because it would take too much time. The ingredients are:

- (graded) reduction: $A \rightsquigarrow \widetilde{A}$,
- compactness of $X$, $Y$.

6.5.2. *Affinoid domains.* Let $X = M(A)$ as before be a $k$-affinoid space.

**Definition 6.35** (Temkin[4])**.** An *affinoid domain* of $X$ is a pair $(V, A_V)$ such that:

- $V \subseteq X$ is closed,
- $A_V$ is a $k$-affinoid algebra with a morphism $A \to A_V$ such that:
  (1) the induced map $\tau\colon M(A_V) \to X$ has image equal to $V$,
  (2) every morphism $Z \xrightarrow{\sigma} X$ of $k$-affinoid spaces such that $\sigma(Z) \subseteq V$ factors uniquely through $\tau$:

$$\begin{tikzcd} Z \arrow[rr, dotted] \arrow[dr, "\sigma"'] & & M(A_V) \arrow[dl, "\tau"] \\ & X & \end{tikzcd}$$

or equivalently:

$$\begin{tikzcd} C & & A_V \arrow[ll, dotted] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

**Example 6.36.** A rational domain is affinoid by Theorem 6.20.

**Exercise.** If $V$ is an affinoid domain, $M(A_V) \to X$ is a monomorphism. *Hint.* Use the universal property above.

**Lemma 6.37.** *Suppose $V \subseteq X$ is an affinoid domain and $V \hookrightarrow X$ is a closed immersion, then:*

(1) *$U = X \setminus V$ is also affinoid,*
(2) *$A \cong A_U \times A_V$,*
(3) *$U, V \subseteq X$ are Weierstrass domains.*

*Sketch of proof.* We know that $A_V = A/I$ for an ideal $I \subseteq A$. We use the universal property of affinoid domains, testing with $A \to A/I^2$, which induces $M(A/I) = M(A/I^2) \to M(A)$ with image $V$. We get a morphism

$$A/I \to A/I^2,$$

so $I = I^2$. Then $I = Ae$ is principle where $e^2 = e$ and $e \neq 0, 1$ (as long as $V \neq \emptyset, X$). Therefore,

$$A \cong A_U \times A_V$$

for $A_U = A(1-e)$, $A_V = Ae$.

Also, $V = \{|e| \leq \frac{1}{2}\}$, $U = \{|(1-e)| \leq \frac{1}{2}\}$. □

[4] Berkovich in [Ber90] had a slightly different definition than Temkin. Eventually, one can show these two definitions are equivalent.

**Corollary 6.38** (Gerntzen–Grauert Theorem)**.** *Any affinoid domain is a finite union of rational domains.*

*Proof.* By Temkin's Theorem 6.33, there is a finite covering $X = X_1 \cup \cdots X_n$ by rational domains such that $V \cap X_i \to X_i$ is a Runge immersion. Therefore, we have:

$$\begin{tikzcd} V \cap X_i \arrow[rr] \arrow[dr, "\substack{\text{closed immersion}\\ \text{of affinoid}}"'] & & X_i \\ & Y_i \arrow[ur, "\text{Weierstrass}"'] & \end{tikzcd}$$

By Lemma 6.37, $V \cap X_i \hookrightarrow Y_i$ is Weierstrass, $V \cap X_i \hookrightarrow X_i$ is also Weierstrass. In particular $V \cap X_i \hookrightarrow X_i$ is rational, so $V \cap X_i \hookrightarrow X$ is rational. □

If $U, V \subseteq X$ are affinoid domains, then $U \cap V$ is also affinoid with

$$A_{U\cap V} \cong A_U \widehat{\otimes}_A A_V.$$

**Example 6.39.** Let $k$ be algebraically closed and trivially valued, $|k^\times| = 1$.

Let $X = E(r)$ for $0 < r < 1$. Then $X \cong [0, r]$ (homeomorphic). A rational domain in this case is the same as a Laurent domain which is a closed interval ($\neq \{0\}$).

Indeed, $X = M(k\{r^{-1}T\})$, $k\{r^{-1}T\} = k[\![T]\!]$ and it only makes sense to use inequalities involving $T$.

If $U, V \subseteq X$ are affinoid domains and $U \cap V = \emptyset$, then $U \cap V$ is affinoid with

$$A_{U\cap V} = A_U \times A_V.$$

**Definition 6.40.** A finite union of affinoid (or rational) domains in $X$ is called *special* (or a *compact analytic domain*).

**Definition 6.41.** For a special domain $V = V_1 \cup \cdots \cup V_n$ where $V_i$ are rational, define

$$A_V = \ker\left(\prod_i A_{V_i} \to \prod_{i,j} A_{V_i \cap V_j}\right).$$

**Proposition 6.42.**

(1) *The ring $A_V$ does not depend on the covering.*
(2) *The assignment $V \mapsto A_V$ satisfies Tate's acyclicity (i.e. the conclusion of Theorem 6.22): if $X = V_1 \cup \cdots \cup V_n$ for special domains $V_i$, then*

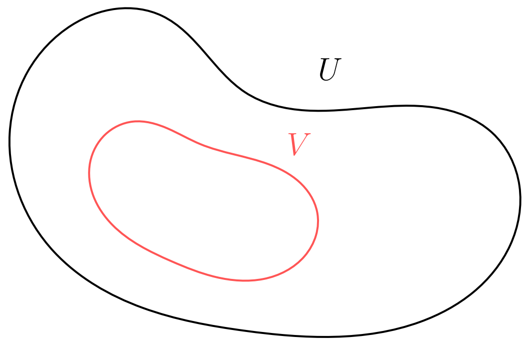
$$0 \longrightarrow A \xrightarrow{(\varphi_i)_i} \prod_i A_{V_i} \xrightarrow{(\varphi_i - \varphi_j)_{i,j}} \prod_{i,j} A_{V_i \cap V_j} \longrightarrow \cdots \qquad (*)$$

*is exact.*
(3) *A special domain $V$ is affinoid if and only if $A_V$ is a $k$-affinoid algebra and* $\mathrm{im}(M(A_V) \to M(A) = X) = V$.

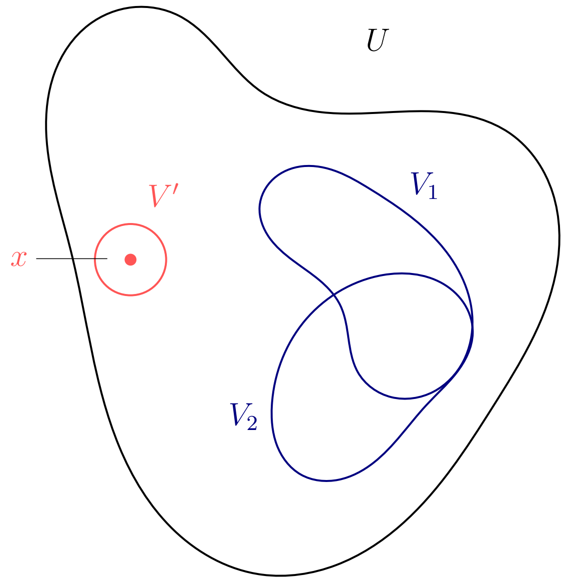
The proof uses Tate's acyclicity Theorem 6.22 and the universal property of affinoid rational domains.

6.6. **The structure sheaf for the Berkovich topology.** Let $U \subseteq X$ open where $X$ is a $k$-affinoid space. Define

$$\mathcal{O}_X(U) = \varprojlim_{\substack{V \subseteq U \\ \text{special}}} A_V.$$




This is a filtered limit: given $V_1, V_2 \subseteq U$ special and $x \in U$, there exists $V \subseteq X$ special such that $V_i \subseteq V$ and $x \in \mathrm{int}V$. Indeed, we may choose a Laurent domain $V'$ whose interior contains $x$ and then set $V = V_1 \cup V_2 \cup V$.



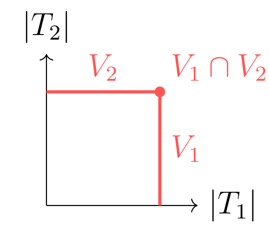

**Corollary 6.43.** *The functor $\mathcal{O}_X$ is a sheaf of $k$-algebras.*

*Proof.* By Proposition 6.42 (2), the sequence in the definition of a sheaf is exact for special domains. Since $\mathcal{O}_X$ is defined as a filtered limit over special subdomains, it is left-exact, and hence the sequence remains exact. ∎

**Example 6.44.** Let $A = \{T_1, T_2\}$ and $X = M(A) = E^2$ be the unit bidisc. Consider the Laurent domains:

$$V_i = \{|T_i| = 1\} \quad \text{for } i = 1, 2.$$

This is a very rough sketch of this situation.




Take $V = V_1 \cup V_2$. Then

$$\begin{aligned} A_{V_1} &= k\{T_1^{\pm 1}, T_2\}, \\ A_{V_2} &= k\{T_1, T_2^{\pm 1}\}, \\ A_{V_1 \cap V_2} &= k\{T_1^{\pm 1}, T_2^{\pm 1}\}. \end{aligned}$$

We have that:

$$A_V = \ker \begin{pmatrix} A_{V_1} \times A_{V_2} & \to & A_{V_1 \cap V_2} \\ (f_1, f_2) & \mapsto & f_1 - f_2 \end{pmatrix} = A$$

but $V \subsetneq X$. This is the analog of *Hartogs phenomenon* from several complex variables.

**Remark 6.45.** We may also define $\mathcal{O}_X(Y)$ for the class of *analytic* subsets $Y \subseteq X$, including special domains and open sets. One should think of a special domain as a compact analytic subset.

## 7. Local rings and residue fields

Suppose $f \in \mathcal{O}_X(U)$ for $U \subseteq X$ open. We can view $f$ as a function

$$f : U \to \coprod_{x \in U} \mathcal{H}(x)$$

via identifying $f(x) \in \mathcal{H}(x)$.

**Definition 7.1.** The *stalk of* $\mathcal{O}_X$ *at* $x$ is the $k$-algebra:

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{x \in U \\ U \subseteq X \text{ open}}} \mathcal{O}_X(U).$$

A priori, this definition is rather inconvenient because $\mathcal{O}_{X,x}$ is defined as a double limit. One can, however, check the following facts.

**Exercise.**

(1) We have that

$$\mathcal{O}_{X,x} \cong \varinjlim_{V} A_V$$

where the limit is over affinoid neighborhoods of $x$ in $X$ (i.e. $x \in V^{\text{int}}$).

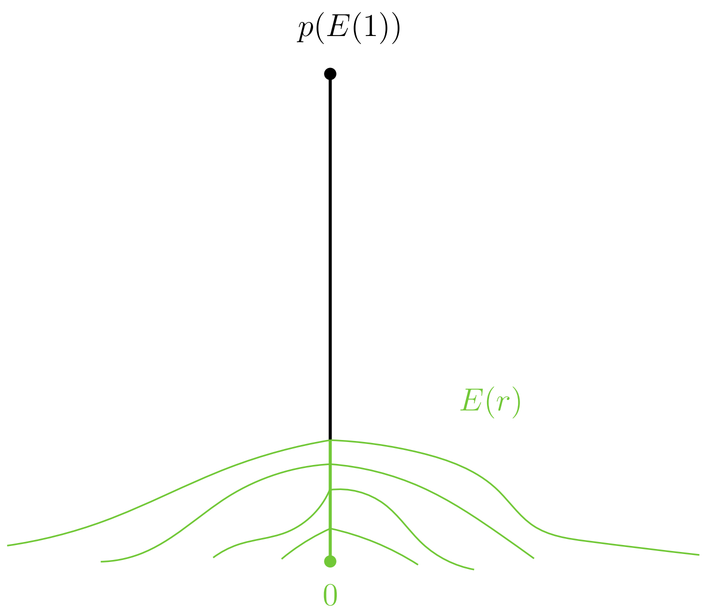
(2) The ring $\mathcal{O}_{X,x}$ is local with maximal ideal
$$\mathfrak{m}_x = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

**Definition 7.2.** The *residue field* $\kappa(x)$ is the valued field $\mathcal{O}_{X,x}/\mathfrak{m}_x$.

In general, $\kappa(x)$ is not complete, but $\kappa(x) \subseteq \mathcal{H}(x)$ is dense.

**Example 7.3.** Let $k$ be algebraically closed, $X = E(1)$, and $x \in X$. The residue field will depend on the *Type* of the point.

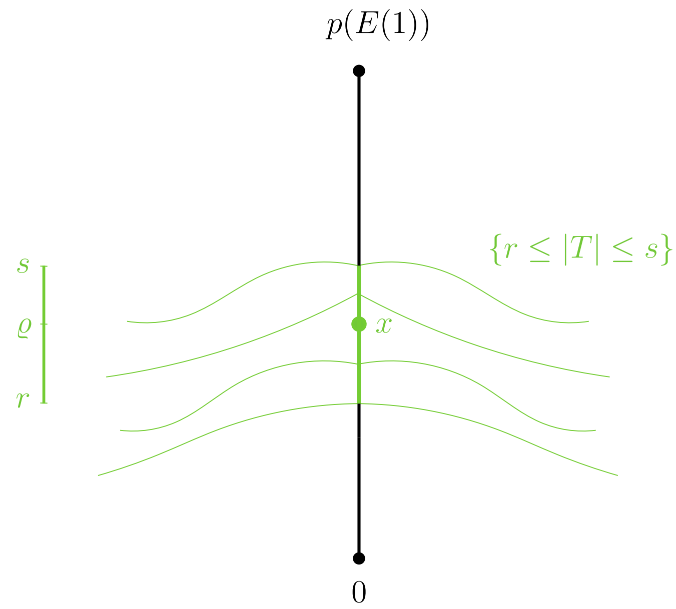
**When $x$ is of Type 1**, assume without loss of generality that $x = 0$. There is a neighborhood basis $E(r)$ for $0 < r \leq 1$.




Then
$$\mathcal{O}_{X,x} = \left\{ f = \sum_0^\infty a_i T^i \;\middle|\; \lim_{i\to\infty} |a_i|^{1/i} < \infty \right\}$$
and hence
$$\kappa(x) = \mathcal{H}(x) = k.$$

**When $x$ is of Type 3**, assume without loss of generality that $x = p(E(\varrho))$ for $\varrho \in (0,1)\backslash|k^\times|$. There is a basis of Laurent neighborhoods of $x$:
$$\{r \leq |T| \leq s\} \qquad \text{for } 0 < r < \varrho < s < 1.$$

We then have that:
$$\mathcal{O}_{X,x} = \left\{ f = \sum_{-\infty}^\infty a_i T^i \;\middle|\; \limsup_{i\to\infty} |a_i|^{1/i} < \varrho^{-1},\ \limsup_{i\to\infty} |a_{-i}|^{1/i} < \infty \right\}.$$




Since this is already a field,

$$\kappa(x) = \mathcal{O}_{X,x}.$$

Finally:

$$\mathcal{H}(x) = \left\{ f = \sum_{-\infty}^{\infty} a_i T^i \mid \lim_{i \to \pm\infty} |a_i| \varrho^i = 0 \right\}.$$

For Type 2 and Type 4 points, it is hard to write down explicitly what the local rings and residue fields are.

7.1. **Kernel map.** Recall the kernel map:

$$\begin{aligned} \ker\colon X = M(A) &\to \mathfrak{X} = \mathrm{Spec}(A) \\ x &\mapsto \ker(|\cdot|_x). \end{aligned}$$

**Example 7.4.** If $X = E(r)$ and $k$ is algebraically closed, then

- if $x$ is of Type 1, $\ker(x) = \mathfrak{m}_x$,
- if $x$ is of Type 2, 3, or 4, $\ker(x) = (0)$ is the generic point of $\mathfrak{X}$.

The idea is to compare $\mathcal{O}_{X,x}$ to $\mathcal{O}_{\mathfrak{X},x}$ using the kernel map.

Theorem 5.36 for strictly $k$-affinoid spaces generalizes to $k$-affinoid spaces.

**Theorem 7.5.** *The map* $\ker\colon X \to \mathfrak{X}$ *is surjective.*

We will work towards the proof of this theorem. This will require some setting up.

7.1.1. *Grusan's results.* To prove this theorem, we will need some results on Banach spaces due to Grusan (a little more than Theorem 4.14).

**Lemma 7.6.** *If $V$ is a NA $k$-Banach space*[5]*, then for any $\alpha > 1$, there is an $\alpha$-orthonormal basis $(v_i)_{i\in I}$ for $V$: for every $v \in V$, there is a unique representation*

$$v = \sum_{i\in I} \lambda_i v_i \quad \lambda_i \in k$$

*such that*

$$\alpha^{-1} \max_{i\in I} |\lambda_i| \|v_i\| \leq \|v\| \leq \max_i |\lambda_i| \|v_i\|.$$

*Proof.* This follows from the Gram-Schimdt algorithm. ☐

**Theorem 7.7** (Grusan)**.**

(1) *If $V$, $W$ are NA $k$-Banach spaces, then the seminorm on $V \otimes_k W$ is a norm is a norm, i.e. $V \otimes_k W \hookrightarrow V \widehat{\otimes}_k W$.*[6]

(2) *If $V' \hookrightarrow V$ and $W' \hookrightarrow W$ are isometric embeddings of NA $k$-Banach spaces, then both*

$$V' \otimes_k W' \hookrightarrow V \otimes_k W$$

*and*

$$V' \widehat{\otimes}_k W' \hookrightarrow V' \widehat{\otimes}_k W$$

*are isometries.*

(3) *If $W$ is an NA $k$-Banach space, then a sequence*

$$V' \to V \to V''$$

*of $k$-Banach spaces is exact (and hence admissible) if and only if*

$$V' \widehat{\otimes}_k W \to V \widehat{\otimes}_k W \to V'' \widehat{\otimes}_k W$$

*is exact.*

We omit the proof of this theorem, but all parts follow by using the basis from Lemma 7.6.

7.1.2. *Finite Banach modules.* If $A$ is a Banach ring, a Banach $A$-module $M$ is *finite* if there is an admissible map $A^n \twoheadrightarrow M$ for some $n \geq 0$.

**Theorem 7.8.** *If $A$ is a Noetherian Banach ring (e.g. $A$ is a $k$-affinoid algebra), then:*

(1) *the forgetful functor*

$$\{\text{finite Banach } A\text{-modules}\} \to \{\text{finite } A\text{-modules}\}$$

*is an equivalence of categories,*

(2) *any $A$-linear morphism $M \to N$ between finite Banach $A$-modules is admissible,*

(3) *if $M$, $N$ are finite Banach $A$-modules, then $M \otimes_A N \xrightarrow{\cong} M \widehat{\otimes}_A N$,*

(4) *if $M$ is a finite Banach $A$-module, and $B$ is a Noetherian Banach $A$-algebra, then $M \otimes_A B \xrightarrow{\cong} M \widehat{\otimes}_A B$.*

[5] A NA $k$-Banach space is the *analog* of a Banach space. This result is the substitute for existence of an orthonormal basis.

[6] This was Theorem 4.14.

We omit the proof of this theorem. It is not hard, but takes a little bit of time to write down.

7.1.3. *Ground field extension.* Let $K/k$ be a NA extension. Let $A$ be a Banach $k$-algebra (e.g. a $k$-affinoid algebra) and $A_K = A\widehat{\otimes}_k K$ (if $A$ was a $k$-affinoid algebra, this is a $K$-affinoid algebra).

The map $k \hookrightarrow K$ induces an isometry $A \hookrightarrow A_K$ by Grusan's Theorem 7.7.

**Lemma 7.9.** *If $A$ is a $k$-affinoid, then the map $A \to A_K$ is faithfully flat (in the algebraic sense).*

*Proof.* We know that $A$ and $A_K$ are Noetherian Banach rings (Proposition 6.7). It suffices to show that a sequence

$$M' \to M \to M'' \tag{3}$$

of finite $A$-modules is exact if and only if

$$M' \otimes_A A_K \to M \otimes_A A_K \to M'' \otimes_A A_K \tag{4}$$

is exact.

Since $A$ is a Noetherian Banach ring, there is a unique structure of finite Banach $A$-modules on $M'$, $M$, $M''$ by Theorem 7.8 (1). Define $M_K = M\widehat{\otimes}_k K$. Then

$$M_K \cong M\widehat{\otimes}_A A_K \cong M \otimes_A A_K$$

where the last isomorphism follows from Theorem 7.8 (4) because $A_K$ is Noetherian. (Since $A^n \twoheadrightarrow M$ is amidssible, $A_K^n \twoheadrightarrow M\widehat{\otimes}_A A_K$ is also admissible.)

The analogous statements hold for $M'$ and $M''$.

Admissibility in (3) and (4) is automatic by Theorem 7.8 (2), since all modules are finite.

The result now follows from Grusan's Theorem 7.7 (3). $\square$

7.1.4. *Kernel is surjective.* We can finally prove that $\ker X \to \mathfrak{X} = \mathrm{Spec}(A)$ is surjective if $X = M(A)$ is a $k$-affinoid space.

*Proof of Theorem 7.5.* Recall that we proved this when $X$ is strictly $k$-affinoid (Theorem 5.36), using Noether Normalization Theorem 5.18. We reduce to this case.

In general, pick $k'/k$ such that $k'$ is non-trivially valued and $A' = A\widehat{\otimes}_k k'$ is strictly $k'$-affinoid. Set $X' = M(A')$ and $\mathfrak{X}' = \mathrm{Spec}(A')$. We have a commutative diagram:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\ker'"'] & X \arrow[d, "\ker"] \\ \mathfrak{X} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

By Theorem 5.36, $X' \to \mathfrak{X}'$ is surjective.

Since the map $A \to A'$ is faithfully flat by Lemma 7.9, $\mathfrak{X}' \to \mathfrak{X}$ is surjective. By the commutativity of the diagram, $X \to \mathfrak{X}$ is also surjective. □

7.2. **Properties of the local rings.** Let $X$ be a $k$-affinoid space and $x \in X = M(A)$. Let $\mathfrak{X} = \mathrm{Spec}(A)$ and $\xi = \ker(x) \in \mathfrak{X}$. We will prove the following results:

(1) $\mathcal{O}_{X,x}$ is Noetherian, faithfully flat over $\mathcal{O}_{\mathfrak{X},\xi}$,
(2) $\mathcal{O}_{X,x}$ is Henselian,
(3) $\mathcal{O}_{X,x}$ is excellent.

We will prove (1) and (2) and likely skip (3).

**Theorem 7.10.** *The ring $\mathcal{O}_{X,x}$ is Noetherian and faithfully flat over $\mathcal{O}_{\mathfrak{X},\xi}$.*

The idea of the proof is to:

- first do the case when $X$ is strictly $k$-affinoid at $x \in X^{\mathrm{rig}}$ is a rigid point,
- use ground field extension to reduce to the above case.

7.2.1. *Local rings at rigid points.* Suppose $|k^\times| \neq \{1\}$ and $X$ is strictly $k$-affinoid. Suppose that $x \in X^{\mathrm{rig}}$ (i.e. $[\mathcal{H}(x) : k] < \infty$). It corresponds to a maximal ideal $\mathfrak{m}$ of $A$.

Recall that

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{V \ni x \\ \text{strictly k-affinoid}}} A_V.$$

Fix a strictly $k$-affinoid neighborhood $V$ of $x$ in $X$. The natural map

$$\varphi\colon A \to A_V$$

induces the map

$$\varphi^*\colon \underbrace{M(A_V)}_{Y} \to X,$$

a homeomorphism onto $V$. Note that $\varphi^*(Y^{\mathrm{rig}}) \subseteq X^{\mathrm{rig}}$.

**Lemma 7.11.**

(1) *We have that $Y^{\mathrm{rig}} = (\varphi^*)^{-1}(X^{\mathrm{rig}})$.*
(2) *If $\mathfrak{n} \subseteq A_V$ is a maximal ideal, and $\mathfrak{m} = \varphi^{-1}(\mathfrak{n}) \subseteq A$, then*
    - *$\mathfrak{n} = \varphi(\mathfrak{m})A_V$ ($= \mathfrak{m}A_V$ by definition),*
    - *for all $j \geq 1$, $A \to A_V$ induces $A/\mathfrak{m}^j \xrightarrow{\cong} A_V/\mathfrak{n}^j$.*

*Sketch.* Part (1) follows from $\mathcal{H}(x) \xrightarrow{\cong} \mathcal{H}(\varphi^*(x))$ and the criteria

$$\begin{cases} x \in X^{\mathrm{rig}} & \text{if and only if } [\mathcal{H}(x) : k] < \infty, \\ y \in Y^{\mathrm{rig}} & \text{if and only if } [\mathcal{H}(y) : k] < \infty. \end{cases}$$

For part (2), we use the universal property twice. We start with the commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\pi"] & A_V \arrow[d, "\varpi"] \arrow[dl, dotted, "\alpha"] \\ A/\mathfrak{m}^j \arrow[r, "\sigma"] & A_V/\mathfrak{m}^j A_V \end{tikzcd}$$

The dotted map $\alpha$ such that the top triangle commutes exists by the universal property, because

$$\mathrm{Im}(\pi^*) = \{x\} \subseteq V.$$

A similar argument shows that the bottom triangle commutes (by uniqueness), using:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\sigma\circ\pi"'] & A_V \arrow[d, dotted] \\ & A_V/\mathfrak{m}_j A_V \end{tikzcd}$$

We play around with this to show that $\sigma$ is an isomorphism. Applying this with $j = 1$ gives the first statement. $\square$

Recall that $x \in X^{\mathrm{rig}} \leftrightarrow$ corresponds to a maximal ideal $\mathfrak{m} \subseteq A$.

**Corollary 7.12.** *We have that:*

1. $\mathfrak{m}_x = \mathfrak{m}\mathcal{O}_{X,x}$,
2. $A \to \mathcal{O}_{X,x}$ *factors through the localization* $A_\mathfrak{m}$,
3. $A/\mathfrak{m}^j \overset{\cong}{\to} (A_\mathfrak{m})(\mathfrak{m}A_\mathfrak{m})^j \overset{\cong}{\to} \mathcal{O}_{X,x}/\mathfrak{m}_x^j$ *for all* $j > 1$.

We look at the completion with respect to maximal ideals. (This has a seminorm $r^{\mathrm{ord}_\mathfrak{m}}$ for $0 < r < 1$.) Write

$$\widehat{A}, \widehat{A_\mathfrak{m}}, \widehat{\mathcal{O}_{X,x}}$$

for the completions.

**Corollary 7.13.** *We have that:*

1. $\widehat{A} \overset{\cong}{\to} \widehat{A_\mathfrak{m}} \overset{\cong}{\to} \widehat{\mathcal{O}_{X,x}}$,
2. $A_\mathfrak{m} \hookrightarrow \widehat{A_\mathfrak{m}}$ *and* $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}_{X,x}}$.

*Proof.* Part (1) follows immediately from Corollary 7.12. In part (2), the assertion for $A_\mathfrak{m}$ follows by Krull's intersection theorem since $A_\mathfrak{m}$ is local and Noetherian.

It remains to check the assertion for $\mathcal{O}_{X,x}$. We must prove that if $f \in \mathcal{O}_{X,x}$ satisfies $f \in \mathfrak{m}^j\mathcal{O}_{X,x}$ for all $j \geq 1$ then $f = 0$. We know that $f \in A_V$ for a strictly affinoid neighborhood $V$ of $x$. Without loss of generality, $V = X$ and $A_V = A$. We know that

$$\mathcal{O}_{X,x}/\mathfrak{m}^j\mathcal{O}_{X,x} \cong A/\mathfrak{m}_j$$

for all $j$. Hence $f \in \mathfrak{m}^j$ for all $j$. This shows that $f = 0$ in $A_\mathfrak{m}$ by Krull. Hence $f = 0$ in $\mathcal{O}_{X,x}$. $\square$

**Corollary 7.14.** *The map* $A \to A_V$ *is flat for any strictly affinoid domain* $V \subseteq X$.

*Proof.* It suffices to show that the map $A_{\mathfrak{m}} \to (A_V)_{\mathfrak{m}A_V}$ is flat for every maximal ideal $\mathfrak{m}$. It then suffices (by Noetherianity) to show this on the completions: $\widehat{A_{\mathfrak{m}}} \to \widehat{(A_V)_{\mathfrak{m}A_V}}$ is flat for all $\mathfrak{m}$. This map is an isomorphism by Corollary 7.13. $\square$

**Theorem 7.15.** *For a strictly $k$-affinoid space $X$, $\mathcal{O}_{X,x}$ is Noetherian for $x \in X^{\mathrm{rig}}$.*

*Sketch of proof.* **Step 1.** We have that $\widehat{\mathcal{O}_{X,x}} \cong \widehat{A}$ is Noetherian since $A$ is Noetherian. We also know that $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}_{X,x}}$.

**Step 2.** Every finitely generated ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,x}$ is $\mathfrak{m}_x$-adically closed. (Without loss of generality, $\mathfrak{a} \subseteq A$ and $\mathcal{O}_{X,x}/\mathfrak{a}\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$ separated.)

**Step 3.** Consider a chain $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subseteq \cdots$ of finitely generated ideals of $\mathcal{O}_{X,x}$. We then have a chain $\widehat{\mathfrak{a}_1} \subset \widehat{\mathfrak{a}_2} \subseteq \cdots \subseteq \widehat{\mathcal{O}_{X,x}}$. Since $\widehat{\mathcal{O}_{X,x}}$ is Noetherian, $\widehat{\mathfrak{a}_{m+1}} = \widehat{\mathfrak{a}_m}$ for $m \gg 0$.

Since $\mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}_{X,x}}$ and $\mathfrak{a}_m$ is closed, we have that $\mathfrak{a}_m = \mathcal{O}_{X,x} \cap \widehat{\mathfrak{a}_m}$, so $\mathfrak{a}_{m+1} = \mathfrak{a}_m$ for $m \gg 0$. $\square$

**Remark 7.16.** The fact that the ring $\mathcal{O}_{X,x}$ is faithfully flat over $\mathcal{O}_{\mathfrak{X},\xi} = A_{\mathfrak{m}}$ follows from $\widehat{\mathcal{O}_{X,x}} \cong \widehat{A_{\mathfrak{m}}}$.

We have completed the proof of Theorem 7.10 in the strictly $k$-affinoid, rigid point case.

7.2.2. *General case.* Suppose $X$ is $k$-affinoid and $x \in X$. Consider $\xi = \ker(x) \in \mathfrak{X}$.

We can pick an extension $k'/k$ and a point $x' \in X' \in M(A\widehat{\otimes}_k k')$ mapping to $x \in X$ such that

- $X'$ is strictly $k'$-affinoid,
- $x' \in (X')^{\mathrm{rig}}$.

We have a commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{X',x'} & \mathcal{O}_{X,x} \arrow[l] \\ \mathcal{O}_{\mathfrak{X}',\xi'} \arrow[u] & \mathcal{O}_{\mathfrak{X},\xi} \arrow[l] \arrow[u] \end{tikzcd}$$

We can now complete the proof of Theorem 7.10: $\mathcal{O}_{X,x}$ is Noetherian and $\mathcal{O}_{\mathfrak{X},\xi} \to \mathcal{O}_{X,x}$ is faithfully flat.

*Proof of Theorem 5.4.* By Lemma 7.9, the maps

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$$
$$\mathcal{O}_{\mathfrak{X}',\xi} \to \mathcal{O}_{\mathfrak{X},\xi}$$

are both faithfully flat. Since we know that $\mathcal{O}_{X',x'} \to \mathcal{O}_{\mathfrak{X}',\xi'}$ is faithfully flat by Remark 7.16, this shows that $\mathcal{O}_{\mathfrak{X},\xi} \to \mathcal{O}_{X,x}$ is faithfully flat by the commutative square above.

We just need to show that $\mathcal{O}_{X,x}$ is Noetherian. We use a similar argument as above, using $\mathfrak{a} = \mathfrak{a}\mathcal{O}_{X',x'} \cap \mathcal{O}_{X,x}$ where $\mathfrak{a}$ is a finitely-generated ideal in $\mathcal{O}_{X,x}$. $\square$

## 8. Global Berkovich spaces

Let $k$ be a NA field.

**Definition 8.1.** The *category of $k$-affinoid spaces* consists of:

- objects: $X = M(A)$ where $A$ is a $k$-affinoid algebra,
- morphisms: $M(B) = Y \to X = M(A)$ induced by a map $A \to B$ of $k$-algebras.

It has

- *fiber products*: $Y \times_X Z = M(B \widehat{\otimes}_A C)$,
- *ground field extensions*: for an extension $K/k$, $X_K = M(A \widehat{\otimes}_k K)$.

**Definition 8.2.** A Berkovich space $X$ has a *structure sheaf* $\mathcal{O}_X$. It is defined by

- $\mathcal{O}_X(V) = A_V$ if $A \subseteq X$ is an affinoid domain,
- $\mathcal{O}_X(V) = \ker \left( \prod_i A_{V_i} \to \prod_{i,j} A_{V_i \cap V_j} \right)$ if $V = V_1 \cup \cdots V_n$ is a special domain,
- $\mathcal{O}_X(V) = \varprojlim_{W \subseteq V \text{ special}} \mathcal{O}_X(W)$ for a general open subset $V \subseteq X$.

In this section, we will *glue* together $k$-affinoid spaces to get *global* Berkovich spaces. There are two **different** approaches:

(1) *Locally ringed spaces*: roughly, a Berkovich space is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to $(M(A), \mathcal{O}_{M(A)})$ where $A$ is a $k$-affinoid algebra.
(2) *Atlases*: roughly, a Berkovich space is a topological space $X$ with an atlas of open subsets $U$ such that each $U$ is homemorphic to $M(A)$ for a $k$-affinoid algebra $A$ and the homeomorphisms are compatible on intersections.

**Example 8.3** (Analytification)**.** If $X$ is an algebraic variety, then there is a Berkovich space $X^{\mathrm{an}}$ associated to it (in the sense of (1)).

**Example 8.4.** Consider $X = \{|T_1| = 1\} \cup \{|T_2| = 1\} \subseteq E^2$. This is an analytic space in the sense of (2) but not in the sense of (1).

We will first follow the second approach.

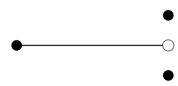
### 8.1. Nets.

We discuss the notion of nets[7].

Let $X$ be a locally Hausdorff space, i.e. a space where every point has a neighborhood which is Hausdorff.

**Example 8.5.** A space which is locally Hausdorff but not Hausdorff is the interval $[0, 1]$ with two copies of 1, i.e.

[7]In the theory of Berkovich spaces, there are two meanings of the word *net*. Sometimes, when one works with Berkovich spaces which are not countable. Then there is a topological notion of a *net*. This is not what we mean here.



There are Berkovich spaces which are only locally Hausdorff, so this is the generality we want to work in.

**Definition 8.6.** A *quasinet* on $X$ is a collection $\tau$ of compact subsets $V \subseteq X$ such that every point $x \in X$ has a neighborhood of the form $V_1 \cup \cdots \cup V_n$ where $V_j \in \tau$ and $x \in V_j$ for all $j$.

**Example 8.7.** If $X = [-1, 1]$, we may take

$$\tau = \{[-1, 0], [0, 1]\}.$$

Even though 0 is not in the relative interior of either of the sets, this set still forms a quasinet.

**Definition 8.8.** A quasinet $\tau$ on $X$ is a *net* if for all $U, V \in \tau$, the collection

$$\tau|_{U \cap V} = \{W \in \tau \mid W \subseteq U \cap V\}$$

is a quasinet on $U \cap V$.

**Example 8.9.** If $X = [-1, 1]$, the quasinet

$$\tau = \{[-1, 0], [0, 1]\}$$

above is not a net, because $\tau|_{[-1,0] \cap [0,1]} = \emptyset$ is not a quasinet on $[-1, 0] \cap [0, 1] = \{0\}$. However, both

$$\tau = \{[-1, 1]\}$$

and

$$\tau = \{[-1, 0], [0, 1], \{0\}\}$$

is a net.

Let $k$ be an NA field and $X$ be locally Hausdorff.

**Definition 8.10.** A *$k$-analytic atlas* $A$ on $X$ consists of:

(1) a net $\tau$ on $X$,
(2) for any $V \in \tau$, a $k$-affinoid algebra $A_V$ with a homeomorphism $V \cong M(A_V)$,
(3) for any $U, V \in \tau$ such that $U \subseteq V$, we have a morphism

$$\alpha_{V/U} \colon A_V \to A_U$$

that identifies $U$ with an affinoid domain in $V$.

**Remark 8.11.** It is automatically ture that if $U \subseteq V \subseteq W$, then

$$\alpha_{W/U} = \alpha_{V/U} \circ \alpha_{W/V}.$$

**Definition 8.12.** The triple $(X, \tau, A)$ is a *$k$-analytic space*.

One can also define a *strictly $k$-analytic space* (at least when $k$ is non-trivially valued).

**Example 8.13.** Suppose $X = M(A)$ is a $k$-affinoid space and let

$$\tau = \{\text{all affinoid domains in } X\}.$$

Then $X$ with the natural atlas is a $k$-analytic space.

**Example 8.14.** Suppose $X = M(A)$ is a $k$-affinoid space and let
$$\tau = \{X\}.$$
Then $X$ with the atlas $A_X = A$ is a $k$-analytic space.

We want these two to be *isomorphic*. We hence must define *morphisms*.

There are several approaches to this as well. The one which is useful in practice is the *maximal atlas* approach. There are two steps:
$$(X, \tau, A) \rightsquigarrow (X, \overline{\tau}, A) \rightsquigarrow (X, \hat{\tau}, A).$$

**Step 1.** Given $\tau$ we define:
$$\overline{\tau} = \{W \subseteq X \text{ compact} \mid W \text{ is an affinoid domain in some } V \in \tau\}$$
It is not completely trivial to check that $(X, \overline{\tau}, A)$ is a $k$-analytic space.

This step will take us from Example 8.13 to Example 8.14. However,
$$\overline{\overline{\tau}} = \overline{\tau},$$
so this will not do anything to Example 8.14).

**Step 2.** We say that $W \subseteq X$ is $\tau$*-special* if $W$ is compact and
$$W = W_1 \cup \cdots \cup W_n$$
such that $W_i, W_i \cap W_j \in \tau$ for all $i, j$ and
$$A_{W_i} \widehat{\otimes} A_{W_j} \twoheadrightarrow A_{W_i \cap W_j}$$
is an admissible epimorphism. We can then check that
$$A_W = \ker\left(\prod_i A_{W_i} \to \prod_{i,j} A_{W_i \cap W_j}\right)$$
is well-defined (independent of the choice of covering). We then get a map $W \to M(A_W)$.

Now, let $\hat{\tau}$ be the collection over $\overline{\tau}$-special subsets $W \subseteq X$ such that

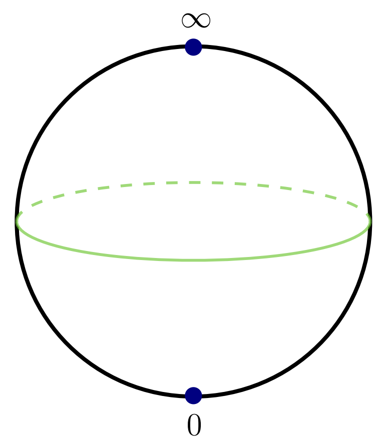
(1) $A_W$ is $k$-affinoid,
(2) $W \overset{\cong}{\to} M(A_W)$,
(3) there is a $\overline{\tau}$-special covering $W = W_1 \cup \cdots \cup W_n$ such that $W_i \subseteq W$ is an affinoid domain for all $i$.

We can check that $(X, \hat{\tau}, A)$ is an atlas and $\hat{\hat{\tau}} = \hat{\tau}$. This is called the *maximal atlas*.

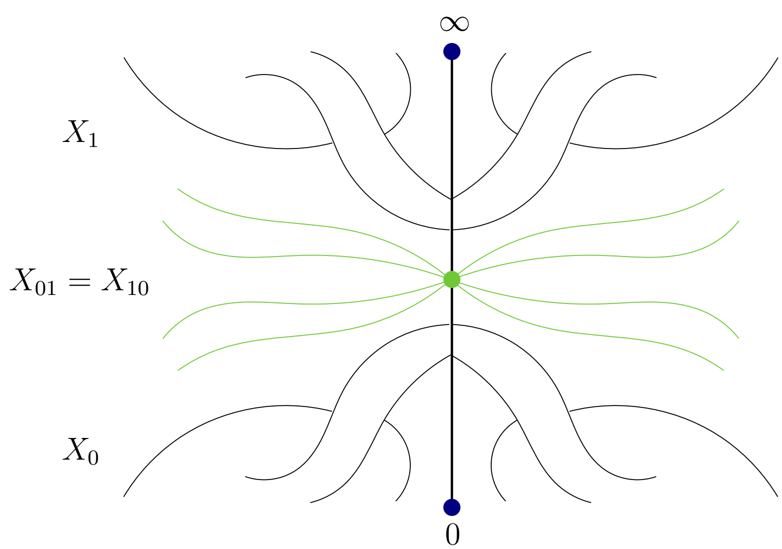
The elements of $\hat{\tau}$ are called *affinoid domains*.

**Example 8.15** (The projective line)**.** As over $\mathbb{C}$, the idea is to glue to copies of the (closed) unit disc along the unit circle.

We describe it as a $k$-analytic space using a (non-maximal) atlas
$$\tau = \{X_1, X_0, X_{10} = X_{01}\}.$$




over $\mathbb{C}$




over $k$

We define

$$\begin{aligned} X_1 &= M(k\{S_1\}) \\ X_0 &= M(k\{S_0\}) \\ X_{10} &= \{|S_1| = 1\} \subseteq X_1 \\ &= M(k\{S_1^{\pm 1}\}) \\ X_{01} &= \{|S_0| = 1\} \subseteq X_0 \\ &= M(k\{S_0^{\pm 1}\}) \end{aligned}$$

where $X_{10}$ is identified with $X_{01}$ by

$$\begin{aligned} X_{10} &\xrightarrow{\cong} X_{01}, \\ S_1 &\mapsto S_0^{-1}. \end{aligned}$$

In this example,

$$\overline{\tau} = \{\text{affinoid domains contained in } X_0 \text{ or } X_1\}$$
$$\hat{\tau} \text{ contains, for example, } \{1.2 \leq |S_1| \leq 2\}$$

**Example 8.16.** One can glue $n+1$ copies of the unit polydisc to construct $\mathbb{P}^n_k$.

**Example 8.17** (Affine space $\mathbb{A}^n_k$)**.** At the beginning of the class, we discuss the Berkovich affine line (Example 1.11). In general, one can similarly define the general affine space and check that it is in fact a Berkovich space with the above definition.

As a **set**,

$$\mathbb{A}^n_k = \{\text{semivaluations on} k[T] = k[T_1, \ldots, T_n] \text{ that extend the valuation on } k\}$$

The **topology** is the weakest so that

$$x \mapsto |f(x)|$$

is continuous for all $f \in k[T]$.

A **net** is given by

$$\tau = \{E(r) \mid r \in (\mathbb{R}_+^\times)^n\}$$

where

$$E(r) = \{x \in \mathbb{A}^n_k \mid |T_i(x)| \leq r_i\}.$$

The **atlas** is given by

$$A_{E(r)} = k\{r^{-1}T\}.$$

A picture was provided in Example 1.11. The maximal atlas $\hat{\tau}$ consists of all compact subsets of $\mathbb{A}^1_k$ that are affinoid domains in some $E(r)$.

## 8.2. **Morphisms.** There are two approaches to this:

(1) localization,
(2) maximal atlases.

### 8.2.1. *Localization approach.*

**Definition 8.18.** A *strong morphism* $\varphi\colon (X, \tau, A) \to (X', \tau', A')$ is a continuous map

$$\varphi\colon X \to X'$$

such that:

- for any $V \in \tau$, there exists $V' \in \tau'$ such that $\varphi(V) \subseteq V'$,
- whenever $\varphi(V) \subseteq V'$, there exists a morphism $A_{V'} \to A_V$,
- the morphisms are compatible:

$$\begin{array}{ccc} U & \longrightarrow & U' \\ \downarrow & & \downarrow \\ V & \longrightarrow & V \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} A_U & \longleftarrow & A'_{U'} \\ \uparrow & & \uparrow \\ A_V & \longleftarrow & A'_{V'} \end{array}$$

We can check that one can compose strong morphisms.

However, we will eventually have to allow a weaker notion of a morphism. Specifically, the first condition above needs to be relaxed.

**Definition 8.19.** A strong morphism $\varphi$ is a *quasi-isomorphism* if

- $\varphi\colon X \to X'$ is a homeomorphism,
- whenever $\varphi(V) \subseteq V'$,
$$\varphi_{V/V'}\colon (V, A_V) \to (V', A_{V'})$$
identifies $V$ with an affinoid domain in $V'$.

**Definition 8.20.** The category of $k$-analytic spaces is the localization of the category of $k$-analytic spaces with strong morphisms with respect to quasi-isomorphisms.

In other words, we formally *invert* quasi-isomorphisms in this category.

**Example 8.21.** If $X = M(A)$ is $k$-affinoid, then the strong morphism
$$(X, \{X\}, A) \to (X, \{\text{all affinoid domains}\}, A)$$
is a quasi-isomorphism.

8.2.2. *Maximal atlases approach.* A *morphism* $(X, \tau, A) \xrightarrow{\varphi} (X', \tau', A')$ where $\tau$ and $\tau'$ are maximal atlases is

- a continuous map $\varphi\colon X \to X'$ such that for any $x \in X$, there exist $V_1, \dots, V_n \in \tau$ and $V_1', \dots, V_m' \in \tau'$ such that
  - $x \in V_1 \cap \cdots \cap V_n$,
  - $x' = \varphi(x) \in V_1' \cap \cdots \cap V_m'$,
  - $V_1 \cup \cdots V_n$ is a neighborhood of $x$ in $X$,
  - $V_1' \cup \cdots V_m'$ is a neighborhood of $x'$ in $X'$,
  - $\varphi(V_i) \subseteq V_i'$,
- a system of compatible morphisms
$$(V, A_V) \to (V', A'_{V'})$$
whenever $V \in \tau$, $V' \in \tau'$ and $\varphi(V) \subseteq V'$.

Furthermore, $\varphi$ is an *isomorphism* if and only if

- $\varphi$ is a homeomorphism $X \to X'$,
- $\varphi(\tau) = \tau'$,
- for all $V \in \tau$, $(V, A_V) \xrightarrow{\cong} (V', A'_{V'})$ where $V' = \varphi(V)$.

**Remark 8.22.** One can check that the functor
$$k\text{-Aff} \to k\text{-An}$$
$$X = M(A) \mapsto (X, \{X\}, A)$$
is fully faithful.

**Definition 8.23.** A $k$-analytic space is *good* if every point has an affinoid neighborhood.

**Example 8.24.** Any $k$-affinoid space is good.

**Example 8.25.** The $k$-analytic spaces $\mathbb{A}_n^k$, $\mathbb{P}_n^k$ and, more generally, any analytification of a variety are all good.

**Example 8.26.** The space $X = \{|T_1| = 1\} \cup \{|T_2| = 1\} \subseteq E^2(1)$ is not good.

8.3. **Analytic domains.**

**Definition 8.27.** A subset $Y$ of a $k$-analytic space $X$ is *analytic* if $\hat{\tau}|_Y$ is a net on $Y$. Equivalently, for any $y \in Y$, there exist affinoid domains $V_1, \dots, V_n$ in $X$ such that:

- $y \in V_i \subseteq Y$ for all $i$,
- $V_1 \cup \dots \cup V_n$ is a neighborhood of $y$ in $Y$.

One sometimes says that we have a *G-covering of* $Y$ (see below).

**Remark 8.28.**

(1) Affinoid domains are analytic.
(2) If $Y_1$, $Y_2$ are analytic, then also $Y_1 \cap Y_2$ is analytic.
(3) If $X' \xrightarrow{\varphi} X$ is a morphism and $Y \subseteq X$ is analytic, then $\varphi^{-1}(Y) \subseteq X'$ is also analytic.
(4) If $Y \subseteq X$ is an analytic domain, then $(Y, \hat{\tau}|_Y, A)$ is also a $k$-analytic space. The map $Y \hookrightarrow X$ has a universal property:

$$\begin{tikzcd} Y \arrow[rr, hook] & & X \\ & Z \arrow[ul, dotted] \arrow[ur] & \end{tikzcd}$$

as the universal property of open immersions.

8.4. *G*-**topology and the structure sheaf.** Any $k$-analytic space carries a G-topology (Grothendieck topology):

- admissible open sets are analytic domains,
- admissible coverings (G-coverings) are quasinets.

We define the structure sheaf $\mathcal{O}_{X_G}$ in the $G$-topology. We define

$$\mathcal{O}_{X_G}(Y) = \mathrm{Hom}_{k\text{-An}}(Y, \mathbb{A}_k^1),$$

**Remark 8.29.** If $Y = V$ is an affinoid domain, then

$$\mathrm{Hom}(Y, A_k^1) = A_V.$$

Hence this definition is consistent with the previous definition.

If $X$ is good, we can restrict $\mathcal{O}_{X_G}$ to open sets to get a sheaf $\mathcal{O}_X$ given by

$$\mathcal{O}_X(Y) = \mathrm{Hom}(Y, \mathbb{A}_k^1)$$

for $Y \subseteq X$ open in the usual topology.

**Fact 8.30** (Ducros)**.** *We have that:*

- $\mathcal{O}_{X_G}$ *is coherent,*
- $\mathcal{O}_X$ *is coherent if* $X$ *is good.*

8.5. **Gluing.** The idea is that we can glue a family $(X_i)_{i\in I}$ of analytic spaces along analytic domains $X_{ij} \overset{\cong}{\to} X_{ji}$ in two cases:

(1) if $X_{ij} \subseteq X_i$ is open for all $i, j$,
(2) if $X_i \subseteq X$ is closed and for all $i$, we have that $X_{i,j} = \emptyset$ almost all $j$.

The properties of the resulting space $X$:

(1) $X_i \subseteq X$ is open for all $i$,
(2) $X_i \subseteq X$ is closed for all $i$; if the $X_i$ are Hausdroff, so is $X$.

We omit the details of this construction.

**Example 8.31** (The projective space, first construction)**.** The idea is to glue copies of $E^n$ to get $\mathbb{P}^n_k$ for $0 \le i \le n$. Specifically:

$$\begin{aligned} X_i &= M(k\{S_0, \ldots, \widehat{S}_i, \ldots, S_n\}), \\ X_{ij} &= \{|S_j| = 1\} && \text{for } i \neq j. \end{aligned}$$

For example, $X_{i0} = M(k\{S_0^{\pm 1}, S_1, \ldots, \hat{S}_i, \ldots, S_n\})$. Moreover:

$$\begin{aligned} X_{ij} &\overset{\cong}{\to} X_{ji} \\ S_j &\mapsto S_i^{-1} \\ S_\ell &\mapsto S_\ell && \text{for } \ell \neq i, j. \end{aligned}$$

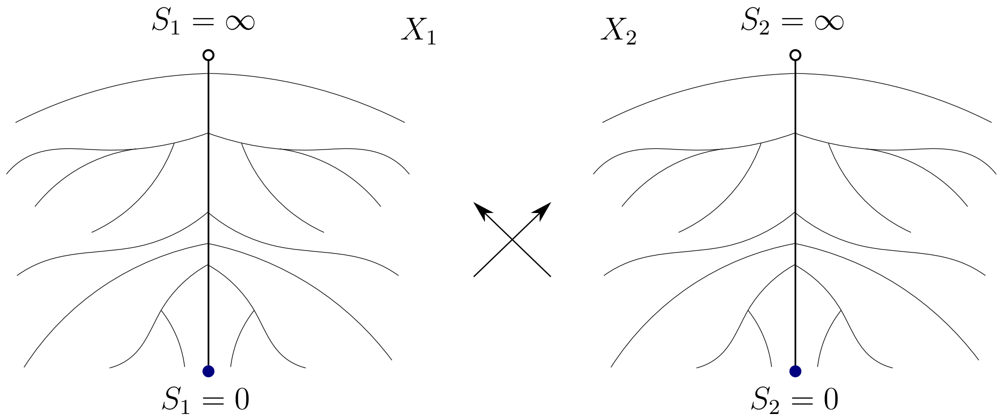
This is exactly what we did in Example 8.15 in the case $n = 1$. A sketch of the projective space and the gluing was shown there.

**Example 8.32** (The projective space, second construction)**.** We glue $n+1$ copies of affine space $\mathbb{A}^n_k$. Specifically:

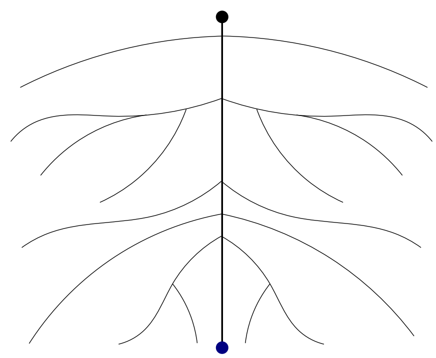
$$\begin{aligned} X_i &= \{\text{semivaluations on } k[T_0, \ldots, \widehat{T}_i, \ldots, T_n] \text{ extending the valuation on } k\} && \text{for } 0 \le i \le n, \\ X_{ij} &= \{S_i \neq 0\} && \text{for } i \neq j \\ &\subseteq X_i && \text{open.} \end{aligned}$$

As before, the identification $X_{ij}$ with $X_{ji}$ is given by $S_i \mapsto S_j^{-1}$.

When $n = 1$, the picture of $X_0$ and $X_1$ is as follows.




They glue together to $\mathbb{P}^1_n$:



**Example 8.33.** Consider a trivially valued field $k$, i.e. $|k^\times| = \{1\}$. For $0 < r < 1$, we glue two copies of
$$E(r) = M(k\{r^{-1}T\}) \cong [0, r]$$
along the open subsets $\{T < r\} \cong [0, r)$. We get:

i.e. Example 8.5 of a space which is locally Hausdorff but not Hausdorff.

8.6. **Fiber products.** Given $Y \to X$ and $X' \to X$, we can define *fiber products* as a pullback in the category of $k$-analytic spaces:

$$\begin{tikzcd} Y \times_X X' \arrow[r] \arrow[d] & X' \\ Y \arrow[r] & X. \end{tikzcd}$$

**Example 8.34.** We have that $\mathbb{A}^m_k \times_{M(k)} \mathbb{A}^n_k = \mathbb{A}^{m+n}_k$.

In the affinoid case, $Y \times_X X' = M(B \widehat{\otimes}_A A')$. In general, we use gluing to construct the fiber products.

8.7. **Ground field extensions.** Given a $k$-analytic space and a NA extension $K/k$, we get a $K$-analytic space $X_K$.

In the affinoid case, if $X = M(A)$, then $X_K = M(A \widehat{\otimes}_k K)$. In general, we construct them by gluing.

8.8. **Complete residue field.** Given $x \in X$, we may associate to it the *complete residue field* $\mathcal{H}(x)$. We may define it by choosing an affinoid neighborhood $x \in V \subseteq X$ and setting

$$\mathcal{H}(x) = \mathcal{H}_V(x).$$

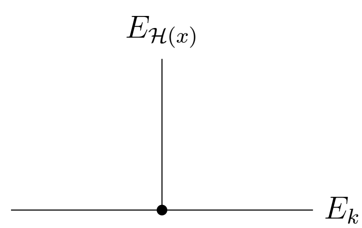
8.9. **Fibers.** Given $\varphi \colon Y \to X$ and $x \in X$, we may view the *fiber* $\varphi^{-1}(x)$ as an $\mathcal{H}(x)$-analytic space. Indeed,

$$\varphi^{-1}(x) = Y_{\mathcal{H}(x)} \times_{X_{\mathcal{H}(x)}} M(\mathcal{H}(x)).$$

**Example 8.35.** Consider the projection onto the first coordinate $\pi \colon E_k^2 \to E_k$. Then

$$\pi^{-1}(x) \cong E_{\mathcal{H}(x)}$$

A very schematic picture is:




8.10. **Finite morphisms and closed immersions.** In the affinoid case, consider a morphism of $k$-affinoid spaces $M(B) \to M(A)$.

We say that $\varphi$ is a $\begin{cases} \text{closed immersion} \\ \text{finite} \end{cases}$ if the map $A \to B$ is $\begin{cases} \text{an admissible epimorphism} \\ \text{is finite.} \end{cases}$
Recall that $A \to B$ is finite if there exists an admissible epimorphism $A^n \twoheadrightarrow B$.

**Proposition 8.36.** *If $\varphi \colon Y \to X$ is a morphism of $k$-analytic spaces, the following are equivalent:*

(1) *for any affinoid domain $V \subseteq X$, the induced map $\varphi^{-1}(V) \to V$ is a finite morphism (respectively, closed immersion) of $k$-affinoid spaces,*
(2) *for any $x \in X$, there exist affinoid domains $V_1, \ldots, V_n \subseteq X$ such that $x \in V_i$ for all $i$, $V_1 \cup \cdots \cup V_n$ is a neighborhood of $x$ in $X$, and $\varphi^{-1}(V_i) \to V_i$ is a finite morphism (respectively, closed immersion) of $k$-affinoid spaces for all $i$.*

**Definition 8.37.** If $\varphi$ satisfies the equivalent properties in the proposition, we say it is *finite* (repsectively, a *closed immersion*).

*Sketch of Proof.* It is clear that (1) implies (2). We give a sketch of the proof that (2) implies (1). The set

$$\tau = \{V \subseteq X \text{ affinoid domain such that } \varphi^{-1}(V) \to V \text{ is finite}\}$$

is a net. Now, suppose $V \subseteq X$ is an affinoid domain. We must show that $\varphi^{-1}(V)$ is an affinoid domain and $A_V \to A_{\varphi^{-1}(V)}$ is finite.

For some $V_i \in \tau$, we have that $V \subseteq V_1 \cup \cdots V_n$. Then

$$V = U_1 \cup \cdots U_m$$

for some affinoid domains $U_i \subseteq V_{j(i)}$ for some $j(i)$. We construct $A_{\varphi^{-1}(V)}$ using Kiehl's Theorem 8.38. □

**Theorem 8.38** (Kiehl)**.** *Suppose:*

- $X = M(A)$,
- $X = V_1 \cup \cdots \cup V_n$ *for affinoid domains* $V_i \subseteq X$,
- *for each* $i$*, we have a finite (Banach)* $A_{V_i}$ *module* $M_i$,
- *for all* $i, j$*, we have isomorphisms* $\alpha_{i,j} \colon M_i \otimes_{A_{V_i}} A_{V_i \cap V_j} \overset{\cong}{\to} M_j \otimes_{A_{V_j}} A_{V_i \cap V_j}$ *such that:*

$$\alpha_{ij}|_W = \alpha_{j\ell}|W \circ \alpha_{\ell i}|W \quad \textit{for every } i, j, \ell, \textit{ where } W = V_i \cap V_j \mathcal{V}_\ell.$$

*Then there is a finite* $A$*-module* $M$ *that induces the* $M_i$ *and the maps* $\alpha_{i,j}$.

The proof of this theorem is similar to Tate's Acyclicity Theorem 6.22: one reduces to a simple covering by two Laurent domains of a special type.

**Definition 8.39.** A map $\varphi \colon Y \to X$ is a *locally closed immersion* if every point $x \in X$ has an open neighborhood $U \ni x$ such that $\varphi^{-1}(U) \to U$ is a closed immersion.

**Definition 8.40.** A map $Y \to X$ is *seperated* (respectively, *locally separated*) if the diagonal map $Y \to Y \times_X Y$ is a closed immersion (respectively, locally closed immersion).

**Proposition 8.41.** *For* $\varphi \colon Y \to X$,

1. *if* $\varphi$ *is separated, the associated map* $\varphi \colon |Y| \to |X|$ *between the underlying topology spaces is Hausdorff,*
2. *the converse is true if* $\varphi$ *is locally separated.*

A map $\varphi \colon Y \to X$ between topological spaces is *Hausdorff* if for any $y_1, y_2 \in Y$ such that $\varphi(y_1) = \varphi(y_2) \in X$, there exist open subsets $U_i \ni y_i$ such that $U_1 \cap U_2 = \emptyset$.

We omit the proof of this proposition.

8.11. **Proper morphisms.** What does it mean for a map $\varphi \colon Y \to X$ to be *proper*?

- In algebraic geometry, a morphism is proper if it is separated and universally closed.
- In complex analysis, a morphism is proper if the preimage of any compact set is compact.
- For Berkovich spaces, we will try to replicate the complex-analytic definition.

**Definition 8.42.** Let $\varphi\colon Y \to X$ be a continuous map of topological spaces. Then $\varphi$ is compact if and only if the preimage of any compact subset of $X$ is compact in $Y$.

**Example 8.43.** The spaces $X = E^1_k, \mathbb{P}^1_k$ are compact, i.e. the maps $X \to M(k)$ are compact.

We want:

- the projective spaces $\mathbb{P}^1_k$ to be proper,
- the disk $E^1_k$ should not be proper; it has boundary.

The following definition is subject to introducing the notion of *relative boundary.*

8.11.1. *Relative interior and boundary.*

**Definition 8.44.** A morphism $\varphi\colon Y \to X$ of $k$-analytic spaces is *proper* if:

(1) $\varphi$ is compact,
(2) $\varphi$ is *boundaryless*, i.e. $\partial(Y/X) = \emptyset$.

Given a morphism $Y \to X$ of $k$-analytic spaces, we want to define:

- $\operatorname{Int}(Y/X) \subseteq Y$ *relative interior*,
- $\partial(Y/X) \subseteq Y$ *relative boundary.*

When $X = M(k)$, we simply write $\operatorname{Int}(Y)$ and $\partial(Y)$.

We first define these notions when $X$, $Y$ are $k$-affinoid. Let us first discuss what properties we want these sets to have.

**Desirable features**:

- $\operatorname{Int}(Y/X) \subseteq Y$ is open and $\partial(Y/X) = Y \setminus \operatorname{Int}(Y/X)$ is closed.
- $\partial(Y/X) = \emptyset$ when $Y \to X$ is a closed immersion or a finite morphism.
- $\partial Y \neq \emptyset$ if $\dim Y > 0$ (although we have not really defined $\dim Y$).
- Transitivity: if $Z \xrightarrow{\psi} Y \to X$ are morphisms, then
$$\operatorname{Int}(Z/X) = \operatorname{Int}(Z/Y) \cap \psi^{-1}(\operatorname{Int}(Y/X)).$$
- As a consequence: if $\varphi\colon Y \to Y$ is an automorphism, then
$$\varphi^{-1}(\operatorname{Int}(Y)) = \operatorname{Int}(Y),$$
$$\varphi^{-1}(\partial(Y)) = \partial(Y).$$

Let us now discuss a motivating example.

**Example 8.45.** Consider $Y = E_k = M(k\{T\})$. What is $\partial Y$?

**Guess 1.** $\partial Y = \{|T| = 1\} \subseteq Y$. However, this is not invariant under $T \mapsto T + a$ for $a \in k^\circ \setminus k^{\circ\circ}$.

**Guess 2.** $\partial Y = \{\text{Gauss points}\}$. This is correct!

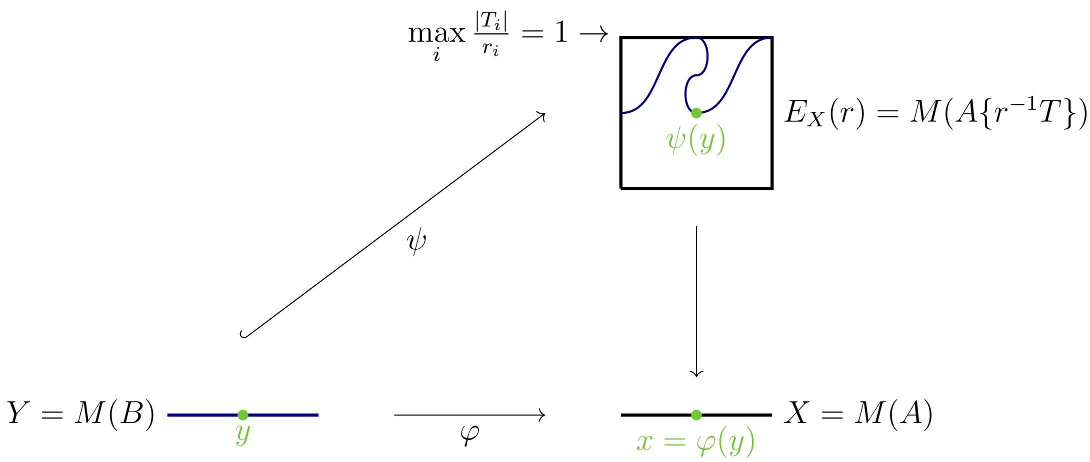
**Definition 8.46.** Consider $\varphi\colon Y \to X$ induced by a map $A \to B$. Then $y \in \operatorname{Int}(Y/X)$ if and only if there is an admissible epimorphism

$$A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow B$$

such that

$$|T_i(\psi(x))| < r_i \quad \text{for } 1 \leq i \leq n.$$

This is illustrated by the following picture:




**Definition 8.47.** The *interior* of $\varphi\colon Y \to X$ is $\operatorname{Int}(Y/X)$. The *relative boundary* of $\varphi\colon Y \to X$ is $\partial(Y/X) = Y \setminus \operatorname{Int}(Y/X)$.

It is clear that the interior $\operatorname{Int}(Y/X)$ is open in $Y$. Therefore, $\partial(Y/X)$ is open in $Y$.

**Example 8.48.** Suppose $k$ is algebraically closed. Let $Y = E_k$ and $X = M(k)$. If $y$ is not the Gauss point, then there exists $a \in k^\times$ such that $|(T-a)(y)| < 1$. We can hence pick $\psi\colon Y \to E_k$ induced by $T \mapsto T - a$ to check that$\partial Y = \{\text{Gauss point}\}$.

**Proposition 8.49.** *Consider $\varphi\colon Y \to X$ associated to a map $A \to B$. Let $y \in Y$. Then the following are equivalent:*

(1) $y \in \operatorname{Int}(Y/X)$,
(2) *for any morphism $A\{r^{-1}T\} \to B$ extending $A \to B$, we can find*

$$P = T^m + a_1T^{m-1} + \cdots + a_m \in A[T]$$

*such that*
- $\varphi(a_i) < r^i$ *for* $i = 1, \dots, m$,
- $|P(\psi(y))| < r^m$, *where $\psi\colon Y \to E_X(r)$ is induced by $A\{r^{-1}T\} \to B$.*

*Moreover, if $X$, $Y$ are strictly $k$-affinoid, then (1) and (2) are also equivalent to*

(3) *the ring $\widetilde{\chi}_y(\widetilde{B})$ is finite over $\widetilde{\chi}_x(\widetilde{A})$, where:*

$$\begin{array}{ccc} B & \xrightarrow{\chi_y} & \mathcal{H}(y) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\chi_x} & \mathcal{H}(x) \end{array} \qquad \begin{array}{ccc} \widetilde{B} & \xrightarrow{\widetilde{\chi}_y} & \widetilde{\mathcal{H}}(y) \\ \uparrow & & \uparrow \\ \widetilde{A} & \xrightarrow{\widetilde{\chi}_x} & \widetilde{\mathcal{H}}(x) \end{array}$$

**Remark 8.50.** There is a version of (3) for general $k$-affinoid spaces. It uses graded reduction.

**Remark 8.51.** If $Y = E_k$, $X = M(k)$, then (2) says that the image of $y$ under $\psi\colon E_k \to E_k(r)$ is not equal to the Gauss point.

8.11.2. *Reduction map and the interior.* Let $X = M(A)$ be a strictly $k$-affinoid space. Consider

$$\widetilde{X} = \operatorname{Spec}(\widetilde{A}).$$

We have a reduction map

$$\operatorname{red}\colon X \to \widetilde{X}$$

defined by: $\chi_x\colon A \to \mathcal{H}(x)$ induces $\widetilde{\chi_x}\colon \widetilde{A} \to \widetilde{\mathcal{H}}(x)$; then $\operatorname{red} x = \ker(\widetilde{\chi_x})$, which is a prime ideal.

**Theorem 8.52** (Berkovich, Tate)**.**

- *The reduction map* $\operatorname{red}\colon X \to \widetilde{X}$ *is surjective.*
- *The reduction map* $\operatorname{red}\colon X^{\mathrm{rig}} \to \widetilde{X}^{\mathrm{cl}}$ *is surjective.*

**Corollary 8.53.** *If $Y = M(B)$ is strictly $k$-affinoid, then $y \in \operatorname{Int}(Y)$ if and only if* $\operatorname{red}(y) \in \widetilde{Y}$ *is a closed point.*

*Proof.* Here, $A = k$, $\widetilde{A} = \widetilde{k}$. Using Proposition 8.49 (3), $y \in \operatorname{Int}(Y)$ if and only if $\widetilde{\chi}_y(\widetilde{B})$ is finite over $\widetilde{k}$. By nullstellensatz, this is equivalent to $\ker(\widetilde{\chi}_y)$ being a maximal ideal. $\square$

**Example 8.54.** Consider $Y = E_k$. Then the branches coming down from the Gauss point are in '1-1' correspondence with the closed points of $\mathbb{A}^1_{\widetilde{k}}$.

8.11.3. *Some further properties.* We have that:

(1) $\operatorname{Int}(Y/X) = Y$ if and only if the morphism $Y \to X$ is finite,
(2) if $Y \hookrightarrow X$ is an affinoid domain, then $\operatorname{Int}(Y/X)$ is equal to the topological interior,
(3) if $Z \xrightarrow{\psi} Y \to X$ are maps, then $\operatorname{Int}(Z/X) = \operatorname{Int}(Z/Y) \cap \psi^{-1}(\operatorname{Int}(Y/X))$,
(4) if $X = \bigcup_i X_i$ is a finite covering by affinoid domains and $Y_i = \varphi^{-1}(X_i)$, then

$$\partial(Y/X) = \bigcup \partial(Y_i/X_i).$$

8.11.4. *Global versions.*

**Definition 8.55.** If $\varphi\colon Y \to X$ is a morphism of $k$-analytic spaces, then $y \in \operatorname{Int}(Y/X)$ if for every affinoid domain $U \subseteq X$ such that $x = \varphi(y) \in U$, there exists an affinoid domain $V \subseteq \varphi^{-1}(U)$ such that $y \in \operatorname{Int}(V/U)$.

We call $\operatorname{Int}(Y/X)$ the *relative interior* and $\partial(Y/X) = Y \setminus \operatorname{Int}(Y/X)$ the *relative boundary.*

**Definition 8.56.** If $\partial(Y/X) = \emptyset$, the morphism $Y \to X$ is *boundaryless* or *closed*.

**Remark 8.57.** The properties (1)–(4) above show that the global definition is compatible with the local ones. Also, some versions of (2)–(4) still hold globally.

**Examples 8.58.** Both $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are boundaryless (where $X = M(k)$).

We can finally state the *proper* definition of proper morphisms.

**Definition 8.59.** A morphism $Y \to X$ of $k$-analytic spaces is *proper* if it is compact and boundaryless.

Note the a proper morphism is separated.

**Example 8.60.** Glue to copies of $M(k\{S\})$, $M(k\{T\})$ of $E$ along the unit circle $\{|S| = 1\} = \{|T| = 1\}$.

If we use the identification $S = T^{-1}$, we get $\mathbb{P}^1_k$. If we use $S = T$, we get a Hausdorff space $X$ which is not separated and not *good*.

## 9. Analytification

There are 3 sources of $k$-analytic spaces:

- gluing,
- analytification,
- formal schemes over the special fiber.

We already discussed the first and we now discuss the second. We will discuss the last one towards the end of the class.

Given a scheme $\mathfrak{X}$ over $k$ of finite time, we want to construct a $k$-analytic space $\mathfrak{X}^{\mathrm{an}}$ associated to $\mathfrak{X}$.

As a **set**,

$$\mathfrak{X}^{\mathrm{an}} = \{(\xi, x) \mid \xi \in \mathfrak{X},\ |\cdot|_x \text{ valuation on } \kappa(\xi) \text{ extending the valuation on } k\}.$$

We have a kernel map

$$\begin{aligned} \ker\colon \mathfrak{X}^{\mathrm{an}} &\to \mathfrak{X}, \\ (\xi, x) &\mapsto \xi. \end{aligned}$$

If $\mathfrak{X} = \mathrm{Spec}(A)$, then

$$\mathfrak{X}^{\mathrm{an}} = \{\text{semivaluations on } A \text{ extending valuations on } k\}.$$

For example, when $\mathfrak{X} = \mathbb{A}^n_k$ so $A = k[T_1, \dots, T_n]$, we have already seen this as the definition of $(\mathbb{A}^n_k)^{\mathrm{an}}$ although we denoted it simply $\mathbb{A}^n_k$.

Alternatively, $\mathfrak{X}^{\mathrm{an}}$ can be defined as:

$$\mathfrak{X}^{\mathrm{an}} = \bigcup_{K/k} \mathfrak{X}(K)/\sim$$

where $K/k$ is a NA extension and $x' \sim x''$ for $x' \in \mathfrak{X}(K')$, $x'' \in \mathfrak{X}(K'')$ if and only if $x'$ and $x''$ come from some point in $\mathfrak{X}(K)$ via $K \hookrightarrow K', K''$.

We endow $\mathfrak{X}^{\mathrm{an}}$ with the weakest **topology** such that

- $\ker\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}$ is continuous,
- for any open affine $U = A \subseteq \mathfrak{X}$ and any $f \in A$,
$$U^{\mathrm{an}} \ni x \mapsto |f(x)| \in \mathbb{R}_+$$
is continuous.

Finally, we discuss the $k$**-analytic space** structure on $\mathfrak{X}^{\mathrm{an}}$.

**Step 1.** Suppose $\mathfrak{X} = \mathbb{A}^n_k$. We have seen that $\mathfrak{X}^{\mathrm{an}}$ is:

- as a topological space, as above,
- the net is given by polydiscs,
- the atlas is given by $A_{E(r)} = k\{r^{-1}T\}$.

**Step 2.** Suppose $\mathfrak{X} = \mathrm{Spec}(A)$ where $A$ is a finitely-generated $k$-algebra:
$$A = k[T_1, \ldots, T_n]/I.$$

- As a topological space, $\mathfrak{X}^{\mathrm{an}}$ is as above.
- The net is given by intersecting with $E^n(r)$,
- The atlas is given by $A_{E^n(r)\cap\mathfrak{X}^{\mathrm{an}}} = k\{r^{-1}T\}/Ik\{r^{-1}T\}$.

One check that this is a functor: given $\mathfrak{X} \to \mathfrak{Y}$, we get an associated morphism $\mathfrak{X}^{\mathrm{an}} \to \mathfrak{Y}^{\mathrm{an}}$. It takes:
$$\{\text{affine } k\text{-schemes of finite type}\} \to \{\text{boundaryless } k\text{-analytic spaces}\},$$
$$\{\text{open immersions}\} \to \{\text{open immersions}\}.$$

**Step 3.** Suppose $\mathfrak{X}$ is a general scheme of finite type. Then $\mathfrak{X}^{\mathrm{an}}$ is constructed by gluing.

The end results is that $\mathfrak{X}^{\mathrm{an}}$ is a good, boundaryless $k$-analytic space. This construction is functorial:
$$\{k\text{-schemes of finite type}\} \to \{k\text{-analytic spaces}\}.$$

**Question.** Is the assignment $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{an}}$ canonical?

**Answer.** Yes, it represents a functor. Given a good $k$-analytic space $Y$, consider
$$F_{\mathfrak{X}}(Y) = \{\text{morphisms } (Y, \mathcal{O}_Y) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})\}.$$
Then $\mathfrak{X}^{\mathrm{an}}$ represents the functor $F_{\mathfrak{X}}(Y)$:
$$F_{\mathfrak{X}}(Y) = \mathrm{Hom}(Y, \mathfrak{X}^{\mathrm{an}}).$$
In fact, this is also try when $Y$ is $K$-analytic for a NA extension $K$ over $k$.

The kernel $\ker\colon \mathfrak{X}^{\mathrm{an}} \to \mathfrak{X}$ is:

- surjective (cf. Theorem 7.5),
- the map $\mathcal{O}_{\mathfrak{X},\xi} \to \mathcal{O}_{\mathfrak{X}^{\mathrm{an}},x}$ where $\xi = \ker(x)$ is faithfully flat (cf. Theorem 7.10).

Also, for any finite extension $K$ of $k$, we have that
$$\mathfrak{X}^{\mathrm{an}}(K) = \mathfrak{X}(K).$$

9.1. **The Proj construction.** Consider a graded ring $A = \bigoplus_{d\geq 0} A_d$ which is a finitely-generated $k$-algebra. Recall the Proj construction from algebraic geometry:
$$\mathrm{Proj}(A) = \{\text{homogenous prime ideals of } A\}.$$

**Definition 9.1.**

(1) A *seminorm* on $A$ is *homogeneous* if
$$\left|\sum_d a_d\right| = \max_d |a_d|$$
where $a_d \in A_d$.

(2) Two homogeneous semivaluations $|\cdot|, |\cdot|'$ are *equivalent* if there exists $C > 0$ such that
$$|f'| = |f| \cdot C^d$$
for $f \in A_d$.

(3) The *trivial* homogeneous valuation is $|\cdot| \equiv 0$ on $\bigoplus_{d>0} A_d$.

Then:
$$\mathrm{Proj}(A)^{\mathrm{an}} = \left\{ \substack{\text{equivalence classes of non-trivial} \\ \text{homogeneous semivaluations on } A} \right\}.$$

This gives an alternative description of the $k$-analytic space $\mathbb{P}^n_k$ as:
$$\mathbb{P}^n_k = (\mathbb{A}^{n+1}_k \setminus \{0\})/\sim$$
where semivaluations $|\cdot|, |\cdot|'$ on $k[T_0, \dots, T_n]$ are equivalent, $|\cdot| \sim |\cdot|'$, if and only if there exists $C > 0$ such that $|f|' = C^d|f|$ for all homogeneous polynomials $f$ of degree $d$.

9.2. **The GAGA principle.** Consider a morphism $\varphi\colon \mathfrak{Y} \to \mathfrak{X}$ of $k$-schemes of finite type. It has an analytification
$$\varphi^{\mathrm{an}}\colon \mathfrak{Y}^{\mathrm{an}} \to \mathfrak{X}^{\mathrm{an}}.$$
We will write $Y = \mathfrak{Y}^{\mathrm{an}}$, $X = \mathfrak{X}^{\mathrm{an}}$.

**Theorem 9.2.** *The morphisms $\varphi$, $\varphi^{\mathrm{an}}$ have the following properties simultaneously,*

| | |
|---|---|
| (1) *flat* | [in the locally ringed sense], |
| (2) *unramified* | [$Y_x = M(K)$ where $K$ is a finite separated $\mathcal{H}(x)$-algebra], |
| (3) *étale* | [unramified and flat], |
| (4) *smooth* | [flat with non-empty fibers of dimension $n$], |
| (5) *separated* | [$Y \to Y \times_X Y$ is a closed immersion], |
| (6) *injective, surjective* | |
| (7) *isomorphism,* | |
| (8) *finite,* | |
| (9) *proper.* | |

The proof of this theorem required quite a lot of work and preparation so we will omit it here. We mention that one key ingredient is to related the fibers:

$$Y_x = (\mathfrak{Y}_\xi \times_{\mathrm{Spec}(K(\xi))} \mathrm{Spec}(\mathcal{H}(x)))^{\mathrm{an}}$$

which follows from the universal property.

**Theorem 9.3.** *Let $\mathfrak{X}$ be a scheme over $k$. Then:*

(1) *$\mathfrak{X}$ is separated if and only if $\mathfrak{X}^{\mathrm{an}}$ is Hausdorff,*
(2) *$\mathfrak{X}$ is proper if and only if $\mathfrak{X}^{\mathrm{an}}$ is compact,*
(3) *$\mathfrak{X}$ is connected if and only if $\mathfrak{X}^{\mathrm{an}}$ is pathwise connected,*

Once again, we omit the proof. We only mention that the ingredients for part (3) include:

- $E^1$ is (uniquely) pathwise connected,
- $E^n$ is pathwise connected (by induction),
- Noether normalization theorem 5.18 etc.

**Theorem 9.4.** *The functor $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{an}}$ is fully faithful on the category of proper $k$-schemes of finite type.*

This is false when $\mathfrak{X}$ is not proper. For example, there are non-polynomial maps $\mathbb{A}_k^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$. However, if $k$ is trivially valued ($|k^\times| = 1$), we can drop the properness assumption.

9.3. **Coherent sheaves.** We can define a notion of coherent $\mathcal{O}_{X,G}$-modules where $X$ is a $k$-analytic space. We can also define coherent $\mathcal{O}_X$-modules if $X$ is good.

If $\mathfrak{X}$ is a scheme of finite type over $k$ and $\mathcal{F}$ is a coherent sheaf on $\mathfrak{X}$, we can define

$$\mathcal{F}^{\mathrm{an}} = \text{coherent } \mathcal{O}_{\mathfrak{X}^{\mathrm{an}}}\text{-module}.$$

**Theorem 9.5.** *If $\mathfrak{X}$ is proper, then there is an equivalence of categories:*

$$\mathrm{Coh}(\mathfrak{X}) \xrightarrow{\cong} \mathrm{Coh}(\mathfrak{X}^{\mathrm{an}}).$$

## 10. Generic fibers of formal schemes

This is like doing *analytic geometry* over the valuation ring $k^\circ$.

To discuss formal schemes, one makes standard definitions:

- A *formal scheme* is a topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ which is locally isomorphic to an *affine formal scheme.*
- An *affine formal scheme* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is ...

We will take these are granted and see the theory in practice.

Consider a (non-trivially valued) NA field $k$ with valuation ring $k^\circ = \{|\cdot| \leq 1\}$, maximal ideal $k^{\circ\circ} = \{|\cdot| < 1\}$, and residue field

$$\widetilde{k} = k^\circ / k^{\circ\circ}.$$

Fix $\pi \in k^{\circ\circ} \setminus \{0\}$, a *uniformizer.*

**Definition 10.1.** A $k^\circ$-algebra $A$ is *topologically of finite presentation* (*tfp*) if

$$A = k^\circ\{T_1, \dots, T_n\}/I$$

where $I$ is a finitely-generated[8] ideal.

**Definition 10.2.** A $k^\circ$ algebra $A$ which is tfp is *admissible* if it is flat over $k^\circ$.

If $A$ is tfp/admissible:

- $A \otimes_{k^\circ} \widetilde{k} = A/k^{\circ\circ}A$ is a finitely generated $\widetilde{k}$-algebra,
- $A \otimes_{k^\circ} k$ is a strictly $k$-affinoid algebra,
- $\mathfrak{X} = \mathrm{Spf}(A)$ is a tpf/admissible affine formal scheme,
- $\mathfrak{X}_s = \mathrm{Spec}(A \otimes_{k^\circ} \widetilde{k})$ is the *special fiber* of $\mathfrak{X}$,
- $\mathfrak{X}_\eta = \mathrm{Spec}(A \otimes_{k^\circ} k)$ is the *generic fiber* of $\mathfrak{X}$.

We have a reduction map $\mathrm{red}\colon \mathfrak{X}_\eta \to \mathfrak{X}_s$. Indeed, any $x \in \mathfrak{X}_\eta$ defines a character $A_{\otimes k^\circ} k \to \mathcal{H}(x)$. We also have a contractive morphism $A \to A \otimes_{k^\circ} k$ such that:

- the image of $A$ lies in $(A \otimes_k^\circ k)^\circ$,
- the image of $k^{\circ\circ}A$ lies ins $(A \otimes_k^\circ k)^{\circ\circ}$.

We get an induced character

$$A/k^{\circ\circ}A \to \mathcal{H}(x),$$

and hence a point

$$\mathrm{red}(x) \in \mathfrak{X}_s.$$

**Example 10.3.** Consider $A = k^\circ/\pi k^\circ$, which is tfp, but not admissible. Then:

$$A \otimes_{k^0} \widetilde{k} = \widetilde{k} \qquad \mathfrak{X}_s = \{\mathrm{pt}\},$$
$$A \otimes_{k^0} k = 0 \qquad \mathfrak{X}_\eta = \emptyset.$$

The reduction map is the unique map $\emptyset \to \{\mathrm{pt}\}$.

**Example 10.4.** Consider $A = k^\circ\{T\}$, which is admissible. Then

$$A \otimes_{k^\circ} \widetilde{k} = \widetilde{k}[T] \qquad \mathfrak{X}_s = \mathbb{A}^1_{\widetilde{k}},$$
$$A \otimes_{k^\circ} k = k\{T\} \qquad \mathfrak{X}_\eta = E^1_k.$$

The reduction map $\mathrm{red}\colon E^1_k \to \mathbb{A}^1_{\widetilde{k}}$ is the usual one.

We now globalize these concepts.

**Definition 10.5.** A formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(k^\circ)$ is *tpf/admissible* if $\mathfrak{X}$ is a locally finite union of open affine formal schemes $\mathfrak{X}_i$ that are tpf/admissible.

We may then glue $(\mathfrak{X}_i)_s$ and $(\mathfrak{X}_i)_\eta$ to obtain a scheme $\mathfrak{X}_s$ over $\widetilde{k}$ which is locally of finite type and a paracompact $k$-analytic space $\mathfrak{X}_\eta$.

The reduction maps glue to

$$\mathrm{red}\colon \mathfrak{X}_\eta \to X_s.$$

[8] Note that $A$ might not be Noetherian, because we did not assume $k^\circ$ is.

**Theorem 10.6** (Reynaud)**.** *Every paracompact strictly $k$-analytic space is a generic fiber of a formal scheme over $k^\circ$.*

**Example 10.7.** Glue $\mathrm{Spf}(k^0\{S\})$, $\mathrm{Spf}(k^0\{T\})$ along the open subsets where $S$, $T$ do not vanish via $S \mapsto T^{-1}$. Then

- $\mathfrak{X}_s$ is two copies of $\mathbb{A}^1_{\widetilde{k}}$ glued along $\mathbb{A}^1_{\widetilde{k}} \setminus \{0\}$, and hence $\mathfrak{X}_s = \mathbb{P}^1_{\widetilde{k}}$,
- $\mathfrak{X}_\eta$ is two copies of $E^1_k$ glued along $|S|, |T| = 1$, and hence $\mathfrak{X}_\eta = \mathbb{P}^1_k$ as a $k$-analytic space.

10.1. **Models of schemes.** Let $X$ be a (separated) scheme of finite type over $k$. Then $X^{\mathrm{an}}$ is the analytification of $X$ (as strictly $k$-analytic space without boundary).

**Goal.** Investigate the properties of $X^{\mathrm{an}}$.

**Tool.** Nice models of $X$ over $k^\circ$.

**Definition 10.8.** A *model* of $X$ (over $k^\circ$) is a flat scheme $\mathfrak{X}$ over $k^\circ$ together with an isomorphism

$$\mathfrak{X}_\eta = \mathfrak{X} \times_{\mathrm{Spec}(k^\circ)} \mathrm{Spec}(k) \xrightarrow{\cong} X$$

(as schemes).

**Example 10.9.** An integral model of $X = \mathbb{P}^n_k$ is $\mathfrak{X} = \mathbb{P}^n_{k^\circ}$.

We are mainly interested in the case when $X$ is proper (or projective) over $k$.

**Theorem 10.10** (Nagata)**.** *If $X$ is proper, then $X$ admits a proper model over $k^\circ$.*

**Remark 10.11.** If $X \subseteq \mathbb{P}^n_k$ is a projective, then we may take $\mathfrak{X}$ as the schematic closure of $X$ in $\mathbb{P}^n_{k^\circ}$.

Models are not unique. There is a partial ordering: $\mathfrak{X}' \geq \mathfrak{X}$ if we have a commutative diagram:

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r] & \mathfrak{X} \\
\mathfrak{X}'_\eta \arrow[u, hook] \arrow[r] \arrow[d, "\cong"] & \mathfrak{X}_\eta \arrow[u, hook] \arrow[d, "\cong"'] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

**Lemma 10.12.** *Any two models can be dominated by a third.*

However, there is no maximal one.

There are 2 types of analytic models. Given an algebraic model $\mathfrak{X}$ of $X$, we can take the formal completion $\hat{\mathfrak{X}}$ along the special fiber $\mathfrak{X}_s = \mathfrak{X} \times_{Speck^\circ} \mathrm{Spec}\,\widetilde{k}$.

Locally, if $\mathfrak{X} = \mathrm{Spec}\,A$, then

$$\hat{\mathfrak{X}} = \mathrm{Spf}\left(\varprojlim A/\pi^n A\right).$$

We now have two $k$-analytic spaces associated to $X$:

- $X^{\mathrm{an}}$,
- $\hat{\mathfrak{X}}_\eta$ (the generic fiber of the formal completion of an algebraic model).

**Fact 10.13.**

(1) *In general, $\hat{\mathfrak{X}}_\eta$ is a closed subspace of $X^{\mathrm{an}}$,*
(2) *Equality holds when $\mathfrak{X}$ is a proper model.*

**Example 10.14.** When $X = \mathbb{P}^1_k$, we saw that $\mathfrak{X} = \mathbb{P}^1_{k^\circ}$ is an integral model. Then

$$\hat{\mathfrak{X}}_\eta = X^{\mathrm{an}}$$

is the Berkovich space $\mathbb{P}^1_k$.

**Example 10.15.** When $X = \mathbb{P}^1_k$ and we take the integral model $\mathfrak{X} = \mathbb{P}^1_k \setminus \{\infty\}$ where $\infty \in \mathbb{P}^1_{\widetilde{k}}$. Then

$$\hat{\mathfrak{X}}_\eta = E^1_k \subsetneq \mathbb{P}^1_k.$$

We have a reduction map

$$\mathrm{red}\colon \hat{\mathfrak{X}}_\eta \to \hat{\mathfrak{X}}_s \cong \mathfrak{X}_s.$$

If $\mathfrak{X}$ is proper, then, we get a reduction map

$$\mathrm{red}_{\mathfrak{X}}\colon X^{\mathrm{an}} \to \mathfrak{X}_s.$$

This does depend on the choice of model $\mathfrak{X}$.

10.2. **Nicer models.** We want to consider nicer models $\mathfrak{X}$ of $X$. We make the following assumptions (for now):

- $k$ is discretely valued,
- $\widetilde{k}$ has characteristic 0 (and hence so does $k$).

**Example 10.16.** Consider $k = \mathbb{C}(\!(t)\!)$ where $\mathbb{C}$ is discretely valued. Then $t \in k^{\circ\circ}$ is a uniformizer.

Assume $X$ is smooth and proper over $k$, of dimension $n \geq 1$. Given a proper smooth model $\mathfrak{X}$ over $k^\circ$, the special fiber $\mathfrak{X}_s$ is the support of the Cartier divisor $\mathrm{div}(t)$.

We write $E_i$, $i \in I$, for the irreducible components of $\mathfrak{X}_s$.

**Definition 10.17.** We say that $\mathfrak{X}$ is an *snc (simple normal crossings) model* if $\mathfrak{X}$ is regular and the divisor $\mathfrak{X}_s = \mathrm{div}(t)$ has simple normal crossings support: for all $\xi \in \mathfrak{X}_s$, let $E_0, \dots, E_p$ be the irreducible components containing $\xi$, then there exist algebraic coordinates $z_0, \dots, z_q$ at $\xi$ such that $E_i = (z_i = 0)$ for $0 \leq i \leq p \leq q$.

**Example 10.18.** Take $k = \mathbb{C}(\!(t)\!)$ and

$$X = \{t(x^3 + y^3 + z^3) + xyz = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}(\!(t)\!)}.$$

Then the model

$$\mathfrak{X} = \{t(x^3 + y^3 + z^3) + xyz = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}[\![t]\!]}$$

has the special fiber:

$$\mathfrak{X}_s = (x = 0) \cup (y = 0) \cup (z = 0),$$

so $\mathfrak{X}$ is snc.

*The class continued remotely from here on, but I stopped typing the notes.*

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