MATH 731: HODGE THEORY

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These are notes from Math 731 taught by Professor Mircea Mustaţă in Fall 2019, LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

I would like to thank David Schwein for sharing his typed notes with me during the times I was away. I indicate within the notes the contributions that were made by him.

This version is from February 27, 2020. Check for the latest version of these notes at

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If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The goal of the course is to give an introduction to the basic results in Hodge theory. The prerequisites are: familiarity with algebraic varieties and sheaf cohomology (no familiarity with scheme theory is required) and with smooth manifolds (the tangent bundle, differential forms, integration).

The course will not follow a single textbook but most of the material covered can be found in [Voi07]. Additional references will be given where appropriate throughout the notes.

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1. The classical topology on a complex algebraic variety

The set up is as follows. Let X be an algebraic variety over $\mathbb{C}$, i.e. (the closed points of) a reduced scheme of finite type over $\mathbb{C}$.

Suppose X is affine. Then there is a closed immersion $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}} = \mathbb{C}^N$. Note that $\mathbb{C}^N$ carries the Euclidean topology. By definition, the *classical topology* on X is the induced subspace topology. This is well-defined: given two closed embeddings

$$\begin{array}{ccc} & & \mathbb{C}^N \\ & \overset{i}{\nearrow} & \\ X & & \\ & \underset{j}{\searrow} & \\ & & \mathbb{C}^{N'} \end{array}$$

there are polynomial functions $\mathbb{C}^N \to \mathbb{C}^{N'}$ and $\mathbb{C}^{N'} \to \mathbb{C}^N$ that make the triangle commute; since polynomial functions $\mathbb{C}^m \to \mathbb{C}^n$ are continuous with respect to the Euclidean topology, the two embeddings induce the same topology on X.

To give some more details, the two closed immersions correspond to surjective maps

$$\begin{array}{ccc} \mathbb{C}[x_1, \ldots, x_N] & & \\ & \searrow & \\ & & \mathcal{O}(X) \\ & \nearrow & \\ \mathbb{C}[y_1, \ldots, y_N] & & \end{array}$$

and we may defined the map $\mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C}[y_1, \ldots, y_N]$ (and vice versa) by taking any lifts from $\mathcal{O}(X)$.

Proposition 1.1.

1. *The classical topology on X is **finer** than the Zariski topology.*
2. *If X is affine and $Z \hookrightarrow X$ is a closed subvariety, then in the classical topology on Z is the subspace topology with respect to the classical topology on X.*
3. *The same is holds for an open subvariety $U \hookrightarrow X$ (if U is affine).*

Proof. It is enough to check (1) for $\mathbb{C}^N$; there, we use the definition of the Zariski topology and the fact that polynomial functions are continuous with respect to the Euclidean topology.

Part (2) follows directly from the definition.

To prove (3), we first note that by covering U by principal affine open subsets, we may assume that U is principal affine:

$$U = \{x \in X \mid f(x) \neq 0\}$$

for some $f \in \mathcal{O}(X)$. Indeed, if $U = U_1 \cup \cdots \cup U_r$ and U_i is principal affine with respect to X (hence also U), then the principal case implies that the topology on U_i is the subspace topology on U_i with respect to both the classical topologies on U and on X, and hence the assertion holds for U.

Given a closed immersion $X \hookrightarrow \mathbb{C}^N$ and $f \in \mathcal{O}(X)$, choose $g \in \mathbb{C}[x_1, \ldots, x_N]$ such that $g|_X = f$ and embed U in $\mathbb{C}^{N+1}$ as $\{(u,t) \mid u \in X, g(u)t = 1\}$:

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{C}^N \\ \uparrow & & \\ U & \hookrightarrow & \mathbb{C}^{N+1} \end{array}$$

The two maps:

- $\mathbb{C}^{N+1} \to \mathbb{C}^N$ given by projecting only the first component,
- $\mathbb{C}^N \setminus V(g) \to \mathbb{C}^{N+1}$ given by $u \mapsto (u, \frac{1}{g(u)})$

are both continuous with respect to the Euclidean topology. Thus the topology on U as a subspace of $\mathbb{C}^N$ and $\mathbb{C}^{N+1}$ coincide. $\square$

We now glue the above construction. Let X be an algebraic variety over $\mathbb{C}$. Take an affine open cover $X = \bigcup_{i=1}^{r} U_i$. Each U_i has the classical topology introduced above.

We know that:

(1) $U_i \cap U_j$ is open in both U_i and U_j with respect to the classical topology,
(2) by covering $U_i \cap U_j$ by affine open subsets, Proposition 1.1 (3) implies that the classical topology on U_i and U_j induce the same topology on $U_i \cap U_j$.

It is now easy to check that in this case there is a unique topology on X such that the subspace topology on each U_i is the classical topology.

By definition, a subset $U \subseteq X$ is classically open if $U \cap U_i$ is open for all i in the classical topology. Note that each U_i is open in X in this topology.

It is easy to see that:

(1) the definition is independent of the choice of cover,
(2) Proposition 1.1 extends to an arbitrary complex variety.

Definition 1.2. We write X^{an} for the topological space X with the classical topology.

Remarks 1.3.

(1) If $f\colon X \to Y$ is a morphism of algebraic varieties, then $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is continuous. Indeed, by covering X and Y by affines, we reduce to X and Y affine. Given closed immersions

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{C}^m \\ {\scriptstyle f}\downarrow & & \vdots\,{\scriptstyle g} \\ Y & \hookrightarrow & \mathbb{C}^n \end{array}$$

there is a polynomial map g that makes the square above commute. Since polynomial maps are continuous, this proves the assertion.

In particular, regular functions on X are continuous with respect to the classical topology.

(2) Every points has a countable basis of open neighborhoods in the classical topology.

(3) If X, Y are complex algebraic variety, then $(X \times Y)^{\mathrm{an}} = X^{\mathrm{an}} \times Y^{\mathrm{an}}$ in the category of topological spaces (i.e. the classical topology of $X \times Y$ is the product of the classical topologies on X and Y).

Indeed, we reduce again to X, Y affine, and using the definition of classical topology, we reduce to $X = \mathbb{C}^m$, $Y = \mathbb{C}^n$ — then the claim follows since the Euclidean topology on $\mathbb{C}^{m+n}$ is the product topology.

Theorem 1.4. *If X is an irreducible complex algebraic variety and $U \subseteq X$ is Zariski open, nonempty, then U is dense in the classical topology.*

We will prove this theorem next time.

Corollary 1.5. *Let X be a complex algebraic variety and $U \subseteq X$ be Zariski open and dense in the Zariski topology, then U is dense in the classical topology.*

Proof. If $X = X_1 \cup \cdots \cup X_r$ is an irreducible decomposition and $U \subseteq X$ is Zariski dense, then $U \cap X_i \neq \emptyset$ for all i. Theorem 1.4 then implies that $U \cap X_i$ is dense in X_i in the classical topology, and hence U is dense in X in the classical topology. □

The proof of Theorem 1.4 follows [Mum99].

Proof of Theorem 1.4. **Step 1.** We may assume that X is affine. Indeed, given an open affine cover $X = U_1 \cup \cdots \cup U_r$ then $U \cap U_i \neq 0$ for all i, and if we know $U \cap U_i$ is dense in $(U_i)^{\mathrm{an}}$, then U is dense in X^{an}.

Step 2. We apply Noether normalization to get a finite surjective map

$$\pi\colon X \to \mathbb{C}^n.$$

We need to show that given and $p \in Z = X \setminus U$, we can fine a sequence $y_m \to p$ with $y_m \in U$.

Let $u = \pi(p)$. Since π is finite, $\pi(Z)$ is a closed proper subset of $\mathbb{C}^n$, there exists $g \neq 0$ in $\mathbb{C}[x_1, \ldots, x_n]$ such that $\pi(Z) \subseteq V(g)$.

Consider

$$\mathbb{R} \xrightarrow{\varphi} \mathbb{C}$$
$$t \mapsto g(tu + (1-t)w)$$

where $w \in \mathbb{C}^n$ is such that $g(w) \neq 0$. Since $\varphi(0) \neq 0$ and φ is a polynomial, φ only vanishes at finite many points. Thus there is a sequence $u_m \to u$, $g(u_m) \neq 0$. After passing to a subsequence, we need to find $y_m \in \pi^{-1}(u_m)$ such that $y_m \to y$. Since $u_m \notin V(g)$, $y_m \in U$, so $p \in \overline{U}$.

Step 3. Finding the y_m's. Recall that we have a map

$$\pi \colon X \to \mathbb{C}^n \ni u_m \to u$$

and we want to find elements in fibers over u_m's. Write

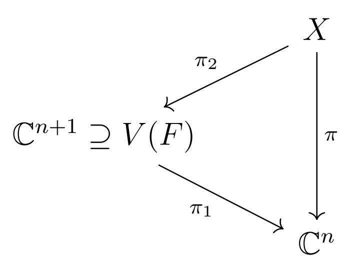
$$\pi^{-1}(u) = \{p = p_1, p_2, \dots, p_r\}.$$

Choose $g \in \mathcal{O}(X)$ such that $g(p) = 0$ but $g(p_j) \neq 0$ for $j \geq 2$. Since π is finite, there exists $F \in \mathcal{O}(\mathbb{C}^n)[t]$ monic such that $F(x, g) = 0$.

Write

$$F = t^d + a_1(x)t^{d-1} + \cdots + a_r(x).$$

Since $\mathcal{O}(X)$ is a domain, we may assume that F is irreducible. The map π factors as



where $\pi_2(x) = (\pi(x), g(x))$. Since π is finite, π_2 is also finite. Since π is surjective, this shows that π_2 is also surjective (otherwise, $\pi_2(X)$ has dimension less than n, and hence so does $\pi(X)$).

Recall that $g(p) = 0$ and hence $a_r(u) = 0$. Since $|a_r(x)|$ is the absolute value of the product of the roots of $F(x, -)$ and $u_m \to u$, we can choose t_m such that $F(u_m, t_m) = 0$ for all m and $t_m \to 0$.

Now choose $y_m \in \pi_2^{-1}(u_m, t_m)$ arbitrarily.

We claim that, after passing to a subsequence, we assume that y_m converges to some y. Since $\pi(y_m) = u_m \to u$, we see that $y \in \pi^{-1}(U)$. Since $g(y) = \lim g(y_m) = \lim t_m = 0$, we have that $y \neq p_j$ for $j \geq 2$. Hence $y = p$.

We thus just need to prove this claim. Choose generators $h_1, \dots, h_s$ of $\mathcal{O}(X)$ to get a closed immersion $X \hookrightarrow \mathbb{C}^s$ given by $(h_1, \dots, h_s)$.

We use the fact that each h_i satisfies a monic equation:

$$t^{d_i} + a_{i,1}(x)t^{d_i-1} + \cdots = 0.$$

We want to show that each $(h_i(y_m))_{m\geq 1}$ is bounded. Since h_i satisfies the monic equation above, we just need to show that the coefficients $(a_{i,j}(\pi(y_m)))_{m\geq 1}$ is bounded. Since $\pi(y_m) = u_m$ is convergent, it is bounded, and hence $a_{i,j}(\pi(y_m))$ is bounded for all i, j. $\square$

Corollary 1.6. *Let X be an algebraic variety over $\mathbb{C}$ and $W \subseteq X$ is a constructible*[1] *subset. Then $\overline{W}^{\mathrm{Zar}} = \overline{W}^{\mathrm{an}}$.*

Proof. Since the classical topology is finer than the Zariski topology, $\overline{W}^{\mathrm{an}} \subseteq \overline{W}^{\mathrm{Zar}}$.

Since W is constructible, there exists $U \subseteq W$ such that U is open and dense in $\overline{W}^{\mathrm{Zar}}$.

By Corollary 1.5, U is dense in $\overline{W}^{\mathrm{Zar}}$ in the classical topology. This shows that $\overline{W}^{\mathrm{Zar}} \subseteq \overline{W}^{\mathrm{an}}$. □

Remark 1.7 (Chevalley's theorem)**.** Let $f\colon X \to Y$ be a finite morphism. The the image of a constructible set under f is constructible.

Theorem 1.8. *Suppose X is a complex algebraic variety. Then*

(1) *X is separated if and only if X^{an} is Hausdorff,*
(2) *X is complete if and only if X^{an} is compact*[2]*,*
(3) *if $f\colon X \to Y$ is a morphism of separated varieties, then f is proper (in the algebraic sense) if and only if $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is proper (i.e. $K \subseteq Y^{\mathrm{an}}$ is compact implies that $f^{-1}(K)$ is compact).*

Proof. We first show (1). Recall that, in general, the diagonal map

$$\Delta\colon X \to X \times X$$

is a locally closed immersion. By definition, X is separated if Δ is a closed immersion, i.e. $\Delta(X)$ is closed in $X \times X$ (in the Zariski topology).

By Corollary 1.6, $\Delta(X)$ is Zariski closed in $X \times X$ if and only if it is closed in the classical topology. But $\Delta(X)$ is closed in $X^{\mathrm{an}} \times X^{\mathrm{an}}$ if and only if X^{an} is Hausdorff.

We will now prove (2). Note first that $(\mathbb{P}^n)^{\mathrm{an}}$ is compact. Indeed, we have a continuous surjective map from the n-sphere which is compact:

$$\left\{ z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{i=0}^{n} |z_i|^2 = 1 \right\} \to (\mathbb{P}^n)^{\mathrm{an}}.$$

Suppose now that X is complete. In particular, it is separated, so we know that X^{an} is Hausdorff by (1). By Chow's lemma, there is a surjective (birational) morphism

$$\pi\colon \widetilde{X} \to X$$

with $\widetilde{X}$ projective, $\widetilde{X} \hookrightarrow \mathbb{P}^n$ Zariski-closed. Then $(\widetilde{X})^{\mathrm{an}}$ is closed in $(\mathbb{P}^n)^{\mathrm{an}}$, hence compact. Therefore, $\pi(\widetilde{X}^{\mathrm{an}}) = X^{\mathrm{an}}$ is compact.

Conversely, suppose X^{an} is compact. In particular, it is Hausdorff and hence X is separated. We need to show that for every algebraic variety Y, the projection map

$$X \times Y \xrightarrow{f} Y$$

is closed in the Zariski topology, i.e. if $Z \subseteq X \times Y$ is Zariski-closed, $f(Z)$ is Zariski-closed.

[1] Recall that a subset is constructible if it is a finite union of locally closed sets.
[2] Recall that compact means quasicompact and Hausdorff.

By Chevalley's theorem 1.7, $f(Z)$ is constructible. By Corollary 1.6, $f(Z)$ is Zariski closed if and only if it is closed in the classical topology. Suppose $y_n \in f(Z)$ is such that $\lim_{n\to\infty} y_n = b \in Y$. Then there exists $x_n \in X$ such that $(x_n, y_n) \in Z$ for all n. Since X^{an} is compact, after passing to a subsequence, we may assume that $x_n \to a \in X$. Since Z is Zariski closed in $X \times Y$, it is closed in the classical topology. Since $(x_n, y_n) \in Z$, $(a, b) \in Z$, and hence $b \in f(Z)$. □

Remark 1.9. A related result is that if X is an irreducible variety over $\mathbb{C}$, then X^{an} is connected. We will come back to this when we discuss holomorphic function. A challenge exercise is to prove this statement directly.

Exercise. Show that if $f\colon X \to Y$ is a morphism of separated algebraic varieties, then f is proper if and only if f^{an} is proper.

Remark 1.10. From now on, all varieties over $\mathbb{C}$ will be assumed to be separated.

2. Holomorphic functions

The reference for this section is [GH94].

2.1. Holomorphic functions in one variable.

First, we consider the case of 1-variable functions.

Setup. Consider an open set $U \subseteq \mathbb{C} = \mathbb{R}^2$. All functions considered will be smooth (C^∞). Coordinate functions on U will be denoted by $z = x + yi$, $\overline{z} = x - yi$. We have

$$dz = dx + idy, \quad d\overline{z} = dz - idy,$$
$$dz \wedge d\overline{z} = (-2i) dx \wedge dy.$$

Dually, we have

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(this is dual to the basis of differentials given by dz, $d\overline{z}$).

These operators acts on z^m by

$$\frac{\partial}{\partial z}(z^m) = mz^{m-1}, \quad \frac{\partial}{\partial \overline{z}} z^m = 0$$

(by product rule, it is enough to check these for $m = 1$).

If $f\colon U \to \mathbb{C}$ is smooth, we write

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}.$$

Exercise. Check that $\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$.

Proposition 2.1 (Cauchy's formula)**.** *Let Δ be an open disc in $\mathbb{C}$. If f is a smooth function on an open neighborhood of $\overline{\Delta}$, then*

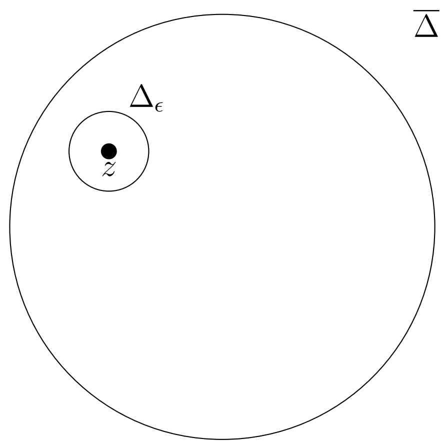
$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i}\int_{\overline{\Delta}} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z}$$

for all $z \in \Delta$.

Remark 2.2.

(1) The loop $\partial\Delta$ is oriented counterclockwise.
(2) Part of the statement is that the second integral is well-defined.
(3) We will define a holomorphic function to be annihilated by $\frac{\partial}{\partial \overline{z}}$. In particular, the second integral vanishes when f is holomorphic.

Proof. Let Δ_ϵ be a disc of radius $0 < \epsilon \ll 1$ around z.



We apply Stokes' formula for

$$\eta = \frac{f(w)}{w-z} dw$$

on $\overline{\Delta} \setminus \Delta_\epsilon$. Note that

$$d\eta = -\frac{\partial}{\partial \overline{w}} \left(\frac{f(w)}{w-z} \right) dw \wedge d\overline{w}.$$

By the quotient rule and

$$\frac{\partial}{\partial \overline{w}} \left(\frac{1}{w-z} \right) = 0,$$

we have that

$$d\eta = -\frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z}.$$

Stokes' theorem then says that

$$-\int\limits_{\overline{\Delta} \setminus \Delta_\epsilon} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z} = \int\limits_{\partial\Delta} \frac{f(w)}{w-z} dw - \int\limits_{\partial\Delta_\epsilon} \frac{f(w)}{w-z} dw.$$

We evaluate the last integral. We change variables to $w = z + \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$ and $dw = \epsilon i e^{i\theta} d\theta$:

$$\int_{\partial\Delta_\epsilon} \frac{f(w)}{w-z} dw = \int_0^{2\pi} \frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} \epsilon i e^{i\theta} d\theta = i \int_0^{2\pi} f(z + \epsilon e^{i\theta}) d\theta.$$

For $\epsilon \to 0$, this converges (by the dominated convergence theorem, for example) to

$$i\int_0^{2\pi} f(z)d\theta = 2\pi i f(z).$$

Finally, we deal with the integral on the left hand side of Stokes' theorem. Again, we change variables to $w = re^{i\theta} + z$ for $r \geq 0$ and $\theta \in [0, 2\pi]$ and

$$\begin{aligned} dw &= e^{i\theta}dr + ire^{i\theta}d\theta, \\ d\overline{w} &= e^{-i\theta}dr - ire^{-i\theta}d\theta, \\ dw \wedge d\overline{w} &= -2irdr \wedge d\theta. \end{aligned}$$

Then

$$\frac{dw \wedge d\overline{w}}{w - z} = -2ie^{-i\theta}dr \wedge d\theta.$$

This is integrable on any compact subset of $\mathbb{C}$ and

$$\int\limits_{\overline{\Delta}} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} = \lim_{\epsilon \to 0} \int\limits_{\overline{\Delta} \setminus \Delta_\epsilon} \frac{\partial f}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z}.$$

This completes the proof. ☐

Definition 2.3. A smooth function $f : U \to \mathbb{C}$ is

- *holomorphic* if $\frac{\partial f}{\partial \overline{z}} = 0$ (if $f = u + iv$, this is equivalent to the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$),
- *analytic* if for any $a \in U$, there exists an open disc $\Delta_r(a)$ centered at a inside U such that

$$f(z) = \sum_{n \geq 0} c_n (z - a)^n$$

  for some $c_n \in \mathbb{C}$ where the convergence is absolute and uniform for $z \in \Delta_r(a)$.

Theorem 2.4. *A function f is holomorphic if and only if it is analytic.*

Proof. We start with the 'only if' implication. Given $a \in U$, let Δ be a disc centered at a such that $\overline{\Delta} \subseteq U$. Cauchy's formula 2.1 then shows that

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial \Delta} \frac{f(w)}{w - z} dw.$$

We write

$$\frac{f(w)}{w - z} = \frac{f(w)}{(w - a) - (z - a)} = \frac{f(w)}{(w - a)\left(1 - \frac{z - a}{w - a}\right)}.$$

If R is the radius of Δ, we fix a disc Δ' centered at a of radius $R' < R$. Then

$$\left|\frac{z - a}{w - a}\right| \leq \frac{R'}{R} < 1$$

for $z \in \Delta'$. Then

$$\frac{f(w)}{w - z} = \sum_{n \geq 0} \frac{f(w)}{(w - a)^{n+1}} (z - a)^n$$

converges absolutely and uniformly for $z \in \Delta'$ and $w \in \partial\Delta$. Therefore,

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n$$

is absolutely and uniformly convergence for $z \in \Delta'$, where

$$c_n = \frac{1}{2\pi i} \int\limits_{\partial\Delta} \frac{f(w)}{(w-a)^{n+1}} dw.$$

Hence f is analytic.

For the 'if' implication, suppose f is analytic and choose around $a \in U$ a small disc Δ such that $\overline{\Delta} \subseteq U$ and

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n$$

converges absolutely and uniformly for $z \in \Delta$.

Since $\frac{\partial P}{\partial \overline{z}} = 0$ for any polynomial P, if P_n is the nth partial sum, by Cauchy's theorem 2.1,

$$P_n(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{P_n(w)}{w-z} dw,$$

and hence

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Delta} \frac{f(w)}{w-z} dw.$$

Therefore:

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2\pi i} \int\limits_{\partial\Delta} \frac{\partial}{\partial \overline{z}} \left(\frac{f(w)}{w-z} \right) dw = 0$$

because the integrand is 0. Hence f is holomorphic. □

Theorem 2.5 ($\overline{\partial}$-lemma in 1 variable)**.** *Let $U \subseteq \mathbb{C}$ and $g \colon U \to \mathbb{C}$ be a smooth function. If Δ is a disc such that $\overline{\Delta} \subseteq U$ and we define*

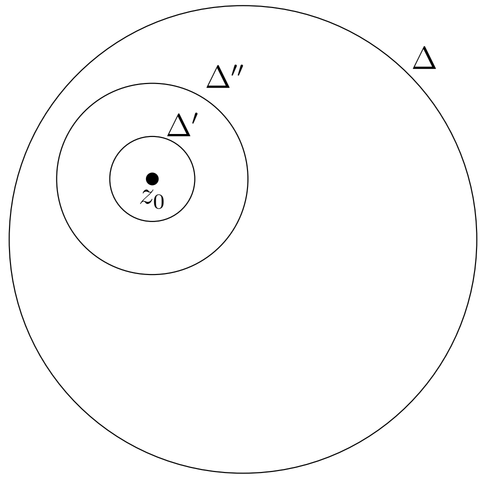
$$f(z) = \frac{1}{2\pi i} \int\limits_{\overline{\Delta}} g(w) \frac{dw \wedge d\overline{w}}{w-z}, \quad \textit{for } z \in \Delta,$$

then f is a smooth function and

$$\frac{\partial f}{\partial \overline{z}} = g \quad \textit{on } \Delta.$$

Remark 2.6. This theorem will later be used to compute Dolbeaux cohomology. See Lemma

Proof. Given $z_0 \in \Delta$, choose discs centered at z_0 such that $\Delta' \subseteq \Delta'' \subseteq \Delta$ (and the closure of the previous is contained in the next).



We can write $g = g_1 + g_2$ with g_1, g_2 smooth on U such that

$$\begin{cases} g_1 = 0 & \text{inside } \Delta', \\ g_2 = 0 & \text{outside } \Delta''. \end{cases}$$

Consider separately

$$f_i(z) = \frac{1}{2\pi i} \int\limits_{\Delta} g_i(w) \frac{dw \wedge d\overline{w}}{w - z}, \quad \text{for } i = 1, 2.$$

For $z \in \Delta'$, f_1 is clearly smooth and

$$\frac{\partial f_1}{\partial \overline{z}} = \frac{1}{2\pi i} \int\limits_{\Delta} \frac{\partial}{\partial \overline{z}} \left(\frac{g_1(w)}{w - z} \right) dw \wedge d\overline{w} = 0.$$

Now note that

$$f_2(z) = \frac{1}{2\pi i} \int\limits_{\mathbb{C}} g_2(w) \frac{dw \wedge d\overline{w}}{w - z} \qquad \text{because } g_2 = 0 \text{ outside } \Delta''$$

$$= \int_0^{2\pi} e^{-i\theta} \int_0^{\infty} g_2(z + re^{i\theta}) dr d\theta \quad \text{where } w = z + re^{i\theta} \text{ and } \frac{dw \wedge d\overline{w}}{w - z} = -2ie^{-i\theta} dr \wedge d\theta$$

This implies that f_2 is smooth on Δ. After going back via the change of variables, we see that

$$\frac{\partial f_2}{\partial \overline{z}} = \frac{1}{2\pi i} \frac{\partial g_2}{\partial \overline{w}} \cdot \frac{dw \wedge d\overline{w}}{w - z}.$$

Cauchy's formula 2.1 for g_2 then shows that

$$g_2(z) = \underbrace{\frac{1}{2\pi i} \int\limits_{\partial \Delta} \frac{g_2(w)}{w - z} dw}_{=0 \text{ since } g_2 = 0 \text{ on } \partial \Delta} + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial g_2}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} = \frac{\partial f_2}{\partial \overline{z}}$$

on Δ. Since $\frac{\partial f_1}{\partial \overline{z}} = 0 = g_1$ on Δ', this shows that

$$\frac{\partial f}{\partial \overline{z}} = g \quad \text{on } \Delta'.$$

This shows that $\frac{\partial f}{\partial \bar{z}} = g$ for any $z \in \Delta$. □

Remark 2.7. The proof also shows that if g is a smooth function of $U_1 \times \cdots \times U_n \subseteq \mathbb{C}^n$, so is f. Moreover, if g is holomorphic (separately) in each of $z_2, \ldots, z_r$, so is f.

2.2. **Holomorphic functions in several variables.** Let $U \subseteq \mathbb{C}^n$ be open with coordinate functions $z_1, \ldots, z_n$, $z_j = x_j + iy_j$.

Definition 2.8. A smooth function $f \colon U \to \mathbb{C}$ is

- *holomorphic* if it is holomorphic in each variable, i.e.
$$\frac{\partial f}{\partial \overline{z_i}} = 0 \quad \text{on } U.$$
- *analytic* if for every $a \in U$, there is a polydisc $B = B_r(a) = \{z \mid |z_j - a_j| < r \text{ for all } j\}$ such that
$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z-a)^\alpha,$$
where we use the multiindex notation:
$$(z-a)^\alpha = \prod_{i=1}^n (z_i - a_i)^{\alpha_i}.$$

Theorem 2.9. *If $f \colon U \to \mathbb{C}$ is a smooth function, the following are equivalent:*

(1) *f is holomorphic,*
(2) *f is analytic,*
(3) *for every polydisc $\Delta = \prod_{i=1}^n \{z_i \mid |z_i - a_i| < \alpha_i\} \subseteq U$,*
$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1)\ldots(w_n - z_n)} dw_1 \wedge \cdots \wedge dw_n,$$
where the integral is over the product of circles with product orientation.

Proof. It is clear that if f is analytic, it is analytic in each variable, hence holomorphic in each variable, i.e. f is holomorphic. This proves that (2) implies (1). To prove (3) implies (2), we argue as in the proof of Theorem 2.4. We get
$$f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta (z-a)^\beta$$
where
$$c_\beta = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w-1-z_1)^{\beta_1+1}\ldots(w_n - z_n)^{\beta_n+1}} dw_1 \wedge \cdots \wedge dw_n.$$

For (1) implies (3), use Cauchy's formula 2.1 for holomorphic functions in each variable:
$$f(z) = \frac{1}{2\pi i} \int\limits_{|z_n - a_n| = \alpha_n} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{z_n - w_n} dw_n = \cdots$$

and use that f is continuous and Fubini's theorem. $\square$

For an open subset $U \subseteq \mathbb{C}^n$, we write

$$\mathcal{O}(U) = \{f\colon U \to \mathbb{C} \mid f \text{ holomorphic}\}.$$

Then the following are true.

- The subset $\mathcal{O}(U) \subseteq C^\infty(U)$ is a $\mathbb{C}$-subalgebra. To prove this, use the fact that $\frac{\partial}{\partial \overline{z_j}}$ are linear (so closed under + and scalar multiplication) and derivations (so closed under product).
- If $f \in \mathcal{O}(U)$ and $f(z) \neq 0$ for all $z \in U$, then $\frac{1}{f} \in \mathcal{O}(U)$. Indeed, $\frac{\partial}{\partial \overline{z_j}}$ satisfies the quotient rule.

Definition 2.10. A function $f = (f_1, \dots, f_m)\colon U \to \mathbb{C}^m$ is *holomorphic* if all f_j are holomorphic.

We start by checking that the composition of holomorphic functions is holomorphic.

Identifying $\mathbb{C}^n = \mathbb{R}^{2n}$ with coordinates $z_1, \dots, z_n$ and $\mathbb{C}^m = \mathbb{R}^{2m}$ with coordinates $z_1', \dots, z_m'$, and $f_j = u_j + iv_j$, we have a map

$$T_p\mathbb{R}^{2n} \xrightarrow{df_p} T_{f(p)}\mathbb{R}^{2m}$$

which can be written explicitly as

$$\begin{aligned}
&\frac{\partial}{\partial x_j}(p) \mapsto \sum_{k=1}^m \frac{\partial u_k}{\partial x_j}(p)\frac{\partial}{\partial x_k'}(f(p)) + \sum_{k=1}^m \frac{\partial v_k}{\partial x_j}(p)\frac{\partial}{\partial y_k'}(f(p))\\
&\frac{\partial}{\partial y_j}(p) \mapsto \cdots
\end{aligned}$$

Exercise. Show that after we tensor with $\mathbb{C}$, we have the formulas

$$\begin{aligned}
\frac{\partial}{\partial z_j}(p) &\mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p)\frac{\partial}{\partial z_k'}(f(p)) + \sum_{k=1}^m \frac{\partial \overline{f_k}}{\partial z_j}(p)\frac{\partial}{\partial \overline{z_k'}}(f(p)),\\
\frac{\partial}{\partial \overline{z_j}}(p) &\mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial \overline{z_j}}(p)\frac{\partial}{\partial z_k'}(f(p)) + \sum_{k=1}^m \frac{\partial \overline{f_k}}{\partial \overline{z_j}}(p)\frac{\partial}{\partial \overline{z_k'}}(f(p)).
\end{aligned}$$

The upshot is that if f is holomorphic, then $\frac{\partial \overline{f_k}}{\partial z_j} = \overline{\frac{\partial f_k}{\partial \overline{z_j}}} = 0$. Therefore

$$\begin{aligned}
\operatorname{span}\left(\frac{\partial}{\partial z_j} \,\middle|\, j\right) &\to \operatorname{span}\left(\frac{\partial}{\partial z_k'} \,\middle|\, k\right)\\
\operatorname{span}\left(\frac{\partial}{\partial \overline{z_j}} \,\middle|\, j\right) &\to \operatorname{span}\left(\frac{\partial}{\partial \overline{z_k'}} \,\middle|\, k\right).
\end{aligned}$$

Consider maps $U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}$. The upshot is that if g is also holomorphic, then $g \circ f$ is also holomorphic. In fact, for any g, we have that

$$\frac{\partial (g \circ f)}{\partial \overline{z_j}}(p) = \sum_{k=1}^{m} \overline{\frac{\partial f_k}{\partial z'_j}} \left(\frac{\partial g}{\partial \overline{z'_k}} \circ g \right).$$

This implies that

- if f, g is holomorphic, then $g \circ f$ is also holomorphic,
- if f is a holomorphic diffeomorphism $f \colon U \to V$ (for $U, V \subseteq \mathbb{C}^n$), $g \circ f$ is holomorphic, and the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}$ is invertible at every point, then g is holomorphic.

Moreover, if both f and $g \colon V \to \mathbb{C}$ are holomorphic, then

$$\frac{\partial (g \circ f)}{\partial z_j} = \sum_k \frac{\partial f_k}{\partial z_j} \left(\frac{\partial g_k}{\partial z'_k} \circ g \right).$$

Remark 2.11. We only assume that $g \colon V \to \mathbb{C}$ to simplify the notation. The above assertions also hold for $g \colon V \to \mathbb{C}^p$ in general.

Let $U \subseteq \mathbb{C}^n$ and $f \colon U \to \mathbb{C}^n$. The next goal is to prove the inverse function theorem. We want to compare the real Jacobian of f with $\det\left(\frac{\partial f_j}{\partial z_j}\right)$, and deduce it from the inverse function theorem for smooth functions.

Write $f = (f_1, \ldots, f_n)$ and $(z_1, \ldots, z_n)$ for the variables on U and $\mathbb{C}^n$. One can compute that:

$$f^*(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n) = \begin{pmatrix} \text{determinant of} \\ \text{real Jacobian} \\ \text{of } f \end{pmatrix} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n,$$

$$dz_j \wedge d\overline{z_j} = (dx_j + i dy_j) \wedge (dx_j - i dy_j) = (-2i) dx_j \wedge dy_j,$$

and hence (after tensoring with $\mathbb{C}$):

$$f^*(dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}) = \begin{pmatrix} \text{determinant of} \\ \text{real Jacobian} \\ \text{of } f \end{pmatrix} dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}.$$

The left hand side is equal to

$$df_1 \wedge d\overline{f_1} \wedge \cdots \wedge df_n \wedge d\overline{f_n}.$$

Recall that

$$df = \sum_j \left(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right) = \sum_j \left(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j. \right).$$

In particular, if f is holomorphic, then each f_k is holomorphic, and hence

$$df_k = \sum_{j=1}^{n} \frac{\partial f_k}{\partial z_j} dz_j,$$

$$d\overline{f_k} = \sum_{j=1}^{n} \overline{\frac{\partial f_k}{\partial z_j}} d\overline{z_j}.$$

Finally, this shows that

$$\begin{aligned} df_1 \wedge d\overline{f_1} \wedge \cdots \wedge df_n \wedge d\overline{f_n} &= \left(\det \frac{\partial f_j}{\partial z_k}\right) \left(\det \overline{\frac{\partial f_j}{\partial z_k}}\right) dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n} \\ &= \left|\det \frac{\partial f_j}{\partial z_k}\right|^2 dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n} \end{aligned}$$

The overall conclusion is that

$$\begin{pmatrix}\text{determinant of the} \\ \text{real Jacobian} \\ \text{matrix of } f\end{pmatrix} = \begin{vmatrix}\text{determinant of the} \\ \text{complex Jacobian} \\ \text{matrix of } f\end{vmatrix}^2.$$

In particular:

- the left hand size is ≥ 0,
- the left hand size is $= 0$ if and only if the right hand side is.

Theorem 2.12 (Holomorphic inverse function theorem)**.** *If $U \subseteq \mathbb{C}^n$ is open and $f\colon U \to \mathbb{C}^n$ is holomorphic. Then for $p \in U$ such that* $\det\left(\frac{\partial f_i}{\partial z_j}(p)\right) \neq 0$*, there are open neighborhoods $U' \subseteq U$ of p and $V' \subseteq \mathbb{C}^n$ of $f(p)$ such that f gives a bijective map $U' \to V'$ and its inverse is holomorphic.*

Proof. By the previous discussion, the hypothesis implies that the determinant of the real Jacobian of f is nonzero at p. The inverse function theorem for smooth maps implies that there are open subsets U', V' as above such that $U' \xrightarrow{f} V'$ is bijective and its inverse is smooth. We may assume that $\det\left(\frac{\partial f_j}{\partial z_k}\right) \neq 0$ on U', and hence g is holomorphic on V since f and $g \circ f$ are. $\square$

Remark 2.13.

(1) If $f\colon U \to \mathbb{C}$ is holomorphic, then $\dfrac{\partial^{|\alpha|} f}{\partial z^\alpha}$ is holomorphic for all α (since $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \overline{z_k}}$ commute).

(2) If $a \in U$ is such that $\dfrac{\partial^{|\alpha|}}{\partial z^\alpha}(a) = 0$ for all α, then $f \equiv 0$ in a neighborhood of a. Indeed, if $B = \{z \mid |z_i - a_i| < \epsilon \text{ for all } i\}$ is such that $\overline{B} \subseteq U$, then

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|z_i - a_i| = \alpha_i} \frac{f(w)}{(w_1 - z_1)\dots(w_n - z_n)} dw_1 \wedge \cdots \wedge dw_n,$$

and using this we got

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha (z-a)^\alpha$$

for $z \in B$, where

$$\begin{aligned} c_\alpha &= \left(\frac{1}{2\pi i}\right)^n \int\limits_{|z_i-a_i|=\alpha_i} \frac{f(w)}{(w-1-z_1)^{\alpha_1+1}\dots(w_n-z_n)^{\alpha_n+1}} dw_1\wedge\cdots\wedge dw_n \\ &= \frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(a) \\ &= 0. \end{aligned}$$

Proposition 2.14. *Suppose $f\colon U\to\mathbb{C}$ is holomorphic and U is connected. If $f=0$ on some $V\subseteq U$ open, then $f=0$.*

Proof. Let $U'=\{z\in U \mid f=0 \text{ on some open neighborhood of } z\}$. This set is non-empty by hypothesis and clearly open. It is enough to show that it is closed. If $z_n\in U'$ converges to a, then for every α

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha}(z_n)=0$$

and hence

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha}(a)=0.$$

This holds for all α, so $f=0$ in a neighborhood of a. Thus $a\in U'$. □

The next goal is to state and prove the maximum modulus principle.

Theorem 2.15 (Maximumu modulus principle)**.** *If $U\subseteq\mathbb{C}^n$ is open and connected, and $f\colon U\to\mathbb{C}$ is a holomorphic function such that $|f|$ has a local max at $a\in U$, then f is constant.*

Proof. By Proposition 2.14, it is enough to show that there is an open neighborhood U_0 of a such that f is constant on U_0.

We first reduce to the case $n=1$. Take U_0 to be an polydisc containing a,

$$U_0=\{z \mid |z_i-a_i|<\epsilon \text{ for all } i\}.$$

For any $z\in U_0$, consider the 1-variable function

$$\mathbb{C}\ni w\mapsto f(wa+(1-w)z)\in\mathbb{C} \quad \text{for } |wa_i-(1-w)z_i-a_i|<\epsilon.$$

This function is defined on an open subset of $\mathbb{C}$ containing 0 and 1. It is a holomorphic function and its has absolute value has a local maximum at $w=1$. The 1-variable case then implies that this is constant, and hence $f(z)=f(a)$.

We now prove the theorem for $n=1$. Let $\Delta=B_R(a)$ be a disc centered at a such that $\overline{\Delta}\subseteq U$. Cauchy's formula 2.1 implies that

$$f(a)=\frac{1}{2\pi i}\int\limits_{\partial\Delta}\frac{f(w)}{w-a}\,dw=\frac{1}{2\pi i}\int_0^1\frac{f(a+Re^{2\pi i\theta})}{Re^{2\pi i\theta}}Re^{2\pi i\theta}\cdot 2\pi i\,d\theta=\int_0^1 f(a+Re^{2\pi i\theta})\,d\theta$$

where $w = a + Re^{2\pi i\theta}$. Therefore,

$$|f(a)| \leq \int_0^1 \underbrace{|f(a+Re^{2\pi i\theta})|}_{\leq |f(a)|}\, d\theta \overset{(*)}{\leq} |f(a)| \int_0^1 d\theta = |f(a)|,$$

assuming that $|f(z) \leq |f(a)|$ in a neighborhood of $\overline{\Delta}$ (this is true for R small enough). Therefore, the above inequalities are all equalities. Since $(*)$ is an equality and f is contunuous, we conclude that $|f(z)| = |f(a)|$ for all $z \in \partial\Delta$.

The same holds for any $0 < R' \leq R$, so $|f(z)|$ is constant in an open neighborhood of a. □

Exercise. Show that if $f = u + iv$ is holomorphic on some open connected subset and $u^2 + v^2$ is constant, then f is constant. (Apply $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and Cauchy–Riemann equations.)

3. Complex manifolds

If $U \subseteq \mathbb{C}^n$ is an open subset, consider the sheaf $\mathcal{O}_U$ of $\mathbb{C}$-algebras on U defined by

$$\mathcal{O}_U(V) = \{f \colon V \to \mathbb{C} \mid f \text{ holomorphic}\}.$$

We check that this is indeed a sheaf:

(1) have restriction maps: if $V_1 \subseteq V_2$ and f is holomorphic on V_2, then f is holomorphic on V_1,
(2) if $V = \bigcup V_i$ and $\varphi_i \colon V_i \to \mathbb{C}$ are holomorphic functions such that $\varphi_i|_{V_i \cap V_j} = \varphi_j|_{V_i \cap V_j}$, then there exists a unique $\varphi \colon V \to \mathbb{C}$ such that $\varphi|_{V_i} = \varphi_i$ for all i; indeed, if φ is such that $\varphi|_{V_i}$ is holomorphic for all i, φ is holomorphic.

Definition 3.1. A *complex manifold* of dimension n is a pair $(X, \mathcal{O}_X)$ where

(1) X is a topological space, assumed Hausdorff and having a countable basis of open subsets,
(2) $\mathcal{O}_X \subseteq \mathcal{C}_{X,\mathbb{C}}$ is a subsheaf of the sheaf of continuous $\mathbb{C}$-valued functions on X,

such that X can be written as

$$X = \bigcup_i U_i, \quad U_i \subseteq X \text{ open}$$

such that each $(U_i, \mathcal{O}_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ for some $V_i \subseteq \mathbb{C}^n$ is open with $\mathcal{O}_{V_i}$ is the sheaf of holomorphic functions on V_i.

Remark 3.2. Suppose $V_1, V_2 \subseteq \mathbb{C}^n$ are open and $f \colon (V_1, \mathcal{O}_{V_1}) \to (V_2, \mathcal{O}_{V_2})$ is an isomorphism, i.e. a homeomorphism $f \colon V_1 \to V_2$ which induces an isomorphism of sheaves: for all $U \subseteq V_2$,

$$\mathcal{O}(V_2) \overset{\cong}{\to} \mathcal{O}(f^{-1}(V_2)),$$
$$\varphi \mapsto \varphi \circ f.$$

This forces f and f^{-1} to be holomorphic functions. The converse is also true.

Definition 3.3.

(1) If $(X, \mathcal{O}_X)$ is a complex manifold, the section of $\mathcal{O}_X$ are the *holomorphic functions on X*.

(2) If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are complex manifolds, then a *holomorphic map*

$$(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$$

is a continuous map $f\colon X \to Y$ that induces a map of sheaves, i.e. for any $V \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(V)$, we have that $\varphi \circ f \in \mathcal{O}(f^{-1}(V))$.

Remark 3.4. If $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ are open subsets, this coincides with the previous definition.

Remark 3.5. If $U \subseteq \mathbb{C}^n$, $p \in U$,

$$\mathcal{O}_{\mathbb{C}^n,p} = \varinjlim_{V \ni p} \mathcal{O}_U(V).$$

To check that this is a local ring, we note that we have a map

$$\mathcal{O}_{U,p} = \varinjlim_{V \ni p} \mathcal{O}_U(V) \twoheadrightarrow \mathbb{C}$$

$$(V, \varphi) \mapsto \varphi(p)$$

whose kernel $\{(V, \varphi) \mid \varphi(p) = 0\} = \mathfrak{m}$ is the unique maximal ideal. Indeed, if $(V, \varphi) \notin \mathfrak{m}$, we may assume that $\varphi(z) \neq 0$ for all $z \in V$, and hence $\frac{1}{\varphi} \in \mathcal{O}(V)$. Hence $(\mathcal{O}_{U,p}, \mathfrak{m})$ is a local ring.

Remark 3.6. All such local rings for manifolds of fixed dimension are isomorphic. This is very different from the algebraic case.

Remark 3.7. One can define complex manifolds using atlases: X is a topological space with suitable properties and $X = \bigcup_i U_i$ is an open cover together with homeomorphisms $\varphi_i\colon U_i \xrightarrow{\cong} V_i \subseteq \mathbb{C}^n$, where $V_i \subseteq \mathbb{C}^n$ are open, such that for all i, j the map

$$\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} \varphi_j(U_i \cap U_j)$$

is biholomorphic.

We identify two such objects $(X, \mathcal{A})$, $(X, \mathcal{A}')$ if $\mathcal{A}$ and $\mathcal{A}'$ are compatible.

Remark 3.8. It is clear from the definition via atlases (Remark 3.7), using that holomorphic maps $\mathbb{C}^m \supseteq U \to \mathbb{C}^n$ are smooth, that every complex manifold of dimension n has an underlying real smooth manifold structure of dimension $2n$. To avoid confusion, we will write $X_{\mathbb{R}}$ for this real smooth manifold (if necessary). We have an inclusion of sheaves

$$\mathcal{O}_X \subseteq \mathcal{C}^\infty_{X,\mathbb{C}}.$$

Next, we will discuss:

- vector bundles in the smooth/holomorphic category,
- submanifolds,
- complex manifold associated to a smooth complex algebraic variety.

3.1. **Vector bundles.** If M is a smooth real manifold, a *real* (or *complex*) *vector bundle* of *rank* r on M is a smooth manifold E with a smooth map $E \to M$ such that for any $x \in M$, $\pi^{-1}(x)$ has the structure of a vector space over $\mathbb{R}$ (respectively $\mathbb{C}$)of dimension r such that there is an open cover $M = \bigcup_i U_i$ such that we have isomorphisms

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\cong} & U_i \times \mathbb{R}^r \\ & \searrow \quad \swarrow & \\ & U_i & \end{array} \qquad \text{(resp. } U_i \times \mathbb{C}^r\text{)}$$

(respectively, $U_i \times \mathbb{C}^r$), inducing linear maps on the fibers.

Given such E, we get a sheaf $\mathcal{E}$ on M such that

$$\mathcal{E}(U) = \{s\colon U \to E \text{ smooth} \mid \pi \circ s = 1_U\}.$$

This gives an equivalence of categories

$$\left\{ \begin{array}{c} \text{real (complex) vector bundles} \\ \text{on } M \text{ (of rank } r\text{)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{locally free sheaves (of rank } r\text{)} \\ \text{of } \mathcal{C}^\infty_{M,\mathbb{R}}\text{-modules } (\mathcal{C}^\infty_{M,\mathbb{C}}\text{-modules)} \end{array} \right\}.$$

We will consider the corresponding notion in the category of complex manifolds. For *complex vector bundles*, we assume that E is a complex manifold, π is holomorphic.

These correspond to locally free sheaves of $\mathcal{O}_M$-modules. Note that associated to such E, we will have: sheaves of smooth sections and sheaves of holomorphic sections.

Definition 3.9. Let X be a complex manifold of dimension n. A *closed submanifold* of X of codimension r is a closed subset $Y \subseteq X$ such that for all $p \in Y$, there is a chart $p \in U \underset{\cong}{\xrightarrow{\varphi}} V \subseteq \mathbb{C}^n$ such that

$$\varphi(U \cap Y) = \{z \in V \mid z_1 = \cdots = z_r = 0\}.$$

It is easy to see that by restricting such charts to Y, we get a holomorphic atlas on Y, making it a complex manifold of dimension $n - r$.

The universal property of submanifolds is: given a holomorphic $g\colon Z \to X$ such that $g(Z) \subseteq Y$, there is a unique holomorphic map $g'\colon Z \to Y$ such that $\mathrm{incl} \circ g' = g$.

Proposition 3.10. *If $U \subseteq \mathbb{C}^n$ is open and $f_1, \ldots, f_r \in \mathcal{O}(U)$ are such that*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(p)\right) = r \leq n$$

for all $p \in U$, then

$$Y = \{z \in U \mid f_1(z) = f_2(z) = \cdots = f_r(z) = 0\}$$

is a closed submanifold of U of codimension r.

Proof. Given $p \in Y$, we may assume that $\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{1\leq i,j\leq n} \neq 0$. Define

$$\varphi\colon U \to \mathbb{C}^n,$$
$$z \mapsto (f_1(z), \ldots, f_r(z), z_{r+1}, \ldots, z_n).$$

Then

$$\det\left(\frac{\partial \varphi_i}{\partial z_j}(p)\right) \neq 0$$

and we apply the Inverse Function Theorem 2.12 to see that φ is biholomorphic in some neighborhood of p. In the neighborhood, φ is the desired chart. ☐

Basic properties of holomorphic functions we discussed extend to this setting. We recall a few of them for completeness. Let X be a complex manifold.

(1) If $f \in \mathcal{O}(X)$ is such that $f|_U = 0$ for some $U \subseteq X$ open, and X is connected, then $f = 0$.
(2) (Maximum modulus principle) If $f \in \mathcal{O}(X)$ is such that $|f|$ has a local max, X is connected, then f is constant.

Corollary 3.11. *If X is a compact connected complex manifold, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.*

Proof. Since X is compact, for any $f \in \mathcal{O}(X)$, $|f|$ has a maximum. Then the maximum modulus principle implies that f is constant. ☐

3.2. **The complex manifold associated to a smooth complex algebraic variety.** Let X be a smooth complex algebraic variety of pure dimension n. Choose an affine open subset $U \subseteq X$ and let $U \hookrightarrow \mathbb{C}^N$ be a closed immersion, $r = N - n$. Since U and $\mathbb{C}^N$ are smooth, can cover $\mathbb{C}^N$ by open subsets V_i (in the Zariski topology) such that if $V_i \cap U \neq$ then $V_i \cap U \hookrightarrow V_i$ is cut out by r equations $f_1, \ldots, f_r \in \mathcal{O}(V_i)$ with

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(p)\right) = r$$

for all $p \in V_i \cap U$.

Applying Proposition 3.10, each $V_i \cap U \hookrightarrow V_i$ is a closed complex submanifold of codimension r.

Exercise. Check that the resulting transition maps are holomorphic, using the fact that rational maps are holomorphic.

Exercise. Show that if $f\colon X \to Y$ is a morphism between smooth complex algebraic varieties, then the induced map $X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is holomorphic.

We now discuss an application.

Theorem 3.12. *If X is a connected complex algebraic variety, then X^{an} is connected.*

We first prove this theorem when X is a smooth connected projective curve over $\mathbb{C}$.

Proof when X is a smooth, projective curve. We first prove this when X is a smooth connected projective curve over $\mathbb{C}$. We know that X^{an} is a 1-dimensional complex manifold, which is compact since X is complete (by Theorem 1.8).

Suppose that $X^{\text{an}} = U \cup V$ is a disjoint union with U, V open in X^{an} and nonempty. Take $P \in U$. If $n \gg 0$ ($n \geq 2 \cdot \text{genus}(X)$), $\mathcal{O}_X(nP)$ is globally generated. Then there exists $s \in \Gamma(X, \mathcal{O}_X(nP))$ which does not vanish at P. Then

$$nP \sim Q_1 + \cdots + Q_n \quad \text{for } Q_i \neq P$$

so there exists $\varphi \in \mathbb{C}(X)^*$ such that $\text{div}(\varphi) = (Q_1 + \cdots + Q_n) - nP$, so φ gives a regular function $X \backslash \{P\} \to \mathbb{C}$. Note that it is holomorphic. By restricting to V, we get a holomorphic map $V \overset{g=\varphi|_V}{\to} \mathbb{C}$. Since V is a compact complext manifold, g is constant by Corollary 3.11. In particular, φ takes the same value infinitely many times, so φ is constant, and hence $\div\varphi = 0$. This is a contradiction. ∎

To reduce the general case to $\dim X = 1$, we use the following result.

Proposition 3.13. *Let X be an algebraic variety over $k = \overline{k}$. For any x_1, $x_2 \in X$, there is an irreducible curve $C \subseteq X$ such that $x_1, x_2 \in C$.*

Proof. We may assume that $n = \dim X \geq 2$.

(1) By Chow's lemma, there is a surjective morphism $\pi \colon \tilde{X} \to X$ where $\tilde{X}$ is irreducible and quasi-projective. If $\tilde{x_1}$, $\tilde{x_2}$ lie above x_1, x_2, it is enough to find a curve $\tilde{C}$ on $\tilde{X}$ through $\tilde{x_1}$, $\tilde{x_2}$ and take $C = \overline{\pi(C)}$. We may hence assume X is quasi-projective.
(2) Choose a locally closed immersion $X \hookrightarrow \mathbb{P}^N$. It is enough to prove the statement for $\overline{X}$. We may hence assume that X is projective.

Consider the blow up of X at $\{x_1, x_2\}$:

$$\begin{array}{ccc} Y = \text{Bl}_{\{x_1,x_2\}}X & \hookleftarrow & E_i = p^{-1}(x_i) \\ \downarrow p & & \\ X & & \end{array}$$

where $\dim E_i = n - 1$.

The variety Y is projective since X is, so we may choose an embedding $Y \hookrightarrow \mathbb{P}^N$. Cut Y with $n-1$ general hyperplane $H_1, \ldots, H_{n-1}$. Since $\dim E_i = n - 1$,

$$E_i \cap H_1 \cap \cdots \cap H_{n-1} \neq \emptyset \quad \text{for } i = 1, 2.$$

If $Z = Y \cap H_1 \cap \cdots \cap H_{n-1}$, the curve $C = p(Z)$ satisfies the requirements. Using Bertini's Theorem: a general hyperplane section of an irreducible projective variety of dimension ≥ 2 is irreducible, and hence Z is irreducible.

We need to assume that $\dim(Z \cap E_i) = 0$ for $i = 1, 2$. This is okay since the H_is are general. ∎

We can now finish the proof of Theorem 3.12.

Proof of Theorem 3.12. We just have to reduce to the smooth, projective curve case from the general case.

First, we may assume that X is irreducible (since by hypothesis we can go from any irreducible component to any other one via points of intersection).

For an irreducible algebraic variety X over $k = \overline{k}$, for any $x, y \in X$, there is an irreducible curve C such that $x, y \in C$ by Proposition 3.13. We may hence assume that X is an irreducible curve.

If $\tilde{X} \to X$ is the normalization, it is enough to show that $\tilde{X}^{\mathrm{an}}$ is connected. We may hence assume that X is smooth.

Finally, let $X \subseteq \overline{X}$ where $\overline{X}$ is a smooth, projective, connected curve. We showed last time that $\overline{X}^{\mathrm{an}}$ is connected.

We now use that if M is smooth real manifold of dimension ≥ 2, $p \in M$, and M is connected, then $M \setminus \{p\}$ is also connected.

Indeed, if $M \setminus \{p\} = U \cup V$ is a disjoint union of open non-empty sets, then $p \in \overline{U} \cap \overline{V}$ because M is connected. Choose a neighborhood W of p such that W is isomorphic to a ball. Then $W \setminus \{p\}$ is disconnected. This is a contradiction, since it is clearly path-connected. □

3.3. **More examples of complex manifolds.** Suppose X is a complex manifold and G is a group acting on X via holomorphic maps. Suppose

(1) for all $x \in X$, there exists an open neighborhood $U \ni x$ such that $U \cap gU \neq \emptyset$ implies $g = e$ (this is sometimes called a *properly discontinuous* action),
(2) for all $x, y \in X$ such that x, y are not in the same orbit, there exist open neighborhoods $U \ni x$, $V \ni y$ such that $gU \cap V = \emptyset$ for all g.

Note that (1) implies that the quotient map $\pi \colon X \to X/G$ is a covering space. Moreover, since the transition maps are holomorphic, there is a unique complex manifold structure on X/G such that π is holomorphic. Condition (2) implies that X/G is Hausdorff.

Example 3.14 (Complex tori)**.** Let V be an n-dimensional complex vector space and $\Lambda \subseteq V$ be a lattice (i.e. a free abelian group of rank $2n$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong V$). The natural action of Λ by translations satisfies (1) and (2) above, and hence

$$V \xrightarrow{\pi} Z = V/\Lambda$$

gives a complex manifold Z. Note that

$$V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (S^1)^{2n},$$

and hence topologically, Z is a $2n$-dimensional torus.

When $n = 1$, the resulting Z is an analytic construction of *elliptic curves*, which are algebraic. We will see that for $n \geq 2$, most of these do not come from algebraic varieties. However, they are still Kähler manifold.

Example 3.15 (Hopf surface)**.** Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{(0,0)\}$, where the generator γ of $\mathbb{Z}$ acts by $(z_1, z_2) \mapsto (2z_1, 2z_2)$.

This clearly satisfies conditions (1) and (2), so we get a complex manifold structure on the quotient.

We have a diffeomorphism:

$$\mathbb{C}^2 \setminus \{(0,0)\} \xrightarrow{\cong} S^3 \times \mathbb{R}$$
$$(z_1, z_2) \mapsto \left(\frac{1}{\sqrt{|z_1|^2 + |z_2|^2}}(z_1, z_2), \log \sqrt{|z_1|^2 + |z_2|^2} \right)$$

under which the action of γ translates to

$$(u, t) \mapsto (u, t + \log 2).$$

Therefore, the Hopf surface is topologically

$$\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z} \cong S^3 \times S^1.$$

We will later see these manifolds are not even Kähler, and hence do not come from algebraic surfaces.

3.4. **Orientation.** If V is a 1-dimensional real vector space, an *orientation* on V is a choice of element in $V/\mathbb{R}^*_{>0}$. Note that an orientation of V is the same as an orientation of V^*.

If V is an n-dimensional vector space, an *orientation* on V is an orientation on $\Lambda^n V$.

If X is a smooth real manifold and E is a real vector bundle on X, an *orientation* on E is a compatible system of orientations on $E_{(x)}$ for all $x \in X$, i.e. locally have trivializations $\pi^{-1}(U) \cong U \times \mathbb{R}^r$ where $\pi \colon E \to X$, preserving the orietations on the fibers.

Note that an orientation on E corresponds to an orientation on E^*.

Definition 3.16. An *orientation* on a smooth real manifold X is an orientation on the tangent bundle TX (or equivalently on the cotangent bundle T^*X).

Giving an orientation is equivalent to giving a system of charts such that for all transition maps

$$f = (f_1, \ldots, f_n) \colon U \to \mathbb{R}^n, \quad \det \frac{\partial f_i}{\partial x_j} > 0.$$

Note that if X is a complex manifold and we consider the smooth manifold structure, we saw that if we take a system of holomorphic charts, then for the transition maps $f = (f_1, \ldots, f_n) \colon U \to \mathbb{C}^n$ where $U \subseteq \mathbb{C}^n$,

$$\det(\text{real Jacobian}) = \left| \det \frac{\partial f_i}{\partial z_j} \right|^2 > 0.$$

Therefore, we have a canonical orientation on X.

By convention: given chart $f \colon U \to \mathbb{C}^n$, the orrientation on U corresponds to the orientation on $\mathbb{C}^n$ given by the top form $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$.

3.5. **The analytic space associated to an algebraic variety.** We first discuss the local model. For $U \subseteq \mathbb{C}^n$ open in the classical topology, $f_1, \ldots, f_r \in \mathcal{O}(U)$, consider

$$Z = \{u \in U \mid f_1(u) = \cdots = f_r(u) = 0\}.$$

Consider on Z the sheaf given by

$\mathcal{O}_Z(V) = \{f \colon V \to \mathbb{C} \mid \text{locally } f \text{ extends to a holomorphic function on an open subset in } \mathbb{C}^n\}.$

If $Z \overset{j}{\hookrightarrow} U$ is the inclusion, we get a map

$$\mathcal{O}_U \twoheadrightarrow j_* \mathcal{O}_Z$$

and the kernel is $\mathcal{I}_{Z/U}$ given by

$$\Gamma(V, \mathcal{I}_{Z/U}) = \{f \colon V \to \mathbb{C} \mid f|_{V \cap Z} = 0\}.$$

Note that $(Z, \mathcal{O}_Z)$ is a locally ringed space.

Definition 3.17. A (reduced) *analytic space* is a locally ringed space $(X, \mathcal{O}_X)$ such that

(1) X is a Hausdorff topological space with a countable basis for the topology,
(2) there is an open cover $X = \bigcup_i W_i$ such that each $(W_i, \mathcal{O}_{W_i})$ is isomorphic as a locally ringed space to a local model as above.

The sections of $\mathcal{O}_X$ are called *holomorphic functions* on X.

Definition 3.18. A *holomorphic map* between analytic spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a continuous map $f \colon X \to Y$ such that for any $V \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(V)$, we have $\varphi \circ f \in \mathcal{O}_X(\varphi^{-1}(V))$.

Examples 3.19.

(1) Every complex manifold is canonically an analytic space.
(2) If X is a separated algebraic variety, we have a sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ on X^{an} that makes it an analytic space. We do it locally. Choose affine open subsets covering X; each such open subspace covering U has a closed immersion $U \hookrightarrow \mathbb{C}^N$ (cut out by finitely many polynomials), so we have a sheaf $\mathcal{O}_{U^{\mathrm{an}}}$ on U^{an} making it an analytic space. It is easy to check that these sheaves are compatible on intersections, so we get a sheaf $\mathcal{O}_{X^{\mathrm{an}}}$ on X^{an}.

We get in this way a functor

$$\{\text{complex algebraic varieties}\} \to \{\text{analytic spaces}\}.$$

However, in this class, we deal with smooth varieties, and hence we only have to work with complex manifolds.

3.6. **Comparison results.** Let X be a complex algebraic variety. As we saw above, it has an associated analytic space X^{an}.

We have a morphism of locally ringed spaces:

$$(\varphi, \varphi^{\#}) \colon (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$$

defined by $\varphi(x) = x$ and

$$\begin{aligned}\varphi^{\#}\colon \mathcal{O}_X &\to \varphi_*\mathcal{O}_{X^{\mathrm{an}}}\\ \mathcal{O}_X(U) &\to \mathcal{O}_{X^{\mathrm{an}}}(U)\\ f &\mapsto f\end{aligned}$$

(since every regular function on U is holomorphic). The corresponding ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is a local homomorphism.

Given an $\mathcal{O}_X$-module $\mathcal{F}$, let

$$\mathcal{F}^{\mathrm{an}} = \varphi^*(\mathcal{F}) = \varphi^{-1}(\mathcal{F}) \otimes_{\varphi^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\mathrm{an}}},$$

which is an $\mathcal{O}_{X^{\mathrm{an}}}$-module.

In particular, for every $x \in X$, we have a canonical isomorphism

$$(\mathcal{F}^{\mathrm{an}})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{\mathrm{an}},x}.$$

We will see later that $\mathcal{O}_{X^{\mathrm{an}},x}$ is a Noetherian ring and the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is flat. In particular, this will imply that the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is exact.

Note that we have canonical maps:

- $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$,
- $H^i(X,\mathcal{F}) \to H^i(X^{\mathrm{an}},\mathcal{F}^{\mathrm{an}})$,
- more generally, $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$.

Theorem 3.20 (GAGA, part 1)**.** *If X is a complete variety, then the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is fully faithful on coherent sheaves. Moreover, for all $\mathcal{F},\mathcal{G}$ coherent, the map*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$$

is an isomorphism.

The theorem is due to Serre when X is projective and due to Grothendieck when X is complete. There is also a relative version for proper morphisms. We will prove this theorem only when X is projective.

The category which is the target of this functor (i.e. which $\mathcal{F}^{\mathrm{an}}$ belongs to) still have to be defined.

Definition 3.21. In general, if $(X,\mathcal{O}_X)$ is a locally ringed space, an $\mathcal{O}_X$-module, $\mathcal{F}$ is *locally finitely generated* if for any $x \in X$, there is an open neighborhood $U \ni x$ and $s_1,\dots,s_n \in \mathcal{F}(U)$ such that

$$s_{1,y},\dots,s_{n,y} \in \mathcal{F}_y$$

generate $\mathcal{F}_y$ over $\mathcal{O}_{X,y}$ for all $y \in U$.

Definition 3.22. An $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if

- it is locally finitely generated,
- for every open subset $U \subseteq X$, $s_1,\dots,s_r \in \mathcal{F}(U)$, the kernel of the induced map

  $$\ker(\mathcal{O}_U^{\oplus r} \to \mathcal{F})$$

  is locally finitely generated.

Exercise. Check that on algebraic varieties, this coincides with the definition in Hartshorne [Har77].

Theorem 3.23 (Oka). *If X is an analytic space, then $\mathcal{O}_X$ is coherent. (In particular, also locally free $\mathcal{O}_X$-modules of finite rank are coherent)*

If X is an algebraic variety over $\mathbb{C}$, then any coherent sheaf on $\mathcal{F}$ on X has a finite presentation, so $\mathcal{F}^{\mathrm{an}}$ is coherent.

Theorem 3.24 (GAGA, part 2). *If X is complete, the functor*

$$\{\text{coherent } \mathcal{O}_X\text{-modules}\} \to \{\text{coherent } \mathcal{O}_{X^{\mathrm{an}}}\text{-modules}\}$$
$$\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$$

is an equivalence of categories.

Remark 3.25. In particular, in this case we have an equivalence of categories

$$\{\text{locally free } \mathcal{O}_X\text{-modules}\} \to \{\text{locally free } \mathcal{O}_{X^{\mathrm{an}}}\text{-modules}\}.$$

To show that $\mathcal{F}$ is locally free if $\mathcal{F}^{\mathrm{an}}$ is, use the fact that $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$ is faithfully flat.

Remark 3.26.

(1) In general, $(\mathcal{O}_X)^{\mathrm{an}} = \mathcal{O}_{X^{\mathrm{an}}}$.
(2) If X is a complex algebraic variety and $E \xrightarrow{\pi} X$ is an algebraic vector bundle with sheaf of sections $\mathcal{E}$, the holomorphic vector bundle $E^{\mathrm{an}} \xrightarrow{\pi^{\mathrm{an}}} X^{\mathrm{an}}$ has the sheaf of sections $\mathcal{E}^{\mathrm{an}}$.
(3) Applying the theorem for coherent ideal sheaves, in the setting of the theorem, every closed analytic subspace of X^{an} is equal to Y^{an} for some closed subvariety $Y \subseteq X$. (When $X = \mathbb{P}^N$, this was known as Chow's Theorem.)
(4) Using (3) and the graph, any morphism $f\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ comes from a morphism $X \to Y$. Therefore, the functor

$$\{\text{complete algebraic varieties}\} \to \{\text{compact analytic spaces}\}$$
$$X \mapsto X^{\mathrm{an}}.$$

3.7. The ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Definition 3.27. An element $f \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$ is *convergent* if there is an R such that

$$f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$$

converges uniformly and absolutely for $|z_i| < R$ for all i. We write

$$\mathbb{C}\{z_1, \ldots, z_n\} = \{f \in \mathbb{C}[\![z_1, \ldots, z_n]\!] \mid f \text{ is convergent}\} \subseteq \mathbb{C}[\![z_1, \ldots, z_n]\!].$$

It is easy to check that $f = \sum a_\alpha z^\alpha$ is convergent if and only if there exists $R > 0$ such that $\{|a_\alpha| R^{|\alpha|}\}_\alpha$ is bounded. This is also equivalent to

$$\limsup_{|\alpha| \to \infty} |a_\alpha|^{1/|\alpha|} < \infty$$

(by the Cauchy-Hadamard Theorem).

We have a map

$$\begin{aligned} \mathcal{O}_{\mathbb{C}^n,0} &\to \mathbb{C}[\![z_1,\dots,z_n]\!] \\ f &\mapsto \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha \qquad \text{where } a_\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|f}}{\partial z^\alpha}(0). \end{aligned}$$

Here $\mathcal{O}_{\mathbb{C}^n,0}$ is the ring of germs of holomorphic functions at 0. Recall that if f is holomorphic at 0, then

$$f(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)z^\alpha$$

converges absolutely and uniformly in a neighborhood of 0. By definition, the image of the above map is hence

$$\mathbb{C}\{z_1,\dots,z_n\}$$

and it is clear it is injective. Moreover, it is clearly a ring homomorphism.

Conclusion. If $p \in M$ where M is a complex manifold, then

$$\mathcal{O}_{M,p} \cong \mathbb{C}\{z_1,\dots,z_n\}$$

where $n = \dim X$.

The next goal is to show that $\mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian. The idea is to proceed by induction and the key ingredient is the Weierstrass Preparation Theorem.

Definition 3.28. A *Weierstrass polynomial with respect to* z_n is an element of $\mathbb{C}\{z_1,\dots,z_n\}$ of the form

$$z_n^d + a_1(z_1,\dots,z_{n-1})z_n^{d-1} + \cdots + a_d(z_1,\dots,z_{n-1})$$

such that $a_0(0) = 0$ for $1 \le i \le d$.

Theorem 3.29 (Weierstrass Preparataion Theorem). *Given* $f \in \mathbb{C}\{z_1,\dots,z_n\}$ *such that* $f(0,\dots,0,z_n) \neq 0$*, there exist unique* $g,h \in \mathbb{C}\{z_1,\dots,z_n\}$ *such that* $h(0) \neq 0$*,* g *is a Weierstrass polynomial, and*

$$f = g\cdot h.$$

Remark 3.30.

(1) If $n = 1$ and $f \in \mathbb{C}\{z\}$, $f \neq 0$, then

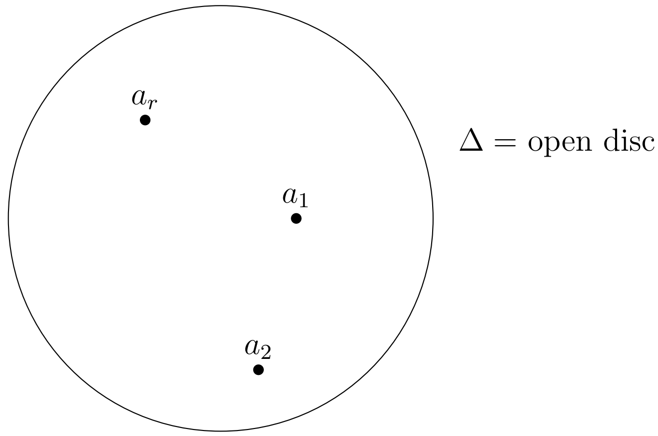
$$f = z^d h$$

where $h(0) \neq 0$. Weierstrass Preparation Theorem 3.29 is a generalization of this statement to more variables.

(2) Note that (1) implies that (still for $n = 1$) if $f \in \mathcal{O}(U)$, the zeroes of f do not accumulate in U.

(3) The condition that $f(0,\dots,0,z_n) \neq 0$ can always be achieved (if $f \neq 0$) by a linear change of variables.

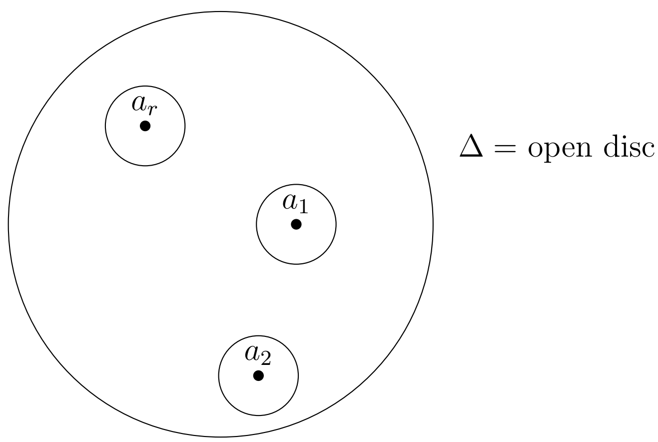
Recall (a special case of) the Residue Theorem. Suppose $f \in \mathcal{O}(U \setminus \{a_1,\dots,a_r\})$ and there is a disc $\overline{\Delta} \subseteq U$ such that $a_i \in \Delta$.



Then

$$\frac{1}{2\pi i}\int_{\partial\Delta}\varphi(z)dz=\sum_{i=1}^{r}\mathrm{Res}_{a_i}(\varphi_i)$$

In fact, we will only need this when φ is meromorphic at a_i with pole of order ≤ 1. Using $d(\varphi(z)dz)=0$ and Stokes' Theorem, we can reduce the computation of the integral to the case $r=1$ by cutting out small discs around $a_1,\ldots,a_r$.



In this case, we may write $\varphi=\frac{\psi}{z-a}$ and then $\mathrm{Res}_a(\varphi)=\psi(a)$. Then

$$\frac{1}{2\pi i}\int_{|w-a|=r}\frac{\psi(w)}{w-a}dw=\psi(a)$$

by Cauchy's formula 2.1.

In our case, we will take $f\in\mathcal{O}(U)$, $\overline{\Delta}\subseteq U$, and consider

$$\frac{1}{2\pi i}\int_{\partial\Delta}z^j\frac{f'(z)}{f(z)}dz$$

where f has no zeroes on $\partial\Delta$. Suppose a is a zero of f and write $f=(z-a)^m h$, $h(a)\neq 0$. Then

$$z^j\frac{f'(z)}{f(z)}=z^j\left(\frac{m}{z-a}+\frac{h'}{h}\right)$$

which implies that

$$\mathrm{Res}_a z^j \frac{f'(z)}{f(z)} = ma^j.$$

Overall, the conclusion is that:

$$\frac{1}{2\pi i}\int_{\partial\Delta} z^j \frac{f'(z)}{f(z)} dz = \lambda_1^j + \cdots + \lambda_m^j \tag{1}$$

if $\lambda_1, \ldots, \lambda_m$ are the roots of f in Δ, listed with multiplicity.

Proof of Weierstrass Preparation Theorem 3.29. Let $z' = (z_1, \ldots, z_{n-1})$ and write

$$f_{z'}(z_n) = f(z', z_n)$$

where f is a holomorphic function on $\mathbb{C}^n \supseteq U \ni 0$. Let $\epsilon_n > 0$ be such that

$$f(0, \ldots, 0, z_n) \neq 0 \text{ for } 0 < |z_n| \leq \epsilon_n.$$

Choose $\epsilon' > 0$ such that if z' satisfies that if $|z_i| < \epsilon'$ for $1 \leq i \leq n-1$ and $|z_n| = \epsilon_n$, then $f(z', z_n) \neq 0$, and

$$\{z \mid |z_i| < \epsilon' \text{ for } i \leq n-1,\ |z_n| < \epsilon_n\} \subseteq U.$$

Otherwise, one can choose $z_i \to 0$ such that $f(z', z_n) = 0$ and continuity of f will contradict the way we chose ϵ_n.

Given z' such that $|z_i| < \epsilon'$ for $i \leq n-1$, let

$$\lambda_1(z'), \ldots, \lambda_m(z')$$

be the zeroes of $f_{z'}$ in

$$\{z_n \mid |z_n| < \epsilon_n\},$$

listed with multiplicities. By equation (1),

$$\sum_{i=1}^m \lambda_i(z')^j = \frac{1}{2\pi i}\int_{|w|=\epsilon_n} w^j \cdot \frac{\frac{\partial f}{\partial z_n}(z', w)}{f(z', w)} dw.$$

Note that the right hand side is a holomorphic function as a function of z'. For $j = 0$, the left hand side is an integer, and hence constant. This shows that

$$m = \mathrm{ord}_{z_n} f(0, \ldots, 0, z_n)$$

by taking $z' = 0$.

If $\sigma_1(z'), \ldots, \sigma_m(z')$ are the symmetric functions of $\lambda_1(z'), \ldots, \lambda_m(z')$, then each σ_i is holomorphic for $|z_j| < \epsilon'$, $j \leq n-1$ and $\sigma_i(0) = 0$ for $1 \leq i \leq m$. Let

$$g = z_n^m - \sigma_1(z') z_n^{m-1} + \cdots + (-1)^m \sigma_m(z')$$

which is a Weierstrass polynomial.

It is clear that the function $\dfrac{f}{g}$ is well-defined and holomorphic in

$$\{z \mid |z_j| < \epsilon' \text{ for } j \leq n-1,\ |z_n| < \epsilon_n\} \setminus \{g = 0\}.$$

For every z', $\dfrac{f(z', -)}{g(z', -)}$ extends to a holomorphic function of z_n for $|z_n| < \epsilon_n$.

Exercise. Check that therefore $h = \dfrac{f}{g}$ is in fact holomorphic in a neighborhood of 0 and $h(0) \neq 0$.

This proves existence.

Uniqueness is straightforward. If $f = g' \cdot h'$ as in the theorem and $g' = z_n^{d'} + \cdots$, we see that $f(0, \dots, 0, z_n) = z_n^{d'} \cdot h(0, \dots, 0, z_n)$ which implies that $d' = m$. For every z', $f(z', -)$ has d roots in $|z_n| < \epsilon_n$, so $g'(z', -)$ vanishes on these with the right multiplicities. For degree reasons, this implies that $g' = g$. □

Corollary 3.31. *For any n, $\mathbb{C}\{z_1, \dots, z_n\}$ is Noetherian.*

Proof. We proceed by induction on $n \geq 0$. When $n = 0$, this ring is a field, which is Noetherian. Suppose $I \subseteq \mathbb{C}\{z_1, \dots, z_n\}$, $I \neq 0$ is an ideal. Let $(f_\lambda)_{\lambda \in \Lambda}$ be a set of generators for I. Fix $\lambda_0 \in \Lambda$ such that $f_{\lambda_0} \neq 0$. Do a linear change of variables to assume that

$$f_{\lambda_0}(0, \dots, 0, z_n) \neq 0.$$

For $\lambda \neq \lambda_0$, if $f_\lambda(0, \dots, 0, z_n) = 0$, replace f_λ by $f_\lambda + f_{\lambda_0}$. We may hence assume that $f_\lambda(0, \dots, 0, z_n) \neq 0$ for all λ.

Now, by Weierstrass Preparation Theorem 3.29, we may write for each λ

$$f_\lambda = (\text{invertible element}) \cdot (\text{element of } \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]).$$

This shows that I is generated (as an ideal) by

$$I \cap \mathbb{C}\{z_1, \dots, z_{n-1}\}[z_n]$$

which is finitely generated by inductive hypothesis and Hilbert's basis theorem (if R is Noetherian, $R[x]$ is also Notherian). □

Remark 3.32. The same proof shows that $\mathbb{C}[\![z_1, \dots, z_n]\!]$ is Noetherian. However, it is easier to see that it is the completion of $\mathbb{C}\{z_1, \dots, z_n\}$ (as shown in the proof of Proposition 3.33), and hence Noetherian.

Proposition 3.33. *If X is a smooth algebraic variety, then the ring homomorphism*

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},x}$$

is (faithfully) flat for every $x \in X$.

Proof. **Step 1.** Suppose $X = \mathbb{A}^n$. Let $R = \mathbb{C}\{z_1, \dots, z_n\} \supseteq \mathfrak{m} = \{f \mid f(0) = 0\}$. It is easy to check that $\mathfrak{m} = (z_1, \dots, z_n)$. Moreover,

$$R/\mathfrak{m}^N \cong \mathbb{C}[z_1, \dots, z_n]/(z_1, \dots, z_n)^N.$$

Therefore,

$$\hat{R} = \varprojlim R/\mathfrak{m}^N \cong \varprojlim \mathbb{C}[z_1, \dots, z_n]/(z_1, \dots, z_n)^N \cong \mathbb{C}[\![z_1, \dots, z_n]\!].$$

We have the following commuting square:

$$\begin{array}{ccc} \mathbb{C}[z_1,\ldots,z_n]_{(z_1,\ldots,z_n)} & \longrightarrow & R \\ \downarrow & & \downarrow \\ \widehat{\mathbb{C}[z_1,\ldots,z_n]}_{(z_1,\ldots,z_n)} & \xrightarrow{\cong} & \hat{R} \end{array}$$

Recall that if $(S,\mathfrak{n})$ is a local Noetherian ring, the map $S \to \hat{S}$ is (faithfully) flat. Since the vertical maps are faithfully flat (as R is Noetherian by 3.31), the top horizontal map is faithfully flat.

(Since $\hat{R}$ is a regular ring of dimension n, R is a regular ring of dimension n.)

Step 2. Prove the following fact.

Exercise. If $X \subseteq Y$ are smooth algebraic varieties where X is defined by the coherent ideal $\mathcal{I}$ and we consider $X^{\mathrm{an}} \subseteq Y^{\mathrm{an}}$, the ideal of $\mathcal{O}_{Y^{\mathrm{an}}}$ vanishing on X^{an} is $\mathcal{I}^{\mathrm{an}}$. (Hint: reduce to the case $(x_1,\ldots,x_r = 0) = X \subseteq \mathbb{C}^n = Y$.)

In general, if X is a smooth algebraic variety, $X \subseteq \mathbb{C}^N$ defined by the ideal $\mathcal{I}$, then the exercise shows that

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}^N,x}/\mathcal{I}\mathcal{O}_{\mathbb{C}^N,x} = \mathcal{O}_{X,x} & \xrightarrow{\varphi} & \mathcal{O}_{X^{\mathrm{an}},x} = \mathcal{O}_{(\mathbb{C}^N)^{\mathrm{an}},x}/\mathcal{I}\mathcal{O}_{(\mathbb{C}^N)^{\mathrm{an}},x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathbb{C}^N,x} & \xrightarrow{\psi} & \mathcal{O}_{(\mathbb{C}^N)^{\mathrm{an}},x} \end{array}$$

Since we have shown that ψ is flat, this shows that φ is also flat. Because this is a local statement, it was enough to consider the case $X \subseteq \mathbb{C}^N$. □

The exercise in the proof of Proposition 3.33 has another consequence. If $i\colon X \hookrightarrow Y$ is a closed immersion of smooth algebraic varieties and $\mathcal{F}$ is a sheaf on X, then the canonical map

$$(i_*(\mathcal{F}))^{\mathrm{an}} \to (i_{\mathrm{an}})_*(\mathcal{F}^{\mathrm{an}})$$

is an isomorphism.

Exercise. Show that we have such a morphism which gives an isomorphism on stalks. (Hint: use the other consequence of the exercise)

We can finally prove a part of GAGA, part 1, 3.20.

Theorem 3.34. *If X is a smooth projective complex algebraic variety. Then:*

1. *the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ is exact,*
2. *if $\mathcal{F},\mathcal{G}$ are coherent on X, we have an isomorphism*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathcal{O}_{X^{\mathrm{an}}}}(\mathcal{F}^{\mathrm{an}},\mathcal{G}^{\mathrm{an}})$$

Assume that:

$$H^i((\mathbb{P}^n)^{\mathrm{an}}, \mathcal{O}_{(\mathbb{P}^n)^{\mathrm{an}}}) = \begin{cases} \mathbb{C}, & i = 0, \\ 0, & i > 0. \end{cases}$$

We will see this later (cf. section 9.3) via Hodge theory when we will compute $H^*((\mathbb{P}^n)^{\mathrm{an}}, \mathbb{C})$.

Proof of Theorem 3.34. We have a closed immersion $X \hookrightarrow \mathbb{P}^n$. We first treat the case $\mathcal{F} = \mathcal{O}_X$. To show that

$$H^i(X, \mathcal{G}) \to H^i(X^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}}),$$

by pushing forward to $\mathbb{P}^N$, we may assume that $X = \mathbb{P}^n$.

Next, suppose $\mathcal{G} = \mathcal{O}_{\mathbb{P}^n}(m)$ and argue by induction on n. The case $n = 0$ is trivial. The key exact sequence to use is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0$$

tensored with $\mathcal{O}(m)$. Since we know the assertion for $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$, then 5-Lemma implies that the assertion holds for $\mathcal{O}_{\mathbb{P}^n}(m)$ if and only if it holds for $\mathcal{O}_{\mathbb{P}^n}(m-1)$. Since we assume we know this for $m = 0$, we know it for all m.

Now, work with general $\mathcal{G}$. We argue by decreasing induction on i that to show that

$$H^i(X, \mathcal{G}) \to H^i(X^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

is an isomorphism. For $i \gg 0$, both are 0, so the assertion is true. For the induction step, given $\mathcal{G}$ have a short exact sequence

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{E}$ is a direct sum of $\mathcal{O}(m)$, so we know the assertion for $\mathcal{E}$. The long exact sequence in cohomology then gives

$$\begin{array}{ccccccccc} H^i(X,\mathcal{G}') & \longrightarrow & H^i(X,\mathcal{E}) & \longrightarrow & H^i(X,\mathcal{G}) & \longrightarrow & H^{i+1}(X,\mathcal{G}') & \longrightarrow & H^{i+1}(X,\mathcal{E}) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\cong} & & \downarrow \\ H^i(X^{\mathrm{an}},(G')^{\mathrm{an}}) & \to & H^i(X^{\mathrm{an}},\mathcal{E}^{\mathrm{an}}) & \to & H^i(X^{\mathrm{an}},\mathcal{G}^{\mathrm{an}}) & \to & H^{i+1}(X^{\mathrm{an}},(G')^{\mathrm{an}}) & \to & H^{i+1}(X^{\mathrm{an}},\mathcal{E}^{\mathrm{an}}) \end{array}$$

By the 5-Lemma, α is surjective for every $\mathcal{G}$. Hence β is surjective as well (applying this to $\mathcal{G}'$ instead of $\mathcal{G}$), which implies by the 5-Lemma that α is injective. This shows that α is an isomorphism, as required.

Finally, we know that for every X smooth projective, any $\mathcal{F}$, $\mathcal{G}$ where $\mathcal{F}$ locally free,

$$H^i(X, \mathcal{G} \otimes \mathcal{F}^\vee) \cong \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}^{\mathrm{an}}, \mathcal{G}^{\mathrm{an}})$$

is an isomorphism. For general $\mathcal{F}$, use increasing induction using

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

for $\mathcal{E}$ locally free. The long exact sequence of Ext together with the 5 lemma implies the result for $\mathcal{F}$ in the same way as above. $\square$

4. Dolbeaux cohomology

4.1. **The tangent bundle of a complex manifold.**

4.1.1. *The complexification of a real vector space.* Let $V_{\mathbb{R}}$ be a finite-dimensional vector space over $\mathbb{R}$. To give a complex vector space structure on $V_{\mathbb{R}}$ is equivalent to giving a linear map $J\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ such that $J^2 = -\mathrm{Id}$ (where J is multiplication by i).

Given such J, we write V for the corresponding $\mathbb{C}$-vector space. This is called a *complexification of* V. We write

$$V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$$

for the extension of scalars. Then J induces

$$\begin{aligned} J_{\mathbb{C}}\colon V_{\mathbb{C}} &\to V_{\mathbb{C}} \\ v \otimes \lambda &\mapsto J(v) \otimes \lambda. \end{aligned}$$

Then $J_{\mathbb{C}}^2 = -\mathrm{Id}$ and we have a decomposition

$$V_{\mathbb{C}} = V' \oplus V''$$

where

$$\begin{aligned} V' &= \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = iv\}, \\ V'' &= \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = -iv\} \end{aligned}$$

are $\mathbb{C}$-subspaces of $V_{\mathbb{C}}$.

Denote by $u \mapsto \overline{u}$ the conjugate-linear map

$$\begin{aligned} V_{\mathbb{C}} &\to V_{\mathbb{C}} \\ v \otimes \lambda &\mapsto v \otimes \overline{\lambda}. \end{aligned}$$

We have an embedding

$$\begin{aligned} V_{\mathbb{R}} &\overset{j}{\hookrightarrow} V_{\mathbb{C}} \\ v &\mapsto v \otimes 1 \end{aligned}$$

such that $V_{\mathbb{R}} = \mathrm{Fix}(u \mapsto \overline{u})$.

We claim that the composition

$$V \overset{j}{\hookrightarrow} V_{\mathbb{C}} \overset{\mathrm{pr}_1}{\to} V' \qquad \text{is a complex isomorphism,}$$
$$V \overset{j}{\hookrightarrow} V_{\mathbb{C}} \overset{\mathrm{pr}_2}{\to} V'' \qquad \text{is a conjugate-linear isomorphism.}$$

If $v \in V$, write $v \otimes 1 = v' + v''$ for $v' \in V'$, $v'' \in V''$. Then

$$Jv \otimes 1 = iv' - iv''.$$

Hence

$$v' = \frac{1}{2}(v - iJ_{\mathbb{C}}(v)),$$
$$v'' = \frac{1}{2}(v + iJ_{\mathbb{C}}(v)).$$

This implies that $v'' = \overline{v'}$.

Exercise. Check that $v \mapsto v'$ is $\mathbb{C}$-linear and $v \mapsto v''$ is conjugate-linear.

By the above formulas, the two maps are injective. Hence they are bijective by dimension considerations. This proves the above claim. Moreover, we note that

$$V'' = \overline{V'}.$$

Let us now describe the decomposition $V_{\mathbb{C}} = V' \oplus V''$ in terms of bases. Suppose $x_1, \ldots, x_n$ give a basis of V over $\mathbb{C}$. This implies that if $y_j = J(x_j)$, then $x_1, \ldots, x_n, y_1, \ldots, y_n$ give a basis of $V_{\mathbb{R}}$. Consider these in $V_{\mathbb{C}}$ via $j\colon V \hookrightarrow V_{\mathbb{C}}$. Let e_j be the V'-component of x_j:

$$e_j = \frac{1}{2}(x_j - iy_j),$$
$$\overline{e_j} = \frac{1}{2}(x_j + iy_j).$$

Then $e_1, \ldots, e_n$ is a basis of V' and $\overline{e_1}, \ldots, \overline{e_n}$ of V''.

Consider now $U = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. This has a complex structure given by

$$(\lambda\varphi)(v) = \varphi(\lambda v) \quad \text{for } \lambda \in \mathbb{C}.$$

By the above, we have a decomposition $U_{\mathbb{C}} = U' \oplus U''$. On the other hand

$$(V_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong U \otimes \mathbb{C}$$

are isomorphisms of complex vector spaces.

Exercise. Check that via these isomorphisms $(J_{V,\mathbb{C}})^*$ corresponds to $J_{U,\mathbb{C}}$.

This implies that $U' = (V')^*$ and $U'' = (V'')^*$.

In the above description of $V_{\mathbb{C}} = V' \oplus V''$ using bases, we see that $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^*$ is a basis of $U = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Moreover, it is a simplex exercise to check that $y_j^* = -J_{U,\mathbb{C}}(x_j^*)$. The bases are then

$$\text{basis of U'}: \quad x_j^* + iy_j^* \quad 1 \leq j \leq n,$$
$$\text{basis of U}: \quad x_j^* - iy_j^* \quad 1 \leq j \leq n.$$

The decomposition $U_{\mathbb{C}} = U' \oplus U''$ induces a decomposition

$$\left(\bigwedge^p U\right)_{\mathbb{C}} = \bigwedge^p (U_{\mathbb{C}}) = \bigwedge^p (U' \oplus U'') = \bigoplus_{i+j=p} \left(\bigwedge^i U' \otimes \bigwedge^j U''\right).$$

The conjugation on $U_{\mathbb{C}}$, which maps U' to U'' and U'' to U', induces a conjugation on $\left(\bigwedge^p U\right)_{\mathbb{C}}$ which maps

$$\bigwedge^i U' \otimes \bigwedge^j U'' \text{ to } \bigwedge^j U' \otimes \bigwedge^i U''.$$

4.1.2. *The complexification of real line bundles.* This globalizes as follows. Suppose M is a smooth real manifold and E is a smooth real vector bundle on M.

Giving a complex structure on E is equivalent to givign a morphism of vector bundles $J\colon E \to E$ such that $J^2 = -\mathrm{Id}$.

In this case, the previous discussion globalizes to a decomposition

$$E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} = E' \oplus E''$$

and we have an isomorphism of complex vector bundles $E \to E'$ and a complex conjugate-linear isomorphism $E \to E''$. We also have a conjugation operator on $E_{\mathbb{C}}$ mapping E' to E''.

The dual E^* also has a complex structure. We get a corresponding decomposition of $\left(\bigwedge^p E^*\right)_{\mathbb{C}}$ etc.

4.1.3. *The tangent bundle.*

Definition 4.1. Let M be a smooth real manifold. An *almost complex structure* on M is a complex structure on the tangent bundle $\mathrm{T}M$, i.e. a morphism $J\colon \mathrm{T}M \to \mathrm{T}M$ of vector bundles such that $J^2 = -\mathrm{Id}$.

Proposition 4.2. *If M is a complex manifold, then M carries a canonical almost complex structure. Moreover, if the corresponding decomposition is*

$$\mathrm{T}M_{\mathbb{C}} = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$$

then $\mathrm{T}^{1,0}M$ is a holomorphic vector bundle.

Proof. It is enough to treat the case of open subsets of $\mathbb{C}^n$ and then show that biholomorphic maps preserve this structure.

If $U \subseteq \mathbb{C}^n$ is an open subset with complex coordinates $z_1, \ldots, z_n$ and $z_j = x_j + iy_j$, then TU is trivialized by

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.$$

Define a complex structure by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

We have a decomposition $\mathrm{T}U_{\mathbb{C}} = \mathrm{T}^{1,0}U \oplus \mathrm{T}^{0,1}U$ where

$$\mathrm{T}^{1,0}U \text{ is trivialized by } \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n},$$

$$\mathrm{T}^{0,1}U \text{ is trivialized by } \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.$$

The key point is that we showed that if $f\colon U \to V$ is holomorphic, then the canonical map

$$\mathrm{T}U_{\mathbb{C}} \to f^*\mathrm{T}V_{\mathbb{C}}$$

maps

$$\mathrm{T}^{1,0}U \text{ to } f^*\mathrm{T}^{1,0}V,$$
$$\mathrm{T}^{0,1}U \text{ to } f^*\mathrm{T}^{0,1}V.$$

In particular, if f is biholomorphic, the isomorphism between $\mathrm{T}U_{\mathbb{C}}$ and $f^*\mathrm{T}V_{\mathbb{C}}$ preserves the decomposition. Hence any biholomorphic map repsects the two compelx structure. Here, we use the following fact: if $\varphi\colon V \to W$ is an $\mathbb{R}$-linear isomorphism between complex vector spaces such that $\varphi \otimes 1\colon V_{\mathbb{C}} \to W_{\mathbb{C}}$ mapsto V' to W' and V'' to W'', then φ is a $\mathbb{C}$-linear isomorphism. Indeed, we have

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow\cong & & \downarrow\cong \\ V' & \xrightarrow{\varphi\otimes 1} & W' \end{array}$$

and since the map $V' \to W'$ is $\mathbb{C}$-linear, so is φ. This proves the first statement.

The second follows, since we saw that if $f\colon V \to W$ is a holomorphic map between open subsets $V \subseteq \mathbb{C}^n$, $W \subseteq \mathbb{C}^m$, then $f^*\mathrm{T}^{1,0}W \to \mathrm{T}^{1,0}V$ is given by the matrix $\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}$. In particular, the transition maps of $\mathrm{T}^{1,0}M$ are holomorphic. □

4.2. **The Dolbeault complex.** Last time, we saw that if M is a complex manifold, then $\mathrm{T}M$ has a canonical complex structure such that

$$\mathrm{T}M_{\mathbb{C}} \cong \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$$

where $\mathrm{T}^{1,0}M$ is a holomorphic vector bundle.

Definition 4.3. The bundle $\mathrm{T}^{1,0}$ is the *holomorphic tangent bundle* of M.

Remark 4.4.

(1) As in the case of the tangent bundle to a smooth manifold, $\mathrm{T}^{1,0}M$ can be described as the derivations on the local rings $\mathcal{O}_{M,x}$ for $x \in M$.
(2) If X is a smooth algebraic variety and $M = X^{\mathrm{an}}$, then

$$\mathrm{T}^{1,0}M \cong (TX)^{\mathrm{an}}.$$

If $f\colon M \to M'$ is a holomorphic map, then we have
$$\begin{array}{ccc} \mathrm{T}M_{\mathbb{C}} & \longrightarrow & f^*\mathrm{T}M'_{\mathbb{C}} \\ \uparrow & & \uparrow \\ \mathrm{T}^{1,0}M & \longrightarrow & f^*\mathrm{T}^{1,0}M' \end{array}$$

Dually, we have a decomposition

$$\mathrm{T}^*M_{\mathbb{C}} = A^{1,0}_M \oplus A^{0,1}_M,$$

where $A_M^{i,j}$ is the dual of $\mathrm{T}^{i,j}M$. Moreover, we also have a corresponding decomposition for

$$\bigwedge^p (T^* M_{\mathbb{C}}).$$

Let $\mathcal{A}_M^m$ be the sheaf of real smooth m-forms on M. Moreover, let $\mathcal{A}_{M,\mathbb{C}}^m = \mathcal{A}_M^m \otimes_{\mathbb{R}} \mathbb{C}$. We have a decomposition

$$\mathcal{A}_{M,\mathbb{C}}^m = \bigoplus_{p+q=m} \mathcal{A}_M^{p,q}$$

where $\mathcal{A}_M^{p,q}$ is the sheaf of (p,q)-forms on M. Note that

$$\overline{\mathcal{A}_M^{p,q}} = \mathcal{A}_M^{q,p}.$$

Note that $\mathcal{A}_M^{p,0}$ is the sheaf of smooth sections of a holomorphic vector bundle. We have a subsheaf $\Omega^p \subseteq \mathcal{A}_M^{p,0}$ of holomorphic sections of a holomorphic vector bundle.

For example,

$$\Omega^1 = \text{sheaf of holomorphic sections of } (T^{1,0}M)^*.$$

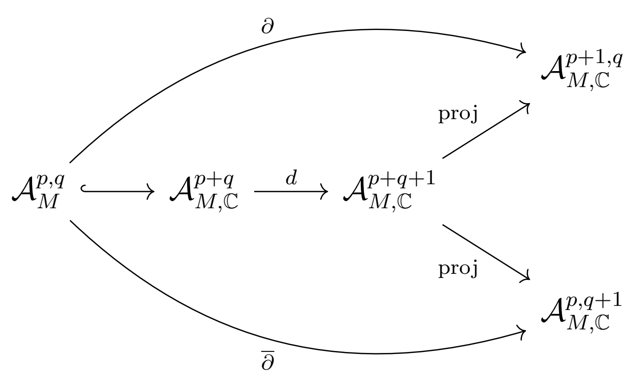
Locally, in a chart with coordinates $z_1, \ldots, z_n$,

$$\mathcal{A}^{p,q} = \text{free } \mathcal{C}_{M,\mathbb{C}}^{\infty}\text{-module, with basis } dz_I \wedge d\overline{z_J} \text{ for } |I| = p, |J| = q,$$

where for I given by $i_1 < \ldots < i_p$ we write

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z_I} = d\overline{z_{i_1}} \wedge \cdots d\overline{z_{i_p}}.$$

Recall that we have a de Rham differential $d\colon \mathcal{A}_M^m \to \mathcal{A}_M^{m+1}$. Given p, q, consider



Proposition 4.5. *We have that $d = \partial + \overline{\partial}$.*

Proof. Let us compute ∂ and $\overline{\partial}$ locally. Consider a chart with coordinates $z_1, \ldots, z_n$. Consider

$$\omega = f\, dz_I \cap d\overline{z_J} \quad \text{for } |I| = p, |J| = q.$$

Then

$$d\omega = df \wedge dz_I \wedge dz_J$$

where

$$
\begin{aligned}
df &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} dy_j \\
&= \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j}_{\partial f} + \underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z_j}} d\overline{z_j}}_{\overline{\partial} f}.
\end{aligned}
$$

This shows that $\partial\omega = \partial f \wedge dz_I \wedge d\overline{z_J}$ and $\overline{\partial}\omega = \overline{\partial} f \wedge dz_I \wedge d\overline{z_J}$. This shows that $d\omega = \partial\omega + \overline{\partial}\omega$. □

Corollary 4.6. *We have that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.*

Proof. Use the fact that $d^2 = 0$ and look in the corresponding graded pieces. □

Corollary 4.7. *The maps ∂ and $\overline{\partial}$ are derivations, i.e.*

$$\partial(\omega_1 \wedge \omega_2) = \partial\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \partial\omega_2$$

and similarly for $\overline{\partial}$.

Proof. Use that d is a derivation and look in the corresponding graded pieces. □

For every p, we have the following complex:

$$0 \longrightarrow \mathcal{A}_M^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_M^{p,n} \longrightarrow 0$$

where n is the dimension of M as a complex manifold.[3]

Definition 4.8. The global sections $\Gamma(M, \mathcal{A}_M^{p,\bullet})$ form the pth *Dolbeault complex* of M. The (p, q) *Dolbeault cohomology* is

$$H^{p,q}(M) = H^q(\Gamma(M, \mathcal{A}_M^{p,\bullet})).$$

Remark 4.9. Note that

$$\ker(\mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}) = \Omega_M^p,$$

i.e. the sheaf of holomorphic of sections of the the sheaf of $(p, 0)$ forms. Indeed, locally, $\omega = \sum_I f_I dz_I$ is in the kernel if and only if $\frac{\partial f_I}{\partial \overline{z_j}} = 0$ for all j and I, which means that f_I is holomorphic for all I.

More generally, suppose E is a holomorphic vector bundle with sheaf of smooth sections $\mathcal{E}$. We claim that we can define a canonical

$$\overline{\partial}_{\mathcal{E}} \colon \mathcal{A}_M^{p,q} \otimes \mathcal{E} \to \mathcal{A}_M^{p,q+1} \otimes \mathcal{E}$$

where the tensor product is over $\mathcal{C}^\infty_{M,\mathbb{C}}$ such that $\overline{\partial}_{\mathcal{E}}^2 = 0$ and $\overline{\partial}_{\mathcal{E}}$ is a derivation.

[3] Note that $\mathcal{A}_M^{p,q} = 0$ whenever p or q is bigger than n.

We prove this claim now. We first work on a chart U such that $\mathcal{E}_{|U}$ has a trivialization by holomorphic sections $s_1, \dots, s_r$. Let ω be a section of $\mathcal{A}_M^{p,q} \otimes \mathcal{E}$ and write it as

$$\omega = \sum_{i=1}^{r} \omega_i s_i.$$

Define

$$\overline{\partial}_{\mathcal{E}}(\omega) = \sum_{i=1}^{r} \overline{\partial}(\omega_i) s_i.$$

This is independent of trivialization. Indeed,

$$\overline{\partial}(f\omega_i) = f\overline{\partial}\omega_i$$

if f is holomorphic (so $\overline{\partial} f = 0$). Therefore, these local maps $\overline{\partial}_{\mathcal{E}}$ glue to give $\overline{\partial}_{\mathcal{E}}$ on M. It is clear from the local description that $\overline{\partial}_{\mathcal{E}}^2 = 0$ and $\overline{\partial}_{\mathcal{E}}$ is a derivation.

Altogether, this gives a *twisted Dolbeault complex*:

$$0 \longrightarrow \mathcal{A}_M^{p,0} \otimes \mathcal{E} \xrightarrow{\overline{\partial}_{\mathcal{E}}} \mathcal{A}_M^{p,1} \otimes \mathcal{E} \xrightarrow{\overline{\partial}_{\mathcal{E}}} \cdots \xrightarrow{\overline{\partial}_{\mathcal{E}}} \mathcal{A}_M^{p,n} \otimes \mathcal{E} \longrightarrow 0.$$

Definition 4.10. The (p,q) *Dolbeault cohomology* of $\mathcal{E}$ is

$$H^{p,q}(M;\mathcal{E}) = H^q(\Gamma(M, \mathcal{A}_M^{p,\bullet} \otimes \mathcal{E})).$$

There are two things to do:

- $\mathcal{A}_M^{p,q}$ is acyclic: $\mathcal{H}^q(\mathcal{A}_M^{p,\bullet}) = 0$ for $q \geq 1$,
- can use this complex to compute $H^q(M, \Omega_M^p)$.

Theorem 2.5 generalizes to functions of several variables.

Proposition 4.11 ($\overline{\partial}$-lemma)**.** *If ω is a (p,q) form on $U \subseteq M$ such that $\overline{\partial}\omega = 0$, $q \geq 1$, then locally, we can find $\beta \in \mathcal{A}_M^{p,q-1}$ such that $\overline{\partial}\beta = \omega$. (Then $\mathcal{A}_M^{p,\bullet}$ is acyclic.)*

Proof. We work locally, so we may assume that we have a chart with coordinates $z_1, \dots, z_n$.

Step 1. Reduce to the case $p = 0$. Write $\omega = \sum_{I,J} d_{I,J} dz_I \wedge d\overline{z_J}$ and assume that $\overline{\partial}\omega = 0$. For every I, consider

$$\omega_I = \sum_J f_{I,J} d\overline{z}_j.$$

Since $\overline{\partial}\omega = 0$, $\overline{\partial}\omega_I = 0$ for all I. If we know the $p = 0$ case, then locally $\omega_I = \overline{\partial}\beta_I$ for some β_I which are $(0, q-1)$ forms. If we take $\beta = \sum_I (-1)^p dz_I \wedge \beta_I$, then $\overline{\partial}\beta = \omega$.

Step 2. Assume $p = 0$. Working locally, in a chart with coordinates $z_1, \dots, z_n$, we may write

$$\omega = \sum_{|J|=q} f_J d\overline{z_J}.$$

Let k be the largest index so that $d\overline{z_k}$ shows up in some $d\overline{z_J}$ with non-zero coefficient. We proceed by increasing induction on k.

First, suppose $\omega \neq 0$ and the smallest k is q. Then

$$\omega = f d\overline{z_1} \wedge \cdots \wedge d\overline{z_q}.$$

Note that $\overline{\partial}\omega = 0$ if and only if $\frac{\partial f}{\partial \overline{z_i}} = 0$ for $i > q$, i.e. f is holomorphic in the variables $z_{q+1}, \ldots, z_n$. By Theorem 2.5 (the one variable $\overline{\partial}$-lemma), locally there is a function g which is smooth and holomorphic with respect to the variables $z_{q+1}, \ldots, z_n$, and

$$\frac{\partial g}{\partial \overline{z_1}} = f.$$

Then

$$\overline{\partial}(g d\overline{z_2} \wedge \cdots \wedge \overline{z_q}) = \omega.$$

For the induction step, write $\omega = \omega_1 + \omega_2 \wedge d\overline{z_k}$ where ω_1 is a $(0, q)$-form and ω_2 is a $(0, q-1)$-for such that ω_1 and ω_2 only involve $d\overline{z_1}, \ldots, d\overline{z_{k-1}}$. Since $\overline{\partial}\omega = 0$, the coefficients of ω_1, ω_2 are holomorphic in $z_1, \ldots, z_{k+1}$. Write

$$\omega_2 = \sum_{|J|=q-1} a_J d\overline{z_J}.$$

Applying Theorem 2.5, we can find locally smooth functions b_J, holomorphic in $z_{k+1}, \ldots, z_n$ such that

$$\frac{\partial b_J}{\partial \overline{z_k}} = a_J.$$

Then

$$\overline{\partial}\underbrace{\left(\sum_{|J|=q-1} b_J d\overline{z_J}\right)}_{\beta'} = \sum_{|J|=q-1} (-1)^{q-1} a_J d\overline{z_J} \wedge d\overline{z_k} + (\text{terms involving only } d\overline{z_1}, \ldots, d\overline{z_{k-1}}).$$

Therefore,

$$\omega - (-1)^{q-1}\overline{\partial}(\beta')$$

only involves $d\overline{z_1}, \ldots, d\overline{z_{k-1}}$. By inductive hypothesis, this must be equal to $\overline{\partial}(\gamma)$ for some γ, and hence

$$\omega = \overline{\partial}(\gamma + (-1)^q \beta').$$

This completes the proof. □

Exercise. Repeat the whole argument when M is a smooth manifold to show that if ω is a p-form for $p \geq 1$ which is closed ($d\omega = 0$), then ω is locally exact.

Corollary 4.12. *For every $p \geq 0$, we have an exact complex of sheaves on M:*

$$0 \longrightarrow \Omega_M^p \hookrightarrow \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n} \longrightarrow 0.$$

More generally, if E is any holomorphic vector bundles, with sheaf of smooth sections $\mathcal{E}$ and sheaf of holomorphic sections $\mathcal{E}^{\text{hol}}$, then we have an exact complex

$$0 \longrightarrow \Omega_M^p \otimes_{\mathcal{O}_M} \mathcal{E}^{\text{hol}} \longrightarrow \mathcal{A}^{p,0} \otimes_{\mathcal{C}^\infty_{M,\mathbb{C}}} \mathcal{E} \xrightarrow{\overline{\partial}_E} \mathcal{A}^{p,1} \otimes_{\mathcal{C}^\infty_{M,\mathbb{C}}} \mathcal{E} \longrightarrow \cdots$$

Exactness follows since locally the complex is isomorphic to a direct sum of $r = \operatorname{rk} E$ copies of the Dolbeault complex.

In the case of smooth manifolds, we have similar exact complexes. We have the *de Rham complex*:

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{A}^0_M \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n_M \longrightarrow 0.$$

4.3. **Soft sheaves on paracompact spaces.** Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. If Z is any subset of X and $i\colon Z \hookrightarrow X$ is the inclusion map, then we define

$$\mathcal{F}(Z) = \Gamma(Z, \mathcal{F}) = \Gamma(Z, i^{-1}\mathcal{F}).$$

to be the set of sections $s\colon Z \to \coprod_{x \in Z} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for all $x \in Z$ and for all $x \in Z$, there is an open neighborhood of x in X and $t \in \mathcal{F}(U)$ such that

$$s(y) = t_y \text{ for all } y \in U \cap Z.$$

Remark 4.13. If $Z' \subseteq Z$, we have natural restriction maps $\mathcal{F}(Z) \to \mathcal{F}(Z')$ which are functorial.

Proposition 4.14. *If $\mathcal{F}$ is as above and $Z_1, \ldots, Z_r$ are closed subsets of X, we have an exact sequence:*

$$0 \longrightarrow \mathcal{F}\left(\bigcup_i Z_i\right) \longrightarrow \prod_i \mathcal{F}(Z_i) \longrightarrow \prod_{i,j} \mathcal{F}(Z_i \cap Z_j)$$

induced by restriction maps.

Proof. Suppose $(s_i)_{1 \le i \le r}$ are sections

$$s_i\colon Z_i \to \coprod_{x \in Z_i} \mathcal{F}_x$$

such that $s_i(x) \in \mathcal{F}_x$ which are compatible, i.e. $s_i(x) = s_j(x)$ for all $x \in Z_i \cap Z_j$. We want to show there is a unique $s\colon \bigcup_i Z_i \to \coprod_{x \in \bigcup Z_i} \mathcal{F}_x$ such that $s(x) \in \mathcal{F}_x$ for all x, and $s|_{Z_i} = s_i$.

Fix $x \in X$. We may replace X be an open neighborhood of x. Since Z_i's are closed, we may assume that $x \in Z_1 \cap \cdots \cap Z_r$ by making this open neighborhood smaller if necessary. Moreover, we may assume that for all i, there exists $t_i \in \mathcal{F}(X)$ such that $(t_i)_y = s_i(y)$ for all $y \in Z_i$. In particular,

$$(t_1)_x = \cdots = (t_r)_x.$$

Further replacing X, we may assume $t_1 = \cdots = t_r = t$. Clearly, $(t)_y = s(y)$ for all y. ☐

Definition 4.15. A topological space X is *paracompact* if

- X is Hausdorff,
- every open cover admits a refinement which is locally finite.

We suppose throughout this section that X is paracompact.

Examples 4.16.

(1) Topological manifolds are paracompact.
(2) CW complexes are paracompact.

Remark 4.17. If Z is closed in X and X is paracompact, then Z is paracompact.

Remark 4.18. If $X = \bigcup_{i \in I} U_i$ is a locally finite open cover and X is paracompact, then there is an open cover $X = \bigcup_i V_i$ such that $\overline{V_i} \subseteq U_i$.

Example 4.19. If $A, B \subseteq X$ are closed, $A \cap B = \emptyset$, there exist U, V open such that $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$. (In other words, X is a *normal space*).

Definition 4.20. A sheaf $\mathcal{F}$ of abelian groups on X is *soft* if for any $Z \subseteq X$ closed, the restriction map $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective.

Compare this to the definition of flasque sheaves. A sheaf $\mathcal{F}$ is *flasque* if for any $U \subseteq X$ open, the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Fact 4.21. *Flasque sheaves are acyclic, i.e. their higher cohomology vanishes. Therefore, one can compute cohomology via flasque resolutions.*

We will see next time that if X is paracompact, then the same holds for soft sheaves. Moreover, we will see that there is a large supply of soft sheaves on complex manifolds.

There are some supplementary notes on the course website

http://www-personal.umich.edu/~mmustata/731-2019.html

covering

- soft sheaves,
- comparison between singular cohomology and sheaf cohomology with constant coefficients.

Proposition 4.22. *Let $\mathcal{F}$ be a sheaf of abelian groups on X. If $Z \subseteq X$ is closed, for any section $s \in \mathcal{F}(Z)$, there exists $U \supseteq Z$ open and $t \in \mathcal{F}(U)$ such that $t|_Z = s$.*

Proof. See the notes on soft sheaves on the course website (Lemma 2.3). □

Corollary 4.23. *If $\mathcal{F}$ is flasque, then $\mathcal{F}$ is soft.*

Proposition 4.24. *If*

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \longrightarrow 0$$

is exact and $\mathcal{F}'$ is soft, then

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0$$

is exact.

Proof. We only need to show that if $s \in \mathcal{F}''(X)$, there exists $\tilde{s} \in \mathcal{F}(X)$ such that $\psi(\tilde{s}) = s$. Since ψ is surjective, there is an open cover $X = \bigcup_i U_i$, $\tilde{s}_i \in \mathcal{F}(U_i)$ such that $\psi(\tilde{s}_i) = s|_{U_i}$.

By paracompactness of X, after passing to some refinement, we may assume this is a locally finite cover. Hence there is an open cover $X = \bigcup_i V_i$ such that $\overline{V_i} \subseteq U_i$.

For $J \subseteq I$, $Z_J = \bigcup_{i \in J} \overline{V_i}$ is closed in X by local finiteness.

Consider pairs (J, t) with $J \subseteq I$ and $t \in \mathcal{F}(Z_J)$ such that $\psi(t) = s|_{Z_J}$. We order the pairs by declaring $(J, t) \leq (J', t')$ if $J \subseteq J'$ and $t'|_{Z_J} = t$. By Zorn's Lemma, we may choose a maximal (J, t).

If $I = J$, we are done. Otherwise, there exists $i \in I \setminus J$, and we will produce a contradiction with maximality of J. We have $t \in \mathcal{F}(Z_J)$ and $\tilde{s}_i|_{\overline{V_i}} \in \mathcal{F}(\overline{V_i})$. Note that

$$\psi(t|_{Z_J \cap \overline{V_i}}) = \psi(\tilde{s}_i|_{Z_J \cap \overline{V_i}})$$

and hence

$$t|_{z_J \cap \overline{V_i}} - \tilde{s}_i|_{Z_J \cap \overline{V_i}} = \varphi(w)$$

for some $w \in \mathcal{F}'(Z_J \cap \overline{V_i})$. Since $\mathcal{F}'$ is soft, there exists $\tilde{w} \in \mathcal{F}'(X)$ such that $\tilde{w}|_{Z_J \cap \overline{V_i}} = w$. Replace

$$\tilde{s}_i|_{U_i} \text{ by } \tilde{s}_i|_{U_i} + \varphi(\tilde{w}|_{U_i})$$

to assume that $t|_{Z_J \cap \overline{V_i}} = \tilde{s}_i|_{Z_J \cap \overline{V_i}}$.

By Proposition 4.14, there exists $t' \in \mathcal{F}(Z_{J \cup \{i\}})$ such that $t'|_{Z_J} = t$ and $t'|_{\overline{V_i}} = \tilde{s}_i|_{\overline{V_i}}$. Then $\psi(t') = s|_{Z_{J \cup \{i\}}}$, contradicting maximality of the pair (J, t). □

Corollary 4.25. *If* $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ *is exact, and* $\mathcal{F}'$ *and* $\mathcal{F}$ *are soft, then* $\mathcal{F}''$ *is soft.*

Proof. Consider $Z \subseteq X$ closed. Then Z is paracompact and $\mathcal{F}'|_Z$ is soft. By Proposition 4.24, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}'(X) & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{F}''(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}'(Z) & \longrightarrow & \mathcal{F}(Z) & \longrightarrow & \mathcal{F}''(Z) & \longrightarrow & 0 \end{array}$$

has exact rows, and hence $\mathcal{F}''(X) \to \mathcal{F}''(Z)$ is surjective. □

We finally show that we can compute cohomology using soft sheaves.

Theorem 4.26.

(1) *If* $\mathcal{E}$ *is a soft sheaf on* X*,* $H^i(X, \mathcal{E}) = 0$ *for all* $i > 0$.
(2) *If* $\mathcal{F}$ *has a resolution*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E}^1 \longrightarrow \cdots$$

with all $\mathcal{E}^i$ soft, then

$$H^i(X,\mathcal{F}) = \mathcal{H}^i(\Gamma(X,\mathcal{E}^\bullet)).$$

Proof. Part (2) follows from part (1) by general reasons. For (1), we argue by induction on i. Consider the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

with $\mathcal{I}$ flasque. In particular, $\mathcal{I}$ is soft by Corollary 4.23. Then, by Corollary 4.25, $\mathcal{G}$ is also soft. The long exact sequence in cohomology then shows that:

- $\underbrace{\Gamma(X,\mathcal{I}) \to \Gamma(X,\mathcal{G})}_{\text{surjective by Prop. 4.24}} \to H^1(X,\mathcal{E}) \to \underbrace{H^1(X,\mathcal{I})}_{=0}$, so $H^1(X,\mathcal{E}) = 0$,
- $H^{i+1}(X,\mathcal{E}) = H^i(X,\mathcal{G}) = 0$ for $i \geq 1$ by inductive hypothesis.

This completes the proof. □

Exercise. Suppose $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ is a morphism of ringed spaces with X, Y paracompact. Let $\mathcal{F}$ be a sheaf on Y and $\mathcal{G}$ be a sheaf on X so that we have a morphism $f^*\mathcal{F} \to \mathcal{G}$. This induces maps $H^i(Y,\mathcal{F}) \longrightarrow H^i(X,f^*\mathcal{F}) \longrightarrow H^i(X,\mathcal{G})$.

Show that if $\mathcal{F} \to \mathcal{E}^\bullet$, $\mathcal{G} \to \mathcal{M}^\bullet$ are soft resolutions and we have induced morphisms

$$\begin{array}{ccc} f^*\mathcal{F} & \longrightarrow & f^*\mathcal{E}^\bullet \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{M}^\bullet \end{array}$$

then we have a commutative diagram

$$\begin{array}{ccc} H^i(Y,\mathcal{F}) & \xrightarrow{\cong} & \mathcal{H}^i(\Gamma(Y,\mathcal{E}^\bullet)) \\ \downarrow & & \downarrow \\ H^i(X,\mathcal{G}) & \xrightarrow{\cong} & \mathcal{H}^i(\Gamma(X,\mathcal{M}^\bullet)) \end{array}$$

Proposition 4.27. *If M is a smooth real manifold, then any $\mathcal{C}^\infty_M$-module is soft.*

Proof. Let $\mathcal{F}$ be a $\mathcal{C}^\infty_M$-module and $Z \subseteq X$ be a closed subset. Consider $s \in \mathcal{F}(Z)$. We want to extend it to a section on X.

By Proposition 4.22, there is an open subset $U \supseteq Z$ and a section $\tilde{s} \in \mathcal{F}(U)$ such that $\tilde{s}|_Z = s$.

Considering $Z \subseteq U$, there is an open subset U_1 such that

$$Z \subseteq U_1 \subseteq \overline{U_1} \subseteq U$$

and an open subset U_2 such that

$$\overline{U_1} \subseteq U_2 \subseteq \overline{U_2} \subseteq U.$$

Then smooth version of Urysohn's Lemma say that there exists a smooth function φ such that

$$\begin{cases} \varphi = 1 & \text{on } \overline{U_1}, \\ \varphi = 0 & \text{on } X \setminus U_2. \end{cases}$$

Consider $\varphi|_U \cdot \tilde{s}$ which is 0 on $U \setminus U_2$. Then there exists a section $s' \in \mathcal{F}(X)$ such that $s'|_{X\setminus\overline{U_2}} = 0$ and $s'|_U = \varphi|_U \tilde{s}|_U$ since $\varphi = 0$ on $X \setminus \overline{U_2}$. Note that

$$s'|_{U_1} = \tilde{s}|_{U_1}$$

since $\varphi = 1$ on U_1. Hence $s'|_Z = s$. □

Applications.

(1) If M is a smooth manifold of dimension n, we have a resolution of $\underline{\mathbb{R}}$ given by

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{A}^0_M \xrightarrow{d} \mathcal{A}^1_M \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n_M \longrightarrow 0,$$

so we recover the de Rham Theorem:

$$H^p(X, \underline{\mathbb{R}}) \cong H^p_{\mathrm{dR}}(X).$$

This gives a simple interpretation of the cup product on cohomology (which is messy to define otherwise) via $\wedge$ of differential forms.

These are also isomorphic to singular cohomology. This is proven in the notes on the course website.

Fact 4.28. *Since M is paracompact and locally contractible,*

$$\underbrace{H^p(M, \mathbb{R})}_{\text{sing. cohomology}} \cong H^p(M, \underline{\mathbb{R}}).$$

(2) If M is a complex manifold of dimension n, for all p, we have an exact complex

$$0 \longrightarrow \Omega^p_M \longrightarrow \mathcal{A}^{p,0}_M \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_M \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,n}_M \longrightarrow 0,$$

which shows that

$$H^{p,q}(M) = \mathcal{H}^q(\Gamma(M, \mathcal{A}^{p,\bullet}_M)) \cong H^q(M, \Omega^p_M)$$

(where the first equality is the definition of Dolbeaux cohomology).

More generally, if E is a holomorphic vector bundle with sheaf of holomorphic sections $\mathcal{E}$, the sheaf of sections is $\mathcal{C}^\infty_M \otimes_{\mathcal{O}_M} \mathcal{E} = \mathcal{E}_{\mathrm{sm}}$. By taking $(*) \otimes_{\mathcal{C}^\infty_M} \mathcal{E}_{\mathrm{sm}} = (*) \otimes_{\mathcal{O}_M} \mathcal{E}$, we get a complex

$$0 \longrightarrow \Omega^p_M \otimes_{\mathcal{O}_M} \mathcal{E} \longrightarrow \mathcal{A}^{p,0}_{\mathcal{E}} \xrightarrow{\bar\partial_E} \mathcal{A}^{p,1}_{\mathcal{E}} \longrightarrow \cdots \xrightarrow{\bar\partial_{\mathcal{E}}} \mathcal{A}^{p,n}_{M,\mathcal{E}} \longrightarrow 0.$$

Then

$$H^{p,q}(M, \mathcal{E}) = \mathcal{H}^q(\Gamma(M, \mathcal{A}^{p,\bullet}_{M,\mathcal{E}})) \cong H^q(M, \Omega^p_M \otimes \mathcal{E})$$

(where the first equality is the definition of Dolbeaux cohomology).

5. Hodge theory of compact, oriented, Riemannian manifolds

We are now done with *introductory* material to the class and we start Hodge theory. During the next few lectures, we discuss Hodge theory for Riemannian manifolds.

5.1. **Linear algebra background: the $*$ operator.** Let V be a finite-dimensional vector space over $\mathbb{R}$.

Definition 5.1. A *scalar product* on V is a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V \times V$ valued in $\mathbb{R}$ which is positive definite, i.e. $\langle v, v \rangle > 0$ for all $v \neq 0$.

Given a scalar product on V, we get an isomorphism

$$V \stackrel{\cong}{\to} V^*$$
$$v \mapsto \varphi_v = \langle v, - \rangle.$$

We put a scalar product on V^* such that

$$\langle \varphi_v, \varphi_w \rangle = \langle v, w \rangle \text{ for all } v, w.$$

Example 5.2. If $e_1, \ldots, e_n$ is an orthonormal basis of V (i.e. $\langle e_i, e_j \rangle = \delta_{i,j}$ for all i, j) and $e_1^*, \ldots, e_n^*$ is the dual basis on V^*, then $\varphi_{e_i} = e_i^*$ and hence $e_1^*, \ldots, e_n^*$ is an orthonormal basis for V^*.

Exercise. Given a scalar product $\langle \cdot, \cdot \rangle$ on V, we have an induced scalar product on each $\bigwedge^p V$ such that

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle).$$

(Hint: use this as the definition and show that if $e_1, \ldots, e_n$ is an orhonormal basis, then we get an orthonormal basis for $\bigwedge^p V$ given by $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \mid I = \{i_1 < \ldots < i_p\} \subseteq \{1, \ldots, n\}\}$.)

Suppose now that, in addition, that on V we also have an orientation (i.e. an orientation on the 1-dimensional top exterior power of V).

In this case, we get a canonical *volume element* $\mathrm{vol} \in \bigwedge^n V$ for $n = \dim V$ by choosing an orthonormal basis $e_1, \ldots, e_n$ such that $e_1 \wedge \cdots \wedge e_n$ is positive and letting

$$\mathrm{vol} = e_1 \wedge \cdots \wedge e_n.$$

This is independent on the choice of basis. If $e_1', \ldots, e_n'$ is another such basis and we write

$$e_i' = \sum a_j a_{i,j} e_j$$

then for $A = (a_{i,j})$ we have $A \cdot A^t = I_n$, so $\det(A)^2 = 1$. Hence $\det(A) = \pm 1$ and we have

$$e_1' \wedge \cdots \wedge e_n' = (\det A) e_1 \wedge \cdots e_n$$

so $\det A = 1$ since $e_1' \wedge \cdots \wedge e_n'$ and $e_1 \wedge \cdots \wedge e_n$ are both positive.

We now define the $*$ operator for $(V, \langle \cdot, \cdot \rangle, \text{orientation})$ where $n = \dim V$.

Proposition 5.3. *For every p, $0 \leq p \leq n$, there is a unique isomorphism*

$$\bigwedge^p V \stackrel{*}{\to} \bigwedge^{n-p} V$$

such that

$$v \wedge (*w) = \langle v, w \rangle \mathrm{vol} \quad \textit{in } \bigwedge^n V$$

for all $v, w \in \bigwedge^p V$.

Proof. Recall that there is a canonical nondegenerate bilinear map

$$\bigwedge^p V \times \bigwedge^{n-p} V \to \bigwedge^n V \cong \mathbb{R}. \tag{2}$$

given by $\wedge$. Given $w \in \bigwedge^p V$, we may consider the map

$$\langle -, w\rangle \mathrm{vol}\colon \bigwedge^p V \to \bigwedge^n V \cong \mathbb{R}.$$

Using the pairing (2), there is an element $*w \in \bigwedge^{n-p} V$ such that

$$\langle -, w\rangle \mathrm{vol} = - \wedge *w.$$

We get a linear map $*\colon \bigwedge^p V \to \bigwedge^{n-p} V$. It is clear this is injective: if $*w = 0$, then for all v, $\langle v, w\rangle = 0$, so $w = 0$ since $\langle -, -\rangle$ is non-degenerate.

By dimension considerations, $*$ is an isomorphism. □

We now describe $*$ via an orthonormal basis. Recall that

$$e_J \wedge (*e_J) = \langle e_J, e_I\rangle \mathrm{vol}.$$

This shows that, by taking $I = J$,

$$*e_I = \epsilon(I, \overline{I}) \cdot e_{\overline{I}} \quad \text{where } \overline{I} = \{1, \ldots, n\} \setminus I$$

where $\epsilon(I, \overline{I})$ is the signature of the permutation $(I, \overline{I})$. In other words,

$$e_I \wedge e_{\overline{I}} = \epsilon(I, \overline{I}) e_1 \wedge \cdots \wedge e_n.$$

Properties of $*$:

(1) $*(\mathrm{vol}) = 1$,
(2) $**\colon \bigwedge^p V \to \bigwedge^p V$ is equal to $(-1)^{p(n-p)}$ because

$$(a_1 \wedge \cdots \wedge a_p) \wedge (b_1 \wedge \ldots \wedge b_{n-p}) = (-1)^{p(n-p)} (b_1 \wedge \cdots \wedge b_{n-p}) \wedge (a_1 \wedge \cdots \wedge a_p).$$

5.1.1. *The global situation.* Let M be a smooth manifold and E be a smooth real vector bundle on M of rank n. Write $\mathcal{E}$ for the sheaf of sections.

Definition 5.4. A *metric* (or *scalar product*) on E is a smoothly varying family of scalar products on the fibers of E. Concretely, for all $p \in M$, we have a scalar product $\langle \cdot, \cdot \rangle$ on E_p such that for sections $s, t \in \mathcal{E}(U)$, the map

$$U \ni p \mapsto \langle s(p), t(p)\rangle \in \mathbb{R}$$

is a smooth function. (Note that it is enough to check this for $s_1, \ldots, s_n$ which trivialize $\mathcal{E}$ over open subsets.)

Example 5.5. If $E = M \times \mathbb{R}^n$, then the standard scalar product on $\mathbb{R}^n$ gives a scalar product on each fiber, which is a matrix on E.

In particular, we always have such metrics locally on any E locally on M. By using partitions of unity, get metrics on E. Hence on every E, we have such a metric.

If, in addition, we have an *orientation* of E (i.e. a compatible system of orientations of all fibers), we get an element $\mathrm{vol} \in \Gamma\left(M, \bigwedge^n \mathcal{E}\right)$ which is everywhere nonzero, belonging to the (positive) orientation.

We get a global $*$ operator

$$*\colon \bigwedge^p \mathcal{E} \to \bigwedge^{n-p} \mathcal{E}$$

globalizing the one on each fiber.

5.1.2. *The tangent bundle.*

Definition 5.6. A *Riemannian metric* on M is a metric on $\mathrm{T}M$.

A Riemannian metrec induces a metric on T^*M and on $\bigwedge^p \mathrm{T}^*M$.

If M is *oriented*, i.e. we have an orientation on $\mathrm{T}M$ (or equivalently on T^*M), we can apply the previous considerations.

In particular, we have an n-form (where n is the dimension of M) dV called the *volume element* which is everywhere nonzero and positively oriented.

We get $*\colon \mathcal{A}_M^p \overset{\cong}{\to} \mathcal{A}_M^{p-1}$ such that $\omega \wedge (*\eta) = \langle \omega, \eta \rangle dV$.

From now on, let M be a **compact** manifold with orientation and a Riemannian structure. Compactness allows us to define a *scalar product* on $\mathcal{A}^p(M)$ by

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle \, dV.$$

It is clearly bilinear, symmetric and *positive-definite*: if $\omega \neq 0$, then $\langle\langle \omega, \omega \rangle\rangle > 0$.

One caveat is that $\mathcal{A}^p(M)$ is not finite-dimensional. Actually, it is not even complete with respect to the metric induced by $\langle\langle \cdot, \cdot \rangle\rangle$.

Definition 5.7. Let $d^*\colon \mathcal{A}_M^p \to \mathcal{A}_M^{p-1}$ be given by

$$\begin{aligned} d^* &= (-1)^{n(p-1)+1} * d * \\ &= (-1)^p *^{-1} d * \end{aligned}$$

Proposition 5.8. *For every p, $d^*\colon \mathcal{A}^{p+1}(M) \to \mathcal{A}^p(M)$ is the* formal adjoint *of $d\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ with respect to $[\![\cdot, \cdot]\!]$. Explicitly,*

$$\langle\langle d\omega, \eta \rangle\rangle = \langle\langle \omega, d^*\omega \rangle\rangle$$

for all $\eta \in \mathcal{A}^{p+1}(M)$, $\omega \in \mathcal{A}^p(M)$.

This is only a *formal adjoint* because these are not really Hilbert spaces (they are not complete).

Remark 5.9. Such a formal adjoint is unique if it exists: if $\overline{d}^*$ is another such operator, then

$$\langle\langle \omega, d^*\eta \rangle\rangle = \langle\langle \omega, \overline{d}^*\eta \rangle\rangle,$$

then for $\omega = d^*\eta - \overline{d}^*\eta$ we have $\langle\langle \omega, \omega \rangle\rangle = 0$, so $\omega = 0$.

Proof. We compute $\langle\langle \omega, d^*\eta\rangle\rangle$ using the definition:

$$\begin{aligned}\langle\langle \omega, d^*\eta\rangle\rangle &= \int_M \langle \omega, (-1)^{p+1} *^{-1} d * \eta\rangle\, dV \\ &= (-1)^{p+1}\int_M \underbrace{\omega\wedge d*\eta}_{(-1)^p(d(\omega\wedge *\eta)-d\omega\wedge(*\eta))} \\ &= -\underbrace{\int_M d(\omega\wedge *\eta)}_{=0 \text{ by Stokes' Theorem}} + \int_M d\omega\wedge *\eta \\ &= \int_M \langle d\omega, \eta\rangle dV \\ &= \langle\langle d\omega, \eta\rangle\rangle,\end{aligned}$$

where we have used that d is a derivation. □

We define the *Laplace–Beltrami operator*:

$$\Delta = dd^* + d^*d \colon \mathcal{A}^p(M) \to \mathcal{A}^p(M).$$

Proposition 5.10. *The operator Δ is formally self-adjoint.*

Proof. We have that

$$\begin{aligned}\langle\langle \Delta\omega, \eta\rangle\rangle &= \langle\langle dd^*\omega + d^*d\omega, \eta\rangle\rangle \\ &= \langle\langle d^*\omega, d^*\eta\rangle\rangle + \langle\langle d\omega, d\eta\rangle\rangle \qquad \text{by Proposition 5.8.}\end{aligned}$$

By symmetry, this is also equal to $\langle\langle \omega, \Delta\eta\rangle\rangle$. □

Definition 5.11. A form $\omega \in \mathcal{A}^p(M)$ is *harmonic* if $\Delta\omega = 0$.

Proposition 5.12. *A form ω is harmonic if and only if $d\omega = 0$ and $d^*\omega = 0$.*

Proof. The 'if' implication is clear from the definition of Δ. For the 'only if' implication, we use the formula $\langle\langle \Delta\omega, \eta\rangle\rangle = \langle\langle d^*\omega, d^*\eta\rangle\rangle + \langle\langle d\omega, d\eta\rangle\rangle$ from the proof of Proposition 5.10 for $\eta = \omega$ such that $d\omega = 0$ to conclude that

$$0 = \|d^*\omega\|^2 + \|d\omega\|^2,$$

so $d\omega = 0$ and $d^*\omega = 0$. □

The goal is to prove that every de Rham cohomology class is represented by a unique harmonic representative. This is the famous Hodge Theorem 5.21. We start with the following lemma.

Lemma 5.13. *For a given de Rham cohomology class, a representative ω is harmonic if and only if $\|\omega\|$ is minimal.*

Proof. Given a p-form ω such that $d\omega = 0$, consider $\omega + d\eta$ for all $(p-1)$-forms η. Then

$$\|\omega + d\eta\|^2 = \|\omega\|^2 + \|d\eta\|^2 + 2\underbrace{\langle\langle \omega, d\eta\rangle\rangle}_{\langle\langle d^*\omega, \eta\rangle\rangle}. \tag{3}$$

Since $d\omega = 0$, ω is harmonic if and only if $d^*\omega = 0$. If this holds, then

$$\|\omega + d\eta\|^2 \geq \|\omega\|^2$$

for all η by formula (3).

Conversely, if $\|\omega\|^2$ is minimal among all $\|\omega + d\eta\|^2$, then

$$\frac{d}{dt}\|\omega + td\eta\|^2|_{t=0} = 0.$$

According to formula (3) derivative is equal to $2\langle\langle\omega, d\eta\rangle\rangle$. For $\eta = d^*\omega$, we get that

$$0 = \langle\langle\omega, dd^*\omega\rangle\rangle = \langle\langle d^*\omega, d^*\omega\rangle\rangle,$$

so $d^*\omega = 0$. □

Note that if ω and ω' are harmonic in the same cohomology class, then $\|\omega\|^2 = \|\omega'\|^2$. If we write $\omega = \omega' + d\eta$, then $\|d\eta\|^2 = 0$ by formula (3), so $\omega = \omega'$. This shows uniqueness.

Proving existence will be much more difficult. The problem is that the space $\mathcal{A}^p(M)$ is not only infinite-dimensional but also not complete. Therefore, there is no *abstract* way to conclude that the desired minimum exists.

Proposition 5.14. *The operators $*$ and Δ commute: $*\Delta = \Delta *$.*

Corollary 5.15. *The form ω is harmonic if and only if $*\omega$ is harmonic.*

Proof of Proposition 5.14. Computing on p-forms:

$$\begin{aligned} *\Delta &= *(dd^* + d^*d) \\ &= (-1)^{n(p-1)+1} * d * d\ \ * + (-1)^{p+1} d * d \end{aligned}$$

and

$$\begin{aligned} \Delta * &= (dd^* + d^*d)* \\ &= (-1)^{n(n-p-1)+1} d * d \underbrace{**}_{(-1)^{p(n-p)}} + (-1)^{n(n-p)+1} * d * d * \\ &= (-1)^{p+1} d * d + (-1)^{n(p-1)+1} * d * d *, \end{aligned}$$

which agrees with $*\Delta$ above. □

Note that if n is even (for example, M is a complex manifold), then the sign becomes simply $d^* = - * d *$, which is easier to keep track of.

We have a formally self-adjoint operator

$$\Delta \colon \mathcal{A}^p(M) \to \mathcal{A}^p(M).$$

If we have a self-adjoint linear map $T \colon V \to V$, where V is a finite-dimensional vector space with a scalar product $\langle\cdot,\cdot\rangle$, then $\ker(T)$ is perpendical to $\operatorname{im}(T)$, because for $Tu = 0$, we have

$$\langle u, Tv\rangle = \langle Tu, v\rangle = 0.$$

Moreover, these have complementary dimension and hence $\langle\cdot,\cdot\rangle$ gives an isomorphism

$$V = \ker(T) \overset{\perp}{\oplus} \operatorname{im}(T).$$

The same holds if T is an operator on a Hilbert space.

However, the spaces $\mathcal{A}^p(M)$ are **not** Hilbert spaces. We hence have to do more work in other to prove such a statement for $T = \Delta$, $V = \mathcal{A}^p(M)$.

5.2. **Differential operators.** Let M be a smooth manifold and $\mathcal{C}_M^\infty$ be the sheaf of real-valued smooth functions on M. Consider

$$\mathcal{D}_M \subseteq \mathcal{E}\mathrm{nd}_{\mathbb{R}}(\mathcal{C}_M^\infty)$$

generated as a sheaf of rings by $\mathcal{C}_M^\infty$ (acting by homotheties) and $\mathcal{D}\mathrm{er}_{\mathbb{R}}(\mathcal{C}_M^\infty)$. Note that this is a sheaf of non-commutative rings.

If $U \subseteq M$ is a chart with coordinates $x_1, \dots, x_n$, then $\mathcal{D}\mathrm{er}(\mathcal{C}_M^\infty)$ is generated over $\mathcal{C}_M^\infty$ by $\partial_1, \dots, \partial_n$, where $\partial_i = \frac{\partial}{\partial x_i}$.

Hence $\mathcal{D}_U$ is free over $\mathcal{C}_M^\infty$ (both as a left and as a right modulo), with basis given by

$$\partial^* = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n).$$

Let $\mathcal{F}_k\mathcal{D}_M \subseteq \mathcal{D}_M$ for $k \geq 0$ be the subsheaf of locally generated (in charts as above) by ∂^α with $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$. We call these differential operators of *order* $\leq k$. For example:

- $\mathcal{F}_0\mathcal{D}_M = \mathcal{C}_M^\infty$,
- $\mathcal{F}_1\mathcal{D}_M = \mathcal{C}_M^\infty + \mathcal{D}\mathrm{er}_{\mathbb{R}}(\mathcal{C}_M^\infty)$.

They satisfy the following obvious properties.

(1) We see that

$$\mathcal{F}_k\mathcal{D}_M \cdot \mathcal{F}_\ell\mathcal{D}_M \subseteq \mathcal{F}_{k+\ell}\mathcal{D}_M$$

using $[\partial_k, g] = \frac{\partial g}{\partial x_k}$. This implies that

$$\operatorname{gr}_{\mathcal{F}} \mathcal{D}_M = \bigoplus_{k \geq 0} \mathcal{F}_k\mathcal{D}_M / \mathcal{F}_{k-1}\mathcal{D}_M$$

has an induced graded ring structure.

(2) We have that $[\mathcal{F}_k\mathcal{D}_M, \mathcal{F}_\ell\mathcal{D}_M] \subseteq \mathcal{F}_{k+\ell-1}\mathcal{D}_M$, so $\operatorname{gr}_F\mathcal{D}_X$ is a sheaf of commutative rings.

Note that

$$\operatorname{gr}_F \mathcal{D}_M = \mathcal{C}_M^\infty \oplus \mathcal{T}_M \oplus \cdots ,$$

where we write $\mathcal{T}_M$ for the sheaf of sections of the tangent bundle $\mathrm{T}M$ and identify it with $\mathcal{D}\mathrm{er}_{\mathbb{R}}(\mathcal{C}_M^\infty)$.

By the universal property of the symmetric algebra, we get a morphism of sheaves of graded commutative $\mathcal{C}_M^\infty$-algebras:

$$\mathcal{S}\mathrm{ym}_{\mathcal{C}_M^\infty}(\mathcal{T}_M) \to \operatorname{gr}_F \mathcal{D}_M.$$

Using the local description of $\mathcal{D}_X$ in a chart, we see that this is an isomorphism. Given an operator, $P \in \Gamma(M, \mathcal{D}_X)$ of order k (i.e. order $\leq k$ but not $\leq k-1$), the *symbol of* P is the corresponding section

$$\sigma_k(P) \in \Gamma(M, \mathcal{S}\mathrm{ym}^k(\mathcal{T}_M)).$$

More generally, suppose E, F are smooth (real) vector bundles on M, with corresponding sheaves $\mathcal{E}$, $\mathcal{F}$. Then

$$\mathrm{Diff}_k(\mathcal{E}, \mathcal{F}) = \left\{ P \in \mathcal{E}\mathrm{nd}(\mathcal{E}, \mathcal{F}) \;\middle|\; \begin{array}{c} \text{locally on open subsets } U \subseteq X \\ \text{such that } \mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r},\ \mathcal{F}_{|U} \cong \mathcal{O}_U^{\oplus s} \\ P \text{ is given by } (P_{i,j}) \text{ with each} \\ P_{i,j} \text{ a differential operator of order } \leq k \end{array} \right\}.$$

Example 5.16. The map $d \colon \mathcal{A}^p_M \to \mathcal{A}^{p+1}_M$ is a differential operator of order 1.

If P is a differential operator of order $\leq k$, we want to define $\sigma_k(P)$, as above. Locally on an open subset $U \subseteq X$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$, $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus s}$, if P is given by $(P_{i,j})_{i,j}$, we consider $(\sigma_k(P_{i,j}))_{i,j}$ where

$$\sigma_k(P_{i,j}) \in \Gamma(U, \mathcal{S}\mathrm{ym}(\mathcal{T}_M)).$$

These glue together to give

$$\sigma_k(P) \in \Gamma(M, \mathcal{S}\mathrm{ym}(\mathcal{T}_M) \otimes \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})).$$

Given $x \in M$, we get a map

$$T^*_x M \to \mathrm{Hom}_{\mathbb{R}}(E_{(x)}, F_{(x)})$$

which is a homogeneous polynomial of degree k.

Definition 5.17. A differential operator $P \in \mathrm{Diff}_k(E, F)$, with $\mathrm{rank}(E) = \mathrm{rank}(F)$, is *elliptic* if for all $x \in M$ and any non-zero $v \in T^*_x M$, $\sigma_k(P)_x(v)$ is an isomorphism.

If $P = \sum\limits_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$, then

$$\sigma_k(P) = \sum_{|\alpha| = k} a_\alpha(x) y^\alpha.$$

Therefore, P is elliptic if and only if for all $(y_1, \ldots, y_n) \neq 0$ and all x,

$$\sum_{|\alpha| = k} a_\alpha(x) y^\alpha \neq 0.$$

Example 5.18 (Main example)**.** The *Laplace-Beltrami operator* Δ, where

$$\Delta = d^* d + d d^* \quad d^* = \pm * d \ * .$$

Recall that $*$ has order 0 and d has order 1, so Δ is a differential operator of order ≤ 2.

Goals.

- Compute Δ on $\mathbb{R}^n$ with the usual metric and orientation.
- Compute $\sigma_2(\Delta)$ in general and show that Δ is elliptic.

Recall that if M is a smooth manifold, X is a vector field, and ω is a p-form, $i_X\omega$ is the *contraction of* ω *with respect to* X given by: if $X_1, \dots, X_{p-1}$ are vector fields, then

$$(i_X\omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1}).$$

For example, $i_X(df) = X(f)$.

Exercise. The contraction along X, i_X, *behaves well* with respect to $\wedge$:

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X(\beta). \tag{4}$$

We write down i_X explicitly in local coordinates. If $\xi_1, \dots, \xi_n$ trivialize TM on U and $\xi_1^*, \dots, \xi_n^*$ is the dual basis of T^*M, we set

$$\xi_I^* = \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^* \quad \text{for } I : i_1 < \cdots < i_p.$$

Then

$$i_{\xi_j}(\xi_I^*) = \begin{cases} 0 & \text{if } j \notin I \\ (-1)^{k-1} \xi_{I \setminus \{j\}}^* & \text{if } j = i_k \end{cases}$$

Lemma 5.19. *Let M be an oriented Riemannian manifold and $\xi_1, \dots, \xi_n$ be an orthonormal positively oriented local basis for TM. Then for all I with $|I| = p$,*

$$*(\xi_j^* \wedge *\xi_I^*) = (-1)^{n(p-1)} i_{\xi_j}(\xi_I^*).$$

Proof. We first make sure that this equality is correct when we ignore the signs. For $j \notin I$, both sides are clearly 0. Otherwise, if $j \in I$, the left hand side is

$$\pm * (\xi_j^* \wedge \pm \xi_{\bar{I}}^*) = \pm * \xi_{\bar{I} \cup \{j\}}^* = \pm \xi_{I \setminus \{j\}}^*.$$

Checking that the signs agree is left as an exercise. ☐

We now compute $\sigma_2(\Delta)$ in general and Δ if $M = \mathbb{R}^n$ with standard metric and orientation.

Consider a p-form ω which may be written in the local coordinates as $\omega = \sum_{|I|=p} f_I \xi_I^*$. Then

$$\begin{aligned} d^*\omega &= (-1)^{n(p-1)+1} * d \; *(\omega) \\ &= (-1)^{n(p-1)+1} * d \left(\sum_{|I|=p} f_I * \xi_I^* \right) \\ &= \underbrace{(-1)^{n(p-1)+1} * \sum_{|I|=p} \sum_{k=1}^{n} \xi_k(f_I)(\xi_k^* \wedge *\xi_I^*)}_{\text{Term A}} + \underbrace{(-1)^{n(p-1)+1} * \sum_{|I|=p} f_I d(*\xi_I^*)}_{\text{Term B}}. \end{aligned}$$

Term B is

- $= 0$ in $\mathbb{R}^n$ with $\xi_i = \frac{\partial}{\partial x_i}$,
- in general, can be ignored in the computation of $\sigma_2(\Delta)$.

We are hence left with computing Term A. We have that

$$\text{Term A} = (-1)^{n(p-1)+1} \sum_{|I|=p} \sum_{k=1}^{n} \xi_k(f_I) \underbrace{*(\xi_k^* \wedge *\xi_I^*)}_{(-1)^{n(p-1)} i_{\xi_k} \xi_I^*}$$
$$= -\sum_I \sum_k \xi_k(f_I) i_{\xi_k}(\xi_I^*).$$

So far, we have shown that

$$d^*\omega = -\sum_I \sum_k \xi_k(f_I) i_{\xi_k}(\xi_I^*) + (\text{linear operator in } \omega)$$

and the linear operator in ω is 0 if $U = \mathbb{R}^n$ with the standard metric and basis. Moreover,

$$d\omega = \sum_{|I|=p} \sum_j \xi_j(f_I) \xi_j^* \wedge \xi_I^*.$$

Putting these together, we see that

$$dd^* = -\sum_I \sum_{j,k} \xi_j \xi_k(f_I) \xi_j^* \wedge i_{\xi_k}(\xi_I^*) + \underbrace{(\text{operator of order } \leq 1 \text{ in } \omega)}_{=0 \text{ in } \mathbb{R}^n},$$
$$d^*d = -\sum_I \sum_{j,k} \xi_k \xi_j(f_I) i_{\xi_k}(\xi_j^* \wedge \xi_I^*) + \underbrace{(\text{operator of order } \leq 1 \text{ in } \omega)}_{=0 \text{ in } \mathbb{R}^n}.$$

and hence

$$\Delta\omega = (dd^*+d^*d)\omega = -\sum_I \sum_{j,k} \xi_k \xi_j(f_I)(\xi_j^* \wedge i_{\xi_k}(\xi_I^*) + i_{\xi_k}(\xi_j^* \wedge \xi_I^*)) + \underbrace{(\text{operator of order } \leq 1 \text{ in } \omega)}_{=0 \text{ in } \mathbb{R}^n}$$

where we use the fact that $[\xi_k, \xi_j]$ is an operator of order ≤ 1. Using the formula (4) for $i_X(\alpha \wedge \beta)$, we see that

$$\Delta\omega = -\sum_I \sum_{j,k} \xi_k \xi_j(f_I) i_{\xi_k}(\xi_j^*) \wedge \xi_I^* - \sum_k \xi_k^2(f_I) \xi_I^* + \underbrace{(\text{operator of order } \leq 1 \text{ in } \omega)}_{=0 \text{ in } \mathbb{R}^n}.$$

Conclusion.

(1) If $M = \mathbb{R}^n$, we get the formula

$$\Delta \left(\sum_I f_I dx_I \right) = -\sum_I \left(\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2} \right) dx_I$$

which is the standard Laplace operator in $\mathbb{R}^n$.

(2) In general, $\sigma_2(\Delta)$ ignores the operators of order ≤ 1, so we get the expression

$$\sigma_2(\Delta)_x \left(\sum_{k=1}^{n} v_k \xi_k^*(x) \right) = -\left(\sum_{k=1}^{n} v_k^2 \right) \cdot \mathrm{Id}$$

so

$$\sigma_2(\Delta)_x(v) = -\|v\|^2 \cdot \mathrm{Id}$$

which is an isomorphism if $v \neq 0$. In particular, Δ is an elliptic operator.

Suppose $P \in \mathrm{Diff}_k(\mathcal{E}, \mathcal{F})$ where M is compact and oriented. If we have a metric on $\mathcal{E}$ and a volume element dV, we can define a scalar product on $\mathcal{E}(M)$ by

$$\langle\langle s, t\rangle\rangle = \int_M \langle s, t\rangle \, dV.$$

Given $P \in \mathrm{Diff}_k(\mathcal{E}, \mathcal{F})$ such that both $\mathcal{E}$, $\mathcal{F}$ carry metrics, there is a *formal adjoint*

$$P^* \in \mathrm{Diff}_k(\mathcal{F}, \mathcal{E})$$

such that

$$\langle\langle Ps, t\rangle\rangle = \langle\langle s, P^* r\rangle\rangle \quad \text{for all } s \in \mathcal{E}(M), t \in \mathcal{F}(M).$$

Moreover, for all $x \in M$, $v \in T^*_X(M)$,

$$\sigma_k(P^*)_x(v) = (\sigma_k(P)_x(v))^*$$

where the right hand side is the adjoint with respect to the scalar product on the fibers.

In particular, if $\mathrm{rank}(E) = \mathrm{rank}(F)$, then P is elliptic if and only if P^* is elliptic.

Theorem 5.20 (Fundamental theorem)**.** *Suppose M is compact and oriented, and E, F are smooth vector bundles of the same rank, with metrics, and we have a volume element dV. For an elliptic differential operator $P \in \mathrm{Diff}_k(E, F)$, we have*

(1) $\dim_{\mathbb{R}} \ker(P) < \infty$ *(so* $\mathrm{codim}_{\mathbb{R}} \mathrm{im}(P) < \infty$*),*
(2) $\mathcal{E}(M) = \ker(P) \overset{\perp}{\oplus} \mathrm{im}(P^*)$*.*

In particular, if P is self-adjoint, then

$$\mathcal{E}(M) = \ker(P) \oplus \mathrm{im}(P).$$

Note that $\ker(P) \perp \mathrm{im}(P^*)$ by adjointness. The assertion in the theorem is that $\mathcal{E}(M)$ is a direct sum of the two.

The subtle issue is that $\mathcal{E}(M)$ and $\mathcal{F}(M)$ are not complete with respect to $\langle\langle \cdot, \cdot\rangle\rangle$, so we cannot apply the usual theory of Hilbert spaces. We have first to enlarge the spaces to suitable spaces of *distributions*. The hard part is then showing that, given an elliptic operator P^* and a section s of $\mathcal{E}$ with coefficients in a distribution, if $P^* s$ is smooth, then s is smooth.

What does this say about Δ? Since Δ is elliptic and self-adjoint, Theorem 5.20 implies that

$$\mathcal{A}^p(M) = \ker(\Delta) \overset{\perp}{\oplus} \mathrm{im}(\Delta). \tag{5}$$

We already know that the kernel is the space of harmonic forms:

$$\ker(\Delta) = \mathcal{H}^p(M, \mathbb{R}) = \{\omega \in \mathcal{A}^p(M) \mid \Delta\omega = 0\}.$$

We need to compute the image. We have that

$$\Delta(\mathcal{A}^p(M)) \subseteq d(\mathcal{A}^{p-1}(M)) + d^*(\mathcal{A}^{p+1}(M)) \tag{6}$$

because $\Delta = dd^* + d^*d$. Note that

(1) $d(\mathcal{A}^{p-1}(M)) \perp d^*(\mathcal{A}^{p+1}(M))$, because

$$\langle\langle d\omega, d^*\eta\rangle\rangle = \langle\langle d^2\omega, \eta\rangle\rangle = 0,$$

(2) $\mathcal{H}^p(M,\mathbb{R})$ is orthogonal to

$$d(\mathcal{A}^{p-1}) + d^*(\mathcal{A}^{p+1}(M)),$$

because for harmonic ω, $d\omega = 0$ and $d^*\omega = 0$, so

$$\langle\langle \omega, d\eta \rangle\rangle = \langle\langle d^*\omega, \eta \rangle\rangle = 0$$

and similarly $\langle\langle \omega, d^*\theta \rangle\rangle = 0$.

The orthogonal decomposition (5) together with the inclusion (6) shows that

$$\Delta(\mathcal{A}^p(M)) = d(\mathcal{A}^{p-1}(M)) \overset{\perp}{\oplus} d^*(\mathcal{A}^{p+1}(M)).$$

What is the kernel $\ker(d\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M))$? It contains $\mathcal{H}^p(M,\mathbb{R})$ and $d(\mathcal{A}^{p-1}(M))$. We claim that

$$\ker(d\colon \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)) = \mathcal{H}^p(M,\mathbb{R}) \overset{\perp}{\oplus} d(\mathcal{A}^{p-1}(M)).$$

For this, it is enough to show that if $d^*\eta \in \ker(d)$, then $d^*\eta = 0$ (using the decomposition (5)). This is clear since $dd^*\eta = 0$ implies that

$$\langle\langle d^*\eta, d^*\eta \rangle\rangle = \langle\langle \eta, dd^*\eta \rangle\rangle = 0,$$

so $d^*\eta = 0$.

Finally, we deduce from this discussion the fundamental theorem of Hodge theory.

Corollary 5.21 (Hodge theorem)**.** *Suppose M is a compact oriented Riemannian manifold. We have a canonical isomorphism*

$$H^p_{\mathrm{dR}}(M,\mathbb{R}) \cong \mathcal{H}^p(M,\mathbb{R}).$$

Corollary 5.22. *We have that* $\dim_{\mathbb{R}} H^p_{\mathrm{dR}}(M,\mathbb{R}) < \infty$.

Proof. This follows from part (1) of Theorem 5.20. □

One can also prove this theorem using triangulation of manifolds. This is, however, not much easier than the method we employed.

Elementary application. Let M be a compact, orientable manifold. We then have a *Poincaré duality*. Put a metric on M and choose an orientation. Then $*$ gives an isomorphism

$$\mathcal{H}^p(M,\mathbb{R}) \cong \mathcal{H}^{n-p}(M,\mathbb{R}).$$

depending on the choice of metric.

Here is a better statement: the pairing

$$H^p_{\mathrm{dR}}(M,\mathbb{R}) \times H^{n-p}_{\mathrm{dR}}(M,\mathbb{R}) \to \mathbb{R}$$
$$(\omega,\eta) \mapsto \int_M \omega \wedge \eta$$

is non-degenerate. To see this, it is enough to show that for every p and every $\alpha \in H^p_{\mathrm{dR}}(M,\mathbb{R})$, there is a $\beta \in H^{n-p}_{\mathrm{dR}}(M,\mathbb{R})$ such that

$$\int \alpha \wedge \beta \neq 0.$$

For this, choose a metric, and choose $\omega \in \mathcal{H}^p(M, \mathbb{R})$ such that $[\omega] = \alpha$. Then take $\beta = [*\omega]$ where $*\omega$ is also harmonic to obtain

$$\int_M \omega \wedge *\omega = \int_M \langle \omega, \omega \rangle \, dV = \langle\langle \omega, \omega \rangle\rangle > 0$$

if $\omega \neq 0$. This implies the non-degeneracy of the Poincaré duality.

6. Hodge theory of complex manifolds

We now discuss Hodge theory for complex manifolds.

6.1. **Linear algebra background.** Let V be a finite-dimensional complex vector space. Then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V' \oplus V''$ with each V', V'' isomorphic to V.

Definition 6.1. A *Hermitian form* on V is a bilinear map

$$V \times \overline{V} \xrightarrow{h} \mathbb{C}$$

such that $h(v, w) = \overline{h(w, v)}$.

Given such h, we write it as $h = S + iA$ where $S, A \colon V \times \overline{V} \to \mathbb{R}$ are bilinear over $\mathbb{R}$ and S is symmetric, A is skew-symmetric. Note that

$$S(iv, w) + iA(iv, w) = i(S(v, w) + iA(v, w))$$

so

$$S(iv, w) = -A(v, w),$$
$$A(iv, w) = S(v, w).$$

It is easy to see that, giving a Hermitian form h on V is equivalent to giving a symmetric bilinear form $S \colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that $S(iv, iw) = S(v, w)$ for all v, w. In this case, A is defined by $A(v, w) = -S(iv, w)$. It is also equivalent to giving a skew-symmetric bilinear form $A \colon V_{\mathbb{R}} \to V_{\mathbb{R}} \to \mathbb{R}$ such that $A(iv, iw) = A(v, w)$. In this case, S is given by $S(v, w) = A(iv, w)$.

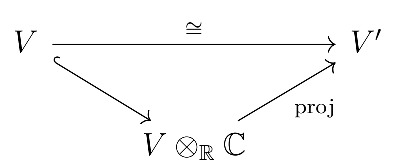
Definition 6.2. The Hermitian form h is a *Hermitian metric* such that $h(v, v) > 0$ for all $v \in V$ non-zero.

In this case, $S = S_h$ given a scalar product on $V_{\mathbb{R}}$. This can be uniquely extended to a Hermitian form S_h on $V_{\mathbb{C}}$. This is again a metric:

$$S_h(v + iw, v + iw) = S_h(v, v) + S_h(w, w) + i(S_h(w, v) - S_h(v, w)) = S_h(v, v) + S_h(w, w) > 0$$

if $v + iw \neq 0$.

Lemma 6.3. *The canonical isomorphism $V \cong V'$ given by*



is compatible with the Hermitian forms (up to a constant scalar factor).

Proof. Recall that if $J\colon V \to V$ is multiplication by i, then

$$\begin{aligned} V &\to V' \\ v &\mapsto \frac{1}{2}(v - iJv). \end{aligned}$$

We then have

$$\begin{aligned} S_h\left(\frac{1}{2}(v-iJv), \frac{1}{2}(w-iJw)\right) &= \frac{1}{4}\left((S_h(v,w) + \underbrace{S_h(Jv,Jw)}_{S_h(v,w)}) - i(S_h(Jv,w) - \underbrace{S_h(v,Jw)}_{-S_h(Jv,w)})\right) \\ &= \frac{1}{2}(S_h(v,w) - iS_h(Jv,w)) \\ &= \frac{1}{2}h(v,w) \end{aligned}$$

This is what we wanted to prove. □

Remark 6.4.

(1) If $h\colon V \times \overline{V} \to \mathbb{C}$ is a Hermitian metric, we get an isomorphism $\overline{V} \cong V^*$ given by

$$v \mapsto h_v = h(-, v).$$

Also, $\overline{h}\colon \overline{V} \times V \to \mathbb{C}$, which gives a Hermitian metric on $\overline{V}$. Combining these, we get a Hermitian metric h^* on V^* such that

$$h^*(h_v, h_w) = \overline{h(v,w)}.$$

(2) Given a Hermitian metric h on V, we get Hermitian metric on all $\bigwedge^p V$ given by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p\rangle = \det(h(v_i, w_j)).$$

Given a Hermitian metric h on V, consider the Hermitian metric S_h on $V_{\mathbb{C}}$ and use it to put a Hermitian metric on all $\bigwedge^p (V_{\mathbb{C}})^*$.

Exercise. This is the same as using the scalar product S_h on V to get a scalar product on $\bigwedge^p V^*$ and then extending this by linearity to a Hermitian linear form on $(\bigwedge^p V^*)_{\mathbb{C}} \cong \bigwedge^p (V^*_{\mathbb{C}})$.

Lemma 6.5. *The decomopsition*

$$\bigwedge^p V^*_{\mathbb{C}} \cong \bigoplus_{i+j=p} \left(\bigwedge^i (V')^* \oplus \bigwedge^j (V'')^*\right)$$

is orthogonal with respect to the Hermitian metric.

Proof. To show this, it is enough to check that the decomposition $V_{\mathbb{C}} = V' \oplus V''$ is orthogonal with respect to S_h. But:

$$\begin{aligned} S_h\left(\frac{1}{2}(v - iJv), \frac{1}{2}(w + iJw)\right) &= \frac{1}{4}\Big((S_h(v,w) + S_h(Jv,Jw)) + i(S_h(v,Jw) + S_h(Jv,w))\Big) \\ &= 0 \end{aligned}$$

as required. $\square$

Suppose (V, h) is as above. We have a scalar product S_h on V. Since V is a complex vector space, we have a canonical orientation on V. Let $n = \dim_{\mathbb{C}} V$. We get a canonical *volume element* $dV \in \bigwedge^{2n} V^* \subseteq \bigwedge^{2n} V_{\mathbb{C}}^*$.

We hence have the Hodge operator

$$*\colon \bigwedge^{p} V^* \xrightarrow{\cong} \bigwedge^{2n-p} V^*$$

which is an isomorphism of $\mathbb{R}$-vector spaces. We extend scalars to $\mathbb{C}$ to get

$$*\colon \bigwedge^{p} V_{\mathbb{C}}^* \xrightarrow{\cong} \bigwedge^{2n-p} V_{\mathbb{C}}^*$$

We have the following analog of Proposition 5.3 which defined $*$.

Lemma 6.6. *For every $\omega, \eta \in \bigwedge^p V_{\mathbb{C}}^*$, we have*

$$\omega \wedge \overline{*}\eta = \langle \omega, \eta \rangle dV.$$

Exercise. Check this, using the fact that we know this if $\omega, \eta \in \bigwedge^p V^*$.

We write $\bigwedge^{p,q} V_{\mathbb{C}}^*$ for $\bigwedge^p V'^* \otimes \bigwedge^q V''^*$.

Corollary 6.7. *The map $*$ maps $\bigwedge^{p,q} V_{\mathbb{C}}^*$ to $\bigwedge^{n-q,n-p} V_{\mathbb{C}}^*$.*

Recall that if $w\colon \bigwedge^{\bullet} V_{\mathbb{C}}^* \to \bigwedge^{\bullet} V_{\mathbb{C}}^*$ is the *de Rham operator* acting by muliplication with $(-1)^m$ on $\bigwedge^m V^*\mathbb{C}$, then

$$** = w.$$

Note that $*$ is defined by scalar extension from $\mathbb{R}$, so it is a real operator, i.e. $\overline{*\omega} = *\overline{\omega}$.

6.2. **Globalization.** Let M be an n-dimensional complex manifold. We can always choose on M a Hermitian metric. The key point is that a real positive function times a Hermitian metric is still a Hermitian metric, and a finite sum of Hermitian metrics is still a Hermitian metric, so we can construct such metrics locally and glue using partitions of unity.

Fix such a metric h. Then $S = \mathrm{Re}(h)$ is a Riemannian metric on M with the standard orientation and we get a *volume element* dV which is a real (n, n)-form on M. The $*$ operator

$$*\colon \mathcal{A}_M^{p,q} \cong \mathcal{A}_M^{n-q,n-p}$$

is the unique map satisfying

$$\langle \omega, \eta \rangle dV = \omega \wedge \overline{*\eta}$$

(as in Lemma 6.6). Note that S also induces Hermitian metrics on all $\mathcal{A}^m_{M,\mathbb{C}}$ such that the (p,q)-components are orthogonal.

From now on, suppose M is compact. We get Hermitian metrics on each $\mathcal{A}^{p,q}(M)$ by

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \langle \omega, \eta \rangle \, dV = \int_M \omega \wedge \overline{*\eta}.$$

Note that $\langle\langle \omega, \omega \rangle\rangle > 0$ unless $\omega = 0$.

It is easy to say that the induced Hermitian metric on $\mathcal{A}^m_{M,\mathbb{C}}(M) = \mathcal{A}^m_M(M) \otimes_{\mathbb{R}} \mathbb{C}$ is the one induced by extending to complexifications of the one we associated before to the Riemannian structure.

6.3. The operators ∂^* and $\overline{\partial}^*$.

We have operators

$$\partial \colon \mathcal{A}^{p,q}_M \to \mathcal{A}^{p+1,q}_M$$
$$\overline{\partial} \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}_M$$

such that $d = \partial + \overline{\partial}$. Recall that $d^* = - * d *$.

Definition 6.8. Define

$$\partial^* = - * \overline{\partial} * \colon \mathcal{A}^{p+1,q}_M \to \mathcal{A}^{p,q}_M,$$
$$\overline{\partial}^* = - * \partial * \colon \mathcal{A}^{p,q+1}_M \to \mathcal{A}^{p,q}_M.$$

Clearly, $d^* = \partial^* + \overline{\partial}^*$.

Proposition 6.9. *The partial (∂, ∂^*) and $(\overline{\partial}, \overline{\partial}^*)$ are formal adjoint pairs.*

Proof. Let $u \in \mathcal{A}^{p,q}(M)$, $v \in \mathcal{A}^{p+1,q}(M)$. Then

$$\begin{aligned}
\langle\langle u, \partial^* v \rangle\rangle &= \int_M u \wedge \overline{*\partial^* v} \\
&= -\int_M u \wedge \overline{* * \overline{\partial} * v} \\
&= (-1)^{p+q+1} \int_M \underbrace{u \wedge \partial * \overline{v}}_{(-1)^{p+q}(\partial(u \wedge * \overline{v}) - \partial u \wedge * \overline{v})} \qquad \text{as } ** = (-1)^{p+q} \\
&= -\int_M \underbrace{\partial(u \wedge * \overline{v})}_{\substack{= d(u \wedge \partial \overline{v}) \\ \text{as } \overline{\partial}(\cdots) = 0}} + \int_M \partial u \wedge * \overline{v} \\
&= 0 + \langle\langle \partial, v \rangle\rangle
\end{aligned}$$

where the first term is 0 by Stokes theorem. The proof of the second adjointness is similar. ☐

Definition 6.10. Let $\Delta' = \partial\partial^* + \partial^*\partial$, $\Delta'' = \overline{\partial\partial}^* + \overline{\partial}^*\overline{\partial}$.

By Proposition 6.9, both Δ' and Δ'' are formally self-adjoint.

Definition 6.11. A form ω is

- *∂-harmonic* if $\Delta'\omega = 0$,
- *$\overline{\partial}$-harmonic* if $\Delta''\omega = 0$.

We write

$$\mathcal{H}^{(p,q)}_{\Delta'}(M) = \{\partial\text{-harmonic } (p,q)\text{-forms}\}$$
$$\mathcal{H}^{(p,q)}_{\Delta''}(M) = \{\overline{\partial}\text{-harmonic } (p,q)\text{-forms}\}$$

As in the case of usual harmonic forms, one shows the following simple properties.

- A form ω is ∂-harmonic if and only if $\partial\omega = 0$ and $\partial^*\omega = 0$,
- A form ω is $\overline{\partial}$-harmonic if and only if $\overline{\partial}\omega = 0$ and $\overline{\partial}^*\omega = 0$,
- We have that $\overline{\partial^*\omega} = \overline{- * \overline{\partial} \ * \omega} = - * \partial \ * \overline{\omega} = \overline{\partial}^*\overline{\omega}$, so $\overline{\Delta'\omega} = \Delta''\overline{\omega}$, and hence
$$\begin{aligned} H^{p,q}_{\Delta'}(M) &\xrightarrow{\cong} \mathcal{H}^{q,p}_{\Delta''}(M) \\ \omega &\mapsto \overline{\omega}, \end{aligned}$$
- Both Δ' and Δ'' commute with $*$: $*\Delta' = \Delta''*$, $*\Delta'' = \Delta'*$. We check the first equality on (p,q)-forms:
$$\begin{aligned} *(\partial\partial^* + \partial^*\partial) &= - * (\partial * \overline{\partial} * + * \overline{\partial} * \partial) \\ &= - * \partial * \overline{\partial} * + (-1)^{p+q+1}\overline{\partial} * \partial \end{aligned}$$
and
$$\begin{aligned} \Delta'' * &= (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) * \\ &= -(\overline{\partial} * \partial * + * \partial * \overline{\partial}) * \\ &= (-1)^{p+q+1}\overline{\partial} * \partial - *\partial * \overline{\partial} *, \end{aligned}$$
so the two agree. Altogether, we get isomorphisms
$$\mathcal{H}^{p,q}_{\Delta'}(M) \xrightarrow[\cong]{*} \mathcal{H}^{n-q,n-p}_{\Delta''}(M) \xrightarrow[\cong]{\text{conj}} \mathcal{H}^{n-p,n-q}_{\Delta'}(M).$$

The same computation we have done for Δ implies that

$$\sigma_2(\Delta')_x(v) = -\frac{1}{2}\|v\|^2 \cdot \mathrm{Id} = \sigma_2(\Delta'')_x(v).$$

Hence, like Δ, the operators Δ' and Δ'' are elliptic operators. We may hence apply the Fundamental Theorem of Elliptic Operators 5.20 to Δ' and Δ''.

Part (1) implies that

$$\mathcal{H}^{p,q}_{\Delta'}(M), \mathcal{H}^{p,q}_{\Delta''}(M)$$

are both finite-dimensional over $\mathbb{C}$. Part (2) for Δ'' gives an orthogonal decomposition

$$\begin{aligned} \mathcal{A}^{p,q}(M) &= \mathcal{H}^{p,q}_{\Delta''}(M) \overset{\perp}{\oplus} \mathrm{Im}(\Delta'' \colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q}(M)) \\ &= \mathcal{H}^{p,q}_{\Delta''}(M) \overset{\perp}{\oplus} \overline{\partial}(\mathcal{A}^{p,q-1}(M)) \overset{\perp}{\oplus} \overline{\partial}^*(\mathcal{A}^{p,q+1}(M)). \end{aligned}$$

Moreover,

$$\ker(\overline{\partial}\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)) = \mathcal{H}^{p,q}_{\Delta''}(M) \overset{\perp}{\oplus} \overline{\partial}(\mathcal{A}^{p,q-1}(M)).$$

The conclusion is that the Dolbeaux cohomology group

$$H^{p,q}(M) = \mathcal{H}^q(\mathcal{A}^{p,\bullet}(M), \overline{\partial}) \cong H^q(M, \Omega^p)$$

is isomorphic to

$$\mathcal{H}^{p,q}_{\Delta''}(M).$$

Here is an application of the above theory. We have a pairing

$$H^q(X, \Omega^p) \times H^{n-p}(X, \Omega^{n-p}) \to \mathbb{C}$$

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

where α is a (p, q)-form such that $\overline{\partial}\alpha = 0$, β is an $(n-p, n-q)$-form such that $\overline{\partial}\beta = 0$. We claim that this is a non-degenerate pairing. To check this, put a metric h on M. Given a non-zero element in $H^q(X, \Omega^p)$, choose a $\overline{\partial}$-harmonic representative α. If $\beta = \overline{*\alpha}$ is harmonic (so $\overline{\partial}\beta = 0$), and

$$\int_M \alpha \wedge \beta = \langle\langle \alpha, \alpha \rangle\rangle > 0.$$

7. Kähler manifolds

7.1. **Linear algebra background:Kähler metrics.** Let V be a finite-dimensional vector space over $\mathbb{C}$. We write J for multiplication by i. We have a decomposition $V_{\mathbb{C}} = V' \oplus V''$.

Giving a Hermitian form h on V is equivalent to giving a bilinear alternating form $A\colon V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that

$$A(u, v) = A(Ju, Jv) \quad \text{for all} u, v.$$

Indeed, given h, we may define $A = \mathrm{Im}(h)$, and given A, we may define $S(u, v) = A(Ju, v)$ and $h = S + iA$.

Given $A \in \bigwedge^2 V^*_{\mathbb{R}}$, let $\tilde{A} \in \bigwedge^2 V^*_{\mathbb{C}}$ be the corresponding alternating bilinear form on $V_{\mathbb{C}}$. By definition, $\widetilde{A}$ is real.

We claim that

$$A(Ju, Jv) = (u, v) \text{ if and only if } \widetilde{A} \text{ is a } (1, 1)\text{-form.}$$

Indeed, $\widetilde{A}$ is a (1,1) form if and only if $\widetilde{A}(V' \times V') = 0$, $\widetilde{A}(V'' \times V'') = 0$. Since $\widetilde{A}(\overline{u}, \overline{v}) = \widetilde{A}(u, v)$, it is enough to check that $\widetilde{A}(V' \times V') = 0$.

Recall that $V' = \{u - iJu \mid u \in V\}$. We have that

$$\widetilde{A}(u - iJu, v - iJv) = (A(u, v) - A(Ju, Jv)) - i(A(Ju, v) + A(u, Jv)).$$

From this, the 'only if' implication is clear and the 'if' implication follows since A takes real values.

Definition 7.1. The *fundamental form* of the Hermitian metric h is the real $(1,1)$ form $\omega_h = -\widetilde{A}$.

We now describe the fundamental form in a basis. Let $x_1, \ldots, x_n$ be a basis of V over $\mathbb{C}$. Let $y_j = Jx_j$ so that

$$x_1, \ldots, x_n, y_1, \ldots, y_n$$

is a basis of $V_{\mathbb{R}}$.

Then a basis of V' is $e_1, \ldots, e_n$ and a basis of V'' is $\overline{e_1}, \ldots, \overline{e_n}$ for

$$e_j = \frac{1}{2}(x_j - iy_j).$$

Given a Hermitian form h, we let $h_{i,j} = h(x_i, x_j)$ so that for $v = \sum v_i e_i$, $w = \sum w_j e_j$,

$$h(v,w) = \sum h_{i,j} v_i \overline{w_j}.$$

Write ω_h as

$$\sum_{j<k} \lambda_{jk} e_j^* \wedge \overline{e_k}^*$$

for some λ_{jk}. We compute it

$$\begin{aligned}
\lambda_{jk} &= \omega_h(e_j, \overline{e_k}) \\
&= -\frac{1}{4}\widetilde{A}(x_j - iy_j, x_k + iy_k) \\
&= -\frac{1}{4}\left((A(x_j,x_k) + \underbrace{A(y_j,y_k)}_{A(x_j,x_k)}) + i(\underbrace{A(x_j,x_k)}_{-S(x_k,x_j)} - \underbrace{A(y_j,x_k)}_{S(x_j,x_k)}) \right) \\
&= -\frac{1}{4} 2 \cdot (A(x_j,x_k) - iS(x_j,x_k)) \\
&= \frac{i}{2}(S(x_j,x_k) + iA(x_j,x_k)) \\
&= \frac{i}{2} h_{j,k}.
\end{aligned}$$

Therefore,

$$\omega_h = \frac{i}{2} \sum_{j<k} h_{j,k} e_j^* \wedge \overline{e_k}^*.$$

Conclusions.

(1) This implies that h is a metric if and only if $-i\omega_h(v,v) > 0$ for all $v \neq 0$.
Hence: giving a Hermitian metric on V is equivalent to giving a real $(1,1)$ form ω with $i\omega(v,v) > 0$ for all v.

(2) Suppose $x_1, \ldots, x_n$ is an orthonormal basis of V. Then $h_{j,k} = \delta_{jk}$, so

$$\omega_h = \frac{i}{2} \sum_{k=1}^{n} e_k^* \overline{e_j}^*.$$

Recall that S gives a top form dV. If $x_1, y_1, \ldots, x_n, y_n$ is a positive orthonormal basis for S,

$$dV = x_1^* \wedge y_1^* \wedge \cdots \wedge x_n^* \wedge y_n^*.$$

On the other hand, the formula above gives

$$\omega_h^n = \left(\frac{i}{2}\right)^n n!\; e_1^* \wedge \overline{e_1}^* \wedge \cdots \wedge e_n^* \wedge \overline{e_n}^*$$

Moreover,

$$e_j^* \wedge \overline{e_j}^* = (x_j^* + iy_j^*) \wedge (x_j^* - iy_j^*) = -2ix_j^* \wedge y_j^*.$$

The conclusion is that

$$\omega_h^n = n! \cdot dV.$$

Definition 7.2. Let M be a complex manifold. A *Hermitian metric* h on M is *Kähler* if the (real $(1,1)$ form) $\omega = \omega_h$ is closed, i.e. $d\omega = 0$.

Example 7.3 (Trivial example)**.** The manifold $\mathbb{C}^n$ with the standard metric is Kähler. With respect to the standard basis which is orthonormal for h,

$$\omega_h = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\overline{z_j}$$

This is clearly closed.

Remark 7.4. The existence of a Kähler metric is a **global** property. The issue is that we cannot glue such metrics using partitions of unity any more. Why? If h is a Kähler metric with form ω and f is a smooth everywhere positive function, then $f \cdot h$ is a metric and $\omega_{fh} = f \cdot \omega_h$. However,

$$d(f\omega_h) = \underbrace{df \wedge \omega_h}_{\text{not necessarily } 0} + \underbrace{fd\omega_h}_{=0}.$$

Example 7.5 (Important example: the Fubini–Study metric on $\mathbb{P}^n$)**.** Let $z_0, \ldots, z_n$ be homogeneous coordinates on $\mathbb{P}^n$ and $U_j = (z_j \neq 0)$. Let

$$\omega_j = \frac{i}{2}\partial\overline{\partial}\log\left(\sum_{k=0}^n \left|\frac{z_k}{z_j}\right|^2\right) \in \mathcal{A}^{1,1}(U_j).$$

These glue to a global $(1,1)$ form. Indeed, on $U_j \cap U_\ell$:

$$\sum \left|\frac{z_k}{z_j}\right|^2 = \left(\sum \left|\frac{z_k}{z_\ell}\right|^2\right) \cdot \left|\frac{z_\ell}{z_j}\right|^2$$

so

$$\log\left(\sum \left|\frac{z_k}{z_j}\right|^2\right) = \log\left(\sum \left|\frac{z_k}{z_\ell}\right|^2\right) + \log\left|\frac{z_\ell}{z_j}\right|^2.$$

It is enough to note that $\partial\overline{\partial}\log|w_j|^2 = 0$ if $w_1, \ldots, w_n$ are coordinates on $\mathbb{C}^n$. This is true because $|w_j|^2 = w_j\overline{w_j}$, so

$$\log|w_j|^2 = \log w_j + \log\overline{w_j},$$

so

$$\partial\overline{\partial}\log|w_j|^2 = 0 \quad \text{ on } \mathbb{C}^n \backslash (w_j = 0).$$

We have hence shown that

$$\omega_j = \frac{i}{2}\partial\overline{\partial}\log\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_j}\right|^2\right)$$

glue to

$$\omega \in \mathcal{A}^{1,1}(\mathbb{P}^n).$$

Since $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$, we see that $\partial\omega = \overline{\partial}\omega = 0$, and since $d = \partial + \overline{\partial}$, $d\omega = 0$. Therefore, ω is closed.

We finally need to check that ω defines a Hermitian metric. Note that

$$\overline{\omega_j} = -\frac{i}{2}\overline{\partial}\partial\log\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_j}\right|^2\right) = \frac{i}{2}\partial\overline{\partial}\log\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_j}\right|^2\right).$$

Let us check that $i\omega(v,v) > 0$ if $v \neq 0$. Work on $U_j \cong \mathbb{C}^n$, $\frac{z_k}{z_j} = w_k$ (and reorder coordinates to assume that $j = 0$). Then

$$\omega_j = \frac{i}{2}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^{n}|w_k|^2\right).$$

Note that

$$\overline{\partial}\log(1 + \sum |w_k|^2) = \sum_{k=1}^{n}\frac{w_k d\overline{w_k}}{1 + \sum_{\ell=1}^{n}|\omega_\ell|^2}$$

and

$$\frac{\partial}{\partial w_j}\left(\frac{w_k}{1 + \sum_\ell |w_\ell|^2}\right) = \frac{\delta_{jk}}{1 + \sum |w_\ell|^2} - \frac{w_k\overline{w_j}}{(1 + \sum |w_j|^2)^2}$$

which shows that

$$\omega_j = \frac{i}{2}\left(\sum_k \frac{dw_k \wedge d\overline{w_k}}{1 + \sum_\ell |w_\ell|^2} - \sum_{k,j}\frac{w_k\overline{w_j}dw_j \wedge d\overline{w_k}}{(1 + \sum |w_\ell|^2)^2}\right).$$

Write

$$a_{jk} = (1 + \sum_\ell |w_\ell|^2)\delta_{jk} - w_k\overline{w_j}.$$

We need to show that if $v \neq 0$, then $v \cdot (a_{jk})\overline{v}^t > 0$. Writing $(\cdot,\cdot)$ for the standard Hermitian metric on $\mathbb{C}^n$, we see that

$$\begin{aligned} v \cdot (a_{jk})\overline{v}^t &= (1 + (w,w))(v,v) - \underbrace{\sum_{j,k} v_j w_k \overline{w_j v_k}}_{(v,w)(w,v)} \\ &= \underbrace{(v,v)}_{>0} + \underbrace{((w,w)\cdot(v,v) - |(v,w)|^2)}_{\geq 0 \text{ by Cauchy–Schwartz inequality}} \\ &> 0. \end{aligned}$$

Therefore, ω is a Kähler metric on $\mathbb{P}^n$.

Remark 7.6. If h is a Hermitian metric on M and $M' \hookrightarrow M$ is a submanifold, then the restriction h' of h to $\mathrm{T}M'$ is a Hermitian metric on M', and $\omega_{h'} = \omega_h|_{M'}$. In particular, if h is Kähler, h' is also Kähler.

Upshot. If X is a smooth quasi-projective complex algebraic variety, we have a locally closed immersion $X \hookrightarrow \mathbb{P}^N$ such that X^{an} is a submanifold of $(\mathbb{P}^N)^{\mathrm{an}}$. By restriction of the Fubini–Study metric to X^{an}, X^{an} has Kähelr metrics.

Example 7.7 (Complex tori)**.** Let $M = V/\Lambda$ where V is an n-dimensional complex vector space and $\Lambda \subseteq V$ is a lattice (i.e. $V \cong \mathbb{C}^n$ and $\Lambda \cong \mathbb{Z}^{2n} \subseteq \mathbb{C}^n$).

If h is the standard Hermitian metric on $\mathbb{C}^n$, with the form $\omega = \frac{i}{2}\sum_{k=1}^{n} z_k \wedge \overline{z_k}$, and $\gamma_\lambda \colon V \to V$ is the translation by some $\lambda \in \Lambda$, then

$$\gamma_\lambda^*(\omega) = \omega.$$

Then h induces a metric on the quotient M, which is again Kähler.

Note that we will later see that for Λ general, M is **not algebraic**. This hence gives an example of Kähler manifolds which are not algebraic.

The next goal is to show that Kähler metrics are not far from the standard one. Suppose $p \in M$. Choose coordinates $z_1, \ldots, z_n$ in a chart around P such that $z_i(P) = 0$ for all i. Suppose h is a Hermitian metric on M, with fundamental form

$$\omega = \frac{i}{2}\sum_{j,k} h_{j,k} dz_j \wedge d\overline{z_k}.$$

We will say that ω *osculates to order 2* to the standard metric at P (in these coordinates) if

$$h_{j,k}(0) = \delta_{jk},$$
$$\frac{\partial h_{j,k}}{\partial z_\ell}(0) = 0,$$
$$\frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(0) = 0.$$

Proposition 7.8. *Given a Hermitian metric h with fundamental form ω, h is Kähler if and only if for all $p \in M$, there is a chart as above such that ω osculates to order 2 with the standard metric.*

Proof. Write $\omega = \frac{i}{2}\sum_{j,k} h_{j,k} dz_j \wedge d\overline{z_k}$. Then

$$d\omega = \frac{i}{2}\sum_{j,k,\ell} \frac{\partial h_{j,k}}{\partial z_\ell} dz_\ell \wedge dz_j \wedge d\overline{z_k} - \frac{i}{2}\sum_{j,k,\ell} \frac{\partial h_{j,k}}{\partial \overline{z_\ell}} dz_j \wedge d\overline{z_\ell} \wedge d\overline{z_k}.$$

It is clear that if $\frac{\partial h_{jk}}{\partial z_\ell}(p) = 0 = \frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(0)$, this implies that $d\omega(p) = 0$. If this holds at every p, ω is closed.

Conversely, suppose $d\omega = 0$. It is easy to see that there is a linear change of variables such that $h_{j,k}(0) = \delta_{jk}$. We will assume that this holds. Let

$$a_{jk\ell} = \frac{\partial h_{j,k}}{\partial z_\ell}(p), \quad a'_{jk\ell} = \frac{\partial h_{jk}}{\partial \overline{z_\ell}}(p).$$

If $d\omega(p) = 0$, we must have

$$a_{jk\ell} = a_{\ell kj}, \quad a'_{jk\ell} = a'_{j\ell k}. \tag{7}$$

Moreover, since ω is real, we have that $h_{jk} = \overline{h_{kj}}$, so

$$\overline{\frac{\partial h_{j,k}}{\partial z_\ell}} = \frac{\partial h_{kj}}{\partial \overline{z_\ell}},$$

which shows that

$$\overline{a_{jk\ell}} = a'_{kj\ell}. \tag{8}$$

Now, do the change of variables

$$w_j = z_j + \frac{1}{2}\sum_{k,\ell=1}^{n} a_{kj\ell} z_k z_\ell.$$

We want to compare ω with $\frac{i}{2}\sum\limits_{j=1}^{n} dw_j \wedge d\overline{w_j}$.

We have that

$$\begin{aligned}
dw_j &= dz_j + \frac{1}{2}\sum_{k,\ell=1}^{n} a_{kj\ell}(z_k dz_\ell + z_\ell dz_k) \\
&= dz_j + \sum_{k,\ell=1}^{n} a_{kj\ell} z_k dz_\ell && \text{by equation (7)} \\
d\overline{w_j} &= d\overline{z_j} + \sum_{k,\ell=1}^{n} a'_{jk\ell}\overline{z_k} d\overline{z_\ell} && \text{by equation (8).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{i}{2}\sum_{j=1}^{n} dw_j \wedge d\overline{w_j} = {} & \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j} \\
& + \frac{i}{2}\sum_{k,\ell,j} \underbrace{a'_{jk\ell}}_{=\frac{\partial h_{j,\ell}}{\partial z_k}(p)} \overline{z_k} dz_j \wedge d\overline{z_\ell} + \frac{i}{2}\sum_{k,\ell,j} \underbrace{a_{kj\ell}}_{\frac{\partial h_{\ell j}}{\partial z_k}}(p) z_k dz_\ell \wedge d\overline{z_j} \\
& + \text{terms vanishis at } p \text{ with order } \geq 2.
\end{aligned}$$

Therefore, $\omega = \frac{i}{2}\sum\limits_{j=1}^{n} dw_j \wedge d\overline{w_j} + \text{terms vanishis at } p \text{ with order } \geq 2$. □

7.2. **Operators on Kähler manifolds.** We have $*, d, \partial, \overline{\partial}$ and the adjoints d^*, ∂^*, $\overline{\partial}^*$. Given the Kähler metric h, with fundamental form ω, the *Lefschetz operator* is

$$L = \omega \wedge - \colon \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q+1}.$$

We also define $\Lambda = *^{-1} L * \colon \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p-1,q-1}$.

Note that since ω is a real form, L is a real operator (i.e. it commutes with conjugation). Since $*$ is a real operator, Λ is a real operator.

Lemma 7.9. *The operator Λ is the adjoint of L, i.e.*

$$\langle L\alpha, \beta \rangle = \langle \alpha, \Lambda\beta \rangle$$

for every (p,q)-form α and $(p+1,q+1)$-form β.

Proof. Recall that $\alpha \wedge \overline{*\beta} = \langle \alpha, \beta \rangle dV$. Then

$$\begin{aligned} \langle \alpha, \Lambda\beta \rangle dV &= \alpha \wedge \overline{*(*^{-1}L*)\beta} \\ &= \alpha \wedge \overline{\omega \wedge *\beta} \\ &= (\alpha \wedge \omega) \wedge \overline{*\beta} \\ &= L\alpha \wedge \overline{*\beta} \\ &= \langle L\alpha, \beta \rangle dV. \end{aligned}$$

□

Theorem 7.10 (Kähler Identities)**.** *We have:*

(1) $[\overline{\partial}^*, L] = i\partial$,
(2) $[\partial^*, L] = i\overline{\partial}$,
(3) $[\Lambda, \partial] = i\partial^*$,
(4) $[\Lambda, \overline{\partial}] = -i\overline{\partial}^*$.

Proof. We only need to prove (1). The other ones follow from the formulas:

$$[P, Q]^* = [Q^*, P^*], \quad (\lambda P)^* = \overline{\lambda} P^*.$$

We prove (1). Suppose first that we deal with a (rescaling by 2) of the standard metric on $\mathbb{C}^n$, i.e.

$$\omega = i \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}.$$

A similar computation to that for d^* gives the expression

$$\overline{\partial}^* \left(\sum_{I,J} f_{I,J} dz_I \wedge d\overline{z_J} \right) = -\sum_{\substack{I,J \\ k}} \frac{\partial f_{I,J}}{\partial z_k} \cdot i_{\frac{\partial}{\partial \overline{z_k}}} (dz_I \wedge d\overline{z_J}).$$

Writing $\eta = \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z_J}$, we have that

$$[\overline{\partial}^*, L]\eta = \overline{\partial}^*(\omega \wedge \eta) - \omega \wedge \overline{\partial}^* \eta.$$

Hence

$$\overline{\partial}^*\eta = -i\sum_{\substack{I,J\\k,j}} \frac{\partial f_{I,J}}{\partial z_k} i_{\frac{\partial}{\partial \overline{z_k}}}(dz_j \wedge d\overline{z_j} \wedge dz_I \wedge d\overline{z_J})$$
$$+ i\sum_{\substack{I,J\\j,k}} \frac{\partial f_{I,J}}{\partial z_k} dz_j \wedge d\overline{z_j} \wedge i_{\frac{\partial}{\partial \overline{z_k}}}(dz_I \wedge d\overline{z_J}).$$

Since $i_{\frac{\partial}{\partial \overline{z_k}}}$ is a derivation, this gives

$$\begin{aligned}\overline{\partial}^*\eta &= -i\sum_{\substack{I,J\\k,j}} \frac{\partial f_{I,J}}{\partial z_k} \underbrace{i_{\frac{\partial}{\partial \overline{z_k}}}(dz_j \wedge d\overline{z_j})}_{-\delta_{jk}dz_j} \wedge dz_I \wedge d\overline{z_J}) \\ &= i\sum_{\substack{I,J\\k}} \frac{\partial f_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z_J} \\ &= i\partial\eta.\end{aligned}$$

This gives the result for the standard metric. In the general case, to check that $[\overline{\partial}^*, L] = i\partial$ at $p \in M$, choose a chart where ω osculates to order 2 with the standard metric (Proposition 7.8), i.e.

$$\omega = i\sum_{j,k} h_{j,k} dz_j \wedge d\overline{z_k}$$

and

$$\begin{aligned}h_{j,k}(P) &= \delta_{jk}, \\ \frac{\partial h_{j,k}}{\partial z_\ell}(P) &= 0, \\ \frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(P) &= 0.\end{aligned}$$

Since both $\overline{\partial}^*$ and ∂ are differential operators of order 1, the difference with respect to the computation for the standard metric will only involve the derivatives $\frac{\partial h_{j,k}}{\partial z_\ell}(P)$, $\frac{\partial h_{j,k}}{\partial \overline{z_\ell}}(P)$. These vanish, which gives the result. □

Corollary 7.11. *If (M, h) is a Kähler manifold, then $\Delta' = \Delta'' = \frac{1}{2}\Delta$.*

Proof. We compute:

$$\begin{aligned}\Delta'' &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \\ &= i(\overline{\partial}(\partial\Lambda - \Lambda\partial)) + i(\partial\Lambda - \Lambda\partial)\overline{\partial} \\ &= i\overline{\partial}\partial\Lambda - i\Lambda\partial\overline{\partial} + i(\partial\Lambda\overline{\partial} - \overline{\partial}\Lambda\partial).\end{aligned}$$

Note that $i\partial\overline{\partial}$ is a real operator, since $\overline{i\partial\overline{\partial}} = -i\overline{\partial}\partial = i\partial\overline{\partial}$. Moreover, L, Λ are both real operators and so is $i(\partial\Lambda\overline{\partial} - \overline{\partial\Lambda\overline{\partial}})$. Overall, this shows that Δ'' is a real operator. Since we know that $\overline{\Delta''} = \Delta'$, this shows that $\Delta' = \Delta''$.

Let us now compute Δ:

$$\begin{aligned}\Delta &= dd^* + d^*d \\ &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}) + (\partial\overline{\partial}^* + \overline{\partial}\partial^* + \partial^*\overline{\partial} + \overline{\partial}^*\partial).\end{aligned}$$

We now show that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. Indeed,

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = i\partial(\partial\Lambda - \Lambda\partial) + i(\partial\Lambda - \Lambda\partial)\partial = -i\partial\Lambda\partial + i\partial\Lambda\partial = 0.$$

Then the above computation shows that

$$\Delta = \Delta' + \Delta'' + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + \overline{(\partial\overline{\partial}^* + \overline{\partial}^*\partial)} = \Delta' + \Delta''.$$

Using $\Delta' = \Delta''$, this completes the proof. □

7.3. **Consequence: the Hodge decomposition.** We finally talk about the consequences. Recall that we have

$$\begin{aligned}\mathcal{H}^m(M, \mathbb{C}) &= \text{space of complex } m\text{-harmonic forms on } M \\ &= \mathcal{H}^m(M) \otimes_{\mathbb{R}} \mathbb{C} \\ &= \text{null}(\Delta_M).\end{aligned}$$

Since $\Delta = 2\Delta' = (2\Delta'')$, the decomposition of $\mathcal{A}^m(M) \otimes_{\mathbb{R}} \mathbb{C}$ into (p, q)-parts, induces a decomposition

$$\mathcal{H}^m(M, \mathbb{C}) = \bigoplus_{p+q=m} \mathcal{H}^{p,q}(M)$$

which are harmonic (p, q)-forms.

We saw (Hodge Theorem 5.21) that the inclusion of $\mathcal{H}^m(M)$ into closed real m-forms induces an isomorphism

$$\mathcal{H}^m(M) \cong H^m_{\text{dR}}(M, \mathbb{R}).$$

Tensoring with $\mathbb{C}$, this gives

$$\mathcal{H}^m(M, \mathbb{C}) \cong H^m_{\text{dR}}(M, \mathbb{C}).$$

If $H^{p,q}(M)$ is the image of $\mathcal{H}^{p,q}(M)$, then we get the *Hodge decomposition*

$$H^m_{\text{dR}}(M, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}.$$

Recall that $\overline{\mathcal{H}^{p,q}_{\Delta'}} = \mathcal{H}^{q,p}_{\Delta''}$. Since $\Delta' = \Delta''$, this shows that

$$\overline{H^{p,q}} = H^{q,p}.$$

We also know that $\mathcal{H}^{p,q}_{\Delta''}(M) \cong H^q(M, \Omega^p)$, so

$$H^{p,q} \cong H^q(M, \Omega^p).$$

Definition 7.12. The *Betti numbers* of M are $b_m = \dim_{\mathbb{C}} H^m(M, \mathbb{C})$.

The *Hodge numbers* of M are $h^{p,q} = \dim_{\mathbb{C}} H^q(M, \Omega^p)$.

Numerically, the above assertions imply that

$$b_m = \sum_{p+q=m} h^{p,q}$$

and the *Hodge symmetry*:

$$h^{p,q} = h^{q,p}.$$

In particular, $b_m(M)$ is even for all m odd.

Therefore, computing these numbers for a manifold might show that it is **not** Kähler.

Example 7.13. Recall the *Hopf surface*

$$M = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{(z_1, z_2) \sim (2z_1, 2z_2)}$$

from Example 3.15. We checked that M is diffeomorphic to $S^3 \times S^1$ and Künneth formula then shows that $b_3(M) = 1$, so M is not Kähler.

7.4. **Independence of metric.** The next goal is to show that the decomposition $H^m(M, \mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}$ is independent of the choice of metric.

Definition 7.14. The *Bott–Chern cohomology* of M is

$$H^{p,q}_{BC}(M) = \frac{\{(p,q)\text{-forms } u \mid \partial u = 0, \overline{\partial} u = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(M)}.$$

Note that $\partial\overline{\partial}v = d(\overline{\partial}v)$, so we have a canonical map

$$H^{p,q}_{BC}(M) \to H^{p+q}_{\mathrm{dR}}(M, \mathbb{C}).$$

Theorem 7.15. *This map gives an isomorphism of $H^{p,q}_{BC}(M)$ and $H^{p,q}(M)$.*

Lemma 7.16 ($\partial\overline{\partial}$-Lemma)**.** *Let M be a Kähler manifold and u be a global form on M such that $\partial u = 0$ and $\overline{\partial} u = 0$. Then the following are equivalent:*

(1) $u \in \mathrm{im}(d)$,
(2) $u \in \mathrm{im}(\partial)$,
(3) $u \in \mathrm{im}(\overline{\partial})$,
(4) $u \in \mathrm{im}(\partial\overline{\partial})$.

Proof. Note that $\partial\overline{\partial}v = d(\overline{\partial}v)$ and $\partial\overline{\partial} = -\overline{\partial}\partial$, so clearly (4) implies (1), (2), (3).

It suffices to prove that (1), (2), (3) imply (4). First, write $u = \partial v$ for some v. By Hodge Theorem 5.21 for $\overline{\partial}$, we may write

$$v = v_1 + \overline{\partial} v_2 + \overline{\partial}^* v_3$$

for some v_1 harmonic and some v_2, v_3. Then

$$u = \partial\overline{\partial} v_2 + \partial\overline{\partial}^* v_3. \tag{9}$$

By Corollary 7.11, $\Delta = \Delta' + \Delta''$ so $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$. Hence $\overline{\partial}u = 0$, equation (9) implies that

$$0 = \underbrace{\overline{\partial}\partial\overline{\partial}v_2}_{=-\overline{\partial}^2\partial v_2=0} + \overline{\partial}\partial\overline{\partial}^* v_3 = \overline{\partial}\partial\overline{\partial}^* v_3.$$

Hence $\overline{\partial}\overline{\partial}^*\partial v_3 = 0$. If $\overline{\partial}\overline{\partial}^*\eta = 0$,

$$0 = \langle\langle \overline{\partial}\overline{\partial}^*\eta, \eta \rangle\rangle = \langle\langle \overline{\partial}^*\eta, \overline{\partial}^*\eta \rangle\rangle$$

so $\overline{\partial}^*\eta = 0$. This shows that $\overline{\partial}^*\partial v_3 = 0$, so

$$u = \partial\overline{\partial}v_2.$$

If $u \in \mathrm{im}(\overline{\partial})$, apply the previous argument for $\overline{u}$ to shows that $u \in \mathrm{im}(\partial\overline{\partial})$.

Finally, if $u = d(w) = \partial w + \overline{\partial}w$, we see that $\partial u = 0$ implies that $\partial\overline{\partial}w = 0$ and $\overline{\partial}\overline{\partial}w = 0$. By the (3) implies (4) implication for $\overline{\partial}w$, we see that $\overline{\partial}w \in \mathrm{im}(\partial\overline{\partial})$. Similarly, $\overline{\partial}u = 0$ implies that $\partial w \in \mathrm{im}(\partial\overline{\partial})$, so $u \in \mathrm{im}(\partial\overline{\partial})$. ☐

Recall that $H^{p,q}_{BC}(M)$, which was defined as

$$\frac{\{\text{global } (p,q)\text{-forms } u \mid \partial u = 0, \overline{\partial}u = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(M)}.$$

We hence have a map

$$\varphi\colon H^{p,q}_{BC} \to H^{p+q}_{DR}(M,\mathbb{C})$$
$$u \mapsto [u].$$

Recall that

$$H^{p,q}(M) = \{\alpha \in H^{p+q}_{\mathrm{dR}}(M,\mathbb{C}) \mid \alpha = [\eta], \eta \text{ is a harmonic } (p,q)\text{-form}\}.$$

We check that $H^{p,q}_{BC}$ lands in this piece of de Rham cohomology. Since $\partial u = 0$, the Hodge Theorem 5.21 for ∂ implies that

$$u = v + \partial w$$

for some harmonic (p,q)-form v. Then $0 = \overline{\partial}u = \overline{\partial}\partial w$ and $\partial\partial w = 0$, so the $\partial\overline{\partial}$-Lemma 7.16 for ∂w shows that $\overline{\partial}w \in \mathrm{im}(d)$, so $[u] = [v]$ in $H^{p+q}_{\mathrm{dR}}(M,\mathbb{C})$.

This shows that $\mathrm{im}(\varphi) \subseteq H^{p,q}(M)$. Moreover, since a harmonic (p,q)-form satisfies $\partial\eta = 0$, $\overline{\partial}\eta = 0$, φ is surjective by Hodge Theorem 5.21.

Finally,φ is injective by the implication (1) implies (4) in $\partial\overline{\partial}$-Lemma 7.16.

Corollary 7.17. *The map φ induces an isomorphism $H^{p,q}_{BC}(M) \cong H^{p,q}(M)$.*

Corollary 7.18.

(1) *If $\alpha \in H^{p,q}(M)$, $\beta \in H^{p',q'}(M)$, $\alpha \cup \beta \in H^{p+p',q+q'}(M)$.*
(2) *If $f\colon M' \to M$ is a holomorphic map of complex manifolds of Kähler type, the pullback maps on cohomology*

$$f^*\colon H^m_{\mathrm{dR}}(M,\mathbb{C}) \to H^m_{\mathrm{dR}}(M',\mathbb{C})$$

map each $H^{p,q}(M)$ to $H^{p,q}(M')$.

Proof. For (1), Corollary 7.17 allows us to write

- $\alpha = [u]$ for a (p,q)-form u such that $\partial u = 0$, $\overline{\partial} u = 0$,
- $\beta = [v]$ for a (p',q')-form v such that $\partial v = 0$, $\overline{\partial} v = 0$.

We then have that

$$\begin{aligned} \partial(u \wedge v) &= \partial u \wedge v \pm u \wedge \partial v = 0, \\ \overline{\partial}(u \wedge v) &= 0, \end{aligned}$$

and $\alpha \cup \beta = [u \wedge v] \in H^{p+p',q+q'}(M)$ by Corollary 7.17.

In (2), for $\alpha \in H^{p,q}(M)$, use Corollary 7.17 to write $\alpha = [u]$ for a (p,q)-form u such that $\partial u = 0$, $\overline{\partial} u = 0$. Then

$$f^*\alpha = [f^*u]$$

and

$$\begin{aligned} \partial(f^*u) &= f^*(\partial u) = 0, \\ \overline{\partial}(f^*u) &= 0. \end{aligned}$$

Using Corollary 7.17, this shows that $f^*\alpha \in H^{p,q}(M')$. □

We prove one more consequence which will be useful later, when we discuss applications to the Kodaira Vanishing Theorem.

Corollary 7.19. *Suppose M is a compact manifold of Kähler type. Consider he map $\mathbb{C} \to \mathcal{O}_M$. The induced morphisms in cohomology*

$$H^q(M, \mathbb{C}) \to H^q(M, \mathcal{O}_M)$$

are surjective.

Proof. Compute the maps in cohomology using soft resolutions (cf. Theorem 4.26). We have

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & (\mathcal{A}^\bullet_M \otimes_{\mathbb{R}} \mathbb{C}, d), \\ \downarrow & & \downarrow \text{proj} \\ \mathcal{O}_M & \longrightarrow & (\mathcal{A}^{0,\bullet}_M, \overline{\partial}). \end{array}$$

For every $\alpha \in H^q(M, \mathcal{O}_M)$, there is a harmonic $(0,q)$-form u such that $\alpha = [u]$. In particular, $du = 0$ implies that $\alpha \in \operatorname{im}(H^q(M,\mathbb{C}) \to H^q(M, \mathcal{O}_M))$. □

7.5. **Hard Lefschetz Theorem.**

7.5.1. *Lefschetz decomposition for differential forms.* Let M be a compact Kähler manifold and ω be the fundamental form of the Kähler metric. We have two operators (that play a role in the Kähler Identities 7.10):

$$\begin{aligned} L\colon \mathcal{A}^{\bullet}_{M,\mathbb{C}} &\to \mathcal{A}^{\bullet+2}_{M,\mathbb{C}} && \text{real and takes } (p,q)\text{-forms to } (p+1,q+1)\text{-forms,} \\ \eta &\mapsto \omega\wedge\eta, \\ \Lambda\colon \mathcal{A}^{\bullet}_{M,\mathbb{C}} &\to \mathcal{A}^{\bullet-2}_{M,\mathbb{C}} && \Lambda = *^{-1}L* \text{ is the adjoint of } L. \end{aligned}$$

Proposition 7.20. *We have* $[L,\Lambda]=H$ *where* $\mathcal{A}^k_{M,\mathbb{C}} \xrightarrow{H} \mathcal{A}^k_{M,\mathbb{C}}$ *is* $(k-n)\mathrm{Id}$ *and* $n=\dim M$.

Remarks 7.21. Both L and Λ are linear operators and we only need to check this pointwise, so it is enough to check it when we have a complex vector space V with a Hermitian metric h and $\omega=-\mathrm{Im}(h)$.

Moreover, these are real operators, so it will be enough to consider their effect on $\bigwedge^m V^*$. The ideas is to

(1) check this for $\dim_{\mathbb{C}} V = 1$,
(2) show that if $V = V' \overset{\perp}{\oplus} V''$ and if we know the assertion for V', V'', then we get it for V.

We remark that L,Λ,H give the generators of the Lie algebra of SL_2. We might discuss this later in the class.

The next three classes were typed by David Schwein.

Proof of Proposition 7.20. Both L and Λ are linear operators and to check this we need only check it pointwise, that is, on a vector space V with Hermitian metric h and $\omega=-\mathrm{Im}(h)$. Moreover, since these operators are real, it is enough to consider their effect on $\bigwedge^m V^*$. We induct on $n=\dim V (=\dim X)$.

When $n=1$, take $x\in V$ such that $h(x,x)=1$, so that x and $y=Jx$ is a basis for the vector space $V_{\mathbb{R}}$. Let x', y' be the dual basis of $\mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$ over $\mathbb{R}$. The volume element is $x'\wedge y'=\omega$. The $*$ operator thus acts as

$$1\mapsto\omega \quad \omega\mapsto 1 \quad x'\mapsto y' \quad y'\mapsto -x'$$

while $\Lambda:\omega\mapsto 1$. Hence $[L,\Lambda]:=L\Lambda-\Lambda L$ acts as follows:

$$\begin{aligned} &\text{on } \mathcal{A}^0 \text{ by } 1\mapsto 1,\\ &\text{on } \mathcal{A}^1 \text{ by } x',y'\mapsto 0,\\ &\text{on } \mathcal{A}^2 \text{ by } \omega\mapsto\omega. \end{aligned}$$

For the induction step, consider an orthogonal decomposition $V=V_1\oplus V_2$ with respect to h:

$$\bigwedge^k V^* = \bigoplus_{k_1+k_2=k}\Big(\bigwedge^{k_1} V_1^*\otimes\bigwedge^{k_2} V_2^*\Big).$$

Hence the volume element in $\bigwedge^{2n} V^*$ is the tensor product of volume elements in $\bigwedge^{n_1} V_1^*$ and $\bigwedge^{n_2} V_2^*$, where $n_i = \dim V_i$. Thus the $*$ operator acts as

$$\bigwedge^{k_1} V_1^* \otimes \bigwedge^{k_2} V_2^* \to \bigwedge^{2n_1-k_1} V_1^* \otimes \bigwedge^{2n_2-k_2} V_2^*$$
$$\eta_1 \otimes \eta_2 \mapsto (-1)^{k_2(2n_1-k_1)} *_1(\eta_1) \otimes *_2(\eta_2).$$

Writing $\omega = \omega_1 + \omega_2$, we see that

$$L(\eta_1 \otimes \eta_2) = L_1(\eta_1) \otimes \eta_2 + \eta_1 \otimes L_2(\eta_2).$$

The formula for $*(\eta_1 \otimes \eta_2)$ now shows that

$$\Lambda(\eta_1 \otimes \eta_2) = \Lambda_1(\eta_1) \otimes \eta_2 + \eta_1 \otimes \Lambda_2(\eta_2).$$

Hence

$$\begin{aligned}
[L, \Lambda](\eta_1 \otimes \eta_2) &= L(\Lambda_1\eta_1 \otimes \eta_2 + \eta_1 \otimes \Lambda_2\eta_2) - \Lambda(L_1\eta_1 \otimes \eta_2 + \eta_1 \otimes L_2\eta_2) \\
&= L_1\Lambda_1\eta_1 \otimes \eta_2 + \Lambda_1\eta_1 \otimes L_2\eta_2 + L_1\eta_2 \otimes \Lambda_2\eta_2 + \eta_1 \otimes L_2\Lambda_2\eta_2 \\
&\quad - \Lambda_1 L_1\eta_1 \otimes \eta_2 - L_1\eta_1 \otimes \Lambda_2\eta_2 - \Lambda_1\eta_1 \otimes L_2\eta_2 - \eta_1\Lambda_2 L_2\eta_2 \\
&= [L_1, \Lambda_1]\eta_1 \otimes \eta_2 + \eta_1 \otimes [L_2, \Lambda_2]\eta_2.
\end{aligned}$$

By induction, this quantity is

$$(k_1 - n_1)\eta_1 \otimes \eta_2 + (k_2 - n_2)\eta_1 \otimes \eta_2 = (k - n)\eta_1 \otimes \eta_2.$$ □

Corollary 7.22. *We have that* $[L^r, \Lambda] = r(k - n + r - 1)L^{r-1}$ *on* $\mathcal{A}^k_{M,\mathbb{C}}$.

Proof. We induct on $r \geq 1$. Proposition 7.20 is the base case $r = 1$. For the induction step, write

$$[L^r, \Lambda] = L^r\Lambda - \Lambda L^r = L(L^{r-1} - \Lambda L^{r-1}) + (L\Lambda - \Lambda L)L^{r-1} = L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1}.$$

By the inductive hypothesis, this expression equals

$$(r - 1)(k - n + r - 2)L^{r-1} + (k + 2(r - 1) - n)L^{r-1}$$

Simplifying this gives the result. □

Proposition 7.23. *Let* $n = \dim M$ *and* $0 \leq k \leq n$. *The map* $L^{n-k} : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^{2n-k}_{M,\mathbb{C}}$ *is an isomorphism. It induces an isomorphism* $\mathcal{A}^{p,q}_M \to \mathcal{A}^{p+n-k,q+n-k}_{M,\mathbb{C}}$ *(where* $p + q = k$*).*

Proof. Evidently L^{n-k} is a morphism of vector bundles of the same rank. (To see this for the bottom map, check that $\binom{n}{p}\binom{n}{q} = \binom{n}{p+n-k}\binom{n}{q+n-k}$.) Hence it suffices to prove injectivity on fibers; in particular, the statement on k-forms implies the statement on (p, q)-forms.

To prove it, induct on $k \geq 0$. The case $k = 0$ follows from our earlier calculation that $\omega^n = n!\mathrm{vol}$. For the induction step, fix $k \geq 1$. We will show by induction on $0 \leq r \leq n - k$ that if $L^r\alpha = 0$ for a k-form α then $\alpha = 0$. The case $r = 0$ is trivial. For the induction step, suppose $L^r\alpha = 0$. Then by Corollary 7.22 above,

$$L^r\Lambda\alpha = [L^r, \Lambda]\alpha = r(k - n + r - 1)L^{r-1}\alpha.$$

So $L^{r-1}(L\Lambda\alpha - r(k - n + r - 1)\alpha) = 0$ and the induction hypothesis implies that

$$L\Lambda\alpha - r(k - n + r - 1)\alpha = 0.$$

Returning to the induction on k, our assumptions on r imply that $k-n+r-1<0$, so that $\alpha = L\beta$ for a $(k-2)$-form β. Hence $L^{n-k}\alpha = 0$ and $L^{n-k+1}\beta = 0$. By the induction hypothesis on k, the operator $L^{n-(k-2)}$ is injective on $(k-2)$-forms. So $\beta = 0$ and thus $\alpha = 0$. □

We will now use this operator to *decompose* the space of differential forms, as in the title.

Definition 7.24. Let $0 \leq k \leq n = \dim M$. A k-form α is *primitive* if $L^{n-k+1}\alpha = 0$. Write $\mathrm{Prim}(\mathcal{A}^k_{M,\mathbb{C}})$ for the space of primitive k-forms.

Since L is real, the form α is primitive if and only if $\overline{\alpha}$ is primitive. Since L has bidegree $(1,1)$, the form α is primitive if and only if all of its (p,q)-components are primitive. Evidently L^i is injective on $\mathcal{A}^{k-2i}$ since by the proposition, $L^{n-(k-2i)}$ is injective and $n-k+2i \geq i$.

Proposition 7.25 (Lefschetz Decomposition for Forms)**.** *There is a decomposition*

$$\mathcal{A}^k_{M,\mathbb{C}} = \bigoplus_{i\geq 0} L^i\,\mathrm{Prim}(\mathcal{A}^{k-2i}_{M,\mathbb{C}}), \qquad (0 \leq k \leq n).$$

Proof. We induct on k. When $k = 0$, the decomposition is clear: all 0-forms are primitive. For the induction step, given a k-form α, the form $\alpha - L\beta$ is primitive for some $(k-2)$-form β if and only if $L^{n-k+1}(\alpha - L\beta) = 0$, or equivalently, $L^{n-k+1}\alpha = L^{n-k+2}\beta$. Since β is a $(k-2)$-form, Proposition 7.23 implies that there is a *unique* $(k-2)$-form β with this property, that is, such that $\alpha = \alpha_0 + L\beta$ with α_0 primitive. To conclude, use the induction hypothesis for β. □

There is a similar decomposition when $k \geq n$, an easy corollary of the decomposition of the proposition:

$$\mathcal{A}^k_{M,\mathbb{C}} = \bigoplus_{i\geq 0} L^{i+k-n}(\mathrm{Prim}\,\mathcal{A}^{2n-k-2i}_{M,\mathbb{C}}).$$

Note that L^{i+k-n} is injective on $\mathcal{A}^{2n-k-2i}_{M,\mathbb{C}}$ since $i+k-n \leq n-(2n-k-2i)$.

7.5.2. *Lefschetz Decomposition for Cohomology Classes.*

Lemma 7.26. *A k-form α is primitive if and only if $\Lambda\alpha = 0$.*

Proof. Note that L is injective on $\mathcal{A}^{<n}_{M,\mathbb{C}}$. Hence $\Lambda = *^{-1}L*$ is injective on $\mathcal{A}^{<n}_{M,\mathbb{C}}$, so we may assume $k \leq n$ without loss of generality. Our earlier computation of $[L^r, \Lambda]$ shows that

$$L^{n-k+1}\Lambda\alpha = \Lambda L^{n-k+1}\alpha.$$

Since $L^{n-k+1}\alpha$ is a $(2n-k+2)$-form and Λ is injective on $\mathcal{A}^{>n}$, the righthand side vanishes if and only if α is primitive. Moreover, since $\Lambda\alpha$ is a $(k-2)$-form and L^{n-k+2} is injective on $(k-2)$-forms, the lefthand side is zero if and only if $\Lambda\alpha = 0$. □

Consequently, if $k \leq n$ and $\alpha = L^i\beta$ with β a primitive $(k-2i)$-form then

$$\Lambda\alpha = \Lambda L^i\beta = L^i\Lambda\beta - [L^i, \Lambda]\beta = i(k-2i-n+i-1)L^{i-1}\beta.$$

Hence

$$\Lambda(L^i\beta) = i(n-k+i+1)L^{i-1}\beta.$$

This equation shows how Λ acts on the Lefschetz decomposition. For example, if α is a k-form with $k \le n$ then $\Lambda\alpha$ is primitive if and only if $\alpha = \alpha_0 + L\alpha_1$ with α_0 and α_1 primitive, and in this case, $\Lambda\alpha = (n-k+2)\alpha_1$.

Lemma 7.27. *We have that* $[\Delta, L] = 0$.

Proof. Recall that $\Delta = 2\Delta''$, $\Delta'' = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, and $[\bar\partial^*, L] = i\partial$, and that if $\mathrm{d}\omega = 0$ then $\partial\omega = 0$ and $\bar\partial\omega = 0$. In this case

$$[\bar\partial, L]\eta = \bar\partial(\omega \wedge \eta) - \omega \wedge \bar\partial\eta = \bar\partial\omega \wedge \eta = 0.$$

So

$$[\Delta'', L] = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)L - L(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \bar\partial[\bar\partial^*, L] + [\bar\partial^*, L]\bar\partial,$$

using that $\bar\partial L = L\bar\partial$. Hence $[\Delta'', L] = \bar\partial i\partial + i\partial\bar\partial = 0$ and $[\Delta, L] = 0$. ☐

Consequently, if α is harmonic then $L\alpha$ is harmonic, and conversely, if $k \le n$ and $i \le n-k$ then whenever $L^i\alpha$ is harmonic, so is α. Hence for $k \le n$, the isomorphism $L^{n-k} : \mathcal{A}^k_{M,\mathbb{C}} \xrightarrow{\sim} \mathcal{A}^{2n-k}_{M,\mathbb{C}}$ induces an isomorphism

$$\mathcal{H}^k_{M,\mathbb{C}} \xrightarrow{\sim} \mathcal{H}^{2n-k}_{M,\mathbb{C}}$$

between spaces of harmonic forms.

Theorem 7.28 (Hard Lefschetz)**.** *For* $k \le n$, *the map*

$$[\omega]^k \cup (-) : \mathrm{H}^k_{\mathrm{dR}}(M, \mathbb{C}) \to \mathrm{H}^{2n-k}_{\mathrm{dR}}(M, \mathbb{C})$$

is an isomorphism.

The fact that the spaces are isomorphic follows from Poincaré duality as well, but Hard Lefschetz is a much more useful statement. For example, it implies for $k \le n-2$ that $b_k(M) \le b_{k+2}(M)$. This statement has consequences in combinatorics: one can show a sequence of numbers to be unimodal by realizing it as the Betti numbers of a Kähler manifold.

Definition 7.29. For $k \le n$, a cohomology class $\alpha \in \mathrm{H}^k_{\mathrm{dR}}(M, \mathbb{C})$ is *primitive* if $[\omega]^{n-k+1} \cup \alpha = 0$.

If we choose a harmonic k-form η such that $\alpha = [\eta]$ then

$$[\omega]^{n-k+1} \cup \alpha = [L^{n-k+1}\eta].$$

Hence α is primitive if and only if η is primitive.

Lemma 7.30. *For* $k \le n$ *and* α *a* k*-form with Lefschetz decomposition* $\alpha = \sum L^i\alpha_i$ *(where* α_i *is primitive), then* α *is harmonic if and only if all* α_i *are harmonic.*

Proof. It is clear that if each α_i is harmonic then so is α; we prove the converse by induction on k, the base case $k = 0$ being trivial. We saw that $[\Delta, L] = 0$, so that $[\Delta, \Lambda] = 0$. Since α is harmonic, $\Delta\alpha = 0$ and therefore $\Lambda\alpha$ is harmonic. We saw earlier that

$$\Lambda\alpha = \sum_{i \ge 1} i(n-k+i+1)L^{i-1}\alpha_i.$$

Induction now shows that α_i is harmonic for all $i \ge 1$; so $L^i\alpha_i$ is harmonic for all $i \ge 1$, and α_0 is harmonic. ☐

Theorem 7.31 (Lefschetz for Cohomology)**.** *For $k \leq n$ there is a decomposition*

$$\mathrm{H}^k_{\mathrm{dR}}(M,\mathbb{C}) = \bigoplus_{i\geq 0} L^i \mathrm{H}^{k-2i}_{\mathrm{prim}}(M,\mathbb{C})$$

where $L := [\omega] \cup (-)$ and $\mathrm{H}^j_{\mathrm{prim}}(M,\mathbb{C}) \subseteq \mathrm{H}^j_{\mathrm{dR}}(M,\mathbb{C})$ is the set of primitive cohomology classes.

Since ω is a real $(1,1)$-form, the Lefschetz decomposition is compatible with the real structure and the decomposition into (p,q)-subspaces.

7.6. **The Hodge-Riemann bilinear relations.** The primitive part of $\mathrm{H}^k(M)$ is the piece not accounted for by lower degree cohomology:

$$\dim H^k_{\mathrm{prim}}(M) = b_k(M) - b_{k-2}(M).$$

In this section, we investigate the sign of some natural bilinear pairings on cohomology. More specifically, let (M,h) be a compact complex manifold of dimension n with Kähler structure and fundamental form ω. Given two k-form α and β, we can consider

$$\int_M \omega^{n-k} \wedge \alpha \wedge \beta;$$

this defines a pairing on the space of k-forms. Our goal is to understand the positivity properties of this pairing.

Definition 7.32. For $k \leq n$ and $\alpha, \beta \in \Gamma(M, \mathcal{A}^k_{M,\mathbb{C}})$ define

$$\mathrm{H}_k(\alpha,\beta) = i^k \int_M \omega^{n-k} \wedge \alpha \wedge \overline{\beta}.$$

We make several observations about this pairing.

(1) The pairing H_k is clearly bilinear in α and Hermitian because

$$\mathrm{H}_k(\beta,\alpha) = i^k \int \omega^{n-k} \wedge \beta \wedge \overline{\alpha} = (-i)^k \int \omega^{n-k} \wedge \overline{\alpha} \wedge \beta = \overline{\mathrm{H}_k(\alpha,\beta)}.$$

(2) The pairing H_k induces a bilinear form on $\mathrm{H}^k_{\mathrm{dR}}(M,\mathbb{C})$, denoted also by H_k: if $\mathrm{d}\alpha = 0$, $\mathrm{d}\beta = 0$, and one of α or β is exact then

$$\int_M \omega^{n-k} \wedge \alpha \wedge \overline{\beta} = 0,$$

by Stokes's Theorem and the fact that $\mathrm{d}\omega = 0$.

(3) The Lefschetz decomposition of $\mathrm{H}^k_{\mathrm{dR}}(M,\mathbb{C})$ is orthogonal with respect to this decomposition: if $\alpha = L^i\gamma$ and $\beta = L^j\delta$ with γ and δ primitive harmonic representatives then

$$\mathrm{H}_k(\alpha,\beta) = 0$$

because

$$\omega^{n-k} \wedge L^i\gamma \wedge L^j\overline{\delta} = L^{n-k+i+j}\gamma \wedge \overline{\delta},$$

so that $L^{n-(k-2i)+1}\gamma = 0$ by primitivity of γ.

(4) The map $L^j : \mathrm{H}^{k-2j}_{\mathrm{prim}}(M,\mathbb{C}) \to \mathrm{H}^k_{\mathrm{dR}}(M,\mathbb{C})$ is an isometry up to a sign:

$$\mathrm{H}_k(L^j\alpha, L^j\beta) = i^k \int_M \omega^{n-k+2j} \wedge \alpha \wedge \overline{\beta}, \quad \mathrm{H}_{k-2j}(\alpha,\beta) = i^{k-2j} \int_M \omega^{n-k+2j} \wedge \alpha \wedge \overline{\beta}.$$

Therefore, to understand the sign of all H_k on $\mathrm{H}^k(M,\mathbb{C})$ it is enough to understand them on the primitive cohomology.

(5) The Hodge decomposition $H^k_{\mathrm{dR}}(M,\mathbb{C}) = \bigoplus_{p+q=k} \mathrm{H}^{p,q}(M)$ is orthgonal with respect to H_k: for α a (p,q)-form and β a (p',q')-form, to have a nonzero integral we need $\omega^{n-k} \wedge \alpha \wedge \overline{\beta}$ to have type (n,n), meaning that $p = p'$ and $q = q'$.

Theorem 7.33 (Hodge-Riemann Bilinear Relations)**.** *For $k \leq n$ and (p,q) with $p+q=k$, the Hermitian form*

$$(-1)^{k(k-1)/2} i^{p-q-k} \mathrm{H}_k$$

is positive-definite on $\mathrm{H}^{p,q}_{\mathrm{prim}}(M)$.

The key ingredient of the proof is a formula for $*\alpha$ when α is primitive.

Proposition 7.34. *Let α be a primitive (p,q)-form with $k = p+q$. Then*

$$*L^j\alpha = (-1)^{k(k+1)/2} i^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-j}\alpha, \qquad j \leq n-k.$$

In particular, $*\alpha = (-1)^{k(k+1)/2} i^{p-q} L^{n-k}\alpha$. We will return to the proof of the proposition after using it to prove the theorem.

Proof of Hodge–Riemann Bilinear Relations 7.33. Let $0 \neq \alpha$ be a primitive (p,q)-form. Then

$$\frac{(-1)^{k(k-1)/2}}{(n-k)!} i^{p-q} \int_M \omega^{n-k} \wedge \alpha \wedge \overline{\alpha} = (-1)^k \int_M (*\alpha) \wedge \overline{\alpha} = \int_M \overline{\alpha} \wedge \overline{(*\overline{\alpha})}$$

$$= \int_M \langle \overline{\alpha}, \overline{\alpha} \rangle \,\mathrm{dvol} = \langle\langle \overline{\alpha}, \overline{\alpha} \rangle\rangle > 0. \qquad \square$$

Proof of Proposition 7.34. This is only a sketch since the proof is similar to earlier ones. Like before, we reduce it to a problem of linear algebra. Consider a complex vector space V of dimension n with a Hermitian metric. From here, the proof proceeds by induction on n. For the base case $n = 1$, one checks by hand the four cases $(j,k) = (0,0), (0,1), (1,0), (1,1)$. For the inductive step, suppose $V = V_1 \oplus V_2$ with $\dim V_2 = 1$ and x, y as before. (That is, x is a basis for V with $h(x,x) = 1$ and $y := Jx$, so that $\{x,y\}$ is a basis for $V_{\mathbb{C}}$.) Then

$$\Big(\bigwedge^k V^*\Big)_{\mathbb{C}} = \Big(\bigwedge^k V_1^*\Big)_{\mathbb{C}} \oplus \Big(\bigwedge^{k-1} V_1^* \otimes_{\mathbb{R}} \bigwedge V_2^*\Big)_{\mathbb{C}} \oplus \Big(\bigwedge^{k-2} V_1^* \otimes_{\mathbb{R}} \bigwedge^2 V_2^*\Big)_{\mathbb{C}}.$$

Consider an element

$$\alpha = \alpha_k + \alpha'_{k-1} \otimes x' + \alpha''_{k-1} \otimes y' + \alpha_{k-2} \otimes \omega$$

where $\{x', y'\}$ is the basis of $V^*_{\mathbb{C}}$ dual to the basis $\{x,y\}$. Applying Λ yields

$$\Lambda\alpha = \Lambda_1\alpha_k + \Lambda_1\alpha'_{k-1} \otimes x' + \Lambda_1\alpha''_{k-1} \otimes y' + \Lambda_1\alpha_{k-2} \otimes \omega + \alpha_{k-2}.$$

The form α is primitive if and only if $\Lambda\alpha = 0$, and by setting the terms of each degree equal to zero, we see that this condition is equivalent to the simultaneous vanishing of $\Lambda_1\alpha'_{k-1}$ and $\Lambda_1\alpha''_{k-1}$, that is, the primitivity of α'_{k-1} and α''_{k-1}. Now compute $*L^j\alpha$ using induction. □

Remark 7.35. For other geometric objects and cohomology theory, one hopes to have the same elements of the "Kähler package" as in the classical case we have studied so far: Poincaré duality, Hard Lefschetz, and the Hodge-Riemann bilinear relations. For example, intersection cohomology (for projective singular algebraic varieties) also satisfies these conditions.

Typing by AH continues.

Theorem 7.36 (Hodge Index Theorem)**.** *If* $\dim M = n = 2m$ *and the form on* $H^n(M, \mathbb{R})$

$$(\alpha, \beta) \overset{Q}{\mapsto} \int_M \alpha \wedge \beta$$

has signature $\sum_{p,q=0}^{n} (-1)^p h^{p,q}$.

Note that the signature is a topological invariant, so this alternating sum of the Hodge numbers is a topological invariant.

Note also that we are considering $H^n(M, \mathbb{R})$, the cohomology with real coefficients. To use the Lefschetz and Hodge decompositions, we will have to tensor with $\mathbb{C}$ and note that spaces such as $H^{m,m}$ and $H^{p,q} + H^{q,p}$ descend to $\mathbb{R}$.

Proof. We consider the Hodge and Lefschetz decomposition:

$$H^n(M, \mathbb{R}) = \bigoplus_{j \geq 0} L^j H^{m-j,m-j}_{\text{prim}}(M)_{\mathbb{R}} \oplus \bigoplus_{\substack{p>q \\ p+q=2m}} \bigoplus_{j \geq 0} L^j \left(H^{p-j,q-j}_{\text{prim}}(M) \oplus H^{q,p}_{\text{prim}}(M) \right)_{\mathbb{R}}.$$

If $\alpha \in H^{m-j,m-j}_{\text{prim}}(M, \mathbb{R})$, then by the Hodge–Riemann Relations 7.33:

$$(-1)^{m-j} \int_M \omega^{n-j} \wedge \alpha \wedge \alpha > 0.$$

If $\alpha \in H^{p-j,q-j}_{\text{prim}}(M, \mathbb{C})$, then by the Hodge–Riemann Relations:

$$\underbrace{(-1)^{m-j} i^{p-q}}_{(-1)^{q-j}} \int_M \omega^{n-j} \alpha \wedge \overline{\alpha} > 0, \quad \text{where } p + q = 2m.$$

If $\beta \in H^{q-j,p-j}_{\text{prim}}(M, \mathbb{C})$, then by the Hodge–Riemann Relations 7.33:

$$\underbrace{(-1)^{p-j}}_{(-1)^{q-j}} \int_M \omega^{n-j} \wedge \beta \wedge \overline{\beta} \quad \text{, where } p + q = 2m.$$

Note that $\alpha+\beta \in (H^{p,q}_{\text{prim}} + H^{q,p}_{\text{prim}})_{\mathbb{R}}$ if and only if $\alpha+\beta = \overline{\beta}+\overline{\alpha}$, i.e. $\beta = \overline{\alpha}$. Hence

$$(-1)^{q-j}\int_M \underbrace{(\alpha+\beta)\wedge(\alpha+\beta)}_{\alpha\wedge\overline{\alpha}+\beta\wedge\overline{\beta}+\alpha\wedge\alpha+\beta\wedge\beta} \wedge\omega^{n-j} = (-1)^{q-j}\int \alpha\wedge\overline{\alpha}\wedge\omega^{n-j} + (-1)^{q-j}\beta\wedge\overline{\beta}\wedge\omega^{n-j}$$
$$> 0$$

if $\alpha \neq 0$. The first equality above follows since $\omega^{n-j}\wedge\alpha\wedge\alpha$ and $\omega^{n-j}\wedge\beta\wedge\beta$ are not (n,n)-forms, so their integrals are 0.

Therefore, we see that the signature of Q is

$$\begin{aligned}
\text{signature}(Q) &= \sum_{j\geq 0}(-1)^{m-j}h^{m-j,m-j}_{\text{prim}} + 2\sum_{\substack{p+q=n\\ p>q}}\sum_{j\geq 0}(-1)^{q-j}h^{p-j,q-j}_{\text{prim}}\\
&= \sum_{p+q=n}\sum_{j\geq 0}(-1)^{q-j}h^{p-j,q-j}_{\text{prim}}\\
&= \sum_{p+q=n}\sum_{j\geq 0}(-1)^{q-j}(h^{p-j,q-j} - h^{p-j-1,q-j-1})\\
&= \sum_{a+b=n-(\text{even},\geq 0)}(-1)^b h^{a,b} + \sum_{a+b=n-2-(\text{even},\geq 0)}(-1)^b h^{a,b}\\
&= \sum_{a+b=\text{even}}(-1)^b h^{a,b} \qquad\qquad h^{a,b} = h^{n-a,n-b}
\end{aligned}$$

Finally:

$$\sum_{a+b=\text{odd}}(-1)^b h^{a,b} = 0$$

since $h^{a,b} = h^{b,a}$ and $(-1)^a = -(-1)^b$ if $a+b$ is odd. □

Example 7.37. Suppose $n = 2$. The signature of

$$\begin{aligned}
H^2(M,\mathbb{R})\times H^2(M,\mathbb{R}) &\to \mathbb{R}\\
(\alpha,\beta) &\mapsto \int_M \alpha\wedge\beta
\end{aligned}$$

is

$$\sum_{a+b \text{ even}}(-1)^b h^{a,b} = \underbrace{h^{0,0}}_{=1} + h^{0,2} + -h^{1,1} + h^{2,0} + \underbrace{h^{2,2}}_{=1} = 2 + 2h^{2,0} - h^{1,1}.$$

One can check that on $\big(H^{2,0}(M)\oplus H^{0,2}(M)\big)_{\mathbb{R}}$, the form $(\alpha,\beta)\mapsto\int\alpha\wedge\beta$ is positive-definitive by the computation in the proof of the Hodge Index Theorem 7.36.

Conclusion. The signature of the intersection form on $H^{1,1}(M)_{\mathbb{R}}$ is $(1, h^{1,1}-1)$.

Hartshorne: part from ample line bundle –¿ everything else contributes to the negative part.

8. Chern classes of line bundles

Let M be a complex manifold (more generally, an analytic space). We have the sheaf $\mathcal{O}_M$ of holomorphic functions on M. Let

$$\mathcal{O}_M^* = \text{sheaf of invertible holomorphic functions on } M, \text{ a group under multiplication.}$$

We have a map

$$\begin{aligned} \mathcal{O}_M &\to \mathcal{O}_M^* \\ f &\mapsto \exp(f) \end{aligned}$$

of sheaves of abelian groups. Locally on $\mathbb{C}^*$, we have a holomorphic inverse of exp, so the map $\mathcal{O}_M \to \mathcal{O}_M^\times$ is surjective.

What is its kernel? If $U \subseteq M$ is open and connected, then $f \in \mathcal{O}(U)$ is such that $\exp(f) = 1$ if and only if $f = 2\pi ik$ for some $k \in \mathbb{Z}$.

We have a short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{O}_M^\times & \longrightarrow & 0 \\ & & n & \longrightarrow & 2\pi in & & & & \\ & & & & f & \longrightarrow & \exp(f). & & \end{array}$$

The long exact sequence in cohomology gives:

$$H^1(M,\mathbb{Z}) \longrightarrow H^1(M,\mathcal{O}_M) \longrightarrow H^1(M,\mathcal{O}_M^\times) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathcal{O}_M).$$

This is an interesting sequence because $H^1(M,\mathbb{Z})$ is topological, $H^1(M,\mathcal{O}_M)$ is coherent cohomology, and $H^1(M,\mathcal{O}_M^\times)$ is the Picard group of M.

Indeed, by definition,

$$\operatorname{Pic}(M) = \left(\frac{\{\text{holomorphic line bundles}\}}{\sim}, \otimes\right)$$

so we have an isomorphism

$$\operatorname{Pic}(M) \cong \check{H}^1(M,\mathcal{O}_M^*).$$

On every topological space, for every sheaf of abelian groups $\mathcal{F}$, there is a canonical isomorphism

$$H^1(M,\mathcal{F}) \cong \check{H}^1(M,\mathcal{F}).$$

(This is an exercise in [Har77].)

We write

$$\begin{aligned} c_1\colon \operatorname{Pic}(M) &\to H^2(M,\mathbb{Z}) \\ L &\mapsto c_1(L) \end{aligned}$$

and call $c_1(L)$ the *Chern class of L*.

Exercise. Check functoriality of Chern classes: if $f\colon M' \to M$ is holomorphic and $L \in \mathrm{Pic}(M)$, then

$$c_1(f^*L) = f^*(c_1(L)).$$

Next , we will check that given L, if we choose a metric on L, we get a closed, real 1-1 form ω on X such that

$$c_1(L) = [\omega].$$

Moreover, if X is algebraic, L is ample, then ω is positive.

Let M be a complex manifold and L be a line bundle on M. Consider a cover $M = \bigcup_j U_j$ such that we have an isomorphism

$$\varphi_j\colon L|_{U_j} \overset{\cong}{\to} \mathcal{O}_{U_j}$$
$$\Gamma(U_j, L) \ni \sigma_j \leftarrow\!\shortmid 1$$

and $g_{jk} \in \mathcal{O}(U_j \cap U_K)^*$ is such that $\varphi_k \circ \varphi_j^{-1}$ is multiplication by g_{jk}.

To give a global section of L is to give $f_j \in \mathcal{O}(U_j)$ such that $f_k = f_j g_{jk}$. Note also that

$$\sigma_j = \sigma_k \cdot g_{jk} \quad \text{on } U_j \cap U_k \text{ for all } j, k.$$

Recall that a Hermitian metric on L is a smoothly varying family of functions on the fibers of L. Given cover as above, consider the smooth function $h_j = h(\sigma_j, \sigma_j)\colon U_j \to \mathbb{R}_{>0}$. This satisfies

$$h_j = |g_{jk}|^2 h_k.$$

Conversely, a family of such functions with this compatibility property induces a Hermitian metric on L.

Suppose h is a metric on L and we are given a cover as above. On each U_j, consider

$$\omega_j = \frac{1}{2\pi i}\partial\overline{\partial}\log h_j.$$

This is a smooth $(1,1)$-form on U_j. Note that on $U_j \cap U_k$:

$$\log h_j = \log|g_{jk}|^2 + \log h_k.$$

Since g_{jk} are holomorphic:

$$\partial\overline{\partial}\log|g_{jk}|^2 = \partial\overline{\partial}(\log g_{jk} + \log\overline{g_{jk}}) = 0.$$

This shows that ω_j glue together to a global $(1,1)$ form ω on M. We sometimes write ω_h for this form to indicate the Hermitian metric h it corresponds to.

Note that:

(1) since $\omega_j = \partial\overline{\partial}(\cdots)$, $d\omega_j = 0$, so ω is closed,
(2) since $i\partial\overline{\partial}$ is a real operator and h_j are real functions, ω is a real form,
(3) hence: ω is Kähler (the fundamental form of a Kähler metric) if and only if $-i\omega(v,v) > 0$ for all $v \neq 0$.

Proposition 8.1. *Given a Hermitian metric h on L with corresponding form ω_h, the image of $c_1(L)$ in $H^2(M,\mathbb{R})$ is $[\omega_h]$.*

Proof. Since M is a smooth manifold, there is an open cover $M = \bigcup_j U_j$ such that all intersections $U_{j_0} \cap \cdots \cap U_{j_k}$ are contractible. Moreover, we can choose this as fine as needed, so we may assume

$$L|_{U_j} \cong \mathcal{O}_{U_j} \quad \text{for all } j.$$

As in the definition of $c_1(L)$, we consider the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O}_M & \longrightarrow & \mathcal{O}_M^\times & \longrightarrow & 0 \\ & & n & \longrightarrow & 2\pi i n & & & & \\ & & & & f & \longrightarrow & \exp(f). & & \end{array}$$

Then $[L] \in \mathrm{Pic}(M)$ corresponds to $(g_{jk}) \in H^1(M, \mathcal{O}_M^\times)$. Since all U_j are contractible, $H^p(U_j, \mathbb{Z}) = 0$ for all $p > 0$. Hence

$$g_{jk} = \exp(\widetilde{g_{jk}})$$

for some holomorphic $\widetilde{g_{jk}}$ on $U_j \cap U_k$. Then

$$\mathbb{Z} \ni n_{jkl} = \frac{1}{2\pi i}(\widetilde{g_{k\ell}} - \widetilde{g_{j\ell}} + \widetilde{g_{jk}}) \quad \text{on } U_{jk\ell},$$

so

$$n_{jk\ell} \in \check{H}^2(\mathcal{U}, \mathbb{Z}).$$

We need to understand the isomorphism

$$\check{H}^2(\mathcal{U}, \underline{\mathbb{R}}) \cong H^2_{\mathrm{dR}}(M).$$

Consider the double complex $C^{\bullet,\bullet}$ where

$$C^{p,q} = \bigoplus_{i_0 < \cdots < i_q} \Gamma(U_{i_0} \cap \cdots \cap U_{i_q}; \mathcal{A}^q_M).$$

If we fix $(i_0, \ldots, i_q)$, the corresponding de Rham complex of $U_{i_0} \cap \cdots U_{i_q}$ is acyclic, with cohomology in degree 0 equal to

$$\Gamma(U_{i_0} \cap \cdots \cap U_{i_q}, \underline{\mathbb{R}}).$$

We get an isomorphism

$$\check{H}^p(\mathcal{U}, \underline{\mathbb{R}}) \cong H^p(\mathrm{Tot}(C^{\bullet,\bullet})).$$

Similarly, if we fix p, we get the Cech complex with respect to $\mathcal{U}$ for $\mathcal{A}^p$. Since $\mathcal{A}^p$ is soft, this is again acyclic, with cohomology in degree 0 equal to

$$\Gamma(M, \mathcal{A}^p).$$

We get an isomorphism:

$$H^p_{\mathrm{dR}}(M) \cong H^p(\mathrm{Tot}(C^{\bullet,\bullet})).$$

Consider

$$\begin{array}{ccccc}
\bigoplus_j \Gamma(U_j, \mathcal{A}^0) & \longrightarrow & \bigoplus_j \Gamma(U_j, \mathcal{A}^1) & \longrightarrow & \bigoplus_j \Gamma(U_j, \mathcal{A}^2) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{j<k} \Gamma(U_j \cap U_k, \mathcal{A}^0) & \longrightarrow & \bigoplus_{j<k} \Gamma(U_j \cap U_k, \mathcal{A}^1) & \longrightarrow & \bigoplus_{j<k} \Gamma(U_j \cap U_k, \mathcal{A}^2)
\end{array}$$

For $\omega_j = \frac{1}{2\pi i}\partial\overline{\partial}\log h_j$, we may write $\omega_j = d\beta_j$ for

$$\beta_j = \frac{1}{2\pi i}\overline{\partial}\log h_j.$$

We write these elements in the corresponding places of the above diagram

$$\begin{array}{ccccc}
(*) & \longrightarrow & (\beta_j) & \longrightarrow & (\omega_j) \\
\downarrow & & \downarrow & & \downarrow \\
(*) & \longrightarrow & (\beta_k|_{U_k\cap U_j}, -\beta_j|_{U_k\cap U_j}) & \longrightarrow & (*)
\end{array}$$

We have that

$$\begin{aligned}
\beta_k - \beta_j &= \frac{1}{2\pi i}\overline{\partial}(\log h_k - \log h_j) \\
&= -\frac{1}{2\pi i}\overline{\partial}(\widetilde{g_{jk}} + \overline{\widetilde{g_{jk}}}) \\
&= -\frac{1}{2\pi i}\overline{\partial}\overline{\widetilde{g_{jk}}} \\
&= \overline{\partial}\overline{\frac{1}{2\pi i}\widetilde{g_{jk}}} \\
&= d\left(\frac{1}{2\pi i}\overline{\widetilde{g_{jk}}}\right).
\end{aligned}$$

Therefore, $[\omega] \in H^2_{\mathrm{dR}}(M)$ maps in $\check{H}^2(\mathcal{U}, \underline{\mathbb{R}})$ to

$$\frac{1}{2\pi i}(\overline{\widetilde{g_{k\ell}}} - \overline{\widetilde{g_{j\ell}}} + \overline{\widetilde{g_{jk}}}) = \overline{n_{jk\ell}} = n_{jk\ell}.$$

This completes the proof. □

Recall that on $\mathbb{P}^n$, we have the Fubini–Study metric (Example 7.5) with corresponding form ω_{FS} given in homogeneous coordinates $z_0, \dots, z_n$, on $U_j = (z_j \neq 0)$ by

$$\omega_{FS}|_{U_j} = \frac{i}{2\pi}\partial\overline{\partial}\log\sum_{k=0}^{n}\left|\frac{z_k}{z_j}\right|^2.$$

We claim that there is a metric h on $\mathcal{O}(1)$ such that $\omega_h = \omega_{FS}$. Indeed, this is okay if

$$h_j = \left(\sum_{k=0}^{n} \left| \frac{z_k}{z_j} \right|^2 \right)^{-1}.$$

Since $g_{jk} = \frac{z_j}{z_k}$, we have that

$$h_j = |g_{jk}|^2 h_k.$$

Therefore, this choice of h_j defines a metric h on $\mathcal{O}(1)$ whose form ω_h gives the Fubini–Study form ω_{FS}.

Remarks 8.2.

(1) If $f\colon M' \to M$ is holomorphic and L is a line bundle on M with Hermitian metric h, we get a Hermitian metric h' on $L' = f^*L$ such that

$$\omega_{h'} = f^*\omega_h.$$

(2) If X is a smooth projective variety with ample line bundle L, there is an $N > 0$ and $X \overset{i}{\hookrightarrow} \mathbb{P}^m$ such that $L^N \cong i^*\mathcal{O}(1)$. Then the metric we had on $\mathcal{O}_{\mathbb{P}^m}(1)$ induces a metric on L^N such that the corresponding form is Kähler. By replacing (h_j) with $(h_j^{1/N})$, we get a metric on L with positive $(1,1)$-form.

The upshot is that we can apply Hard Lefschetz Theorem 7.28 or Lefschetz decomposition for $c_1(L)$ when L is an ample line bundle on a smooth projective algebraic variety.

Theorem 8.3 (Kodaira's Embedding Theorem)**.** *If X is a compact Kähler manifold whose fundamental class is the Chern class of a line bundle $L \in \operatorname{Pic}(M)$, then X is algebraic and L is ample.*

Remark 8.4 (Lefschetz Theorem on $(1,1)$-classes)**.** Let M be a complex manifold. Recall the exponential sequence:

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O}_M \xrightarrow{\exp(2\pi i -)} \mathcal{O}_M^\times \longrightarrow 0$$

whose long exact sequence in cohomology gives the *Chern class map*

$$c_1\colon \operatorname{Pic} M \cong H^1(M, \mathcal{O}_M^*) \to H^2(M, \mathbb{Z}).$$

If M is compact and Kähler and $L \in \operatorname{Pic} M$, the image of $c_1(L)$ in $H^2(M, \mathbb{C})$ lies in $H^{1,1}(M)$.

Conversely, if $\alpha \in H^2(M, \mathbb{Z})$ whose image in $H^2(M, \mathbb{C})$ lies in $H^{1,1}$, then there exists $L \in \operatorname{Pic}(M)$ such that $\alpha = c_1(L)$.

Indeed, it is enough to show that the image of α in $H^2(M, \mathcal{O}_M)$ is 0. However,

$$\begin{array}{ccc} \text{image of } \alpha & & H^2(M, \mathbb{C}) \\ \big\downarrow & & \big\downarrow {\scriptstyle \text{projection with respect to Hodge decomposition induced by } \mathbb{C} \to \mathcal{O}_M} \\ 0 & & H^2(M, \mathcal{O}_M) = H^{0,2}(M) \end{array}$$

Example 8.5 (Hodge Decomposition for tori)**.** Let V be a complex vector space of dimension n and $L \subseteq V$ be a lattice, i.e. $L \cong \mathbb{Z}^{2n}$ and $L \otimes_{\mathbb{Z}} \mathbb{R} \cong V$. Then

$$M = V/L \text{ is a compact Kähler manifold of dimension } n.$$

Let $\pi\colon V \to M$ be the projection map. This is a covering space by a simply connected space, so

$$\pi_1(M) \cong L.$$

Hence

$$H_1(M, \mathbb{Z}) \cong \pi_1(M)^{\text{ab}} \cong L,$$

and

$$H^1(M, \mathbb{Z}) \cong L^* = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}).$$

We have a canonical map

$$\bigwedge^{*} H^1(M, \mathbb{Z}) \to H^*(M, \mathbb{Z})$$

given by the cup product. This is an isomorphism by Küneth since since M is homeomorphic to $(S^1)^n$.

Now,

$$H^1(M, \mathbb{R}) = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \otimes \mathbb{R} = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R}) = V_{\mathbb{R}}^*$$

and hence

$$H^1(M, \mathbb{C}) = V_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = V^* \oplus \overline{V^*}.$$

The group action on M allows us to trivialize the tangent bundle:

$$\mathcal{T}_M = \mathrm{T}_0 M \otimes_{\mathbb{R}} \mathcal{C}_M^{\infty} = V \otimes_{\mathbb{R}} \mathcal{C}_M^{\infty}.$$

For the cotangent bundle:

$$\mathcal{A}_M^m \cong \bigwedge^{m} V^* \otimes_{\mathbb{R}} \mathcal{C}_M^{\infty}$$

$$\mathcal{A}_M^{p,q} \cong \bigwedge^{p} V^* \otimes_{\mathbb{R}} \bigwedge^{q} \overline{V^*} \otimes_{\mathbb{C}} \mathcal{C}_{M,\mathbb{C}}^{\infty}.$$

Choosing an isomorphism $V \cong \mathbb{C}^n$ and the standard metric on $\mathbb{C}^n$ which is translation invariant, we get a metric on M.

Let us see the Hodge decomposition for H^1. Clearly, $V^* \subseteq H^{1,0} \subseteq \Gamma(M, \mathcal{A}_M^{1,0})$. Since the sheaf of holomorphic forms is also trivial,

$$\dim H^{1,0} = \dim H^0(M, \Omega_M) = \dim_{\mathbb{C}} V^*.$$

This shows that

$$H^{1,0} = V^*,$$

and

$$H^{0,1} = \overline{V^*}.$$

This also implies that the Hodge decomposition for $H^m(M, \mathbb{Z})$ has

$$H^{p,q} = \bigwedge^{p} V^* \otimes \bigwedge^{q} \overline{V^*}.$$

Indeed, the $\supseteq$ inclusion follows from the Hodge decomposition for H^1, and since we know where each of these pieces lies, we must have equality.

Remark 8.6. If L is general and $n \geq 2$, then M is not algebraic.

We present a sketch of the proof for $n = 2$. Let z_1, z_2 be coordinates on $\mathbb{C}^2$ and ω be the form on M induced by $dz_1 \wedge dz_2$. This is a holomorphic 2-form.

If C is a smooth algebraic curve and $f\colon C \to M$ is a non-constant holomorphic map, then $f^*(\omega) = 0$ since $\dim C = 1$.

Equivalently, considering $[C] \in H_2(C, \mathbb{Z})$, we have that

$$f_*([C]) \in H_2(M, \mathbb{Z})$$

The the assertion that $f^*(\omega) = 0$ is equivalent to

$$[\omega] \cap f_*([C]) = 0.$$

(Here, we can think of $[\omega]$ as a homology class via Poincaré duality).

General fact. If M is algebraic, then $f_*([C]) \neq 0$.[4]

If we write it in terms of the basis of $H_2(M, \mathbb{Z}) = \bigwedge^2 L$ and if L is described by a matrix $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$ (where the lattice L is generated by the columns of this matrix), using the fact that

$$[\omega] \cap (\text{generators of } H_2(M, \mathbb{Z}) \cong \bigwedge^2 L)$$

is given by the 2×2 minors of the matrix, there is an non-trivial relalation with $\mathbb{C}$-coefficients between these 6 minors. Indeed, $f_*([C]) \neq 0$ and $[\omega] \cup f_*([C]) = 0$ is such a relation.

However, for $a_1, \cdots, a_n, b_1, \ldots, b_n$ general in $\mathbb{C}$, these 6 minors are linearly independent. Therefore, M is not algebraic.

9. Algebraic cycles and some cohomology computations

A detailed exposition of this topic together with complete proofs is on the course website:

http://www-personal.umich.edu/~mmustata/SingularCohomology.pdf

9.1. **Two theorems in algebraic topology.** Recall that a map $\pi\colon M_1 \to M_2$ is*locally trivial with fiber F* if there is an open cover $M_2 = \bigcup_i U_i$ such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\text{homeomorphism}} & U_i \times F \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

In our setup, the cover will always be finite.

[4] We will soon discuss such statements in more generality, so we restrict ourselves to this statement for now.

Theorem 9.1 (Leray–Hirsch)**.** *Suppose* $\pi\colon X \to Y$ *is locally trivial with fiber* F*. Suppose all* $H^i(F,\mathbb{Z})$ *are free, finite-generated, and we have* $\alpha_1,\dots,\alpha_n \in H^*(X,\mathbb{Z})$ *such that for all* $y \in Y$*, their restriction to* $H^*(\pi^{-1}(y),\mathbb{Z})$ *is a basis, we have an isomorphism*

$$H^*(Y)\times\mathbb{Z}^n \to H^*(X),$$
$$(\beta,(m_1,\dots,m_n)) \mapsto \pi^*(\beta)\cup(m_1\alpha_1+\cdots+m_n\cup\alpha_n).$$

Proof. If $X = Y\times F$, this follows from Künneth. If we have a finite trivial cover, we argue by induction on the number of sets in the cover, using Mayer–Vietoris.

We do not prove this in more generality. □

Next, suppose $E \to X$ is oriented, real vector bundle of rank r. Write $E_x = \pi^{-1}(x)$.

The choice of orientation is equivalent to a choice of a compatible system of generators for each $H^r(E_x, E\setminus\{0\},\mathbb{Z}) \cong \mathbb{Z}$.

Theorem 9.2.

(1) *There is a unique cohomology class* $\eta_E \in H^r(E, E \setminus X, \mathbb{Z})$ *(where* $X \hookrightarrow E$ *is the 0-section) such that for all* $x \in X$*, the restriction of* η_E *to* $H^r(E_x, E_x \setminus \{0\}, \mathbb{Z})$ *is the chosen orientation.*

(2) *For every closed subset* $W \subseteq X$*, we have an isomorphism*

$$H^{j-r}(X, X \setminus W, \mathbb{Z}) \xrightarrow{\cong} H^j(E, E \setminus W, \mathbb{Z}),$$
$$\beta \mapsto \pi^*(\beta) \cup \eta_E.$$

Proof. First, treat the case $E = M \times \mathbb{R}^n$ using Künneth. Then, proceed by induction on the number of elements in a trivial cover of E using Mayer–Vietoris.

We do not prove this in more generality. □

Definition 9.3. The unique cohomology class η_E in the theorem is the *Thom class* of E. The isomorphism in part (2) of the theorem is the *Thom isomorphism*.

Suppose Y is a smooth real manifold of dimension n and X is a smooth real closed submanifold of dimension m. Suppose X and Y are both oriented. Since

$$\det(N_{X/Y}) = \det(T_Y)|_X \otimes \det(T_X)^{-1},$$

$N_{X/Y}$ is oriented of rank $n - m$.

Theorem 9.4 (Tubular Neighborhood Theorem)**.** *There is an open neighborhood of* X *in* Y *and a retraction* $r\colon U \to X$ *such that*

$$\begin{array}{ccccc} X & \hookrightarrow & U & \xrightarrow{r} & X \\ \downarrow{=} & & \downarrow{\simeq} & & \downarrow{=} \\ X & \hookrightarrow & N_{X/Y} & \xrightarrow{\pi} & X, \end{array}$$

i.e. $N_{X/Y}$ *is homeomorphic to* U*.*

Let $r = n - m$. For $W \subseteq X$ closed, we have that

$$\begin{aligned} H^j(X, X \setminus W, \mathbb{Z}) &\cong H^{j+r}(N_{X/Y}, N_{X/Y} \setminus W, \mathbb{Z}) && \text{Theorem 9.2 (2)} \\ &\cong H^{j+r}(U, U \setminus W, \mathbb{Z}) && \text{Theorem 9.4} \\ &\cong H^{j+r}(Y, Y \setminus W, \mathbb{Z}) && \text{excision.} \end{aligned}$$

In particular, for $W = X$, we get an isomorphism

$$H^j(X, \mathbb{Z}) \cong H^{j+r}(Y, Y \setminus X, \mathbb{Z}).$$

Then the long exact sequence in cohomology becomes

$$H^{p-1}(Y \setminus X, \mathbb{Z}) \to H^{p-r}(X, \mathbb{Z}) \to H^p(Y, \mathbb{Z}) \to H^p(Y \setminus X, \mathbb{Z}).$$

The map

$$H^{p-r}(X, \mathbb{Z}) \to H^p(Y, \mathbb{Z})$$

is denoted by i_* if $i\colon X \hookrightarrow Y$ is the inclusion and called the *Gysin homomorphism*.

Recall that if X is compact, oriented, smooth manifold of dimension n, there is a fundamental class $\mu_x \in H_n(X,\mathbb{Z})$ and the map

$$\begin{aligned} H^p(X,\mathbb{Z}) &\to H_{n-p}(X,\mathbb{Z}) \\ \alpha &\mapsto \alpha \cup \mu_x \end{aligned}$$

is an isomorphism. This is the strongest statement of Poincaré duality.

If $f\colon X \to Y$ is a smooth map between compact oriented smooth manifolds and $n = \dim Y$, $m = \dim X$, $r = n - m$, we have:

$$\begin{array}{ccc} H^p(X,\mathbb{Z}) & \xrightarrow{f_*} & H^{p+r}(Y,\mathbb{Z}) \\ {\scriptstyle \mathrm{PD}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{PD}}\downarrow{\scriptstyle \cong} \\ H_{m-p}(X,\mathbb{Z}) & \xrightarrow{f_*} & H_{n-p-r}(Y,\mathbb{Z}) \end{array}$$

The map on cohomology is also called the *Gysin map*.

Fact 9.5. *If f is the inclusion of a closed submanifold, this agreed with the previous definition of the Gysin map.*

9.2. **Special features for algebraic varieties.** Before moving on to the computations of cohomology groups, we make a few general statements about the special features for algebraic varieties.

(1) If X is a complete complex algebraic variety of dimension n, X^{an} has a triangularization with simplices of dimension at most $2n$. Moreover, if $Y \subseteq X$ is a closed subvariety, we can find a compatible triangulizations for X^{an} and Y^{an}.
(2) (Nagata's Theorem) If W is any complex algebraic variety, there is an open immersion $W \hookrightarrow X$ to a complete algebraic variety X.
Using (1), one can show that all $H^p(W,\mathbb{Z})$, $H_p(W,\mathbb{Z})$ are finitely-generated.

Given any complete irreducible algebraic variety X, of dimension n, we have a fundamental class $\mu_X \in H_{2n}(X,\mathbb{Z})$.

- If X is smooth, this is clear.
- In general, use resolution of singularities to construct $\pi\colon Y \to X$ which is proper, birational, with Y smooth, and take $\mu_X = \pi_*\mu_Y$.
  We claim that this does not depend on π. If we had $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$, we may construct some $\widetilde{Y}$ mapping to both Y and Y' (the *Hiranaka hat*) and show that μ_Y and $\mu_{Y'}$ are both pushforwards of $\mu_{\widetilde{Y}}$.
  It is hence enough to show that if $W \to Y$ is proper, birational, with W, Y smooth and complete, then $f_*\mu_W = \mu_Y$. Since $H_{2n}(Y) = \mathbb{Z}\mu_Y$, where $n = \dim Y$, we certainly know that $f_*\mu_W = d \cdot \mu_Y$ for some $d \in \mathbb{Z}$. We just need to show that $d = 1$.
  To check this, tensor with $\mathbb{R}$ and use de Rham cohomology. We have that

$$\begin{aligned} H_{2n}(Y,\mathbb{R}) &\cong H^{2n}_{\mathrm{dR}}(Y)^* \\ \mu_Y &\mapsto \left([\omega] \mapsto \int_Y \omega\right). \end{aligned}$$

To check that $f_*\mu_W = \mu_Y$, modulo these identification, it is enough to check that if ω is a $2n$-form on Y, then

$$\int_Y \omega = \int_W f^*(\omega).$$

This is clear since f is a diffeomorphism outside a measure 0 subset.

One can argue similarly to show that: if $f\colon W \to X$, with W, X are smooth, complete with f surjective and generically finite, then

$$f_*\mu_W = \deg(f) \cdot \mu_X.$$

Exercise. Use the projection formula[5] to show that

$$f_* f^* \alpha = \deg(f) \cdot \alpha$$

on $H^*(X)$.

In particular, f^* is injective with $\mathbb{Q}$-coefficients, and f^* is injective with $\mathbb{Z}$-coefficients if f is birationals.

Here is a more general result.

Fact 9.6. *If $f\colon X \to Y$ is a surjective holomorphic map of compact complex manifolds, with X Kähler,*

$$f^*\colon H^i(Y, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

is injective.

A proof of this is given in the notes on singular cohomology on the website.

Remark 9.7. If X is a smooth, complete variety, and $Y \subseteq X$ is an irreducible closed subvariety with

$$i\colon Y \hookrightarrow X$$

with $\dim X = n$, $\dim Y = m$, $r = n - m$, then

$$i_*\mu_Y \in H_{2m}(X, \mathbb{Z}) \overset{\text{PD}}{\cong} H^{2r}(Y, \mathbb{Z}).$$

This is the *cohomology class of* Y, denoted $[Y]$.

We can extend this to cycles. If

$$\alpha = \sum_{i=1}^{r} n_i Y_i$$

and $\operatorname{codim}(Y_i) = r$, then we define

$$[\alpha] = \sum n_i [Y_i] \in H^{2r}(X, \mathbb{Z}).$$

[5] The projection formula is $f_*(f^*(\alpha) \cup \beta) = \alpha \cup f_*(\beta)$

Example 9.8. Suppose X is a smooth complete variety and D is a smooth divisor on X. Then
$$[D] = c^1(\mathcal{O}_X(D)).$$
This is true even if D is just a prime divisor.

More generally, suppose X is a smooth complete variety and $D_1, \dots, D_r$ are smooth divisors, intersecting transversely. For $Y = D_1 \cap \cdots \cap D_r$, induction on r shows that
$$[Y] = c^1(\mathcal{O}(D_1)) \cup \cdots \cup c_1(\mathcal{O}(D_r)).$$

As a special case, take $r = n = \dim X$. Under the isomorphism $H^{2n}(X) \cong \mathbb{Z}$,
$$c^1(\mathcal{O}(D_1)) \cup \cdots \cup c_1(\mathcal{O}(D_n)) = \#Y.$$

Example 9.9. If $X = \mathbb{P}^n$, $h = c^1(\mathcal{O}(1))$,
$$h^r = [L_r]$$
where $L_r \subseteq \mathbb{P}^n$ is a linear subspace of codimension r.

In particular, $h^n = 1$ via $H^{2n}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$.

Exercise. Let X be a smooth, complete complex algebraic variety. Let $Y \subseteq X$ be a smooth irreducible closed subvariety of codimension r. Let $j \colon Y \hookrightarrow X$. We then have maps

$$\begin{array}{ccc} H^p(X) & \xrightarrow{j^*} & H^p(Y) \\ & & \downarrow{\scriptstyle j_*} \\ & & H^{p+2r}(X) \end{array}$$

Show that

(1) $j_* j^* \alpha = \alpha \cup [Y]$ for all $\alpha \in H^*(X)$,
(2) $j^* j_* \beta = \beta \cup j^*[Y]$ for all $\beta \in H^*(Y)$.

Hint. For (1), use the description of j_* via Poincaré duality and the projection formula. For (2), use the description of j_* via the Thom isomorphism.

Remark 9.10. If $f \colon Y \to X$ is a holomorphic map of compact Kähler manifolds, we defined
$$f_* \colon H^p(Y) \to H^{p+2r}(X)$$
where $r = \dim X - \dim Y$ via Poincaré duality. Using the behavior of Poincaré duality with respect to the Hodge composition, after tensoring with $\mathbb{C}$, we get that
$$f_*(H^{i,j}(Y)) \subseteq H^{i+r,j+r}.$$
In particular, if X is a smooth complete algebraic variety and $Y \subseteq X$ is an irreducible closed subvariety of codimension r, then
$$[Y] \in H^{2r}(X)$$
and the image of $[Y]$ in $H^{2r}(X, \mathbb{C})$ lies in $H^{r,r}(X)$.

Indeed, if $\widetilde{Y} \to Y$ is a resolution of singularities and $m = \dim Y$, then
$$\mu_{\widetilde{Y}} \in H_{2n}(\widetilde{Y}) \overset{\text{PD}}{\cong} H^0(\widetilde{Y})$$

is of type (0,0).

Definition 9.11. If X is a smooth projective complex algebraic variety, the set of *Hodge classes* is defined as

$$\mathrm{Hdg}^p(X) = \{\alpha \in H^{2p}(X, \mathbb{Q}) \mid \text{the image of } \alpha \text{ in } H^{2p}(X, \mathbb{C}) \text{ lies in } H^{p,p}\}.$$

Let $Z_p(X)_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector space with basis given by codimension p irreducible subvarieties.

By Remark 9.10, we have a linear map

$$\begin{aligned} Z^p(X)_{\mathbb{Q}} &\to \mathrm{Hdg}^p(X) \\ \sum n_i Y_i &\mapsto \sum_i n_i [Y_i]. \end{aligned}$$

Conjecture (Hodge Conjecture)**.** This is a surjective map.

Remark 9.12. The Hodge Conjecture 9.2 holds for $p = 1$, even with $\mathbb{Z}$-coefficients. Indeed, we showed that if $\alpha \in H^2(X, \mathbb{Z})$ has image in $H^{1,1}$, then $\alpha = c_1(L)$ for some $L \in \mathrm{Pic}(X)$. If $L = \mathcal{O}(D)$ with $D = \sum n_i D_i$, $\alpha = \sum n_i [D_i]$.

Remark 9.13. However, the Hodge Conjecture 9.2 is not true in general with $\mathbb{Z}$-coefficients. Atiyah–Hirzebruch constructed a counterexample using a torsion class. Later, Kollar gave a non-torsion example.

As a parenthesis, we mention the following result.

Theorem 9.14 (Lefschetz Hyperplane Theorem)**.** *Let X be a smooth projective complex algebraic variety and $D \subseteq X$ be a smooth divisor such that $L = \mathcal{O}(D)$ is ample. If $j \colon D \hookrightarrow X$ is the inclusion, then*

$$j^* \colon H^p(X, \mathbb{Z}) = H^p(D, \mathbb{Z})$$

is an isomorphism for $p \leq n - 2$ and injective for $p \leq n - 1$, where $n = \dim X$.

Sketch of proof. Recall that $j_* j^* \alpha = [D] \cup \alpha = c_1(L) \cup \alpha$. Hard Lefschetz 7.28 implies that this is injective on $H^p(X, \mathbb{Q})$ for $p < n$. Hence j^* is also injective on $H^p(X, \mathbb{Q})$ for $p \leq n - 1$.

Proving the isomorphism for $p \leq n - 2$ is harder. The key ingredient of the proof (due to Andreotti-Frankel) is the following result: If Y is an **affine** complex algebraic variety of dimension n, Y has the homotopy type of a CW complex of dimension $\leq n$. The proof uses *Morse theory.* In particular, for $p > n$:

$$\begin{aligned} H_p(Y, \mathbb{Z}) &= 0, \\ H^p(Y, \mathbb{Z}) &= 0. \end{aligned}$$

In our setting, L is ample, so $X \setminus D$ is affine, hence this result applies. The long exact sequence in cohomology gives:

$$H^{p-1}(X \setminus D, \mathbb{Z}) \longrightarrow H^{p-2}(D, \mathbb{Z}) \xrightarrow{j_*} H^p(X, \mathbb{Z}) \longrightarrow H^p(X \setminus D, \mathbb{Z}),$$

so $j_*\colon H^{p-2}(D) \to H^p(X)$ is an isomorphism if $p-1>n$ and surjective if $p>n$. Hence

$$j^* H^{2n-p}(X) \to H^{2n-p}(D)$$

is an isomorphism if $p \geq n+2$, i.e. $2n-p \leq n-2$, and injective if $p \geq n+1$. $\square$

9.3. **Some cohomology computations.**

Example 9.15 (The projective space $\mathbb{P}^n$)**.** We show that

$$H^i(\mathbb{P}^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } 0 \leq i \leq 2n \text{ and } i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

We use induction on n. The case $n=0$ is trivial. For the inductive step, consider $\mathbb{P}^{n-1} \cong H \subseteq \mathbb{P}^n$ such that

$$\mathbb{P}^n \setminus H \cong \mathbb{C}^n$$

is contractible. The long exact sequence for $i>0$ gives

$$0 \longrightarrow H^{i-1}(\mathbb{P}^n \setminus H) \longrightarrow H^{i-2}(\mathbb{P}^{n-1}) \longrightarrow H^i(\mathbb{P}^n) \longrightarrow H^i(\mathbb{P}^n \setminus H) = 0.$$

Hence for $i \geq 2$, we have that $H^i(\mathbb{P}^n) \cong H^{i-2}(\mathbb{P}^{n-1})$. Moreover, $H^1(\mathbb{P}^n) = 0$ and $H^0(\mathbb{P}^n) = H^0(\mathbb{P}^n \setminus H) \cong \mathbb{Z}$.

Clearly, the Hodge decomposition is such that

$$H^{2k}(\mathbb{P}^n, \mathbb{C}) = H^{k,k}(\mathbb{P}^n)$$

(by Hodge symmetry). In particular,

$$H^q((\mathbb{P}^n)^{\mathrm{an}}, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^n}) = 0$$

for $q>0$. (Recall that we assumed this when proving GAGA 3.34. We finally arrived at a proof.)

In fact, we have that

$$H^*(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}[h]/(h^{n+1})$$

where $h = c^1(\mathcal{O}(Q))$. Why? We note that

$$H^{2n}(\mathbb{P}^n, \mathbb{Z}) \xrightarrow{\cong} H_0(\mathbb{P}^n, \mathbb{Z}),$$
$$h^n \mapsto 1,$$

since we have n hyperplanes intersecting transversely in 1 point. Since $H^{2k}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ we have that $h^k \in H^{2k}(\mathbb{P}^n, \mathbb{Z})$ is nonzero. For every $\alpha \in H^{2k}(\mathbb{P}^n, \mathbb{Z})$, we can write $\alpha = u \cdot h^k$ for some $u \in \mathbb{Q}$, whence

$$\alpha \cup h^{n-k} \mapsto \deg(\alpha u h^{n-k}) \in \mathbb{Z}$$
$$uh^n \mapsto u$$

and $\alpha \cup h^{n-k} = uh^n$, showing that $u \in \mathbb{Z}$.

We may summarize the example as follows:

$$\begin{array}{ccc} H^*(\mathbb{P}^n,\mathbb{Z}) & \xleftarrow[\cong]{} & \mathbb{Z}[x]/(x^{n+1}) \\ c^1(\mathcal{O}(q))^i & \longleftarrow & x^i \end{array}$$

By the Leray–Hirsch Theorem 9.1, if E is a rank $r+1$ vector bundle on any X and

$$\pi\colon \mathbb{P}(E) \to X$$

is the associated map, then

$$H^*(X)\otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{r+1}) \overset{\cong}{\to} H^*(\mathbb{P}(E))$$
$$\alpha x^i \mapsto \pi^*(\alpha)\cup c^1(\mathcal{O}(1))^i.$$

In particular,

$$H^{p,q}(\mathbb{P}(E)) \cong \bigoplus_{i=0}^{r} H^{p-i,q-i}(X).$$

Remark 9.16. The isomorphism $H^*(X)\otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{r+1}) \overset{\cong}{\to} H^*(\mathbb{P}(E))$ is a group isomorphism, **not a ring isomorphism**, In general, $c^1(\mathcal{O}(1))^{r+1} \neq 0$. By looking at its coefficients in $H^*(X)$, we get Chern classes of $c_i(E)$.

Exercise. Suppose X is compact. Show that for $0 \le i \le r$:

$$\pi_*(\pi^*(\alpha)\cup c^1(\mathcal{O}(1))^i) = \begin{cases} 0 & \text{if } i < r \\ \alpha & \text{if } i = r. \end{cases}$$

We now compute the cohomology of the blow-up.

Proposition 9.17. *Let X be a smooth projective complex variety and $Y \subseteq X$ be a smooth closed subvariety of codimension r. Let:*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ {\scriptstyle\iota}\uparrow & & \uparrow \\ E & \xrightarrow{\varphi} & Y \end{array}$$

be the blow-up along Y, where E is the exceptional divisor. Then, for all p,

$$H^p(\widetilde{X},\mathbb{Z}) \cong H^p(X,\mathbb{Z}) \oplus \bigoplus_{i=1}^{r-1} H^{p-2i}(Y,\mathbb{Z})$$
$$\pi^*\alpha_0 + \iota_* \sum_{i=1}^{r-1} c^1(\mathcal{O}_E(1))^{i-1} \leftarrow\!\shortmid (\alpha_0,\ldots,\alpha_{r-1}).$$

Sketch of proof. We have long exact sequences (suppressing the coefficient ring, $\mathbb{Z}$, from notation):

$$\begin{array}{ccccccc}
H^{p-1}(X \setminus Y) & \longrightarrow & H^{p-2r}(Y) & \longrightarrow & H^p(X) & \longrightarrow & H^p(X \setminus Y) \\
\big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \varphi_p} & & \big\downarrow{\scriptstyle \pi^*} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \pi^*} \\
H^{p-1}(X \setminus Y) & \longrightarrow & H^{p-2}(E) & \longrightarrow & H^p(\widetilde{X}) & \longrightarrow & H^p(\widetilde{X} \setminus E)
\end{array}$$

We know that E is a projective bundle over Y of rank r and hence

$$H^{p-2}(E) \cong \bigoplus_{i=1}^{r} H^{p-2i}(Y).$$

Note that the map φ_p is not given explicitly. One can, however, show the following.

Exercise. We have that $\varphi_* \circ \varphi_p = \mathrm{Id}_{H^{p-2r}(Y)}$. Equivalently, using the previous exercise, the projection onto the last component of φ_p is the identity on $H^{p-2r}(Y)$.

The the conclusion follows by a diagram chase. (Check this! For example, for surjectivity, one sees that any class in $H^p(\widetilde{X})$ is a sum of a class in $H^p(X)$ and $H^{p-2}(E)$, and the result follows from the exericse.) $\square$

10. Pure Hodge structures

We now discuss an abstract (linear algebra) setting which Hodge theory fits into.

Definition 10.1. A *pure (integral) Hodge structure of weight m* is given by:

(1) a finitely-generated free abelian group H (also written $H_{\mathbb{Z}}$),
(2) a decomposition $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=m} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$ for any p, q.

There are some variants of this definition. A *rational* or *real* pure Hodge structure starts with a finite-dimensional vector space over $\mathbb{Q}$ or over $\mathbb{R}$, respectively.

Example 10.2 (Main example)**.** If X is a compact Kähler manifold, then $H^i(X, \mathbb{Z})/(\text{torsion})$ is endowed with a canonical pure Hodge structure of weight i.

Example 10.3 (Tate Hodge structure)**.** We define the *Tate Hodge structures* are:

- $\mathbb{Z}(1) = (2\pi i)\mathbb{Z} \subseteq \mathbb{C}$, pure of type $(-1, -1)$, (weight -2),
- $\mathbb{Z}(1) = \frac{1}{2\pi i}\mathbb{Z} \subseteq \mathbb{C}$, pure of type $(1, 1)$, (weight 2).

Proposition 10.4. *Giving a pure Hodge structure of weight m is equivalent to giving a free finitely-generated abelian group H together with a finite, decreasing filtration $(F^p H_{\mathbb{C}})_{p\in\mathbb{Z}}$ on $H_{\mathbb{C}}$ such that $F^p \oplus \overline{F}^{m+1-p} = H_{\mathbb{C}}$.*

While this seems to make things more complicated, it is actually much more convenient to work with. This will become apparent when we discuss the Hodge–de Rham spectral sequence (the filtration will be given algebraically while the Hodge decomposition is only given analytically) and the behavior of Hodge structures in families.

Proof. Suppose we have a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$ as in the definition of a Hodge structure. Then we define

$$F^p H_{\mathbb{C}} = \bigoplus_{\substack{i+j=p \\ i\geq p}} H^{i,j}.$$

This is clearly a decreasing, finite filtration. Note also that

$$\overline{F}^{m+1-p} = \bigoplus_{\substack{i+j=m \\ i\geq m+1-p}} \overline{H^{i,j}} = \bigoplus_{\substack{i+j=m \\ j\leq p-1}} H^{j,i}$$

and hence the condition $F^p \oplus \overline{F}^{m+1-p} = H_{\mathbb{C}}$ also holds. A similar computation shows that

$$H^{p,q} = F^p \cap \overline{F}^q,$$

which motivates the proof of the converse.

Suppose $F^\bullet H$ is a finite, decreasing filtration on H which satisfies $F^p \oplus \overline{F}^{m+1-p} = H_{\mathbb{C}}$. Define

$$H^{p,q} = F^p \cap \overline{F}^q.$$

- Suppose $\sum_{p+q=m} \alpha_{p,q} = 0$ for $\alpha_{p,q} \in H^{p,q}$. If not all $\alpha_{p,q}$ are 0, choose minimal p such that $\alpha_{p,q} \neq 0$. Then
$$\alpha_{p,q} = -\sum_{p'>p} \alpha_{p',q'} \in F^{p+1} \cap \overline{F}^q = 0$$
by the assumption that $F^p \oplus \overline{F}^{m+1-p} = H_{\mathbb{C}}$.
- Let us show that every $0 \neq u \in H$ lies in $\sum_{p+q=m} H^{p,q}$. Suppose $u \in F^p$. We argue by decreasing induction on p. (If $p \gg 0$, $u = 0$, so there is nothing to prove.) We know that $F^{p+1} \oplus \overline{F}^{m-p} = H_{\mathbb{C}}$, so $u = v_1 + v_2$ for $v_1 \in F^{p+1}$ and $v_2 \in \overline{F}^{m-p}$. By the inductive hypothesis, $v_1 \in \sum H^{p',q'}$. Moreover,
$$v_2 = u - v_1 \in F_p \cap \overline{F}_{m-p},$$
so $v_2 \in H^{p,q} \subseteq \sum_{p'+q'=m} H^{p',q'}$. Hence
$$u \in \sum_{p',q'} H^{p',q'}.$$

In fact, we see that $F^p \subseteq \bigoplus_{\substack{p'+q'=m \\ p'\geq p}} H^{p',q'}$ and the reserve inclusion is clear. Hence we recover the equality

$$F^p = \bigoplus_{p'\geq p} H^{p',q'}$$

which shows that these are inverse constructions. □

Definition 10.5. If A and B are pure Hodge structures of weights m and $m + 2r$, then a *morphism of pure Hodge structures of type* (r, r) is a morphism of abelian groups $f\colon A \to B$ such that

$$f_{\mathbb{C}}(A^{p,q}) \subseteq B^{p+r,q+r} \quad \text{for all } p,\ q$$

or, equivalently, using Proposition 10.4,

$$f(F^pA) \subseteq F^{p+r}B \quad \text{for all } p.$$

If $r = 0$, we simply call f a *morphism of pure Hodge structures.*

Example 10.6. If $f\colon X \to Y$ is a morphism of compact Kähler manifolds, then

$$f^*\colon H^i(Y, \mathbb{Z})/(\text{torsion}) \to H^i(X, \mathbb{Z})/(\text{torsion})$$

is a morphism of Hodge structures for all i.

Operations with pure Hodge structures.

(1) We have finite direct sums of pure Hodge structures of the same weight.
(2) We have kernels and cokernels. If $f\colon A \to B$ is a morphism between Hodge structures of weights m and $m + 2r$, then

$$\ker(f) \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=m} \ker(A^{p+q} \to A^{p+r,q+r}),$$

so $\ker(f)$ is a pure Hodge structure of weight m.

Similarly, $\operatorname{coker}(f)$/torsion is a pure Hodge structure of weight $m + 2r$:

$$\operatorname{coker}(f) \otimes_{\mathbb{Z}} \mathbb{C} = \operatorname{coker}(A^{p,q} \to B^{p+r,q+r}).$$

These satisfy the usual universal properties.

One can then check that the category of pure Hodge structures of weight m is an abelian category.
(3) If A and B are Hodge structures of weights m and n, then $A_{\mathbb{Z}} \otimes_{\mathbb{Z}} B_{\mathbb{Z}}$ is a finitely-generated free abelian group, and

$$A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}} = \left(\bigoplus_{p+qm} A^{p,q}\right) \otimes \left(\bigoplus_{p'+q'=n} B^{p',q'}\right),$$

and hence $A \otimes B$ is a pure Hodge structure of weight $m + n$:

$$(A \otimes B)^{p,q} = \bigoplus_{\substack{i+i'=p \\ j+j'=q}} A^{i,j} \otimes B^{i',j'}.$$

In particular, the category of pure Hodge structures of weight m does not have a monoidal structure, while the category of pure Hodge structures does have one. However, this latter category does not have direct sums so it is not an abelian category.

Example 10.7. Recall the Tate Hodge structure $\mathbb{Z}(1) = (2\pi i)\mathbb{Z}$ of type $(-1, -1)$ (weight -2). Then

$$\mathbb{Z}(m) = (2\pi i)^m \mathbb{Z}$$

is a Hodge structure of type $(-m, -m)$ (weight $-2m$), and

$$\mathbb{Z}(i) \otimes \mathbb{Z}(j) = \mathbb{Z}(i + j).$$

In general, if A is a pure Hodge structure of weight m, then

$$A(i) = A \otimes \mathbb{Z}(i)$$

is a pure Hodge structure of weight $m - 2i$.

Example 10.8. If X is a compact Kähler manifold, then

$$H^i(X,\mathbb{Z})/(\text{torsion}) \otimes H^j(X,\mathbb{Z})/(\text{torsion}) \to H^{i+j}(X,\mathbb{Z})/(\text{torsion})$$

is a morphism of pure Hodge structures of weight $i+j$.

(4) The dual of a pure Hodge structure A of weight m is

$$A^* = \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$$

with the grading

$$\operatorname{Hom}_{\mathbb{C}}(A_{\mathbb{C}},\mathbb{C}) = \bigoplus_{p+q=m} \underbrace{\operatorname{Hom}(A^{p,q},\mathbb{C})}_{(A^*)^{-p,-q}},$$

which is a pure Hodge structure of weight $-m$.

Example 10.9. We have that $\mathbb{Z}(-m) = \mathbb{Z}(m)^*$.

Example 10.10. Let X be a compact Kähler manifold. Then

$$H_i(X,\mathbb{Z})/(\text{torsion}) \cong (H^i(X,\mathbb{Z})/(\text{torsion}))^*$$

by the Universal Coefficients Theorem, so $H_i(X,\mathbb{Z})/(\text{torsion})$ carries a pure Hodge structure of weight $-i$.

Let X be a compact Kähler manifold of dimension n.

(1) We have that:

$$H^{2n}(X,\mathbb{Z}) \to \mathbb{Z}(-n)$$
$$\omega \mapsto (2\pi i)^{-n}$$

where ω is the element such that $\omega \cup \mu_X = 1 \in H_0(X,\mathbb{Z})$.

(2) We have that

$$(H^k(X,\mathbb{Z})/\text{torsion})^* \xrightarrow{\cong} (H^{2n-k}(X,\mathbb{Z})/(\text{torsion}))(n)$$

via Poincaré duality.

(3) Similarly,

$$H^k(X,\mathbb{Z})/\text{torsion} \xrightarrow{\text{PD}} (H_{2n-k}(X,\mathbb{Z})/\text{torsion})(-n)$$

is an isomorphism of pure Hodge structures.

(4) Consider the Gysin maps. Let $f\colon X \to Y$ be a holomorphic map of compact Kähler manifolds of dimensions n and m, respectively. For $d = m-n$, we get the *Gysin map*:

$$H^p(X,\mathbb{Z})/(\text{torsion}) \xrightarrow{f_*} (H^{p+2d}(Y,\mathbb{Z})/(\text{torsion}))(d)$$

10.1. Polarized Hodge structures.

Suppose A is a pure Hodge structure of weight m.

Definition 10.11. A *polarization on* A is given by a bilinear form

$$A_{\mathbb{Z}} \times A_{\mathbb{Z}} \xrightarrow{Q} \mathbb{Z}$$

which is:

- symmetric if m is even,

- skew-symmetric if m is odd,

(equivalently, $(u,v) \mapsto i^m Q(u,v)$ is a Hermitian form) such that, after tensoring with $\mathbb{C}$,

(1) $Q(A^{p,q}, A^{p',q'}) = 0$ unless $p = q'$, $q = p'$,
(2) the Hermitian form

$$(u,v) \mapsto i^{p-q}(-1)^{\frac{m(m-1)}{2}} Q(u, \overline{v})$$

is positive definite on $A^{p,q}$.

The last condition should remind the reader of the Hodge–Riemann bilinear relations 7.33.

In particular, Q is non-degenerate, so it induces an isomorphism $A(m) \xrightarrow{\cong} A^*$.

Example 10.12 (Main example)**.** Let X be a smooth projective variety and $L \in \operatorname{Pic}(X)$ be ample. For $k \leq n = \dim X$, we define the *primitive cohomology*

$$P\overline{H}^k(X, \mathbb{Z}) = \{\alpha \in H^k(X, \mathbb{Z})/(\text{torsion}) \mid c^1(L)^{n-k+1} \cup \alpha = 0 \mod (\text{torsion})\}$$

Here $\overline{H}$ stands for killing the torsion.[6] This is a polarized Hodge structure of weight k by the Hodge–Riemann bilinear relations 7.33.

Moreover, we get a polarization of $\overline{H}^k(X, \mathbb{Z})$ by using the Lefschetz decomposition 7.28:

$$\overline{H}^k(X, \mathbb{Z}) = \bigoplus_{i \geq 0} P\overline{H}^{k-2i}(X, \mathbb{Z})(-i).$$

The advantage of a polarized pure Hodge structures is the following semisimplicity result.

Proposition 10.13. *If V is a polarized pure rational Hodge structure of weight m and $W \subseteq V$ is a pure sub Hodge structure of V, the restriction of Q (a polarization on V) to W gives a polarization of W, and there is a polarized pure sub Hodge structure W' such that*

$$V \cong W \oplus W'.$$

Proof. Take $W' = W^{\perp}$ with respect to Q. □

11. Algebraic de Rham cohomology

We next discuss algebraic de Rham cohomology. It will related to Hodge cohomology via a spectral sequence, so we start with a review of spectral sequences.

11.1. Introduction to spectral sequences.

Let $\mathcal{C}$ be an abelian category and $A^\bullet$ be a complex. Suppose $(F^p A^\bullet)_{p \in \mathbb{Z}}$ is a decreasing filtration such that

(1) there is an m such that $F^m A^\bullet = A^\bullet$,
(2) for all i, there is a p such that $F^p A^i = 0$.

[6] In fact, we already dealt with primitive cohomology with $\mathbb{C}$-coefficients; the only novelty here is to deal with the torsion.

We get an induced filtration on $H^*(A^\bullet)$ by

$$F^pH^i(A^\bullet) = \mathrm{im}(H^i(F^pA^\bullet) \to H^i(A^\bullet)).$$

The goal is to compute the graded pieces of this filtration:

$$\frac{F^pH^i(A^\bullet)}{F^{p+1}H^i(A^\bullet)}.$$

This is done by the spectral sequence associated to this filtered complex:

$$(E_r^{p,q}, d_r^{p,q})_{r\geq 1}$$

where $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$, together with isomorphisms:

$$E_{r+1}^{p,q} \cong \ker(d^{p,q})/\,\mathrm{im}(d_r^{p-r,q+r-1}).$$

(In other words, for fixed r we have a *page* of the spectral sequence $E_r^{p,q}$ and its cohomology with respect to the maps $d_r^{p,q}$ is the next *page* of the spectral sequence, $E_{r+1}^{p,q}$.)

Example 11.1. We have that

$$E_1^{p,q} = H^{p+q}(F^pA^\bullet/F^{p+1}A^\bullet)$$

for all p, q with

$$d_1^{p,q}\colon H^{p+q}(F^pA^\bullet/F^{p+1}A^\bullet) \to H^{p+q+1}(F^{p+1}A^\bullet/F^{p+2}A^\bullet)$$

with the boundary homomorphisms corresponds to the exact sequence of complexes

$$0 \longrightarrow F^{p+1}A^\bullet/F^{p+1}A^\bullet \longrightarrow F^pA^\bullet/F^{p+2}A^\bullet \longrightarrow F^pA^\bullet/F^{p+1}A^\bullet \longrightarrow 0.$$

Theorem 11.2 (Main result)**.** *We have $E_1^{p,q} \Rightarrow H^*(A^\bullet)$, meaning:*

1. *for all p, q, $E_r^{p,q}$ stabilizes for $r \gg 0$ to $E_\infty^{p,q}$,*
2. *there is a canonical isomorphism $F^pH^{p+q}(A^\bullet)/F^{p+1}H^{p+q}(A^\bullet) \cong E_\infty^{p,q}$.*

11.1.1. *Hypercohomology.* Let $\mathcal{C}$, $\mathcal{D}$ be two abelian categories and assume that $\mathcal{C}$ has enough injectives. Let $T\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor.

If $A^\bullet$ is a complex in $\mathcal{C}$, bounded to the left, there is a complex $I^\bullet$, consisting of injective objects, bounded to the left, and a morphism of complexes $A^\bullet \to I^\bullet$, which is a quasi-isomorphism. Define

$$R^qT(A^\bullet) = \mathcal{H}^q(T(I^\bullet)) \in \mathrm{Ob}(\mathcal{D}).$$

This can be made functorial.

Suppose now we have a filtration $(F^pA^\bullet)_{p\in\mathbb{Z}}$ on $A^\bullet$ as before. One can construct $A^\bullet \to I^\bullet$, as before, such that we have a decreasing filtration $F^pI^\bullet$ of $I^\bullet$ with each F^pI^m/F^pI^{m+1} (hence also F^pT^m) injective, and for all p:

$$F^pA^\bullet \to F^pI^\bullet \qquad \text{is a quasi-isomorphism,}$$

$$F^pA^\bullet/F^{p+q}A^\bullet \to F^pI^\bullet/F^{p+1}I^\bullet \qquad \text{is a quasi-isomorphism.}$$

Applying T to $I^\bullet$, we get a filtration on $T(I^\bullet)$ by setting

$$F^pT(I^\bullet) = (T(F^pI^\bullet) \hookrightarrow T(I^\bullet)).$$

Hence we get a spectral sequence with

$$E_1^{p,q} = H^{p+q}(\underbrace{T(F^pI^\bullet)/T(F^{p+1}I^\bullet)}_{T(F^pI^\bullet/F^{p+1}I^\bullet)}) = R^{p+q}T(F^pA^\bullet/F^{p+1}A^\bullet)$$

and

$$E_1^{p,q} \Rightarrow R^{p+q}T(A^\bullet).$$

Example 11.3 (Key example)**.** Consider $A^\bullet$ with the *naïve filtration*[7], i.e.

$$F^pA^\bullet = \overset{p-1}{0} \to \overset{p}{A^p} \to \overset{p+1}{A^{p+1}} \to \cdots .$$

Then

$$F^pA^\bullet/F^{p+1}A^\bullet = A^p[-p].$$

This gives the *first hypercohomology spectral sequence*:

$$E_1^{p,q} = R^qT(A^p) \Rightarrow R^{p+q}T(A^\bullet).$$

Remark 11.4. We used the notation $C^\bullet[m]$ to mean that $(C^\bullet[m])^i = C^{i+m}$.

11.2. **De Rham cohomology.** Let $k = \overline{k}$ be a field and X be a smooth algebraic variety over k. We have the *de Rham complex*:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \longrightarrow 0$$

where $n = \dim X$. Note that Ω_X^p is coherent for all i, but d is not $\mathcal{O}_X$-linear. Here, $\Omega_X^i = \bigwedge^i \Omega_X$.

Definition 11.5. The *de Rham cohomology* of X is

$$H^i_{\mathrm{dR}}(X) = \mathbb{H}^i(X, \Omega_X^\bullet) = R^i\Gamma(X, \Omega_X^\bullet).$$

Theorem 11.6 (Grothendieck)**.** *If X is a complete smooth variety over $\mathbb{C}$, there is a canonical isomorphism*

$$H^i_{\mathrm{dR}}(X) \cong H^i(X^{\mathrm{an}}, \mathbb{C}).$$

Remark 11.7. This result also holds if X is not complete (this is a result due to Deligne, which has a different proof).

Proof of Theorem 11.6. Consider the corresponding analytic de Rham complex

$$0 \longrightarrow \mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{d} \Omega_{X^{\mathrm{an}}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X^{\mathrm{an}}} \longrightarrow 0$$

obtain via the analytification functor from $\Omega_X^\bullet$.

The key fact is that this is acyclic with 0-cohomology equal to $\underline{\mathbb{C}}_{X^{\mathrm{an}}}$.

Indeed, recall that we have a double complex $(\mathcal{A}^{\bullet,\bullet}_{X^{\mathrm{an}}}, \partial, \overline{\partial})$ and the $\partial\overline{\partial}$-Lemma 7.16 implies that

$$(\mathcal{A}^{p,\bullet}_{X^{\mathrm{an}}}, \overline{\partial}) \text{ is an acyclic complex with 0-cohomology } \Omega^p_{X^{\mathrm{an}}}.$$

[7]The *naïve* filtration is sometimes also called the *stupid* filtration

Therefore, the inclusion

$$\Omega^\bullet_{X^{\mathrm{an}}} \hookrightarrow \mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}_{X^{\mathrm{an}}})$$

is a quasi-isomorphism, and

$$\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}_{X^{\mathrm{an}}}) = \text{de Rham complex on } X^{\mathrm{an}} \text{ of smooth } \mathbb{C}\text{-forms.}$$

It is acyclic with 0-cohomology $\underline{\mathbb{C}}_{X^{\mathrm{an}}}$.

Therefore,

$$\mathbb{H}^i(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) \cong H^i(X, \underline{\mathbb{C}}_{X^{\mathrm{an}}}) = H^i(X^{\mathrm{an}}, \mathbb{C}).$$

Hence it is enough to show that the canonical map

$$\mathbb{H}^i(X, \Omega^\bullet_X) \to \mathbb{H}^i(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$$

is an isomorphism. GAGA 3.34 would imply this result if the map d was $\mathcal{O}_X$-linear (i.e. if it was a map of coherent sheaves). We just need a small argument to deal with this technical issue.

The hypercohomology spectral sequence for $\Omega^\bullet_X$ gives

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{dR}}(X).$$

We also have a corresponding spectral sequence for $\Omega^\bullet_{X^{\mathrm{an}}}$:

$$\widetilde{E}_1^{p,q} = H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}) \Rightarrow \mathbb{H}^{p,q}(X, \Omega^\bullet_{X^{\mathrm{an}}}).$$

Functoriality of the hypercohomology spectral sequence then gives a morphism of spectral sequences

$$E_r^{p,q} \to \widetilde{E}_r^{p,q}.$$

This is an isomorphism for $r = 1$ by GAGA 3.34, which implies that it is an isomorphism for all r. By convergence of the spectral sequences, we see that for all p, q

$$F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_X)/F^{p+1}\mathbb{H}^{p+q}(X, \Omega^\bullet_X) \xrightarrow{\cong} F^p\mathbb{H}^{p+q}(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})/F^{p+1}\mathbb{H}^{p+q}(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}).$$

Since these are finite filtrations, the induced map

$$\mathbb{H}^i(X, \Omega^\bullet_X) \to \mathbb{H}^i(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$$

is an isomorphism. □

Remark 11.8. To prove the more general statement, we first embed the variety in a complete variety. One then shows that the log-de Rham complex gives a resolution of the constant sheaf. Then one shows that the log-de Rham complex gives an isomorphism as above.

11.3. **Hodge-to-de Rham spectral sequence.** Suppose k is an algebraically closed field and X is a smooth projective variety over k. We have the algebraic de Rham complex $\Omega^\bullet_X$ on X. Then *naïve filtration* on $\Omega^\bullet_X$ gives the *Hodge-to-de Rham spectral sequence*:

$$E_1^{p,q} = H^q(X, \Omega^p_X) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_X).$$

Theorem 11.9. *If* $\mathrm{char}(k) = 0$*, the spectral sequence collapses on its first page.*

Proof. It is enough to show this when $k = \mathbb{C}$. Theorem 11.6 gave an isomorphism

$$\mathbb{H}^m(X, \Omega_X^\bullet) \cong H^m(X^{\mathrm{an}}, \mathbb{C}).$$

We want to show that $d_r^{p,q} = 0$ for all p, q and all $r \geq 1$. In general, $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ and $E_{r+1}^{p,q}$ for all p, q if and only if $d_r^{p,q} = 0$ for all p, q. Hence the spectral sequence collapses on the first page if and only if

$$\dim F^p\mathbb{H}^n / F^{p+1}\mathbb{H}^n = \dim E_1^{p,n-p}$$

for all p. This is equivalent to:

$$\dim \mathbb{H}^n(X, \Omega_X^\bullet) = \sum_{p+q=n} \underbrace{\dim E_1^{p,q}}_{h^{p,q}(X)}.$$

Since $\dim \mathbb{H}^n(X, \Omega_X^\bullet) = \dim H^n(X^{\mathrm{an}}, \mathbb{C})$, this follows from the Hodge decomposition. $\square$

Remark 11.10. The naïve filtration on $\Omega_X^\bullet$ gives a filtration on $\mathbb{H}^*(X, \Omega_X^\bullet)$ which via the identification $\mathbb{H}^*(X, \Omega_X^\bullet) \cong H^*(X^{\mathrm{an}}, \mathbb{C})$ is the **Hodge filtration**.

Step 1. It is enough to prove the corresponding statement for $\Omega_{X^{\mathrm{an}}}^\bullet$. One checks this by similar argument to the proof of Theorem 11.6.

Step 2. We saw that the inclusion $\Omega_{X^{\mathrm{an}}}^\bullet \hookrightarrow \mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet$ is a quasi-isomorphism. Moreover,

$$F^p\Omega_{X^{\mathrm{an}}}^\bullet \hookrightarrow F^p\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet = \bigoplus_{\substack{i+j=\bullet \\ i \geq p}} \mathcal{A}^{i,j}$$

is a quasi-isomorphism again, by the $\partial\overline{\partial}$-Lemma 7.16. Since these are soft sheaves, we have that

$$F^p\mathbb{H}^*(X^{\mathrm{an}}, \Omega_{X^{\mathrm{an}}}^\bullet) = \mathrm{Im}(\mathcal{H}^\bullet\Gamma(F^p\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet) \to \mathcal{H}^\bullet\Gamma(\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet)).$$

Choose a Kähler metric on X^{an} so that we have harmonic forms. We have:

$$\begin{array}{ccc}
\mathcal{H}^m\Gamma(F^p\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet) & \longrightarrow & \mathcal{H}^m\Gamma(\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet) \\
\text{natural map} \nwarrow & & \nearrow \text{isom. onto } F^p(\cdots) \text{ by the Hodge decomposition} \\
& \bigoplus_{\substack{i+j=m \\ i \geq p}} \mathcal{H}^{i,j}(X^{\mathrm{an}}) &
\end{array}$$

Hence it is enough to show that the natural map in the diagonal is an isomorphism.

We argue by decreasing induction on p. We write

$$\mathcal{C}_p = \Gamma(F^p\mathcal{A}_{X^{\mathrm{an}},\mathbb{C}}^\bullet).$$

We have a short exact sequence

$$0 \longrightarrow \mathcal{C}_{p+1} \longrightarrow \mathcal{C}_p \longrightarrow \mathcal{C}_p/\mathcal{C}_{p+1} = \Gamma(X^{\mathrm{an}}, \mathcal{A}_{X^{\mathrm{an}}}^{p,\bullet}) \longrightarrow 0.$$

The long exact sequence gives

$$\begin{array}{ccccccccc}
(*) & \xrightarrow{\delta} & \mathcal{H}^m(\mathcal{C}_{p+1}) & \longrightarrow & \mathcal{H}(\mathcal{C}_p) & \longrightarrow & \mathcal{H}^m\Gamma(X^{\mathrm{an}}, \mathcal{A}^{p,\bullet}_{X^{\mathrm{an}}}) & \xrightarrow{\delta} & \mathcal{H}^{m+1}(\mathcal{C}_{p+1}) \\
 & & \uparrow{\scriptstyle \varphi_{p+1}} & & \uparrow{\scriptstyle \varphi_p} & & \cong \uparrow \text{Hodge decomposition} & & \\
0 & \longrightarrow & \bigoplus_{\substack{i+j=m \\ i\geq p}} \mathcal{H}^{i,j} & \longrightarrow & \bigoplus_{\substack{i+j=m \\ i\geq p+1}} \mathcal{H}^{i,j} & \longrightarrow & \mathcal{H}^{p,m-p} & \longrightarrow & 0
\end{array}$$

Note that $\delta = 0$, since we can lift a harmonic representation by a harmonic form, which is both ∂ and $\overline{\partial}$ closed.

Hence we have a morphism of short exact sequences. Since φ_{p+1} is an isomorphism by the inductive hypothesis, φ_p is also an isomorphism. This completes the proof of the assertion.

Remark 11.11. One can ask whether the degeneration of the Hodge-to-de Rham spectral sequence also holds in characteristic $p > 0$. It turns out it is false.

However, Deligne and Illusie give an algebraic proof in characteristic 0 by reducing modulo p.

Alas, this degeneracy of the spectral sequence **does not** imply the Hodge decomposition, i.e. that $H^m(X, \mathbb{C}) = F^p \oplus \overline{F}^{m+1-p}$. The Hodge symmetry does not follow from this either.

12. Introduction to Variations of Hodge structures

The motivation for this topic comes from the following result in differential topology.

Theorem 12.1 (Ehresman)**.** *If $\pi\colon X \to B$ is a smooth map between smooth (real) manifolds which is proper and submersive, then for all $b_0 \in B$, there is an open neighborhood U of b_0 and a diffeomorphism $\pi^{-1}(U) \cong U \times X_{b_0}$ over U.*

Proof. Consider

$$\begin{array}{ccc}
X_0 = X_{b_0} & \hookrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\{b_0\} & \hookrightarrow & B
\end{array}$$

Let $n = \dim X$, $m = \dim B$. Since π is a submersion, $X_0 \hookrightarrow X$ is a submanifold of dimension $n - m$.

The Tubular Neighborhood Theorem implies that there is an open neighborhood W of X_0 and a retraction $r\colon W \to X_0$ of the inclusion $X_0 \hookrightarrow W$.

Define $\varphi\colon W \to B \times X_0$ by

$$\varphi(x) = (\pi(x), r(x)).$$

If $p \in X_0$, $\ker(d\pi)_p = \mathrm{T}_p X_0$ and $(dr)_p|_{\mathrm{T}_p X_0} = \mathrm{Id}$.

Hence $d\varphi_p$ is an isomorphism, showing that φ is a local diffeomorphism at every point of X_0. Moreover, $\varphi|_{X_0}$ is injective and X_0 is compact since π is proper.

Therefore, there is an open neighborhood W' of X_0 such that φ is injective and locally a diffeomorphism on W'. Hence φ is an open embedding on W'.

Hence there is an open neighborhood U' of b_0 such that $\pi^{-1}(U') \subseteq \varphi(W')$.

Replacing B by U' an W' by $W' \cap \pi^{-1}(U')$, we may assume that φ is surjective. Take $U = B \setminus \pi(X \setminus W')$. Note that $\pi(X \setminus W')$ is closed because π is proper. Hence U is an open neighborhood of b_0 such that φ gives a diffeomorphism $\pi^{-1}(U) \cong U \times X_{b_0}$. □

Corollary 12.2. *Let $f\colon X \to B$ be a smooth projective morphism of complex algebraic varieties. If B is connected, then maps $B \ni t \mapsto h^{p,q}(X_t)$ are constant.*

Proof. By Ehresman's Theorem 12.1, we immediately see that the maps $B \ni t \mapsto b_i(X_t)$ are locally constant, hence constant, since B is connected.

Note that if $B' \to B$ is a resolution of singularities, we may replace f by f' where

$$\begin{array}{ccc} X' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

and thus we may assume that B (and hence also X) is smooth.

Each fiber is smooth and projective, so the Hodge decomposition shows that

$$b^i(X_t) = \sum_{p+q=i} h^{p,q}(X_t).$$

We will apply the Semicontinuity Theorem to $\Omega^p_{X/B}$. These are locally free $\mathcal{O}_X$-modules and f is flat, so they are flat over B. Therefore, the functions

$$t \mapsto h^{p,q}(X_t)$$

is upper-semicontinuous (can only go up under specialization). Since the sums of $h^{p,q}$ for $p+q$ constant are constant, each of these functions is constant. □

Remark 12.3. We get more by using Grauert's Theorem. If B is reduced, since all $h^q(X_t, \Omega^p_{X_t})$ are constant, all $R^q f_* \Omega^p_{X/B}$ are locally free and commute with base change.

In particular, for all $t \in B$, the canonical map

$$(R^q f_* \Omega^p_{X/B})_{(t)} \to H^q(X_t, \Omega^p_{X_t})$$

is an isomorphism.

Definition 12.4. Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. Then $\mathcal{F}$ is *locally constant* if for all $x \in X$, there is an open neighborhood U of x such that $\mathcal{F}|_U \cong \underline{A}_U$ for some abelian group A.

In particular, note that $A \cong \mathcal{F}_x$ for all x.

Definition 12.5. A sheaf $\mathcal{F}$ of vector spaces (over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$) which is locally constant with finite-dimensional stalks over a field is a *local system.*

Corollary 12.6. *Let $f\colon X \to B$ be a smooth map between smooth real manifolds. If f is a proper submersion, then $R^* f_* \underline{A}_x$ is locally constant for all abelian groups A.*

Proof. For $x \in B$, let $U \ni x$ be an open neighborhood such that $\pi^{-1}(U) \cong U \times F$ over U (as in the conclusion of Ehresman's Theorem 12.1.) Then $R^k f_* \underline{A}_x|_U$ is isomorphic to the sheaf associated to the presheaf $V \mapsto H^k(V \times F, \underline{A})$.

By taking V to be a basis of contracting open neighborhoods of x, we see that

$$R^k f_* \underline{A}_x|_U \cong \underline{H^k(F, A)}_U.$$

This completes the proof. □

12.1. **Overview of Riemann–Hilbert correspondence.** Let X be a complex manifold and $\mathcal{E}$ be a locally free sheaf on X.

Definition 12.7. A *connection* ∇ on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla\colon \mathcal{E} \to \Omega_X \otimes \mathcal{E}$ which satisfies the *Leibnitz rule*:

$$\nabla(fs) = f\nabla(s) + df \otimes s$$

if f is a local section of $\mathcal{O}_X$ and s is a local section of $\mathcal{E}$.

Given any connection ∇, we get induced $\mathbb{C}$-linear maps

$$\nabla\colon \Omega_X^p \otimes \mathcal{E} \to \Omega_X^{p+1} \otimes \mathcal{E}$$

by

$$\nabla(\eta \otimes s) = \eta \wedge \nabla(s) + d\eta \otimes s.$$

Definition 12.8. A connected ∇ is *flat* (or *integrable*) if $\nabla \circ \nabla = 0$ as a map $\mathcal{E} \to \Omega_X^2 \otimes \mathcal{E}$. Equivalently, $\nabla \circ \nabla = 0$ as a map $\Omega^p \otimes \mathcal{E} \to \Omega^{p+2} \otimes \mathcal{E}$ for all p.

The sheaf of *flat sections* of ∇ is $\mathcal{E}^\nabla = \ker(\nabla\colon \mathcal{E} \to \Omega^1 \otimes \mathcal{E})$.

Hence, given such a vector bundle with flat connection $(\mathcal{E}, \nabla)$, we get the *de Rham complex* $\mathrm{DR}_x(\mathcal{E}, \nabla)$:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega_X \otimes \mathcal{E} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^{\dim X} \otimes \mathcal{E} \longrightarrow 0.$$

Example 12.9 (Basic example)**.** If $\mathcal{E} = \mathcal{O}_X$ and we take $d\colon \mathcal{O}_X \to \mathcal{O}_X$ as the flat connection, the corresponding de Rham complex is the holomorphic de Rham complex of X.

More generally, suppose $\mathcal{L}$ is a local system of $\mathbb{C}$-vector spaces on X. Take $\mathcal{E} = \mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$ which is locally free and

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \Omega_X \otimes \mathcal{E} \\ \Big\downarrow{=} & & \Big\downarrow{=} \\ \mathcal{L} \otimes \mathcal{O}_X & \xrightarrow{1 \otimes d} & \mathcal{L} \otimes \Omega_X \end{array}$$

defines a flat connection on $\mathcal{E}$.

Then corresponding de Rham complex is

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}_X \longrightarrow \mathcal{L} \otimes \Omega_X \longrightarrow \cdots \longrightarrow \mathcal{L} \otimes \Omega_X^{\dim X} \longrightarrow 0.$$

This is quasi-isomorphic to $\mathcal{L}$.

Note that $\mathcal{L} = \mathcal{E}^\nabla = \ker(\nabla\colon \mathcal{E} \to \Omega^1 \otimes \mathcal{E})$.

Theorem 12.10 (The Riemann–Hilbert correspondence)**.** *There is an equivalence of categories*

$$\left\{\begin{matrix}\text{local systems of}\\ \mathbb{C}\text{-vector spaces}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{vector bundles with}\\ \text{integrable connection}\end{matrix}\right\}$$

with inverse given by $(\mathcal{E}, \nabla) \to \mathcal{E}^{\nabla}$.

Idea of proof. To show that $\mathcal{E}^{\nabla}$ is a local system, we need to show that for all $x \in M$ and $u \in \mathcal{E}_{(x)}$, locally there is a unique flat section s of $\mathcal{E}$ such that $s(x) = u$.

This is a local statement, so we may assume that we have section $e_1, \ldots, e_r$ which trivialize $\mathcal{E}$ and coordinates $x_1, \ldots, x_n$ on X. Then

$$\begin{aligned} \nabla\colon \mathcal{E} &\to \Omega \otimes \mathcal{E}, \\ e_j &\mapsto \sum_{i=1}^{n} \sum_{k=1}^{r} \Gamma_{ij}^{k}\, dx_i \otimes e_k. \end{aligned}$$

where Γ_{ij}^k are some functions, called *Chrstoffel coefficients*.

For a section $s = \sum_{j=1}^{r} s_j e_j$,

$$\nabla(s) = \sum_{j=1}^{r} s_j \sum_{i=1}^{n} \sum_{k=1}^{r} \Gamma_{ij}^{k}\, dx_i \otimes e_k + \sum_{k=1}^{r} \sum_{i=1}^{n} \frac{\partial s_k}{\partial x_i} dx_i \otimes e_k.$$

Then s is flat if and only if

$$\frac{\partial s_k}{\partial x_i} = -\sum_{j=1}^{r} \Gamma_{ij}^{k} s_j$$

for all i, k.

The key point is that ∇ is integrable if and only if this system of linear PDEs is integrable. In this case, we get (local) existence and uniqueness given an initial condition. □

The next two classes were typed by David Schwein.

Let $\pi : X \to B$ be a smooth projective morphism of smooth complex varieties. We saw that $\mathrm{R}^k\pi_*\underline{\mathbb{C}}$ is a local system with stalks at b given by

$$(\mathrm{R}^k\pi_*\underline{\mathbb{C}})_b \cong \mathrm{H}^k(X_b, \mathbb{C}).$$

By the Riemann–Hilbert correspondence 12.10, this local system corresponds to an analytic vector bundle with integrable connection, called the *Gauss–Manin connection*.

Our first goal is to describe the vector bundle supporting the Gauss–Manin connection. This requires a relative version of the de Rham complex. On X we have $\Omega_X^{\bullet}$, but also $\Omega_{X/B}^{\bullet}$ (with $\Omega_{X/B}^p := \bigwedge^p \Omega_{X/B}$). Neither complex is $\mathcal{O}_X$-linear, but the complex $\Omega_{X/B}^{\bullet}$ is $\pi^{-1}\mathcal{O}_B$-linear, that is, each $\Omega_{X/B}^p$ is a $\pi^{-1}\mathcal{O}_B$-module. Let

$$\mathcal{H}^k := \mathrm{R}^k\pi_*(\Omega_{X/B}^{\bullet}),$$

an $\mathcal{O}_B$-module. The naive filtration $F^\bullet\Omega^\bullet_{X/B}$ induces a filtration on $\mathcal{H}^k$:

$$F^p\mathcal{H}^k = \mathrm{Im}(\mathrm{R}^k\pi_*F^p\Omega^\bullet_{X/B} \to \mathrm{R}^k\pi_*\Omega^\bullet_{X/B}).$$

This filtration gives rise to a spectral sequence

$$E_1^{p,q} = \mathrm{R}^q\pi_*\Omega^p_{X/B} \implies \mathcal{H}^{p+q}.$$

We know that each $E_1^{p,q}$ is locally free and there is a canonical base-change isomorphism $(E_1^{p,q})_{(b)} \simeq \mathrm{H}^q(X_b, \Omega^p_{X_b})$. (Recall that $(\Omega_{X/B})_{(b)} \simeq \Omega_{X_b}$; this is a consequence of a general compatibility between sheaves of differentials and pullbacks.) We know that for every $b \in B$, the Hodge-to-de-Rham spectral sequence degenerates on the first page (cf. Theorem 11.9); hence for every $b \in B$, the spectral sequence $(E_\bullet)_{(b)}$ degenerates at level one. Hence the original spectral sequence degenerates on the first page, and the graded pieces are

$$F^p\mathcal{H}^k/F^{p+1}\mathcal{H}^k \simeq \mathrm{R}^{k-p}\pi_*\Omega^p_{X/B}.$$

We deduce that each $F^p\mathcal{H}^k$ is a subbundle of $\mathcal{H}^k$. Moreover, for every $b \in B$,

$$F^p\mathcal{H}^k_{(b)} = F^p\mathbb{H}^k(\Omega^\bullet_{X_b}) \simeq F^p\mathrm{H}^k(X_b, \mathbb{C}).$$

Remark 12.11. We can run similar arguments for $\mathrm{R}^k\pi_*\Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}}$. Since $\Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}} = (\Omega^\bullet_{X/B})^{\mathrm{an}}$ there is a canonical map

$$(\mathcal{H}^k)^{\mathrm{an}} \to \mathrm{R}^k\pi_*\Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}}$$

and this maps turns out to be an isomorphism. Using the spectral sequence associated to the second naive filtration, which degenerates on the first page, to prove that this is an isomorphism it is enough to show that for all p and q, the map

$$(\mathrm{R}^q\pi_*\Omega^p_{X/B})^{\mathrm{an}} \to \mathrm{R}^q\pi_*\Omega^p_{X^{\mathrm{an}}/B^{\mathrm{an}}}$$

is an isomorphism. This is a consequence of a general fact due to Deligne, relative GAGA Theorem, but in our setting, we can see it by taking fibers at each b and using the isomorphism

$$\mathrm{H}^p(X_b, \Omega_{X_b}) \to \mathrm{H}^p(X_b^{\mathrm{an}}, \Omega^{\mathrm{an}}_{X_b})$$

from usual GAGA Theorem 3.34.

Note now that we have a canonical morphism $\underline{\mathbb{C}}_X \to \Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}}$, which gives rise to a morphism $\mathrm{R}^k\pi_*\underline{\mathbb{C}}_X \to \mathrm{R}^k\pi_*(\Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}})$, and then a morphism

$$(\mathrm{R}^k\pi_*\underline{\mathbb{C}}_X) \otimes_{\mathbb{C}} \mathcal{O}_{B^{\mathrm{an}}} \to \mathrm{R}^k\pi_*(\Omega^\bullet_{X^{\mathrm{an}}/B^{\mathrm{an}}}).$$

This morphism is an isomorphism because it is an isomorphism on fibers. To conclude, the analytic vector bundle associated to $\mathrm{R}^k\pi_*\underline{\mathbb{C}}_X$ is $(\mathcal{H}^k)^{\mathrm{an}}$.

12.2. **The Gauss–Manin Connection.** As we will see, the connection on $(\mathcal{H}^k)^{\mathrm{an}}$ comes from a connection $\nabla : \mathcal{H}^k \to \Omega_B \otimes \mathcal{H}^k$ on $\mathcal{H}^k$. Its main property is the following.

Theorem 12.12 (Griffiths Transversality). *We have:*

$$\nabla(F^p\mathcal{H}^k) \subseteq \Omega_B \otimes F^{p-1}\mathcal{H}^k.$$

Therefore ∇ induces a map $\overline{\nabla} : F^p\mathcal{H}^k/F^{p+1}\mathcal{H}^k \to \Omega_B \otimes F^{p-1}\mathcal{H}_k/F^p\mathcal{H}^k$. Although ∇ is not $\mathcal{O}_X$-linear, $\overline{\nabla}$ is $\mathcal{O}_B$-linear because

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s,$$

so that $\overline{\nabla}(f\bar{s}) = f\overline{\nabla}(\bar{s})$. Our goal is to describe $\overline{\nabla}$ at the level of fibers. For $b \in B$, we have a map

$$\overline{\nabla}_b : \mathrm{H}^q(X_b, \Omega^p_{X_b}) \to T^*_b B \otimes \mathrm{H}^{q+1}(X_b, \Omega^{p-1}_{X_b}), \qquad q = k - p.$$

Theorem 12.13 (Griffiths)**.** *For every $b \in B$ and $u \in \mathrm{T}_b B$, the map $\mathrm{H}^q(X_b, \Omega^p_{X_b}) \to \mathrm{H}^{q+1}(X_b, \Omega^{p-1}_{X_b})$ induced by $\overline{\nabla}_b$ and u is the cup product with the Kodaira–Spencer class of u, a certain element of $\mathrm{H}^1(X, \mathrm{T}_X)$.*

That is, cupping with the Kodaira–Spencer class gives a map $\mathrm{H}^q(\Omega^p_{X_b}) \to \mathrm{H}^{q+1}(\Omega^p_{X_b} \otimes \mathrm{T}_{X_b})$, and we then compose this map with the contraction map $\mathrm{H}^{q+1}(\Omega^p_{X_b} \otimes \mathrm{T}_{X_b}) \to \mathrm{H}^{q+1}(\Omega^{p-q}_{X_b})$.

Recall that there is a short exact sequence

$$0 \to \mathrm{T}_{X_b} \to \mathrm{T}_X|_{X_b} \to \mathrm{N}_{X_b/X} (= \mathrm{T}_b B \otimes \mathcal{O}_{X_b}) \to 0$$

The long exact sequence in cohomology yields a map

$$T_b B = \mathrm{H}^0(N_{X_b/X}) \to \mathrm{H}^1(X_b, \mathrm{T}_{X_b}),$$

called the *Kodaira–Spencer map*. The significance of the map comes from deformation theory: for Y a smooth projective variety, the space $\mathrm{H}^1(Y, \mathrm{T}_Y)$ parameterizes order-one deformations of Y, that is, Cartesian diagrams

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow{\scriptstyle \text{flat}} \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{C}[\varepsilon] \end{array}$$

(Here $\varepsilon^2 = 1$.) Given $\pi : X \to B$ and $u \in \mathrm{T}_b B$, we get a deformation

$$\begin{array}{ccc} \widetilde{X}_b & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}\mathbb{C}[\varepsilon] & \xrightarrow{u} & B \end{array}$$

The image of u under the Kodaira–Spencer map corresponds to $\widetilde{X}_b$.

12.3. **Algebraic Description of the Gauss–Manin Connection (Katz–Oda).** Let $\pi\colon X \to B$ be as in the previous section. Recall the short exact sequence

$$0 \to \pi_*\Omega_B \to \Omega_X \to \Omega_{X/B} \to 0.$$

Consider $\Omega^\bullet_X$ with the filtration

$$G^i\Omega^p_X = \mathrm{Im}(\pi^*\Omega^i_B \otimes \Omega^{p-i}_X \to \Omega^p_X).$$

Evidently this filtration is compatible with the de Rham differential. Its graded pieces are

$$G^i\Omega^p_X/G^{i+1}\Omega^p_X \simeq \pi^*\Omega^i_B \otimes \Omega^{p-i}_{X/B}.$$

The filtration $(G^i\Omega^\bullet_X)_{i\geq 0}$ induces a spectral sequence with respect to π_*:

$$E_1^{p,q} = \mathrm{R}^{p+q}\pi_*(\underbrace{G^p\Omega^\bullet_X/G^{p+1}\Omega^\bullet_X}_{\pi^*\Omega^p_B\otimes\Omega^{\bullet-p}_{X/B}}) = \mathrm{R}^q\pi_*(\pi^*\Omega^p_B\otimes\Omega^\bullet_{X/B}) = \Omega^p_B\otimes\mathrm{R}^q\pi_*\Omega^\bullet_{X/B}.$$

In particular, we have a map $E_1^{0,q}\to E_1^{1,q}$, that is, $\mathrm{H}^q\to\Omega_B\otimes\mathcal{H}^q$. This is the Gauss–Manin connection.

Let's see how this description implies Griffiths Transversality 12.12. We need to understand $\mathrm{d}_1^{0,q}$: it is the boundary map in the long exact sequence associated to the short exact sequence

$$0\to G^1\Omega^\bullet_X/G^2\Omega^\bullet_X\to\Omega^\bullet_X/G^2\Omega^\bullet_X\to\Omega^\bullet_X/G^1\Omega^\bullet_X\to 0,$$

that is, the sequence

$$0\to\pi^*(\Omega_B)\otimes\Omega^{\bullet-1}_X\to\Omega^\bullet_X/G^2\Omega^\bullet_X\to\Omega^\bullet_{X/B}\to 0.$$

So $\mathrm{d}_1^{0,q}$ is the connection

$$\mathrm{R}^q\pi_*\Omega^\bullet_{X/B}\to\mathrm{R}^{q+1}\pi_*(\pi^*\Omega_B\otimes\Omega^{\bullet-1}_{X/B}) = \Omega_B\otimes\mathrm{R}^q\pi_*\Omega^\bullet_{X/B}.$$

Consider now the naive filtration

$$0\to\pi^*\Omega_B\otimes F^{p-1}\Omega^\bullet_X\to F^p(\Omega^\bullet_X/G^2\Omega^\bullet_X)\to F^p(\Omega^\bullet_{X/B})\to 0.$$

From the resulting morphism of short exact sequences we get a commutative diagram

$$\begin{array}{ccc}\mathrm{R}^q\pi_*F^p\Omega^\bullet_{X/B} & \longrightarrow & \Omega_B\otimes\mathrm{R}^q\pi_*F^{p-1}\Omega^\bullet_{X/B}\\ \downarrow & & \downarrow\\ \mathrm{R}^q\pi_*\Omega^\bullet_{X/B} & \xrightarrow{\ \nabla\ } & \Omega_B\otimes\mathrm{R}^q\pi_*\Omega^\bullet_{X/B}.\end{array}$$

12.4. **General Definition of Variation of Hodge Structures.**

Definition 12.14. An (analytic) *variation of* (rational) *Hodge structures* on a complex manifold B is given by the following data.

(1) A vector bundle with integrable connection $(\mathcal{E},\nabla)$.
(2) ($\mathbb{Q}$-structure) A local system $\mathcal{L}_\mathbb{Q}$ of $\mathbb{Q}$-vector spaces together with an isomorphism $(\mathcal{L}_\mathbb{Q})\otimes_\mathbb{Q}\mathbb{C}\simeq\mathcal{E}^\nabla$.
(3) (Hodge filtration) A finite decreasing filtration $F^\bullet\mathcal{E}$ on $\mathcal{E}$ by subbundles that satisfy Griffiths Transversality 12.12,

$$\nabla(F^p\mathcal{E})\subseteq\Omega\otimes F^{p-1}\mathcal{E},$$

and such that for all $b\in B$, the filtration $(F^p\mathcal{E}_{(b)})$ gives a Hodge structure on $\mathcal{L}_b$.

The variations of Hodge structure that come from smooth morphisms are called *geometric*. These are usually the easiest ones to work with.

12.5. **Period Maps.** Let $\pi\colon X \to B$ be as before. Fix $b \in B$. To understand the behavior near b, choose a contractible open neighborhood $U \ni b$ such that $\pi^{-1}(U) \simeq U \times X_b$ over U. Given any $b \in U$, we have canonical isomorphisms

$$\begin{array}{ccc} & \mathrm{H}^k(\pi^{-1}(U), \mathbb{C}) & \\ {}^{\text{restr}}\swarrow & & \searrow^{\text{restr}} \\ \mathrm{H}^k(X_b, \mathbb{C}) & \xrightarrow{\ \cong\ } & \mathrm{H}^k(X_{b_0}, \mathbb{C}). \end{array}$$

Fix p. For every $b \in U$, we get a linear subspace $F^p\mathrm{H}^k(X_b, \mathbb{C}) \subseteq \mathrm{H}^k(X_b, \mathbb{C}) \simeq \mathrm{H}^k(X_{b_0}, \mathbb{C})$. In this way, we get a period map

$$\mathcal{P}^{k,p} : U \to G(r, V)$$

where G is a Grassmannian (parameterizing subspaces), $V = \mathrm{H}^k(X_{b_0}, \mathbb{C})$, and $r = \sum h^{p',q'}$, the sum taken over those p' and q' with $p' + q' = k$ and $p' \geq p$.

Proposition 12.15. *The map $\mathcal{P}^{k,p}$ is holomorphic.*

Proof. We have a subbundle $F^p\mathcal{H}^k|_U \subseteq \mathcal{H}^k|_U = V \otimes \mathcal{O}_U$. This defines a holomorphic map $U \to G(r, V)$ which is precisely $\mathcal{P}^{k,p}$. □

To globalize the period map, one must take the quotient of the image in $G(r, V)$ by a certain discrete group arising from monodromy. The resulting map is extremely useful in studying and constructing moduli spaces.

Our final goal is to relate $\mathrm{d}\mathcal{P}^{k,p} : \mathrm{T}_b B \to \mathrm{T}_{[W]} G$ to the Gauss-Manin connection.

Let $[W] \in G(r, V) =: G$. We can think of the tangent space at $[W]$ as the collection of free $\mathbb{C}[\varepsilon]$-modules $\mathcal{W} \subseteq V \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ of rank r and with free cokernel such that $\mathcal{W}/\varepsilon\mathcal{W} = W$. Given such W, for each $w \in W$ there is $w + \varepsilon\widetilde{w} \in \mathcal{W}$. Defining $\varphi : W \to V/W$ by $w \mapsto \widetilde{w} \bmod W$ gives an isomorphism

$$\mathrm{T}_{[W]} G \simeq \mathrm{Hom}_{\mathbb{C}}(W, V/W).$$

Explicitly, choose a basis $e_1, \ldots, e_n$ for V and suppose we are in the chart of G with subspaces generated by vectors of the form $e_i + \sum_{j=r+1}^n a_{i,j} e_j$. The coefficients $a_{i,j}$ are independent from each other and give an isomorphism of this chart to affine space with coordinates $A_{i,j}$. Suppose that

$$W = \left\langle e_i + \sum_{j=r+1}^{n} a_{i,j} e_j \;\middle|\; i \right\rangle,$$

so that a tangent vector has the form

$$u = \sum_{i,j} b_{i,j} \frac{\partial}{\partial A_{i,j}}\bigg|_{[W]}.$$

The corresponding map $W \to V/W$ maps $e_i + \sum\limits_{j=r+1}^{n} a_{i,j} e_j$ to $\overline{\sum b_{i,j} e_j}$.

Consider now the sheaf of sections $\mathcal{T} \to V \otimes \mathcal{O}_G$ of the universal subbundle. Given $[W] \in G$, consider the trivial vector bundle $V \otimes \mathcal{O}_G$ with the connection $1 \otimes \mathrm{d}$. On our chart, $\mathcal{T}$ is generated by the sections

$$\sum_{j=r+1}^{n} e_i + A_{i,j} e_j.$$

Then

$$(1 \otimes \mathrm{d})(s_i) = \sum_{j=r+1}^{n} \mathrm{d}A_{i,j} \otimes e_j.$$

Letting $u \in \mathrm{T}_{[W]}G$ corresond to $\varphi \in \mathrm{Hom}(W, V/W)$, this calculation implies that

$$\overline{\nabla_u(s_i)} = \varphi(s_i([W]))$$

where the bar stands for the image in V/W. Using this, we see the following: given the period map $\mathcal{P}^{k,p} : U \to G$ associated to $\pi\colon X \to B$, for every $b \in U$ and every $u \in \mathrm{T}_b U$, if $[W] = \mathcal{P}^{k,p}(b)$ then $\mathrm{d}\mathcal{P}^{k,p}_b(u) = (\nabla|_{F^p\mathcal{H}^k})_{b,u}\colon W \to V/W$.

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