Motivic action conjectures

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Harder's 85th birthday: Motives and Automorphic forms
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MOTIVES AND AUTOMORPHIC FORMS

Langlands program.

$$\left\{\begin{matrix}\text{automorphic}\\ \text{forms/reps}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{Galois}\\ \text{representations}\end{matrix}\right\} \xleftarrow[\substack{\text{étale}\\ \text{real.}}]{} \{\text{motives}\}$$

$\underline{G = GL_2}$:

- f MF wt 2 $\longmapsto H^1_{ét}(X_0(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \longleftarrow$ elliptic curve $/\mathbb{Q}$
- f MF wt 1 $\longmapsto \rho : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell) \longleftarrow \rho : Gal(L/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}})$ odd Artin rep.

$F = \mathbb{Q}(\sqrt{d})$

$\underline{G = GL_{2,F}}$:

- f Hilbert MF, wt 2 $\longmapsto H^2_{ét}(X_0(n)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)_f \xleftarrow{\text{(roughly)}}$ elliptic curve $/F$
- f Hilbert MF, wt 1 $\longmapsto \rho : G_F \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell) \longleftarrow \rho : Gal(L/F) \to GL_2(\overline{\mathbb{Q}})$ odd Artin rep.

$K = \mathbb{Q}(\sqrt{-d})$

$\underline{G = GL_{2,K}}$:

- f Bianchi MF, wt 2 $\longmapsto \rho : G_F \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell) \longleftarrow$ elliptic curve $/K$

$\underline{G = GSp_4}$:

- f Siegel MF, wt 2 $\longmapsto \rho : G_{\mathbb{Q}} \longrightarrow GSp_4(\overline{\mathbb{Q}}_\ell) \longleftarrow$ abelian surface $/\mathbb{Q}$

ALGEBRAIC CYCLES & LANGLANDS PROGRAM

Broad question: How does the theory of algebraic cycles interact with the Langlands program?

Example. Motivic action conjectures. (See Kartik Prasanna's talk for a different example.)

$$\begin{Bmatrix}\text{automorphic}\\ \text{forms/reps}\end{Bmatrix} \quad \begin{matrix} f \longmapsto M(f) \\ \longrightarrow \end{matrix} \quad \{\text{motives } M\}$$

(Related to algebraic cycles)

Where does f occur in cohomology of locally symm. space?

$\delta :=$ defect

	$H_f^{i_{min}}$	$H_f^{i_{min}+1}$	$\cdots$	$H_f^{i_{min}+j}$	$\cdots$	$H_f^{i_{min}+\delta}$
dim	1	δ	$\cdots$	$\binom{\delta}{j}$	$\cdots$	1

coincidence??

Motivic cohomology

$H^1_{\mathcal{M}} := H^1_{\mathcal{M}}(\mathrm{Ad}M(f)_{\mathbb{Z}}, \mathbb{Q}(1))$

Dimension

$\delta = \dim_{\mathbb{Q}} H^1_{\mathcal{M}}$

$\dim \Lambda^j H^1_{\mathcal{M}} = \binom{\delta}{j}$

Today: Examples:

$G = GL_{2,\mathbb{Q}}$ (Harris-Venkatesh, H.)

$G = GL_{2,F}$ (H.)

any G, π tempered, cohomological

$G = GL_{2,k}$ (Prasanna-Venkatesh, Venkatesh, Galatius-Venkatesh)

$G = GSp_{4,\mathbb{Q}}$ (Oh, ongoing: H.-Prasanna)

EXAMPLE 1 : $G = GL_{2,\mathbb{Q}}$

$$\left\{\begin{array}{c}\text{automorphic}\\ \text{forms/reps}\end{array}\right\} \longrightarrow \{\text{motives } M\}$$

f MF wt 2 $\qquad$ elliptic curve/$\mathbb{Q}$

$\rightsquigarrow \delta = 0 \Rightarrow$ no motivic action

EXAMPLE 1 : $G = GL_{2,\mathbb{Q}}$

[Harris–Venkatesh '16]
[Horawa '22]

$\left\{\begin{array}{c}\text{automorphic}\\ \text{forms/reps}\end{array}\right\} \longrightarrow \{\text{motives } M\}$

f MF wt 1

$\rho : Gal(L/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}})$ odd Artin rep.

X = modular curve $/\mathbb{Q}$, ω = sheaf of wt 1

$\left.\begin{array}{l} - f \in H^0(X,\omega)_f \\ - \omega_f^\infty := f(-\bar{z}) \cdot y^{-1} d\bar{z} \in H^1(X_{\mathbb{C}},\omega)_f \end{array}\right\} \delta = 1$

Motivic cohomology

$H^1_{\mathcal{M}} \cong U_f$ Stark unit group

$U_f := \mathrm{Hom}_{Gal(L/\mathbb{Q})}(Ad^0\rho, \mathcal{O}_L^\times)$ rank 1 ✓

$Ad^0\rho$-isotypic component

Def. $U_f^\vee \otimes \mathbb{C} \ni c$ explicit element $\Rightarrow U_f^\vee \otimes \mathbb{C} \circlearrowright H^*(X_{\mathbb{C}},\omega)_f$ by

$$c * f = \omega_f^\infty$$

I.e. $H^*(X_{\mathbb{C}},\omega)_f$ = free $\Lambda^* U_f^\vee \otimes \mathbb{C}$ - module of rk 1.

Conj. This action descends to $\Lambda^* U_f^\vee \circlearrowright H^*(X,\omega)_f$.

Explicitly: $u_f^\vee \in U_f^\vee$ acts by
$$\begin{array}{ccc} H^0(X,\omega) & \longrightarrow & H^1(X,\omega) \\ & & \cup \\ f & \longmapsto & \dfrac{\omega_f^\infty}{\log|u_f|} \end{array}$$

Prop. This is equiv. to Stark's conjecture on $L(Ad^0\rho, 1)$.

Archimedean analog of mod p^n conjecture of Harris–Venkatesh ('16).

EXAMPLE 2 : $G = GL_{2,F}$ $\quad F = \mathbb{Q}(\sqrt{d})$ [Horawa '22]

$$\left\{\begin{matrix}\text{automorphic}\\ \text{forms/reps}\end{matrix}\right\} \longrightarrow \{\text{motives } M\}$$

f Hilbert modular form wt $(1,1)$ $\qquad$ $\rho : \mathrm{Gal}(L/F) \to GL_2(\overline{\mathbb{Q}})$ odd Artin rep.

$X = \begin{matrix}\text{Hilbert modular}\\ \text{surface}\end{matrix}/\mathbb{Q}$, ω = sheaf of wt $(1,1)$

- $f \in H^0(X, \omega)$
- $\omega_f^1, \omega_f^2 \in H^1(X_{\mathbb{C}}, \omega)$
- $\omega_f^{1,2} \in H^2(X_{\mathbb{C}}, \omega)$

$\delta = 2$

Motivic cohomology

$H^1_{\mathcal{M}} \cong \mathcal{U}_f$ Stark unit gp

$\mathcal{U}_f := \mathrm{Hom}_{\mathrm{Gal}(L/F)}(\mathrm{Ad}^0\rho, \mathcal{O}_L^\times)$ rank 2 ✓

$\mathrm{Ad}^0\rho$-isotypic component

Def. $\mathcal{U}_f^\vee \otimes \mathbb{C} \ni c_1, c_2$ explicit elements $\Rightarrow$ $\mathcal{U}_f^\vee \otimes \mathbb{C} \circlearrowright H^*(X_{\mathbb{C}}, \omega)_f$ by

$$c_1 * f = \omega_f^1 \quad , \quad c_2 * f = \omega_f^2$$

I.e. $H^*(X_{\mathbb{C}}, \omega)_f$ = free $\Lambda^* \mathcal{U}_f^\vee \otimes \mathbb{C}$ - module of rk 1.

Conj. (H). This action descends to $\Lambda^* \mathcal{U}_f^\vee \circlearrowright H^*(X, \omega)_f$.

Explicitly, $\exists$ units $u_{11}, u_{12}, u_{21}, u_{22} \in \mathcal{O}_L^\times$ s.t.

(1) $$\frac{-\log|u_{21}|\,\omega_f^1 + \log|u_{22}|\,\omega_f^2}{\log|u_{11}|\log|u_{22}| - \log|u_{12}|\log|u_{21}|} \in H^1(X, \omega)_f \quad (\text{\& another similar class})$$

(2) $$\frac{\omega_f^{1,2}}{\log|u_{11}|\log|u_{22}| - \log|u_{12}|\log|u_{21}|} \in H^2(X, \omega)_f .$$

Results (H).

(1) Numerical evidence in base change case.

(2) Equivalent to Stark conj. for $L(\mathrm{Ad}^0\rho, 1)$.

EXAMPLE 3 : $G = GL_{2,K}$ $\quad K = \mathbb{Q}(\sqrt{-d})$

[Prasanna-Venkatesh '21]
[Venkatesh '19]
[Galatius-Venkatesh '18]

$$\left\{\begin{matrix}\text{automorphic} \\ \text{forms/reps}\end{matrix}\right\} \longrightarrow \{\text{motives } M\}$$

f_K Bianchi modular form wt $(2,2)$

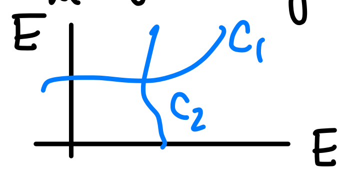
X_K = real threefold (Bianchi mod. threefold)

$$\left.\begin{array}{l} - \ \omega^1_{f_K} \in H^1_B(X_K, \mathbb{C})_{f_K} \\ - \ \omega^2_{f_K} \in H^2_B(X_K, \mathbb{C})_{f_K} \end{array}\right\} \delta = 1$$

E = elliptic curve / K

Motivic cohomology

$\alpha \in H^1_\mu$ given by ...



- $C_i \subseteq E \times E$
- φ_i = meromorphic on C_i
- $\sum \operatorname{div}(\varphi_i) = 0$

Def. $(H^1_\mu)^\vee \otimes \mathbb{C} \ni \delta^\vee$ explicit element, corresp. to $\delta^\vee := \omega^\sigma \otimes \overline{\omega^\sigma} + \overline{\omega^\sigma} \otimes \omega^\sigma$
$\omega^\sigma \in H^0(E^\sigma, \Omega^1_{E^\sigma})$

$\rightsquigarrow (H^1_\mu)^\vee \otimes \mathbb{C} \curvearrowright H^*_B(X, \mathbb{C})_{f_K}$, $\quad \delta^\vee : H^1_B(X_K, \mathbb{Q})_f \longrightarrow H^2_B(X_K, \mathbb{C})_f$
(rational class) $\longmapsto \omega^2_{f_K}$

Conj. (Prasanna-Venkatesh). $(H^1_\mu)^\vee \curvearrowright H^*_B(X_K, \mathbb{Q})_{f_K}$

Explicitly : for $\alpha \in H^1_\mu$ as above :

$$\frac{\omega^2_{f_K}}{\sum_i \int_{C_i(\mathbb{C})} \log|\varphi_i| \, \delta^\vee} \in H_B(X^2_K, \mathbb{Q})_{f_K}$$

Thm (Prasanna-Venkatesh). Equivalent to Beilinson's conjecture for $L(f, \mathrm{Ad}, 1)$.

[Tilouine-Urban : integral version in BC case ; $\Leftrightarrow$ Bloch-Kato.]

EXAMPLE 4: $G = GSp_4$

[Horawa–Prasanna, in progress]
[Ch, '22]

$$\left\{\begin{matrix}\text{automorphic}\\ \text{forms/reps}\end{matrix}\right\} \longrightarrow \left\{\text{motives } M\right\}$$

f Siegel modular form wt $(2,2)$

A = abelian surface / $\mathbb{Q}$

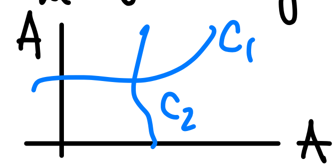
X = Siegel modular threefold / $\mathbb{Q}$, $\mathcal{E}_{2,2}$ sheaf

- $[f] \in H^0(X, \mathcal{E}_{2,2})_f$ "holomorphic", rational
- $[f^W] \in H^1(X_{\mathbb{C}}, \mathcal{E}_{2,2})_f$ "Whittaker"

$\delta = 1$

Motivic cohomology

$\alpha \in H^1_{\mathcal{M}}$ given by ...



- $C_i \subseteq A \times A$ divisors
- φ_i = meromorphic on C_i
- $\sum \operatorname{div}(\varphi_i) = 0$

Def. $(H^1_{\mathcal{M}})^\vee \otimes \mathbb{C} \ni \delta^\vee$ explicit element, corresp. to $(\omega_1 \otimes \overline{\omega_2} + \overline{\omega_2} \otimes \omega_1) - (\omega_2 \otimes \overline{\omega_1} + \overline{\omega_1} \otimes \omega_2)$, $\omega_1, \omega_2 \in H^0(A, \Omega^1)$ basis

$\rightsquigarrow (H^1_{\mathcal{M}})^\vee \otimes \mathbb{C} \circlearrowright H^*(X_{\mathbb{C}}, \mathcal{E}_{2,2})_f$ $\quad \delta^\vee : H^0(X, \mathcal{E}_{2,2})_f \longrightarrow H^1(X_{\mathbb{C}}, \mathcal{E}_{2,2})_f$, $\quad [f] \longmapsto [f^W]$

Conj. (H.–Prasanna). $(H^1_{\mathcal{M}})^\vee \circlearrowright H^*(X, \mathcal{E}_{2,2})_f$

Explicitly: for $\alpha \in H^1_{\mathcal{M}}$ as above:

$$\frac{[f^W]}{\sum_i \int_{C_i(\mathbb{C})} \log|\varphi_i| \, \delta^\vee} \in H^1(X, \mathcal{E}_{2,2})_f$$

Thm (H.–Prasanna).

(1) Equivalent to Beilinson for $L(f, \mathrm{Ad}, 1)$

(2) Our conj. $\Rightarrow$ [PV] for Bianchi m.f. f_K

$$\begin{array}{ccc} H^1(X_K, \mathbb{Q})_{f_K} & \xrightarrow{\theta_1} & H^0(X, \mathbb{Q})_f \\ {\scriptstyle [PV]}\downarrow & \circlearrowright & \downarrow{\scriptstyle [HP]} \\ H^2(X_K, \mathbb{Q})_{f_K} & \xrightarrow{\theta_2} & H^1(X, \mathbb{Q})_f \end{array}$$