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Motivic action on coherent cohomology of Hilbert modular varieties

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(Hilbert) modular forms

Modular form of level N, weight 1: holomorphic function $f\colon \mathcal{H} = \{z = x+iy \in \mathbb{C} \mid y > 0\} \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)f(z) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ with } c \equiv 0 \mod N.$$

Let F be a totally real degree n extension of $\mathbb{Q}$ and $\sigma_1, \dots, \sigma_n\colon F \hookrightarrow \mathbb{R}$ be the real embeddings. For $a \in F$, write $a_j = \sigma_j(a) \in \mathbb{R}$.

Hilbert modular form of level N, weight 1: holomorphic function $f\colon \mathcal{H}^n \to \mathbb{C}$ such that

$$f\left(\frac{a_1z_1+b_1}{c_1z_1+d_1}, \dots, \frac{a_nz_n+b_n}{c_nz_n+d_n}\right) = \prod_{j=1}^n (c_jz_j+d_j) f(z_1, \dots, z_n)$$

$$\text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \text{ with } c \equiv 0 \mod N.$$

Modular curve: $n = 1$.

$$f \in H^0(X_{\mathbb{Q}}, \omega)$$
$$\Big\downarrow u_f^\vee$$
$$\frac{\omega_f^\infty}{\log u_f} \overset{?}{\in} H^1(X_{\mathbb{Q}}, \omega)$$

Theoretical evidence

Theorem. *We have that*

$$\frac{\omega_f^{\sigma_1, \dots, \sigma_n}}{\prod_{j=1}^n \log(u_{f,j})} \in H^n(X_{\mathbb{Q}}, \omega).$$

Hence the action of top degree elements $u_{f,1}^\vee \wedge \cdots \wedge u_{f,n}^\vee \in \bigwedge^n U_f^\vee \otimes \mathbb{Q}$ is rational:

$$\begin{array}{ccc} H^0(X_{\mathbb{C}}, \omega) & \xrightarrow{u_1^\vee \wedge \cdots \wedge u_{f,n}^\vee} & H^n(X_{\mathbb{C}}, \omega) \\ \uparrow & & \uparrow \\ H^0(X_{\mathbb{Q}}, \omega) & \xrightarrow{u_{f,1}^\vee \wedge \cdots \wedge u_{f,n}^\vee} & H^n(X_{\mathbb{Q}}, \omega) \end{array}$$

Corollary. *The conjecture is true for modular curves ($n = 1$).*

For the sake of this poster, we have assumed that $f \in H^0(X_{\mathbb{Q}}, \omega)$. In general, there is a finite extension $E/\mathbb{Q}$ such that $f \in H^0(X_{\mathbb{Q}}, \omega) \otimes_{\mathbb{Q}} E$. In that case, the analogous result is conditional on Stark's conjecture.

Modular forms as sections of line bundles

A modular form f of weight 1 may be interpreted as a section of a line bundle ω over the modular curve $X_{\mathbb{C}} := \Gamma_0(N)\backslash\mathcal{H}$:

$$\gamma(z, \tau) = \left(\frac{az+b}{cz+d}, (cz+d)\tau\right) \qquad \omega := \Gamma_0(N)\backslash\mathcal{H} \times \mathbb{C} \qquad (z, f(z))$$
$$\downarrow \qquad\qquad \uparrow$$
$$\gamma z = \frac{az+b}{cz+d} \qquad X_{\mathbb{C}} := \Gamma_0(N)\backslash\mathcal{H} \qquad z$$

We write $f \in H^0(X_{\mathbb{C}}, \omega)$. Similarly, a Hilbert modular form is a section $f \in H^0(X_{\mathbb{C}}, \omega)$ of a line bundle ω over a Hilbert modular variety $X_{\mathbb{C}}$.

The Hilbert modular variety $X_{\mathbb{C}}$ has a model $X_{\mathbb{Q}}$ over $\mathbb{Q}$ and the line bundle ω is defined over $\mathbb{Q}$. We assume $f \in H^0(X_{\mathbb{Q}}, \omega)$.

The Main Conjecture

Partial complex conjugation. Given a Hilbert modular form f, we may consider the differential form

$$\omega_f^{\sigma_j} = f(z_1, \dots, -\overline{z_j}, \dots, z_n) y_j^{-1} (dz_j \wedge d\overline{z_j}) \in H^1(X_{\mathbb{C}}, \omega).$$

More generally, there are differential forms

$$\omega_f^J \in H^{|J|}(X_{\mathbb{C}}, \omega) \quad \text{for } J \subseteq \{\sigma_1, \dots, \sigma_n\}.$$

Question. Does a multiple of $\omega_f^J \in H^{|J|}(X_{\mathbb{C}}, \omega)$ belong to the rational structure $H^{|J|}(X_{\mathbb{Q}}, \omega)$? If so, what multiple?

Conjecture. *Recall that there is a decomposition $U_f \cong \bigoplus_{j=1}^n U_{f,j}$. For any $u_{f,j} \in U_{f,j}$, $j \in J$, we have that:*

$$\frac{\omega_f^J}{\prod_{j \in J} \log(u_{f,j})} \in H^{|J|}(X_{\mathbb{Q}}, \omega) \subseteq H^{|J|}(X_{\mathbb{C}}, \omega).$$

In particular, the exterior algebra $\bigwedge^* U_f^\vee$ acts on the space $H^*(X_{\mathbb{Q}}, \omega)$.

Relationship to other conjectures

Cohomology theory	Over $\mathbb{C}$	Over $\mathbb{Z}/p^n\mathbb{Z}$	Over $\mathbb{Q}_p$
Betti/étale cohomology	Prasanna–Venkatesh [PV16]	Venkatesh [Ven16]	Venkatesh [Ven16]
Coherent cohomology			
Modular curves	Horawa [Hor20]	Harris–Venkatesh [HV17]	???
Hilbert modular varieties	Horawa [Hor20]	Horawa [Hor20]	???

References

[Hor20] Aleksander Horawa, *Motivic action on coherent cohomology of Hilbert modular varieties*, 2020, arXiv:arXiv:2009.14400.

[HV17] Michael Harris and Akshay Venkatesh, *Derived Hecke algebra for weight one forms*, 2017, arXiv:arXiv:1706.03417.

[PV16] Kartik Prasanna and Akshay Venkatesh, *Automorphic cohomology, motivic cohomology, and the adjoint L-function*, 2016, arXiv:arXiv:1609.06370.

[Ven16] Akshay Venkatesh, *Derived hecke algebra and cohomology of arithmetic groups*, 2016, arXiv:arXiv:1608.07234.

Galois representations and motivic cohomology

According to the Langlands program, a Hilbert modular form f of weight 1 corresponds to an (odd) 2-dimensional **Artin representation**:

$$\varrho\colon \mathrm{Gal}(L/F) \to \mathrm{GL}(V),$$

where L/F is a Galois extension and V is a 2-dimensional vector space over $\mathbb{Q}$.

The space $\mathrm{Ad}^0 V = \{\varphi\colon V \to V \mid \mathrm{trace}(\varphi) = 0\}$ is the **trace 0 adjoint representation** of $\mathrm{Gal}(L/F)$ via the conjugation action on $\mathrm{End}(V)$.

Motivic cohomology group:

$$U_f := U_L[\mathrm{Ad}^0 V] = \mathrm{Hom}_{\mathrm{Gal}(L/F)}(\mathrm{Ad}^0 V, U_L \otimes \mathbb{Q}).$$

This is sometimes known as a *Stark unit group*.

Proposition. *The vector space U_f has dimension n and naturally decomposes into 1-dimensional subspaces $U_f \cong \bigoplus_{j=1}^n U_{f,j}$.*

Hilbert modular surface: $n = 2$.

$$f \in H^0(X_{\mathbb{Q}}, \omega)$$
$$\swarrow u_{f,1}^\vee \qquad u_{f,2}^\vee \searrow$$
$$\frac{\omega_f^{\sigma_1}}{\log(u_{f,1})} \overset{?}{\in} H^1(X_{\mathbb{Q}}, \omega) \qquad H^1(X_{\mathbb{Q}}, \omega) \overset{?}{\ni} \frac{\omega_f^{\sigma_2}}{\log(u_{f,2})}$$
$$\searrow u_{f,2}^\vee \qquad u_{f,1}^\vee \swarrow$$
$$\frac{\omega_f^{\sigma_1, \sigma_1}}{\log(u_{f,1})\log(u_{f,2})} \overset{?}{\in} H^2(X_{\mathbb{Q}}, \omega)$$

Numerical evidence

When $n = 2$, it remains to verify that

$$\frac{\omega_f^{\sigma_i}}{\log(u_{f,j})} \overset{?}{\in} H^1(X_{\mathbb{Q}}, \omega) \subseteq H^1(X_{\mathbb{C}}, \omega).$$

Idea. Consider the restriction to a modular curve $\iota\colon Y_{\mathbb{Q}} \hookrightarrow X_{\mathbb{Q}}$. It is then enough to check that:

$$\int_{Y(\mathbb{C})} f(-\overline{z}, z) y^{-1} dz \wedge d\overline{z} \overset{?}{=} c \cdot \log(u_{f,\sigma_1}) \qquad \text{for } c \in \mathbb{Q}.$$

We checked this numerically (up to 15 digits) in some cases:

F	level N	constant c	time taken
$\mathbb{Q}(\sqrt{5})$	23	2	00:09:34
$\mathbb{Q}(\sqrt{5})$	31	-4	00:13:36
$\mathbb{Q}(\sqrt{13})$	23	8	00:10:19
$\mathbb{Q}(\sqrt{13})$	31	-2	00:49:47