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A CLASS OF CODIMENSION-TWO FREE BOUNDARY PROBLEMS*

S. D. HOWISON†, J. D. MORGAN‡, AND J. R. OCKENDON†

Abstract. This review collates a wide variety of free boundary problems which are characterized by the uniform proximity of the free boundary to a prescribed surface. Such situations can often be approximated by mixed boundary value problems in which the boundary data switches across a "codimension-two" free boundary, namely, the edge of the region obtained by projecting the free boundary normally onto the prescribed surface. As in the parent problem, the codimension-two free boundary needs to be determined as well as the solution of the relevant field equations, but no systematic methodology has yet been proposed for nonlinear problems of this type. After presenting some examples to illustrate the surprising behavior that can sometimes occur, we discuss the relevance of traditional ideas from the theories of moving boundary problems, singular integral equations, variational inequalities, and stability. Finally, we point out the ways in which further refinement of these techniques is needed if a coherent theory is to emerge.

Key words. codimension-two free boundary problem

AMS subject classifications. 35R35, 76S05, 76B10

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1. Introduction. Several books and monographs [14, 20, 27, 54, 69, 70], proceedings [6, 10, 16, 22, 32, 51, 58, 61, 79], and bibliographies [17, 77] have appeared during the past thirty years on the mathematical theory of free boundary problems. Such problems are defined as differential equations that must be solved in domains of dimension n, some of whose boundaries, of dimension $(n-1)$, are unknown a priori. The inherent nonlinearity of these problems has prompted theoretical investigations into questions of existence, uniqueness, regularity of the boundary, numerical algorithms, stability, and asymptotic behavior. Certain mathematical techniques have emerged as widely applicable, including the use of weak solutions and scaling arguments. In cases where there is enough structure for weak or variational formulations to be found, great unification has been achieved both analytically and numerically. At the other extreme, when very irregular or unstable free boundary morphology can occur, much theoretical work remains to be done and justifiable numerical algorithms are only in their infancy.

Despite all this mathematical activity, there remains a widely-occurring, but little-studied, subclass of free boundary problems in which the free boundary has dimension $(n-2)$ and for which only two tentative attempts at unification have been made [39, 60]. Nonetheless, the number of case studies of this type that have appeared in the literature is now so large that it is appropriate to write a review with the aim of stimulating further mathematical study of these important problems. They form a subset of what is now known as "codimension-two free boundary problems," although this term is also commonly applied to models of, say, a curve in $\mathbb{R}^3$, as might be the case for vortex dynamics in a fluid or superconductor. In these latter cases, the motion of the curve is often governed, to lowest order, by a purely local partial differential

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†OCIAM, Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB, United Kingdom (howison@maths.ox.ac.uk, ejam@maths.ox.ac.uk).

‡Gas Research Centre, British Gas plc, Ashby Road, Loughborough LE11 3QU, United Kingdom.
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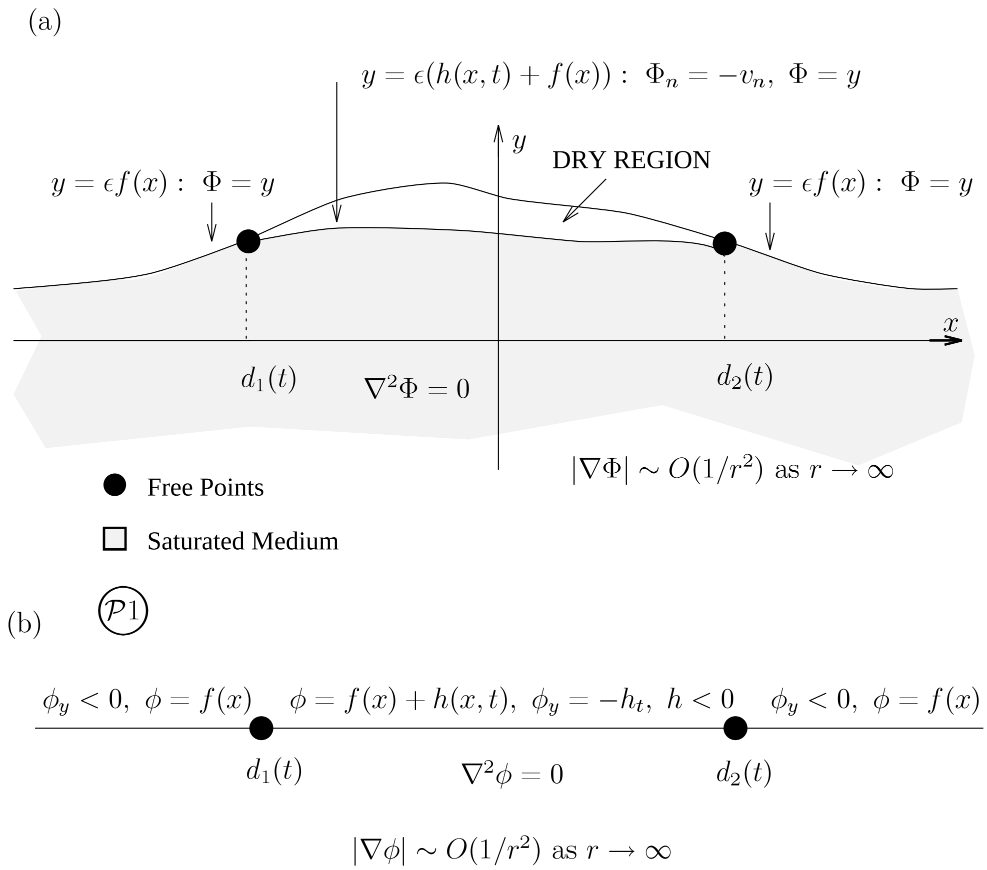

FIG. 1. *The shallow dam:* (a) *the codimension-one problem;* (b) *the codimension-two problem* $\mathcal{P}1$.

equation, and hence the model loses the distinctive global attribute of free boundary problems.

The way in which our class of codimension-two problems arises is exemplified by referring to the well-known dam problem [57, 64]. The two-dimensional, codimension-one version of this problem is shown in Figure 1(a) with a dam of infinite extent (as may be a model for a sandbank). It involves percolation in a saturated region where liquid flows with dimensionless velocity $-\nabla\Phi$, where $\Phi = p + y$ and the pressure p is Laplacian. This region is separated from a dry region by a codimension-one free boundary at which two free boundary conditions are imposed, namely, a kinematic condition and the condition that p be atmospheric (zero). Here, and in other models, we denote the normal free boundary speed by v_n. Now suppose, as in [1], that $\epsilon \ll 1$ so that the top of the dam, $y = \epsilon f(x)$, is nearly flat; the boundary conditions can then be linearized onto the x-axis. By scaling $\Phi = \epsilon\phi$ (we shall use lowercase letters for all our codimension-two problems) and with the codimension-one free boundary represented by $y = \epsilon(f(x) + h(x,t))$, it is plausible, but almost impossible to prove, that the leading order model[1] is that given in Figure 1(b). Thus we have a mixed boundary

[1]Formally, ϕ and h should be expanded in asymptotic expansions in powers of ϵ, but here, and in other cases, it is always the leading-order term which is of interest, and so for ease of reading we shall omit this step.

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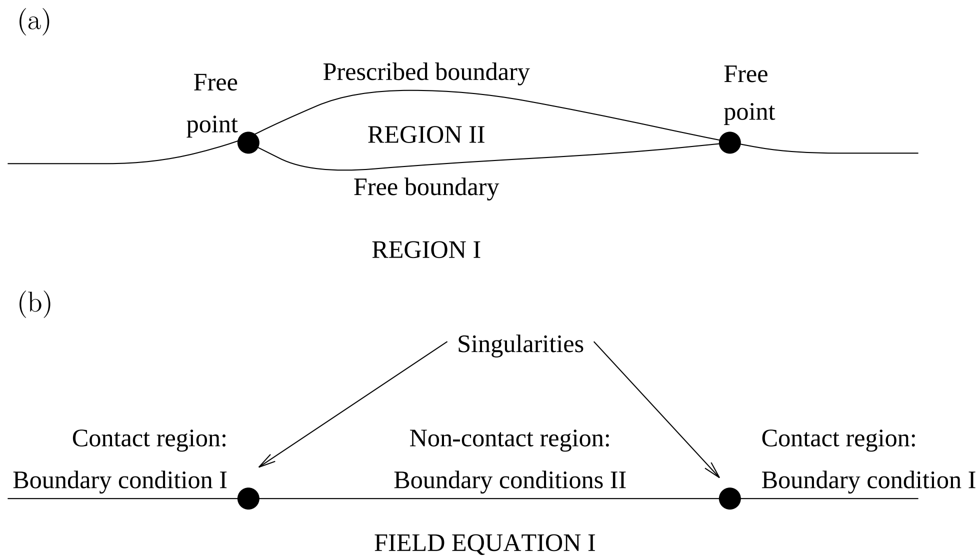

Fig. 2. (a) *The geometry needed for a codimension-two free boundary problem;* (b) *the resulting mixed boundary value problem.*

value problem in which the only geometrical unknowns are the “free points” which mark the points at which the free boundary meets the top of the dam. We shall use the term “free points” throughout this paper and “free curves” in a three-dimensional problem.

We remark that there is a “complement” of our class of codimension-two problems in which a codimension-one free boundary lies near a known boundary, but the field equations only need to be solved in the thin intervening region, as in the flow of a thin film on a rigid base. In this case, a local partial differential equation can also often be derived to describe the approximate position of the free boundary.

More familiar codimension-two configurations occur in solid mechanics where the codimension-two free boundary could be a crack tip or the perimeter of a Hertzian contact region; these situations often pose challenging modelling problems and we shall mention them again later.

With this motivation we shall discuss a general scenario for our codimension-two problems as limiting cases of conventional free boundary problems when, as in Figure 2(a), the free boundary is separated from a known boundary by a thin region (labelled Region II) in which the solution of the relevant field equations is either unnecessary, as in the case of one-phase free boundary problems, or easy to approximate. In either case we expect that, except near the free points, we can linearize the free boundary conditions onto the prescribed boundary; if this boundary is nearly straight, we can linearize it onto a straight line, as in Figure 2(b). This leaves the free points as the only parts of the free boundary to be determined. In this configuration we call the region of the prescribed boundary between the free points the “noncontact” region because there is no contact between this boundary and Region I; the complement of this region on the prescribed boundary is termed the “contact region.” As indicated in Figure 2(b), there will be different boundary conditions on the two regions and, in general, one more condition on the noncontact region because the codimension-one free boundary is unknown.

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The codimension-two problems we shall be addressing are characterized by four pieces of information: the field equations, the boundary conditions on the contact and noncontact regions, and some specification of the behavior of the solution near the free points. It soon becomes apparent that it is this latter piece of information that is the most difficult to prescribe. One reason for this is that, if we wish to think of our scenario in terms of matched asymptotic expansions, this singular behavior is determined by matching with the far field of an "inner" problem that is inevitably a codimension-one free boundary problem, albeit on an infinite domain. Such a matched expansion analysis can in fact be carried out for the problem in Figure 1 (see [1] and the discussion at the end of section 2).

More generally, we shall see that constraints on the severity of the singularities at the free points can often be obtained by functional analysis or index arguments independently of the availability of matched asymptotic expansions. Nonetheless, there remain many cases where there is no alternative to physical intuition if the singularity is to be identified plausibly.

In order to illustrate the ubiquity of our class of codimension-two problems, in Table 1 we list twelve examples that have appeared in the literature. They are classified in the format of Figure 2(b) where, for simplicity, we have mostly specialized to cases where the field equation is to be solved in a half-plane. To produce a manageable list we have deliberately restricted ourselves to two-dimensional problems for either Laplace's equation or the biharmonic equation. Where possible, the conjectured asymptotic behavior of the field variable (almost always called ϕ) at the free points is characterized in terms of the local radial distance from the free point, and similarly in the final column for the behavior at infinity. We also cite some inequalities that have been suggested, often on physical grounds, to supplement the mixed boundary conditions; in exceptional cases, it may be necessary for technical reasons to relax these inequalities to allow equality. In some cases they may not be essential for the mathematical formulation of the model, but they can be shown to restrict the possible singularities at the free points, and indeed sometimes to fix them. However, a full mathematical analysis is not available in the majority of the twelve cases.

In addition to this list we can, as mentioned earlier, cite more general elastic contact problems and elastic crack propagation as perhaps the prototypes of our class of codimension-two free boundary problems. Assuming the displacement in the contact region or at the crack face is sufficiently small, the perimeter of the contact region is the only geometric unknown. However, each of these problems has such a vast literature that we shall not present a survey here except to state that whereas contact problems can be formulated as variational inequalities in the absence of friction [20, 23], the vexed question of the elastic/plastic/cohesive behavior of solids near crack tips has stood in the way of a unifying mathematical theory for crack propagation [25, 46]. We shall make some further comments about this theory at the end of section 3.

We could have also cited other problems whose study from their codimension-two point of view is still in its infancy. Examples include the rise of a bubble under a nearly horizontal inclined plane [52, 55], a model for toner deposition in photocopiers (see [26, p. 156]), and the evolution of thin fingers in the Muskat problem [55].

The attempt to unify the different models in Table 1 immediately raises the following questions:

1. In what precise sense are these models limits of traditional codimension-one free boundary problems?
2. What can be said about existence, uniqueness, and regularity of solutions to the models as stated? In particular, is any of the information redundant?

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TABLE 1
Codimension-two free boundary problems. Suitable initial conditions are needed in problems $\mathcal{P}1$, $\mathcal{P}2$, $\mathcal{P}4$, $\mathcal{P}5$, $\mathcal{P}7$, and $\mathcal{P}11$.

	Physical problem	Field equation	Boundary conditions on: Contact	Boundary conditions on: Noncontact	Free point	∞
$\mathcal{P}1$ [1]	Percolation in a sandbank[a]	$\nabla^2\phi = 0$	$\phi = f$ $\phi_y < 0$	$\phi_y = -h_t$ $\phi = f + h,\ h < 0$	$r^{\alpha(\dot{d})}$	$1/r$
$\mathcal{P}2$ [42, 48]	Water entry	$\nabla^2\phi = 0$	$\phi_y = -1$ $\phi_t < 0$	$\phi = 0,\ \phi_y = h_t$ $h < f(x) - t$	$r^{1/2}$	$1/r$
$\mathcal{P}3$ [9, 41]	Patch cavitation[b]	$\nabla^2\phi = 0$	$\phi_y = 0$ $\phi_x < f$	$\phi_x = f,\ \phi_y = h_x$ $h > 0$	$r^{5/2}$, $r^{1/2}$	$1/r$
$\mathcal{P}4$ [35]	Surface tension driven sintering in slow flow[c]	$\nabla^4\phi = 0$	$\phi_{yy} = 0$ $\phi = 0$ $\sigma_{22} > 0$	$\phi_x + h_t = 0$ $3\phi_{xxy} + \phi_{yyy} = 0$ $\phi_{xx} = \phi_{yy},\ h_t < 0$	$r^{1/2}$	r
$\mathcal{P}5$ [67]	Hele–Shaw flow	$\nabla^2\phi = 0$	$\phi_y = 0$ $\phi < 0$	$\phi_y = -h_t$ $\phi = 0,\ h > 0$	$r^{1/2}$	see text
$\mathcal{P}6$ [2]	Steady electropainting	$\nabla^2\phi = 0$	$\phi = 0$ $\phi_n < j_0$	$h\phi_n = \phi$ $\phi_n = j_0,\ h > 0$	$r^{3/2}$	n/a
$\mathcal{P}7$ [31]	Dislocations on a single slip plane	$\nabla^2\phi = 0$	$\phi_y = 0$ $\phi_x < 0$	$U = 1 + \phi_x$ $\phi_{yt} = -[U\phi_y]_x$ $h_t = -Uh_x,\ h > 0$	$r^{3/2}$	$1/r$
$\mathcal{P}8$ [12]	Thermistor with discontinuous conductivity	$\nabla^2\phi = 0$ $\nabla^2\psi = 0$	$\psi_y = 0$ $\phi = 0$ $\psi < 1$ $\phi_y > 0$	$\psi = 1 + \phi^2/2$ $\psi_y = 0$ $h = \phi/\phi_y$ $\phi > 0,\ h > 0$	$r^{1/2}$ (for ϕ)	n/a
$\mathcal{P}9$ [63]	Flow over a downward step ($x = 0$)	$\nabla^2\phi = 0$	$\phi_y = 0$ $x\phi_x < 0$	$(8\phi_x + h^2)_x = 0$ $\phi_y = h_x,\ h > 0$	$r^{3/2}$	$\log r$
$\mathcal{P}10$ [20]	Elastic contact (the displacement is (u_1, u_2))	$\dfrac{\partial\sigma_{ij}}{\partial x_j} = 0$	$\sigma_{12} = 0$ $u_2 = f$ $\sigma_{22} > 0$	$\sigma_{12} = 0,\ \sigma_{22} = 0$ $h = u_2$ $u_2 > f(x_1)$	$r^{1/2}$	n/a
$\mathcal{P}11$ [78]	Slender viscous inclusion[c]	$\nabla^4\phi = 0$	$\phi = 0$ $\phi_{yy} = 0$ $\sigma_{22} > 0$	$u = x + \phi_y,\ \phi = 0$ $\phi_{yy} + 4(hu_x)_x = 0$ $h_t = -(uh)_x,\ h > 0$	$r^{3/2}$	$1/r$
$\mathcal{P}12$ [5]	Elasto-hydrodynamic lubrication	$\dfrac{\partial\sigma_{ij}}{\partial x_j} = 0$	$(h^3p_x)_x = h_x$ $\sigma_{22} = p,\ \sigma_{12} = 0$ $h > 0, p > 0$	$\sigma_{12} = 0,\ \sigma_{22} = 0$ $h = u_2 + f$ $h > 0$	$r^{3/2}$	$1/r$

[a]In the steady case $\alpha = 3/2$ and $\phi \sim O(\log r)$ at infinity.
[b]The singularities at the leading and trailing edges are shown.
[c]The normal component of the traction in the contact region is denoted by σ_{22}.

3. What methodology is available to solve these problems explicitly?
4. Is there a possibility of generalizing the models either to make their analysis easier or to make them easier to solve numerically?
5. If solutions exist, are they stable to perturbations in the direction parallel to the free curve?

There is one situation where the answer to the first question can be given at once, namely, when the progenital codimension-one problem has an explicit analytical solution. Unfortunately, such solutions are known only for $\mathcal{P}4$, $\mathcal{P}5$, and $\mathcal{P}7$, and the details of their relevant limits are given in [55].

With the above questions in mind, the remainder of this paper comprises a review of the kind of behavior and difficulties that can be anticipated whenever these codimension-two free boundary problems are encountered. In section 2 we begin by deriving some examples that we will subsequently need for illustrative purposes. A primary aim is to show that there are many mathematical tools available for the explicit solution of our codimension-two problems. As we shall see in section 3, this is

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because problems such as that in Figure 2(b) are much more likely to possess explicit solutions than are their parent codimension-one problems; this is a consequence of the well-developed theory of mixed boundary value problems [28, 75]. Another interesting contrast comes when we try to consider codimension-two problem formulations that are more general than the mixed boundary value problems of Table 1. We recall that weak formulations, as developed for the theory of conservation laws, have played a key role in the theory of codimension-one problems, the theory of shock waves being the standard example. No such theory has yet been proposed for these codimension-two problems any more than has been suggested for other codimension-two free singularities such as vortices in an inviscid fluid [71]. However, the unification brought about by variational inequalities in codimension-one problems does, as we shall see, sometimes carry over to codimension-two cases with or without the need for preliminary smoothing transformations, and this will form the subject of section 4, where we will also discuss the importance of this reformulation for obtaining numerical solutions. As mentioned earlier, weak and variational formulations of codimension-one free boundary problems have been a great boon to numerical analysts, as have front tracking and other fixed domain formulations [14]; more recent approaches include the level set method of [73]. However, except in situations where variational inequalities can be utilized, there is as yet no catalogue of algorithms for our class of free boundary problems.

Finally, in section 5, we will discuss the relatively unexplored possibilities of applying perturbation theory to codimension-two problems. Even linear stability theory poses serious challenges at the formal level, and the exploitation of small parameters in the initial or fixed boundary conditions can lead to unexpected new models. This is an important issue since, as we shall see, there is clear evidence for the possible irregular evolution of codimension-two free boundaries. Also, the interesting question of the relationship between the stability of our codimension-two problems and their codimension-one progenitors will be mentioned.

2. Derivation of some codimension-two problems. This section contains brief derivations of five members of our class of codimension-two models starting from their codimension-one counterparts. Apart from the Hele–Shaw problem, they all describe commonly occurring practical situations and we will use them, and the shallow dam problem described in the introduction, to illustrate the ideas put forward in later sections. In each case, the question of the local behavior near the free points will be discussed from a physical viewpoint only, with the mathematical implications left to section 3. The first four are "one-phase" problems, beginning with a model for water entry, which produces one of the simplest codimension-two models. The next example, of a type of cavitation, leads to a more complicated model and demonstrates that there may be different singularities at different free points. Then the small-time sintering of viscous cylinders is introduced as a problem in which an exact codimension-one solution is available but which is not governed by Laplace's equation. We also mention Hele–Shaw flow as another free boundary problem where precise comparison can be made between the codimension-two problem and its codimension-one parent. The last example, of electropainting, is introduced to illustrate a situation where the geometry is not a half-space and the underlying problem has two phases.

2.1. Water entry of a blunt body. In the simplest water entry problems, a uniformly smooth body $Y = f(\epsilon X)$, where f is even, moves with unit speed in the negative Y-direction into water, which is initially at rest in $Y < 0$. The effects of gravity, viscosity, and surface tension are neglected. As shown in [42], for $\epsilon \ll 1$ the codimension-one configuration is modelled as in Figure 3(a): the free surface "turns

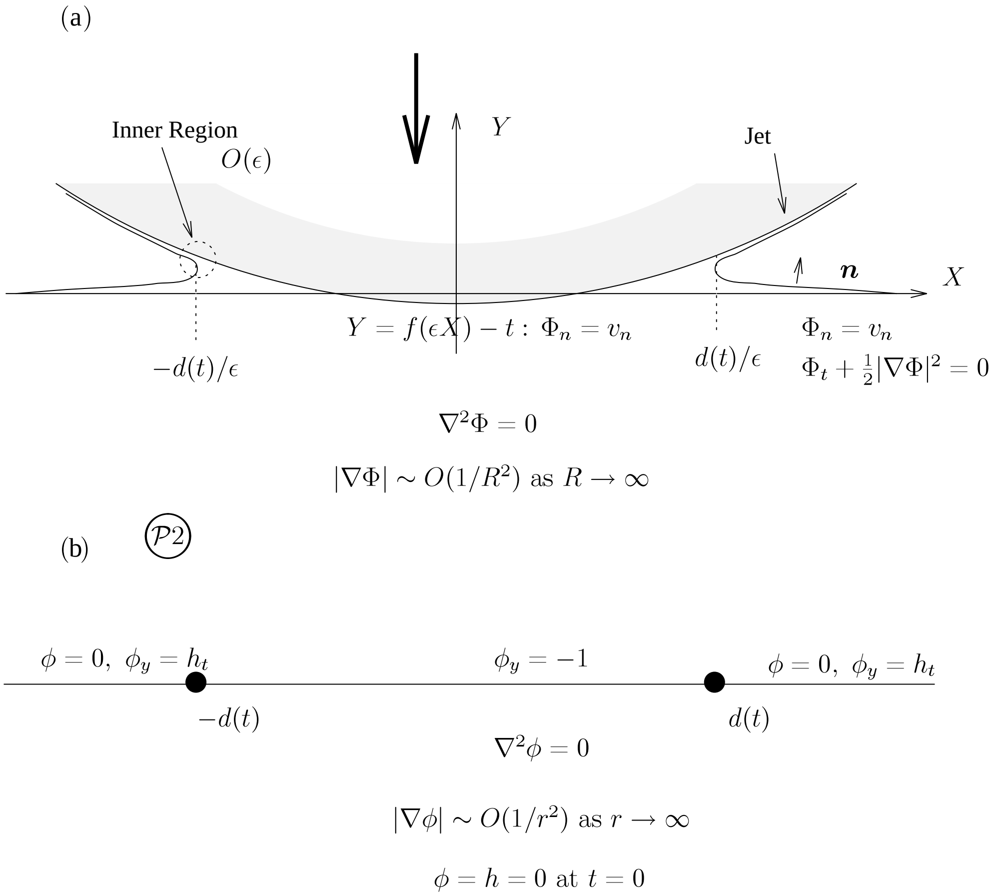

FIG. 3. *The ship impact model:* (a) *codimension-one problem;* (b) *codimension-two problem* $\mathcal{P}2$.

over" and forms two jets along the impacting body and these "turn-over" points are found to lie within $O(\epsilon)$ of $(\pm d(t)/\epsilon, f(\pm d(t)) - t)$. The model consists of inviscid, irrotational flow with velocity potential $\Phi(X, Y, t)$ and with Bernoulli and kinematic conditions on the free surface $Y = H(X, t)$. We have not been specific about the details of the jet flow because, as shown in [42], it exerts only a second-order influence on the codimension-two model. Relative to an $O(1)$ lengthscale for f, the turn-over points have an $O(1/\epsilon)$ lateral separation, and we therefore rescale distances via $x = \epsilon X$, $y = \epsilon Y$ and write the free surface $Y = H(X, t)$ as $y = \epsilon h(x, t)$; the body is at $y = \epsilon(f(x) - t)$ and the free (turn-over) points at $x = \pm d(t)$ (the jet roots are now at $y = \epsilon(f(\pm d(t)) - t) + O(\epsilon^2)$). The kinematic condition demands that we also rescale $\phi(x, y, t) = \epsilon\Phi(X, Y, t)$. It is now reasonable to linearize the boundary conditions for ϕ onto the x-axis and to ignore the jets altogether so that $h(d(t), t) = f(d(t)) - t$; to lowest order, the problem is as shown in Figure 3(b), where the condition $\phi = 0$ for $|x| > d(t)$ is an integration of Bernoulli's equation.

We note the following physically reasonable inequalities:

$$h(x, t) \leq f(x) - t \quad \text{for} \quad |x| > d(t), \tag{1}$$

$$\phi \leq 0 \quad \text{for} \quad |x| < d(t). \tag{2}$$

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$\Phi \to U_\infty X$ as $X^2 + Y^2 \to \infty$

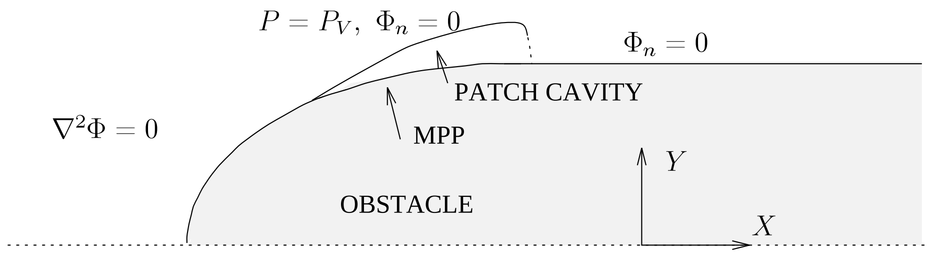

FIG. 4. *The codimension-one patch cavity model.*

The latter equation is an integration of Bernoulli's equation, assuming that the pressure beneath the body is positive.

We also note that the water *exit* problem can be formulated in the same way by reversing the sign of $\phi_y(x, 0, t)$ for $|x| < d(t)$. However, we shall see later that the solution properties are very different in this case.

2.2. Patch cavitation. Cavitation in liquids occurs because they cannot sustain indefinitely low pressures and instead vaporize at the vapor pressure P_V, which imposes a lower bound on the pressure. In inviscid irrotational flow, the lowest pressures must occur on the boundary of the flow domain [4], and [13] records observations of thin "patch" cavities on the surface of an axisymmetric obstacle in a uniform mainstream flow. The cavities form near the point on the body at which the pressure would take its minimum value, P_M, in their absence (we call this the minimum pressure point, or MPP), and we let the tangential fluid speed at this point be U. They are observed to be long and thin when $P_V - P_M \ll \rho U^2$ and to persist for a time much greater than the local residence time of fluid particles. Therefore, as in [41], we initially restrict ourselves to steady two-dimensional flows and neglect viscosity, gravity, and surface tension; the codimension-one model is then outlined in Figure 4.

The small parameter that we exploit here is

$$\epsilon^2 = \frac{(P_V - P_M)}{\frac{1}{2}\rho U^2}, \tag{3}$$

and it can be argued [41] that the thickness of the cavity is $O(\epsilon^3)$ and its length is $O(\epsilon)$.

Because the cavity is small compared to the obstacle, it is controlled only by the local flow near the MPP. Unlike the earlier problems we have described, the codimension-two model now arises as a *local* model for the flow near the cavity and the flow in this local region is matched to the mainstream outer flow by using the theory of matched asymptotic expansions. Since the aspect ratio of the cavity is $O(\epsilon^{-2})$, all the variables must be expanded to second order in ϵ. Thus, when we transform to local coordinates (x, y) tangential and normal to the obstacle at the MPP as shown in Figure 5(a), the body is approximately $y = \epsilon k_1 x^2 + \epsilon^2 k_2 x^3$, where $k_1 < 0$. The thickness of the cavity is written as $\epsilon^2 h(x)$, the leading edge of the cavity is at $x = d_1$, and the trailing edge is at $x = d_2$.

With a suitable scaling the potential in the codimension-two region is written as

$$x + 2\epsilon k_1 xy + \epsilon^2 \left(\phi(x, y) - k_2(y^3 - 3yx^2) + c_0(x^3 - 3xy^2)\right), \tag{4}$$

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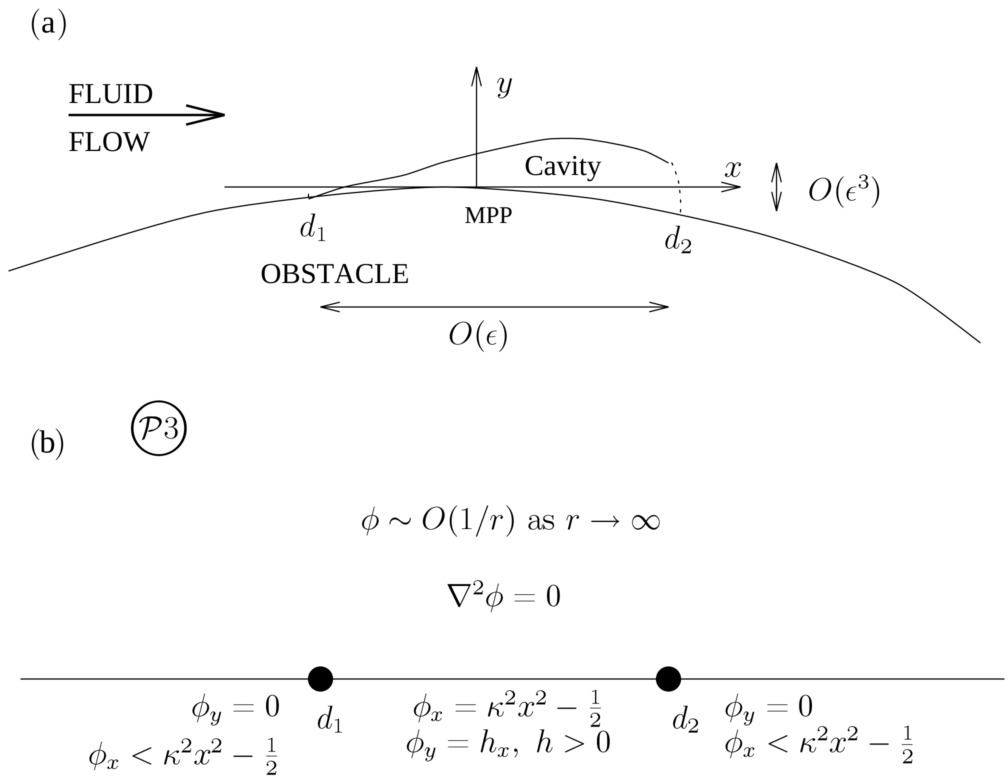

FIG. 5. *The codimension-two patch cavity problem:* (a) *geometry;* (b) *mixed boundary value problem* $\mathcal{P}3$.

where c_0 is a constant known from the outer flow (it is not possible to determine c_0 from a *local* knowledge of the obstacle shape near the MPP). In (4), $\phi(x, y)$, the potential in the codimension-two problem, is to be determined; $\phi = 0$ corresponds to unseparated flow past the obstacle in which, as shown in [41], the pressure near the MPP can be written as

$$p \sim p_V + \epsilon^2 \left(\kappa^2 x^2 - \frac{1}{2} \right), \tag{5}$$

where $\kappa^2 = -3c_0 - 4k_1^2$ and it is necessary that $c_0 < -4k_1^2/3$.

Linearization of the boundary conditions onto the x-axis gives the mixed boundary problem shown in Figure 5(b), with $h(d_1) = h(d_2) = 0$ and where the condition $\phi_x = \kappa^2 x^2 - \frac{1}{2}$ expresses the constancy of the pressure in the cavity. Pressure considerations also dictate the physically reasonable inequalities on $y = 0$, $x < d_1$, and $x > d_2$. The geometric parameter k_2 does not appear in the model $\mathcal{P}3$ because, to the order of magnitude considered, it does not affect the pressure on the obstacle.

2.3. Sintering of viscous cylinders. We now consider a benchmark problem whose codimension-one parent possesses an explicit analytical solution. The mechanical sintering of viscous drops and cylinders is a process of some technological importance in, for example, optical fiber manufacture. Both [35] and [68] describe methods whereby exact solutions can be obtained that describe the coalescence under the action of surface tension of two equal circular cylinders of viscous liquid that initially

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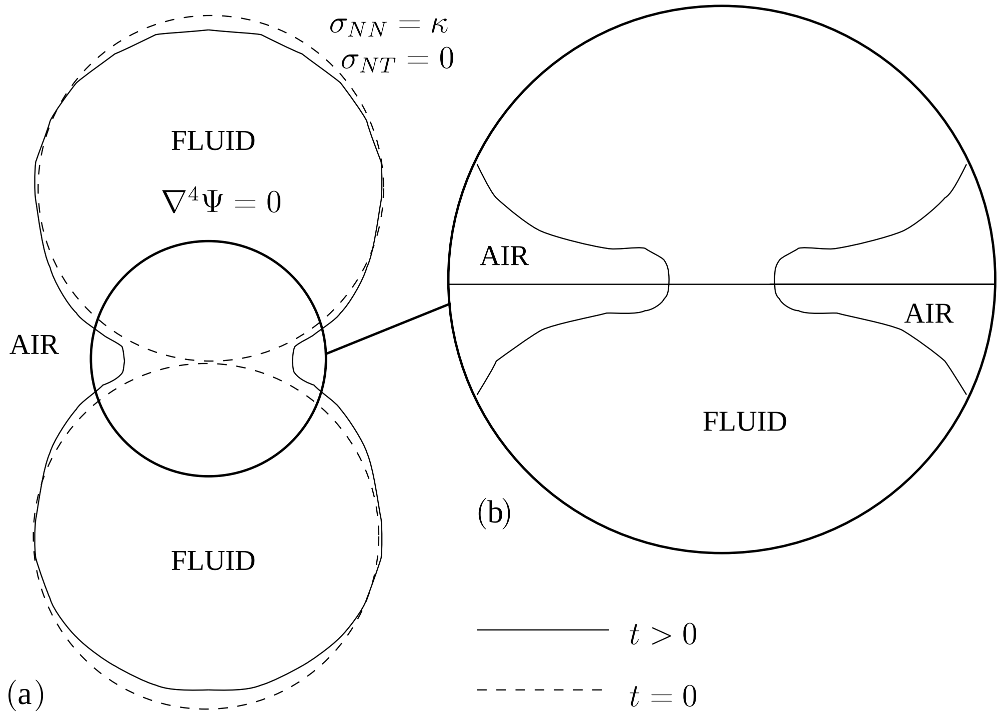

FIG. 6. (a) *The coalescence of two viscous cylinders under surface tension;* (b) *the codimension-two region.*

touch along a common generator, as in Figure 6(a). The full model is Stokes flow, with a kinematic boundary condition on the free boundary as well as the conditions that the normal stress be equal to the curvature, $\sigma_{NN} = \kappa$, and that the tangential stress be zero, $\sigma_{NT} = 0$. The initial motion near the origin is rapid, and it is this regime that we analyze within our codimension-two framework as shown in Figure 6(b); we give only a brief résumé of the description in [55].

We work on a timescale over which ϵ, defined to be the order of magnitude of the lateral extent of the contact region between the two cylinders, is small. For the contact region to be of this size, it turns out that time T and the stream function Ψ must be scaled with $-\epsilon/\log\epsilon$ and $-\epsilon^2\log\epsilon$, respectively. Therefore, to retain a nontrivial kinematic condition the full scalings for the codimension-two region (lower case) are

$$(X, Y) = \epsilon(x, y), \quad \Psi = -\epsilon^2(\log\epsilon)\psi, \quad T = \left(\frac{-\epsilon}{\log\epsilon}\right)t, \quad H = \epsilon^2 h. \tag{6}$$

(Some of these scalings, it must be admitted, would have been very difficult to ascertain without comparison with the exact solution in [35].) The logarithm reflects the fact that in contrast to, say, the water entry problem, there is no local similarity solution involving a power of t. On this timescale, the free boundary far away from the origin only moves by $o(\epsilon)$, and so the outer problem away from the origin is to solve the biharmonic equation in a circle with the stress conditions mentioned earlier and a singularity at the origin. This singularity produces a flow toward the origin, and this is reflected in the condition at infinity in the codimension-two problem, where

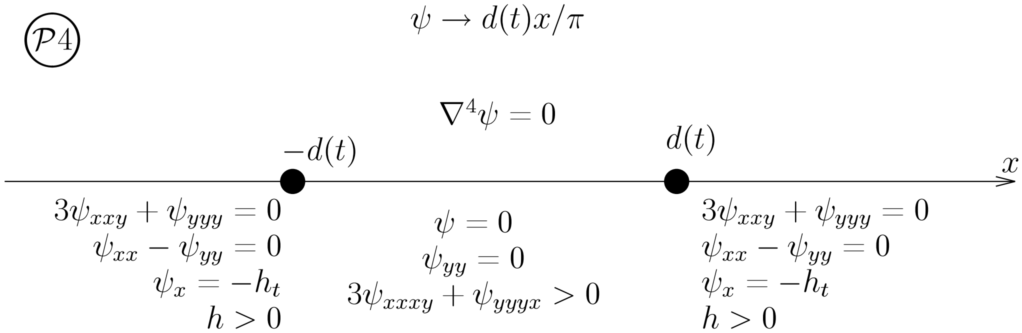

FIG. 7. *The codimension-two coalescence problem* $\mathcal{P}4$.

the initial free boundary is $y = \frac{1}{2}x^2$. In the codimension-two region the air gap thickness $2h = O(\epsilon)$, and so linearization onto the y-axis yields the mixed boundary value problem for ψ shown in Figure 7, in which symmetry has been exploited to formulate the problem in the upper half-plane. Again, the inequalities are based on the physically realistic assumptions that the fluid is in a state of tension and the sintering cylinders do not interpenetrate. Note that even in this region surface tension forces are dominated by viscous ones, except in the immediate vicinity of the free points $x = \pm d(t)$.

2.4. Hele–Shaw free boundary flows. A Hele–Shaw cell [72] consists of two parallel plates between which a viscous liquid flows, under the influence of injection or suction at the edges of the cell or through holes in the plates. When an effectively inviscid fluid such as air is also introduced, interfaces can form between the two fluids. For large aspect ratios (small gaps), these free surfaces can be modelled as curves in the plane of the cell. Moreover, the slow flow equations reduce (in suitable dimensionless variables) to a two-dimensional potential flow. The fluid velocity $\boldsymbol{u}(x, y, t)$ and pressure $p(x, y, t)$ satisfy

$$\boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0,$$

and writing $\Phi = -p$ we have $\nabla^2\Phi = 0$ in the fluid region. On free boundaries, we use the simple "zero surface tension" model $\Phi = 0$ and the kinematic condition

$$\Phi_n = v_n.$$

These equations also model two-dimensional flow in a porous medium [64] and are relevant to several other physical situations such as electrochemical machining [49, 53] and, even more importantly, they are a special case of the Stefan model (see section 4). We study the Hele–Shaw problem because complex variable methods lead to an unparalleled variety of explicit solutions (see [33] for a review), many of which are suitable tests of the validity of the codimension-two approach. We mention three such situations.

The first has a geometry analogous to that in $\mathcal{P}4$. If fluid is injected symmetrically into a Hele–Shaw cell through point sources at $(0, \pm 1)$, the fluid region first consists of two expanding circles centred on the sources. Eventually they sinter at the origin and the domain is thereafter simply connected and tends to a circle as $t \to \infty$ (the exact solution is given in [67] and sketched in Figure 8(a)). The codimension-two approximation is valid for times immediately after the circles touch, and it is shown

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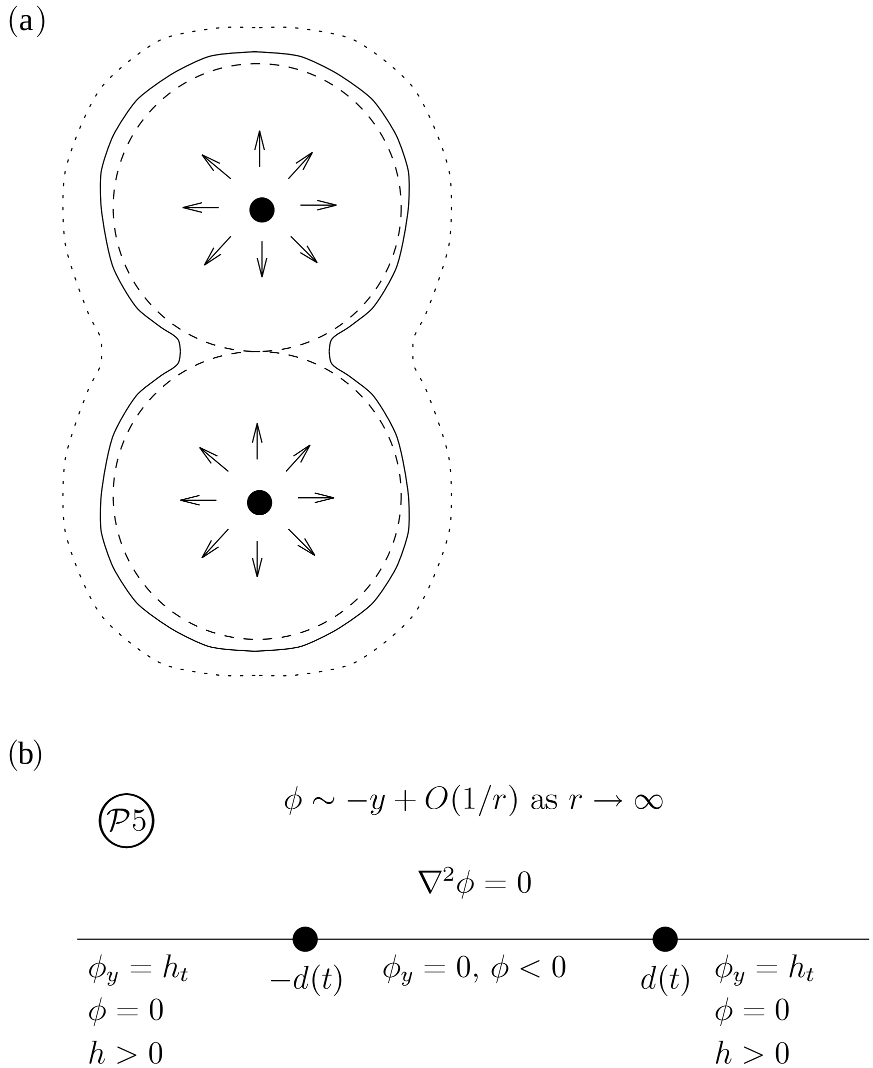

FIG. 8. *The Hele–Shaw cell with two equal sources:* (a) *codimension-one problem;* (b) *the codimension-two problem* $\mathcal{P}5$.

in Figure 8(b). There the air gap has thickness $2h(x,t)$ in suitable local coordinates, with $h(x,0) = \frac{1}{2}x^2$, and the inequality $\phi < 0$ on the contact region follows from the fact that $p > 0$ throughout the fluid region (symmetry has been exploited to formulate the problem in the upper half-plane).

A second configuration concerns the evolution of a long thin bubble in an infinite cell with an appropriate driving mechanism at infinity. The driving mechanism might be either uniform suction or injection [37], in which case $\phi \sim Q \log r$ as $r \to \infty$, $Q > 0$ corresponding to injection and $Q < 0$ to suction, or a dipole flow [21] with $\phi \sim A(x^2 - y^2)$. In the former case the area of the bubble increases or decreases at a rate Q, while in the latter it remains constant. The codimension-two formulation is very similar to that of Figure 8(b).

Our final example concerns cusp formation in the free boundary. It is well known that the suction problem is ill posed and that for almost all initial value problems there

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is finite-time blow-up involving a singularity at the free boundary. For a large class of such blow-up solutions, it can be shown that the free boundary develops a 3/2-power cusp, and the late stages of cusp formation are also modelled by a codimension-two free boundary problem very similar to that in Figure 8(b). At the moment of blow-up, the fluid velocity at the cusp tip is infinite, and the solution does not exist thereafter. This highly unstable scenario presents an extreme test of the robustness of the codimension-two approach. In fact, the traditional linear stability analysis of planar solutions to the codimension-one Hele–Shaw problem [59, 72] leads to a problem for the perturbation potential in which the noncontact boundary condition in $\mathcal{P}5$ is replaced by $\phi = \pm h$, $\phi_y = h_t$, i.e., $\phi_y = \pm\phi_t$ in the injection (+) and suction (−) cases, respectively. (Of course, the linearization is the same as in the derivation of a codimension-two problem; the distinctive feature of the latter is the switch in boundary conditions at the free points.) As discussed in [40], this linearized stability problem can be interpreted in terms of motion of singularities of the analytic continuation of the perturbation potential, and blow-up can result if such a singularity reaches the free boundary. Although the introduction of a codimension-two free boundary changes this scenario substantially, it still seems likely that codimension-two problems suffer ill posedness via blow-up well away from the free points, just as the codimension-one problem does. (We consider the stability of the codimension-two free boundary later.)

We note that cusp formation can also occur in Stokes flow [43, 66] and that flows that very nearly realize these cusps can be observed in experiments [45]. However, these flows do not suffer ill posedness in the same way as Hele–Shaw flows do. To be more precise, while the solutions to slow flow free boundary problems with suction generally develop singularities such as cusps in finite time, the linear growth rate of small perturbations to a planar interface is algebraic rather than exponential, and the time to blow-up is correspondingly larger than in the Hele–Shaw case.

2.5. Electropainting. In our final example, the underlying codimension-one problem is two phase and the geometry is more complicated than in the problems considered hitherto. A metal object, the workpiece, is to be painted electrically by being placed in a bath of paint particles (positive ions) in solution, as indicated in Figure 9. A potential difference is applied between the workpiece and the anode, which is the bath itself; the resulting current drives the paint toward the workpiece. However, the paint only adheres to bare metal if the current density normal to the surface exceeds a critical value j_0; otherwise no painting of bare metal occurs, and if paint is already present, it dissolves. The resulting paint layer is thin because its conductivity is small compared to that of the solution.

The codimension-one model [2] assumes an electric potential which satisfies Laplace's equation in the solution and in the paint, with continuity of the potential and the current at the paint surface. Because the paint layer is so thin, the potential varies approximately linearly across it in the direction normal to the workpiece. Then, in suitable dimensionless units, the potential ϕ at the paint surface is related to the current density there by Ohm's law so that $\phi = h\phi_n$, where h is the paint thickness (which is also proportional to the resistance of the layer). This boundary condition is linearized onto the workpiece so that

$$\phi = \begin{cases} h\phi_n, & \text{where } h > 0, \\ 0, & \text{where } h = 0, \end{cases} \tag{7}$$

the second condition stating that the workpiece is grounded. As discussed in [2], the other boundary condition on the noncontact region describes the kinetics of the paint

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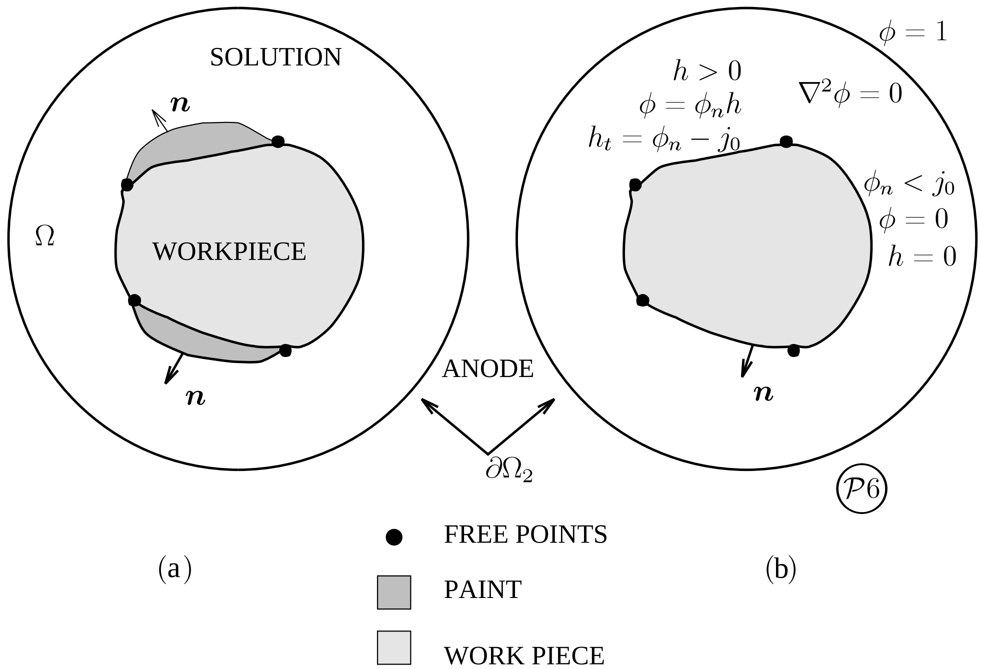

FIG. 9. *The electropainting problem:* (a) *codimension-one geometry;* (b) *the codimension-two problem* $\mathcal{P}6$*.*

growth. There are several possible models, of which the simplest is

$$h_t = \begin{cases} 0, & \text{where } h = 0 \text{ and } \phi_n < j_0, \\ \phi_n - j_0 & \text{elsewhere.} \end{cases} \tag{8}$$

The relevant mixed boundary value problem and associated inequalities are shown in Figure 9(b).

2.6. Discussion. With the above prototypical problems in mind, we can make some preliminary observations about the first three questions raised at the end of section 1. First we note that in all cases there is a nonuniformity in the approximation of the codimension-one problem by the codimension-two problem in the neighborhood of a free point. This nonuniformity can be best understood if matched asymptotic expansions can be constructed linking the codimension-two problem to a local inner problem near each free point. Indeed, such "inner" solutions can be found for $\mathcal{P}1$–$\mathcal{P}5$ in terms of a "hatted" coordinate system denoting variables in an inner region near a free point[2] as follows.

$\mathcal{P}1$. For the dam problem, the singularity at the free point is so weak that, to leading order, an inner region is not needed. However, it is instructive to look at the relevant singularity in a general steady dam problem in the limit as the dam becomes flat. It is well known [64] that for general dam problems there is a singularity in which $\hat{\phi} \sim O(\hat{r}^{(3\pi-2\alpha)/2\pi})$ at points where the free boundary meets a seepage face making an angle $\alpha < \pi/2$ with the horizontal and that the intersection is tangential, as in

[2]Here and elsewhere we temporarily use r in the outer unhatted coordinate system to denote distance from the free point.

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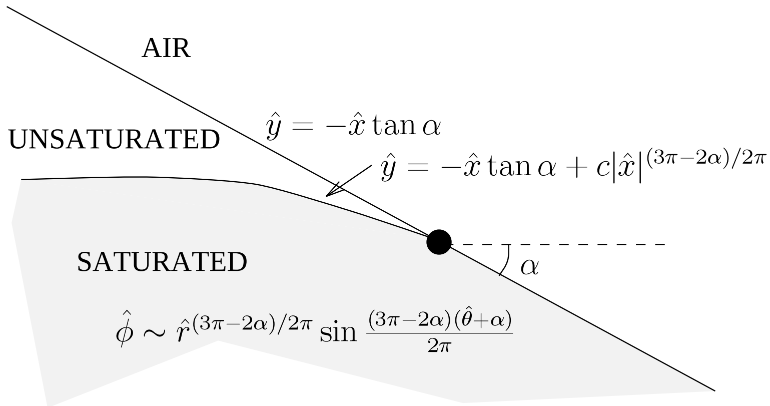

FIG. 10. *The local problem in a general dam problem.*

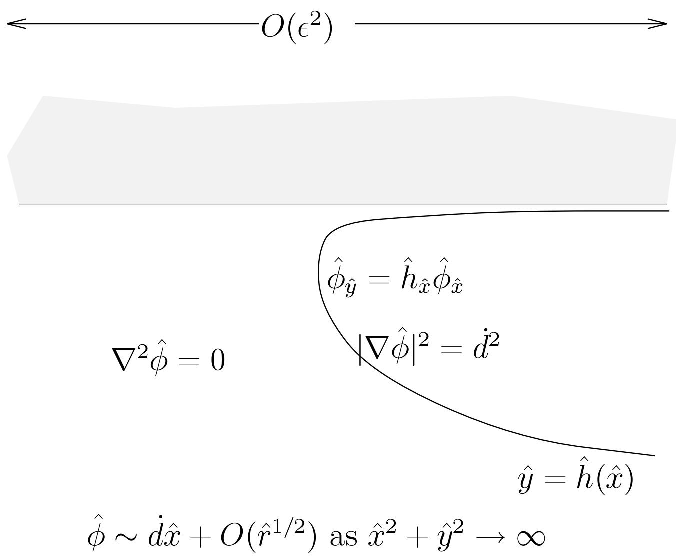

FIG. 11. *The inner region in the water impact problem.*

Figure 10. It is thus suggested in [1] that the singularity at $x = d$ in $\mathcal{P}1$ for which $\alpha \sim O(\epsilon)$ should be one in which, near the free point, $\phi \sim O(r^{3/2})$ as $r \to 0$, with $h \sim (d-x)^{3/2}$.

$\mathcal{P}2$. We have already mentioned that the local solution near the turn-over points in Figure 3 implies the existence of two thin jets that travel up the sides of the impacting body. In [42] it is shown that the formation of these jets can be described locally by the solution of a Helmholtz flow in a region where $(x-d)^2 + y^2 = O(\epsilon^4)$, as shown in Figure 11, in which $\hat{h}$ is parabolic as $\hat{r} \to \infty$. Hence we expect that in $\mathcal{P}2$ both ϕ and h should have square root singularities at the free point.

$\mathcal{P}4$. This is in principle one of the simplest cases since the codimension-one problem has an explicit solution, but it is interesting to note that the "travelling wave" that describes the local behavior near $x = d$ only appeared [36] after the

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original work [35]. It is only in this travelling wave region that surface tension is strong enough to balance the viscous forces; the free boundary is exactly parabolic and the stream function $\hat{\psi} \sim O(\hat{r}^{1/2})$ as $\hat{r} \to \infty$. Hence, after a scaling in which $(x-d)^2 + y^2 = O(\epsilon^4)$, we expect matching with such a travelling wave to imply that $\psi \sim O(r^{1/2})$ as $r \to 0$ and $h \sim (x-d)^{1/2}$.

In the Hele–Shaw problems $\mathcal{P}5$, the local behavior has $\phi \sim O(r^{1/2})$ and $h \sim (x-d)^{1/2}$; again the relevant inner solution, which can be found exactly [44], is a travelling wave with parabolic free boundary.

In $\mathcal{P}3$ the behavior near $x = d_1$ may differ from that near $x = d_2$, and it is argued in [55] that, at the trailing edge, the inner region is the same as that in the water impact problem. For problem $\mathcal{P}6$, no local solution near the free points has been proposed, but it is likely on physical grounds that $|\nabla\phi|$ is bounded there.

The question immediately arises as to the relationship between the singular behavior near the free points and the auxiliary inequalities that have been listed. It remains unclear whether the specification of either implies the other. For the moment we simply remark that if a codimension-two problem can be formulated as a variational inequality, the complementarity statement of the auxiliary inequalities can be used to ascertain the regularity of the solution at the free points [20, 50]. We shall return to this point in section 4.

Concerning the more general question of well posedness of codimension-two problems, we have already cited cusp formation in Hele–Shaw cells with suction as an instance where the codimension-two formulation inherits the ill posedness of its codimension-one progenitor. Despite its time reversibility, the codimension-two Hele–Shaw suction problem ($\mathcal{P}5$ with $\phi \sim -y$ as $r \to \infty$) can be shown to be ill posed (see [34] and references therein) and almost all initial value problems exhibit finite-time blow-up. The close analogy between $\mathcal{P}2$ and $\mathcal{P}5$ then suggests that the former is ill posed in the water *exit* case, a conjecture which will be supported by the stability analysis of section 5. Furthermore, the contrast between ill posedness and well posedness extends to $\mathcal{P}1$. This problem is only time reversible if the sign of gravity is also changed, but as shown in [59], the codimension-one dam free boundary problem is linearly unstable either if the normal to the free boundary points downward, or, if the normal points upward, the free boundary moves downward sufficiently fast. Hence, if we were to formulate a codimension-two problem for a dam that was saturated except for a thin dry patch adjacent to its horizontal, impermeable base, we should expect this problem to be ill posed. Similarly, instability probably results if fluid is removed sufficiently fast from $y = -\infty$ in the configuration of $\mathcal{P}1$.

We shall discuss ill posedness further after a more detailed examination of some problems that are, we hope, well posed. Meanwhile, spurred on by the observation that several of our prototype codimension-two problems involve the solution of mixed boundary value problems in a half-space for field equations that are linear, we proceed to discuss the implications of the theory of singular integral equations. This will immediately necessitate a mathematical consideration of the strengths of the singularities at the free points; however, it will not usually give a prescription for solving the problem completely because the relevant integral equations are nonlinear in their dependence on the position of these points.

3. Application of the theory of mixed boundary value problems. We observe that in all the codimension-two models we have considered, boundary data for the field equation is prescribed on the contact region and the noncontact region;

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moreover, information about h is also prescribed on the noncontact region. This suggests that if we can solve the relevant mixed boundary value problem for any arbitrary position of the free points, we can then use the information about h to try to find them. However, we shall soon see that this simple strategy is inadequate.

We note another common attribute of the problems listed in section 2, namely, that they are mixed boundary value problems for Laplace's equation or the biharmonic equation. In fact, we have deliberately chosen these examples because a well-developed method exists for finding the solution to such problems [28, 75]. The theory exploits the fact that they can be written in terms of Riemann problems for analytic functions and, using the Plemelj formulae, as equivalent first-kind singular integral equations with Cauchy kernels, the region of integration being either the contact or noncontact region. Hence an existence/uniqueness theory can be developed that generalizes the Fredholm theory for nonsingular integral equations. It is based on the observation that, when a Cauchy integral equation is posed on a smooth closed curve Γ in the Argand diagram, the solution is uniquely determined in terms of the boundary values of a function analytic away from Γ when the index of the related Riemann problem vanishes. Although this is a very simple concept, it becomes cumbersome to apply to situations in which, for instance, Γ is the real axis and the behavior at infinity varies from problem to problem. Fortunately, our examples are sufficiently simple that the relevant uniqueness results can be found in practice by noting that if a function $w(z)$ analytic in $y < 0$ has, say, a prescribed real part on $y = 0$, $|x| > d$ and an imaginary part on $y = 0$, $|x| < d$ then $w/(z^2 - d^2)^{1/2}$ satisfies a Dirichlet problem on the whole real axis and can be written down as a Fourier integral together with eigensolutions that, by inspection, have a Laurent expansion in powers of $(z + d)$ and $(z - d)$. Hence the coefficients in this Laurent expansion can be juggled to satisfy the relevant singularity conditions at the free points $y = 0$, $x = \pm d$ and give the necessary behavior at infinity. As stated above, our procedure will be to try to write down conditions at the free points that ensure that $\phi = \Re\{w\}$ is uniquely determined as a function of d and then use our extra information about h to find d. However, it will sometimes turn out to be the case that the problem for w must be deliberately overdetermined so that d emerges as an eigenvalue for this problem. In any event, we will also have to ensure that the relevant inequalities for h are satisfied and that the solution is compatible with any conditions that could be determined on the basis of matched asymptotic expansions near the free points.

We begin by analyzing some steady state situations.

3.1. Steady states. $\mathcal{P}1$. Steady states in which seepage faces exist can only be maintained in dam problems when there is some inflow of liquid as is the case when hydrostatic pressure is applied at, say, $x = 0$, $y < 0$ as in Figure 12.

Then, after a conformal map into a half-plane in which we denote all variables by a tilde, we obtain the codimension-two problem shown in Figure 13. We also demand that $h(0) + f(0) = 0$ because a simple physical argument shows that there can be no seepage face in the vicinity of the origin in Figure 12, either on the x- or y-axis.

In the quarter-plane, ϕ is linear in the radial variable r near the origin, which becomes $O(\tilde{r}^{1/2})$ after the conformal map. Similarly at infinity in the quarter-plane $\nabla\phi$ decays as $O(r^{-1})$, which means that $\tilde{\nabla}\tilde{\phi} \sim O(\tilde{r}^{-1})$ in the half-plane. Concerning the crucial question of the behavior near the free point $\tilde{x} = d^2$, $\tilde{y} = 0$, we already suggested in section 2 that $\tilde{\phi} \sim O(\tilde{r}^{3/2})$ is the appropriate singularity specification

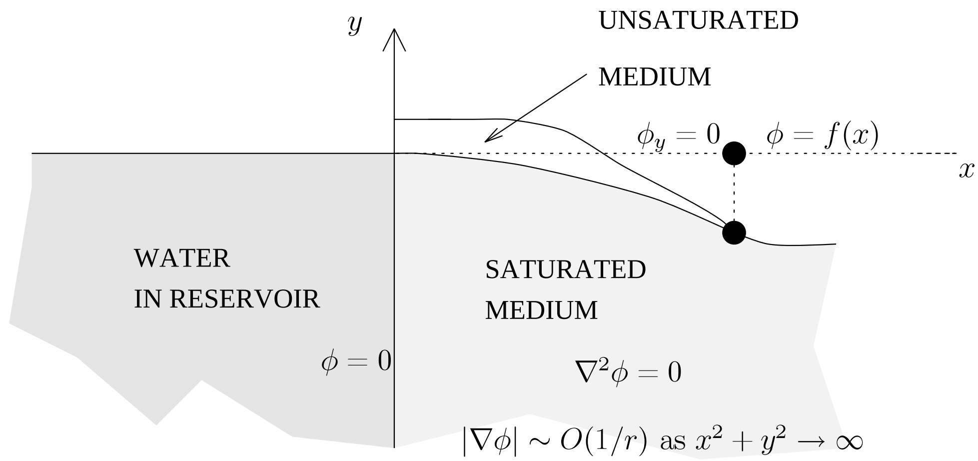

FIG. 12. *The quarter-plane shallow dam problem with boundary conditions linearized onto the x-axis in the codimension-two approximation.*

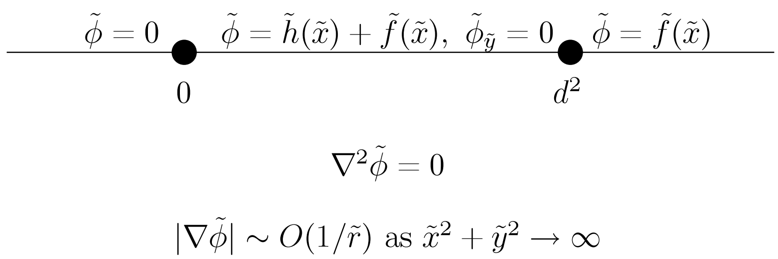

FIG. 13. *The mixed boundary value problem for the steady quarter-plane shallow dam problem, after a preliminary conformal map.*

in terms of distance from the free point $\tilde{r}$. In fact, the change from Dirichlet to Neumann data at this point implies[3] that $|\tilde{\nabla}\tilde{\phi}| \sim O(\tilde{r}^{n+\frac{1}{2}})$, $n \in \mathbb{Z}$, and it is easy to see from the argument advanced at the beginning of this section that $n = -1$ guarantees uniqueness for $\tilde{\phi}$. However, if we wrote down this solution and solved for h, we would find d to be undetermined and the inequalities on either the contact or noncontact region to be violated. Hence we deliberately overdetermine our problem for $\tilde{\phi}$ by insisting, as in section 2, that $\tilde{\phi} \sim O(\tilde{r}^{3/2})$ at the free point, leaving d to be determined from a solvability condition. This implies that near the free point $h \sim O(d - x)^{3/2}$ and the fluid velocity is finite here; it is also consistent with the inequalities in Table 1 because $h < 0$ in $0 < x < d$ and $\phi_y < 0$ in $x > d$. We can now take Fourier/Hilbert transforms to deduce the singular integral equation

$$0 = \int_0^{d^2} \frac{\tilde{h}'(s) + \tilde{f}'(s)}{\tilde{x} - s} ds + \int_{d^2}^{\infty} \frac{\tilde{f}'(s)}{\tilde{x} - s} ds, \tag{9}$$

[3] This kind of estimate can of course be made more precise by using the theory of Sobolev spaces [76]. If we assume on physical grounds that $|\tilde{\nabla}\tilde{\phi}|^2$ is locally integrable, then $\tilde{\phi} \in \mathcal{H}^1(\Omega)$ where Ω is $\tilde{y} < 0$ so that the trace theorem [74] implies the boundary data is an element of $\mathcal{H}^{1/2}(\partial\Omega)$, thereby ruling out any cases where $\tilde{\phi}$ is infinite at a free point.

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with relevant solution for $0 < \tilde{x} < d^2$

$$\tilde{h}'(\tilde{x}) + \tilde{f}'(\tilde{x}) = \frac{1}{\pi}\sqrt{\frac{d^2-\tilde{x}}{\tilde{x}}}\int_0^{d^2} K(s)\sqrt{\frac{s}{d^2-s}}\frac{ds}{\tilde{x}-s}, \tag{10}$$

where

$$K(s) = \int_{d^2}^{\infty} \frac{\tilde{f}'(\xi)}{s-\xi}d\xi.$$

Integrating and using the fact that $\tilde{h}(0) + \tilde{f}(0) = \tilde{h}(d^2) = 0$, after some simplification we obtain the required solvability condition in the form

$$\tilde{f}(d^2) = \frac{1}{\pi}\int_0^{d^2}\sqrt{\frac{s}{d^2-s}}\int_{d^2}^{\infty}\frac{\tilde{f}'(\xi)}{s-\xi}\,d\xi ds. \tag{11}$$

Thus we have been able, on purely theoretical grounds, to solve our codimension-two problem uniquely. It is gratifying to note that the conditions for a unique solution are in accordance with the physically motivated requirement that $|\nabla\phi(d,0)|$ be finite.

We also note that if, in the configuration of Figure 1, an impermeable base is introduced, there are infinitely many steady state solutions; they all have $\phi \equiv 0$, and $f + h$ is any constant less than or equal to the smallest value of the height of the upper surface of the dam. Thus in these solutions the fluid region lies entirely below the top of the dam and there are no seepage faces. We return to this indeterminacy of steady states for codimension-two problems below.

$\mathcal{P}3$. In this problem we have less insight into the behavior near the free points than in $\mathcal{P}1$, although the photographs in [13] (similar experiments are described in the more accessible [9]) suggest that h is much smoother at the leading edge $x = d_1$ than at $x = d_2$. The singularities at both these points are again $\phi \sim O(r^{n_i+\frac{1}{2}})$ as $r \to 0$; however, negative values of n_i give unbounded values of h and the values $n_i = 1, 3, 5, \ldots$ violate the inequalities in Table 1. When we again solve the problem for ϕ by the procedure outlined at the beginning of this section (suitably modified to cater for asymmetry), we find a unique solution for given d_i when $n_1 = n_2 = 0$. However, this choice would lead to an acceptable formula for h for arbitrary d_i and we now have to "doubly" overdetermine the problem for ϕ. Choosing $n_1 = 2$, $n_2 = 0$ because of the photographic evidence mentioned above (these singularities are in agreement with the Brillouin condition [7]), we find it is a simple matter to solve the integral equation for h in the form

$$h_x = \frac{1}{\pi}\sqrt{\frac{x-d_1}{d_2-x}}\int_{d_1}^{d_2}\frac{\frac{1}{2}-\kappa^2t^2}{x-t}\sqrt{\frac{d_2-t}{t-d_1}}dt. \tag{12}$$

Integrating (12) with the correct singularity at $x = d_1$ and the condition that $h(d_1) = h(d_2) = 0$ gives two conditions on the free points, and we find that

$$d_1 = -\sqrt{(2/5)}/\kappa, \quad d_2 = -5d_1. \tag{13}$$

Again we note the concordance of the mathematical and physical requirements for d_1 and d_2 to be determined uniquely. Indeed, we find that we can construct an inner solution near $x = d_2$ using matched asymptotic expansions, and the solution is the same as in the jet formation region in $\mathcal{P}2$ described above. Moreover, the singularity at $x = d_1$ is so weak that no inner expansion is necessary to lowest order. Furthermore, the comparison of this solution with experimental evidence that is presented in [13] is sufficiently encouraging to suggest that an attempt to generalize the model to axisymmetric or even fully three-dimensional flow would be justified.

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3.2. Evolution problems. Our strategy immediately becomes more complicated when we must solve an evolution problem for $d(t)$. Fortunately, there are some models where we can do this explicitly.

$\mathcal{P}2$. When we use the same singularity arguments as above, the condition at infinity is such that there is a unique solution for ϕ if $\phi \sim O(r^{1/2})$ as $r \to 0$, where r is the distance from either free point; hence

$$\phi = -y + \Re\left\{\sqrt{(x+iy)^2 - d^2(t)}\right\}. \tag{14}$$

In this case we do not need to overspecify the problem for ϕ as we did in the steady states described above. In fact, we find $h(x,t)$ by a simple integration of the kinematic condition $\phi_y = h_t$ on $y = 0$, $|x| > d(t)$, and the condition that the free surface meets the impacting body at $x = d(t)$ is

$$f(d(t)) = \int_0^t \frac{d(t)}{\sqrt{d^2(t) - d^2(\tau)}} d\tau. \tag{15}$$

Solving this Abel integral by setting $d(t) = x$, $d(\tau) = \xi$, we find that

$$d^{-1}(t) = \frac{2}{\pi}\int_0^t \frac{f(\xi)}{\sqrt{t^2 - \xi^2}} d\xi. \tag{16}$$

It is, of course, fortunate that this solution can be written down explicitly, and again it gives us an easy comparison with experimental results [42]. Moreover, when the impacting body is a wedge, a rigorous analytical verification of (16) is also possible [24]. For asymmetric or three-dimensional impacts, or if the ship has a large enough horizontal velocity, the equation corresponding to (15) becomes more complicated and must be solved numerically [55].

We also note that there is no unique steady state for this problem as posed. Indeed, $\phi \equiv 0$ satisfies $\mathcal{P}2$ whenever $V = 0$, and h can then be arbitrary. This situation is reminiscent of the indeterminacy of steady states for $\mathcal{P}1$.

$\mathcal{P}4$. In principle, biharmonic mixed boundary value problems can also be solved as Riemann–Hilbert problems, but the codimension-two problem in Figure 7 can, like the water entry problem, be solved by inspection. Using the index theory described in [28, p. 256], we expect there to be a unique solution for the problem for ψ when $\psi = O(r^{1/2})$ near the free points and $\psi = O(r)$ as $r \to \infty$. Thus, combining functions of the form $\sqrt{z \pm d}$ and $(\overline{z} \pm d)/\sqrt{z \pm d}$, where $z = x + iy$, gives

$$\psi = \Re\left\{\frac{d(t)}{2\pi}\frac{z^2 + z\overline{z} - 2d^2(t)}{\sqrt{z^2 - d^2}}\right\}, \tag{17}$$

and the solvability condition that $h(d) = 0$ gives, after a calculation similar to that leading to (16) (again with no overspecification in the problem for ψ), $d(t) = \sqrt{2}\,t$, in accordance with the small-time expansion[4] of the exact solution given in [35].

$\mathcal{P}5$. The Hele–Shaw problems are all mathematically very similar to the ship impact problem $\mathcal{P}2$ and can be solved in the same way; in fact, a simple transformation can be used to turn the symmetrical "two-source" problem into exactly $\mathcal{P}2$. Using the explicit solutions of [67] for the coalescing circles, or of [37] for the self-similar

[4] In the unscaled variables, $\sqrt{2}t$ becomes $\sqrt{2}T/\log T$.

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growth of an elliptical bubble, or of [21] for the evolution of an ellipse placed in the straining dipole flow $\phi = A(x^2 - y^2)$, gives limiting agreement in all cases. It is especially worth mentioning that in the latter case the codimension-one free boundary is elliptical for all time and its semimajor axis tends to infinity *in finite time*; this behavior is reproduced exactly by the codimension-two approximation (the theory of long thin morphologies in Hele–Shaw flows is developed further in [34] and references therein). Finally, the local analysis of cusps also retains all the essential features of the full problem with the generic 3/2-power blow-up.

$\mathcal{P}1$. We may now dispense with the pumping mechanism introduced earlier and revert to the problem in Figure 1. We are not fortunate enough to be able to find explicit solutions for this problem and the singularity at a free point is time dependent, being $O(r^\alpha)$ where $\tan \alpha\pi = -1/\dot{d}$ and $1 \le \alpha \le 2$. Such time-dependent exponents also occur in models of crack propagation [25, p. 174] and in electropainting [8]. However, the same solution procedure can be followed as in the steady case to give the generalization of (9) as

$$h_t = \frac{1}{\pi}\int_{d_1(t)}^{d_2(t)} h_\xi(\xi,t)\frac{d\xi}{x-\xi} + \frac{1}{\pi}\int_{-\infty}^{\infty} f'(\xi)\frac{d\xi}{x-\xi}. \tag{18}$$

However, the possibility now arises that d_1 and d_2 instantaneously move to infinity. Indeed if we assume that, at least for some sufficiently small positive t, $d_2 = -d_1 = \infty$ then (18) states that $h + f$ is an analytic function of $x + it$, and so

$$h(x,t) = \frac{t}{\pi}\int_{d_1(0)}^{d_2(0)} \frac{h(\xi,0)}{t^2 + (x-\xi)^2}d\xi + \frac{t}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{t^2+(x-\xi)^2}d\xi - f(x) \tag{19}$$

satisfies (18). The inequality $h < 0$ can easily be verified in certain cases, for example, when $f = 0$ and the initial free boundary is $h = 0$ for $|x| > 1$ and $h = x^2 - 1$ for $|x| < 1$, but it is interesting to consider the precise conditions on f and h at $t = 0$ for the free boundary to remain below the dam surface. If it does not, then the *nonlinear* singular integro-differential equation (18) must be solved under the side conditions $h(d_i) = 0$ and may require, say, a generalization of the numerical techniques used in [63]. If it does, then the open question of whether the solution of (19) can tend to any of the "horizontal" steady states mentioned in section 2.1 arises.

3.3. Further remarks. We have discussed $\mathcal{P}1$–$\mathcal{P}6$ to try to see how the standard theory of mixed boundary value problems can shed light on a representative group of codimension-two free boundary problems. This theory is, of course, of much less value when the geometry is more complicated or the codimension-two field equations or boundary conditions are nonlinear. The latter situation occurs in [63], where a nonlinear version of (9) is encountered; as discussed there, such situations need to be considered on an ad hoc basis, with even more reliance on physical intuition than was necessary above. Nonetheless, we believe the following general methodological framework can form a basis for tackling a wide class of codimension-two problems. State the problem in its naive form as in $\mathcal{P}1$–$\mathcal{P}6$ and

1. attempt to use asymptotic methods and/or physical arguments to obtain bounds on the allowable singularities at the free points,
2. use theoretical arguments to select those singularities that guarantee a unique solution for the relevant mixed boundary value problem for any arbitrarily prescribed free boundary position,

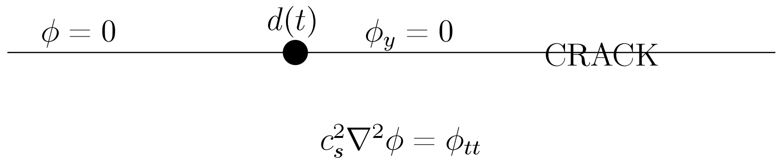

FIG. 14. *Half-plane problem for a propagating crack. Suitable conditions are prescribed at a finite or infinite distance.*

3. either
   (a) use this solution, together with hitherto unexploited information about h to find the free points and hence h or
   (b) if the information about h contained in the boundary data is insufficient, prescribe singularities that overdetermine the problem for ϕ as a function of the free point and hence find the free point as an eigenvalue.

However, we have failed to find any rules for resolving the determinacy problem when applying this methodology to codimension-two problems in general. Moreover, the whole question of the determinacy of steady states seems to be as open as it is for codimension-one problems.

We conclude this section with a very brief description of a mixed boundary formulation of a codimension-two problem that has perhaps been studied more intensively than any other. This is the theory of rectilinear crack propagation in elastic materials, with the implicit assumption that the crack width $2h \geq 0$. In the static case the problem of fracture is usually posed only for a prescribed crack geometry but several theories are also available for crack tip motion [25]. The displacements are all small, and we shall denote them in the case of a type III crack by $\phi(x, y, t)$; we also denote the shear wave speed by c_s.

In the static case it is traditional to assume a switch from zero traction to zero strain at the crack tip in the limit as h tends to zero. Unless some regularizing mechanism is introduced this inevitably leads to singularities at the tip such that

$$\phi \sim \Re \left\{ K(x + iy)^{1/2} \right\}, \tag{20}$$

where the constant K is the "stress intensity factor" and depends on the global geometry and applied tractions. Also, the local solution will be such that $h \sim O(x^{1/2})$ as $x \to 0$.

Now suppose we consider an evolution problem in which the tip $x = d(t)$, $y = 0$ advances with a speed that is not known a priori. One procedure would be to compute K as a function of the tip position and assert that this stress intensity factor must equal some critical value imposed by external considerations such as an energy balance for a locally parabolic tip [29]. Alternatively, we could assume that the tip ultimately becomes cuspidal and that its motion is governed by a cohesive force $g(x, t)$ that acts over a small region in the vicinity of the tip [3]. Then, using matched asymptotic expansions, it is possible to equate the critical stress intensity factor to an integral of g over the cohesive region which is found to be proportional to $(1 - \dot{d}^2/c_s^2)^{1/2}$ [56]. However, neither of these procedures is satisfactory for predicting more commonly occurring practical situations where motion is initiated at a critical value of K and the tip speed then rapidly increases to a value near c_s.

When the tip speed is comparable to the elastic wave speed, we have the codimension-two problem shown in Figure 14 and, unless we are considering a subsonic trav-

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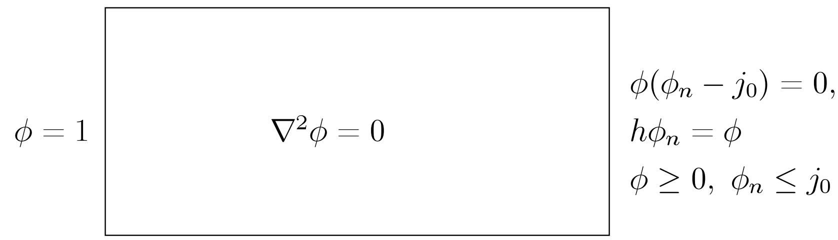

FIG. 15. *Electropainting the inside of a rectangle Ω. The boundary conditions on the right-hand side of the rectangle also hold on the upper and lower faces.*

elling wave solution, we cannot reduce the field equation to Laplace's equation. However, all the available solutions exhibit a local behavior in which

$$\phi \sim \Re\left\{K^*(1 - \dot{d}^2/c_s^2)^{1/2}(x - d + iy)^{1/2}\right\} \tag{21}$$

near the tip for some constant K^*. This again allows the tip velocity to be read off if the "dynamic stress intensity factor" is known, and such an approach often suggests that the tip speed is close to c_s. However, if cohesion ideas are used as a velocity selection mechanism, the matching between the linear travelling wave cohesion problem[5] and (21) gives $K^*(1-\dot{d}^2/c_s^2)^{1/2}$ to equal the aforementioned $(1-\dot{d}^2/c_s^2)^{1/2}$ multiplied by an integral of g. Hence cohesion cannot be used to select the velocity on the basis of a codimension-two analysis. This is just one reason why we shall not discuss these difficult modelling issues further here.

4. Generalized solutions and numerical methods. The theory of some codimension-two free boundary problems is intimately related to that of variational inequalities. Indeed, as mentioned in the introduction, the study of elastic contact problems was a principal motivation for a large part of the general theory of free boundary problems. The prototypical Signorini problem, listed in Table 1 as $\mathcal{P}10$, can easily be formulated as the problem of minimizing the strain energy over displacements that are such that there is no interpenetration of the contacting bodies. This follows because the "stress-free" boundary conditions that apply outside the contact area are the natural boundary conditions for this variational problem. It was the precise connection between this minimization statement and the free boundary problem cited in Table 1 that led to the introduction of variational inequalities [23]; however, it is the minimization formulation that is so useful for rapid and relatively accurate computations.

Variational inequality ideas can be shown to apply directly to $\mathcal{P}1$ and $\mathcal{P}6$ and indirectly to $\mathcal{P}2$ and $\mathcal{P}5$. Consider, for example, the steady state of $\mathcal{P}6$ for a geometry shown in Figure 15, which represents the process of painting the interior of a rectangular box Ω. The Neumann data is the natural boundary condition for the Dirichlet integral and so, instead of trying to solve a mixed boundary value problem as in Figure 9, we simply minimize

$$\frac{1}{2}\iint_\Omega |\nabla\phi|^2 dxdy + j_0\int_\Gamma \phi\, ds, \tag{22}$$

[5]When linear friction is introduced, this becomes another situation where the free point singularity exponent is time dependent as in $\mathcal{P}1$.

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where $\phi = 1$ on the left-hand side of the rectangle but on the workpiece Γ, which consists of the remaining three edges of the rectangle, $\phi(\phi_n - j_0) = 0$ with $\phi \geq 0$ and $\phi_n \leq j_0$. The proof of the equivalence of statements (22) and Figure 15 uses the variational inequality

$$\iint_\Omega \nabla\phi \cdot \nabla(v - \phi)dxdy + j_0 \int_\Gamma (v - \phi)ds \geq 0 \tag{23}$$

for suitable test functions v, as explained in [2].

We can similarly formulate $\mathcal{P}1$ (Figure 1) as the minimization [19] of

$$\iint_{y\leq 0} |\nabla v|^2 dxdy$$

over test functions $v \leq f$ on $y = 0$ having suitable behavior at infinity. In this case it is even possible to formulate the evolution problem as the parabolic variational inequality

$$\frac{1}{2}\iint_{y\leq 0} \nabla\phi \cdot \nabla(v - \phi) + \int_{y=0} \phi_t(v - \phi) \geq 0 \tag{24}$$

for suitable test functions v.

It is clear that the possibility of the formulation of a codimension-two problem as a variational inequality demands that both the field equation be the Euler–Lagrange equation of a minimization problem and the boundary conditions be natural on either the contact or noncontact region. However, it is also crucial that the solution be sufficiently regular at the free boundary for the existence of the estimates that are necessary for a variational inequality to be relevant. Indeed, the key mathematical distinction between the fracture problem for a crack in tension mentioned at the end of the last section and the contact problem (or the closing of a crack in compression) is that the degree of freedom offered by the presence of the free boundary in the contact problem allows the singularity there to be weaker than in the corresponding fracture problem. While problems $\mathcal{P}1$ and $\mathcal{P}6$ have this flexibility, $\mathcal{P}2$ and $\mathcal{P}5$ do not, and their Dirichlet integrals are unbounded at the free points; their free boundary motion, like that of crack tips, is determined by something other than a smoothness condition on the field variable. However, using a smoothing transformation that was introduced in [47] and is reminiscent of a traditional "Baiocchi" transformation,[6] this difficulty can be overcome. When we define Φ^* to be the negative of the displacement potential,

$$\Phi^*(\boldsymbol{x}, t) = -\int_0^t \phi(\boldsymbol{x}, \tau)d\tau, \tag{25}$$

we find that Φ^* satisfies the codimension-two problem in Figure 16. There is now sufficient smoothness at $x = \pm d$ for us to be able to assert that this problem has the variational formulation as the minimization of

$$\frac{1}{2}\iint_{y\leq 0} |\nabla\Phi^*|^2 + \int_{y=0} (t - f)\Phi^* dx, \tag{26}$$

with $\Phi^* \leq 0$ on $y = 0$.

[6] In fact, a Baiocchi transform can also be used to good effect for the codimension-one Hele–Shaw problem [20], but not for the codimension-one water entry problem.

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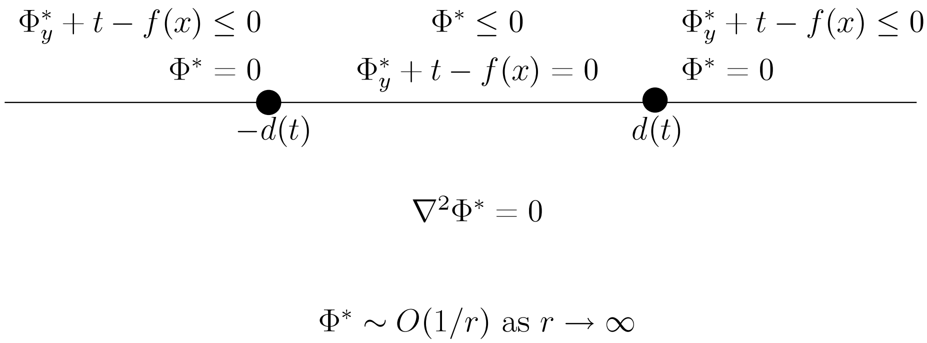

FIG. 16. *Codimension-two problem for the transformed impact problem.*

It is also interesting to study the behavior of the so-called weak formulation of some codimension-one free boundary problems in the codimension-two limit. Suppose, for example, that we consider the classical two-phase codimension-one Stefan problem for the temperature ϕ in the geometry of Figure 17(a). In this configuration, the phase boundary $\phi = 0$ lies at $y = \epsilon h(x,t)$, and we also assume that $y = 0$ is a thermally insulating boundary. Then consider the distributional formulation

$$E_t = \nabla^2 \phi,$$

where the enthalpy E is related to ϕ and the latent heat L^* by

$$E = \begin{cases} \phi + L^* & \text{in Phase II,} \\ \phi & \text{in Phase I.} \end{cases}$$

Assuming that the temperature gradients are initially smaller than $O(1/\epsilon)$, the codimension-two limit is appropriate when the latent heat is large, specifically, $L^* = L/\epsilon$. (If L^* is smaller, say $O(1)$, a rescaling of time shows that the evolution of the phase boundary is entirely governed by the gradient of the initial temperature, which does not change before Phase II disappears.) A formal asymptotic calculation then gives that $E \sim \epsilon L^* h$ in Phase II, and the codimension-two problem is as in Figure 17(b). Again $|\nabla \phi|$ is unbounded at the free points, but we can adopt the same smoothing as in [18] to write

$$\Phi(\boldsymbol{x}, t) = \int_0^t \phi(\boldsymbol{x}, \tau)\, d\tau.$$

The resulting codimension-two problem is shown in Figure 17(c), where ϕ_0 is the initial temperature (note that the free points move inward). This problem admits a parabolic variational inequality formulation and is, in fact, the codimension-two limit of the variational formulation introduced for the classical Stefan problem in [18]. In the special case when the codimension-one problem has an explicit similarity solution with an elliptical free boundary [30, 38], the codimension-two approximation is confirmed as correct.

Numerical algorithms for most of the above variational formulations have been implemented in the references cited. It seems that in each case reasonably reliable answers have been obtained whenever it has been possible to compare numerical solutions with explicit analytical solutions. However, few error estimates are available,

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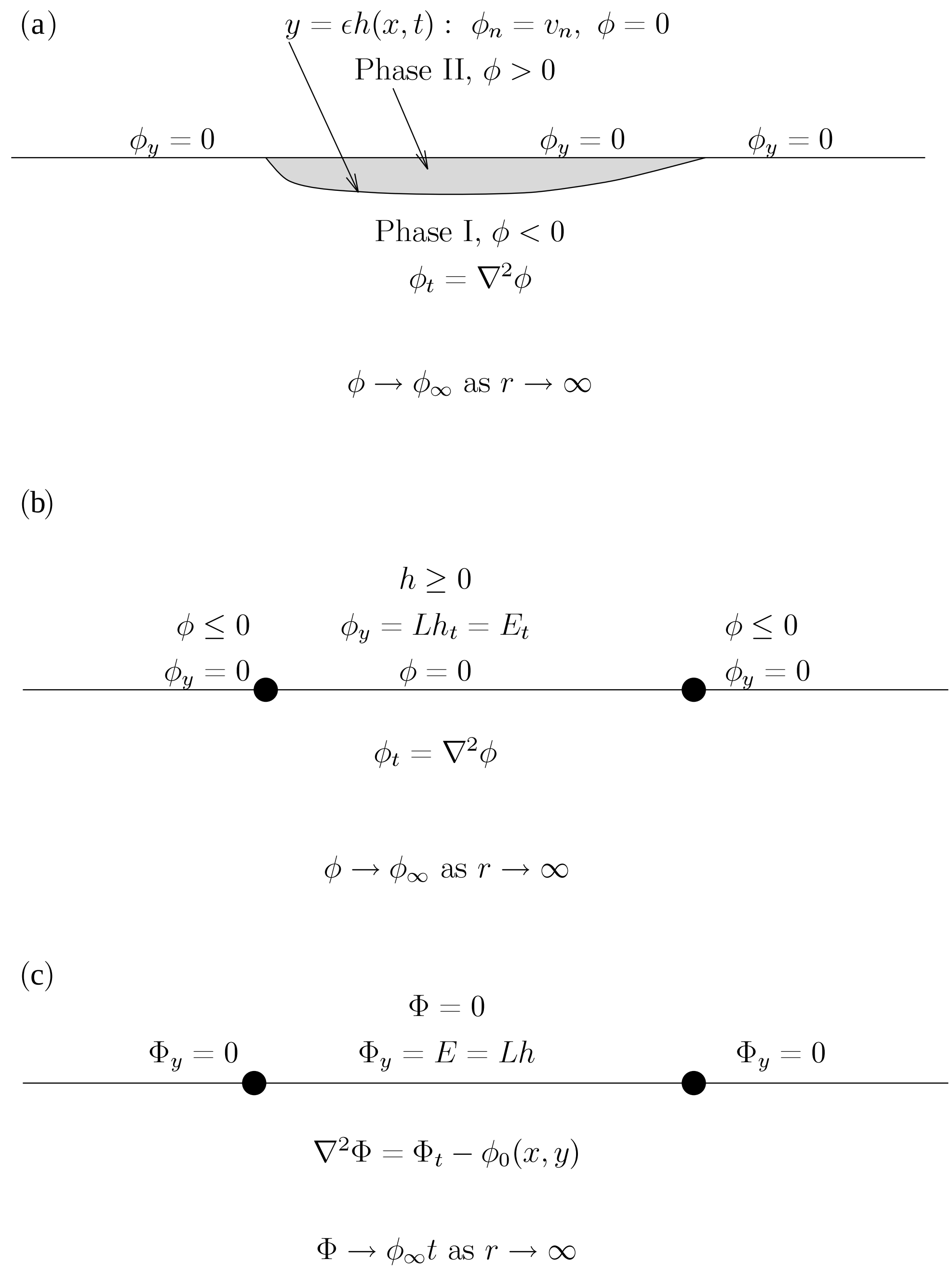

FIG. 17. *The Stefan problem:* (a) *codimension-one version;* (b) *codimension-two formulation;* (c) *codimension-two formulation in the transformed variables.*

especially concerning the position of the free boundary. Of course, there is no point in obtaining any accuracy greater than that of the power of ϵ that appears in the spatial scaling demanded by a matched asymptotic expansion procedure in the vicinity of the codimension-two free point.

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When a variational inequality is unavailable, direct discretizations can also be attempted for any singular integral equation representation that may be available. The best philosophy seems to be to invert any singular integral terms that involve derivatives of h and then integrate with respect to arc length so as to build in all the available information about h before, say, making a piecewise-constant approximation. Other ad hoc discretizations have been constructed for evolution problems; e.g., it is possible to time-step the unsteady version of $\mathcal{P}6$ and obtain results that tend to the correct steady state [2, 8].

5. Perturbation theory and stability. Perturbation methods have provided great insight into the structure of many venerable codimension-one free boundary problems. For the purposes of this article, we shall consider perturbation expansions only in terms of a small *geometric* parameter, there being two basic regimes in which such expansions have been successful for codimension-one problems. The first involves linearization about a simple explicit classical solution and is traditional, for example, in the theory of small amplitude surface gravity waves or the initiation of morphological instabilities in Stefan problems. The second considers long wavelength, but not necessarily small amplitude, variations in the free boundary location, as say in hydraulics or the Dupuit approximation in aquifer flows. No systematic extension of these ideas to codimension-two problems appears to be available, so we will just describe two relatively simple cases from which some surprising features emerge.

5.1. Linear stability theory. It is important to consider how the stability of the solutions of codimension-two problems reflect those of their progenitors, especially as it is well known that many codimension-one solutions can easily switch from being linearly stable to linearly unstable when, say, the direction of motion of the free boundary is reversed [59]. In section 2, we commented on this state of affairs in the context of two-dimensional models, and our objective here is to point out a new undesirable attribute that a codimension-two problem which is modulated along the free curve can acquire when the direction of motion of the free boundary is reversed. Hence $\mathcal{P}2$ seems a good example to study because it has a simple explicit two-dimensional solution and we know from experience that the flows generated by the entry and exit of rigid bodies in liquid half-spaces are not time reversals of each other. However, the mathematical model, as in Figure 3(b), *is* time reversible, and we seek to explain this situation in terms of a linear stability analysis in the same spirit as that used to distinguish between injection and suction in Hele–Shaw flows [59].

As is often the case for codimension-one problems, it is much easier to consider a *local* stability analysis of the solution near a free point in a coordinate system (here denoted by hatted variables) moving with the free point at speed V ($V > 0$ for entry and $V < 0$ for exit), so we shall only consider perturbations to the solution of

$$\nabla^2\hat{\Phi} = 0 \quad \text{in} \quad \hat{y} < 0, \tag{27}$$

$$\hat{\Phi}_{\hat{y}} = 0 \quad \text{on} \quad \hat{y} = 0, \quad \hat{x} < 0, \tag{28}$$

$$\left.\begin{aligned} -V\hat{h}_{\hat{x}} + \hat{h}_{\hat{t}} &= \hat{\Phi}_{\hat{y}} \\ \hat{\Phi} &= 0 \end{aligned}\right\} \quad \text{on} \quad \hat{y} = 0, \quad \hat{x} > 0, \tag{29}$$

$$\hat{h}(0, \hat{t}) = 0, \tag{30}$$

where we have assumed that variations in the $\hat{z}$-direction are going to be sufficiently rapid so that the velocity of the impacting body can be neglected in (28). The steady

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solution to (27)–(30), which matches with the outer solution in (14), is

$$\hat{\Phi} = A\hat{r}^{1/2}\sin(\hat{\theta}/2), \tag{31}$$

$$\hat{h} = -\frac{A}{V}\hat{x}^{1/2}, \tag{32}$$

where A and V are constants determined by the global motion. Note that $A/V > 0$ since $\hat{h}(\hat{x}) < 0$.

Now small oscillations are imposed on the free curve in the $\hat{z}$-direction. A perturbation of (27)–(30) is sought where the free curve is situated at $\hat{x} = \delta e^{\sigma\hat{t}}\cos n\hat{z} + o(\delta)$, where $\delta \ll 1$ and $n > 0$ are given and σ is to be determined. The relevant perturbations to the potential and free boundary are

$$\hat{\Phi} \sim A\hat{r}^{1/2}\sin(\hat{\theta}/2) + \delta B\frac{\sin(\hat{\theta}/2)}{\hat{r}^{1/2}}e^{-n\hat{r}+\sigma\hat{t}}\cos n\hat{z} + \cdots, \tag{33}$$

$$\hat{h} \sim -\frac{A}{V}\hat{x}^{1/2} + \delta\hat{h}^{(1)}(\hat{x})e^{\sigma\hat{t}}\cos n\hat{z} + \cdots. \tag{34}$$

Substituting $\hat{h}$ and $\hat{\Phi}$ into the kinematic boundary condition (29) gives a differential equation for $\hat{h}^{(1)}$, namely,

$$\frac{Be^{-n\hat{x}}}{2\hat{x}^{3/2}} = -V\frac{d\hat{h}^{(1)}}{d\hat{x}} + \sigma\hat{h}^{(1)}, \tag{35}$$

which can be integrated to give

$$\begin{aligned}\hat{h}^{(1)} = \frac{B}{V}e^{\sigma\hat{x}/V}\Bigg[&\hat{x}^{-1/2}\exp\left\{-\left(\frac{\sigma}{V}+n\right)\hat{x}\right\} \\ &+ \left(n+\frac{\sigma}{V}\right)\int_0^{\hat{x}} s^{-1/2}\exp\left\{-\left(\frac{\sigma}{V}+n\right)s\right\}ds + K\Bigg],\end{aligned} \tag{36}$$

where K is a constant. Physically acceptable solutions have $\hat{h}^{(1)}(\hat{x}) \sim o(\hat{x}^{1/2})$ as $\hat{x} \to \infty$. If $\sigma/V \le 0$, this is always true and so, although σ is undetermined, entry problems ($V > 0$) are stable and exit problems are unstable. We still must deal with the possibility $\sigma/V > 0$, and we claim that this leads to a contradiction and is thus impossible.

If $\sigma/V > 0$, then a physically acceptable solution for $\hat{h}^{(1)}$ must have

$$K = -\left(n+\frac{\sigma}{V}\right)\int_0^{\infty}\frac{e^{-(n+\sigma/V)s}}{\sqrt{s}}ds = -\pi^{1/2}\left(n+\frac{\sigma}{V}\right)^{1/2}. \tag{37}$$

Now, as suggested in [55], an inner region near the free point must be constructed. Although this calculation is rather long and cumbersome, it reveals that the free boundary condition is violated unless $K = 0$, whence $\sigma/V = -n < 0$, contradicting the assumption $\sigma/V > 0$. Recently, a global linear stability analysis was presented in [11].

In summary, although we have shown that σ should have the opposite sign to V, we have been unable to obtain a dispersion relation of the conventional type. Clearly much further work needs to be carried out in this area if a sensible comparison is to be made with the relatively straightforward stability analysis of the codimension-one water entry problem which, in this parameter regime, gives neutral stability.

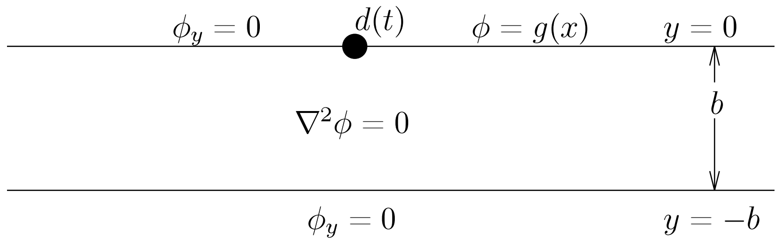

FIG. 18. *Long wavelength approximation for the shallow dam.*

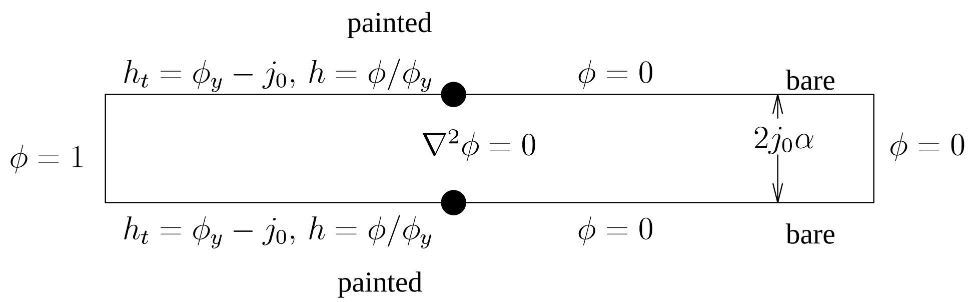

FIG. 19. *Electropainting in a long thin box.*

5.2. Long wavelength approximations. We recall two illustrations of this kind of approximation. The first concerns $\mathcal{P}1$ when an impermeable base is introduced much nearer to the dam surface than d, as in Figure 18. Then, as in [39],

$$\phi \sim \phi_0(x,t) - \frac{1}{2}\left((y+b)^2\phi_{0xx} + b^2\phi_2(x,t)\right) + \cdots \quad \text{as } b \to 0 \tag{38}$$

and, after a rescaling in time with $\tau = bt$, the boundary condition yields

$$\phi_{0\tau} = \phi_{0xx} \quad \text{for} \quad x < d(\tau), \tag{39}$$

with $\phi_0(d(t),t) = -g(d(t))$ and $\phi_{0x}(d(t),0,t) = -g'$. Writing $\psi = \phi_0 + g(x)$, we have

$$\begin{aligned} \psi_\tau = \psi_{xx} + g''(x) \quad &\text{for} \quad x < d(\tau), \\ \psi = \psi_x = 0 \quad &\text{for} \quad x \geq d(\tau). \end{aligned}$$

Since $\psi \geq 0$, ψ satisfies a version of the so-called oxygen consumption problem [15] with consumption when $g'' < 0$ and replenishment when $g'' > 0$. This permits discontinuous motion of d for certain g. Also, as remarked in [39], the fact that the well posedness of the oxygen problem does not depend on the sign of $\dot{d}$ is evidence that the codimension-two dam model in Table 1 is well posed either in the presence of inflow or outflow.

Second, we may again follow [39] and consider $\mathcal{P}6$ in a long thin box as in Figure 19 when $j_0 \ll 1$ and the edges of the box are at $\pm j_0\alpha$ for some $O(1)$ constant α.

The analogue of (38) is now

$$\phi \sim \phi_0(x,t) - \frac{1}{2}y^2\phi_{0xx} + j_0^2\phi_2(x,t) + \cdots .$$

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It is again necessary to rescale time via $t = j_0^{-1}\tau$, and then the boundary conditions on $y = \pm\alpha j_0$ lead to the third-order system

$$h_\tau = \alpha\phi_{0xx} - 1, \quad \alpha h \phi_{0xx} = \phi_0 \tag{40}$$

for $0 < x < d(t)$, where there is paint, and $\phi_0 = \phi_{0x} = h = 0$ at $x = d(t)$. In contrast to (39), this is an almost unstudied codimension-one free boundary problem because the presence of third-order space derivatives precludes any standard treatment.

6. Conclusions. We have tried to present as comprehensive an account as possible of the current status of the theory of codimension-two free boundary problems. We have primarily restricted our attention to two-dimensional problems where the field equation is either Laplace's equation or the biharmonic equation; even within this class, only a few codimension-one progenitor problems have exact solutions, but in *every* case these confirm the correctness of the codimension-two approximation. However, the scarcity of exact solutions in itself indicates the need for a more thorough theoretical investigation. In particular, we cite the following deficiencies and requirements:

1. The lack of a theory of weak solutions when the problem does not have any obvious variational formulation.
2. The need for the development of numerical algorithms that can locate the codimension-two free points accurately. In the absence of a variational formulation, such algorithms have been proposed only on an ad hoc basis.
3. A rationalization of the methodology outlined at the end of section 3, especially concerning the question of the degree of indeterminacy of the solution of the relevant mixed boundary value problem.
4. The lack of an existence, uniqueness, and regularity theory for evolution problems.

While there is a vital practical need for answers to the second and third of these, we regard the fourth as the most challenging and important theoretical question. Throughout our review, we have come across manifestations of singular behavior; there is the likely ill posedness of $\mathcal{P}1$, $\mathcal{P}2$, and $\mathcal{P}5$ for many classes of driving mechanism, the possible instantaneous movement of the free point in $\mathcal{P}1$ as described by (18) or under approximation (39), and finally the instability demonstrated in section 5. All of these points lead us to pose the question of the relationship between the stability of codimension-two problems and that of their codimension-one progenitors. In particular, despite the absence of any kind of traditional stability theory, as in [60], we make the following conjectures:

- If a codimension-one problem is stable or unstable, then the corresponding codimension-two problem will have the same behavior.
- If a codimension-one problem is neutrally stable, then the stability of the corresponding codimension-two problem can be categorized in terms of the dynamics of the contact region. In certain cases, if the contact region is expanding, the problem is stable, and if it is contracting, the problem is unstable.

Beyond these concerns, there is the more fundamental possibility that if any general theory is to emerge it will have to encompass configurations that have not been mentioned in this review. For example, codimension-*three* problems could be contemplated, say, when two sessile lenses meet at a point [65]. An even more fascinating possibility is illustrated by the inviscid version of $\mathcal{P}4$. The configuration (in three dimensions) is described in [62], together with some experimental and numerical ev-

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idence concerning the codimension-one formulation. But a glance at the proposed multiply-connected morphology in that problem suggests how conservative we have been in our description of the whole subject, at least as far as the number of free points and the nature of their singularities is concerned.

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