2 Models for thin domains

2.1 Introduction

Many manufacturing technologies involve processing thin regions of material. Examples include the "hot rolling" process to produce thin sheets of steel, as well as the processing of glass to produce slender optical fibres or thin sheets of glass for windows or television screens. In such situations, the geometry is "thin" in the sense that one dimension is very much smaller than another; for example the thickness of a glass tablet screen is very much smaller than a typical width (by a factor of at least 10^{-2}). It can be computationally challenging to simulate problems with such a vast separation of length-scales. However, we can instead *exploit* the thinness of the geometry to derive simplified models which are much more amenable to computation, analysis and (if we are very lucky) analytic solution.

A very well-known example is *lubrication theory*, which gets its name from its application to the thin films of oil that lubricate moving parts in an engine. Lubrication theory applies when a viscous fluid flows in a thin layer over a rigid boundary, and the thinness of the geometry permits a huge simplification of the governing Navier–Stokes equations. Another example is the flow of heat along a thin, slowly-varying rod; we will show how the three-dimensional heat equation may be simplified and the resulting model will be applied to a crystal growing process. Our final example concerns *extensional flow* of a thin sheet or fibre of viscous fluid, with applications to the stretching of glass sheets or optical fibres.

2.2 Lubrication theory

Industrial problem: hot rolling of steel

Thin sheets of steel are manufactured using a "hot rolling" process. A sheet of steel is heated up and then squeezed between rollers to reduce its thickness. A schematic of the process is shown in Figure 2.1. Here a sheet with initial thickness $2h_0$ is fed at speed U between two rollers: only the upper half us shown, with symmetry assumed about the x-axis. The rollers have radius R and rotate with angular speed Ω, so that the surface speed is ΩR. The gap between the rollers (the "nip") is h_{m}. The aim is to determine how the final sheet thickness h_1 depends on the input thickness h_0 and the control parameters U, Ω, R and h_{m}.

Newtonian lubrication theory

Lubrication theory is an approximation that holds when the geometry of the problem is thin, i.e. when the length-scale in the y-direction is much smaller than that in the x-direction in the setup illustrated in Figure 2.1. This will be true provided the rollers are large compared with the sheet thickness, i.e. provided $h_0 \ll R$. In this case, the surface $y = h(x)$ of the roller

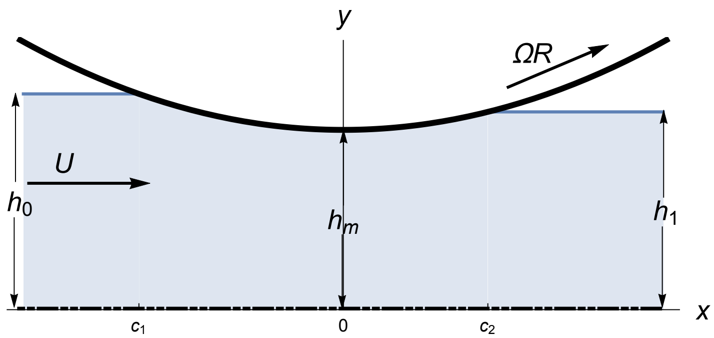

Figure 2.1: Schematic of a hot rolling process.

is approximately parabolic, with

$$h(x) = h_{\mathrm{m}} + R - \sqrt{R^2 - x^2} \sim h_{\mathrm{m}} + \frac{x^2}{2R} + O\left(\frac{x^4}{R^3}\right). \tag{2.1}$$

Therefore the incoming sheet will first make contact at

$$x = c_1 \sim \sqrt{2R(h_0 - h_{\mathrm{m}})}, \tag{2.2}$$

and the *aspect ratio* of the nip region is of order

$$\epsilon = \sqrt{\frac{h_0}{R}}. \tag{2.3}$$

The lubrication approximation arises when we take the limit $\epsilon \to 0$.

For simplicity, we assume that the problem is purely two-dimensional and that the hot steel may be modelled as an incompressible Newtonian viscous liquid. In reality, the rheology is likely to be more complicated, and to be strongly temperature dependent, but this simple theory will illustrate the general method and should give qualitatively reasonable results.

The velocity $\boldsymbol{u} = (u, v)$ and pressure p in the steel then satisfy the Navier–Stokes equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \qquad \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right) = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{u}, \tag{2.4}$$

where ρ is the density and μ is the viscosity (both assumed constant).

The incoming sheet has uniform thickness $2h_0$ and horizontal velocity U, and is at atmospheric pressure p_{a}. Similarly, the outgoing sheet has constant thickness $2h_1$ (which is to be determined) and horizontal velocity given by Uh_0/h_1 (by net mass conservation). Hence, denoting the top surface of the sheet by $y = h(x)$, we have

$$h \to h_0, \quad \boldsymbol{u} \to (U, 0), \quad p \to p_a \qquad \text{as } x \to -\infty, \tag{2.5a}$$

$$h \to h_1, \quad \boldsymbol{u} \to (Uh_0/h_1, 0), \quad p \to p_a \qquad \text{as } x \to +\infty. \tag{2.5b}$$

The flow is assumed to be symmetric about the x-axis, so we impose the symmetry conditions

$$v = \frac{\partial u}{\partial y} = 0 \quad \text{at } y = 0. \tag{2.6}$$

The conditions on the top surface $y = h(x)$ switch depending on whether or not the sheet is in contact with the roller. Where there is contact, the velocity of the steel must equal that of the roller; where the surface is free of the roller, the stress on it and the normal velocity must be zero. Therefore the conditions on $y = h(x)$ are

$$v = uh'(x), \quad u + h'(x)v = \Omega R\sqrt{1 + h'(x)^2} \qquad c_1 < x < c_2, \tag{2.7a}$$

$$v = uh'(x), \quad \sigma_{xy} = h'(x)\sigma_{xx}, \quad \sigma_{yy} = h'(x)\sigma_{xy} \qquad x < c_1 \text{ and } x > c_2, \tag{2.7b}$$

where for a Newtonian fluid the stress components are given by

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \qquad \sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y}. \tag{2.8}$$

The function $h(x)$ is given by equation (2.1) for $c_1 < x < c_2$ but in principle is to be determined in $x < c_1$ and in $x > c_2$.

In general this is a formidable mixed boundary value problem. However, when we exploit the lubrication approximation things become much simpler. The first step is to non-dimensionalise the problem is follows:

$$h = h_0\tilde{h}, \qquad (x, y) = h_0\left(\epsilon^{-1}\tilde{x}, \tilde{y}\right), \qquad (u, v) = U\left(\tilde{u}, \epsilon\tilde{v}\right), \qquad p = p_a + \left(\frac{\mu U}{\epsilon h_0}\right)\tilde{p}. \tag{2.9}$$

The scalings (2.9) may be obtained by seeking dominant balances in the governing equations (2.4). Note here that we have exploited the slenderness of the geometry by imposing that $x \gg y$ and $v \ll u$. Also note that the scaling for p is larger by a factor of ϵ^{-1} than we would expect for classical two-dimensional viscous flow: this reflects the large pressures generated by compressing the sheet through a very thin gap.

So, we use (2.9) rescale the governing equations (2.4) and the boundary conditions (2.7a) under the roller, and then let the small parameter $\epsilon \to 0$. The problem involves one more dimensionless parameter, namely the *reduced Reynolds number*

$$\mathrm{R}^* = \frac{\epsilon\rho U h_0}{\mu}, \tag{2.10}$$

which is assumed to be negligible. (This means that the effects of inertia, represented by the left-hand side of equation (2.4b), are negligible, as is typical in lubrication problems.)

This procedure results in the following leading-order model:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0 \qquad 0 < y < h(x), \ -\alpha < x < \beta, \tag{2.11a}$$

$$v = \frac{\partial u}{\partial y} = 0 \qquad y = 0, \ -\alpha < x < \beta, \tag{2.11b}$$

$$u = \gamma, \quad v = \gamma h'(x), \qquad y = h(x), \ -\alpha < x < \beta, \tag{2.11c}$$

where

$$h(x) = \eta + \frac{x^2}{2}, \tag{2.12}$$

and the dimensionless parameters remaining in the problem are

$$\alpha = -\frac{c_1}{\sqrt{h_0 R}}, \qquad \beta = \frac{c_2}{\sqrt{h_0 R}}, \qquad \gamma = \frac{\Omega R}{U}, \qquad \eta = \frac{h_{\mathrm{m}}}{h_0}. \tag{2.13}$$

From the specified incoming sheet thickness we have $h(-\alpha) = 1$ (in dimensionless variables), and hence

$$\alpha = \sqrt{2(1-\eta)}, \tag{2.14}$$

but β remains to be determined.

It is often helpful in these problems to use an equation representing net conservation of mass. Here, by integrating the first equation of (2.11a) from $y = 0$ to $y = h(x)$ and then applying the boundary conditions (2.11b) and (2.11c), we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{h(x)} u(x,y)\,\mathrm{d}y\right) = 0. \tag{2.15}$$

The integral in brackets must therefore be a constant, corresponding to the net flux of fluid through the nip. This must equal the flux fed in from the left (which is 1 in dimensionless variables), and therefore

$$\int_0^{h(x)} u(x,y)\,\mathrm{d}y \equiv 1. \tag{2.16}$$

Now integration of the remaining equations leads to

$$u = \gamma - \frac{1}{2}\frac{\mathrm{d}p}{\mathrm{d}x}\left(h^2 - y^2\right), \tag{2.17}$$

where now $p = p(x)$ is independent of y. Thus equation (2.16) becomes

$$\gamma h - \frac{h^3}{3}\frac{\mathrm{d}p}{\mathrm{d}x} = 1, \tag{2.18}$$

which is a version of *Reynolds' equation*. Since $h(x)$ is given by (2.12), this is just an ODE for $p(x)$. It remains to deduce the relevant boundary conditions.

As the sheet first enters the nip, the viscous stress caused by the roller will be of order $\mu U/h_0$. This is a factor ϵ smaller than the scaling for p introduced in (2.9), and hence the leading-order matching condition for our lubrication problem is $p = 0$ at $x = -\alpha$. Similarly, the leading-order pressure must be zero as the sheet exits the nip at $x = \beta$, so the boundary conditions for equation (2.18) are

$$p(-\alpha) = p(\beta) = 0. \tag{2.19}$$

It follows that

$$\int_{-\alpha}^{\beta}\left(\frac{\gamma}{h^2} - \frac{1}{h^3}\right)\mathrm{d}x = 0. \tag{2.20}$$

In principle this determines the position of the exit point $x = \beta$, and hence also the final sheet thickness $h(\beta)$, in terms of the control parameters γ and η.

It transpires that, given η, physical solutions for β exist only for a range of values of γ — see Exercise 1. This suggests that the process will fail unless the roller rotation speed is carefully set; for example, the sheet will jam if the roller speed is too slow compared with the feed speed. This also should prompt us to examine whether the assumptions made in the model are uniformly valid.

In particular, we have assumed that no deformation of the sheet occurs before it enters the rollers. If the sheet really was made of Newtonian viscous liquid, then it would stretch under the tensile force of the rollers, and the value of h where it first makes contact would be slightly less than the input value h_0. In reality, hot steel is not Newtonian, but more like a "viscoplastic" material, which only flows appreciably when the stress is sufficiently large. This explains why there is negligible deformation until the high-stress zone under the roller is encountered.

Power law fluid

As a first step towards a more realistic viscoplastic model for the sheet, we now present a relatively simple model that captures some of the important features. The basic idea is to treat the metal as a viscous liquid whose viscosity is not constant but varies with the shear rate, i.e.

$$\mu = f\big(||\boldsymbol{D}||\big), \qquad \text{where} \quad D_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{2.21}$$

is the rate-of-strain tensor. The relevant norm is defined such that

$$||\boldsymbol{D}||^2 = \frac{1}{2}\sum_{i,j} D_{ij}D_{ij} = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \tag{2.22}$$

in two dimensions.

A *power law fluid* corresponds to the particular choice

$$f\big(||\boldsymbol{D}||\big) = K||\boldsymbol{D}||^{n-1}, \tag{2.23}$$

where n (dimensionless) and K (dimensions $\mathrm{Pa\,s}^n$) are positive parameters characterising the rheological behaviour of the fluid. Increasing K corresponds to increasing the overall viscosity, while n measures the sensitivity of the viscosity to the shear rate. The case $n = 1$ gives a constant-viscosity Newtonian fluid; if $n > 1$ then the fluid is *shear thickenning*; if $n < 1$ then the fluid is *shear thinning*.

If n is small, then the viscosity becomes extremely large, so the material is effectively rigid, at low shear rates. This behaviour is called *pseudo-plastic*: it mimics the desired behaviour of effectively no flow occuring unless the applied stress is sufficiently high.

It is relatively easy to extend the lubrication theory model for the hot rolling problem to the case of a power law fluid: see Exercise 2.

2.3 Heat flow in a slowly varying domain

Industrial problem: the Czochralski method for crystal growth

The Czochralski method is a technique for growing semiconductor crystals for the electronics industry. The basic setup is illustrated in Figure 2.2. A small seed crystal is dipped into a pool of molten material and then withdrawn very slowly and smoothly. When the technique is successful, a large crystal may be grown with few defects, but this relies on extremely careful control of the temperature and the drawing speed V, which may be varied with time.

A model is required to determine how the size and shape of the crystal depend on the drawing protocol. It is also dequired to evaluate the temperature profile in the crystal, since the resuling thermal stress can lead to the formation of undesirable defects.

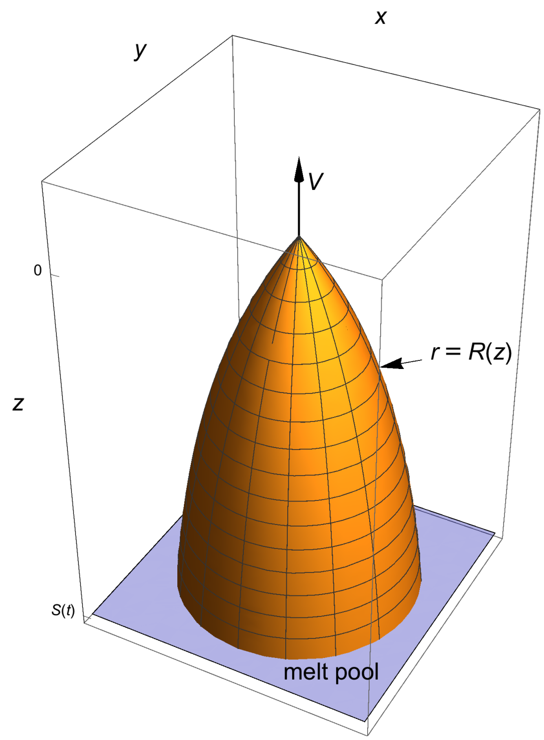

Figure 2.2: Schematic of the Czochralski method for crystal growth.

Heat flow in a thin rod

In practice, a crystal grown using the Czochralski method resembles a thin rod, with a radius $R(z)$ which is much smaller than its length and varies slowly with distance z along the axis of the crystal. This thinness of the geometry may be exploited to derive a simple model for heat flow along the rod, in a similar manner to the lubrication approximation from §2.2.

We use cylindrical polar coordinates (r, θ, z), with the origin at the top of the crystal as shown in Figure 2.2, so the melt pool as at $z = S(t)$, where $\dot{S} = V$, the draw speed. We assume radial symmetry, so that the temperature $T(r, z, t)$ in the rod satisfies the heat equation in the form

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T, \qquad \text{where } \nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}, \tag{2.24}$$

and the density ρ, specific heat c and thermal conductivity k are assumed constant (these assumptions can of course be relaxed).

At the boundary of the rod, given by $r = R(z)$, there is a transfer of heat to the surrounding atmosphere, which is assumed to be at constant temperature T_0. This is typically modelled using an empirical law called *Newton's law of cooling*, namely

$$-k\frac{\partial T}{\partial n} = h(T - T_0) \quad \text{at } r = R(z), \tag{2.25}$$

where h is the *heat transfer coefficient*, which depends on the properties of the surrounding atmosphere. In practice h is small (in a sense defined below), meaning that the rod loses heat slowly to the surrounding atmosphere. This is consistent with the crystal growing to be long and thin before it cools down to the ambient temperature. More complicated versions of (2.25) may be formulated to include nonlinear effects such as radiative cooling. At the axis of the rod $r = 0$, we only require the temperature to be bounded so that

$$r\frac{\partial T}{\partial r} \to 0 \quad \text{as } r \to 0. \tag{2.26}$$

We expect the crystal radius to decrease to zero at the top, i.e. $R(z) \to 0$ as $z \to 0$. We then anticipate that no boundary condition for T will be required at $z = 0$, beyond saying that T must be bounded. The other end of the rod is where the crystal meets the melt pool. We assume for the moment that this solid-liquid interface is flat and given by $z = S(t)$. Therefore we impose the usual Stefan conditions

$$T = T_{\mathrm{m}}, \quad q - k\frac{\partial T}{\partial z} = \rho L\frac{\mathrm{d}S}{\mathrm{d}t} \qquad \text{at } z = S(t), \tag{2.27}$$

where T_{m} is the melting temperature and L is the latent heat, while q is the heat flux coming from the melt pool. In principle, q is determined by a complicated problem for the turbulent flow and the temperature in the melt pool. Rather than trying to tackle this difficult problem here, instead we assume that q is a given constant, which in principle could be varied by tweaking the operating conditions.

Now we nondimensionalise the problem. We denote a typical draw speed by V, a typical length-scale for the rod by ℓ and a typical radius by $\epsilon\ell$. The rod is thin if $\epsilon \ll 1$, and we will see how the problem may be simplified in this limit. Specific choices for these parameters may be made by balancing relevant terms in the equations and boundary conditions. The Stefan condition (2.27) suggests a length-scale and a velocity-scale, namely

$$\ell = \frac{k\,(T_{\mathrm{m}} - T_0)}{q}, \qquad\qquad V = \frac{q}{\rho L}. \tag{2.28}$$

We choose ϵ such that the cooling right-hand side of the heat transfer law (2.25) is smaller than the left-hand side by a factor of ϵ^2. This choice leads to the definition

$$\epsilon = \frac{h\ell}{k} = \frac{h\,(T_{\mathrm{m}} - T_0)}{q}, \tag{2.29}$$

and the thin-rod approximation carried out below then relies on ϵ being small: it is in this sense that the heat transfer coefficient is assumed to be small. The reasons for the particular choice (2.29) will become clearer below: it leads to a dominant balance between cooling at the surface of the rod and thermal diffusion in the axial direction.

We then define dimensionless variables as follows:

$$(r, z)z = \ell\,(\epsilon\tilde{r}, \tilde{z})\,, \qquad R = \epsilon\ell\tilde{R}, \qquad t = \frac{\ell}{V}\,\tilde{t}, \qquad T = T_0 + (T_{\mathrm{m}} - T_0)\,u, \tag{2.30}$$

so the governing equations and boundary conditions become (with tildes now dropped)

$$\epsilon^2\mathrm{St}\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \epsilon^2\frac{\partial^2 u}{\partial z^2} \qquad 0 < r < R(z),\ 0 < z < S(t), \tag{2.31a}$$

$$\frac{\partial u}{\partial r} - \epsilon^2 R'(z)\frac{\partial u}{\partial z} = -\epsilon^2 u\sqrt{1 + \epsilon^2 R'(z)^2} \qquad r = R(z),\ 0 < z < S(t), \tag{2.31b}$$

$$u = 1, \quad \frac{\partial u}{\partial z} = 1 + \dot{S}(t) \qquad z = S(t),\ 0 < r < R\big(S(t)\big), \tag{2.31c}$$

where

$$\mathrm{St} = \frac{c(T_\mathrm{m} - T_0)}{L} \tag{2.32}$$

is the usual Stefan number.

Now we expand u as an asymptotic expansion in powers of ϵ^2, i.e.

$$u(r,z,t) \sim u_0(r,z,t) + \epsilon^2 u_1(r,z,t) + \cdots \qquad \text{as } \epsilon \to 0. \tag{2.33}$$

At leading order we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right) = 0 \tag{2.34a}$$

subject to

$$\frac{\partial u_0}{\partial r} = 0 \quad \text{at } r = R(z). \tag{2.34b}$$

It follows that $\partial u_0/\partial r \equiv 0$, so that u_0 is a function only of z and t: the temperature is approximately uniform in each cross-section of the rod. Otherwise, there does not appear to be any way to determine $u_0(z,t)$: apparently any function that satisfies the boundary conditions

$$u_0 = 1, \quad \frac{\partial u_0}{\partial z} = 1 + \dot{S}(t) \qquad \text{at } z = S(t) \tag{2.35}$$

will do.

To obtain the governing equation for the leading-order temperature u_0, it is necessary to proceed to higher order in the expansions. At order ϵ^2 we find that u_1 satisfies the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r}\right) = \mathrm{St}\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial z^2} \qquad r < R(z), \tag{2.36a}$$

$$\frac{\partial u_1}{\partial r} = R'(z)\frac{\partial u_0}{\partial z} - u_0 \qquad r = R(z). \tag{2.36b}$$

Note that this is an inhomogeneous version of the problem (2.34) satisfied by u_0. We already know that (2.34) admits nontrivial eigensolutions, namely any function independent of r. Therefore the Fredholm Alternative implies that the inhomogeneous problem (2.36) will have no solutions for u_1 unless a *solvability condition* is satisfied. This solvability condition will give us the governing equation for u_0 that we are looking for.

For the problem (2.36) it is easy to find the solvability condition just by integrating equation (2.36a) with respect to r and then applying the boundary condition (2.36b):

$$\int_0^{R(z)} \left(\mathrm{St}\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial z^2}\right) r\,\mathrm{d}r = \left[r\frac{\partial u_1}{\partial r}\right]_{r=0}^{r=R(z)} \tag{2.37}$$

$$\Rightarrow \quad \left(\mathrm{St}\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial z^2}\right)\frac{R(z)^2}{2} = \left(R'(z)\frac{\partial u_0}{\partial z} - u_0\right)R(z), \tag{2.38}$$

which can be rearranged to

$$\mathrm{St}\frac{\partial}{\partial t}\left(\pi R^2 u_0\right) = \frac{\partial}{\partial z}\left(\pi R^2\frac{\partial u_0}{\partial z}\right) - 2\pi R u_0. \tag{2.39}$$

Equation (2.39) is the leading-order governing equation for heat flow in a thin rod. Each term has a clear physical interpretation. The left-hand side is the normalised rate-of-change of thermal energy, where the effective heat capacity of the rod is proportional to the cross-sectional area. The first term on the right-hand side describes the heat flux along the rod, with an effective thermal conductivity again proportional to the cross-sectional area. The final term measures the loss of heat to the surrounding atmosphere, at a rate proportional to the circumference of the rod. Our definition (2.29) of ϵ was made precisely so that the two terms on the right-hand side of equation (2.39) would balance.

The derivation given here illustrates the use of a solvability condition for the higher-order terms in an asymptotic expansion to derive an amplitude equation for the leading-order solution. This situation arises frequently, for example in the method of multiple scales; a simple example from linear algebra is given in Exercise 3. As in lubrication theory, one can use an integrated equation representing net energy conservation as a shortcut to obtaining the leading-order governing equation: see Exercise 4.

Modelling of the Czochralski method

Now drop the zero subscript on the leading-order normalized temperature $u(z,t)$. Our final model is therefore

$$\mathrm{St}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + \frac{2R'(z)}{R(z)}\frac{\partial u}{\partial z} - \frac{2u}{R(z)}, \qquad 0 < z < S(t), \tag{2.40a}$$

$$u = 1, \quad \frac{\partial u}{\partial z} = 1 + \dot{S}(t) \qquad z = S(t). \tag{2.40b}$$

As pointed out above, we assume that $R(z) \to 0$ as $z \to 0$, and therefore do not expect to require any boundary conditions there other than ensuring that u is bounded as $z \to 0$. The initial conditions are simply $S(0) = 0$; Exercise 5 demonstrates that this is sufficient to define the solution as $t \to 0$.

The problem (2.40) is a slightly strange-looking inverse Stefan problem. If we specify the required shape $R(z)$ of the crystal we would like to grow, then both the temperature and the free boundary $S(t)$ can in principle determined by solving (2.40). The solution for $S(t)$ then tells us the time-dependent drawing protocol that will lead to the specified crystal shape $R(z)$. However, it transpires that not all shapes are possible. The main limitation is the requirement that $\dot{S} > 0$ for all t — solutions with negative drawing speed are unphysical. This restricts the maximum crystal radius that may be achieved using this process; see Exercise 5.

We have assumed here that the crystal meets the melt pool at a flat free boundary $z = S(t)$. By analysing an inner region near $z = S$, one can show that the surface is not exactly flat but has small parabolic variations of the form

$$z \sim S(t) + \epsilon^2 s(t)\left[R\big(S(t)\big)^2 - r^2\right]. \tag{2.41}$$

The function $s(t)$ is determined in terms of the leading-order solution for $S(t)$, and the leading-order boundary conditions (2.40b) are not affected.

2.4 Extensional flow

Industrial problem: fibre drawing

A typical drawdown process in the manufacture of optical fibres is illustrated in Figure 2.3. A large cylinder ("blank") of solid glass with radius R_{in} is fed in to the top of a furnace at

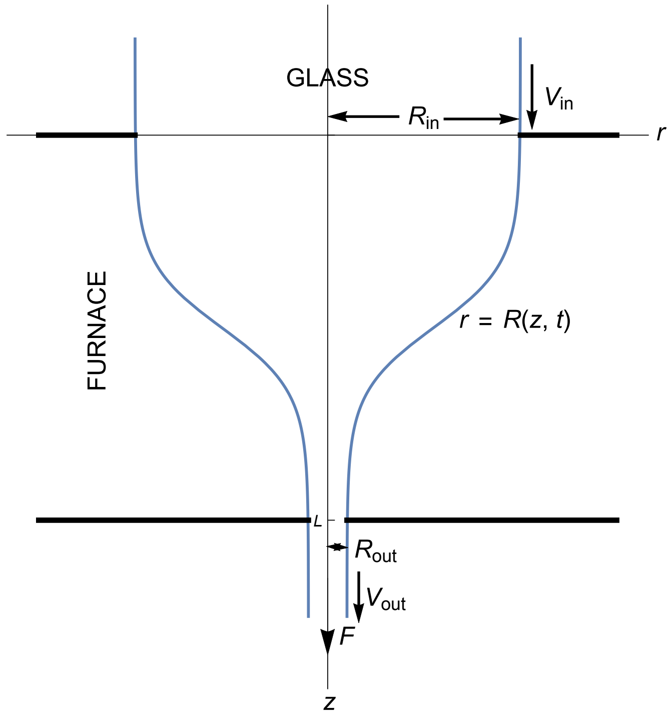

Figure 2.3: Schematic of a fibre drawing process.

initial speed V_{in}. The high temperature in the furnace melts the glass, which then stretches under a force F applied at the bottom of the furnace. The fibre is drawn off at an exit velocity V_{out}. A model is required to determine how the radius R_{out} of the final product is affected by control parameters (e.g. the temperature in the furnace). We also want to know whether any operating conditions might cause the process to become unstable.

Model for a thin viscous sheet

To illustrate the procedure, we start by considering the stretching of a two-dimensional sheet of viscous fluid, as illutrated in Figure 2.4. The simpler geometry makes the algebra slightly easier, and it transpires that the leading-order model is essentially identical to that for a stretching fibre. Consider the two-dimensional flow of an incompressible Newtonian viscous fluid between free surfaces at $y = \pm(1/2)h(x,t)$, so that $h(x,t)$ is the thickness of the sheet. We assume symmetry about $y = 0$ as indicated in Figure 2.4. We assume that inertia is negligible so that the velocity $\boldsymbol{u} = (u, v)$ and pressure p satisfy the two-dimensional Stokes equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{\nabla} p = \mu \nabla^2 \boldsymbol{u}, \tag{2.42}$$

where μ is the viscosity (assumed constant).

We impose symmetry conditions on the x-axis:

$$v = \frac{\partial u}{\partial y} = 0 \quad \text{at } y = 0. \tag{2.43}$$

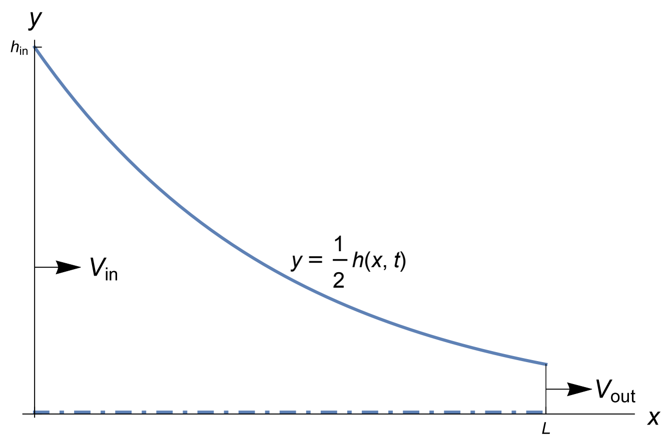

Figure 2.4: Schematic of a stretching sheet of viscous fluid.

At the free surface $y = h(x,t)/2$ we impose the kinematic boundary condition and the condition of zero traction, namely

$$v = \frac{1}{2}\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right), \qquad \sigma_{xy} = \frac{\sigma_{xx}}{2}\frac{\partial h}{\partial x}, \qquad \sigma_{yy} = \frac{\sigma_{xy}}{2}\frac{\partial h}{\partial x}, \tag{2.44}$$

where the stress components in a Newtonian viscous fluid are given by

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \qquad \sigma_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y}. \tag{2.45}$$

We also specify the inlet and outlet values of the velocity and the inlet sheet thickness:

$$u(0) = V_{\text{in}}, \quad h(0) = h_{\text{in}}, \qquad u(L) = V_{\text{out}}, \tag{2.46}$$

where L is the length of the stretching zone. In principle, the Stokes equations (2.42) need one more boundary condition at the two ends $x = 0$ and $x = L$, but we will see that this is no longer required if the sheet is very thin.

Now we non-dimensionalise the problem to exploit the assumed thinness of the geometry. Specifically we assume that the aspect ratio of the domain,

$$\epsilon = \frac{h_{\text{in}}}{L}, \tag{2.47}$$

is very small — in a real fibre drawdown process ϵ is typically smaller than 10^{-3}. Dominant balances in the governing equations and boundary conditions (2.42)–(2.46) suggest the scalings

$$(x,y) = L\left(\tilde{x}, \epsilon\tilde{y}\right), \qquad h = \epsilon L\tilde{h} \qquad (u,v) = V_{\text{in}}\left(\tilde{u}, \epsilon\tilde{v}\right), \qquad t = \frac{L}{V_{\text{in}}}\tilde{t}, \qquad p = \frac{\mu V_{\text{in}}}{L}\tilde{p}, \tag{2.48}$$

and the dimensionless problem then reads (with tildes dropped)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.49a}$$

$$\epsilon^2 \frac{\partial p}{\partial x} = \epsilon^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{2.49b}$$

$$\frac{\partial p}{\partial y} = \epsilon^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{2.49c}$$

in $0 < y < h(x,t)$, $0 < x < 1$, suject to

$$v = \frac{\partial u}{\partial y} = 0 \qquad y = 0, \tag{2.49d}$$

$$v = \frac{1}{2}\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right) \qquad y = \tfrac{1}{2}h(x,t), \tag{2.49e}$$

$$\frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x} = \frac{\epsilon^2}{2}\frac{\partial h}{\partial x}\left(-p + 2\frac{\partial u}{\partial x}\right) \qquad y = \tfrac{1}{2}h(x,t), \tag{2.49f}$$

$$-p + 2\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial y} + \epsilon^2 \frac{\partial v}{\partial x}\right) \qquad y = \tfrac{1}{2}h(x,t), \tag{2.49g}$$

and

$$h = u = 1 \quad \text{at } x = 0, \qquad u = D \quad \text{at } x = 1, \tag{2.49h}$$

where

$$D = \frac{V_{\text{out}}}{V_{\text{in}}} \tag{2.50}$$

is called the *draw ratio.*

Now we write the dependent variables as asymptotic expansions in powers of ϵ^2; this includes the position $h(x,t)$ of the free boundary, which is not speficied but must be determined as part of the solution, i.e.

$$u(x,y,t) \sim u_0(x,y,t) + \epsilon^2 u_1(x,y,t) + \cdots, \tag{2.51}$$

and similarly for v, p and h.

At leading order, equation (2.49b) and boundary conditions (2.49d), (2.49f) imply that u_0 satisfies the problem

$$\frac{\partial^2 u_0}{\partial y^2} = 0 \qquad 0 < y < h_0(x,t), \tag{2.52a}$$

$$\frac{\partial u_0}{\partial y} = 0 \qquad y = 0 \text{ and } y = \tfrac{1}{2}h_0(x,t). \tag{2.52b}$$

This is a homogeneous Neumann problem, which is satisfied by any u_0 that is independent of y, i.e.

$$u_0 = u_0(x,t). \tag{2.53}$$

This is called an *extensional flow*, with the axial velocity uniform across the sheet to leading order. It may be contrasted with a lubrication flow like (2.17), where the u is a parabolic function of y. The difference is that in a lubrication flow at least one of the surfaces is rigid; here the surfaces of the sheet are both free and unable to withstand any shear stress.

It is straightforward to solve the remaining equations for v_0 and p_0:

$$v_0 = -y\frac{\partial u_0}{\partial x}, \qquad p_0 = -2\frac{\partial u_0}{\partial x}, \tag{2.54}$$

and the kinematic condition (2.49e) then implies that u_0 and h_0 satisfy

$$\frac{\partial h_0}{\partial t} + \frac{\partial}{\partial x}(u_0 h_0) = 0. \tag{2.55}$$

Equation (2.55) represents net conservation of mass in the sheet. However, we have now used all of the leading-order equations and still only have one equation for the two unknowns $u_0(x,t)$ and $h_0(x,t)$. The situation is analogous to the heat flow problem analysed in §2.3: the leading-order problem is insufficient to determine the leading-order solution uniquely, and to close the problem we must proceed to higher order and seek a solvability condition.

Here, by considering (2.49b), (2.49d) and (2.49f) at order ϵ^2, we find that $u_1(x,y,t)$ satisfies the inhomogeneous Neumann problem

$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial p_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} \qquad 0 < y < \tfrac{1}{2}h_0(x,t), \tag{2.56a}$$

$$\frac{\partial u_1}{\partial y} = 0 \qquad y = 0, \tag{2.56b}$$

$$\frac{\partial u_1}{\partial y} = -\frac{\partial v_0}{\partial x} + \frac{1}{2}\frac{\partial h_0}{\partial x}\left(-p_0 + 2\frac{\partial u_0}{\partial x}\right) \qquad y = \tfrac{1}{2}h_0(x,t). \tag{2.56c}$$

The solvability condition for (2.56) is simply

$$\int_0^{h_0(x,t)/2} \frac{\partial^2 u_1}{\partial y^2}\,\mathrm{d}y = \left[\frac{\partial u_1}{\partial y}\right]_0^{h_0(x,t)/2}, \tag{2.57}$$

that is,

$$\left(\frac{\partial p_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2}\right)\frac{h_0}{2} = \left[-\frac{\partial v_0}{\partial x} + \frac{1}{2}\frac{\partial h_0}{\partial x}\left(-p_0 + 2\frac{\partial u_0}{\partial x}\right)\right]_{y=h_0/2}. \tag{2.58}$$

Rearrangement and simplification leads to the equation

$$\frac{\partial}{\partial x}\left(4h_0\frac{\partial u_0}{\partial x}\right) = 0, \tag{2.59}$$

where the significance of the factor of 4 will be explained shortly.

Equations (2.55) and (2.59) give us a closed system of equations, called the *Trouton model*, for the leading-order sheet thickness $h_0(x,t)$ and axial velocity $u_0(x,t)$. As pointed out above, equation (2.55) represents net mass conservation, while (2.59) is a force balance. To see this, note that the leading-order (dimensionless) axial stress in the sheet is given by

$$\sigma_{xx0} = -p_0 + 2\frac{\partial u_0}{\partial x} = 4\frac{\partial u_0}{\partial x}. \tag{2.60}$$

Hence the net leading-order tension in the sheet is given by

$$F_0 = \int_{-h_0/2}^{h_0/2} \sigma_{xx0}\,\mathrm{d}y = 4h_0\frac{\partial u_0}{\partial x}, \tag{2.61}$$

and equation (2.59) implies that F is independent of x. In dimensional variables, this reads

$$F = 4\mu h\frac{\partial u}{\partial x} \qquad \text{(dimensional)}. \tag{2.62}$$

The factor 4 is called the *Trouton ratio* between the shear viscosity μ and the *extensional viscosity* 4μ (which relates the rate of stretch $\partial u/\partial x$ to the axial stress σ_{xx}). The Trouton ratio depends on the dimensionality of the problem; in particular, for the case of a thin axisymmetric fibre, one gets exactly the same Trouton model (2.55), (2.59) with h replaced by the cross-sectional area A of the fibre and with a Trouton ratio of 3 instead of 4 (see Exercise 6).

In deriving the model (2.55) and (2.59), we have made several simplifying assumptions. In addition to taking the thin-sheet limit $\epsilon \to 0$, we have also neglected various physical effects, including inertia, gravity and surface tension, all of which could be significant under certain operating regimes. We have also assumed that the viscosity μ is constant in the furnace; in practice the viscosity of glass varies dramatically with temperature. With a little bit more effort, all of these effects can be incorporated by following a procedure analogous to that followed above.

Alternative derivation of the Trouton model

With the benefit of hindsight, one can derive the leading-order extensional flow model a bit more directly. Since u is independent of y to lowest order, the shear stress σ_{xy} in the sheet is zero to lowest order. By insisting on leading-order balances in both the incompressibility condition and the Stokes equations (2.42), we find that the stress components must be scaled as follows:

$$\sigma_{xx} = \frac{\mu V_{\text{in}}}{L}\,\tilde{\sigma}_{xx}, \qquad \sigma_{xy} = \frac{\epsilon\mu V_{\text{in}}}{L}\,\tilde{\sigma}_{xy}, \qquad \sigma_{yy} = \frac{\epsilon^2\mu V_{\text{in}}}{L}\,\tilde{\sigma}_{yy}. \tag{2.63}$$

These stress components then satisfy the dimensionless equations and boundary conditions (with tildes dropped)

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0, \quad \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0 \qquad 0 < y < \tfrac{1}{2}h(x,t), \tag{2.64a}$$

$$\sigma_{xy} = 0 \qquad y = 0, \tag{2.64b}$$

$$\sigma_{xy} = \frac{\sigma_{xx}}{2}\frac{\partial h}{\partial x}, \quad \sigma_{yy} = \frac{\sigma_{xy}}{2}\frac{\partial h}{\partial x} \qquad y = \tfrac{1}{2}h(x,t), \tag{2.64c}$$

and the constitutive relations (2.45) read

$$\sigma_{xx} = -p + 2\frac{\partial u}{\partial x}, \qquad \epsilon^2\sigma_{xy} = \frac{\partial u}{\partial y} + \epsilon^2\frac{\partial v}{\partial x}, \qquad \epsilon^2\sigma_{yy} = -p + 2\frac{\partial v}{\partial y}. \tag{2.65}$$

In addition we have the incompressibility condition (2.49a) and the kinematic conditions on $y = 0$ and $y = h/2$, which we reproduce here:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 0 < y < \tfrac{1}{2}h(x,t), \tag{2.66a}$$

$$v = 0 \qquad y = 0, \tag{2.66b}$$

$$v = \frac{1}{2}\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right) \qquad y = \tfrac{1}{2}h(x,t). \tag{2.66c}$$

First by integrating the PDEs (2.64) and (2.66) from $y = 0$ to $y = h(x,t)/2$ and applying the boundary conditions, we get the *exact* equations representing conservation of mass and axial momentum, namely

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}h\right) = 0, \qquad \frac{\partial}{\partial x}\left(h\bar{\sigma}_{xx}\right) = 0, \tag{2.67}$$

where $^-$ denotes the cross-sectional average:

$$\bar{u}(x,t) = \frac{2}{h(x,t)}\int_0^{h(x,t)/2} u(x,y,t)\,\mathrm{d}y, \quad \bar{\sigma}_{xx}(x,t) = \frac{2}{h(x,t)}\int_0^{h(x,t)/2} \sigma_{xx}(x,y,t)\,\mathrm{d}y. \tag{2.68}$$

Now it is only necessary to let $\epsilon \to 0$ in the constitutive relations (2.65) to get leading-order approximations for u and σ_{xx} and thus close the problem. By following this approach (and figuring out the correct scalings (2.63) for the stress components!) we have removed the need to proceed to order ϵ^2.

Steady drawdown

Now consider a drawdown process as sketched in Figure 2.3. As noted above (and shown in Exercise 6), the cross-sectional area $A(z,t)$ and axial velocity $w(z,t)$ satisfy the Trouton model

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(wA) = 0, \qquad \frac{\partial}{\partial z}\left(3A\frac{\partial w}{\partial z}\right) = 0. \tag{2.69}$$

The normalized boundary conditions are

$$A(0,t) = w(0,t) = 1, \qquad w(1,t) = D, \tag{2.70}$$

where we recall that D is the draw ratio.

In steady state, with $A = A(z)$ and $w = w(z)$, this problem is easy to solve. From (2.69) we get $A = 1/w$ and then

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{F}{3}\,w, \tag{2.71}$$

where F is the tension in the fibre, which is constant but unknown *a priori*. Now apply the boundary conditions (2.70) to get the solutions

$$w(z) = \mathrm{e}^{z\log D} = D^z, \qquad A(z) = \mathrm{e}^{-z\log D} = D^{-z}, \qquad F = 3\log D. \tag{2.72}$$

Hence the steady velocity and area profiles are exponential functions functions of z. The normalised outlet area is $A(1) = 1/D$, which could have been predicted from net mass conservation. Finally, we get a prediction of the force that must be applied to achieve a given draw ratio.

Linear stability: draw resonance

In practice it is found that drawdown may become unstable, with oscillations in the area profile, if the draw ratio is too large. This phenomenon, known as *draw resonance*, may be explained by performing a linear stability analysis of the Trouton model.

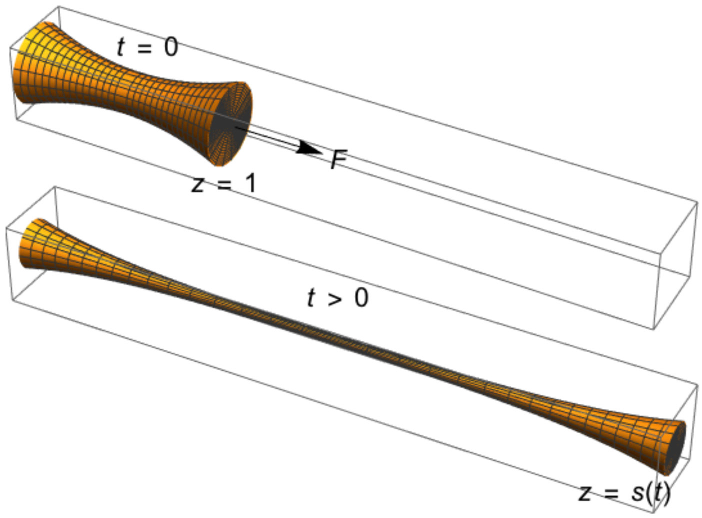

Figure 2.5: Schematic of a fibre tapering process.

We linearise about the steady solutions obtained above by setting

$$w(z,t) = \mathrm{e}^{mz}\left(1+\tilde{w}(z)\mathrm{e}^{\sigma t}\right), \quad A(z,t) = \mathrm{e}^{-mz}\left(1+\tilde{A}(z)\mathrm{e}^{\sigma t}\right), \quad F(t) = 3m\left(1+\tilde{F}\mathrm{e}^{\sigma t}\right), \tag{2.73}$$

where the tildes indicate small perturbations, σ is the linear growth rate, and we have introduced $m = \log D$ as shorthand. Now linearising we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\tilde{A}+\tilde{w}\right) + \sigma \mathrm{e}^{-mz}\tilde{A} = 0, \qquad \frac{\mathrm{d}\tilde{w}}{\mathrm{d}z} + m\left(\tilde{A}+\tilde{w}\right) = m\tilde{F}, \tag{2.74a}$$

which are subject to the boundary conditions

$$\tilde{A}(0) = \tilde{w}(0) = 0, \qquad \tilde{w}(1) = 0. \tag{2.74b}$$

This is an eigenvalue problem. The linear homogeneous boundary value problem (2.74) is always satisfied by the trivial solution $\tilde{A} = \tilde{w} = \tilde{F} = 0$, but will admit nontrivial solutions if and only if σ is equal to one of a set of specific eigenvalues. If any one of those eigenvalues has a positive real part then the problem is linearly unstable: small perturbations to the steady solution will grow exponentially.

As shown in Exercise 7, this problem may be reduced to a transcendental algebraic equation between λ and D. The loss of stability occurs via a Hopf bifurcation, where λ is complex and the sign of its real part changes from negative to positive. It follows that λ is pure imaginary precisely at the critical value $D = D_{\text{crit}}$, and one thus finds that $D_{\text{crit}} \approx 20.218$. Indeed it is observed experimentally, and in time-dependent simulations, that drawdown suffers an oscillatory instability if the draw ratio exceeds this critical value.

Industrial problem: fibre tapering

Optical fibres are often manufactured using a process called *fibre tapering*, which is slightly different from drawdown. A glass blank is heated and then the ends are pulled apart, resulting in a long thin filament, like the stretching of a piece of toffee or chewing gum; this is illustrated schematically in Figure 2.5. Unlike drawdown, this process is inherently unsteady. A model is needed to determine how the stretching of the fibre depends on the applied force F, and how the area profile of the final product depends on the profile of the initial blank used and the stretch applied.

A model for fibre tapering

We again assume that inertia and gravity are negligible, and that viscosity is constant, so that the cross-sectional area $A(z,t)$ and velocity $w(z,t)$ satisfy the Trouton model (2.69). We suppose that one end $z=0$ of the fibre is held fixed while the other is stretched to a given length $s(t)$ at time t. The boundary conditions are therefore

$$w(0,t)=0, \qquad w\big(s(t),t\big)=\dot{s}(t). \tag{2.75}$$

We also specify the initial length and area profile of the blank, i.e.

$$s(0)=1, \qquad A(z,0)=A_0(z). \tag{2.76}$$

Remarkably, the Trouton model may be solved analytically subject to the boundary and initial conditions (2.75), (2.76). We write the governing equations (2.69) in the form

$$\frac{\partial A}{\partial t}+w\frac{\partial A}{\partial z}=-A\frac{\partial w}{\partial z}=-\frac{F(t)}{3}, \tag{2.77}$$

where as above F denotes the dimensionless tension in the fibre. Now we transform to Lagrangian variables (ζ,τ), such that $t=\tau$ and $z(\zeta,\tau)$ satisfies

$$\frac{\partial z}{\partial \tau}=w(z,\tau), \qquad z(\zeta,0)=0. \tag{2.78}$$

The idea is that the Lagrangian variables "follow the flow", so a fixed value of ζ follows a fixed material point as it moves along the fibre under the applied stretching. In Lagrangian variables, the left-hand side of equation (2.77) is just the derivative $\partial A/\partial\tau$. Furthermore, the boundary conditions (2.75) imply that the two ends $z=0$ and $z=s(t)$ of the fibre are both material boundaries corresponding to fixed values of $\zeta=0$ and $\zeta=1$ respectively. Therefore the moving domain $z\in[0,s(t)]$ is mapped to the fixed domain $\zeta\in[0,1]$, and the problem restated in Lagrangian variables reads

$$-\frac{F(\tau)}{3}=\frac{\partial A}{\partial \tau}=-\,A\frac{\partial w}{\partial \zeta}\Big/\frac{\partial z}{\partial \zeta}=-\,A\frac{\partial^2 z}{\partial\zeta\partial\tau}\Big/\frac{\partial z}{\partial \zeta}, \tag{2.79}$$

subject to the boundary and initial conditions

$$z(0,\tau)=0, \qquad z(1,\tau)=s(\tau), \tag{2.80}$$
$$z(\zeta,0)=\zeta, \qquad A(\zeta,0)=A_0(\zeta). \tag{2.81}$$

We can integrate (2.79a) directly with respect to τ to get

$$A(\zeta,\tau)=A_0(\zeta)-f(\tau), \qquad \text{where } f(\tau)=\frac{1}{3}\int_0^\tau F\left(\tau'\right)\,\mathrm{d}\tau'. \tag{2.82}$$

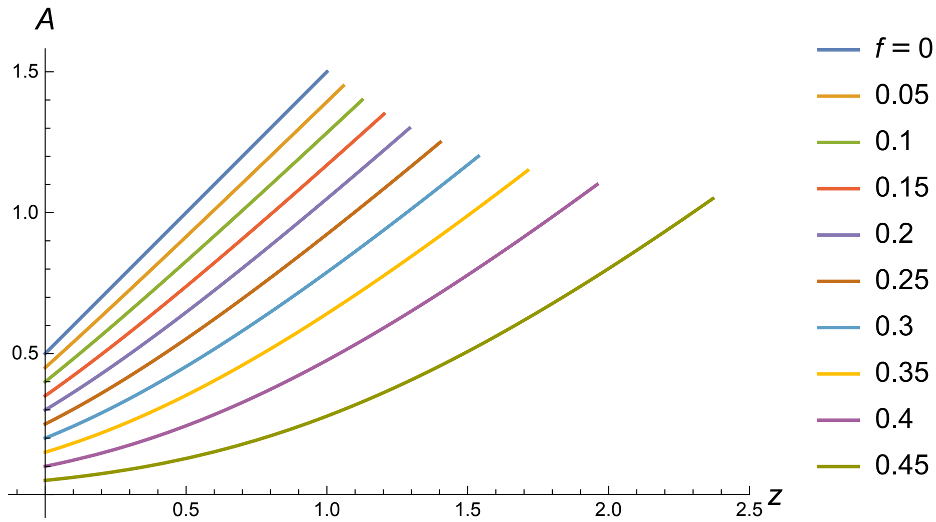

Figure 2.6: Cross-sectional area A versus axial position z for an initially linear profile, with different values of the integrated force f. Here A and z are related parametrically by equation (2.88) with $\alpha = 1$.

On the other hand, multiplication of (2.79b) by $\partial z/\partial \tau$ leads to

$$\frac{\partial}{\partial \tau}\left(A\frac{\partial z}{\partial \zeta}\right) = 0. \tag{2.83}$$

Hence the term in brackets is a function of ζ alone, which may be determined from the initial conditions (2.80):

$$A\frac{\partial z}{\partial \zeta} = A_0(\zeta). \tag{2.84}$$

By using the solution (2.82) for A, we get the following expression for z:

$$z(\zeta,\tau) = \int_0^{\zeta} \frac{A_0\left(\zeta'\right)\,\mathrm{d}\zeta'}{A_0\left(\zeta'\right) - f(\tau)} = \zeta + f(\tau)\int_0^{\zeta}\frac{\mathrm{d}\zeta'}{A_0\left(\zeta'\right) - f(\tau)}. \tag{2.85}$$

Once the initial area profile $A_0(\zeta)$ is specified, the applied force $F(\tau)$ determines $f(\tau)$, and then equations (2.82) and (2.85) determine A and z as functions of ζ and τ. Thus A is related to z and t parametrically (parameterised by ζ). On the other hand, if the length $s(t)$ of the fibre is specified, instead of the applied force, then $f(\tau)$ must be determined from the equation

$$z(1,\tau) = s(\tau) = 1 + f(\tau)\int_0^1 \frac{\mathrm{d}\zeta}{A_0\left(\zeta\right) - f(\tau)}. \tag{2.86}$$

As a very simple example, suppose the initial area profile is linear, so that

$$A_0(\zeta) = 1 + \alpha\left(\zeta - \frac{1}{2}\right), \tag{2.87}$$

where $-2 < \alpha < 2$. Then (2.82) and (2.85) lead to

$$A(\zeta,\tau) = 1 - f(\tau) + \alpha\left(\zeta - \frac{1}{2}\right), \quad z(\zeta,\tau) = \zeta + f(\tau)\log\left(1 + \frac{\alpha\zeta}{1 - f(\tau) - \alpha/2}\right). \tag{2.88}$$

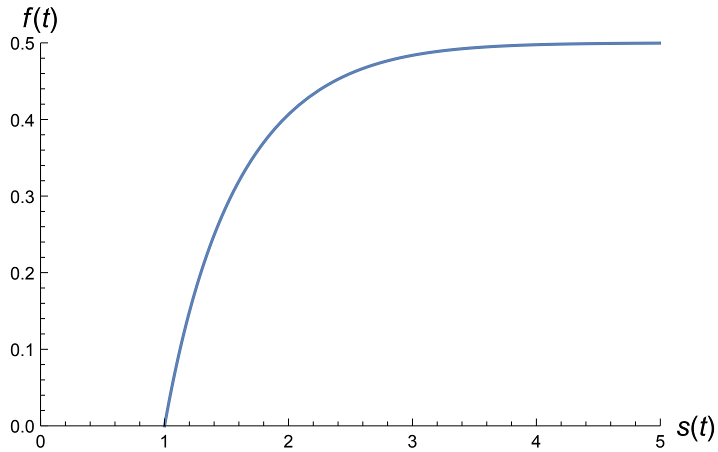

Figure 2.7: Integrated force $f(t)$ versus fibre length $s(t)$ for an initially linear area profile. The relation is defined by equation (2.89), with $\alpha = 1$ here.

The resulting behaviour of A versus z is plotted in Figure 2.6, with $\alpha = 1$ and various values of f. As the fibre lengthens, the thinner parts tend to stretch more than the thicker parts. This leads to a very long thin filament near $z = 0$. The area tends to zero at $z = 0$ at a finite critical value $f_{\text{crit}} = 1 - \alpha/2 = 1/2$, while simultaneously the length of the fibre tends to infinity.

In this example, the relation between f and the length s of the fibre is given by

$$z(1,\tau) = s(\tau) = 1 + f(\tau)\log\left(1 + \frac{\alpha}{1 - f(\tau) - \alpha/2}\right). \tag{2.89}$$

This relationship is plotted in Figure 2.7, again with $\alpha = 1$. This illustrates how f increases as the fibre lengthens; the concavity of the graph indicates that it gets easier to stretch the fibre as it becomes thinner, and the length tends to infinity at a finite value of f (namely 1/2 here).

Exercises

1. Derive the leading-order model (2.11) for the hot rolling problem [*particularly if you haven't previously encountered lubrication theory*].

   Write the relation (2.20) between β, γ and η in the form

$$\gamma = \frac{F_2\left(\dfrac{\beta}{\sqrt{2\eta}}\right) + F_2\left(\sqrt{\dfrac{1-\eta}{\eta}}\right)}{\eta\left[F_1\left(\dfrac{\beta}{\sqrt{2\eta}}\right) + F_1\left(\sqrt{\dfrac{1-\eta}{\eta}}\right)\right]},$$

where

$$F_1(X) = \int_0^X \frac{\mathrm{d}s}{(1+s^2)^2} = \frac{1}{2}\left(\frac{X}{1+X^2} + \tan^{-1} X\right),$$

$$F_2(X) = \int_0^X \frac{\mathrm{d}s}{(1+s^2)^3} = \frac{1}{8}\left(\frac{X\left(5+3X^2\right)}{(1+X^2)^2} + 3\tan^{-1} X\right).$$

Hence plot γ versus β for different values of η. Show that generically two positive roots for β exist provided γ lies in some interval (which depends on η). We assume that the smaller positive root is the physical one. Show that, as η and γ both tend to 1, this root is given asymptotically by

$$\beta \sim \sqrt{\frac{1-\eta}{2}}\left(1-\sqrt{9-4\Gamma}\right), \qquad \text{where } \Gamma = \frac{3(\gamma-1)}{(1-\eta)},$$

and Γ must lie in the range $2 < \Gamma < 9/4$.

2. Re-do the lubrication model for the steel hot-rolling problem for a power-law fluid. [*It is not necessary to start from the full Navier–Stokes equations: the assumptions of the lubrication approximation will lead to the simpler system*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \qquad \frac{\partial p}{\partial y} = 0,$$

in dimensional variables.]

Show that, following suitable non-dimensionalisation, the Reynolds equation (2.18) is modified to

$$\gamma h(x) - \frac{n}{2n+1} p'(x) \left|p'(x)\right|^{-1+1/n} h(x)^{2+1/n} = 1,$$

and deduce from the boundary conditions $p(-\alpha) = p(\beta) = 0$ the relation

$$\int_{-\alpha}^{\beta} \left(\frac{\gamma}{h^{1+1/n}} - \frac{1}{h^{2+1/n}}\right) \left|\frac{\gamma}{h^{1+1/n}} - \frac{1}{h^{2+1/n}}\right|^{n-1} \mathrm{d}x = 0.$$

3. [*Relevant if you're unused to Fredholm Alternative, solvability conditions etc.*]

Consider the matrix equation

$$(\boldsymbol{A} - c\boldsymbol{I})\boldsymbol{x} = \boldsymbol{b},$$

where A is a real symmetric $n \times n$ matrix, $\boldsymbol{I}$ is the identity matrix, $c \in \mathbb{R}$ and $\boldsymbol{b} \in \mathbb{R}^n$. Provided c is not equal to an eigenvalue of A, the determinant of the left-hand-side is nonzero and there is a unique solution for $\boldsymbol{x} \in \mathbb{R}^n$. But what happens if c is *close* to an eigenvalue of A?

Let $c = \lambda + \epsilon$, where λ is a simple eigenvalue of A. Show that a regular perturbation expansion of the form $\boldsymbol{x} \sim \boldsymbol{x}_0 + \epsilon \boldsymbol{x}_1 + \cdots$ fails (apart from the exceptional case where $\boldsymbol{b} \in \mathrm{im}(A - \lambda I)$).

Instead rescale $\boldsymbol{x} = \epsilon^{-1}\boldsymbol{X}$ — this reflects the fact that the amplitude of the solution will increases as c approaches λ. Now seek the solution as a regular perturbation expansion for $\boldsymbol{X}$. Note how the *amplitude* of the leading-order solution $\boldsymbol{X}_0$ is determined from the *solvability condition* for the first-order perturbation $\boldsymbol{X}_1$.

4. From the heat equation (2.31a) and the boundary condition (2.31b), obtain the *exact* net conservation equation

$$\frac{\partial}{\partial t}\int_0^{R(z)} \mathrm{St}\, ur\,\mathrm{d}r = \frac{\partial}{\partial z}\int_0^{R(z)} \frac{\partial u}{\partial z}\, r\,\mathrm{d}r - R(z)\sqrt{1+\epsilon^2 R'(z)^2}\, u\big|_{r=R(z)}.$$

[*Now it is only necessary to establish that* $u \sim u(z,t)$ *to get the leading-order governing equation* (2.39).]

5. **Small-time behaviour of the Czochralski method**

Assume that $R(z)$ is analytic at $z=0$, so that

$$R(z) \sim R_1 z + R_2 z^2 + R_3 z^3 + \cdots \quad \text{as } z \to 0.$$

Seek a small-t solution to the model (2.40) in the form

$$u(z,t) \sim 1 + t f_1(\eta) + t^2 f_2(\eta) + \cdots, \qquad S(t) \sim tS_1 + t^2 S_2 + \cdots,$$

where $\eta = z/t$. [*You may find it helpful to use a symbolic manipulation package such as* mathematica.] Hence show that

$$S_1 = \frac{1-R_1}{R_1}, \qquad S_2 = -\frac{(1-R_1)}{6R_1^3}\left(\frac{1-R_1}{\mathrm{St}} + 1 + 4R_2\right),$$

and so on. Deduce that R_1 must be less than 1 — there is a maximum slope achievable at the tip of the crystal. In the limiting case where $R_1 \nearrow 1$, deduce further that $R_2 < -1/4$. By continuing this process, show that the maximal achievable crystal shape takes the form

$$R_{\mathrm{m}}(z) \sim z - \frac{z^2}{4} + \frac{z^3}{24} + \cdots \quad \text{as } z \to 0.$$

[*In fact it can be shown that* $R_{\mathrm{m}}(z) = 2\left(1 - \mathrm{e}^{-z/2}\right)$. *You can verify that this is consistent with the above expansion and satisfies the problem* (2.40) *in the limit* $\dot{S} \to 0$.]

6. **Trouton model for a fibre**

The Stokes equations for axisymmetric slow viscous flow with respect to cylindrical polar coordinates (r,z) read

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} &= 0,\\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rr}\right) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} &= 0,\\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right) + \frac{\partial \sigma_{zz}}{\partial z} &= 0,
\end{aligned}$$

where (for a Newtonian viscous fluid) the stress components are related to the velocity $\boldsymbol{u} = u\boldsymbol{e}_r + w\boldsymbol{e}_z$ by the constitutive relations

$$\begin{aligned}
\sigma_{rr} &= -p + 2\mu\frac{\partial u}{\partial r}, & \sigma_{zz} &= -p + 2\mu\frac{\partial w}{\partial z},\\
\sigma_{\theta\theta} &= -p + 2\mu\frac{u}{r}, & \sigma_{rz} &= \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).
\end{aligned}$$

Consider an axisymmetric jet of viscous fluid occupying the region $0 \leq r < R(z,t)$, $0 < z < L$, with zero traction applied at the free surface $r = R(z,t)$. Write down the boundary conditions on $r = R(z,t)$. By integrating the above equations with respect to r, obtain the exact conservation equations

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}\left(\bar{w}A\right) = 0, \qquad\qquad \frac{\partial}{\partial z}\left(A\bar{\sigma}_{zz}\right) = 0,$$

where $A(z,t) = \pi R(z,t)^2$ is the cross-sectional area and $\bar{}$ denotes the cross-sectional average

$$\bar{f}(z,t) = \frac{1}{A}\int_0^{R(z,t)} f(r,z,t)\, 2\pi r\,\mathrm{d}r.$$

Now nondimensionalise the constitutive relations appropriately [*using* (2.63) *as a guide*]. In the limit $\epsilon \to 0$, show that $w = w(z,t)$ and obtain a leading-order expression for σ_{zz}. Hence show that $A(z,t)$ and $w(z,t)$ satisfy the Trouton model with Trouton ratio equal to 3.

7. Show that the change of variables

$$\tilde{w}(z) = \tilde{F}\big(1 - \phi(x)\big), \qquad\qquad x = \frac{\sigma}{m}\,\mathrm{e}^{-mz}$$

transforms the eigenvalue problem (2.74) into

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \left(\frac{1}{x} - 1\right)\frac{\mathrm{d}\phi}{\mathrm{d}x} + \frac{\phi}{x} = 0, \tag{$\star$}$$

subject to

$$\phi(\lambda) = \lambda\phi'(\lambda) = \phi\left(\lambda/D\right) = 1,$$

where $\lambda = \sigma/m$ and $D = \mathrm{e}^m$ is the draw ratio.

Show that $\phi(x) = x - 1$ satisfies equation ($\star$) and hence find the general solution. Show that, if $\lambda = \mathrm{i}\omega$, then ω and D must satisfy the equation

$$F(\omega, D) = \int_{\omega/D}^{\omega} \frac{\mathrm{e}^{\mathrm{i}\xi}\,\mathrm{d}\xi}{\xi(1-\mathrm{i}\xi)^2} + \mathrm{e}^{\mathrm{i}\omega}\left(\frac{1}{1-\mathrm{i}\omega/D} - \frac{1}{1-\mathrm{i}\omega}\right) = 0.$$

The real and imaginary parts of this expression give two simultaneous transcendental equations for ω and D, namely

$$\mathrm{Ci}(\omega) - \mathrm{Ci}(\omega/D) + \frac{\big[\cos(\omega) - \cos(\omega/D)\big] - (\omega/D)\big[\sin(\omega) - \sin(\omega/D)\big]}{1 + \omega^2/D^2} = 0,$$
$$\mathrm{Si}(\omega) - \mathrm{Si}(\omega/D) + \frac{(\omega/D)\big[\cos(\omega) - \cos(\omega/D)\big] + \big[\sin(\omega) - \sin(\omega/D)\big]}{1 + \omega^2/D^2} = 0,$$

where Ci and Si denote the Cosine Integral and Sine Integral functions. By numerically finding the roots of these simultaneous equations (as illustrated in Figure 2.8) show that the smallest critical draw ratio is given by $D_{\mathrm{crit}} \approx 20.218$.

[*You could also try to compute* D_{crit} *by discretising the eigenvalue problem* (2.74) *or by numerical solution of the unsteady Trouton model* (2.69).]

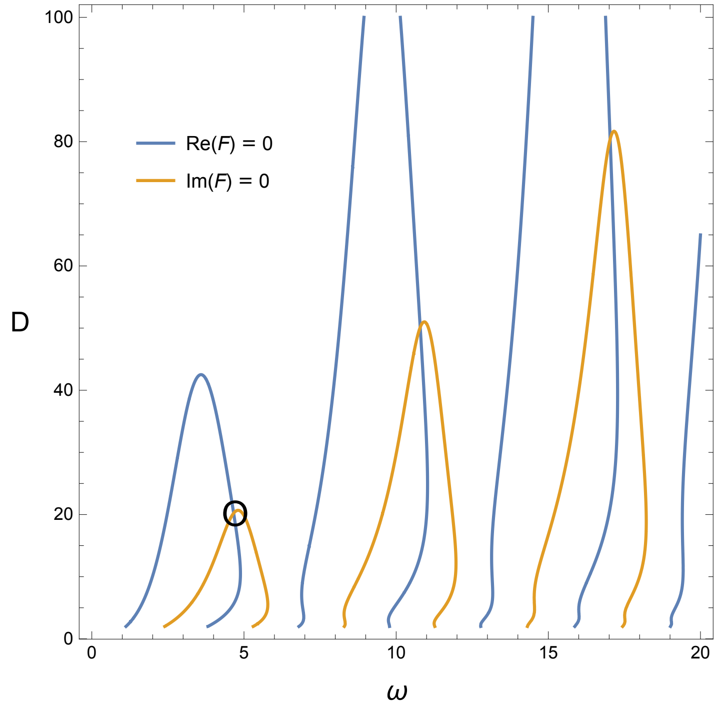

Figure 2.8: The zero contours of the real and imaginary parts of the function $F(\omega, D)$; "O" marks the smallest root for $D \approx 20.218$.

8. Calculate $A(\zeta, \tau)$ and $z(\zeta, \tau)$ from equation (2.85) when the initial are profile is quadratic, with

$$A_0(x) = 1 + \alpha \left(x - \frac{1}{2} \right)^2$$

and $\alpha > 0$. Explore how the profile stretches as f increases from 0 towards 1. Show that the length s tends to infinity as f tends to 1, with

$$f \sim 1 - \frac{\pi^2}{\alpha s^2} \quad \text{as } s \to \infty.$$

9. When inertia is included, the Trouton model (2.69) is modified to

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(wA) = 0, \qquad \mathrm{R}\, A \left(\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} \right) = \frac{\partial}{\partial z} \left(3A \frac{\partial w}{\partial z} \right),$$

where

$$\mathrm{R} = \frac{\rho V_{\mathrm{in}} L}{\mu}$$

is the Reynolds number. [*You could try to derive this from the axisymmetric Navier–Stokes equations if you like.*]

Show that

$$\left(\frac{\partial}{\partial t} + w \frac{\partial}{\partial z} \right) \left(\frac{\mathrm{R}}{3} w + \frac{1}{A} \frac{\partial A}{\partial z} \right) = 0.$$

Consider a fibre which is initially at rest with a uniform cross-section, so that

$$w = 0, \quad A = 1 \quad \text{at } t = 0.$$

For $t > 0$, one end $z = 0$ is held fixed, while the other end is drawn off at a constant speed equal to 1 in dimensionless variables, so that

$$w = 0 \quad \text{at } z = 0, \qquad\qquad w = 1 \quad \text{at } z = 1.$$

Deduce that in the subsequent motion, the cross-sectional area satisfies the *heat equation*

$$\frac{\mathrm{R}}{3}\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial z^2},$$

subject to

$$\begin{aligned} \frac{\partial A}{\partial z} &= 0 & & z = 0, \\ \frac{\partial A}{\partial z} + \frac{\mathrm{R}\,A}{3} &= 0 & & z = 1, \\ A &= 1 & & t = 0. \end{aligned}$$

Show that the solution takes the form

$$A(z,t) = \sum_{n=1}^{\infty} c_n \cos(k_n x) \mathrm{e}^{-3k_n^2 t/\mathrm{R}},$$

where

$$c_n = \frac{2\sin(k_n)}{k_n + \sin(k_n)\cos(k_n)},$$

and k_n is the nth positive root of the transcendental equation

$$k\tan(k) = \frac{\mathrm{R}}{3}.$$

Plot the solution and explore how the qualitative behaviour depends on R.