3 Slender body theory

3.1 Introduction

In §2 we considered problems where the geometry of the domain in which the equations are posed is slender, and showed how the slenderness of the geometry can be expointed to derive simplified models. Now we consider a dual class of problems in which the equations are posed *outside* a slender region. The canonical example is flow past a slender obstacle, and the slenderness of the obstacle may again be exploited to simplify the problem. The basic idea of *slender body theory* is to approximate the effect of the obstacle on the flow by a distribution of singularities, whose strength must be determined by imposing the required boundary conditions. Since the singularities are chosen to satisfy the field equations identically, this approach removes the need to solve any PDEs. Inspead, one typically ends up with integro-differential equations for the singularity distribution functions. We will show how the same general approach applies for example to porous medium flow flow outside a thin porous tube and to viscous flow outside a thin deforming bubble.

3.2 Slender body theory in potential flow

Model problem: flow past a thin projectile

Consider a thin rigid radially symmetric projectile moving at constant speed U through an inviscid fluid. The projectile is assume to be thin in the sense that the aspect ratio $\epsilon = a/L$ between its typical radius a and its half-length L is small. This situation might model a javelin or a rocket moving through air, or a submarine moving through water. We adopt cylindrical polar coordinates (r, θ, z) in a frame moving at speed U. As shown schematically in Figure 3.1, the boundary of the projectile is then a fixed axisymmetric surface $r = S(z)$ and the surrounding fluid has uniform velocity $U\boldsymbol{e}_z$ plus a small perturbation due to the projectile. Assuming that the flow is inviscid, incompressible and irrotational, we can describe the disturbance using a velocity potential ϕ which satisfies Laplace's equation, i.e.

$$\boldsymbol{u} = U\boldsymbol{e}_z + \boldsymbol{\nabla}\phi, \qquad \text{where} \quad \nabla^2\phi = 0. \tag{3.1}$$

On the boundary of the projectile, the normal velocity of the fluid must be zero, i.e.

$$\frac{\partial\phi}{\partial r} = \left(U + \frac{\partial\phi}{\partial z}\right) S'(z) \quad \text{at } r = S(z), \; -L < z < L. \tag{3.2}$$

We also require the fluid to revert to the given uniform flow in the far field, i.e.

$$\boldsymbol{\nabla}\phi \to \boldsymbol{0} \quad \text{as } \boldsymbol{r} \to \infty. \tag{3.3}$$

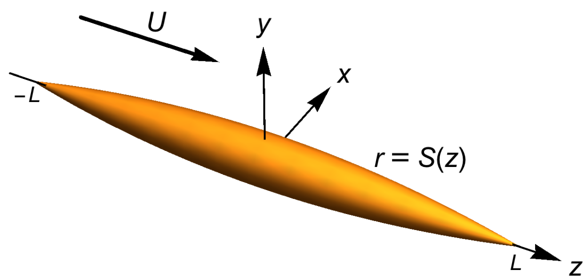

Figure 3.1: Schematic of uniform flow past a thin projectile.

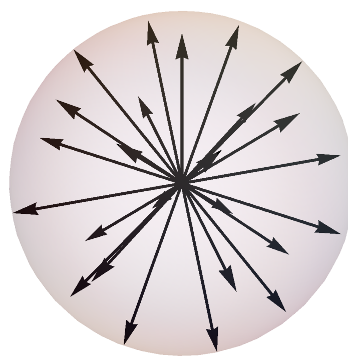

Figure 3.2: Schematic of the velocity field due to a point source.

We start by normalising the problem as follows:

$$(r, z) = L\left(\tilde{r}, \tilde{z}\right), \qquad S = a\tilde{S}, \qquad \phi = \frac{Ua^2}{L}\,\tilde{\phi}, \tag{3.4}$$

so the governing equations and boundary conditions (3.1)–(3.3) are transformed to (with tildes now dropped)

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{3.5a}$$

$$\epsilon\frac{\partial\phi}{\partial r} = \left(1 + \epsilon^2\frac{\partial\phi}{\partial z}\right)S'(z) \qquad \text{at } r = \epsilon S(z) \qquad , \tag{3.5b}$$

$$\boldsymbol{\nabla}\phi \to \mathbf{0} \qquad \text{as } r^2 + z^2 \to \infty, \tag{3.5c}$$

where as above

$$\epsilon = \frac{a}{L}. \tag{3.6}$$

Our aim is to solve the problem (3.5) asymptotically in the limit $\epsilon \to 0$.

As a first step, consider the velocity potential

$$\phi_{\mathrm{s}} = -\frac{q}{4\pi|\boldsymbol{r}|} = -\frac{q}{4\pi\sqrt{r^2+z^2}}, \tag{3.7}$$

where q is a constant. One can readily verify that the function ϕ_{s} satisfies Laplace's equation (3.5a) exactly everywhere except at the origin. Indeed it's the *fundamental solution* or *Green's function* for Laplace's equation in $\mathbb{R}^3$, satisfying

$$\nabla^2\phi_{\mathrm{s}} = \delta(x)\delta(y)\delta(z), \tag{3.8}$$

where δ is the Dirac delta-function. Physically, (3.7) is the velocity potential due to a point source of strength q at the origin. As illustrated in Figure 3.2, the velocity field points radially outwards from the origin, and the net flux through any surface containing the origin is q, i.e.

$$\iint_{\partial D} \boldsymbol{\nabla}\phi\cdot \mathrm{d}\boldsymbol{S} = \begin{cases} q & \text{if } \mathbf{0}\in D \\ 0 & \text{otherwise.}\end{cases} \tag{3.9}$$

Now we seek an approximate solution to the problem (3.5) in the form

$$\phi(r,z) = \int_{-1}^{1} \frac{-q(s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}}. \tag{3.10}$$

Assuming q is sufficiently smooth, this proposed potential automatically satisfies Laplace's equation outside the projectile (because (3.7) does). Physically, (3.10) represents a distribution of point sources along the z-axis, with source strength $q(s)\mathrm{d}s$ at $z=s$. The idea is to try and choose the function $q(s)$ such that the source distribution approximately mimics the effect of the slender object inserted into the flow.

Now let us examine the behaviour of the potential (3.10) in the limit $r\to 0$. If we just naïvely try to set $r=0$, then the integral on the right-hand side of (3.10) is divergent when $z\in[-1,1]$. Therefore we have to be more careful and split up the range of integration before letting $r\to 0$. We decompose ϕ into two integrals:

$$-4\pi\phi(r,z) = \underbrace{\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] \frac{q(s)\,\mathrm{d}s}{\sqrt{r^2+(z-s)^2}}}_{=I_1(r,z;\delta)} + \underbrace{\int_{z-\delta}^{z+\delta} \frac{q(s)\,\mathrm{d}s}{\sqrt{r^2+(z-s)^2}},}_{=I_2(r,z;\delta)} \tag{3.11}$$

where δ is a parameter chosen such that $r\ll\delta\ll 1$.

In the first integral, since the singularity at $s=z$ has now been removed, we can just let r tend to zero to get

$$I_1(r,z;\delta) \sim \left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] \frac{q(s)\,\mathrm{d}s}{|z-s|} - \frac{r^2}{2}\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] \frac{q(s)\,\mathrm{d}s}{|z-s|^3} + O\left(r^4\right). \tag{3.12}$$

In the second integral, we make the change of variables $s=z+rt$ to get

$$I_2(r,z;\delta) = \int_{-\delta/r}^{\delta/r} \frac{q(z+rt)\,\mathrm{d}t}{\sqrt{1+t^2}} \sim 2q(z)\int_0^{\delta/r}\frac{\mathrm{d}t}{\sqrt{1+t^2}} + r^2q''(z)\int_0^{\delta/r}\frac{t^2\,\mathrm{d}t}{\sqrt{1+t^2}}, \tag{3.13}$$

where we have exploited the symmetry in the integrands about $t = 0$. Now we evaluate the integrals and exploit the assumed largeness of δ/r to get

$$\int_0^{\delta/r} \frac{\mathrm{d}t}{\sqrt{1+t^2}} = \sinh^{-1}\left(\frac{\delta}{r}\right) \sim \log\left(\frac{2\delta}{r}\right) + \frac{r^2}{4\delta^2} + O\left(\frac{r^4}{\delta^4}\right) \qquad \text{as } \frac{r}{\delta} \to 0, \tag{3.14a}$$

$$\int_0^{\delta/r} \frac{t^2\,\mathrm{d}t}{\sqrt{1+t^2}} = \frac{\delta}{2r}\sqrt{1+\frac{\delta^2}{r^2}} - \frac{1}{2}\sinh^{-1}\left(\frac{\delta}{r}\right) \sim \frac{\delta^2}{2r^2} + \frac{1}{4} - \frac{1}{2}\log\left(\frac{2\delta}{r}\right) + O\left(r^2/\delta^2\right) \qquad \text{as } \frac{r}{\delta} \to 0. \tag{3.14b}$$

Now, when we add the contributions from I_1 and I_2, the terms involving δ should all cancel: the asymptotic expansion should be independent of the choice of δ, provided it is in the range $r \ll \delta \ll 1$. Here we find that

$$\begin{aligned} I_1 + I_2 \sim \left\{ \left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] \frac{q(s)\,\mathrm{d}s}{|z-s|} + 2q(z)\log\delta + 2q(z)\log\left(\frac{2}{r}\right) + O\left(\delta^2\right)\right\} \\ + r^2\left\{-\frac{1}{2}\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)\,\mathrm{d}s}{|z-s|^3} + \frac{q(z)}{2\delta^2} - \frac{q''(z)}{2}\log\delta \right. \\ \left. + \frac{q''(z)}{4} - \frac{q''(z)}{2}\log\left(\frac{2}{r}\right) + O\left(\delta^2\right)\right\} + O\left(r^4\log r\right). \end{aligned} \tag{3.15}$$

In the first line of (3.15), the leading integral and the term proportional to $\log\delta$ are both singular as $\delta \to 0$. However, the singularities cancel (as they must if we have chosen δ appropriately) and hence this leading term approaches a well defined finite value as $\delta \to 0$.

One way to evaluate this limit is to note that (by Leibniz' rule)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] q(s)\,\mathrm{sgn}(z-s)\log|z-s|\,\mathrm{d}s \\ = \left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)\,\mathrm{d}s}{|z-s|} + q(z-\delta)\log\delta + q(z+\delta)\log\delta, \end{aligned} \tag{3.16}$$

where $\mathrm{sgn}(z)$ denotes the sign of z. Taking the limit $\delta \to 0$, we therefore get

$$\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)\,\mathrm{d}s}{|z-s|} + 2q(z)\log\delta \to \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1} q(s)\,\mathrm{sgn}(z-s)\log|z-s|\,\mathrm{d}s. \tag{3.17}$$

One other way to deal with the singular integral in equation (3.15) is to subtract off a suitable function to remove the singularity, as follows:

$$\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)\,\mathrm{d}s}{|z-s|} = \left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)-q(z)}{|z-s|}\,\mathrm{d}s + q(z)\left[-2\log\delta + \log\left(1-z^2\right)\right]. \tag{3.18}$$

Hence equation (3.17) may alternatively be written in the form

$$\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right]\frac{q(s)\,\mathrm{d}s}{|z-s|} + 2q(z)\log\delta \to \int_{-1}^{1}\frac{q(s)-q(z)}{|z-s|}\,\mathrm{d}s + q(z)\log\left(1-z^2\right) \quad \text{as } \delta \to 0. \tag{3.19}$$

In either case, we define the regular part of the integral in equation (3.10) as follows:[1]

$$\fint_{-1}^{1} \frac{q(s)\,\mathrm{d}s}{|z-s|} = \lim_{r\to 0}\left\{\int_{-1}^{1} \frac{q(s)\,\mathrm{d}s}{\sqrt{r^2+(z-s)^2}} - 2q(z)\log\left(\frac{2}{r}\right)\right\} \tag{3.20a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\int_{-1}^{1} q(s)\,\mathrm{sgn}(z-s)\log|z-s|\,\mathrm{d}s \tag{3.20b}$$

$$= \int_{-1}^{1} \frac{q(s)-q(z)}{|z-s|}\,\mathrm{d}s + q(z)\log\left(1-z^2\right). \tag{3.20c}$$

Finally, then, we arrive at

$$\phi(r,z) \sim \frac{q(z)}{2\pi}\log\left(\frac{r}{2}\right) - \frac{1}{4\pi}\fint_{-1}^{1}\frac{q(s)\,\mathrm{d}s}{|z-s|} + O\left(r^2\log r\right) \quad \text{as } r\to 0. \tag{3.21}$$

The leading term proportional to $\log r$ shows that the posed distribution of point sources looks like a line source of strength q as $r \to 0$. In principle, the next correction can be evaluated in a similar way. Considering the coefficient of r^2 in equation (3.15), the singularity in the integral cancels the factors involving δ^{-2} and $\log\delta$ as $\delta \to 0$: see Exercise 1.

Now we are ready to apply the boundary condition (3.5b). We neglect the $O\left(\epsilon^2\right)$ correction on the right-hand side and use (3.21) to evaluate $\partial\phi/\partial r$. Thus we determine the required strength of the effective source distribution, namely

$$q(z) = 2\pi S(z)S'(z) = \frac{\mathrm{d}A}{\mathrm{d}z}, \tag{3.22}$$

where $A(z) = \pi S(z)^2$ is the cross-sectional area of the projectile. Perhaps we could have anticipated this dependence of the source strength on the rate-of-change of area.

Now that q has been determined, the disturbance potential is given by equation (3.10):

$$\phi(r,z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{A'(s)\,\mathrm{d}s}{\sqrt{r^2+(z-s)^2}}. \tag{3.23}$$

In particular, in the very far field the projectile looks like a point source:

$$\phi(r,z) \sim -\frac{Q}{4\pi\sqrt{r^2+z^2}} \quad \text{as } r^2+z^2\to\infty, \tag{3.24}$$

where

$$Q = \int_{-1}^{1} A'(s)\,\mathrm{d}s = \big[A(s)\big]_{-1}^{1}. \tag{3.25}$$

Hence this net source strength is identically zero for any projectile that is pointed at both ends, like the one depicted in Figure 3.1.

There are still some technical points to be cleared up. First, the application of the boundary condition (3.5b) required us to equate terms that are apparently of different order. This slightly awkward step may be justified more systematically by analysing a boundary layer where $r = O(\epsilon)$ — see Exercise 2. Second, it should be noted that the decomposition

[1]note that this definiton of the regular part is not unique: for example, the factor of $-2q(z)\log 2$ has here been incorporated into the definition of $\fint$ for convenience

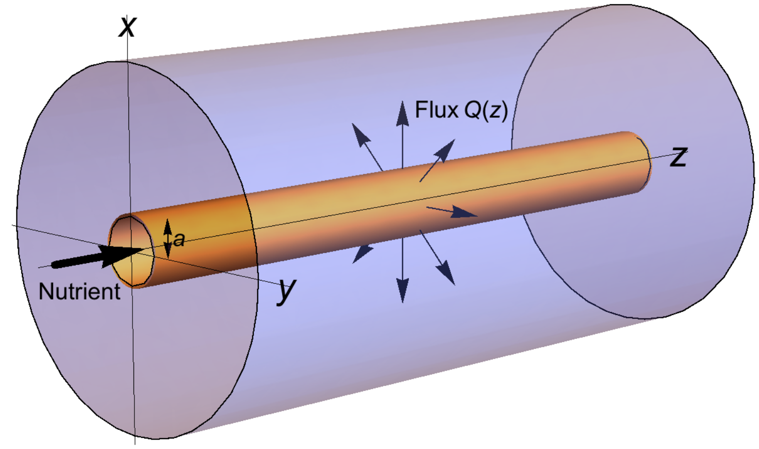

Figure 3.3: Schematic of flow through a porous tube in a bioreactor.

(3.11) assumes that z is not too close to ± 1, i.e. to either either end of the projectile. The asymptoptic structure is slightly different in neighbourhoods of these end points, and requires somewhat more careful treatment.

Finally, we recall the relation between $\log r$ and the fundamental solution for Laplace's equation in two dimensions, i.e.

$$\nabla^2 \left(\frac{1}{2\pi} \log r \right) = \delta(x)\delta(y), \tag{3.26}$$

where $r = \sqrt{x^2 + y^2}$. Hence the velocity potential (3.10), with the logarithmic singularity as $r \to 0$ given by equation (3.21), may also be viewed as the solution of the singular Poisson equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \nabla^2 \phi = q(z)\delta(x)\delta(y). \tag{3.27}$$

This again has the obvious interpretation of a distribution of sources along the z-axis.

Industrial problem: nutrient flow in a bioreactor

This problem concerns the growth of cells in a bioreactor for tissue engineering applications. The cells are seeded in a saturated porous medium and are fed via small porous tubes through the medium through which nutrient solution is injected. A mathematical model is needed to determine how the nutrient solution spreads through the porous medium and to ensure that the concentration is reasonably uniform throughout the bioreactor: any parts not reached will result in cell death and poor performance of the reactor.

A simple version of the process is illustrated schematically in Figure 3.3. Here nutrient solution is injected through a single porous tube of radius a, which is aligned with the z-axis. The solution flows out through the wall of the tube into the surrounding porous medium at a rate $Q(z)$ which is to be determined. This setup could also model flow through a porous underground pipe, for example in oil recovery, or a simple filtration process.

Model for flow through a porous tube

Consider the setup illustrated in Figure 3.3. Flow through the porous medium surrounding the tube is governed by Darcy's law relating the liquid flux $\boldsymbol{q}$ to the pressure p:

$$\boldsymbol{q} = -\frac{k}{\mu}\boldsymbol{\nabla} p, \tag{3.28}$$

where k is the permeability of the medium and μ is the viscosity of the liquid (both assumed constant). The specific value of k depends on the detailed geometry of the microstructure; however, k is generally of order d^2, where d is a typical pore radius. In (3.28) we have neglected te effects of gravity, which is certainly negligible on the scale of a typical bioreactor but might be important in groundwater applications. If the porosity of the medium is also constant, then conservation of mass leads to $\boldsymbol{\nabla}\cdot\boldsymbol{q} = 0$ and hence p satisfies Laplace's equation

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial^2 p}{\partial z^2} = 0 \tag{3.29}$$

in cylindrical polar coordinates (r, z).

For simplicity we suppose in the first instance that the porous medium surrounding the tube is infinite so the far-field condition for (3.29) is simply

$$p \to P_{\mathrm{a}} \quad \text{as } r \to \infty, \tag{3.30}$$

where P_{a} is some constant ambient pressure. The flux $Q(z)$ entering the porous medium through the tube wall corresponds to the boundary condition

$$-\frac{k}{\mu}\frac{\partial p}{\partial r} = Q(z) \quad \text{at } r = a. \tag{3.31}$$

Now, assuming that the tube is long and thin, we can model the flow inside it using lubrication theory. In cylindrical polars, the lubrication equations for the pressure P and liquid velocity $\boldsymbol{u} = u\boldsymbol{e}_r + w\boldsymbol{e}_z$ read

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \qquad \frac{\partial P}{\partial r} = 0, \qquad \frac{\partial P}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right). \tag{3.32a}$$

At the tube wall we specify zero slip and now relate the radial velocity to the flux through the wall:

$$w = 0, \quad u = Q(z) \qquad \text{at } r = a. \tag{3.32b}$$

(It is known that flow past a porous boundary can experience an effective slip boundary condition, but this effect is small if the permeability of the medium is relatively small, and we will neglect it.)

It is straightfoward to integrate the problem (3.32) and hence find that the pressure $P(z)$ in the tube satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{\pi a^4}{8\mu}\frac{\mathrm{d}P}{\mathrm{d}z}\right) = -2\pi a Q(z). \tag{3.33}$$

The term in brackets on the left-hand side of (3.33) is the flux of fluid along the tube, which is related to the pressure gradient through the *Poiseuille law*; the right-hand side represents fluid loss through the tube wall. Equation (3.33) is easily rearranged to

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2} = \frac{16\mu Q(z)}{a^3}. \tag{3.34}$$

Finally, the problem is closed by equating the pressures at the tube wall, i.e.

$$p(a, z) = P(z). \tag{3.35}$$

This assumes that the wall of the tube has no intrinsic resistance to flow. One could easily incorporate a semi-permeable tube wall by modifying (3.35) to

$$Q(z) = \lambda\big(P(z) - p(a, z)\big), \tag{3.36}$$

where λ is the effective permeability of the tube wall; then (3.35) is just the limiting case of a completely permeable wall with $\lambda \to \infty$.

We will consider for the moment a semi-infinite tube, with a specified inlet pressure P_{in} at the end $z = 0$. We also expect the pressure eventually to equilibrate a long way downstream. We therefore apply the boundary conditions

$$P(0) = P_{\text{in}}, \qquad\qquad P(z) \to P_{\text{a}} \quad \text{as } z \to \infty. \tag{3.37}$$

on the tube pressure $P(z)$.

Slender body approximation

Now we non-dimensionalise the problem as follows:

$$(r, z) = \ell\,(\tilde{r}, \tilde{z})\,, \quad p = P_{\text{a}} + (P_{\text{in}} - P_{\text{a}})\tilde{p}, \quad P = P_{\text{a}} + (P_{\text{in}} - P_{\text{a}})\tilde{P}, \quad Q = \frac{k(P_{\text{in}} - P_{\text{a}})}{\mu a}\,\tilde{Q}, \tag{3.38}$$

where ℓ is an intrinsic length-scale for the problem. A balance in both equations (3.31) and (3.34). suggests the choice

$$\ell = \frac{a^2}{4\sqrt{k}}. \tag{3.39}$$

The corresponding dimensionless tube radius is given by

$$\epsilon = \frac{a}{\ell} = \frac{4\sqrt{k}}{a} = O\left(\frac{d}{a}\right), \tag{3.40}$$

where we recall that d is a typical pore radius. We expect the pores in the porous medium to be much smaller than the tube drilled through it, and this is consistent with ϵ being a small parameter.

The dimensionless pressure in the porous medium thus satisfies the problem (with tides now dropped)

$$\nabla^2 p = 0 \qquad\qquad r > \epsilon, \tag{3.41a}$$

$$p = P(z), \qquad -\epsilon\frac{\partial p}{\partial r} = Q(z) \qquad\qquad r = \epsilon, \tag{3.41b}$$

$$p \to 0 \qquad\qquad r \to \infty. \tag{3.41c}$$

Since we are considering here a semi-infinite tube, we have $Q(z) = 0$ for $z < 0$.

By comparing with equations (3.10) and (3.21), we can simply read off the solution

$$p(r, z) = \int_0^\infty \frac{Q(s)\,\mathrm{d}s}{2\sqrt{r^2 + (z - s)^2}}, \tag{3.42}$$

as well as the following relation between $P(z)$ and $Q(z)$:

$$P(z) = Q(z) \log\left(\frac{2}{\epsilon}\right) + \frac{1}{2} \int_0^\infty \frac{Q(s)}{|z - s|}\,\mathrm{d}s. \tag{3.43}$$

We also have the dimensionless version of (3.34)

$$\frac{\mathrm{d}^2 P}{\mathrm{d}z^2} = Q(z). \tag{3.44}$$

The slender-body model for the flow consists of the coupled integro-differential equations (3.43) and (3.44), which are to be solved subject to the boundary conditions

$$P(0) = 1, \qquad\qquad P(z) \to 0 \quad \text{as } z \to \infty. \tag{3.45}$$

The slender body approach removes the need to solve any PDEs: the only remaining unknowns are functions only of z. However, the pressure outside the tube *does* satisfy Laplace's equation, subject to global boundary conditions. This non-local effect is captured by the integral term in equation (3.43): the pressure at a point z depends on the *entire* flux profile, not just the local behaviour of $Q(z)$.

In general we should solve the problem (3.43)–(3.45) numerically. A leading-order approximation may be obtained by letting $\epsilon \to 0$ so that the first term on the right-hand side of equation (3.43) dominates. We thus obtain the leading-order governing equation

$$\frac{\mathrm{d}^2 P_0}{\mathrm{d}z^2} = \nu^2 P_0, \tag{3.46}$$

where

$$\nu^2 = \frac{1}{\log(2/\epsilon)} \ll 1, \tag{3.47}$$

and leading-order solution is thus

$$P_0(z) = \mathrm{e}^{-\nu z}, \qquad\qquad Q_0(z) = \nu^2 \mathrm{e}^{-\nu z}. \tag{3.48}$$

This gives qualitatively reasonable behaviour, with the tube pressure P_0 decaying in z as the fluid leaks out of the tube. However, this solution is unlikely to be very quantitatively accurate unless ϵ is extremely small.

The leading-order approximation (3.48) is also asymptotically nonuniform as $z \to 0$ and as $z \to \infty$. To see this, we can calculate the first correction to the flux by plugging the leading-order solution (3.48) back into equation (3.43) to get

$$Q(z) \sim \nu^2 \mathrm{e}^{-\nu z} - \frac{\nu^4}{2} \int_0^\infty \frac{\mathrm{e}^{-\nu s}\,\mathrm{d}s}{|z - s|} + O\left(\nu^6\right), \tag{3.49}$$

where

$$\begin{aligned}\fint_0^\infty \frac{\mathrm{e}^{-\nu s}\,\mathrm{d}s}{|z-s|} &= \frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty \mathrm{e}^{-\nu s}\,\mathrm{sgn}(z-s)\log|z-s|\,\mathrm{d}s \\ &= \mathrm{e}^{-\nu z}\big(\mathrm{Ei}(\nu z) - 2\gamma - 2\log\nu\big), \end{aligned} \tag{3.50}$$

where γ is the Euler–Mascheroni constant, and Ei denotes the exponential integral function. Since

$$\mathrm{e}^{-\nu z}\big(\mathrm{Ei}(\nu z) - 2\gamma - 2\log\nu\big) \sim \begin{cases} \log\left(\dfrac{z}{\nu}\right) - \gamma & \text{as } z \to 0, \\ \dfrac{1}{\nu z} & \text{as } z \to \infty, \end{cases} \tag{3.51}$$

it is clear that the second term in the expansion (3.49) becomes larger than the first, so the expansion becomes nonuniform, both as $z \to 0$ and as $z \to \infty$.

3.3 Slender body theory in Stokes flow

Model problem: slow flow past a slender projectile

Now let us consider the flow problem depicted in Figure 3.1 in the opposite limit where the external fluid is extremely viscous (in the sense that the Reynolds number is small), so that the velocity $\boldsymbol{u}$ and pressure p satisfy the Stokes equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{\nabla}p = \mu\nabla^2\boldsymbol{u}, \tag{3.52}$$

where μ is the viscosity (assumed constant). Again we impose a uniform flow with speed U in the far field, so that

$$\boldsymbol{u} \to U\boldsymbol{e}_z \quad \text{as } \boldsymbol{r} \to \infty, \tag{3.53}$$

where $\boldsymbol{e}_z$ is a unit vector in the z-direction, which is also parallel to the axis of the projectile. In a viscous fluid, we must impose the no-slip boundary condition of zero velocity on the projectile, i.e.

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } r = S(z), \quad -L < z < L. \tag{3.54}$$

In contrast with the inviscid problem analysed in §3.2, it is no longer possible to linearise about the uniform flow. Since the velocity is zero on the boundary of the projectile, it is not true that $\boldsymbol{u} - U\boldsymbol{e}_z$ is everywhere small.

After suitable non-dimensionalisation, the problem reads

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{3.55a}$$

$$\boldsymbol{\nabla}p = \nabla^2\boldsymbol{u}, \tag{3.55b}$$

$$\boldsymbol{u} \to \boldsymbol{e}_z \qquad \text{as } \boldsymbol{r} \to \infty, \tag{3.55c}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } r = \epsilon S(z), \quad -1 < z < 1, \tag{3.55d}$$

where $\epsilon \ll 1$ is the aspect ratio of the projectile.

Assuming the flow is axisymmetric, with $\boldsymbol{u} = u(r,z)\boldsymbol{e}_r + w(r,z)\boldsymbol{e}_z$, we can write out (3.55a) in the form

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{3.56}$$

and it follows that there exists a potential function $\psi(r,z)$ (called the Stokes streamfunction) such that

$$u = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad w = \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{3.57}$$

or, equivalently,

$$\boldsymbol{u} = \boldsymbol{\nabla}\wedge\left(\frac{\psi}{r}\,\boldsymbol{e}_\theta\right), \tag{3.58}$$

where $\boldsymbol{e}_\theta = \boldsymbol{e}_z\wedge\boldsymbol{e}_r$ is the unit basis vector in the θ-direction with respect to cylindrical polar coordinates (r,θ,z). Then by taking the curl of (3.55b) we find that ψ must satisfy

$$\operatorname{curl}^4\left(\frac{\psi}{r}\,\boldsymbol{e}_\theta\right) = \boldsymbol{0}. \tag{3.59}$$

Now, a direct calculation shows that

$$\operatorname{curl}^2\left(\frac{\psi}{r}\,\boldsymbol{e}_\theta\right) = -\frac{\mathcal{L}[\psi]}{r}\,\boldsymbol{e}_\theta, \tag{3.60}$$

(this is the *vorticity* of the flow) where $\mathcal{L}$ denotes the linear differential operator

$$\mathcal{L}[\psi] = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}. \tag{3.61}$$

Hence we deduce that $\psi(r,z)$ must satisfy the fourth-order linear PDE

$$\mathcal{L}^2\psi = 0, \tag{3.62a}$$

and the boundary conditions corresponding to (3.55c) and (3.55d) are

$$\psi \sim \frac{r^2}{2} \qquad \text{as } \boldsymbol{r}\to\infty, \tag{3.62b}$$

$$\psi = \frac{\partial\psi}{\partial r} = 0 \qquad \text{at } r=\epsilon S(z),\quad -1<z<1. \tag{3.62c}$$

(Here we have fixed an arbitrary integration constant by setting the constant value of ψ on the projectile to zero.)

In §3.2, in the case of inviscid flow, we were able to satisfy (at least approximately) the boundary condition of zero normal velocity on the projectile by adding a suitable distribution of point sources along the z-axis. We recall that the velocity potential for an inviscid point source at the origin is given by (3.7), with corresponding velocity field

$$\boldsymbol{u}_{\mathrm{s}} = \frac{q}{4\pi}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = \frac{q}{4\pi}\frac{r\boldsymbol{e}_r + z\boldsymbol{e}_z}{(r^2+z^2)^{3/2}}. \tag{3.63}$$

For viscous flow, there is no velocity potential (because the vorticity is generally not zero), but we can easily find the Stokes streamfunction corresponding to (3.63), namely

$$\psi_{\mathrm{s}} = \frac{-qz}{4\pi\sqrt{r^2+z^2}}. \tag{3.64}$$

By analogy with the inviscid case, we might similarly try to solve the problem (3.62) with a source distribution, i.e.

$$\psi(r,z) = \frac{r^2}{2} + \epsilon^2 \int_{-1}^{1} \frac{-q(s)(z-s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}}. \tag{3.65}$$

(We have anticipated that, as in §3.2, the source strength will be $O\left(\epsilon^2\right)$ compared with the undisturbed uniform flow.) Unfortunately this approach does not work. The source distribution function $q(z)$ may be chosen to make the *normal* component of velocity equal to zero at the projectile boundary, but the *tangential* component will still be nonzero. Equivalently, we can use the function $q(z)$ to impose *one* of the boundary conditions (3.62c), namely $\psi = 0$ at $r = \epsilon S(z)$, but generally $\partial\psi/\partial r$ will remain nonzero. To have a chance of satisfying *both* boundary conditions, we must include a second distribution of singularities along the z-axis.

To motivate the approach taken, consider the problem of slow flow at speed U past a sphere of radius a. The corresponding problem for the Stokes streamfunction $\psi(r,z)$ is

$$\mathcal{L}^2[\psi] = 0 \qquad r^2+z^2 > a^2, \tag{3.66a}$$

$$\psi \sim \frac{Ur^2}{2} \qquad z^2+z^2 \to \infty, \tag{3.66b}$$

$$\psi = \frac{\partial\psi}{\partial r} = 0 \qquad r^2+z^2 = a^2. \tag{3.66c}$$

The solution is given by

$$\psi(r,z) = \frac{Ur^2}{2} + \frac{f}{8\pi\mu}\left(\frac{r^2}{\sqrt{r^2+z^2}} - \frac{a^2r^2}{3\left(r^2+z^2\right)^{3/2}}\right), \tag{3.67}$$

where

$$f = -6\pi\mu U a. \tag{3.68}$$

The easiest way to obtain this solution is to use spherical polar coordinates: see Exercise 6. Alternatively, one can easily verify that the streamfunction (3.67) identically satisfies Laplace's equation for $r > 0$ and also satisfies the boundary conditions (3.66c) if f is given by (3.68).

Far from the sphere, when r^2+z^2 is large, the first bracketed term (of order $|\boldsymbol{r}|$) in (3.67) dominates the final term (which is of order $|\boldsymbol{r}|^{-1}$). Therefore the first correction to the outer flow due to the presence of the sphere takes the form

$$\psi_{\mathrm{St}} = \frac{fr^2}{8\pi\mu\sqrt{r^2+z^2}}, \tag{3.69}$$

and the corresponding pressure and velocity components are given by

$$p_{\mathrm{St}} = \frac{fz}{4\pi\left(r^2+z^2\right)^{3/2}}, \qquad u_{\mathrm{St}} = \frac{frz}{8\pi\mu\left(r^2+z^2\right)^{3/2}}, \qquad w_{\mathrm{St}} = \frac{f\left(r^2+2z^2\right)}{8\pi\mu\left(r^2+z^2\right)^{3/2}}. \tag{3.70}$$

These may be rearranged to

$$p_{\mathrm{St}} = \frac{\boldsymbol{f}\cdot\boldsymbol{r}}{4\pi\mu|\boldsymbol{r}|^3}, \qquad \boldsymbol{u}_{\mathrm{St}} = \frac{(\boldsymbol{f}\cdot\boldsymbol{r})\boldsymbol{r} + |\boldsymbol{r}|^2\boldsymbol{f}}{8\pi\mu|\boldsymbol{r}|^3}, \tag{3.71}$$

where $\boldsymbol{f} = f\boldsymbol{e}_z$. This is the flow due to a point singularity at the origin called a *Stokeslet*. As we have just shown, it is the leading-order far-field correction caused by a spherical obstacle in a uniform flow. It may also be interpreted as the flow due to a point force of magnitude f in the z-direction: Exercise 7 shows that the flow (3.71) satisfies the forced Stokes equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{St}} = 0, \qquad \boldsymbol{\nabla}p_{\mathrm{St}} - \mu\nabla^2\boldsymbol{u}_{\mathrm{St}} = \boldsymbol{f}\delta(\boldsymbol{x}), \tag{3.72}$$

where δ denotes the three-dimensional delta-function.

Therefore, the spherical obstacle effectively applies a force $\boldsymbol{f}$ to the flow, and the flow exerts an equal and opposite force $-f\boldsymbol{e}_z$ on the obstacle, where f is given by equation (3.68). This is called the *Stokes drag* experienced by a spherical obstacle moving at low Reynolds number through a viscous fluid.

Now, returning to flow past a slender projectile, it seems reasonable that the obstacle should exert an effective tangential force on the fluid, as well as an effective source. Therefore we seek a solution to the problem (3.62) in the form

$$\psi(r,z) = \frac{r^2}{2} + \epsilon^2\int_{-1}^{1}\frac{-q(s)(z-s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}} + \int_{-1}^{1}\frac{f(s)r^2\,\mathrm{d}s}{8\pi\sqrt{r^2+(z-s)^2}}, \tag{3.73}$$

where both functions $q(z)$ and $f(z)$ are to be determined by satisfying (at least approximately) both boundary conditions (3.62c). The corresponding pressure and velocity components are given by

$$p = \int_{-1}^{1}\frac{f(s)(z-s)\,\mathrm{d}s}{4\pi\left(r^2+(z-s)^2\right)^{3/2}}, \tag{3.74a}$$

$$u = \int_{-1}^{1}\frac{\epsilon^2 q(s)r\,\mathrm{d}s}{4\pi\left(r^2+(z-s)^2\right)^{3/2}} + \int_{-1}^{1}\frac{f(s)r(z-s)\,\mathrm{d}s}{8\pi\left(r^2+(z-s)^2\right)^{3/2}}, \tag{3.74b}$$

$$w = 1 + \int_{-1}^{1}\frac{\epsilon^2 q(s)(z-s)\,\mathrm{d}s}{4\pi\left(r^2+(z-s)^2\right)^{3/2}} + \int_{-1}^{1}\frac{f(s)\left(r^2+2(z-s)^2\right)\,\mathrm{d}s}{8\pi\left(r^2+(z-s)^2\right)^{3/2}}. \tag{3.74c}$$

These are solutions of the inhomogeneous Stokes equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \epsilon^2 q(z)\delta(x)\delta(y), \qquad \boldsymbol{\nabla}p - \nabla^2\boldsymbol{u} = f(z)\delta(x)\delta(y)\boldsymbol{e}_z, \tag{3.75}$$

where we now have a distribution of mass sources *and* point forces along the z-axis.

To apply the no-slip boundary conditions on the projectile boundary, we need to calculate the asymptotic behaviour of the integrals in equation (3.73) as $r\to 0$. By analogy with equation (3.21), we find that

$$\int_{-1}^{1}\frac{-q(s)(z-s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}} \sim \int_{-1}^{1}\frac{q(s)\,\mathrm{sgn}(s-z)}{4\pi}\,\mathrm{d}s + O\left(r^2\log r\right) \tag{3.76a}$$

$$\int_{-1}^{1}\frac{f(s)r^2\,\mathrm{d}s}{8\pi\sqrt{r^2+(z-s)^2}} \sim \frac{r^2}{8\pi}\left(2f(z)\log\left(\frac{2}{r}\right) + \fint_{-1}^{1}\frac{q(s)\,\mathrm{d}s}{|z-s|}\right) + O\left(r^4\log r\right), \tag{3.76b}$$

for $z\in(-1,1)$. Now we can evaluate ψ and $\partial\psi/\partial r$ on the projectile boundary $r = \epsilon S(z)$, and we find that

$$\psi|_{r=\epsilon S} \sim \epsilon^2\left\{C + \frac{S^2}{2}\left[1 + 2A + 2B\log\left(\frac{\epsilon S}{2}\right)\right]\right\}, \tag{3.77a}$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=\epsilon S} \sim \epsilon S\left[B + 1 + 2A + 2B\log\left(\frac{\epsilon S}{2}\right)\right], \tag{3.77b}$$

where we use the shorthand

$$A = \frac{1}{8\pi}\fint_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{|z-s|}, \qquad B = -\frac{f(z)}{4\pi}, \qquad C = \frac{1}{4\pi}\int_{-1}^{1} q(s)\,\mathrm{sgn}(s-z)\,\mathrm{d}s. \tag{3.78}$$

Therefore the boundary conditions (3.62c) lead to the relations

$$C = \frac{BS^2}{2}, \qquad\qquad A = -\frac{1}{2} - \frac{B}{2} - B\log(\epsilon S/2), \tag{3.79}$$

which may be rearranged to

$$q(z) = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}z}\left(f(z)S(z)^2\right), \tag{3.80a}$$

$$\left[2\log\left(\frac{2}{\epsilon S(z)}\right) - 1\right] f(z) + \fint_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{|z-s|} = -4\pi. \tag{3.80b}$$

Given the projectile radius profile $S(z)$, in principal we can solve the integral equation (3.80b) for the force distribution $f(z)$, and the source distribution $q(z)$ is then determined by equation (3.80a). In the limit $\epsilon \to 0$, the first term in equation (3.80b) formally dominates the second, which suggests the leading-order approximation

$$f(z) \sim \frac{-4\pi}{2\log\left(\frac{2}{\epsilon S(z)}\right) - 1}. \tag{3.81}$$

This approximation makes errors of order $1/\log(1/\epsilon)$ and therefore is unlikely to be very accurate unless ϵ is extremely small. However, it does suggest a possible route to solve (3.80b) numerically, namely to iterate on the equation

$$f(z) = \frac{-4\pi}{2\log\left(\frac{2}{\epsilon S(z)}\right) - 1}\left(1 + \frac{1}{4\pi}\fint_{-1}^{1} \frac{f(s)\,\mathrm{d}s}{|z-s|}\right), \tag{3.82}$$

with (3.81) as the first iteration.

Generalisations

We have assumed so far that the projectile is stationary (in our reference frame) and that the outer flow is purely in the z-direction, i.e. parallel to the axis of the projectile. We could also consider transverse motion of the outer flow and/or of the obstacle by including force distributions in the x- and y-directions, with corresponding Stokeslet distributions analogous to (3.71).

We can also consider cases where the obstacle is not a rigid projectile but is itself deforming. For example, consider the axisymmetric slow viscous flow caused by a deforming filament aligned with the z-axis whose boundary is given by $r = \epsilon S(z,t)$ with respect to dimensionless cylindrical polar coordinates (r,z). Let the normalised axial velocity of the filament be given by $W(z,t)$, so that the fluid velocity $\boldsymbol{u} = u\boldsymbol{e}_r + w\boldsymbol{e}_z$ must satisfy the boundary conditions

$$w(r,z,t) = W(z,t), \quad u(r,z,t) = \epsilon\frac{\partial S}{\partial t} + \epsilon W\frac{\partial S}{\partial z} \qquad \text{at } r = \epsilon S(z,t). \tag{3.83}$$

We again suppose that the effect of the filament is modelled by a distribution of sources and Stokeslets along the z-axis as in equation (3.73), i.e.

$$\psi(r,z,t) = \epsilon^2 \int_{-\infty}^{\infty} \frac{-q(s,t)(z-s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}} + \int_{-\infty}^{\infty} \frac{f(s,t)r^2\,\mathrm{d}s}{8\pi\sqrt{r^2+(z-s)^2}}. \tag{3.84}$$

The effective source and Stokeslet strengths q and f now depend on time t as well as z; both will be zero outside a finite interval if the filament is of finite extent.

As shown in Exercise 9, the source and Stokeslet distribution functions $q(z,t)$ and $f(z,t)$ are related to the filament velocity $W(z,t)$ and radius $S(z,t)$ by the equations

$$4\pi W = \left[2\log\left(\frac{2}{\epsilon S}\right) - 1\right] f + \fint_{-\infty}^{\infty} \frac{f(s,t)}{|z-s|}\,\mathrm{d}s \tag{3.85a}$$

$$q = \frac{1}{4}\frac{\partial}{\partial z}\left(fS^2\right) + \frac{\partial}{\partial t}\left(\pi S^2\right) + \frac{\partial}{\partial z}\left(W\pi S^2\right). \tag{3.85b}$$

If the deformation of the filament is specified, then in principle (3.85) determines $f(z,t)$ and $q(z,t)$. The dimensionless drag force per unit length exerted on the filament by the fluid is then found to be given by $-f(z,t)\boldsymbol{e}_z$, as might have been anticipated.

Industrial problem: bubble deformation in glass

In glass manufacture, air bubbles may become entrained at various stages as the liquid glass is formed and processed. The presence of even small bubbles may cause an unacceptable reduction in both strength and optical integrity, and lead to expensive and wasteful quality control failures. It is therefore important to understand how small entrained bubbles behave as they are convected and deformed by the flowing liquid glass.

A small bubble will convect with the bulk glass flow, and therefore experience to leading order a linear straining flow. To see this, suppose we are in a frame that moves with the centroid of the bubble, which we assume is convected with the local glass velocity. Therefore, relative to this moving frame, the glass velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ is zero at $\boldsymbol{x}=\boldsymbol{0}$. By Taylor expansion, the local velocity field experienced by the bubble is therefore given by

$$\boldsymbol{u}(\boldsymbol{x},t) \sim (\boldsymbol{x}\cdot\boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{0},t) + O\left(|\boldsymbol{x}|^2\right). \tag{3.86}$$

The right-hand side gives a locally linear flow as anticipated. If we assume that the glass is undergoing an extensional flow, as would be the case in a stretching fibre or sheet for example, then this local velocity will be a straining flow of the form

$$\boldsymbol{u}(\boldsymbol{x},t) \sim \begin{pmatrix} a(t)x \\ b(t)y \\ c(t)z \end{pmatrix} \tag{3.87}$$

with respect to suitably chosen axes, where $a+b+c\equiv 0$ by incompressibility.

For additional simplicity, we will suppose further that the flow is axisymmetric, so that $a=b$ and

$$\boldsymbol{u}(\boldsymbol{x},t) \sim \alpha(t)\begin{pmatrix} -x/2 \\ -y/2 \\ z \end{pmatrix} = -\frac{\alpha(t)r}{2}\,\boldsymbol{e}_r + \alpha(t)z\,\mathrm{e}_z, \tag{3.88}$$

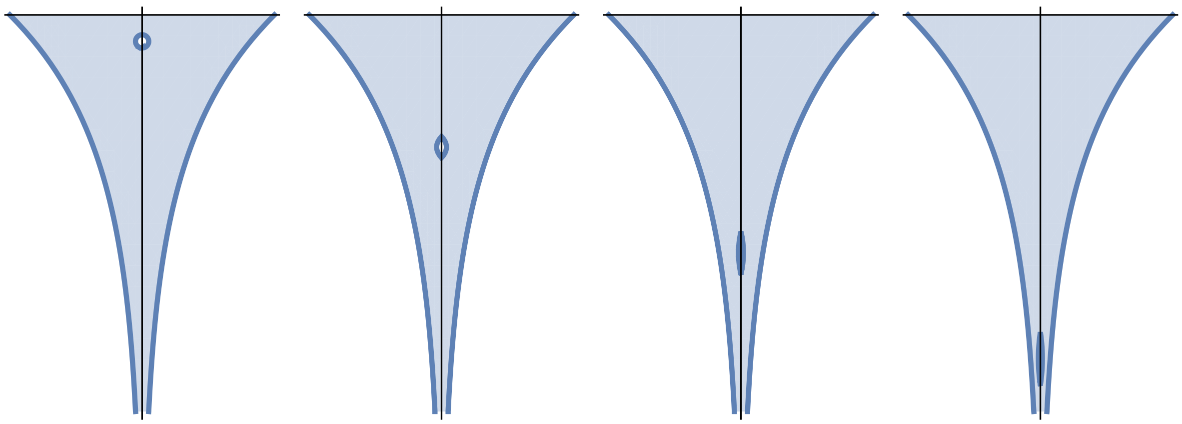

Figure 3.4: Schematic of a small bubble being convected and elongated in an extensional flow.

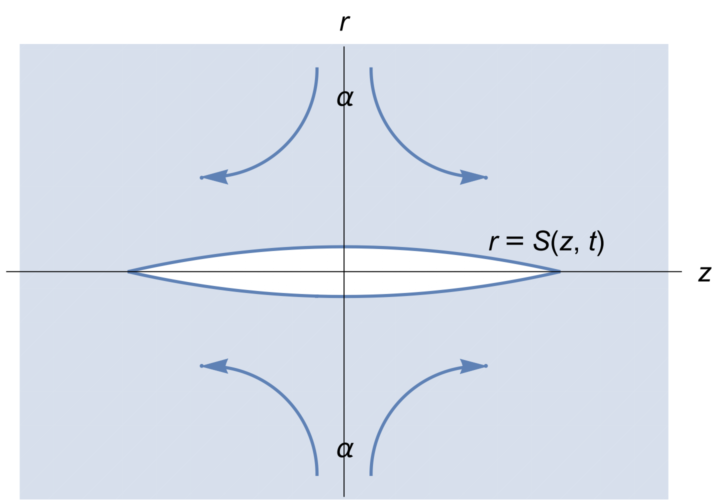

Figure 3.5: Schematic of a thin bubble subject to a straining flow of strength α.

with respect to cylindrical polar coordinates (r, z). Then $c(t) = \alpha(t)$ is the axial strain rate. This sort of local flow would occur, for example, in the case of a bubble being convected along the axis of a stretching fibre, as shown schematically in Figure 3.4.

It is a familiar observation that a small bubble subjected to such a stretching flow rapidly becomes elongated, thin and "pointy". For example, if a jet of honey is dripped from a spoon, one can easily observe that small bubbles become stretched out as they are convected by the flow. Hence we can try to describe the behaviour illustrated in Figure 3.4 using a version of slender body theory.

Axisymmetric model of a thin deforming bubble

We will consider the model problem illustrated in Figure 3.5. A thin axisymmetric bubble along the z-axis is subject to a straining flow of strength α. The flow in the surrounding fluid is governed by the Stokes equations (3.52). The velocity approaches the specified linear flow in the far field, i.e.

$$\boldsymbol{u} \sim -\frac{\alpha(t)r}{2}\,\boldsymbol{e}_r + \alpha(t)z\,\mathrm{e}_z \quad \text{as } \boldsymbol{r} \to \infty. \tag{3.89}$$

This corresponds to a far-field Stokes streamfunction

$$\psi \sim \frac{\alpha r^2 z}{2} \quad \text{as } \boldsymbol{r} \to \infty. \tag{3.90}$$

On the bubble surface $r = S(z,t)$ we have the kinematic and dynamic boundary conditions

$$u = \frac{\partial S}{\partial t} + w\frac{\partial S}{\partial z} \qquad\qquad \boldsymbol{\sigma}\cdot\boldsymbol{n} = (-P + \gamma\kappa)\boldsymbol{n}, \tag{3.91}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $P(t)$ is the pressure inside the bubble, γ is the surface tension; $\boldsymbol{n}$ and κ are the unit normal and curvature of the bubble surface, given by

$$\boldsymbol{n} = \frac{\boldsymbol{e}_r - S_z\,\boldsymbol{e}_z}{\sqrt{1+S_z^2}}, \qquad\qquad \kappa = \frac{1}{S\sqrt{1+S_z^2}} - \frac{S_{zz}}{(1+S_z^2)^{3/2}}. \tag{3.92}$$

(Here we have used the shorthand $S_z = \partial S/\partial z$, $S_{zz} = \partial^2 S/\partial z^2$.)

The bubble pressure $P(t)$ is determined by conservation of the gas inside the bubble. If this gas is incompressible, then the net volume of the bubble is specified, i.e.

$$\int_{-\ell}^{\ell} \pi S(z,t)^2\,\mathrm{d}z \equiv V \equiv \frac{4\pi a^3}{3}, \tag{3.93}$$

where ℓ is the half-length of the bubble. Here for later convenience we have introduced the radius a of the sphere with the same volume as the bubble. One can easily generalise (3.93) to the case of a compressible gas, with the pressure and volume related through Boyle's Law (for example).

Now we non-dimensionalise the problem to exploit the assumed slenderness of the bubble. Suppose that the bubble length and radius are of order L and ϵL respectively, where $\epsilon \ll 1$. We also assume that the (possibly time dependent) strain rate α is of order $\bar{\alpha}$. We then non-dimensionalise the spatial cordinates $\boldsymbol{r}$, time t, bubble radius S, streamfunction ψ, pressure p and stress tensor $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{r} = L\,\tilde{\boldsymbol{r}}, \qquad t = \bar{\alpha}^{-1}\,\tilde{t}, \qquad S = \epsilon L\,\tilde{S}, \tag{3.94a}$$

$$\psi = \bar{\alpha}L^3\,\tilde{\psi}, \qquad p = \mu\bar{\alpha}\,\tilde{p}. \qquad \boldsymbol{\sigma} = \mu\bar{\alpha}\,\tilde{\boldsymbol{\sigma}}. \tag{3.94b}$$

As shown in Exercise 11, the boundary conditions (3.91) on the bubble surface are transformed to (with tildes now dropped)

$$u = \epsilon\frac{\partial S}{t} + \epsilon w\frac{\partial S}{\partial z}, \quad \sigma_{rr} = -P + \frac{1}{\mathrm{Ca}\,S} + O\left(\epsilon^2\right), \quad \sigma_{rz} = \epsilon\frac{\partial S}{\partial z}(\sigma_{zz} - \sigma_{rr}) + O\left(\epsilon^3\right) \tag{3.95}$$

at $r = \epsilon S(z,t)$, where $P(t)$ is the dimensionless bubble pressure and

$$\mathrm{Ca} = \frac{\mu\bar{\alpha}\epsilon L}{\gamma}$$

is a scaled *capillary number*. Next considering the volume constraint (3.93) in dimensionless form we obtain

$$\int_{-\ell}^{\ell} S(z,t)^2 \,\mathrm{d}z \equiv \frac{4a^3}{3\epsilon^2 L^3}, \tag{3.96}$$

where the bubble half-length ℓ is made dimensionless with the length-scale L.

Now we can choose our scalings L and ϵL for the bubble length and radius respectively such that both Ca and the right-hand side of equation (3.96) are identically equal to 1. This choice leads to the definitions

$$L = \left(\frac{4}{3}\right)^{1/3} a\,\epsilon^{-2/3}, \qquad \epsilon = \frac{3}{4}\left(\frac{\gamma}{\mu\bar{\alpha}a}\right)^3. \tag{3.97}$$

Hence our assumption that the bubble is long and thin is valid provided $\mu\bar{\alpha}a/\gamma$ is large. This dimensionless parameter measures the ability of the viscous stress caused by the straining flow to overcome surface tension and stretch out the bubble. If this parameter is not large, then surface tension dominates, and the bubble is not deformed signifcantly by the flow but remains approximately spherical.

From equation (3.95), we see that the shear stress σ_{rz} at the bubble surface is of order ϵ. On the other hand, Exercise 9 shows that σ_{rz} is of order $f(z,t)/\epsilon$ at $r = \epsilon S$, where $f(z,t)$ is the Stokeslet distribution strength (in the notation of equation (3.73)). Hence the Stokeslet distribution must be of order ϵ^2, and we therefore seek solutions for the streamfunction and pressure of the forms

$$\psi(r,z,t) = \frac{\alpha r^2 z}{2} + \epsilon^2 \int_{-\ell}^{\ell} \frac{-q(s,t)(z-s)\,\mathrm{d}s}{4\pi\sqrt{r^2+(z-s)^2}} + \epsilon^2 \int_{-\ell}^{\ell} \frac{f(s,t)r^2\,\mathrm{d}s}{8\pi\sqrt{r^2+(z-s)^2}}, \tag{3.98a}$$

$$p(r,z,t) = \epsilon^2 \int_{-\ell}^{\ell} \frac{f(s,t)(z-s)\,\mathrm{d}s}{4\pi\left(r^2+(z-s)^2\right)^{3/2}}. \tag{3.98b}$$

By expanding the velocity and stress as $r \to 0$ and applying the boundary conditions (3.95), as shown in Exercise 11, we then find that the bubble radius $S(z,t)$ satisfies the equation

$$\frac{\partial S}{\partial t} + \alpha z \frac{\partial S}{\partial z} + \left(\alpha - \frac{P}{2}\right) S + \frac{1}{2} = 0, \tag{3.99}$$

with respect to suitable dimensionless variables. The dimensionless bubble pressure $P(t)$ is then determined by the volume requirement (3.96), which reads

$$\int_{-\ell}^{\ell} S(z,t)^2 \,\mathrm{d}z \equiv 1, \tag{3.100}$$

when the definitions (3.97) are adopted.

Once the strain rate $\alpha(t)$ is specified, equation (3.99) gives a linear first-order PDE for $S(z,t)$. However, the function $P(t)$ is unknown *a priori* and must be found as part of the solution from the nonlinear constraint (3.100). This coupled problem appears formidable in general, but actually it admits a lot of analytical solutions. In particular, it is easily verified that (3.99) may be satisfied identically whenever $S(z,t)$ is a polynomial in z with time-dependent coefficients, i.e.

$$S(z,t) = \sum_{n=0}^{N} c_n(t) z^{2n}, \tag{3.101}$$

where we have assumed symmetry about $z = 0$ and therefore ignored odd powers of z. Substitution of (3.101) into (3.99) yields a system of ODEs for the coefficients $c_n(t)$.

For example, consider the simple case where $N = 1$ so that $S(z,t)$ is just a quadratic function of z. Since $S(z,t) = 0$ at the two ends $z = \pm\ell(t)$, it must take the form

$$S(z,t) = c(t)\left(\ell^2 - z^2\right). \tag{3.102}$$

We then find from the volume constraint (3.100) that $c(t) = \sqrt{15}/4\ell^{5/2}$, i.e.

$$S(z,t) = \frac{\sqrt{15}}{4\ell^{5/2}}\left(\ell^2 - z^2\right). \tag{3.103}$$

Finally, plugging (3.103) into equation (3.99), we find that

$$P(t) = \alpha(t) + \sqrt{\frac{5\ell(t)}{3}}, \tag{3.104}$$

where $\ell(t)$ satisfies the ODE

$$\frac{\dot{\ell}(t)}{\ell(t)} = \alpha(t) - \sqrt{\frac{\ell(t)}{15}}. \tag{3.105}$$

In the steady case where α is constant, we therefore predict a steady state bubble length $\ell = 15\alpha^2$. This shows how the bubble becomes increasingly elongated as the strength α of the applied straining flow increases. Also note that the quadratic solution (3.103) gives pointed tips at the two ends $z = \pm\ell$ of the bubble, in agreement with observations of bubbles in honey or syrup, for example.

Exercises

1. Evaluate the coefficient of r^2 in the expansion (3.15) in the limit $\delta \to 0$, and hence show that

$$\phi(r,z) \sim \frac{q(z)}{2\pi}\log\left(\frac{r}{2}\right) - \frac{1}{4\pi}\,J(z) + r^2\left\{-\frac{q''(z)}{8\pi}\,\log\left(\frac{r}{2\mathrm{e}}\right) + \frac{1}{16\pi}\frac{\mathrm{d}^2 J}{\mathrm{d}z^2}\right\}$$

as $r \to 0$, where $J(z)$ is shorthand for

$$J(z) = \fint_{-1}^{1} \frac{q(s)\,\mathrm{d}s}{|z-s|}.$$

Reach the same conclusion by seeking a solution of Laplace's equation (3.5a) with behaviour

$$\phi(r,z) \sim \frac{q(z)}{2\pi}\log\left(\frac{r}{2}\right) - \frac{1}{4\pi}\,J(z) + C_1(z)r^2\log r + C_2(z)r^2 + \cdots \quad \text{as } r \to 0.$$

2. Consider the problem (3.5) in a boundary layer of thickness order ϵ near the projectile. Show that the rescaling

$$r = \epsilon R, \qquad \phi(r,z) = \Phi(R,z)$$

results in the leading-order inner problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) = 0 \qquad R > S(z),$$
$$\frac{\partial \Phi}{\partial R} = S'(z) \qquad R = S(z).$$

Deduce that the boundary condition (3.5b) can be replaced by the matching condition

$$\phi(r,z) \sim S(z)S'(z)\log r \quad \text{as } r \to 0.$$

3. The disturbance velocity potential $\phi(r,z)$ for flow past a slender projectile with boundary $r = \epsilon S(z)$ satisfies the problem (3.5). Show that the velocity at the surface of the projectile is of magnitude

$$1 + \epsilon^2 v_{\mathrm{s}} \qquad \text{where} \quad v_{\mathrm{s}} = \left.\frac{\partial \phi}{\partial z}\right|_{r=\epsilon S(z)} + \frac{S'(z)^2}{2}.$$

Suppose the projectile is ellipsoidal, with dimensionless radius profile $r = \epsilon S(z)$ where

$$S(z) = \sqrt{1-z^2}.$$

Use equations (3.21) and (3.22) to evaluate $q(z)$ and the resulting behaviour of $\phi(r,z)$ as $r \to 0$. Hence show that the surface velocity correction in this case is given by

$$v_{\mathrm{s}} = \log\left(\frac{2}{\epsilon}\right) - \frac{2-z^2}{2\,(1-z^2)}.$$

[*This predicts infinite velocity at $z = \pm 1$: the slender body limit fails in neighbourhoods of these two end points.*]

4. We wish to calculate the electric potential $\phi(r,z)$ due to a thin charged metal rod with dimensionless radius profile $r = \epsilon S(z)$ for $-1 < z < 1$. In dimensionless variables, ϕ satisfies the normalized problem

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0,$$
$$\phi = 1 \qquad \text{at } r = \epsilon S(z) \qquad ,$$
$$\phi \to 0 \qquad \text{as } r^2 + z^2 \to \infty,$$

and the normalised capacitance of the rod is then given by

$$C = -2\pi \int_{-1}^{1} r \left.\frac{\partial \phi}{\partial r}\right|_{r=\epsilon S(z)} \mathrm{d}z.$$

Seek a solution for ϕ in the form (3.10). Show that

$$C = -\int_{-1}^{1} q(z)\,\mathrm{d}z,$$

where $q(z)$ satisfies the integral equation

$$2q(z)\log\left(\frac{2}{\epsilon S(z)}\right) + \fint_{-1}^{1} \frac{q(s)\,\mathrm{d}s}{|z-s|} = -4\pi. \qquad (\star)$$

Deduce that

$$q(z) \sim -\frac{2\pi}{\log(2/\epsilon)} + O\left(\frac{1}{\log^2(2/\epsilon)}\right) \quad \text{as } \epsilon \to 0.$$

Show that the leading-order approximation is exact for the case of an ellipsoid where $S(z) = \sqrt{1-z^2}$. [*This logarithmic approximation will not be very accurate in general unless ϵ is extremely small! For moderately small ϵ it is usually better to solve the integral equation*

$$\int_{-1}^{1} \frac{q(s)\,\mathrm{d}s}{\sqrt{\epsilon^2 S(z)^2 + (z-s)^2}} = -4\pi$$

numerically.]

5. Consider a porous bioreactor consisting of a single tube of dimensionless radius ϵ, length L and variable permeability $\lambda(z)$. The walls $z=0$ and $z=L$ of the porous medium are impermeable. The dimensionless input and outlet pressures are equal to 1 and $P_{\text{out}} \in (0,1)$ (where the pressure as $r \to \infty$ has been normalised to zero).

   The dimensionless pressure $p(r,z)$ in the porous medium satisfies the problem

$$\begin{aligned}
\nabla^2 p &= 0 && r>\epsilon,\ 0<z<L,\\
-r\frac{\partial p}{\partial r} &= Q(z) && r=\epsilon,\ 0<z<L,\\
p &\sim -q\log(r) + o(1) && r\to\infty,\ 0<z<L,\\
\frac{\partial p}{\partial z} &= 0 && r>\epsilon,\ z=0 \text{ and } z=L,
\end{aligned}$$

   where $Q(z)$ is the flux out of the tube and $q = \dfrac{1}{L}\displaystyle\int_0^L Q(z)\,\mathrm{d}z$.

   The tube pressure $P(z)$ satisfies the problem

$$\begin{aligned}
P''(z) = Q(z) = \lambda(z)\big[P(z) - p(\epsilon,z)\big] & && 0<z<L,\\
P = 1 & && z=0,\\
P = P_{\text{out}} & && z=L.
\end{aligned}$$

   (a) Consider the case $\lambda = \text{constant}$. By separating the variables, show that

$$\begin{aligned}
Q(z) &= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi z}{L}\right),\\
p(\epsilon,z) &= \frac{c_0}{2}\log\left(\frac{1}{\epsilon}\right) + \sum_{n=1}^{\infty}\left(\frac{\mathrm{K}_0(n\pi\epsilon/L)}{(n\pi\epsilon/L)\,\mathrm{K}_1(n\pi\epsilon/L)}\right) c_n \cos\left(\frac{n\pi z}{L}\right),\\
P(z) &= 1 + mz + \frac{c_0 z^2}{4} + \sum_{n=1}^{\infty} \frac{L^2 c_n}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi z}{L}\right)\right],
\end{aligned}$$

for some constants c_n, where K denotes the modified Bessel function of the second kind and where $m = P'(0)$ is related to the outlet pressure by the equation

$$1 + mL + \frac{c_0 L^2}{4} + \sum_{n=1}^{\infty} \frac{L^2 c_n}{n^2 \pi^2} \left[1 - (-1)^n\right] = P_{\text{out}}.$$

Hence obtain a linear system of equations for the coefficients c_n and solve the problem numerically by truncating at a large but finite value of n.

(b) Now consider the case where the permeability λ is a function of z. Suppose we wish to design the permeability profile $\lambda(z)$ such that the flux into the porous medium is uniform, i.e. Q is independent of z.

For what permeability profile $\lambda(z)$ is $Q(z)$ constant? For what range of parameter values does such a profile exist?

6. **Slow flow past a sphere**

Transform the problem (3.66) to spherical polar coordinates $(R, \phi) \in [0, \infty) \times [0, \pi]$ such that $z = R\cos\phi$, $r = R\sin\phi$. Show that, if ψ is of the form

$$\psi(R, \phi) = f(R)\sin^2\phi,$$

then

$$\mathcal{L}[\psi] = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{2f}{R^2}\right)\sin^2\phi,$$

and deduce that solutions exist of the form $f(R) \propto R^k$, with $k = -1$, 1, 2 or 4. Hence show that the solution of the problem (3.66) is

$$\psi = U\sin^2\phi\left(\frac{R^2}{2} - \frac{3aR}{4} + \frac{a^3}{4R}\right) = \frac{Ur^2}{2} - \frac{3Uar^2}{4\sqrt{r^2+z^2}} + \frac{Ua^3r^2}{4\left(r^2+z^2\right)^{3/2}}.$$

7. **Stokeslet as a point force**

Here we use a three-dimensional Fourier transform, defined for a (suitably integrable) function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$\widehat{F}(\boldsymbol{k}) = \iiint_{\mathbb{R}^3} F(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{k} = (k, \ell, m)$ is the vector of transform variables. The corresponding inverse transform is

$$F(\boldsymbol{x}) = \frac{1}{8\pi^3}\iiint_{\mathbb{R}^3} \widehat{F}(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}.$$

Consider Stokes flow due to a point force at the origin of strength $f\boldsymbol{e}_z$, so the velocity $\boldsymbol{u}$ and pressure p satisfy equations (3.72). Show that the Fourier transformed pressure and velocity components are given by

$$\frac{\mu\widehat{u}}{f} = \frac{-km}{\left(k^2+\ell^2+m^2\right)^2}, \qquad \frac{\mu\widehat{v}}{f} = \frac{-\ell m}{\left(k^2+\ell^2+m^2\right)^2},$$

$$\frac{\mu\widehat{w}}{f} = \frac{k^2+\ell^2}{\left(k^2+\ell^2+m^2\right)^2}, \qquad \frac{\widehat{p}}{f} = \frac{-\mathrm{i}m}{k^2+\ell^2+m^2}.$$

Given [*or you could try to calculate this if you like*] that the appropriate inverse transform for p is

$$p = \frac{fz}{4\pi\left(x^2+y^2+z^2\right)^{3/2}},$$

deduce that the corresponding velocity components are given by

$$u = \frac{fxz}{8\pi\mu\left(x^2+y^2+z^2\right)^{3/2}}, \quad v = \frac{fyz}{8\pi\mu\left(x^2+y^2+z^2\right)^{3/2}}, \quad w = \frac{f\left(x^2+y^2+2z^2\right)}{8\pi\mu\left(x^2+y^2+z^2\right)^{3/2}},$$

which are equivalent to (3.70).

8. Show that, for an ellipsoidal projectile, with $S(z) = \sqrt{1-z^2}$, the integral equation (3.80b) is solved exactly by a constant force distribution

$$f = \frac{-4\pi}{2\log(2/\epsilon)-1},$$

and find the corresponding source distribution $q(z)$. Try to solve (3.80b) numerically for general $S(z)$, and use this exact solution as a test case. [*You may find it easier to formulate the problem in the form*

$$f(z) - \int_{-1}^{1}\frac{f(s)\,\mathrm{d}s}{\sqrt{\epsilon^2S(z)^2+(z-s)^2}} = 4\pi.]$$

9. Show that, as $r \to 0$, the leading-order behaviour of the streamfunction ψ in equation (3.84) is given by

$$\psi(r,z,t) \sim A(z,t)r^2 + B(z,t)r^2\log(r/2) + \epsilon^2C(z,t),$$

where the functions A, B, C are as in equation (3.78), i.e.

$$A(z,t) = \frac{1}{8\pi}\int_{\infty}^{\infty}\frac{f(s,t)}{|z-s|}\,\mathrm{d}s, \quad B(z,t) = -\frac{f(z,t)}{4\pi}, \quad C(z,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}q(s,t)\,\mathrm{sgn}(s-z)\,\mathrm{d}s.$$

Show that the velocity components and pressure are locally given by

$$\begin{aligned} u &\sim -\frac{\partial A}{\partial z}r - \frac{\partial B}{\partial z}r\log(r/2) - \frac{\epsilon^2}{r}\frac{\partial C}{\partial z}, \\ w &\sim 2A + B + 2B\log(r/2), \\ p &\sim -2\frac{\partial A}{\partial z} - 2\frac{\partial B}{\partial z}\log(r/2). \end{aligned}$$

By evaluating the velocity at the filament surface $r = \epsilon S$, show that

$$W = 2A + B + 2B\log\left(\frac{\epsilon S}{2}\right), \qquad \frac{\partial}{\partial z}\left(BS^2 - 2C\right) = \frac{\partial}{\partial t}\left(S^2\right) + \frac{\partial}{\partial z}\left(WS^2\right),$$

and hence obtain the relations (3.85).

Evaluate the stress components and deduce that the leading-order dimensionless stress at the filament surface $r = \epsilon S(z,t)$ is given by

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} \sim \left(\frac{2}{S^2}\frac{\partial C}{\partial z} - \frac{2}{S}\frac{\partial}{\partial z}(BS)\right)\boldsymbol{e}_r + \frac{2B}{\epsilon S}\boldsymbol{e}_z.$$

Deduce that the net drag force per unit length exerted on the filament by the fluid is given by

$$\boldsymbol{D} = \int_0^{2\pi} \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \epsilon S \, \mathrm{d}\theta = -f(z,t)\boldsymbol{e}_z.$$

[*Thus the drag force distribution by the fluid on the filament is equal and opposite to the force distribution by the filament on the fluid, as expected.*]

10. **Viscous drag in fibre drawing**
Return to the fibre-drawing problem described in §2.4. Consider a viscous fibre with radius $S(z,t)$ and axial velocity $w(z,t)$ (made dimensionless as in §2.4). Now suppose that the external atmosphere exerts a shear stress $(\epsilon \mu V_{\text{in}}/L)\tau(z,t)$ at the boundary $r = S(z,t)$ (so that τ is dimensionless).

Show that the Trouton model (2.69) is modified to

$$\frac{\partial}{\partial t}\left(S^2\right) + \frac{\partial}{\partial z}\left(wS^2\right) = 0, \qquad \frac{\partial}{\partial z}\left(3S^2\frac{\partial w}{\partial z}\right) = -2S\tau.$$

Now suppose that the fibre is surrounded by a Newtonian fluid with much smaller viscosity $\tilde{\mu}$, such that

$$\frac{\tilde{\mu}}{\mu} = \epsilon^2\lambda,$$

with $\lambda = O(1)$. Assuming that inertia effects are negligible, deduce that the cross-sectional area $A(z,t)$, extensional velocity $w(z,t)$ and net drag force $f(z,t)$ satisfy the coupled equations

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(wA) = 0, \qquad \frac{\partial}{\partial z}\left(3A\frac{\partial w}{\partial z}\right) = \lambda f,$$

$$f \log\left(\frac{4\pi}{\epsilon^2 \mathrm{e} A}\right) + \int\!\!\!\!\!\!-_{\infty}^{\infty} \frac{f(s,t)}{|z-s|}\,\mathrm{d}s = 4\pi w.$$

11. Consider the deformation of a slender bubble in a viscous fluid subject to the linear straining flow (3.89). Perform the non-dimensionalisation (3.94) and hence obtain the approximate boundary conditions (3.95).

Now suppose that the streamfunction and pressure are written in the form (3.98). By using the results of Exercise 9, deduce that

$$u \sim -\frac{\alpha r}{2} + \frac{\epsilon^2 q(z)}{2\pi r}, \qquad w \sim \alpha z, \qquad \sigma_{rr} \sim -\alpha - \frac{\epsilon^2 q(z)}{\pi r^2}$$

as $r \to 0$. By applying the boundary conditions (3.95), show that the bubble radius $S(z,t)$ satisfies equation (3.99).

12. Show that the substitution

$$\ell(t) = 60\left(\frac{\dot{\phi}(t)}{\phi(t)}\right)^2$$

transforms equation (3.105) to

$$\dddot{\phi}(t) - \frac{\alpha(t)}{2}\,\dot{\phi}(t) = 0.$$

Consider the steady drawdown of a fibre, with axial velocity $w(z)$ given by equation (2.72), i.e.

$$w(z) = \mathrm{e}^{mz},$$

where $m = \log D$ and $D > 1$ is the draw ratio. Suppose a small bubble starts at $t = 0$ at the top of the fibre axis, i.e. $r = z = 0$, and is then convected along the fibre by the flow. Show that the distance z moved by the bubble along the fibre and the axial strain rate α experienced by the bubble at time t are given by

$$z = \frac{1}{m}\log\left(\frac{1}{1-mt}\right). \qquad \alpha(t) = \frac{m}{1-mt}.$$

Show that the general solution of equation (3.105) in this case is given by

$$\ell(t) = \frac{15m^2}{(1-mt)\left(k-\sqrt{1-mt}\right)^2},$$

where the integration constant k is related to the initial bubble length $\ell_0 = \ell(0)$ by

$$k = 1 + m\sqrt{\frac{15}{\ell_0}}.$$