Continuum Methods in Industry

Exercises #1

1(a) If the density isn't constant, and the velocity is $V(x,t)$, then the 1-D heat eq^{n} should be

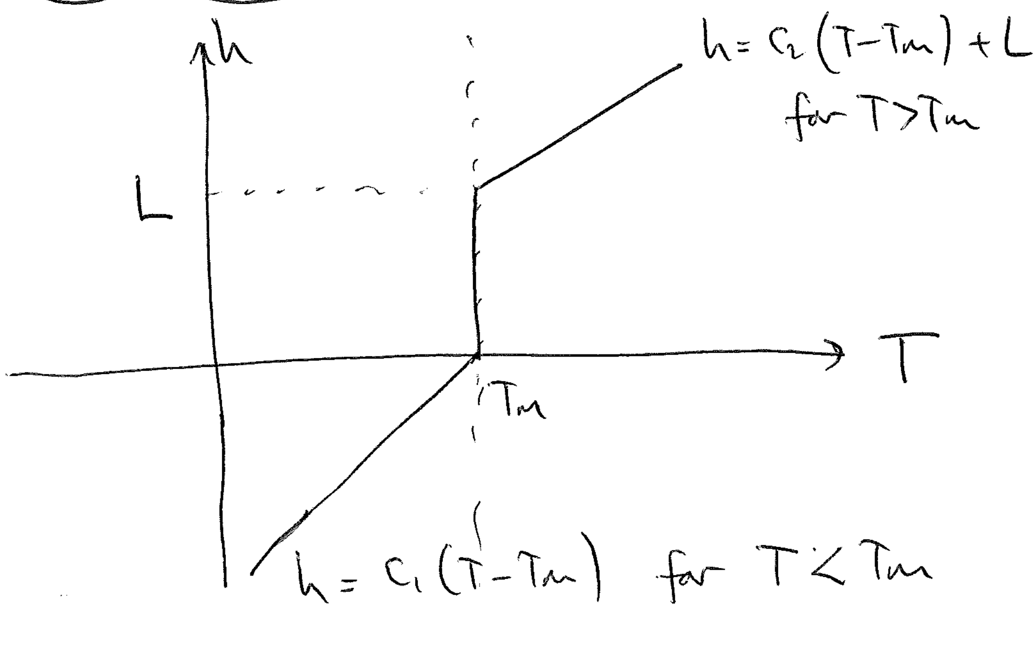
$$\boxed{\frac{\partial}{\partial t}(\rho c T) + \frac{\partial}{\partial x}\left[\rho c V T - k\frac{\partial T}{\partial x}\right] = 0}$$

[the extra flux term is energy flux through convection]

Note also that conservation of mass gives

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0} \quad (*1)$$

To include a phase change, it's better to work with the enthalpy h — the total energy per unit mass including latent heat:



Then the energy equation can be written

$$\boxed{\frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}\left[\rho v h - k\frac{\partial T}{\partial x}\right] = 0} \quad \text{*2}$$

The jump conditions across a moving boundary $x = s(t)$

are $\boxed{\left[\rho(v-\dot{s})\right]_-^+ = 0}$ conservation of mass

& $\boxed{\left[\rho h(v-\dot{s}) - k\frac{\partial T}{\partial x}\right]_-^+ = 0}$ conservation of energy

[NB you can get these either from physical conservation arguments or as the Rankine - Hugoniot conditions for the two conservation equations *1 & *2]

Since $\rho(v-\dot{s})$ is continuous, we can write

$$\rho(v-\dot{s})[h]_-^+ = \left[k\frac{\partial T}{\partial x}\right]_-^+$$

and hence $\boxed{\pm\rho L(v-\dot{s}) = \left[k\frac{\partial T}{\partial x}\right]_-^+}$

+ if liquid on right, solid on left.
− if liquid on left, solid on right.

1(b) We have

$$\frac{\sqrt{\pi}\,\beta\,e^{St\beta^2/4}\,\mathrm{erf}\left(\frac{\beta\sqrt{St}}{2}\right)}{2\sqrt{St}} = 1$$

Let $\boxed{\frac{\beta\sqrt{St}}{2} = \xi}$

Then $\frac{\sqrt{\pi}\,\xi}{St}\,e^{\xi^2}\,\mathrm{erf}(\xi) = 1$

$$\therefore \quad \boxed{St = \sqrt{\pi}\,\xi\,e^{\xi^2}\,\mathrm{erf}(\xi)}$$

& then $\beta = \frac{2\xi}{\sqrt{St}} = \frac{2\xi}{\pi^{1/4}\sqrt{\xi}\,e^{\xi^2/2}\sqrt{\mathrm{erf}(\xi)}}$

$$\Rightarrow \quad \boxed{\beta = \frac{2\sqrt{\xi}\,e^{-\xi^2/2}}{\pi^{1/4}\sqrt{\mathrm{erf}(\xi)}}}$$

Now let $\xi \to 0$: $\mathrm{erf}\,\xi \sim \frac{2}{\sqrt{\pi}}\left(\xi - \frac{\xi^3}{3} + \dots\right)$

So
$$\left.\begin{aligned} St &\sim 2\xi^2 + \frac{4}{3}\xi^4 + \dots \\ \beta &\sim \sqrt{2}\left(1 - \frac{\xi^2}{3} + \dots\right) \end{aligned}\right\} \text{ as } \xi \to 0.$$

Eliminate ξ: $\boxed{\beta \sim \sqrt{2}\left(1 - \frac{St}{6} + \dots\right) \text{ as } St \to 0}$

Now let $\xi \to \infty$: $\quad \operatorname{erf} \xi \sim 1 - \dfrac{e^{-\xi^2}}{\sqrt{\pi}\,\xi} + \cdots$

So $\quad St \sim \sqrt{\pi}\,\xi\, e^{\xi^2} \left(1 + \text{exponentially small}\right)$

$$\beta \sim \frac{2\sqrt{\xi}\, e^{-\xi^2/2}}{\pi^{1/4}} \left(1 + \text{exponentially small}\right)$$

Recall $\beta\sqrt{St} = 2\xi$ & now $\log\left(\dfrac{St}{\sqrt{\pi}}\right) \sim \log \xi + \xi^2$ + exponentially small

To lowest order, $\xi^2 \gg \log \xi$ & hence $\xi \sim \sqrt{\log\left(\dfrac{St}{\sqrt{\pi}}\right)}$

& then $\boxed{\beta \sim \dfrac{2}{\sqrt{St}} \sqrt{\log\left(\dfrac{St}{\sqrt{\pi}}\right)}}$

[Can improve on this by successive approximation:

$$\xi = \sqrt{\log\left(\frac{St}{\sqrt{\pi}}\right)} \left[1 - \frac{\log \xi}{\log\left(\frac{St}{\sqrt{\pi}}\right)}\right]^{\frac{1}{2}}$$ ← iterate on this.]

Alternatively, start from

$$\boxed{\begin{gathered} f'' + \frac{St}{2}\eta f' = 0 \\ f(0) = 1, \quad f(\beta) = 0, \quad f'(\beta) = -\beta/2 \end{gathered}}$$

As $St \to 0$, get to lowest order

$$f'' = 0 \Rightarrow f(\eta) = A\eta + B$$

Then BCs give $B = 1$, $A\beta + B = 0$, $A = -\beta/2$

$$\therefore f(\eta) = 1 - \frac{\beta\eta}{2} \quad \text{and} \quad \beta^2 = 2$$

which gives required result.

$St \to \infty$ is harder!

First rescale $\boxed{\eta = \frac{z}{\sqrt{St}}, \quad \beta = \frac{b}{\sqrt{St}}}$ to get

$$\boxed{\begin{gathered} F'' + \frac{z}{2}F' = 0 \\ F(0) = 1, \quad F(b) = 0, \quad F'(b) = -\frac{b}{2St} \end{gathered}}$$ where $f(\eta) = F(z)$

Letting $St \to \infty$ we get $F_0'(b) = 0$ to lowest order.

But, there are no non trivial solutions to the resulting homogeneous BVP, i.e. inevitably $F_0(z) \equiv 0$.

The best we can do is $F_0(z) \to 0$ as $z \to \infty$.

This results in the solution

$$\boxed{F_0(z) = \operatorname{erfc}\left(\frac{z}{2}\right)}$$

and then
$$F_0(z) \sim \frac{2e^{-z^2/4}}{\sqrt{\pi}\, z}\left(1 - \frac{2}{z^2} + \dots\right)$$
as $z \to \infty$

So suppose there is an inner region at $z = b \gg 1$.

Let $\boxed{z = b + \delta\zeta}$ where both b and δ are to be determined.

The outer solution in inner variables reads

$$F_0(z) \sim \frac{2}{\sqrt{\pi}\, b} \exp\left[-\frac{b^2}{4} - \frac{b\delta\zeta}{2} - \frac{\delta^2\zeta^2}{4}\right]\left(1 + \dots\right)$$

which suggests the scaling $\boxed{F(z) = \frac{2e^{-b^2/4}}{\sqrt{\pi}\, b} \cdot G(\zeta)}$

Plug into ODE:

$$G'' + \frac{\delta}{2}(b + \delta\zeta)\, G' = 0$$

For dominant balance choose $\boxed{\delta = 1/b}$

& then to leading order

$$\boxed{G'' + \frac{1}{2} G' = 0}$$

subject to $\boxed{G(\xi) \sim e^{-\xi/2} \quad \text{as } \xi \to -\infty}$ (match with outer)

& the BCs at $z = b$ become

$$\boxed{G(0) = 0, \quad \frac{2 e^{-b^2/4}}{\sqrt{\pi}\, b\, \delta} G'(0) = -\frac{b}{2 St}}$$

Hence leading-order solution is

$$\boxed{G(\xi) = e^{-\xi/2} - 1}$$

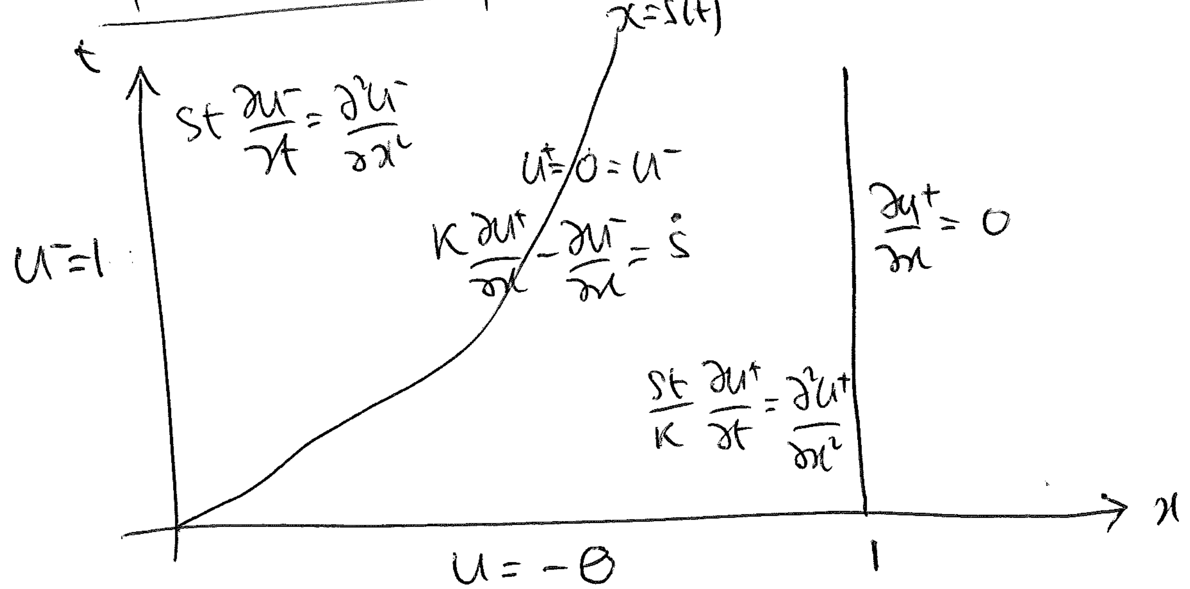
& $\boxed{G'(0) = -\frac{1}{2} = -\frac{b\sqrt{\pi}}{4 St} e^{b^2/4}}$

So $\boxed{\frac{St}{\sqrt{\pi}} \approx \frac{b e^{b^2/4}}{2}}$ & this gives

$$b \sim 2\sqrt{\log\left(\frac{St}{\sqrt{\pi}}\right)}$$

as before. whew!

2) 2-phase Stefan problem:



As $St \to 0$, leading-order solution is $\begin{cases} u^- = 1 - \frac{x}{s} \\ u^+ = 0 \end{cases}$

$$\& \quad \underline{s = \sqrt{2t}}$$

but this doesn't equal $-\theta$ at $t = 0$.

Perform suggested scalings: u^+ varies on $O(1)$ length-scale, while u^- varies on much smaller length-scale $O(\sqrt{St})$.

Problem in $x > s(t) = \sqrt{St}\, S(\tau)$:

$$\boxed{\begin{array}{ll} \frac{\partial u^+}{\partial \tau} = K \frac{\partial^2 u^+}{\partial x^2}, & x > \sqrt{St}\, S(\tau) \\ \frac{\partial u^+}{\partial x} = 0 & x = 1 \end{array}}$$

Problem in $X < S(\tau)$:

$$\boxed{\begin{array}{ll} \delta t \dfrac{\partial U}{\partial \tau} = \dfrac{\partial^2 U}{\partial X^2} & \text{in } 0 < X < S(\tau) \\ U = 1 \quad \text{at } X = 0 & \end{array}}$$

Free boundary conditions

$$\begin{cases} K \dfrac{\partial u^+}{\partial x}\left(\sqrt{\delta t}\, S(\tau), \tau\right) - \dfrac{1}{\sqrt{\delta t}} \dfrac{\partial U}{\partial X}\left(S(\tau), \tau\right) = \dfrac{1}{\sqrt{\delta t}} \dot{S}(\tau) \\ u^+\left(\sqrt{\delta t}\, S(\tau), \tau\right) = U\left(S(\tau), \tau\right) = 0 \end{cases}$$

Now just let $\delta t \to 0$ to get required model.

In $X < S(\tau)$: as $\delta t \to 0$: $\underline{U = 1 - \dfrac{X}{S}}$

$$\& \quad \underline{\dot{S} = \frac{1}{S}} \Rightarrow \boxed{S = \sqrt{2\tau}}$$

which agrees with outer.

In $x > 0$: separation of vars gives

$$u(x,\tau) = \sum_{n=0}^{\infty} c_n \sin\left(\left(n+\tfrac{1}{2}\right)\pi x\right) e^{-(n+\frac{1}{2})^2 \tau / K}$$

where by Fourier series $c_n = -2\theta \displaystyle\int_0^1 \sin\left(\left(n+\tfrac{1}{2}\right)\pi x\right) dx$

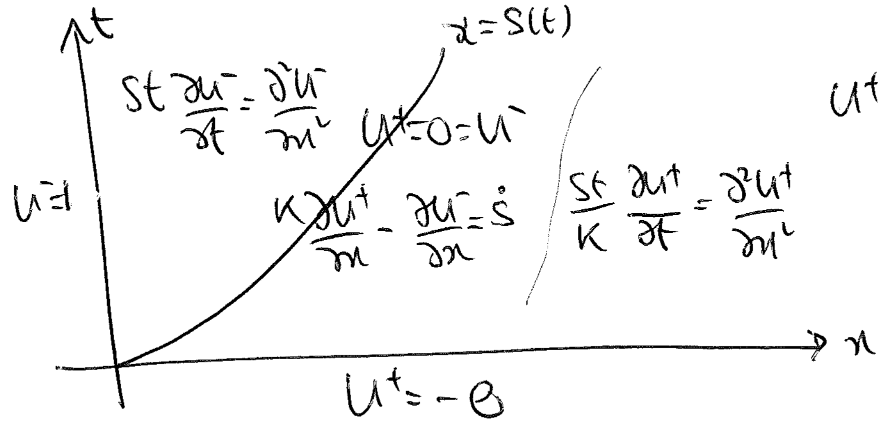
$= \ldots\ldots$

In any case, $u \to 0$ (exponentially) as $\tau \to \infty$

which matches with outer ☐

3. Now let $t \to 0$ (with $St = O(1)$).

When $t \ll 1$ and $x \ll 1$, problem is effectively semi-infinite:



$u^+ \to -\theta$ as $x \to \infty$.

So can seek similarity solution of the form given.

Problems for f & g:

$$\boxed{\begin{array}{ll} f'' + \frac{St\,\eta}{2} f' = 0 & \eta < \beta \\ g'' + \frac{St\,\eta}{2K} g' = 0 & \eta > \beta \\ f(0) = 1 & \\ g \to -\theta \text{ as } \eta \to \infty & \\ f(\beta) = g(\beta) = 0 & \\ K g'(\beta) - f'(\beta) = \frac{\beta}{2} & \end{array}}$$

Thus easily get given solution.

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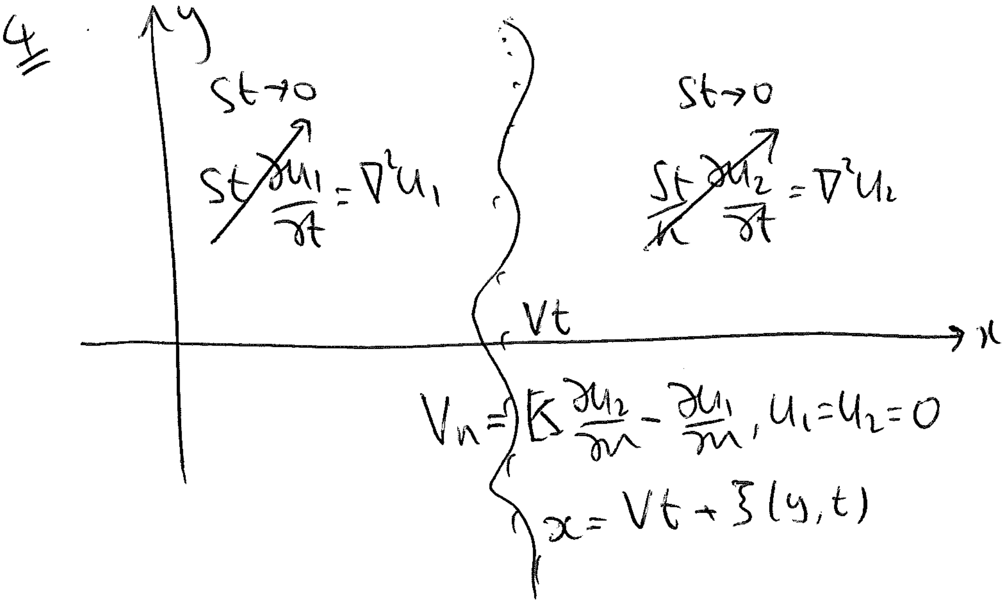


Note: If free boundary is $x = f(y,t)$, then unit normal is

$$\underset{\sim}{n} = \frac{\left(1, -\frac{\partial f}{\partial y}\right)}{\sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

Normal derivative:
$$\frac{\partial u}{\partial n} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2}} \left(\frac{\partial u}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial u}{\partial y} \right)$$

Normal velocity
$$V_n = \frac{\partial f / \partial t}{\sqrt{1 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

So free boundary conditions are

$$\begin{cases} V + \dfrac{\partial \xi}{\partial t} = K \left(\dfrac{\partial u_2}{\partial x} - \dfrac{\partial \xi}{\partial y} \dfrac{\partial u_2}{\partial y} \right) - \left(\dfrac{\partial u_1}{\partial x} - \dfrac{\partial \xi}{\partial y} \dfrac{\partial u_1}{\partial y} \right) \\ u_1 = u_2 = 0 \end{cases}$$

at $x = Vt + \xi(y,t)$.

Now let $\begin{cases} u_1 = -\lambda_1 (x - Vt) + \tilde{u}_1 \\ u_2 = -\lambda_2 (x - Vt) + \hat{u}_2 \end{cases}$

& linearize to get

$$\boxed{\begin{array}{ll} \nabla^2 \tilde{u}_1 = 0 & x < Vt \\ \nabla^2 \tilde{u}_2 = 0 & x > Vt \end{array}}$$

& BCs ... at leading order $\boxed{V = \lambda_1 - K \lambda_2}$

at first order ... note that (e.g.)

$$u_1 (Vt + \xi, y, t) \sim -\lambda_1 \xi + \tilde{u}_1 (Vt, y, t)$$

+ non linear terms

so linearized BCs are

$$\boxed{\begin{array}{l} \hat{u}_1 - \lambda_1 \xi = \hat{u}_2 - \lambda_2 \xi = 0 \\ \dfrac{\partial \xi}{\partial t} = K \dfrac{\partial \hat{u}_2}{\partial x} - \dfrac{\partial \hat{u}_1}{\partial x} \end{array}}$$

at $x = Vt$.

Seek solutions of Laplace's equation which decay as $x \to \pm\infty$:

$$\boxed{\begin{array}{l} \hat{u}_1 = A e^{\sigma t + iky + k(x - Vt)} \\ \hat{u}_2 = B e^{\sigma t + iky - k(x - Vt)} \\ \xi = C e^{\sigma t + iky} \end{array}}$$

Then BCs become:

$$A = \lambda_1 C, \quad B = \lambda_2 C, \quad \sigma C = -k\bar{K} B - kA$$

i.e. $$\begin{pmatrix} 1 & 0 & -\lambda_1 \\ 0 & 1 & -\lambda_2 \\ k & k\bar{K} & \sigma \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \underline{0}$$

Non trivial solutions $\Leftrightarrow$ det (LHS) = 0

$$\Rightarrow \quad \sigma + \lambda_2 k \bar{K} + \lambda_1 k = 0$$

$$\therefore \quad \sigma = -k(\lambda_1 + \bar{K}\lambda_2)$$

$$\therefore \quad \boxed{\frac{\sigma}{kV} = -\left(\frac{\lambda_1 + \bar{K}\lambda_2}{\lambda_1 - \bar{K}\lambda_2}\right)}$$

B

5/ Non-dimensionalize as follows:

$$x = a\tilde{x}, \quad t = \frac{\rho L a^2 \tilde{t}}{k(T_m - T_0)}, \quad T = T_m + (T_m - T_0)u$$

to get given equations & BCs. Indeed find that

$$q = \frac{a^2 J^2}{\sigma k (T_m - T_0)}$$

& relation between current and voltage: $V = IR$

$$\left.\begin{aligned} I &= JA \\ R &= \frac{a}{\sigma A} \end{aligned}\right\}$$

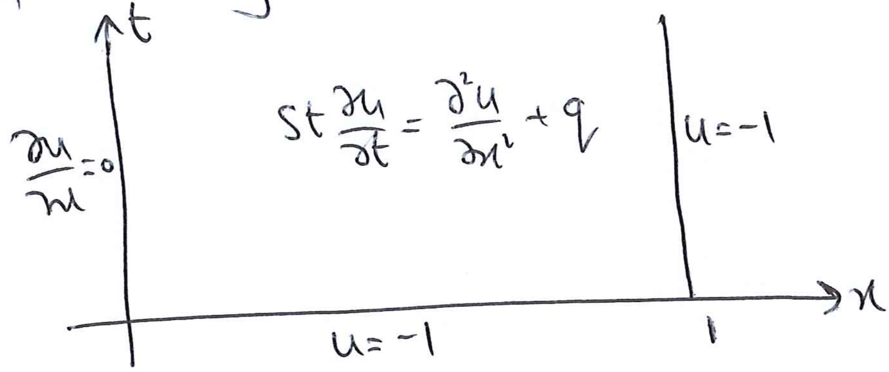
(A = x-sectional area)

gives $J = \frac{\sigma}{a} V$

& hence $q = \frac{\sigma V^2}{k(T_m - T_0)}$

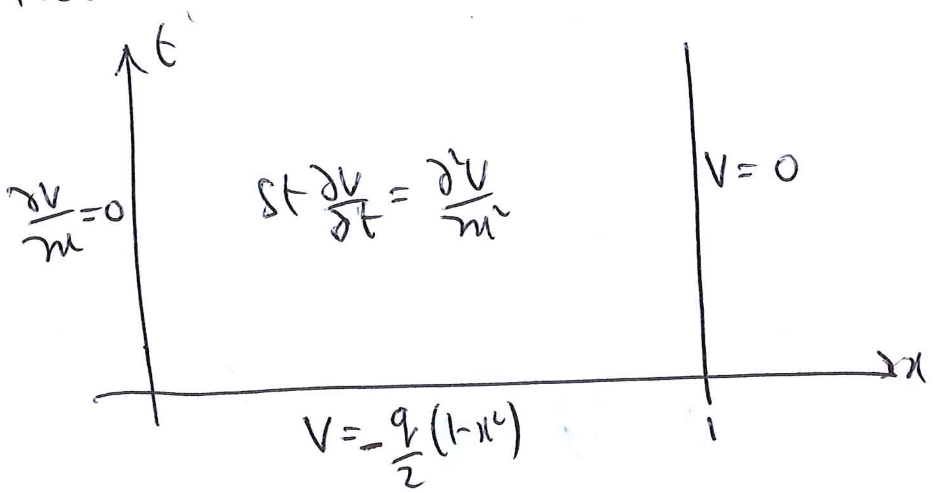
q must be at least $O(1)$ to have any hope of melting the plate, i.e. we need $\boxed{V \gtrsim \sqrt{\frac{k(T_m - T_0)}{\sigma}}}$

Before melting occurs, we have



First find steady solution: $u_s(x) = -1 + \frac{q}{2}(1-x^2)$

Then $v(x,t) = u(x,t) - u_s(x)$ satisfies



which is easily solved by separation of variables:

$$v(x,t) = -\sum_{n=0}^{\infty} c_n \cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right) e^{-(n+\frac{1}{2})^2\pi^2 t/St}$$

& by Fourier series $c_n = q\int_0^1 (1-x^2)\cos\left(\left(n+\tfrac{1}{2}\right)\pi x\right)dx$

$$\therefore \boxed{c_n = \frac{2q(-1)^n}{\left(n+\frac{1}{2}\right)^3\pi^3}}$$

Hottest point will be at $x=0$:

$$u(0,t) = -1 + \frac{q}{2} - 2q \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(n+\frac{1}{2})^2 \pi^2 t/St}}{(n+\frac{1}{2})^3 \pi^3}$$

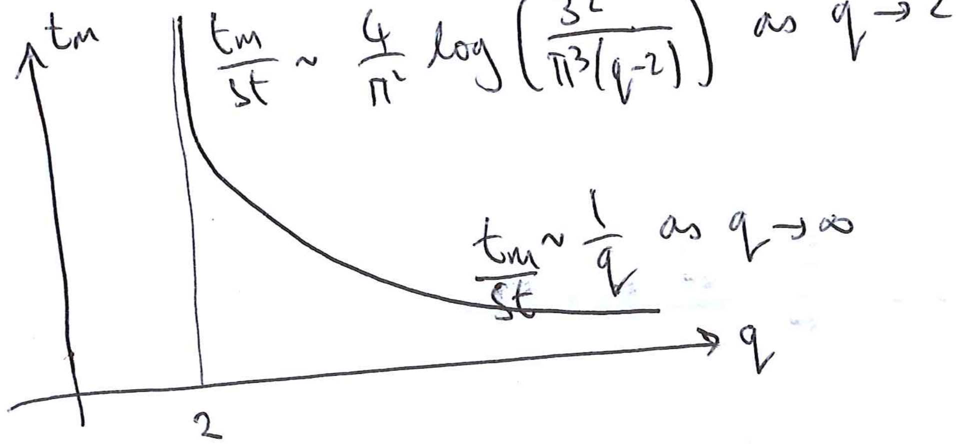
$\to -1 + \frac{q}{2}$ as $t \to \infty$. The plate will have melted (i.e. $u>0$) provided

$\boxed{q>2}$. In this case, value t_m such that

$u(0,t_m)=0$ is given by

$$\boxed{q = \left[\frac{1}{2} - 2 \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(n+\frac{1}{2})^2 \pi^2 t_m/St}}{(n+\frac{1}{2})^3 \pi^3}\right]^{-1}}$$

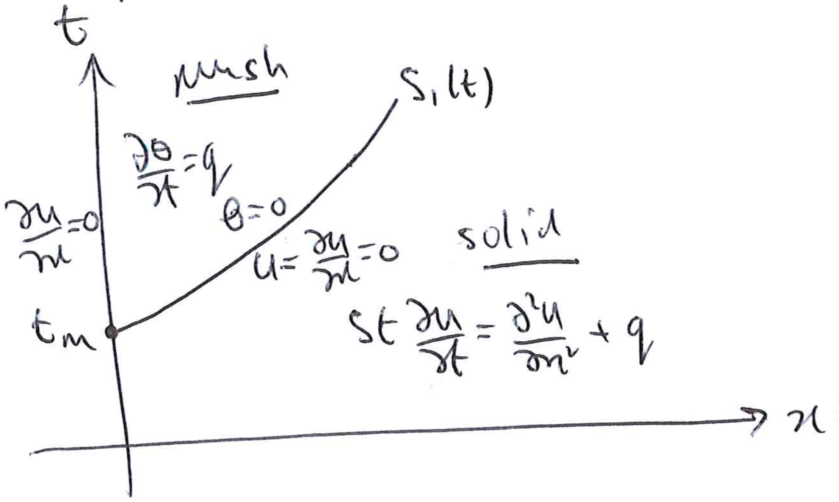
The behaviour looks like



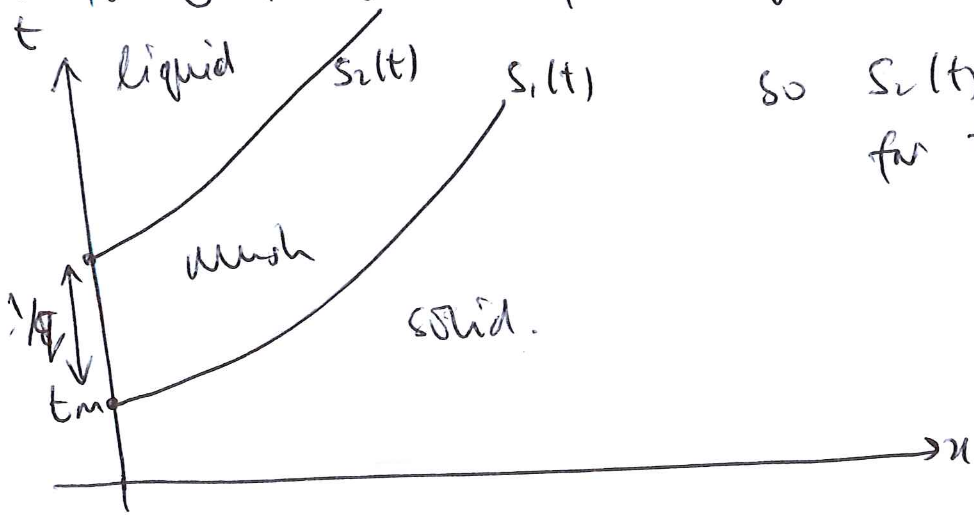
i.e. when $q \gg 1$, plate melts v. quickly;

when $q-2 \ll 1$, —— " —— slowly.

For $q>2$ & $t>t_m$:



It takes a time $t=1/q$ for $\theta(0,t)$ to increase from $\theta=0$ to $\theta=1$ so that pure liquid region starts, i.e.



so $S_2(t)$ only starts for $t>t_m+1/q$.

Alternatively note that $\theta(0,t)=q(t-t_m)$ for $t>t_m$

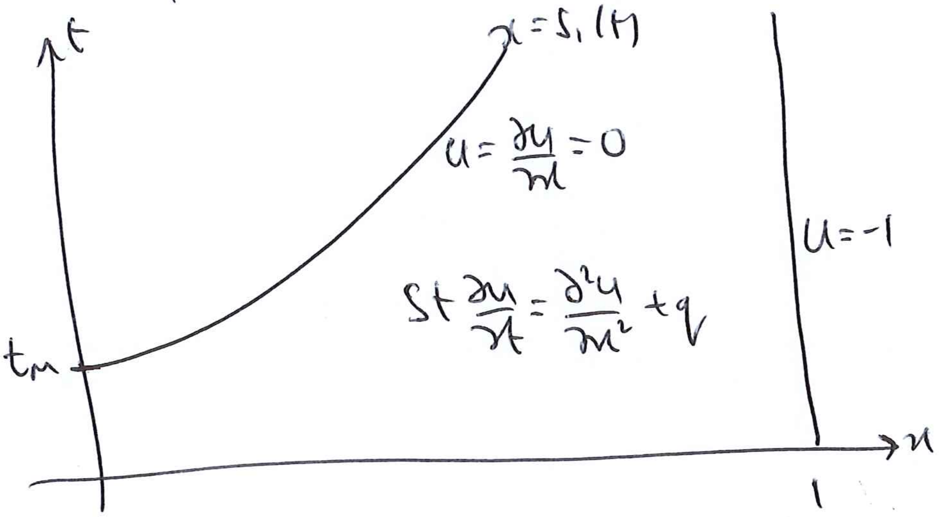
Now, Stefan condition on $x=S_2(t)$ is

$$-\frac{\partial u^-}{\partial x}=(1-\theta^+)\dot{S}_2$$

But $\frac{\partial u^-}{\partial x}=0$ at $x=0$, so this implies that

$\dot{S}_2=0$ until $\theta^+(0,t)=1$, i.e. until $t=t_m+1/q$

Consider problem in solid $x > s_1(t)$:



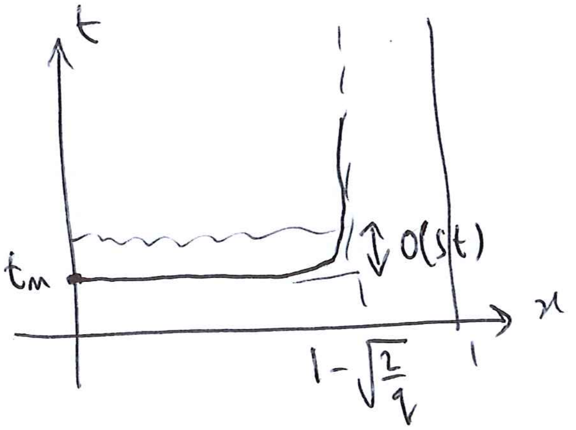
This gives us enough BCs to determine $u(x,t)$ and $x = s_1(t)$ completely without solving for anything in $x < s_1(t)$.

As $t \to \infty$, approach steady state with

$$u = -\frac{q}{2}(x - s_1)^2 \quad \text{and} \quad -1 = -\frac{q}{2}(1 - s_1)^2$$

$$\therefore \; s_1 = 1 - \sqrt{\frac{2}{q}}$$

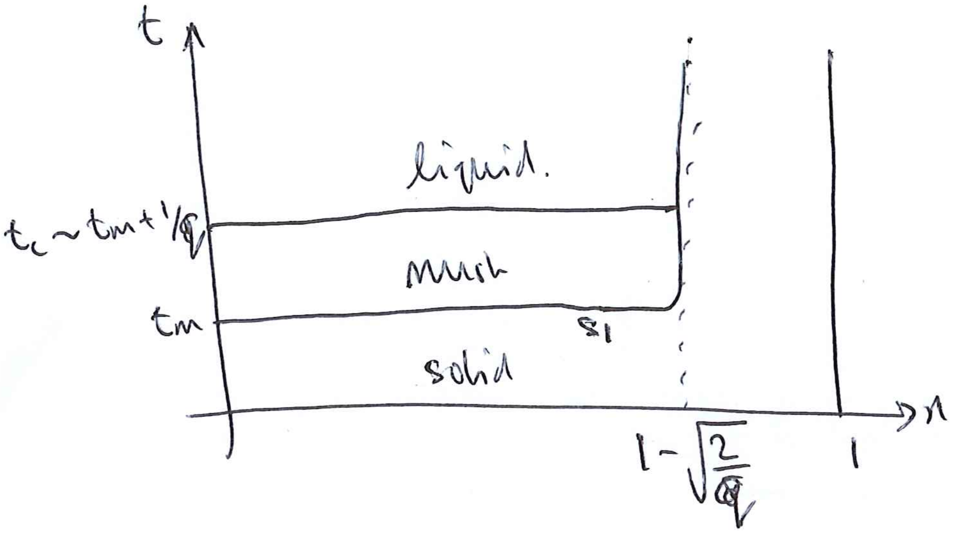
So over a time-scale of order $t = O(St)$, get $s_1 \to 1 - \sqrt{\frac{2}{q}}$:



So problem in mush is (to leading order)

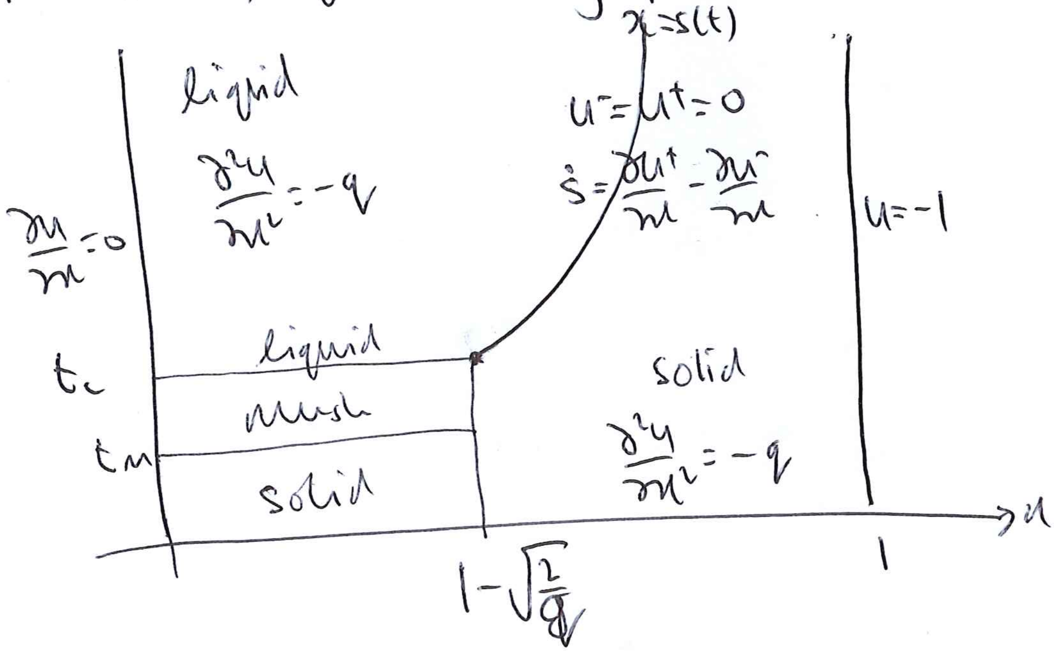
$$\frac{\partial \theta}{\partial t} = q \quad \text{with} \quad \theta = 0 \; \text{at} \; t = t_m \quad (+O(St))$$

$\therefore \; \boxed{\theta = q(t - t_m)}$ and the entire mushy region has melted when $\boxed{t \sim t_m + 1/q} + O(St)$



For $t > t_c$, quasi-steady problem with $St \to 0$ is

(mush has gone)



so
$$\begin{cases} u^- = \frac{q}{2}\left(s^2 - x^2\right) \\ u^+ = (s-x)\left[\frac{1}{1-s} - \frac{q}{2}(1-x)\right] \end{cases}$$

& then
$$\dot{s} = \left[\frac{\partial u^+}{\partial x} - \frac{\partial u^-}{\partial x}\right]_{x=s} = -\frac{1}{1-s} + \frac{q}{2}(1-s) + qs$$

$$\therefore \boxed{\dot{s} = \frac{q}{2}(1+s) - \frac{1}{1-s}}$$

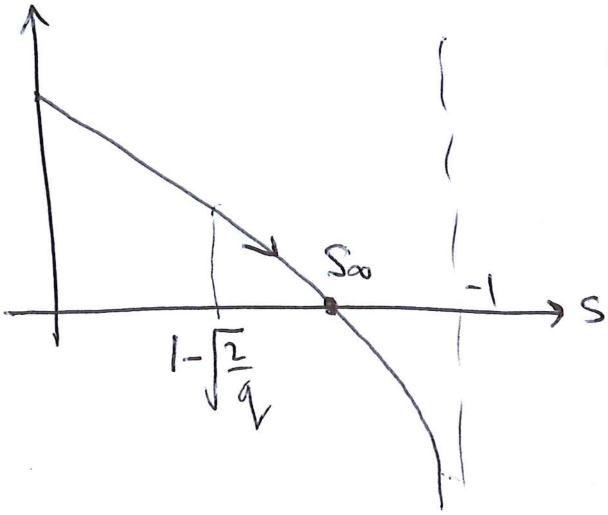
$$\& \quad \boxed{s(t_c) = 1 - \sqrt{\frac{2}{q}}}$$

Also note that $\left.\dfrac{\partial u^+}{\partial x}\right|_{x=s} = \left[\dfrac{q}{2} - \dfrac{1}{(1-s)^2}\right](1-s)$

$$\leqslant 0 \text{ for } s \geqslant 1-\sqrt{\frac{2}{q}}$$

So solid metal is now ~~not~~ superheated ahead of $x = s(t)$.

Now



root of $\dot{s}=0$ is

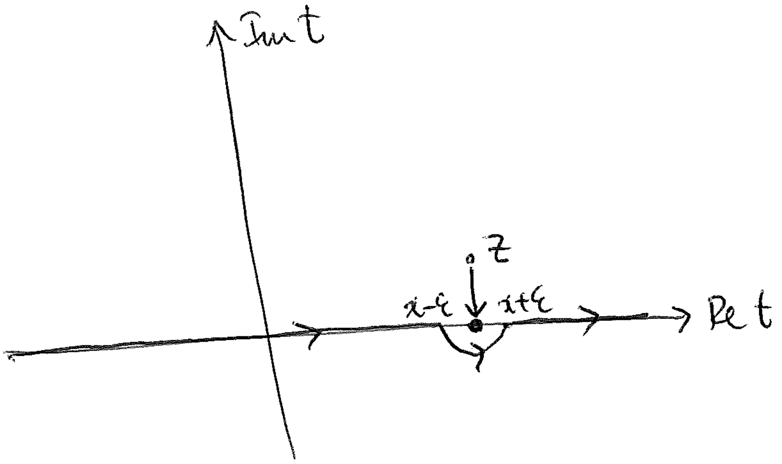
$$s_\infty = \sqrt{1-\frac{2}{q}} > 1-\sqrt{\frac{2}{q}}$$

as $t\to\infty$, $s(t)\to s_\infty = \sqrt{1-2/q}$.

6: Just verify that this form is consistent with the governing PDE in each region and with the BCs between them.

7_{ii}

$$\boxed{I(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-z}}$$



By contour deformation

$$\lim_{y\to 0} I(x+iy) = I_+(x) = \frac{1}{\pi i}\int_{-\infty}^{x-\varepsilon} \frac{f(t)}{t-x}\,dt + \frac{1}{\pi i}\int_{-\pi}^{0} \frac{f(x+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta}\,d\theta$$

$$+ \frac{1}{\pi i}\int_{x+\varepsilon}^{\infty} \frac{f(t)\,dt}{t-x}$$

Now let $\varepsilon \to 0$

$$\boxed{I_+(x) = \frac{1}{\pi i} ⨍_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt + f(x)}$$

Alternatively, let $z = x+iy$ where $0<y\ll 1$:

$$I(x+iy) = \frac{1}{\pi i}\left[\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right] \frac{f(t)\,dt}{t-x-iy} + \frac{1}{\pi i}\int_{x-\varepsilon}^{x+\varepsilon} \frac{f(t)\,dt}{t-x-iy}$$

where $0<y\ll\varepsilon\ll 1$

In first integral, expand for small y; in 2nd integral make substitution $t = x+ys$

$$I(x+iy) \sim \frac{1}{\pi i}\left[\int_{\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right] \frac{f(t)}{(t-x)}\left[1 + \frac{iy}{t-x} + \dots\right] dt$$

$$+ \underbrace{\frac{1}{\pi i}\int_{-\varepsilon/y}^{\varepsilon/y} \frac{f(x+ys)\,\cancel{y}\,ds}{\cancel{y}(s-i)}}_{\sim \frac{1}{\pi i}\int_{-\varepsilon/y}^{\varepsilon/y} \frac{f(x)+sy f'(x)+\dots}{s-i}\,ds}$$

For this 2nd integral, either use $\log(s-i)$ & be careful with branches of log! or..

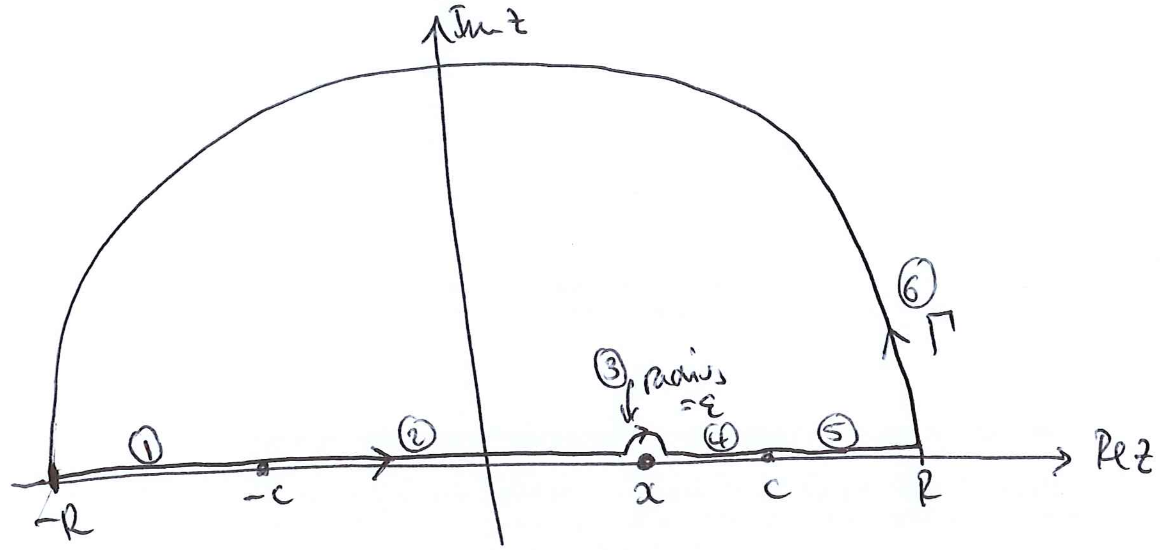
$$\int_{-\varepsilon/y}^{\varepsilon/y} \frac{ds}{s-i} = \int_{-\varepsilon/y}^{\varepsilon/y} \frac{s+i}{s^2+1}\,ds = 2i\int_0^{\varepsilon/y} \frac{ds}{s^2+1}$$

$$= 2i\tan^{-1}(\varepsilon/y).$$

$$\therefore I(x+iy) \sim \frac{1}{\pi i}\left[\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right] \frac{f(t)}{t-x}\,dt + \frac{2f(x)}{\pi}\tan^{-1}(\varepsilon/y) + O(y)]$$

Now let $\varepsilon \to 0$ & $\varepsilon/y \to \infty$ to get required result

8_{11}



$$\oint_\Gamma \frac{dz}{(x-z)(1+z^2)\sqrt{c^2-z^2}} = \underbrace{\int_{-R}^{-c} \frac{dt}{(x-t)(1+t^2)\sqrt{c^2-t^2}}}_{\text{① imaginary}} + \underbrace{\int_{-c}^{x-\varepsilon} \frac{dt}{(x-t)(1+t^2)\sqrt{c^2-t^2}}}_{\text{② real}}$$

$$+ \underbrace{\int_{-\pi}^{0} \frac{i\varepsilon e^{i\theta}\,d\theta}{-\varepsilon e^{i\theta}\left[1+(x+\varepsilon e^{i\theta})^2\right]\sqrt{c^2-(x+\varepsilon e^{i\theta})^2}}}_{③} + \underbrace{\int_{x+\varepsilon}^{c} \frac{dt}{(x-t)(1+t^2)\sqrt{c^2-t^2}}}_{\text{④ real}}$$

$$+ \underbrace{\int_{c}^{R} \frac{dt}{(x-t)(1+t^2)\sqrt{c^2-t^2}}}_{\text{⑤ imaginary}} + \underbrace{\int_{0}^{\pi} \frac{iRe^{i\theta}\,d\theta}{(x-Re^{i\theta})(1+R^2e^{2i\theta})\sqrt{c^2-R^2e^{2i\theta}}}}_{⑥}$$

As $\varepsilon \to 0$, ③ $\to \dfrac{i\pi}{(1+x^2)\sqrt{c^2-x^2}}$ (i.e. $\frac{1}{2}\times$ contribution from residue of pole at $z=x$) which is pure imaginary.

As $R \to \infty$, ⑥ $\to 0$.

So as $\varepsilon \to 0$ and $R \to \infty$,

$$\mathrm{Re} \oint_\Gamma \frac{dz}{(x-z)(1+z^2)\sqrt{c^2-z^2}} \to \lim_{\varepsilon \to 0} \left[\int_{-c}^{x-\varepsilon} + \int_{x+\varepsilon}^{c} \right] \frac{dt}{(x-t)(1+t^2)\sqrt{c^2-t^2}}$$

$$= ⨍_{-c}^{c} \text{——}\,''\,\text{——}\, dt.$$

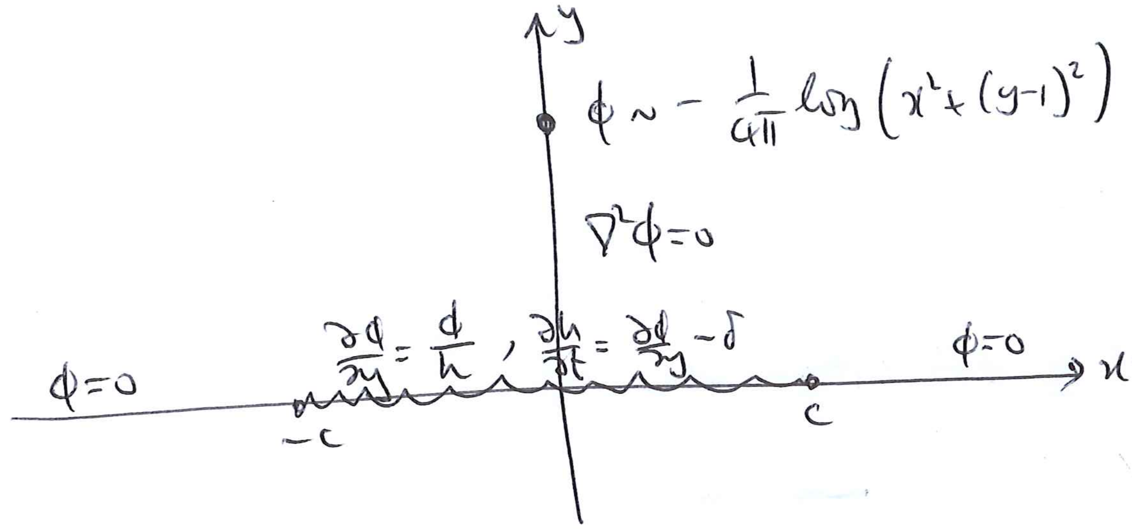
Now compute LHS using Cauchy's residue theorem:

Only pole inside Γ is at $z = i$

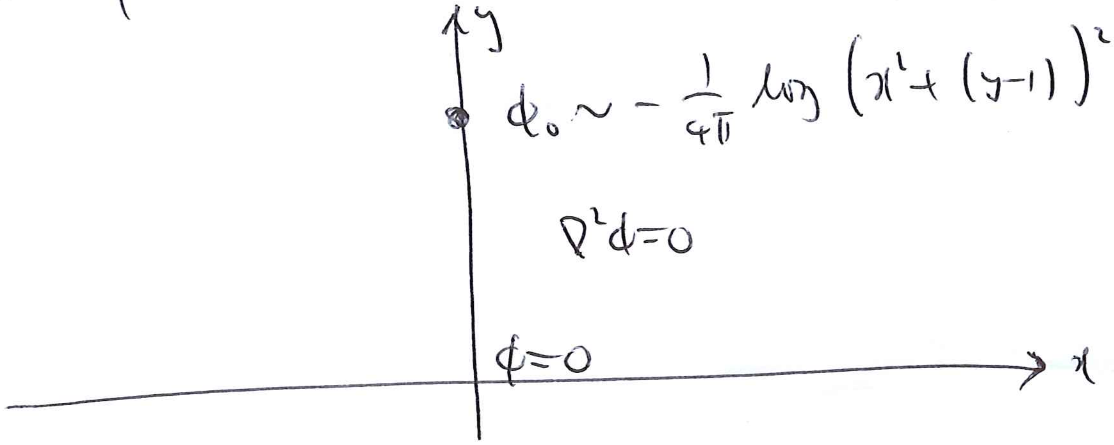
$$\oint_\Gamma \frac{dz}{(x-z)(1+z^2)\sqrt{c^2-z^2}} = \frac{2\pi i}{(x-i)(2i)\sqrt{c^2+1}} = \frac{\pi(x+i)}{(x^2+1)\sqrt{c^2+1}}$$

$$\therefore \operatorname{Re} \oint_\Gamma \frac{dz}{(x-z)(1+z^2)\sqrt{c^2-z^2}} = \frac{\pi x}{(x^2+1)\sqrt{c^2+1}} \quad \boxtimes$$

9_{ii}



At $t=0$ we have $h=0$ everywhere, and hence the initial potential satisfies



& solution is $$\boxed{\phi_0 = \frac{1}{4\pi}\log\left[\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right]}$$

(by images) initial growth rate of layer is thus given

by $$\left.\frac{\partial h}{\partial t}\right|_{t=0} = \left.\frac{\partial\phi_0}{\partial y}-\delta\right|_{y=0} \quad \text{for } |x|<c_0$$

$$= \frac{1}{\pi(1+x^2)}-\delta$$

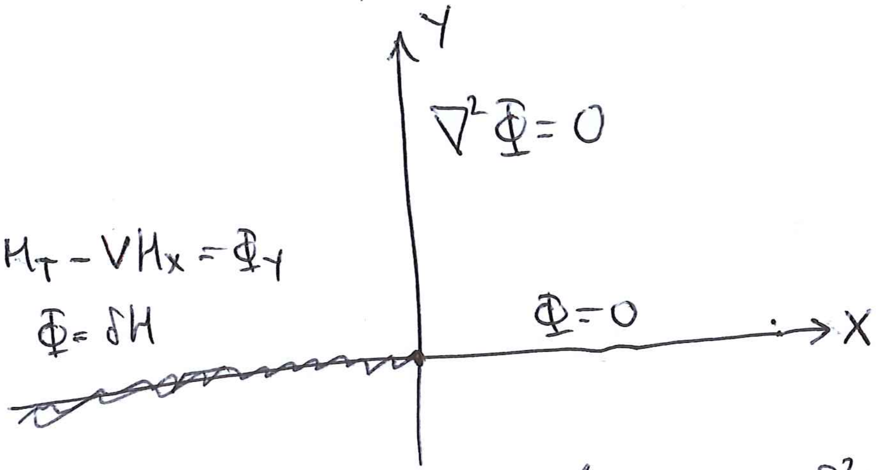
$\therefore$ $h \sim h_1 t$ where $\boxed{h_1(x) = \frac{1}{\pi(1+x^2)}-\delta}$

& this is valid where $h_1>0$, i.e. $\boxed{|x|<c_0=\sqrt{\frac{1}{\delta\pi}-1}}$ $\left(\delta<1/\pi\right)$

The inner problem near $x=c$ looks like this:

Let $t=\varepsilon T$, $x-c(t)=\varepsilon X$, $y=\varepsilon Y$, $\phi-\delta y=\varepsilon^2\Phi$, $h=\varepsilon^2 H$

to get inner problem (as $\varepsilon\to 0$) with $c(\varepsilon T)\sim c_0+\varepsilon VT+\dots$



& matching with outer:

$$\begin{cases} \Phi\sim-2\delta^2 c_0\pi XY & \text{as } |X|\to\infty \\ H(X)\sim-2\delta^2 c_0\pi T(X+VT) & \text{as } X\to-\infty \end{cases}$$

Then, the problem for $H(X,T)$ can be reduced to a singular integro-differential equation, something like

$$\boxed{\frac{\partial H}{\partial T}-V\frac{\partial H}{\partial X}=-2\delta^2 c_0\pi X+\frac{1}{\pi}\int_{-\infty}^{0}\frac{\delta H'(s)\,ds}{X-s}}$$

for $X<0$.