Continuum Methods for Industry

Exercises 3

1.

Coefficient of r^2 in (3.15) is

$$-\frac{1}{2}\left[\int_{-1}^{z-\delta}+\int_{z+\delta}^{1}\right]\frac{q(s)\,ds}{|z-s|^3}+\frac{q(z)}{2\delta^2}-\frac{q''(z)}{2}\log\delta$$

$$+\frac{q''(z)}{4}-\frac{q''(z)}{2}\log\left(\frac{2}{r}\right) = F, \quad \text{say.}$$

Recall $J(z) = ⨍_{-1}^{1}\frac{q(s)\,ds}{|z-s|} = \frac{d}{dz}\int_{-1}^{1}q(s)\,\mathrm{sgn}(z-s)\log|z-s|\,ds$

$$=\lim_{\delta\to 0}\left\{\frac{d}{dz}\left[\int_{-1}^{z-\delta}q(s)\log(z-s)\,ds-\int_{z+\delta}^{1}q(s)\log(s-z)\,ds\right]\right\}$$

$$=\lim_{\delta\to 0}\left\{\int_{-1}^{z-\delta}\frac{q(s)}{z-s}\,ds+\int_{z+\delta}^{1}\frac{q(s)}{s-z}\,ds+\left[q(z-\delta)+q(z+\delta)\right]\times\log\delta\right\}$$

$$\therefore \frac{dJ}{dz}=\lim_{\delta\to 0}\left\{-\int_{-1}^{z-\delta}\frac{q(s)\,ds}{(z-s)^2}+\int_{z+\delta}^{1}\frac{q(s)\,ds}{(s-z)^2}+\frac{q(z-\delta)-q(z+\delta)}{\delta}+\left[q'(z-\delta)+q'(z+\delta)\right]\log\delta\right\}$$

$$\frac{d^2J}{dz^2}=\lim_{\delta\to 0}\left\{2\int_{-1}^{z-\delta}\frac{q(s)\,ds}{(z-s)^3}+2\int_{z+\delta}^{1}\frac{q(s)\,ds}{(s-z)^3}-\frac{q(z-\delta)+q(z+\delta)}{\delta^2}+\frac{q'(z-\delta)-q'(z+\delta)}{\delta}+\left[q''(z-\delta)+q''(z+\delta)\right]\log\delta\right\}$$

(NB assuming lim and d/dz commute...)

Let $\delta \to 0$ and keep only terms that don't $\to 0$ ... (2)

$$\frac{d^2 J}{dz^2} = \lim_{\delta \to 0} \left\{ 2\left[\int_{-1}^{z-\delta} + \int_{z+\delta}^{1}\right] \frac{q(s)\,ds}{|z-s|^3} - \frac{2q(z)}{\delta^2} - q''(z) - 2q''(z) + 2q''(z)\log\delta \right\}$$

Hence

$$\lim_{\delta\to 0} F = -\frac{1}{4}\frac{d^2 J}{dz^2} - \frac{1}{2}q''(z) - \frac{1}{2}q''(z)\log\left(\frac{2}{r}\right)$$

$$= -\frac{1}{4}\frac{d^2J}{dz^2} + \frac{1}{2}q''(z)\log\left(\frac{r}{2e}\right)$$

$\therefore$ (plugging into eq$^{\underline{n}}$ for ϕ):

$$\phi(r,t) \sim \frac{q(z)}{2\pi}\log\left(\frac{r}{2}\right) - \frac{J(t)}{4\pi} + r^2 \overbrace{\left\{-\frac{q''(t)}{8\pi}\log\left(\frac{r}{2e}\right) + \frac{J''(t)}{16\pi}\right\}}^{-F/4\pi}$$

Now try $\phi \sim \frac{q}{2\pi}\log\left(\frac{r}{2}\right) - \frac{J}{4\pi} + C_1 r^2 \log r + C_2 r^2$

Into Laplace's equation:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} \; ; \quad \left\{ \begin{array}{l} \nabla^2 \log r = 0 \\ \nabla^2 (r^2 \log r) = 4(1+\log r) \\ \nabla^2 (r^2) = 4 \end{array} \right.$$

$$\therefore \nabla^2\phi \sim \frac{q''}{2\pi}\log\left(\frac{r}{2}\right) - \frac{J''}{4\pi} + 4C_1(1+\log r) + 4C_2 + O(r^2\log r)$$

$$\Rightarrow \underline{C_1 = -\frac{q''}{8\pi}}, \quad C_2 = -C_1 + \frac{J''}{16\pi} + \frac{q''}{8\pi}\log 2 = \frac{q''}{8\pi} + \frac{J''}{16\pi} + \frac{q''}{8\pi}\log 2$$

$$\therefore\ \phi \sim \frac{q}{2\pi}\log\left(\frac{r}{2}\right) - \frac{J}{4\pi} + r^2\left\{\frac{J''}{16\pi} + \frac{q''}{8\pi}\left[1+\log 2-\log r\right]\right\}$$ 3

which reproduces previous result. ∎

2ii Let $r = \varepsilon R$, $\phi(r,z) = \Phi(R,z)$; problem (3-5) becomes ④

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \varepsilon^2 \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\frac{\partial \Phi}{\partial R} = \left(1 + \varepsilon^2 \frac{\partial \Phi}{\partial z}\right) S'(z) \quad \text{at } R = S(z).$$

So to leading order $\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) = 0$

$$\Rightarrow \quad \Phi = A \log R + B \qquad (A \,\&\, B \text{ fns of } z)$$

B.C. $\quad \frac{\partial \Phi}{\partial R} = \frac{A}{R} = S'(z) \quad \text{at } R = S(z).$

$$\therefore \quad A = S(z) S'(z)$$

$$\therefore \quad \Phi = S(z) S'(z) \log R + B$$

matching: $\quad \Phi(R, z) = S(z) S'(z) \log(R) + O(1)$ as $R \to \infty$

$$\Phi(r/\varepsilon, z) \sim S(z) S'(z) \log\left(\frac{r}{\varepsilon}\right) + O(1) \text{ as } r/\varepsilon \to \infty.$$

$$\therefore \quad \phi(r,z) \sim S(z) S'(z) \log r + O(1) \quad \text{as } r \to 0$$

3.ii Dimensionless velocity $\underset{\sim}{u} = \underline{e}_z + \varepsilon^2 \nabla \phi$

$$\therefore \; |\underline{u}|^2 = \left(1 + \varepsilon^2 \frac{\partial \phi}{\partial z}\right)^2 + \varepsilon^4 \left(\frac{\partial \phi}{\partial r}\right)^2$$

on $r = \varepsilon S(z)$, $\displaystyle \frac{\partial \phi}{\partial r} = \frac{S'(z)}{\varepsilon} + O(\varepsilon)$

$$\therefore \; |\underline{u}|^2 = 1 + 2\varepsilon^2 \frac{\partial \phi}{\partial z} + \varepsilon^2 S'(z)^2 + O(\varepsilon^4)$$

on $r = \varepsilon z$

$$\therefore \; \boxed{|\underline{u}| \sim 1 + \varepsilon^2 V_s}$$

where $\boxed{V_s = \left. \frac{\partial \phi}{\partial z} + \frac{1}{2} S'(z)^2 \right|_{r = \varepsilon S(z)}}$

Now let $S(z) = \sqrt{1 - z^2}$; then

$$q(z) = 2\pi S(z) S'(z) = 2\pi \sqrt{1 - z^2} \cdot \frac{-z}{\sqrt{1 - z^2}}$$

$$\therefore \; \boxed{q(z) = -2\pi z}$$

Now we need $\displaystyle -\frac{1}{4\pi} \int_{-1}^{1} \frac{q(s)\,ds}{|z - s|} = \frac{1}{2} \int_{-1}^{1} \frac{s\,ds}{|z - s|}$

$$= \frac{1}{2} \int_{-1}^{1} \frac{s - z}{|z - s|}\,ds + \frac{1}{2} z \log(1 - z^2)$$

$$= \frac{1}{2} \int_{-1}^{1} \operatorname{sgn}(s - z)\,ds + \frac{1}{2} z \log(1 - z^2)$$

$$= -z + \frac{1}{2} z \log(1 - z^2)$$

$$\therefore \boxed{\phi(r,z) \sim -z\log\left(\frac{r}{2}\right) - z + \frac{1}{2}z\log(1-z^2) \quad \text{as } r\to 0}$$

$$\therefore \quad \frac{\partial\phi}{\partial z} \sim -\log\left(\frac{r}{2}\right) - 1 + \frac{1}{2}\log(1-z^2) - \frac{z^2}{1-z^2}$$

on $r = \varepsilon\sqrt{1-z^2}$, $\quad \dfrac{\partial\phi}{\partial z} = -\log\left(\dfrac{\varepsilon}{2}\sqrt{1-z^2}\right) + \dfrac{1}{2}\log(1-z^2) - \dfrac{1}{1-z^2}$

$$\left.\frac{\partial\phi}{\partial z}\right|_{r=\varepsilon S} = \log\left(\frac{2}{\varepsilon}\right) - \frac{1}{1-z^2}$$

$$\therefore \quad V_s = \log\left(\frac{2}{\varepsilon}\right) - \frac{1}{1-z^2} + \frac{1}{2}\left(\frac{-z}{\sqrt{1-z^2}}\right)^2$$

$$= \log\left(\frac{2}{\varepsilon}\right) - \frac{1}{1-z^2} + \frac{z^2}{2(1-z^2)}$$

$$\therefore \quad \boxed{V_s = \log\left(\frac{2}{\varepsilon}\right) - \frac{(2-z^2)}{2(1-z^2)}}$$

$\square$

4. To solve:

$$\boxed{\begin{array}{ll} \nabla^2\phi = 0 & \\ \phi = 1 & r = \varepsilon S(t) \\ \phi \to 0 & |\underline{r}| \to \infty \end{array}}$$

Try $\phi(r,t) = \int_{-1}^{1} \frac{-q(s)\,ds}{4\pi\sqrt{r^2+(z-s)^2}}$

By (3.21): $\phi(r,z) \sim \frac{q(z)}{2\pi}\log\left(\frac{r}{2}\right) - \frac{1}{4\pi}⨍_{-1}^{1}\frac{q(s)\,ds}{|z-s|} + O(r^2\log r)$

as $r \to 0$.

$\therefore$ we get $\boxed{\frac{q(z)}{2\pi}\log\left(\frac{\varepsilon S}{2}\right) - \frac{1}{4\pi}⨍_{-1}^{1}\frac{q(s)\,ds}{|z-s|} = 1}$

up to $O(\varepsilon^2\log\varepsilon)$

then $r\frac{\partial\phi}{\partial r}\Big|_{r=\varepsilon S} = \frac{q(z)}{2\pi} + O(\varepsilon^2\log\varepsilon)$

$$\therefore \boxed{C = -2\pi\int_{-1}^{1} r\frac{\partial\phi}{\partial r}\Big|_{r=\varepsilon S} dz = -\int_{-1}^{1} q(z)\,dz}$$

Now write $\frac{q(t)}{2\pi}\log\left(\frac{2}{\varepsilon}\right) = \frac{q(z)}{2\pi}\log(S) - 1 - \frac{1}{4\pi}⨍_{-1}^{1}\frac{q(s)\,ds}{|z-s|}$

$$q(z) = \frac{-2\pi}{\log(2/\varepsilon)}\left[1 - \frac{q(t)}{2\pi}\log(S) - \frac{1}{4\pi}⨍_{-1}^{1}\frac{q(s)\,ds}{|z-s|}\right] \quad (*)$$

So we get $q = -\dfrac{2\pi}{\log(2/\varepsilon)} + O\left(\dfrac{1}{\log^2(2/\varepsilon)}\right)$

where in principle the higher-order terms could be determined by iterating on (*).

Now suppose q is constant. Then

$$\int_{-1}^{1} \frac{q(s)\,ds}{|t-s|} = \int_{-1}^{1} \frac{q(s)-q(t)}{|t-s|}\,ds + q(t)\log(1-t^2)$$

$$= q\log(1-t^2) \qquad \text{if } q = \text{const.}$$

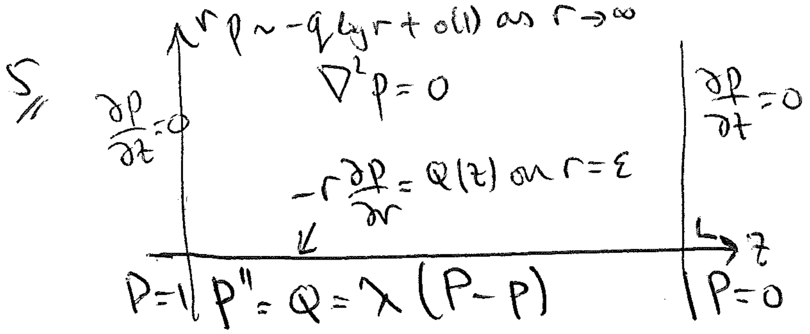
∴ Integral equation becomes...

$$\frac{q}{2\pi}\log\left(\frac{\varepsilon S}{2}\right) - \frac{q}{4\pi}\log\left(1-t^2\right) = 1$$

If $S(t) = \sqrt{1-t^2}$ then this reduces to

$$\frac{q}{2\pi}\log\left(\frac{\varepsilon}{2}\right) = 1 \Rightarrow \boxed{q = \frac{-2\pi}{\log(2/\varepsilon)}}$$

when $S(t) = \sqrt{1-t^2}$ ∎



First consider problem for $p(r,z)$. BCs at $z=0, L$ suggest separable solutions of the form

$$p(r,z) = f(r)\cos\left(\frac{n\pi z}{L}\right)$$

$$\therefore\quad \nabla^2 p = f''(r) + \frac{1}{r}f'(r) - \frac{n^2\pi^2}{L^2}f(r) = 0.$$

Solution is $f(r) = A K_0\left(\frac{n\pi}{L}r\right) + B I_0\left(\frac{n\pi}{L}r\right)$

where K_0 and I_0 are modified Bessel functions. Solution which decays as $r\to\infty$ is K_0.

$\therefore$ general linear combination of separable solutions:

$$\boxed{p(r,z) = \frac{b_0\log r}{2} + \sum_{n=1}^{\infty} b_n K_0\left(\frac{n\pi r}{L}\right)\cos\left(\frac{n\pi z}{L}\right)}$$

where $b_0 = -2q$

[NB additive constant = 0 by far field condition]

[Note, $K_0(\xi) \sim -\gamma + \log\left(\frac{2}{\xi}\right) + O\left(\xi^2\log\xi\right)$ as $\xi\to 0$

i.e. has a logarithmic singularity which is consistent with the BC $-r\frac{\partial p}{\partial r} = Q$ on $r=\varepsilon$]

$$\frac{\partial p}{\partial r} = \frac{b_0}{2r} - \sum_{n=1}^{\infty} b_n\cdot\frac{n\pi}{L}K_1\left(\frac{n\pi r}{L}\right)\cos\left(\frac{n\pi}{L}z\right)$$

$$\therefore\quad Q(z) = -r\frac{\partial p}{\partial r}\bigg|_{r=\varepsilon} = -\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n\cdot\frac{\varepsilon n\pi}{L}K_1\left(\frac{\varepsilon n\pi}{L}\right)\cos\left(\frac{n\pi z}{L}\right)$$

$$\therefore \boxed{Q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi z}{L}\right)} \qquad [c_0 = 2q]$$

where coeffs in p expansion are related to c_n by

$$\boxed{b_0 = -c_0 \,, \quad b_n = \frac{c_n}{\left(\frac{n\pi\epsilon}{L}\right) K_1\left(\frac{n\pi\epsilon}{L}\right)}} \qquad (n \geq 1)$$

$$\therefore \boxed{p(\epsilon, z) = \frac{c_0}{2} \log\left(\frac{1}{\epsilon}\right) + \sum_{n=1}^{\infty} \frac{K_0\left(\frac{n\pi\epsilon}{L}\right)}{\left(\frac{n\pi\epsilon}{L}\right) K_1\left(\frac{n\pi\epsilon}{L}\right)} c_n \cos\left(\frac{n\pi z}{L}\right)}$$

We also have $P''(z) = Q(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi z}{L}\right)$

$$\therefore P'(z) = m + \frac{c_0 z}{2} + \sum_{n=1}^{\infty} \frac{L c_n}{n\pi} \sin\left(\frac{n\pi z}{L}\right)$$

where $\boxed{m = P'(0)}$

$$\therefore \boxed{P(z) = 1 + mz + \frac{c_0 z^2}{4} + \sum_{n=1}^{\infty} \frac{L^2 c_n}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi z}{L}\right)\right]}$$

(applying BC $P(0) = 1$)

Put $z = L$:

$$\boxed{1 + mL + \frac{c_0 L^2}{4} + \sum_{n=1}^{\infty} \frac{L^2 c_n}{n^2\pi^2}\left[1 - (-1)^n\right] = P_{out}}$$

Finally, $\underline{Q = \lambda\left(P - p(\epsilon, z)\right)}$

Write z and z^2 terms in P as Fourier series:

$$z = \frac{L}{2} + 2L \sum_{n=1}^{\infty} \frac{\left[-1 + (-1)^n\right]}{n^2\pi^2} \cos\left(\frac{n\pi z}{L}\right)$$

$$z^2 = \frac{L^2}{3} + 4L^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi z}{L}\right)$$

Finally, BC $q(z) = \lambda\left[P(z) - p(\epsilon, z)\right]$ leads to...

$$\frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi z}{L}\right) = \lambda\left\{1 + \frac{mL}{2} + 2mL \sum_{n=1}^{\infty} \frac{\left[-1+(-1)^n\right]}{n^2\pi^2}\cos\left(\frac{n\pi z}{L}\right)\right.$$

$$+ \frac{c_0 L^2}{12} + c_0 L^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi z}{L}\right) + L^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi z}{L}\right)\right]$$

$$\left. - \frac{c_0}{2}\log\left(\frac{1}{\epsilon}\right) - \sum_{n=1}^{\infty} \frac{K_0\left(\frac{n\pi\epsilon}{L}\right) c_n}{\left(\frac{n\pi\epsilon}{L}\right) K_1\left(\frac{n\pi\epsilon}{L}\right)} \cos\left(\frac{n\pi z}{L}\right)\right\}$$

Now we just (!) need to collect the terms...

$$\frac{c_0}{2} - \lambda\left\{1 + \frac{mL}{2} + \frac{c_0 L^2}{12} + L^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2\pi^2} - \frac{c_0}{2}\log\left(\frac{1}{\epsilon}\right)\right\} = 0$$

$$c_n - \lambda\left\{\frac{2mL}{n^2\pi^2}\left[-1+(-1)^n\right] + \frac{c_0 L^2 (-1)^n}{n^2\pi^2} - \frac{L^2 c_n}{n^2\pi^2} - \frac{K_0\left(\frac{n\pi\epsilon}{L}\right) c_n}{\left(\frac{n\pi\epsilon}{L}\right) K_1\left(\frac{n\pi\epsilon}{L}\right)}\right\} = 0 \quad (n \geq 1)$$

& recall

$$1 + mL + \frac{c_0 L^2}{4} + L^2 \sum_{n=1}^{\infty} \frac{c_n}{n^2\pi^2}\left[1 - (-1)^n\right] = P_{out}$$

In principle this linear system of equations determines

$c_0, \ldots, c_n$ <u>and</u> m.

```
In[1]:= (* First set up the linear system of
         equations for m and the coefficients c[n] *)

In[2]:= tab[la_, L_, e_, Pout_, max_] :=
         Join[{c[0] / 2 - la (1 + m L / 2 + c[0] L^2 / 12 + L^2 Sum[c[n] / n^2 / Pi^2, {n, 1, max}] -
               c[0] / 2 Log[1 / e]), 1 + m L + c[0] L^2 / 4 +
            L^2 Sum[c[n] (1 - (-1)^n) / n^2 / Pi^2, {n, 1, max}] - Pout},
          Table[c[n] - la (2 m L (-1 + (-1)^n) / n^2 / Pi^2 +
               c[0] L^2 (-1)^n / n^2 / Pi^2 - L^2 c[n] / n^2 / Pi^2 -
               BesselK[0, n Pi e / L] c[n] / (n Pi e / L) / BesselK[1, n Pi e / L]), {n, 1, max}]]

In[3]:= (* Solve the system and evaluate the tube pressure P(z) and flux Q(z) *)

In[4]:= sol[la_, L_, e_, Pout_, max_] := (ss = Solve[
            Evaluate[tab[la, L, e, Pout, max] == 0], Join[{m}, Table[c[n], {n, 0, max}]]];
          Q = c[0] / 2 + Sum[c[n] Cos[n Pi z / L], {n, 1, max}] /. ss[[1]];
          P = 1 + m z + c[0] z^2 / 4 +
             L^2 Sum[c[n] / n^2 / Pi^2 (1 - Cos[n Pi z / L]), {n, 1, max}] /. ss[[1]];)

In[5]:= sol[1, 1, 0.1, 0.1, 50]; P50 = P; Q50 = Q;

In[6]:= sol[1, 1, 0.1, 0.1, 100]; P100 = P; Q100 = Q;

In[7]:= (* Now plot P(z) and Q(z) for some particular paremeter values.
          Here we test the effect truncating the system at n=50 and n=100 *)

In[8]:= Plot[{P50, P100}, {z, 0, 1}, AxesLabel → {z, "P"}]
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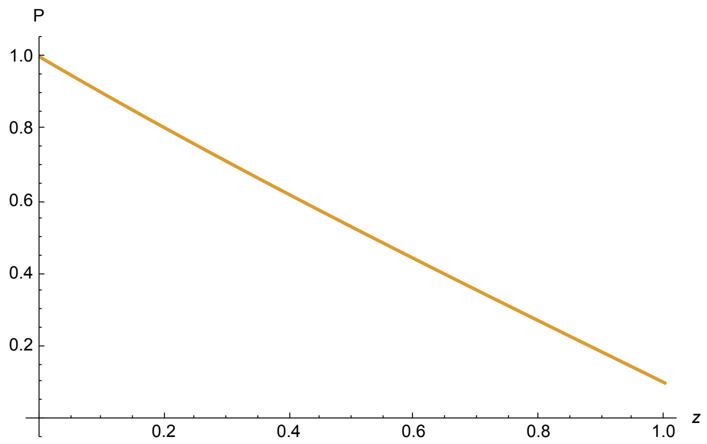
Out[8]=

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In[9]:= Plot[{Q50, Q100}, {z, 0, 1}, AxesLabel → {z, "Q"}]
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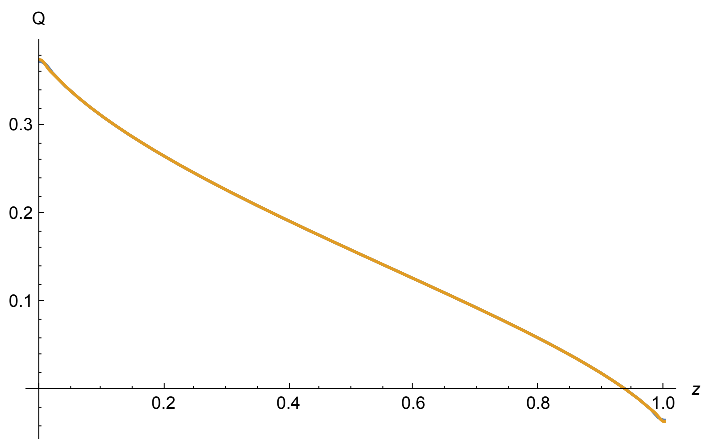
Out[9]=

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In[10]:= (* The differences are negligible so the scheme has converged well *)
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In[11]:= (* Next compute the net flux q *)

In[12]:= qsol[la_, L_, e_, Pout_, max_] :=
          c[0] / 2 /. Solve[Evaluate[tab[la, L, e, Pout, max] == 0],
              Join[{m}, Table[c[n], {n, 0, max}]]][[1]]

In[13]:= (* Test the effect on q of varying ϵ *)

In[14]:= Table[{1 / (Log[2] - k Log[10]), qsol[1, 1, 10^k, 0.1, 100]}, {k, -20, -1, 0.1}];

In[15]:= qp1 = ListPlot[%, Joined → True, AxesLabel → {1 / Log[2 / ϵ], q}];

In[16]:= qp2 = Plot[0.55 z, {z, 0, 0.1}, PlotStyle → Dashed];

In[17]:= Show[qp1, qp2]
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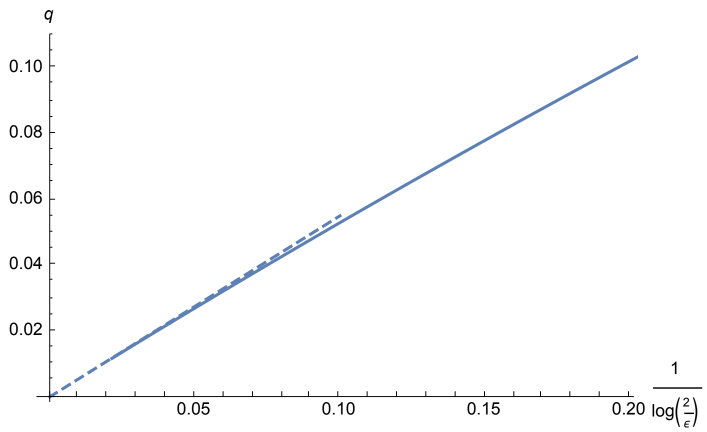
Out[17]=



In[18]:= (* This shows that the solutions agrees with the asymptotic behaviour

$q \sim \frac{(1+P_{out})}{2\log(2/\epsilon)}$ as $\epsilon \to 0$.

*)

We want $\underline{Q = \text{constant}}$

$\therefore \quad P'' = Q$ with $P(0) = 1$, $P(L) = P_{out}$ leads to

$$\therefore \quad \boxed{P = 1 - (1 - P_{out})\frac{z}{L} - \frac{1}{2}Qz(L-z)}$$

Now outer pressure satisfies
$$\begin{cases} \nabla^2 p = 0 & r > \varepsilon \\ -r\frac{\partial p}{\partial r} = Q \text{ (constant)} & r = \varepsilon \\ p \sim -q\log r + o(1) & r \to \infty \\ \frac{\partial p}{\partial z} = 0 & z = 0, L \end{cases}$$

& the solution is $\underline{\text{independent of } z}$: $\boxed{p = -Q\log r}$ ($\& q = Q$).

So $\quad Q = \lambda\left(P - p(\varepsilon, z)\right) = \lambda\Big[1 - \frac{(1 - P_{out})z}{L} - \frac{1}{2}Qz(L-z) - Q\log\left(\frac{1}{\varepsilon}\right)\Big]$

$\therefore$ permeability should be of the form

$$\boxed{\lambda = \frac{Q}{1 - \frac{(1-P_{out})}{L}z - \frac{1}{2}Qz(L-z) - Q\log\left(\frac{1}{\varepsilon}\right)}}$$

This only makes sense if $\underline{\underline{\text{denominator} > 0}}$

$\forall z \in [0, L]$

Assume that $\underline{P_{out} \leq 1}$ and $\underline{Q \geq 0}$

then the denominator $> 0 \quad \forall z \in [0, L] \Leftrightarrow$

$$\boxed{P_{out} - Q \log\left(\frac{1}{\varepsilon}\right) > \begin{cases} 0 & \text{if } QL^2 > 2(1-P_{out}) \\ \dfrac{\left(QL^2 - 2(1-P_{out})\right)^2}{8QL^2} & \text{if } QL^2 < 2(1-P_{out}) \end{cases}}$$

6. Let $z = R\cos\phi$, $r = R\sin\phi$

Then chain rule: $\begin{cases} \dfrac{\partial}{\partial r} = \sin\phi \dfrac{\partial}{\partial R} + \dfrac{\cos\phi}{R}\dfrac{\partial}{\partial \phi} \\ \dfrac{\partial}{\partial z} = \cos\phi \dfrac{\partial}{\partial R} - \dfrac{\sin\phi}{R}\dfrac{\partial}{\partial \phi} \end{cases}$

leads to (eventually...) $L\psi = \dfrac{\partial^2\psi}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + \dfrac{\partial^2\psi}{\partial z^2}$

$$\rightarrow \boxed{L\psi = \frac{\partial^2\psi}{\partial R^2} + \frac{\sin\phi}{R^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\phi}\frac{\partial\psi}{\partial\phi}\right)}$$

(it's easier to get this by using spherical polars from the start!)

Then problem (3-65) is
$$\boxed{\begin{array}{ll} L^2[\psi] = 0 & R > a \\ \psi \sim \dfrac{U}{2}R^2\sin^2\phi & R \to \infty \\ \psi = \dfrac{\partial\psi}{\partial R} = 0 & R = a \end{array}}$$

Try $\psi(R,\phi) = f(R)\sin^2\phi$.

Then BCs are $\boxed{\begin{array}{l} f(a) = f'(a) = 0 \\ \& \ f(R) \sim \dfrac{UR^2}{2} \text{ as } R \to \infty \end{array}}$

$$L\psi = \sin^2\phi f'' + \frac{\sin\phi}{R^2}\frac{\partial}{\partial\phi}\left[2f\cos\phi\right] = \left[f'' - \frac{2f}{R^2}\right]\sin^2\phi$$

$$\therefore L^2\psi = \left(\frac{d^2}{dR^2} - \frac{2}{R^2}\right)^2 f(R)\,.\,\sin^2\phi$$

Try $f(R) = R^k$. Then $\left(\frac{d^2}{dR^2} - \frac{2}{R^2}\right) f(R) = [k(k-1) - 2] R^{k-2}$

$$= (k-2)(k+1) R^{k-2}$$

$$\therefore \left(\frac{d^2}{dR^2} - \frac{2}{R^2}\right)^2 f(R) = (k-2)(k+1)(k-4)(k-1) R^{k-4}$$

So roots are $\underline{k = -1, 1, 2, 4}$.

Solution satisfying far-field conditions is

$$\underline{f(R) = \frac{U}{2} R^2 + AR + \frac{B}{R}}$$ where A & B are constants.

BCs on $R = a$:
$$\left.\begin{aligned} \frac{Ua^2}{2} + Aa + \frac{B}{a} &= 0 \\ Ua + A - \frac{B}{a^2} &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} A = -\frac{3}{4} Ua \\ B = \frac{Ua^3}{4} \end{cases}$$

$$\therefore \underline{f(R) = \frac{U}{2} R^2 - \frac{3}{4} UaR + \frac{Ua^3}{4R}}$$ and given result follows.

7. (3.71) in component form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial p}{\partial x} - \mu \nabla^2 u = 0$$

$$\frac{\partial p}{\partial y} - \mu \nabla^2 v = 0$$

$$\frac{\partial p}{\partial z} - \mu \nabla^2 w = f\delta(\underline{x})$$

Fourier transform

① $ik\hat{u} + il\hat{v} + im\hat{w} = 0$

② $ik\hat{p} + \mu(k^2+l^2+m^2)\hat{u} = 0$

③ $il\hat{p} + \mu(k^2+l^2+m^2)\hat{v} = 0$

④ $im\hat{p} + \mu(k^2+l^2+m^2)\hat{w} = f$

eliminate $\hat{\underline{u}}$ by taking $k②+l③+m④$:

$$i(k^2+l^2+m^2)\hat{p} = mf$$

$$\therefore \boxed{\hat{p} = \frac{-imf}{k^2+l^2+m^2}}$$

then get $\hat{\underline{u}}$ by plugging back into ②, ③, ④ :

$$\boxed{\hat{u} = \frac{-kmf}{\mu(k^2+l^2+m^2)^2}, \quad \hat{v} = \frac{-lmf}{\mu(k^2+l^2+m^2)^2}, \quad \hat{w} = +\frac{(k^2+l^2)f}{\mu(k^2+l^2+m^2)^2}}$$

Now, $\frac{\partial\hat{p}}{\partial k} = \frac{2ikmf}{(k^2+l^2+m^2)^2} \Rightarrow \hat{u} = +\frac{i}{2\mu}\frac{\partial\hat{p}}{\partial k} = \frac{1}{2\mu}\widehat{xp}$

So $\boxed{u = \frac{xp}{2\mu}}$ and similarly, $\boxed{v = \frac{yp}{2\mu}}$

<u>Given</u> $\boxed{p = \frac{fz}{4\pi(x^2+y^2+z^2)^{3/2}}}$, this produces given results for u & v.

Now
$$i\frac{\partial\hat{p}}{\partial m} = \frac{f\,(k^2+l^2-m^2)}{(k^2+l^2+m^2)^2} = \widehat{zp}$$

$$\Rightarrow \mathcal{F}\left[\frac{z^2}{4\pi\,(x^2+y^2+z^2)^{3/2}}\right] = \frac{k^2+l^2-m^2}{(k^2+l^2+m^2)^2}$$

& by symmetry:
$$\mathcal{F}\left[\frac{x^2}{4\pi\,(x^2+y^2+z^2)^{3/2}}\right] = \frac{l^2+m^2-k^2}{(k^2+l^2+m^2)^2}$$

$$\mathcal{F}\left[\frac{y^2}{4\pi\,(x^2+y^2+z^2)^{3/2}}\right] = \frac{m^2+k^2-l^2}{(k^2+l^2+m^2)^2}$$

$$\therefore \mathcal{F}\left[\frac{x^2+y^2+2z^2}{8\pi\,(x^2+y^2+z^2)^{3/2}}\right] = \frac{k^2+l^2}{(k^2+l^2+m^2)^2} = \frac{\mu}{f}\hat{\omega}$$

which produces required result for w.

8. (3.79b):

$$\left[2\log\left(\frac{2}{\varepsilon S(z)}\right)-1\right]f(z)+\int_{-1}^{1}\frac{f(s)\,ds}{|z-s|}=-4\pi$$

If f is constant, $\displaystyle\int_{-1}^{1}\frac{f\,ds}{|z-s|}=f\log(1-z^2)$

so, with $S(z)=\sqrt{1-z^2}$, (3.79b) becomes

$$\left[2\log\left(\frac{2}{\varepsilon}\right)-\cancel{\log(1-z^2)}-1\right]f+f\cancel{\log(1-z^2)}=-4\pi$$

$$\therefore\ \boxed{f=\frac{-4\pi}{2\log(2/\varepsilon)-1}}$$

To find $q(z)$: $\displaystyle q(z)=\frac{1}{4}\frac{d}{dz}\left(fS^2\right)$

$$=\frac{f}{4}\frac{d}{dz}\left[1-z^2\right]$$

$$\boxed{q(z)=-\frac{fz}{2}}$$

For numerical solution, could proceed as follows:

let $f\left((n+\tfrac{1}{2})\delta\right) \approx f_n$ for $n=-N,\ldots,N-1$ and $\delta = 1/N$.

& let $$\int_{-1}^{1} \frac{f(s)\,ds}{\sqrt{\varepsilon^2 S(z)^2 + (z-s)^2}} = \sum_{m=-N}^{N-1} I_{nm}$$ where

$$I_{nm} = \int_{m\delta}^{(m+1)\delta} \frac{f(s)\,ds}{\sqrt{\varepsilon^2 S\left((n+\tfrac{1}{2})\delta\right)^2 + \left((n+\tfrac{1}{2})\delta - s\right)^2}}$$

which can be approximated using (e.g.)

$$I_{nm} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f\left((m+\tfrac{1}{2})\delta + \delta\xi\right)\,d\xi}{\sqrt{\frac{\varepsilon^2}{\delta^2} S\left((n+\tfrac{1}{2})\delta\right)^2 + (\xi+m-n)^2}}$$

$$\simeq f\left((m+\tfrac{1}{2})\delta\right)\left\{\sinh^{-1}\left[\frac{\delta\left(\tfrac{1}{2}+m-n\right)}{\varepsilon S\left((n+\tfrac{1}{2})\delta\right)}\right] - \sinh^{-1}\left[\frac{\delta\left(-\tfrac{1}{2}+m-n\right)}{\varepsilon S\left((n+\tfrac{1}{2})\delta\right)}\right]\right\}$$

[Need to be careful when $m \approx n$!]

So we get a linear system of equations

$$\boxed{f_n - \sum A_{nm} f_m = 4\pi}$$ where...

$$\boxed{A_{nm} = \sinh^{-1}\left[\frac{\delta\left(m-n+\tfrac{1}{2}\right)}{\varepsilon S_n}\right] - \sinh^{-1}\left[\frac{\delta\left(m-n-\tfrac{1}{2}\right)}{\varepsilon S_n}\right]}$$

and $S_n = S\left((n+\tfrac{1}{2})\delta\right)$.

```
In[1]:= (* This is going to solve the
         discretised integral equation for Exercise 3.8 *)

In[2]:= (* First set up and solve the linear system *)

In[3]:= A[n_, m_, d_, e_] := ArcSinh[d (m - n + 1 / 2) / e / S[(n + 1 / 2) d]] -
          ArcSinh[d (m - n - 1 / 2) / e / S[(n + 1 / 2) d]]

In[4]:= prob[e_, max_] := Table[
          f[n] - Sum[A[n, m, 1 / max, e] f[m], {m, -max, max - 1}] == 4 Pi, {n, -max, max - 1}]

In[5]:= sol[e_, max_] := Table[{(n + 1 / 2) / max, f[n]}, {n, -max, max - 1}] /.
          Evaluate[Solve[prob[e, max], Table[f[n], {n, -max, max - 1}]][[1]]]

In[6]:= (* Print solution and compare with leading-order approximation
           f(z) ~ -4Pi/(2log(2/ϵ)-1) *)

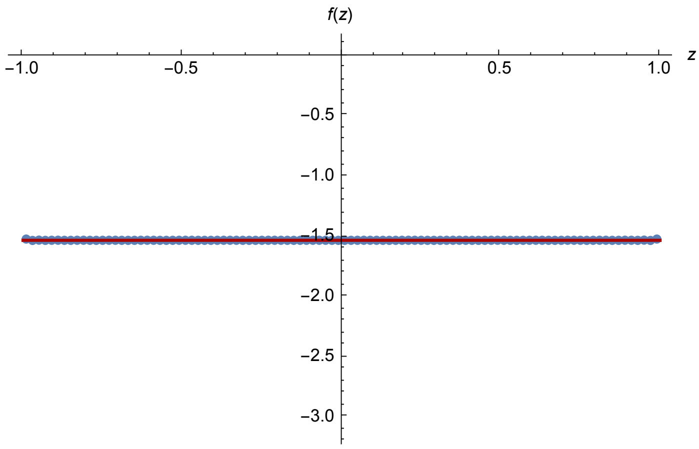
In[7]:= printsol[e_, max_] :=
         Show[Plot[-4 Pi / (2 Log[2 / e] - 1), {z, -1, 1}, PlotStyle → Darker[Red],
            PlotRange → Full], ListPlot[Evaluate[sol[e, max]]],
          Plot[-4 Pi / (2 Log[2 / e] - 1), {z, -1, 1}, PlotStyle → Darker[Red]],
          AxesLabel → {z, f[z]}]

In[8]:= (* First example: ellipsoid *)

In[9]:= S[z_] := Sqrt[1 - z^2]

In[10]:= printsol[0.02, 50]

Out[10]=
```



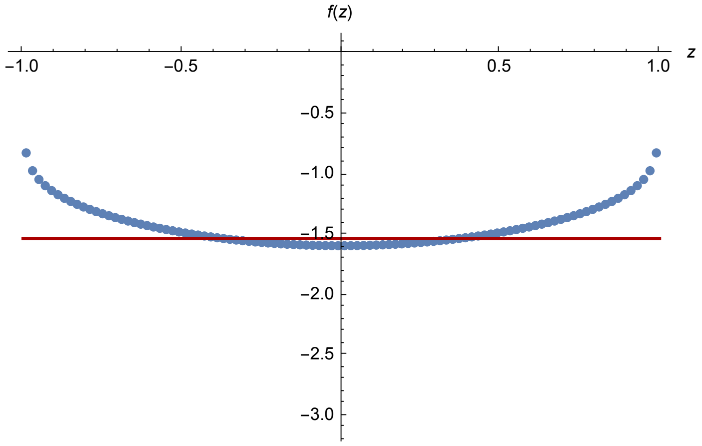

```
In[11]:= (* Pretty good. Approximation should be exact in this case *)

In[12]:= (* Another example *)

In[13]:= S[z_] := Cos[Pi z / 2]
```

In[14]:=
```
printsol[0.02, 50]
```

Out[14]=



In[15]:=
```
(* Approximation not so close but still gives right order of magnitude *)
```

9// Stream function from (3.83):

$$\Psi = \varepsilon^2 \underbrace{\int_{-\infty}^{\infty} \frac{-q(s,t)(z-s)\,ds}{4\pi\sqrt{r^2+(z-s)^2}}}_{I_1} + \underbrace{\int_{-\infty}^{\infty} \frac{f(s,t)\,r^2\,ds}{8\pi\sqrt{r^2+(z-s)^2}}}_{I_2}$$

when $r = O(\varepsilon) \to 0$:

$$I_1 \sim \left[\int_{-\infty}^{z-\delta} + \int_{z+\delta}^{\infty}\right] \frac{q(s,t)\,\mathrm{sgn}(s-z)}{4\pi}\left[1 - \frac{r^2}{2(z-s)^2} + \dots\right] ds$$

$$+ r\int_{-\delta/r}^{\delta/r} \frac{q(z+r\xi,t)\,\xi\,d\xi}{4\pi\sqrt{1+\xi^2}}$$

let $\delta, r \to 0$ with $0 < r \ll \delta \ll 1$:

$$I_1 \sim \frac{1}{4\pi}\int_{-\infty}^{\infty} q(s,t)\,\mathrm{sgn}(s-z)\,ds + O(r^2 \log r)$$

$$\frac{I_2}{r^2} \sim \left[\int_{-\infty}^{z-\delta} + \int_{z+\delta}^{\infty}\right] \frac{f(s,t)}{8\pi|z-s|}\left[1 - \frac{r^2}{2(z-s)^2} + \dots\right] ds$$

$$+ \frac{1}{8\pi}\int_{-\delta/r}^{\delta/r} \frac{f(z+r\xi,t)\,d\xi}{8\pi\sqrt{1+\xi^2}}$$

exactly as in (3.11)... read off result from (3.21) (with $q \mapsto -f/2$):

$$\frac{I_2}{r^2} \sim -\frac{f(z,t)}{4\pi}\log\left(\frac{r}{2}\right) + \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{f(s,t)\,ds}{|z-s|} + O(r^2 \log r)$$

$$\therefore\ \underbrace{\Psi \sim A r^2 + B r^2 \log\left(\tfrac{r}{2}\right) + \varepsilon^2 C}\ + O(\varepsilon^4 \log \varepsilon)$$ when $r = O(\varepsilon)$

where $$A = \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{f(s,t)\,ds}{|z-s|}, \quad B = -\frac{f(z,t)}{4\pi}$$

$$C = \frac{1}{4\pi}\int_{-\infty}^{\infty} q(s,t)\,\mathrm{sgn}(s-z)\,ds$$ as required.

$$\therefore\ \boxed{\begin{aligned} u &= -\frac{1}{r}\frac{\partial \Psi}{\partial z} \sim -\frac{\partial A}{\partial z} r - \frac{\partial B}{\partial z} r \log\left(\tfrac{r}{2}\right) - \frac{\varepsilon^2}{r}\frac{\partial C}{\partial z} \\ w &= \frac{1}{r}\frac{\partial \Psi}{\partial r} \sim 2A + 2B\log\left(\tfrac{r}{2}\right) + B \end{aligned}}$$

We get the pressure from (3.73a):

$$p = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{f(s,t)(z-s)\,ds}{\left[r^2 + (z-s)^2\right]^{3/2}} = -\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \frac{f(s,t)\,ds}{4\pi\sqrt{r^2 + (z-s)^2}}$$

$$\therefore\ \boxed{p = -\frac{2}{r^2}\frac{\partial I_2}{\partial z}}$$ where as above I_2 is the Stokeslet contribution to Ψ.

Recall $\dfrac{I_2}{r^2} \sim A + B\log(r/2)$ as $r \to 0$

$$\therefore\ \boxed{p \sim -2\frac{\partial A}{\partial z} - 2\frac{\partial B}{\partial z}\log(r/2)}$$

At the filament surface $r = \varepsilon S(z,t)$, we have $w = W(z,t)$ and $u = \varepsilon\left[\frac{\partial S}{\partial t} + W\frac{\partial S}{\partial z}\right]$. Plug in leading-order approximations for u & w:

$$\boxed{W = 2A + B + 2B\log\left(\frac{\varepsilon S}{2}\right)}$$

and

$$-\frac{\partial A}{\partial z}S - \frac{\partial B}{\partial t}S\log\left(\frac{\varepsilon S}{2}\right) - \frac{1}{S}\frac{\partial C}{\partial z} = \frac{\partial S}{\partial t} + W\frac{\partial S}{\partial z}$$

Now, $\frac{\partial W}{\partial z} = 2\frac{\partial A}{\partial z} + \frac{\partial B}{\partial z} + 2\frac{\partial B}{\partial z}\log\left(\frac{\varepsilon S}{2}\right) + \frac{2B}{S}\frac{\partial S}{\partial z}$

$$\therefore 2\frac{\partial S}{\partial t} + 2W\frac{\partial S}{\partial t} = -S\frac{\partial W}{\partial t} + S\frac{\partial B}{\partial t} + 2B\frac{\partial S}{\partial z} - \frac{2}{S}\frac{\partial C}{\partial z}$$

$$\therefore \boxed{\frac{\partial}{\partial t}(S^2) + \frac{\partial}{\partial z}(WS^2) = \frac{\partial}{\partial z}\left[S^2B - 2C\right]}$$

Sub in expressions for A & B into expression for W:

$$\boxed{W = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{f(s,t)\,ds}{|z-s|} + \left[2\log\left(\frac{2}{\varepsilon S}\right) - 1\right]\frac{f(z,t)}{4\pi}}$$

Recall $4\pi C = \int_{-\infty}^{\infty} q(s,t)\,\mathrm{sgn}(s-z)\,ds = \int_{-\infty}^{z} -q(s,t)\,ds + \int_{z}^{\infty} q(s,t)\,ds$

$$\therefore 4\pi\frac{\partial C}{\partial z} = -2q(z,t)$$

[Alternatively, $\frac{\partial}{\partial z}\int_{-\infty}^{\infty} q(s,t)\,\mathrm{sgn}(s-z)\,ds$

$$= \int_{-\infty}^{\infty} q(s,t)\left(-2\delta(s-z)\right)ds = -2q(z,t).]$$

So, plugging in B & $\frac{\partial C}{\partial z}$ gives

$$\frac{\partial}{\partial t}(S^2) + \frac{\partial}{\partial z}(WS^2) + \frac{\partial}{\partial z}\left(\frac{fS^2}{4\pi}\right) = \frac{q}{\pi}$$

ie. $$\boxed{q = \frac{1}{4}\frac{\partial}{\partial z}(fS^2) + \frac{\partial}{\partial t}(\pi S^2) + \frac{\partial}{\partial z}(W\pi S^2)}$$

The stress components are given by (in dimensional form)

$$\sigma_{rr} = -p + 2\mu\frac{\partial u}{\partial r}, \quad \sigma_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z}, \quad \sigma_{\theta\theta} = -p + \frac{2\mu u}{r}.$$

Make dimensionless with $\frac{\mu U}{L}$:

$$\begin{cases} \sigma_{rr} = -p + 2\frac{\partial u}{\partial r}, & \sigma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \sigma_{zz} = -p + 2\frac{\partial w}{\partial z}, & \sigma_{\theta\theta} = -p + \frac{2u}{r} \end{cases}$$

& plug in asymptotic expressions for p, u, w as $r \to 0$ ...

$$\sigma_{rr} \sim 2\frac{\partial A}{\partial z} + 2\frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) + 2\left[-\frac{\partial A}{\partial z} - \frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) - \frac{\partial B}{\partial z} + \frac{\varepsilon^2}{r^2}\frac{\partial C}{\partial z}\right]$$

$$\therefore \boxed{\sigma_{rr} \sim -2\frac{\partial B}{\partial z} + \frac{2\varepsilon^2}{r^2}\frac{\partial C}{\partial z}}$$

$$\sigma_{rz} \sim -\frac{\partial^2 A}{\partial z^2}r - \frac{\partial^2 B}{\partial z^2}r\log\left(\frac{r}{2}\right) - \frac{\varepsilon^2}{r}\frac{\partial^2 C}{\partial z^2} + \frac{2B}{r} \leftarrow$$ final term dominates by a factor of $\varepsilon^2 \log \varepsilon$

$$\therefore \boxed{\sigma_{rz} \sim \frac{2B}{r}}$$

$$\sigma_{zz} \sim 2\frac{\partial A}{\partial z} + 2\frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) + 2\left[2\frac{\partial A}{\partial z} + 2\frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) + \frac{\partial B}{\partial z}\right]$$

$$\therefore \boxed{\sigma_{zz} \sim 6\frac{\partial A}{\partial z} + 2\frac{\partial B}{\partial z}\left[1 + 3\log\left(\frac{r}{2}\right)\right]}$$

$$\sigma_{\theta\theta} \sim 2\frac{\partial A}{\partial z} + 2\frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) + 2\left[-\frac{\partial A}{\partial z} - \frac{\partial B}{\partial z}\log\left(\frac{r}{2}\right) - \frac{\varepsilon^2}{r^2}\frac{\partial C}{\partial z}\right]$$

$$\therefore \boxed{\sigma_{\theta\theta} \sim -\frac{2\varepsilon^2}{r^2}\frac{\partial C}{\partial z}}$$

on $r = \varepsilon S(z,t)$, the unit normal is given by

$$\underset{\sim}{n} = \frac{\underset{\sim}{e}_r - \varepsilon \frac{\partial S}{\partial z} \underset{\sim}{e}_z}{\sqrt{1+\varepsilon^2 \left(\frac{\partial S}{\partial z}\right)^2}}$$

so the components of $\underset{\approx}{\sigma} \cdot \underset{\sim}{n}$ are:

$$\underset{\sim}{e}_r : \frac{\sigma_{rr} - \varepsilon \frac{\partial S}{\partial z} \sigma_{rz}}{\sqrt{1+\varepsilon^2 \left(\frac{\partial S}{\partial z}\right)^2}} \quad ; \quad \underset{\sim}{e}_z : \frac{\sigma_{rz} - \varepsilon \frac{\partial S}{\partial z} \sigma_{zz}}{\sqrt{1+\varepsilon^2 \left(\frac{\partial S}{\partial z}\right)^2}}$$

$$\sim -2\frac{\partial B}{\partial z} + \frac{2}{S^2}\frac{\partial C}{\partial z} - \frac{2B}{S}\frac{\partial S}{\partial z} \qquad \sim \frac{2B}{\varepsilon S}$$

$$\therefore \boxed{\underset{\approx}{\sigma} \cdot \underset{\sim}{n} \sim \left[\frac{2}{S^2}\frac{\partial C}{\partial z} - \frac{2}{S}\frac{\partial}{\partial z}(BS)\right] \underset{\sim}{e}_r + \frac{2B}{\varepsilon S} \underset{\sim}{e}_z}$$

Now note that $\underset{\sim}{e}_r = \cos\theta \, \underset{\sim}{e}_x + \sin\theta \, \underset{\sim}{e}_y$ so that

$$\int_0^{2\pi} \underset{\sim}{e}_r \, d\theta = \underset{\sim}{0} .$$

$$\therefore \boxed{\int_0^{2\pi} \underset{\approx}{\sigma} \cdot \underset{\sim}{n} \, \varepsilon S \, d\theta = 4\pi B \underset{\sim}{e}_z = - f(z,t) \underset{\sim}{e}_z}$$

10// Repeat procedure from exercise 2.6:

Dimensionless equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

Constitutive relations

$$\varepsilon^2\sigma_{rr} = -p + 2\frac{\partial u}{\partial r}, \qquad \sigma_{zz} = -p + 2\frac{\partial w}{\partial z}$$

$$\varepsilon^2\sigma_{\theta\theta} = -p + \frac{2u}{r}, \qquad \varepsilon^2\sigma_{rz} = \varepsilon^2\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

BCs

$$\left.\begin{aligned} u &= \frac{\partial S}{\partial t} + w\frac{\partial S}{\partial z} \\ \sigma_{rr} - \frac{\partial S}{\partial z}\sigma_{rz} &= \tau\frac{\partial S}{\partial z} \\ \sigma_{rz} - \frac{\partial S}{\partial z}\sigma_{zz} &= \tau \end{aligned}\right\}$$

at $r = S(z,t)$, where $\tau(z,t)$ is dimensionless applied shear stress.

Net mass conservation

$$[ru]_0^S + \int_0^S r\frac{\partial w}{\partial z}\,dr = 0$$

$$\therefore S\left(\frac{\partial S}{\partial t} + w\frac{\partial S}{\partial z}\right)\Big|_{r=S} + \frac{\partial}{\partial z}\int_0^S wr\,dr - Sw\frac{\partial S}{\partial z}\Big|_{r=S} = 0$$

$$\therefore \boxed{\frac{\partial}{\partial t}(S^2) + \frac{\partial}{\partial z}\int_0^S 2wr\,dr = 0}$$ (exact)

Net axial force balance

$$\left[r \sigma_{rz} \right]_0^S + \int_0^S r \frac{\partial \sigma_{zz}}{\partial z}\, dr = 0$$

$$\therefore\ S\left(\tau + \frac{\partial S}{\partial z}\sigma_{zz}\right)\Big|_{r=S} + \frac{\partial}{\partial z}\int_0^S \sigma_{zz}\, r\, dr - S\frac{\partial S}{\partial z}\sigma_{zz}\Big|_{r=S} = 0$$

$$\therefore\ \boxed{\frac{\partial}{\partial z}\int_0^S \sigma_{zz}\, r\, dr = -S\tau} \qquad \text{(exact)}$$

Now let $\varepsilon \to 0$: $\quad \frac{\partial w}{\partial r} = 0 \Rightarrow \boxed{w = w(z,t)}$ (extensional)

$$\text{Then } \frac{\partial}{\partial r}(ru) = -r\frac{\partial w}{\partial z}$$

$$\to\ ru = -\frac{r^2}{2}\frac{\partial w}{\partial z} \qquad (\because u \text{ bounded as } r \to 0)$$

$$\therefore\ u = -\frac{r}{2}\frac{\partial w}{\partial z}.$$

$$\therefore\ p = 2\frac{\partial u}{\partial r} = \frac{2u}{r} = -\frac{\partial w}{\partial z}$$

$$\text{Then } \sigma_{zz} = -p + 2\frac{\partial w}{\partial z} = 3\frac{\partial w}{\partial z} \qquad \text{(Trouton ratio!)}$$

Just plug into exact integral relations above:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}(S^2) + \frac{\partial}{\partial z}(wS^2) = 0 \\ \&\quad &\frac{\partial}{\partial z}\left(3S^2\frac{\partial w}{\partial z}\right) = -2S\tau \end{aligned}}$$

∎

Now use results of previous question. Recall that stress was made dimensionless with $\frac{\bar{\mu} U}{L} = \frac{\varepsilon^2 \lambda \mu U}{L}$

So stress on fibre due to external fluid is given by

$$\frac{\mu U}{L} \cdot \left\{ \varepsilon^2 \lambda \left(\frac{2}{S^2}\frac{\partial c}{\partial z} - \frac{2}{S}\frac{\partial}{\partial t}(BS) \right) \underline{e}_r + \frac{2\varepsilon \lambda B}{S} \underline{e}_z \right\}$$

Thus the BCs for the Trouton model are modified to

$$\left.\begin{aligned} \sigma_{rr} - \frac{\partial S}{\partial z}\sigma_{rz} &= \lambda\left(\frac{2}{S^2}\frac{\partial c}{\partial z} - \frac{2}{S}\frac{\partial}{\partial z}(BS)\right) \\ \sigma_{rz} - \frac{\partial S}{\partial z}\sigma_{zz} &= \frac{2\lambda B}{S} \end{aligned}\right\} \text{ at } r = S(z,t).$$

Following previous analysis, momentum equation becomes

$$\frac{\partial}{\partial z}\left(3S^2 \frac{\partial w}{\partial z}\right) = -4\lambda B$$

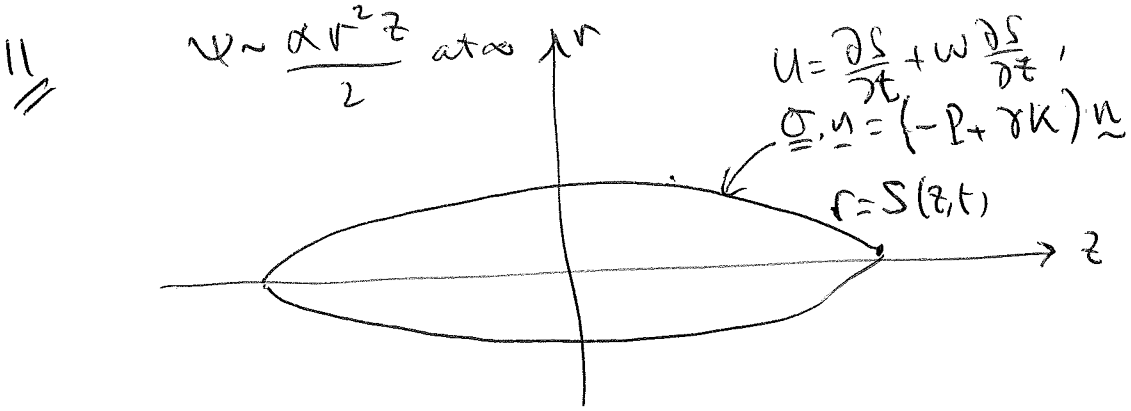
ie.

$$\boxed{\frac{\partial}{\partial z}\left(3S^2 \frac{\partial w}{\partial z}\right) = \frac{\lambda f}{\pi}}$$

& from (3-84)

$$4\pi w = \left[\log\left(\frac{4}{\varepsilon^2 S^2}\right) - 1\right] f + \int_{-\infty}^{\infty} \frac{f(s,t)}{|z-s|}\, ds$$

$$\therefore \boxed{4\pi w = f \log\left(\frac{4\pi}{\varepsilon^2 e S^2}\right) + \int_{-\infty}^{\infty} \frac{f(s,t)}{|z-s|}\, ds}$$



Dynamic BCs give

$$\begin{pmatrix} \sigma_{rr} & \sigma_{rz} \\ \sigma_{rz} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\partial S}{\partial z} \end{pmatrix} = (-P + \gamma K) \begin{pmatrix} 1 \\ -\frac{\partial S}{\partial z} \end{pmatrix} \quad \text{on } r = S$$

Non-dimensionalize as in (3-93):

$$\begin{pmatrix} \sigma_{rr} & \sigma_{rz} \\ \sigma_{rz} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 1 \\ -\varepsilon\frac{\partial S}{\partial z} \end{pmatrix} = \left(-P + \frac{K}{Ca}\right) \begin{pmatrix} 1 \\ -\varepsilon\frac{\partial S}{\partial z} \end{pmatrix}$$

where dimensionless curvature is now

$$K = \frac{1}{S\sqrt{1+\varepsilon^2\left(\frac{\partial S}{\partial z}\right)^2}} - \frac{\varepsilon^2\frac{\partial^2 S}{\partial z^2}}{\left(1+\varepsilon^2\left(\frac{\partial S}{\partial z}\right)^2\right)^{3/2}} = \frac{1}{S} + O(\varepsilon^2)$$

$$\therefore \quad \sigma_{rr} - \varepsilon\frac{\partial S}{\partial z}\sigma_{rz} = -P + \frac{1}{Ca\,S} + O(\varepsilon^2)$$

$$\sigma_{rz} - \varepsilon\frac{\partial S}{\partial z}\sigma_{zz} = -\varepsilon\frac{\partial S}{\partial z}\left(-P + \frac{1}{Ca\,S}\right) + O(\varepsilon^3)$$

$\therefore \; \sigma_{rz} = O(\varepsilon)$ & hence

$$\boxed{\sigma_{rr} = -P + \frac{1}{Ca\,S} + O(\varepsilon^2)}$$

Sub back into 2nd be...

Then $\boxed{\sigma_{rz} = \varepsilon \frac{\partial S}{\partial z}\left(\sigma_{zz} - \sigma_{rr}\right) + O(\varepsilon^3)}$

(both at $r = \varepsilon S$).

Hence forth $\underset{\sim}{Ca \equiv 1}$ by choice (3-96) of L and ε.

Write solution as

$$\Psi = \frac{\alpha r^2 z}{2} + \varepsilon^2 \int_{-\ell}^{\ell} \frac{-q(s,t)(z-s)\,ds}{4\pi\sqrt{r^2+(z-s)^2}} + \varepsilon^2 \int_{-\ell}^{\ell} \frac{f(s,t)r^2\,ds}{8\pi\sqrt{r^2+(z-s)^2}}$$

$$P = \varepsilon^2 \int_{-\ell}^{\ell} \frac{f(s,t)(z-s)\,ds}{4\pi\left(r^2+(z-s)^2\right)^{3/2}}$$

Use results from Exercise 9; note that we have replaced f by $\varepsilon^2 f$, so $A \mapsto \varepsilon^2 A$ and $B \mapsto \varepsilon^2 B$ in the notation of Exercise 9. We also have to include the contribution from the straining flow:

$$u \sim -\frac{\alpha r}{2} - \varepsilon^2 r \frac{\partial A}{\partial z} - \varepsilon^2 r \log\left(\frac{r}{2}\right)\frac{\partial B}{\partial z} - \frac{\varepsilon^2}{r}\frac{\partial C}{\partial z}$$

$$\Rightarrow \boxed{u \sim -\frac{\alpha \varepsilon S}{2} - \frac{\varepsilon}{S}\frac{\partial C}{\partial z}} + O(\varepsilon^3 \log \varepsilon) \quad \text{on } r = \varepsilon S$$

$$w \sim \alpha z + 2\varepsilon^2 A + 2\varepsilon^2 B \log\left(\frac{r}{2}\right) + \varepsilon^2 B$$

$$\boxed{w \sim \alpha z} + O(\varepsilon^2) \quad \text{on } r = \varepsilon S$$

$$\sigma_{rr} \sim -\alpha - 2\varepsilon^2 \frac{\partial B}{\partial z} + \frac{2\varepsilon^2}{r^2}\frac{\partial C}{\partial t}$$

$$\Rightarrow \boxed{\sigma_{rr} \sim -\alpha + \frac{2}{S^2}\frac{\partial C}{\partial t}} + O(\varepsilon^2 \log \varepsilon) \quad \text{on } r = \varepsilon S.$$

$$\sigma_{rz} \sim -\frac{\varepsilon^2}{r}\frac{\partial^2 C}{\partial z^2} + \frac{2\varepsilon^2 B}{r}$$

$$\Rightarrow \boxed{\sigma_{rz} \sim -\frac{\varepsilon}{S}\frac{\partial^2 C}{\partial z^2} + \frac{2\varepsilon}{S} B} + O(\varepsilon^3 \log \varepsilon) \quad \text{on } r = \varepsilon S$$

$$\sigma_{zz} \sim 2\alpha + 6\varepsilon^2 \frac{\partial A}{\partial z} + 2\varepsilon^2 \frac{\partial B}{\partial z}\left[1 + 3\log\left(\frac{r}{2}\right)\right]$$

$$\Rightarrow \boxed{\sigma_{zz} \sim 2\alpha} + O(\varepsilon^2 \log \varepsilon) \quad \text{on } r = \varepsilon S$$

Apply BCs:

kinematic: $$-\frac{\alpha S}{2} - \frac{1}{S}\frac{\partial C}{\partial z} = \frac{\partial S}{\partial t} + \alpha z \frac{\partial S}{\partial z} \qquad (1)$$

dynamic: $$-\alpha + \frac{2}{S^2}\frac{\partial C}{\partial z} = -P + \frac{1}{S} \qquad (2)$$

& $$-\frac{1}{S}\frac{\partial^2 C}{\partial z^2} + \frac{2B}{S} = \frac{\partial S}{\partial z}\left[3\alpha - \frac{2}{S^2}\frac{\partial C}{\partial z}\right] \qquad (3)$$

Eliminate C from (1) & (2) to get equation for $S(z,t)$.
(3) then determines B a posteriori.

$$\frac{2}{S}\frac{\partial C}{\partial z} = 1 + (\alpha - P)S = -\alpha S - 2\frac{\partial S}{\partial t} - 2\alpha z \frac{\partial S}{\partial z}$$

$$\Rightarrow \boxed{\frac{\partial S}{\partial t} + \alpha z \frac{\partial S}{\partial z} + \left(\alpha - \tfrac{1}{2}P\right)S + \frac{1}{2} = 0}$$

12// Eqⁿ (3.104): $\boxed{\dfrac{\dot{\ell}}{\ell} = \alpha - \sqrt{\dfrac{\ell}{15}}}$

Let $\ell = 60\left(\dfrac{\dot{\phi}}{\phi}\right)^2$

$$\therefore \dot{\ell} = 60\left[\frac{2\ddot{\phi}\dot{\phi}}{\phi^2} - \frac{2\dot{\phi}^3}{\phi^3}\right]$$

$$\therefore \frac{\dot{\ell}}{\ell} = 2\left[\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{\phi}}{\phi}\right] = \alpha - 2\left(\frac{\dot{\phi}}{\phi}\right)$$

[assuming $\dfrac{\dot{\phi}}{\phi} \geq 0$]

$$\therefore \boxed{\ddot{\phi} - \frac{\alpha}{2}\dot{\phi} = 0}$$

Now, given $w(z) = e^{mz}$.

If bubble starts from $z=0$ at $t=0$ & is then convected with the fibre velocity, then

$$\frac{dz}{dt} = e^{mz}, \qquad z(0)=0$$

$$\therefore -\frac{1}{m}e^{-mz} = t - \frac{1}{m}$$

$$\therefore e^{-mz} = 1 - mt$$

$$\therefore \boxed{z = \frac{1}{m}\log\left(\frac{1}{1-mt}\right)}$$

The local strain rate is

$$\alpha = \frac{dw}{dz} = m e^{mz}$$

$$\therefore \quad \boxed{\alpha = \frac{m}{1-mt}}$$

So

$$\frac{\ddot{\phi}}{\dot{\phi}} = \frac{m}{2(1-mt)}$$

$$\therefore \quad \log \dot{\phi} = -\frac{1}{2}\log(1-mt) + \text{Const.}$$

$$\therefore \quad \dot{\phi} = A/\sqrt{1-mt} \qquad A = \text{const.}$$

$$\therefore \quad \boxed{\phi = B\sqrt{1-mt} + C} \qquad B, C = \text{const.}$$

$$\therefore \quad \frac{\dot{\phi}}{\phi} = \frac{-\frac{mB}{2\sqrt{1-mt}}}{B\sqrt{1-mt}+C} = \frac{m}{2\sqrt{1-mt}\left(-\frac{C}{B} - \sqrt{1-mt}\right)}$$

Recall that we require $\frac{\dot{\phi}}{\phi} \geq 0$ and $m > 0$ if $D > 1$,

so define $\underbrace{-\frac{C}{B} = k > 1}$

then

$$\boxed{\ell = 60\left(\frac{\dot{\phi}}{\phi}\right)^2 = \frac{15\, m^2}{(1-mt)\left(k-\sqrt{1-mt}\right)^2}}$$

& $\ell_0 = \dfrac{15 m^2}{(k-1)^2} \Rightarrow \boxed{k = 1 + m\sqrt{\dfrac{15}{\ell_0}}}$ $(\because k > 1)$