Exercises for Mathematical Models of Financial Derivatives

January 24, 2000

1. It is customary for shares in the UK to have prices between 100p and 1,000p (in the US, between $10 and $100), perhaps because then typical daily changes are of the same sort of size as the last digit or two, and perhaps so that average purchase sizes for retail investors are a sensible number of shares. A company whose share price rises above this range will usually issue new shares to bring it back. This is called a **scrip issue** in the UK, a **stock split** in the US. What is the effect of a one-for-one issue (i.e. one additional, newly-created, share for each old one) on the share price? How should option contracts be altered? What will be the effect on option prices? Illustrate with a suitable example using data from the Financial Times 'London Traded Options' section (in 'Companies and Markets'). Repeat for a two-for-one issue.

2. A stock price is S just before a dividend D is paid. What is the price immediately after the payment?

3. **Simple and compound interest.** Under simple interest at a rate r_s, over a time period τ a sum of money M grows to $M(1 + r_s\tau)$. What is the value after n periods if the interest is reinvested (compounded) after each individual period? Now consider a fixed period $t = n\tau$ and let $n \to \infty$ keeping r_s constant. Show that you retrieve continuosly compounded interest and find the corresponding interest rate.

4. Should the value of a call and put options increase with uncertainty? Why?

5. "If taxes and transaction costs are ignored, options transactions are a zero-sum game." What is meant by this?

6. Today's date is 9th January 2000 and XYZ's share price stands at $10. On 8th November 2000 there is to be a Presidential election and you believe that, depending on which party is elected, XYZ's share price will either rise or fall by approximately 10%. Construct a portfolio of options which will do well if you are correct. Calls and puts are available with expiry dates in March, June, September, December and with strike prices of $10 plus or minus 50¢. Draw the payoff diagram and describe the payoff mathematically.

7. Draw the expiry payoff diagrams for each of the following portfolios:
   (a) Short one share, long two calls with exercise price E (this combination is called a **straddle**);
   (b) Long one call and one put, both with exercise price E (this is also a straddle: why?);
   (c) Long one call and two puts, all with exercise price E (a **strip**);
   (d) Long one put and two calls, all with exercise price E (a **strap**);
   (e) Long one call with exercise price E_1 and one put with exercise price E_2. Compare the three cases $E_1 > E_2$ (known as a **strangle**), $E_1 = E_2$ and $E_1 < E_2$.
   (f) As (e) but also short one call and one put with exercise price E (when $E = \frac{1}{2}(E_1 + E_2)$, this is called a **butterfly spread**).

Use real market data to calculate the cost of an example of each portfolio. What view about the market does each strategy express?

8. Use arbitrage arguments to prove the following simple bounds on European call options on an asset that pays no dividends:

   (a) $C \leq S$;

   (b) $C \geq S - Ee^{-r(T-t)}$;

   (c) If two otherwise identical calls have exercise prices E_1 and E_2 with $E_1 < E_2$, then

   $$0 \leq C(S, t; E_1) - C(S, t; E_2) \leq E_2 - E_1;$$

   (d) If two otherwise identical call options have expiry times T_1 and T_2 with $T_1 < T_2$, then

   $$C(S, t; T_1) \leq C(S, t; T_2).$$

9. Suppose that a share price S is currently \$100, and that tomorrow it will be either \$101, with probability p, or \$99, with probability $1 - p$. A call option, with value C, has exercise price \$100. Set up a Black–Scholes hedged portfolio and hence find the value of C. (Ignore interest rates.)

   Now repeat the calculation for a cash-or-nothing call option with payoff \$100 if the final asset price is above \$100, zero otherwise. What difference do you notice?

   This very simple discrete model is the basis of the **binomial method**.

10. In a typical step of the binomial model the asset can go from value S to $S_u = uS$ or $S_d = dS$ with probability p, $1 - p$ respectively. If the timestep is δt and $u = 1/d = e^{\sigma\sqrt{\delta t}}$, work out the mean and variance of $\delta S/S$ (which of these depend(s) on p?); compare them with the continuous-time model $dS/S = \mu\, dt + \sigma\, dX$. Calculate the hedge ratio (number of assets held, i.e. δ) for one option value $V(S, t)$ and derive a recurrence relation for V in terms of $V_u = V(S_u, t + \delta t)$ and V_d. (Don't forget to discount the hedged portfolio by r.) Draw the whole tree and show how today's option value can be found from the payoff.

11. Consider the binomial method with with $S_u = uS$, $S_d = S/u$, $u = e^{\sigma\sqrt{\delta t}}$ where δt is the timestep. Expand the recurrence relation to $O(\delta t)$ (use Taylor series about the point (S, t)) to derive the Black–Scholes equation in the limit $\delta t \to 0$.

12. If $dS = \sigma S\, dX + \mu S\, dt$, and A and n are constants, find the stochastic differential equations satisfied by

    (a) $f(S) = AS$,

    (b) $f(S) = S^n$.

13. Use Itô's lemma to confirm that

    $$\int_0^t X(\tau)\, dX(\tau) = \tfrac{1}{2}X^2(t) - \tfrac{1}{2}t$$

    where $X(t)$ is the usual Brownian motion. You can assume the usual rule that for any smooth function f

    $$f(X) = f(0) + \int_0^t df(X).$$

14. Derive the probability density function for future values of S from the probability density function for $\log S$, under the aasumption that

    $$dS = \mu S\, dt + \sigma S\, dX,$$

    with $S(0) = S_0$.

15. Suppose that S follows the geometric Brownian motion

$$\frac{dS}{S} = \mu\, dt + \sigma\, dX.$$

Show that this can be written as

$$d(\log S) = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dX$$

and hence deduce that the expect value of S at time t, given $S = S_0$ at time zero, is

$$\mathcal{E}[S|S_0] = S_0 e^{\mu t}.$$

Show that if $R = S^n$, for constant n, then R follows the geometric Brownian motion

$$\frac{dR}{R} = n\left(\mu + \frac{1}{2}(n-1)\sigma^2\right)\, dt + n\sigma\, dX.$$

Suppose that S denotes the USD/YEN exchange rate, so $1/S$ denotes the YEN/USD exchange rate, and that today's rate is S_0. You are told that both the expected USD/YEN *and* the YEN/USD exchange rates a year ahead are $2S_0$. Is this possible, and if so how? (Hint: what are the drift and volatility of $R = 1/SD$?)

16. Consider the stochastic differential equation

$$dS = A(S)\, dX + B(S)\, dt.$$

Use Itô's lemma to show that it is theoretically possible to find a function $f(S)$ which itself follows a random walk but with zero drift.

17. There are n assets satisfying the following stochastic differential equations:

$$dS_i = \sigma_i S_i\, dX_i + \mu_i S_i\, dt \quad \text{for} \quad i = 1, \ldots, n.$$

The Wiener processes dX_i satisfy

$$\mathcal{E}[dX_i] = 0, \quad \mathcal{E}[dX_i^2] = dt$$

as usual, but the asset price changes are correlated with

$$\mathcal{E}[dX_i\, dX_j] = \rho_{ij}\, dt$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$.

'Derive' Itô's lemma for a function $f(S_1, \ldots, S_n)$ of the n assets $S_1, \ldots, S_n$.

18. Show by substitution that two exact solutions of the Black–Scholes equation are

    (a) $V(S,t) = AS$,

    (b) $V(S,t) = Ae^{rt}$,

    where A is an arbitrary constant. What do these solutions represent and what is the Δ in each case?

19. Show that the explicit formulæ for a call and a put also satisfy the Black–Scholes equation with the relevant boundary conditions (one at each of $S = 0$ and $S = \infty$) and final conditions at $t = T$. Show also that they satisfy put-call parity.

20. Sketch the graphs of the Δ for the European call and put. Suppose that the asset price now is $S = E$ (each of these options is at-the-money). Convince yourself that it is plausible that the delta-hedging strategy is self-financing for each option, in the two cases that the option expires in-the-money and out-of-the-money; look at the contract from the point of view of the writer. That is, sketch an asset price evolution that rises fairly steadily and consider how the delta-hedge would evolve; repeat for one that falls.

21. Find the most general solution of the Black–Scholes equation that has the special form

    (a) $V = V(S)$;

    (b) $V = A(t)B(S)$.

    These are examples of 'similarity solutions'. Time-independent options as in (a) are called **perpetual** options.

22. Show that the solution to the initial value problem below is unique provided that it is sufficiently smooth and decays sufficiently fast at infinity, as follows:

    Suppose that $u_1(x, \tau)$ and $u_2(x, \tau)$ are both solutions to the initial value problem

    $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,$$

    with

    $$u(x, 0) = u_0(x).$$

    Show that $v(x, \tau) = u_1 - u_2$ is also a solution with $v(x, 0) = 0$.

    Show that if

    $$E(\tau) = \int_{-\infty}^{\infty} v^2 \, dx,$$

    then

    $$E(\tau) \geq 0, \quad E(0) = 0,$$

    and, by integrating by parts, that

    $$\frac{dE}{d\tau} \leq 0;$$

    thus $E(\tau) \equiv 0$, hence $v(x, \tau) \equiv 0$.

23. Show that $\sin nx \; e^{-n^2\tau}$ is a solution of the forward diffusion equation, and that $\sin nx \; e^{n^2\tau}$ is a solution of the backward diffusion equation. Now try to solve the initial value problem for the forward and backward equations in the interval $-\pi < x < \pi$, with $u = 0$ on the boundaries and $u(x, 0)$ given, by expanding the solution in a Fourier series in x with coefficients depending on τ. What difference do you see between the two problems? Which is well-posed? (The former is useful for double knockout options; see exercise 75.)

24. Verify that

    $$u_\delta(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/4\tau}$$

    does satisfy the diffusion equation.

25. Suppose that $u(x, \tau)$ satisfies the following initial value problem on a semi-infinite interval:

    $$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \; \tau > 0,$$

    with

    $$u(x, 0) = u_0(x), \; x > 0, \quad u(0, \tau) = 0, \; \tau > 0.$$

Define a new function $v(x,\tau)$ by reflection in the line $x = 0$, so that

$$v(x,\tau) = u(x,\tau) \quad \text{if } x > 0,$$

$$v(x,\tau) = -u(-x,\tau) \quad \text{if } x < 0.$$

Show that $v(0,\tau) = 0$, and that

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_0^\infty u_0(s) \left(e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau} \right) ds.$$

The function multiplying $u_0(s)$ here is called the **Green's function** for this initial-boundary value problem. This solution is applicable to barrier options.

26. Suppose that a and b are constants. Show that the parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu$$

can always be reduced to the diffusion equation. Use a change of time variable to show that the same is true for the equation

$$c(\tau)\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where $c(\tau) > 0$. Suppose that σ^2 and r in the Black–Scholes equation are both functions of t, but that r/σ^2 is constant. Derive the Black–Scholes formulæ in this case.

27. Show that the Black–Scholes equation can also be reduced to the diffusion equation by writing $S = Ee^x$, $t = T - \tau/\frac{1}{2}\sigma^2$, $V(S,t) = v(x,\tau)$, and then setting

$$v(x,\tau) = e^{-k\tau}V(\xi,\tau),$$

where

$$\xi = x + (k-1)\tau.$$

What disadvantages might there be to this change of variables?

28. If $C(S,t)$ and $P(S,t)$ are the values of a European call and put with the same exercise and expiry, show that $C - P$ also satisfies the Black–Scholes equation, with the particularly simple final data $C - P = S - E$ at $t = T$. Deduce that $S - Ee^{-r(T-t)}$ is also a solution of the Black–Scholes equation; interpret these results financially.

29. Use the explicit solution of the diffusion equation to derive the Black–Scholes value for a European put option without using put-call parity.

30. Calculate the gamma ($\partial^2 V/\partial S^2$), theta ($\partial V/\partial t$), vega ($\partial V/\partial \sigma$) and rho ($\partial V/\partial r$) for European call and put options.

31. Show how to construct a gamma- and delta-neutral portfolio of a derivative hedged with the asset *and* another derivative. (Gamma-neutral means that the gamma of the portfolio is zero.) Suppose that the portfolio is not rehedged until a small time interval δt has elapsed. What is the advantage of this portfolio over one that is merely delta-hedged? (You will need to calculate an approximation to the random walk over a small but not infinitesimal timestep δt, using the "Taylor series" approach to Itô's formula. Use this to calculate the small change in V, and fix the hedge by removing the leading-order randomness in the portfolio; remember that $\delta X^2 \neq \delta t$ for non-infinitesimal intervals.)

32. Use Maple (or any other computer algebra package) to plot out the functions of exercise 30. Use the `plot3d` command to generate three-dimensional plots of call and put options as functions of two variables, for example S and t.

33. What is the random walk followed by a European call option?

34. If $u(x,0)$ in the initial value problem for the heat equation on an infinite interval (exercise 22) is positive, then so is $u(x,\tau)$ for $\tau > 0$. Show this, and deduce that any option whose payoff is positive always has a positive value.

35. Consider the following initial value problem on an infinite interval:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(x,\tau)$$

with

$$u(x,0) = 0 \quad \text{and} \quad u \to 0 \quad \text{as} \quad x \to \pm\infty.$$

It can be shown (for example by the Green's function representation of the solution) that if $f(x,\tau) \geq 0$ then $u(x,\tau) \geq 0$. Why is this physically reasonable? Use this result to show that if $C_1(S,t)$ and $C_2(S,t)$ are the values of two otherwise identical calls with different volatilities σ_1 and $\sigma_2 < \sigma_1$, then $C_2 < C_1$. Is the same result true for puts? Why?

36. In dimensional variables, heat conduction in a bar of length L is modelled by

$$\rho c \frac{\partial U}{\partial T} = k \frac{\partial^2 U}{\partial X^2}$$

for $0 < X < L$, where $U(x,\tau)$ is the dimensional temperature, ρ is the density, c is the specific heat, and k is the thermal conductivity. Suppose that U_0 is a typical value for temperature variations, either of the initial temperature $U_0(X)$, or of the boundary values at $X = 0, L$; make the equation dimensionless.

37. What is the value of a digital call? Here are 4 ways of calculating it.

- Solve B–S (via the heat equation).
- Note that the Δ of a call has the right value at expiry, $\mathcal{H}(S - E)$. Differentiate B–S; you will have to do something about the coefficients. (See 1995/C1/HH1.)
- Write the value of a call in the form $C(S,t;E) = Ec(S/E,t)$. Hence work out a relation between $\partial C/\partial S$ and $\partial C/\partial E$. Then write down the final value problem for $\partial C/\partial E$ (differentiate B–S and the payoff), and use the known value of $\partial C/\partial S$. (You can also consider the limit as $n \to \infty$ of n times a standard bull spread struck at E and $E + 1/n$.)
- Write down the risk-neutral random walk for $\log S$. Write the value of the call as the discounted expected value of the payoff. Show that this is easily interpreted in terms of the cumulative distribution of the normal distribution.

What is the value of an option with payoff $\mathcal{H}(E - S)$? (What is put-call parity for digital options?)

38. The European **asset-or-nothing** call pays S if $S > E$ at expiry, and nothing if $S \leq E$. What is its value? (Relate it to a vanilla call.)

39. What *is* the probability that a European call will expire in the money?

40. An option has a general payoff $\Lambda(S)$, and its value is $V(S,t)$. Show how to synthesize it from vanilla call options with varying exercise prices; that is, how to find the 'density' $f(E)$ of calls, with the same expiry, exercise price E and price $C(S,t;E)$, such that

$$V(S,t) = \int_0^\infty f(E)C(S,t;E)\,dE.$$

Verify that your answer is correct

(a) When $\Lambda(S) = \max(S - E, 0)$ (a vanilla call);

(b) When $\Lambda(S) = S$. (What is the synthesizing portfolio here?)

Repeat the exercise using cash-or-nothing calls as the basis. (Hint: it is sufficient to synthesize the payoff from call payoffs: why? You may need the result that $x\delta(x) = 0$, where $\delta(x)$ is the delta function.)

41. Suppose that European calls of all exercise prices are available. Regarding S as fixed and E as variable, show that their price $C(E,t)$ satisfies the partial differential equation

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 E^2 \frac{\partial^2 C}{\partial E^2} - rE\frac{\partial C}{\partial E} = 0.$$

42. 'If an asset has zero volatility, then its future path is deterministic, and specified completely by μ. Therefore, we can calculate exactly the value of a call option on the asset, and it too must depend explicitly on μ. However, it is repeatedly stated above that this is *not* the case.' Why is this not a contradiction?

43. What is the put-call parity relation for options on an asset that pays a constant continuous dividend yield?

44. What is the delta for the call option with continuous and constant dividend yield?

45. Find a transformation that reduces the Black–Scholes equation with a constant continuous dividend yield to the diffusion equation. What is the transformed payoff for a call? How many dimensionless parameters are there in the problem?

46. Show that the value of a European call option on an asset that pays a constant continuous dividend yield lies below the payoff for large enough values of S. Show also that the call on an asset with dividends is less valuable than the call on an asset without dividends.

47. Calculate the value of a put option for both continuous and discrete dividend yields (one payment). What is the put-call parity relation in the latter case? Do the dividends increase or decrease the value of the put? Why?

48. Calculate the value of a call on an asset that pays out *two* dividends during the lifetime of the option.

49. Another model for dividend structures is to assume that the dividend payment will be a fixed amount D paid at time t_d. Work out the jump condition for a derivative product, and calculate the value of a call and put. What possible disadvantages might this model have?

50. Suppose that a forward contract had the additional condition that a premium Z had to be paid on entering into the contract. How would the forward price be affected?

51. What is the random walk followed by the futures price F? What if dividends are paid? What is the Black–Scholes equation for options on futures (including dividends)?

52. Derive the put-call parity result for the forward/future price in the form

$$C - P = (F - E)e^{-r(T-t)}.$$

What is the corresponding version when the asset pays a constant continuous dividend yield?

53. Show that vanilla call and put futures options with the same strike E are related by

$$C(F,t;E) = (F/E)P(E^2/F,t).$$

Is this true if dividends are paid?

54. What is the forward price for an asset that pays a single dividend $d_y S(t_d)$ at time t_d?

55. Analyse the **range forward** contract, which has the following features. There are two exercise prices, E_1 and E_2, with $E_1 < E_2$. The holder of a long position must purchase the asset for E_1 if at expiry $S < E_1$, for S if $E_1 \le S \le E_2$, and for E_2 if $S > E_2$. The exercise prices are to be chosen so that the initial cost is zero.

56. What is the model for derivative products on an asset that pays a time-varying dividend yield $D(t)S$? Show how to incorporate this variation into the time-dependent version of the Black–Scholes model; how are the functions $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ modified?

57. The **instalment option** has the same payoff as a vanilla call or put option; it may be European or American. Its unusual feature is that, as well as paying the initial premium, the holder must pay 'instalments' during the life of the option. The instalments may be paid either continuously or discretely. The holder can choose at any time to stop paying the instalments, at which point the contract is cancelled and the option ceases to exist. When instalments are paid continuously at a rate $L(t)$ per unit time, derive the differential equation satisfied by the option price. What new constraint must it satisfy? Formulate a free boundary problem for its value.

58. Consider American vanilla call and put options, with prices C and P. Derive the following inequalities (the second part of the last inequality is the version of put-call parity result appropriate for American options):

$$P \ge \max(E - S, 0), \qquad C \ge S - Ee^{-r(T-t)},$$

$$S - E \le C - P \le S - Ee^{-r(T-t)}.$$

Also show that, without dividends, it is never optimal to exercise an American call option. Show that this is not true for options on futures.

59. Find the explicit solution to the obstacle problem when the obstacle is $f(x) = \frac{1}{2} - x^2$. Try to find the solution when $f(x) = \frac{1}{2} - \sin^2(\pi x/2)$; the free boundaries now have to be found numerically.

60. A space of functions $\mathcal{K}$ is said to be **convex** if, for any functions $u \in \mathcal{K}$ and $v \in \mathcal{K}$, $(1-\lambda)u + \lambda v \in \mathcal{K}$ for all $0 \le \lambda \le 1$. Show that the space $\mathcal{K}$ of all piecewise continuously differentiable functions $v(x)$ $(-1 \le x \le 1)$ satisfying $v \ge f$ and $v(\pm 1) = 0$ is convex. (These functions are called **test functions.**)

The obstacle problem may also be formulated as: find the function u that minimises the energy

$$E[v] = \int_{-1}^{1} \tfrac{1}{2}(v')^2\,dx$$

over all $v \in \mathcal{K}$. (This is the usual energy minimisation but with the constraint incorporated.) If u is the minimiser, and v is any test function, use the fact that $E[(1-\lambda)u + \lambda v] - E[u] \ge 0$ for all λ to show that

$$\int_{-1}^{1} u'(v-u)'\,dx \ge 0.$$

This is the **variational inequality** for the obstacle problem.

61. Set up the American call with dividends as a linear complementarity problem.

62. Transform the American cash-or-nothing call into a linear complementarity problem for the diffusion equation and show that the transformed payoff is

$$g(x,\tau) = be^{\frac{1}{4}(k+1)^2\tau + \frac{1}{2}(k-1)x}\mathcal{H}(x), \tag{1}$$

where $b = B/E$.

Since in this case the free boundary is always at $x = 0$, the problem can be solved explicitly: do this. (Hint: put $u(x,\tau) = be^{\frac{1}{4}(k+1)^2\tau}X(x) + w(x,\tau)$ and choose $X(x)$ appropriately, then use images. Alternatively, use Laplace transforms or Duhamel's theorem.)

63. Set up the American call and put problems as linear complementarity problems using the original (S,t) variables.

64. What is the put-call parity result for compound options?

65. One might approach the compound option by considering the underlying option as the asset on which the compound option depends. Why is this not a good idea from the point of view of valuation?

66. It is possible to derive explicit formulæ for European compound and chooser options. With $C_{BS}(S,T_1)$ as the value at time T_1 of a European call option with strike price E_2 that expires on T_2, write the payoff function for a call on a call explicitly in terms of S. Sketch this payoff as a function of the underlying asset. Now use the explicit solution of the heat equation to give an expression for the value of the compound option. Simplify this expression using the function for the cumulative distribution of a bivariate normal distribution.

67. The chooser option with $E_1 = 0$ has a particularly simple value: what is it? (I.e. the intermediate payoff is $\max(C,P)$ where C and P are the underlying vanilla call and put options.)

68. The underlying call and put options for a chooser have different strikes. Show that the chooser is equivalent to a package of compound options.

69. Discuss how one could define and value an American compound option, that is, one where the underlying option could be bought at any time between the initiation of the compound option, $t = 0$, and its expiry $t = T_1$. What happens if the underlying option is American and is exercised?

70. If $V(S,t)$ satisfies the Black–Scholes equation, show that for constant B, $U(S,t) = (S/B)^{2\alpha}V(B^2/S,t)$ satisfies

$$\frac{\partial U}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - (r + (2\alpha - 1)\sigma^2)S\frac{\partial U}{\partial S} - (1 - 2\alpha)\left(r + \tfrac{1}{2}\alpha\sigma^2\right)U = 0.$$

What is special about the case $2\alpha = 1 - r/\frac{1}{2}\sigma^2$? Given a solution of the Black–Scholes equation, use this result to show how to generate a second solution 'reflected' in a barrier B. Hence find the formulæ for the values of the down-and-out call option (barrier below the strike) and the up-and-out put option (barrier above the strike). (Write both in terms of vanilla call values.)

71. Extend the idea of the previous exercise to value a down-and-out call option when the strike is below the barrier, writing the answer in terms of vanilla options and standard digital options. (Truncate the call payoff at the barrier and then reflect.) Hence value the corresponding down-and-in option.

72. How is an out boundary condition changed if an out option pays a rebate of Z if the barrier is triggered? Find the value of a down-and-out call in terms of vanilla options, together with standard European and American digital options.

73. How is an in final condition changed if a rebate of Z is paid if the barrier is never triggered? Value a down-and-in option in this case.

74. Describe the evolution of the delta-hedge of written down-and-in and down-and-out barrier call options, considering separately the cases in which the barrier is and is not triggered. (Compare with Exercise 20.)

75. The **double knockout** call or put option expires worthless if the asset price *either* rises to an upper barrier value X_2 *or* falls to a lower barrier X_1; otherwise its payoff is that of a vanilla call or put. What is its value?

76. By seeking solutions of the Black–Scholes equation which are independent of time, show that there are 'perpetual' barrier options, i.e. ones whose values are independent of t. These options have no expiry date ($T = \infty$). Find their explicit formulæ and include a continuously paid constant dividend yield on the underlying.

77. How would you value a barrier option that pays \$1 if an asset price first rises to some given level X_0 *and then* falls to another level X_1 before a time T, and otherwise pays nothing?

78. Repeat the analysis of the European continuously sampled geometric average rate option when the average is measured discretely. Derive explicit formulæ for the value of such options.

79. We have seen the equation satisfied by the European continuously sampled arithmetic average rate call option. In special circumstances it can be known with certainty before expiry that the option will expire in the money. What are these circumstances? Find an explicit formula for the value of the option in this case.

80. What is the partial differential equation for the value of an option that depends on

$$\left(\int_0^T \Big(S(\tau) \Big)^n d\tau \right)^{1/n} ?$$

81. The average strike foreign exchange option has the payoff

$$\max\left(1 - \frac{1}{ST} \int_0^T S(\tau)\, d\tau, 0\right),$$

where S is an exchange rate. What is the partial differential equation satisfied by this option? (Remember the interest payments on the foreign currency.) Is there a similarity reduction?

82. Recall the jump conditions for the discretely sampled arithmetic average strike option. In this case the option price has a similarity reduction of the form $V(S, I, t) = SH(S/I, t)$. Write the jump conditions in terms of $H(R, t)$, where $R = I/S$.

83. Find similarity variables for the discretely and continuously sampled *geometric* Asian options. What form must the payoff function take?

84. Derive the explicit formula for the value of a European lookback put.

85. Find explicit formulæ in the following cases, all with continuous sampling of the maximum or minimum:

    (a) lookback call, with payoff $\max(S - J, 0)$, where J is the asset price minimum;

    (b) lookback calls and puts with constant dividend yield;

86. There is no reason why sampling dates must be evenly spaced. How do you expect lookback option prices to be affected by the structure of the sampling?

87. Generalise the analysis of transaction costs to include costs which have three components: a fixed cost at each transaction, a cost proportional to the number of assets traded and a cost proportional to the value of the assets traded (only the last is included in our analysis here).

88. Continuing with the theme of a more general cost structure, show that under this general cost structure option prices can become negative. Is this financially reasonable? In the light of this, can the model be improved?

89. If $K = \kappa/\sigma\sqrt{\delta t}$ is small then it is possible to solve the nonlinear diffusion equation incorporating transaction costs by iteration. How would you do this?

90. Suppose you hold a portfolio of options all expiring at time T. This portfolio has value $V(S,t)$ satisfying the Black–Scholes equation with transaction costs, and with payoff $V(S,T)$ at expiry. The opportunity arises to issue a new contract having payoff $\Lambda(S)$ at time T. Find the equation satisfied by the *marginal* value of the new option, assuming that this new contract has only a small value compared with the initial portfolio.

91. Verify the local analysis of the bond pricing equation near $r = \beta/\alpha$.

92. Suppose that a bond pays a coupon $K(r,t)$. Show that the bond pricing equation is modified to

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + K = 0.$$

93. In practice a swap contract entails the exchange of interest payments at discrete times, usually every quarter. How does this affect the swap pricing partial differential equation?

94. Consider a swap having one discrete exchange of payments at time T only. This contract satisfies

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0$$

with

$$V(r,T) = r - r^*.$$

The final condition represens an exchange of fixed and floating payments. Find explicit solutions to this problem for each of the interest rate models we have described here.

95. Find the Taylor series expansion of the zero-coupon bond about the maturity date for the case

$$dr = w(r)dX + u(r)dt.$$

It is common market practice to use the Black–Scholes formulæ to value options on bonds. What are the advantages and disadvantages of such an approach?

96. Suppose that an interest rate derivative has the payoff at $t = T$

$$\max(r - r^*, 0).$$

Draw this function. On the same graph sketch the value of the contract at various times up to expiry. What is the behaviour of the contract for large r?

97. Another interest rate model (Cox, Ingersoll & Ross 1990) has

$$u - \lambda w = ar^2$$

and

$$w = br^{3/2}.$$

Write down the zero-coupon bond pricing equation. Does this equation have any special properties?