Monte Carlo valuation of American options

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An American option is a contract giving its holder the right to buy (call option) or sell (put option) one unit of an underlying security of value S for a prearranged amount. This right can be exercised at any time prior to the expiration date T. In contrast, a European option can be exercised only at the expiry. Define the amount paid to the holder of an American option at the moment of exercise, the payoff, as $\Psi(S,t) \geq 0$; a standard contract is a put option where $\Psi = \max(K - S, 0)$ and K is the strike price. The discounted exercise value of the option is $Z(t) = \Psi(t)/B(t)$, where $B(t)$ is the value at time t of \$1 invested in a riskless money market account at $t = 0$. American option valuation can be characterised as an optimal stopping problem. The time 0 value of an American option is given by

$$V(0) = \sup_{0 \leq \tau \leq T} \mathrm{E}[Z(\tau)] \tag{1}$$

where the supremum is taken over all the possible stopping times τ less than the expiration date T, and the expectation is taken over the risk-neutral probability density. This is the *primal* problem.

The overwhelming majority of traded options are of American type. Yet their valuation, even in the standard case of a lognormal process for the underlying asset, remains a topic of active research. In general it is not possible to find explicit formulae for American option prices, and numerical techniques or approximation schemes are required for option evaluation. The literature concerning the numerical solution of (1) is vast; for a summary see [2] and references therein. While simulation techniques have been used extensively to price European style derivatives, only recently have there been attempts to extend the method to price American-style claims [6]. The problem lies in the estimation of the exercise boundary; the Monte Carlo (MC) method entails the simulation of the evolution of the asset prices forward in time, but the determination of the optimal exercise policy requires a backward style algorithm. To make an exercise decision for a specific price path at a specific time, one needs to know the holding value of the option, i.e. the discounted expected value from one time period ahead, which is not directly provided by the method.

In general terms, many papers use simulation in some way to derive a stopping rule by comparing the current value of stopping with some estimate (based on simulated paths) of the value of waiting. This will provide a lower bound to the price, since this stopping rule is almost surely sub-optimal. In contrast, we investigate a new approach using MC techniques reported by Rogers [5] which makes no attempt to determine an approximate exercise policy, but instead gives an upper bound for the true price. Two other recent papers have independently used the same theory to obtain an upper bound, but differ in their implementation [1, 4]. In §1 we introduce this method to obtain an upper bound to the price. This approach is investigated in §2 where we demonstrate methods to improve the upper bound. Finally, we summarise our work in §3.

1 Theory

For an arbitrary martingale $M(t)$, we define a *dual* function $F(t, M)$ as

$$\frac{F(t,M)}{B(t)} = \mathrm{E}\left[\max_{t\le\tau\le T}\left(Z(\tau) - M(\tau)\right)\right] + M(t). \tag{2}$$

The dual problem is to minimise the dual function at time 0 over all martingales $M(t)$. Let $U(0)$ denote the optimal value of the dual problem, so that

$$U(0) = \inf_M F(0,M) \;=\; \inf_M \mathrm{E}\left[\max_{0\le\tau\le T}\left(Z(\tau) - M(\tau)\right)\right] + M(0). \tag{3}$$

The main result is that the optimal values of the dual and primal problems coincide.

Proof: For an arbitrary (adapted) martingale $M(t)$, we have

$$\begin{aligned} V(0) = \sup_{0\le\tau\le T} \mathrm{E}\left[Z(\tau)\right] \;&=\; \sup_{0\le\tau\le T} \mathrm{E}\left[Z(\tau) - M(\tau) + M(\tau)\right] \\ &= \sup_{0\le\tau\le T} \mathrm{E}\left[Z(\tau) - M(\tau)\right] + M(0) \\ &\le \mathrm{E}\left[\max_{0\le\tau\le T}\left(Z(\tau) - M(\tau)\right)\right] + M(0) \end{aligned}$$

where the second equality follows from the optional sampling theorem. Since $M(t)$ was an arbitrary martingale, the inequality will hold after taking the infimum, implying $V(0) \le U(0)$.

The "duality gap" will be zero if the upper bound holds with equality. The discounted value of an American option is a supermartingale, due to the loss of exercise rights as time progresses. The supermartingale property of $V(t)/B(t)$ allows for a Doob-Meyer decomposition of the form

$$\frac{V(t)}{B(t)} = M(t) - A(t), \tag{4}$$

where $M(t)$ is a martingale, and $A(t)$ is a previsible integrable increasing process with $A(0) = 0$. Using this martingale in the dual problem gives

$$U(0) \leq \mathrm{E}\left[\max_{0\leq t\leq T}\left(\frac{\Psi(t)}{B(t)} - \frac{V(t)}{B(t)} - A(t)\right)\right] + V(0). \tag{5}$$

Since $V(t) \geq \Psi(t)$ for all t, we conclude that $V(0) \geq U(0)$. Therefore $V(0) = U(0)$ when $M(t)$ is taken to be the martingale component of the discounted American option price process $V(t)/B(t)$. When the optimal martingale is used, both the expectation and the variance of the lookback option are equal to zero, i.e.

$$\mathrm{E}\left[\max_{0\leq\tau\leq T}(Z(\tau) - M(\tau))\right] = 0 \qquad \text{and} \qquad \mathrm{Var}\left[\max_{0\leq\tau\leq T}(Z(\tau) - M(\tau))\right] = 0.$$

The variance may provide an empirical measure of the distance to optimality for a given martingale [3].

2 Implementation

This theorem demonstrates that an upper bound on the price of an American option can be constructed by evaluating the dual function using an arbitrary martingale $M(t)$,

$$F(0, M) = \mathrm{E}\left[\max_{0\leq t\leq T}(Z(t) - M(t))\right] + M(0) \;\geq\; V(0). \tag{6}$$

The choice of martingale is crucial, since the tightness of the upper bound will depend on this. To obtain a good upper bound, (5) suggests that a suitable choice of $M(t)$ is one that approximates the martingale component of the discounted price of the American option. A sensible place to start is to consider the martingale part of the corresponding European option; in his paper, Rogers reports some encouraging results with errors in the region of 1–2%. The martingale can be refined by including a weighting coefficient, which is determined by a numerical optimisation procedure on an initial subsample of size N_1, followed by a simulation of N_2 paths. The main difficulty in this approach is choosing an appropriate martingale. In contrast, [1, 4] form a martingale from a calculation of the lower bound. This approach is more general, but is computationally more intensive. The Rogers approach is quick to calculate, but requires a careful choice of martingale. We direct our efforts towards choosing such a martingale.

We assume that the asset price S satisfies the lognormal risk-neutral process $dS = rS\ dt + \sigma S\ dW$, where σ is the volatility. For each sample path, the process is simulated at Q equally spaced discrete times. We base our first investigations on the American 1D put option, but this approach is extensible to options on multiple assets. There are two natural ways of defining a martingale based on the European put:

$$M^A(t) = B(t)^{-1} V_{euro}[S(t), K, \sigma, T-t, r] - V_{euro}[S(0), K, \sigma, T, r], \quad (7a)$$
$$M^B(t) = V_{euro}[S(t), K, \sigma, T-t, r] - B(t) V_{euro}[S(0), K, \sigma, T, r], \quad (7b)$$

where V_{euro} is the Black–Scholes value of the European put, and we have assumed a deterministic short-rate r.

To understand this method more clearly, we investigate where the pathwise maximum (pwm) occurs. Specifically, we are interested in the shape of the surface $Z(t) - M(t)$, and where the pwm occurs for simulations of the asset price since this ultimately determines the option price, see (6). Graphs of the surface $Z(t) - M(t)$ for both martingales, together with the location of the pwm for each simulation, can be seen in Fig. 1. From these it is apparent that the M^B surface has a higher value than M^A around $S = 70$, giving rise to a greater value of the mean pwm. This implies M^A is a better martingale than M^B.

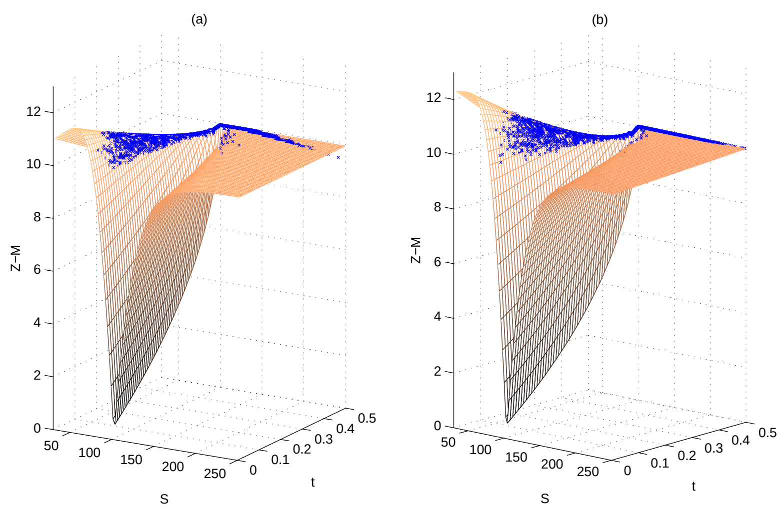

Fig. 1. Plot of $Z(t) - M(t)$ for (a) M^A, giving $V = 9.976$ and (b) M^B, giving $V = 10.092$. Each cross represents the position of the pwm during a simulation of the asset price. $N_1 = 300$, $N_2 = 5000$, $Q = 50$. Parameter values were $K = 100$, $r = 0.06$, $T = 0.5$ and $\sigma = 0.4$. The true American option price $= 9.9458$.

2.1 Adding additional European martingales

In this section we consider creating the martingale from multiple European contracts. We consider

$$M = \sum_{i=0}^{n} \lambda_i \pi_i^{euro}, \tag{8}$$

where π_i^{euro} is the martingale part of a European option calculated using (7a). We take π_0^{euro} to be a European option with the same contract parameters as the American option, and by adding n extra martingales we seek to improve the martingale and reduce the upper bound. The λ_i's are found by numerical optimisation to minimise the sum of the pwms.

We add one extra martingale ($n = 1$), based on either a put or a binary put, but with a different strike price K_1. By looking at the surface plot of $Z - M$ in Fig. 1(a), it seems sensible to choose the additional martingale contract with a strike close to 70 since this is the region where the pwms have their highest value. If we can reduce the value of the pwms, we can lower the value of the upper bound. The optimal strike K_1 for the European put was found to be close to 74, which is consistent with our initial estimate. The addition of the European contract lowers the surface near $S = K_1 = 74$, decreasing the mean of the pwms. In Table 1 we compare the MC value using a martingale based on 1 or 2 European put options. The addition of the extra contract within the martingale leads to an improved upper bound in each case.

Table 1. Calculation of upper bound for 1D American put option using a martingale based on 1 option : $M(t) = \lambda_0 \pi_0^{euro}$, or a combination of 2 options : $M(t) = \lambda_0 \pi_0^{euro} + \lambda_1 \pi_1^{euro}$. Parameter values as before, with $N_1 = 5000$, $N_2 = 50000$, $Q = 50$, $K_1 = 74$.

$S(0)$	American (True)	MC : 1 option	MC : 2 options
80	21.6059	21.6689	21.6102
85	18.0374	18.0854	18.0396
90	14.9187	14.9559	14.9209
95	12.2314	12.2591	12.2333
100	9.9458	9.9674	9.9482
105	8.0281	8.0428	8.0289
110	6.4352	6.4447	6.4351
115	5.1265	5.1330	5.1265
120	4.0611	4.0645	4.0601

2.2 Analytic approximation

As discussed in §1, the value of the optimal martingale at $t = 0$ is equal to the value of the American option itself. This implies that if we have found the optimal martingale (or one close to it), then evaluating this at time zero will provide a good approximation to the American option price.[1] Having determined a martingale suitable for a specific value of $S(0)$ using MC simulation, we then have an analytic expression for the martingale at $t = 0$ and can calculate M as a function of S. This provides an approximation to the American option value at asset prices near $S(0)$, without having to perform a full MC simulation each time the asset price changes slightly. In this manner, it is possible to provide a very quick approximation to the American option price over a range of S values once the martingale has been determined.

3 Concluding remarks

The quality of the upper bound depends on the martingale chosen. We have taken steps towards improving the determination of a 'good' martingale, but have yet to find an efficient framework for valuing any high-dimensional American option. An area of further research would be to develop a recursive method to iteratively reduce the greatest pwms by gradually altering the martingale used. This could possibly be accomplished using Green's functions, by incorporating a cash payment within the martingale designed to reduce the value of the $Z - M$ surface in the area of the greatest pwms. To reduce the variance of the final result, it may also be appropriate to increase the value of the smallest pwms during this process.

References

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[1] The values obtained using this method are no longer upper bounds, since we are just evaluating the martingale at $t = 0$.