A matched asymptotic expansions approach to continuity corrections for discretely sampled options. Part 2: Bermudan options.

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Abstract

We discuss the 'continuity correction' that should be applied to connect the prices of discretely sampled American put options (i.e. Bermudan options) and their continuously-sampled equivalents. Using a matched asymptotic expansions approach we compute the correction and relate it to that discussed by Broadie, Glasserman & Kou (*Mathematical Finance* **7**, 325 (1997)) for barrier options. In the Bermudan case, the continuity correction is an order of magnitude smaller than in the corresponding barrier problem. We also show that the optimal exercise boundary in the discrete case is slightly higher than in the continuously sampled case.

1 Introduction

In an earlier article [7], we discussed the continuity correction that should be applied to discretely sampled barrier options when the number of sampling (reset) dates is large. Broadie, Glasserman & Kou [1, 2] (referred to as BGK) showed that the discretely sampled option can be valued approximately as if it were continuously sampled but with a barrier level B adjusted up or down (for up and down barrier contracts respectively) by multiplying it by $e^{\pm\beta\sigma\sqrt{T/m}}$, where the + sign is taken for an up option and the − sign for a down option; here σ is the asset price volatility, m is the number of resets including the start date, T is the life of the option and $\beta = -\zeta(\frac{1}{2})/\sqrt{2\pi} \approx 0.5826$. The error in their approximation is $O(1/m)$. We revisited the problem using the method of matched asymptotic expansions; this confirmed that the BGK approximation is correct not only to the order stated but, in certain circumstances, is accurate to $O(1/m)$ (numerical tests suggest that it is more accurate still).

We now apply the same technique to the case of a Bermudan option, specifically a put option that can be exercised at any one of a specified set of equally spaced dates, with a reset interval of ΔT. With just one reset, a Bermudan put

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option can be reformulated as a compound option, but with more resets this approach rapidly becomes less feasible. We are interested in the case when the reset interval is small, which is also considered from a random-walk perspective in [9], building on earlier work [4, 5] on sequential analysis. Our work is a new, and potentially more accurate and more general, approach to these problems.

As in [7], we develop inner and outer expansions in terms of the small parameter $\epsilon = \sigma\sqrt{\Delta T}$. Unlike the barrier case, where the outer correction is $O(\epsilon)$, in the Bermudan case the correction is $O(\epsilon^2)$, and we calculate this term explicitly. Such an expansion is of use in its own right, albeit rather less so than for barrier options, because in the American case there is no explicit formula for the continuously sampled option (in the barrier case, we were able to give explicit expressions for the correction terms in the outer expansion in terms of vanilla option values). However, there is also considerable interest in Monte-Carlo approaches to American option valuation. Many of these necessarily use a fairly small number of timesteps, in between which the asset price may be simulated accurately; however, the optimal exercise policy is only applied at the end of each timestep. The current results may, therefore, be used to 'reverse-engineer' an accurate approximation to the continuously sampled price from the discretely sampled one. Furthermore, the small magnitude of the correction may be a reason why, even with a small number of timesteps, the Monte-Carlo approximations are remarkably good.

We shall consider an American put with strike K, on an asset with constant volatitity σ paying a dividend yield q in the standard Black–Scholes continuous-time model with constant interest rate r. As in [7], we write $V_{\mathrm{cont}}(S,t)$ (resp. $V_{\mathrm{d}}(S,t)$) for the continuously (resp. discretely) sampled contract. We recall that $V_{\mathrm{cont}}(S,t)$ satisfies the Black–Scholes equation

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \qquad S^*_{\mathrm{cont}}(t) < S < \infty, \quad 0 < t < T,$$

where $S^*_{\mathrm{cont}}(t)$ is the optimal exercise price of the contract, at which the smooth-pasting conditions

$$V_{\mathrm{cont}} = K - S, \quad \frac{\partial V_{\mathrm{cont}}}{\partial S} = -1$$

are applied. A schematic of the value surface is shown in Figure 1 (in which the time axis measures time back from expiry).

We construct inner and outer expansions for the value of the discrete contract; the outer expansion has the form

$$V_{\mathrm{d}}(S,t) \sim V_{\mathrm{cont}}(S,t) + \epsilon^2 V_2(S,t) + O(\epsilon^3),$$

where $V_2(S,t)$ is the value of a barrier-style contract that pays nothing at expiry, but pays a known negative amount, which we find, if the asset reaches the optimal exercise price of the continuously sampled option (thus, as expected, the discretely sampled option with its smaller set of exercise opportunities is worth less than the continuously exercisable contract). We also show that the correction $\epsilon^2 V_2(S,t)$ can be written in terms of the American put value and its

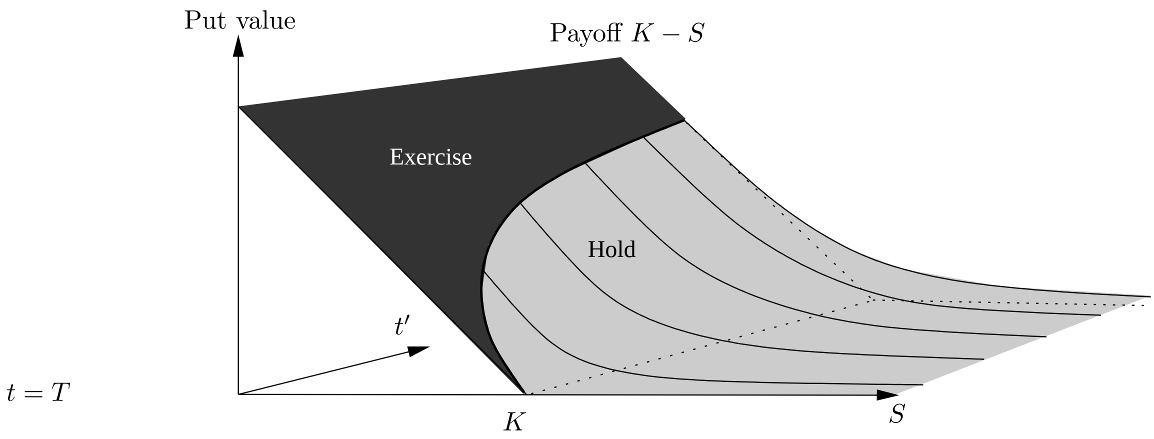

Figure 1: Schematic of the hold and exercise regions, and the value surface, for an American put. See Section 2 for the definition of t'.

Delta (or its strike derivative), and we show how to generalise these two results to other payoff structures. Lastly we construct the inner expansion, and in so doing we make extensive use of a certain function $h(x,\tau)$ of the inner variables (x,τ) defined in Section 2, and for convenience we summarise these properties here. This function was introduced in [7] and further details can be found there.

1.1 The Spitzer function $h(x,\tau)$

The Spitzer function $h(x,\tau)$ is the unique solution of the heat equation

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < \tau < 1,$$

with $h(x,0)=0$, $x<0$, satisfying the 'repeating condition'

$$h(x,1) = h(x,0) = H(x), \quad \text{say}, \quad x > 0,$$

and the growth conditions

$$h(x,\tau) \to 0, \quad x \to -\infty, \qquad h(x,\tau) \sim x + O(1) \quad \text{as} \quad x \to +\infty.$$

If, at time $\tau = 1$, the values $h(x,1-)$ for $x<0$ are discarded and replaced with 0, then $h(x,\tau)$ can thereby be extended to a periodic function in τ with period 1. In fact, the asymptotic behaviour at $x=+\infty$ is more precisely determined as $h(x,\tau) \sim x + \beta + o(1)$, where β is defined above, a fact which was crucial in the analysis of [7]. We shall need the fact that $h(x,0)$ has a jump discontinuity, from 0 to $1/\sqrt{2}$, at $x=0$.

We shall also need the function

$$h^{(1)}(x,\tau) = \int_{-\infty}^{x} h(\xi,\tau)\, d\xi,$$

which is a solution of the heat equation and has the asymptotic behaviour

$$h^{(1)}(x,\tau) \sim \frac{1}{2}(x+\beta)^2 + \frac{1}{2}\tau - \frac{1}{8}$$

with an error of $o(1)$; Note that $h^{(1)}$ is not periodic, but instead increases by $\frac{1}{2}$ over a period.

2 Problem formulation

We assume that the discretely sampled option is exercisable at N equally spaced reset times $t_1, t_2 = t_1 + \Delta T, \dots T_N = T - \Delta T$, separated by an interval ΔT. (If the interval from the current time until the first exercise date is also ΔT, we have $\Delta T = T/(N+1)$, but we do not assume this.) At exercise dates, the optimal behaviour for the option holder is to take the more valuable of the continuation value and the payoff; in this way the value can be calculated by a backward induction procedure. This translates into the condition

$$V_{\rm d}(S, t_i-) = \max(V_{\rm d}(S, t_i+), K - S).$$

That is, as the Black–Scholes equation is solved backwards from expiry, when we reach each exercise date we discard those option values below the payoff and replace them with $K - S$. We write $S_{\rm d}^{i*}$ for the discrete optimal exercise boundary; where it is not necessary, we suppress the superscript i. Note that $V_{\rm d}(S,t)$ is continuous at $S = S_{\rm d}^{i*}$, $t = t_i$, although its S derivative (Delta) is not. The discrete value surface is sketched in Figure 2.

As in [7], we first make the preliminary scaling

$$t = T - t'/\sigma^2,$$

so that the time t' is measured back from expiry and scaled with σ^2.

The Black–Scholes equation to be solved is then

$$\frac{\partial V}{\partial t'} = \frac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \alpha_1 S\frac{\partial V}{\partial S} - \alpha_2 V, \qquad \alpha_1 = (r-q)/\sigma^2, \quad \alpha_2 = r/\sigma^2. \qquad (1)$$

We recall that we have defined the small scaled exercise interval to be

$$\epsilon^2 = \sigma^2 \Delta T.$$

By differentiating the conditions

$$V_{\rm cont}(S^*_{\rm cont}(t'), t') = K - S^*_{\rm cont}(t'), \qquad \frac{\partial V_{\rm cont}}{\partial S}(S^*_{\rm cont}(t'), t') = -1$$

with respect to t' and using (1), we establish the useful results that

$$\left.\frac{\partial V_{\rm cont}}{\partial t'}\right|_{S=S^*_{\rm cont}(t')} = 0. \qquad \left.\frac{\partial^2 V_{\rm cont}}{\partial S^2}\right|_{S=S^*_{\rm cont}(t')} = \frac{2\left(\alpha_2 K - (\alpha_2-\alpha_1)S^*_{\rm cont}\right)}{S^{*2}};$$

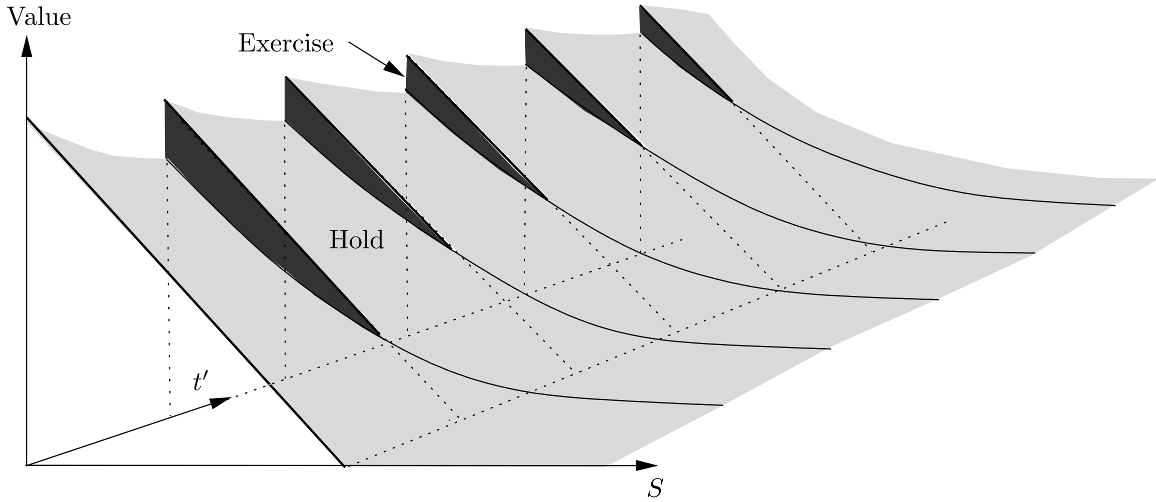

Figure 2: Schematic of the hold and exercise regions, and the value surface, for a Bermudan put.

we denote the latter quantity, the Gamma of the continuously sampled option at the exercise boundary,[1] by $\gamma^*_{\text{cont}}(t')$.

3 Approximate solution

The approximate solution consists of an outer expansion, valid far above $S = S^*_{\text{cont}}(t')$, and an inner expansion near $S = S^*_{\text{cont}}(t')$ (we should in principle include another outer region far below $S^*_{\text{cont}}(t')$, but in view of the lack of practical interest in this region we omit it). This enables us to compute the 'effective boundary conditions' for the outer solution, from which we can subsequently calculate the continuity correction to the Black–Scholes value. The timescale for the inner region is $O(\epsilon^2)$, and thus the price-scale in this region must be $O(\epsilon)$ times the scale for S. We therefore define the inner variables (x, τ) near $S = S^*_{\text{cont}}(t'_i)$ and near a typical reset time t'_i by

$$S/S^*_{\text{cont}}(t') = 1 + \epsilon x, \quad t' = t'_i + \epsilon^2 \tau,$$

which we use throughout. Note that $S^*_{\text{cont}}(t')$ is not constant.

[1]Note that, provided that $r > q$ so that the optimal exercise boundary tends to K as $t \to T$, we have $\int_{-\infty}^{\infty} \Gamma(S,t)\,\mathrm{d}S = 1$ for all $t < T$; the limiting Gamma at expiry is $\delta(S-K)$ just as for the corresponding European option. If $r < q$ the exercise boundary finishes at $S^*(T) = rK/q$ and $\Gamma(S,T)$ has two delta-function components.

3.1 Outer expansion

Away from the exercise boundary we pose the outer expansion

$$V_{\rm d}(S,t') \sim V_{\rm cont}(S,t') + \epsilon V_1(S,t') + \epsilon^2 V_2(S,t) + O(\epsilon^3),$$

which we expect to be valid for $S/S^*_{\rm cont}(t') - 1 \gg O(\epsilon)$. Each of these functions satisfies the Black–Scholes equation, and all except $V_{\rm cont}$ vanish at expiry.

By a Taylor expansion, the behaviour of this near the inner region is

$$\begin{aligned}
&V_{\rm cont} + \epsilon V_1 + \epsilon^2 V_2 \\
&\quad \sim K - S^*_{\rm cont}(t') - \big(S - S^*_{\rm cont}(t')\big) + \frac{1}{2}\gamma^*_{\rm cont}\big(S - S^*_{\rm cont}(t')\big)^2 \\
&\qquad + \epsilon\big(V_1^* + \delta_1^*(S - S^*_{\rm cont}(t'))\big) + \epsilon^2 V_2^* + O(\epsilon^3) \\
&\quad \sim K - S^* + \epsilon\left(-S^* x + V_1^*\right) \\
&\qquad + \epsilon^2\left(-\dot{S}^*\tau + \frac{1}{2}S^{*2}\gamma^* x^2 + S^*\delta_1^* x + V_2^*\right) + O(\epsilon^3), \qquad (2)
\end{aligned}$$

where we have written

$$\begin{aligned}
S^* = S^*_{\rm cont}(t_i'), &\quad \dot{S}^* = \frac{{\rm d}S^*_{\rm cont}}{{\rm d}t'}(t_i'), \\
V_1^* = V_1(S^*_{\rm cont}(t_i'), t_i'), &\quad \delta_1^* = \frac{\partial V_1}{\partial S}(S^*_{\rm cont}(t_i'), t_i'), \\
\gamma^* = \gamma^*_{\rm cont}(t'), &\quad V_2^* = V_2(S^*_{\rm cont}(t_i'), t_i').
\end{aligned}$$

The origin of the term $-\dot{S}^*\tau$ is in the expansion of $K - S = K - S^*_{\rm cont}(t')(1+\epsilon x)$ correct to $O(\epsilon^2)$; apart from this, when we neglect $O(\epsilon^3)$, it is sufficient to treat $S^*_{\rm cont}(t')$ and other such quantities as constants, equal to their value at t_i', in this expansion. Were we to go to one more term, we would have to expand these quantities as well. Note also that we do *not* perturb $S^*_{\rm cont}(t')$, which is assumed known.

3.2 Inner expansion

Changing to the inner variables (x,τ), and writing $V_{\rm d}(S,t') = v(x,\tau)$, we have the equation

$$\frac{\partial v}{\partial \tau} - \epsilon\frac{\dot{S}^*}{S^*}(1+\epsilon x)\frac{\partial v}{\partial x} = \frac{1}{2}(1+\epsilon x)^2\frac{\partial^2 v}{\partial x^2} + \epsilon\alpha_1(1+\epsilon x)\frac{\partial v}{\partial x} - \epsilon^2\alpha_2 v, \qquad -\infty < x < \infty.$$

This equation is not exact, as the coefficient

$$\frac{1}{S^*_{\rm cont}(t')}\frac{{\rm d}S^*}{{\rm d}t'}(t')$$

has been replaced by $\dot{S}^*/S^*$, its value at t_i'; the error is $O(\epsilon^3)$, however, and is neglected. We expand the solution in the form

$$v(x,\tau) \sim v_0(x,\tau) + \epsilon v_1(x,\tau) + \epsilon^2 v_2(x,\tau) + O(\epsilon^3).$$

We now turn to the conditions on $v(x,\tau)$ at $\tau = 0$ and $\tau = 1$. Recall that (see Figure 2) at optimal exercise price, v switches continuously from the exercise value (equal to the payoff) to the hold value. We anticipate that the discrete exercise value is close to the continuous one and write it as

$$x = x^* \sim x_0^* + \epsilon x_1^* + \cdots,$$

where x^* depends parametrically on t' but not on τ. We shall only[2] compute x_0^*. We therefore have that, for $x < x_0^*$,

$$v(x,0) = K - S^* - \epsilon S^* x,$$

this being the expansion of $K - S$ in inner variables at $\tau = 0$.

As for the corresponding inner barrier problem [7], we shall impose a kind of periodicity on the inner solution. Except for the term $-\epsilon^2 \dot{S}^* \tau$, which is due to the coordinate change from (S, t') to (x, τ), all the terms with which $v(x,\tau)$ matches in (2) are independent of τ. Hence we require that, at $\tau = 1$ and for $x > x_0^*$, the corresponding terms in the expansion of v return to their values at $\tau = 0$. This is why the Spitzer function is necessary for the inner solution.

The problem for $v_0(x,\tau)$ is

$$\frac{\partial v_0}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v_0}{\partial x^2}, \qquad -\infty < x < \infty,$$

with $v_0(x,0) = K - S^*$, $x < x_0^*$, and $v_0 \to K - S^*$ as $x \to \infty$. The unique solution with periodic behaviour is the constant solution $v_0(x,\tau) = K - S^*$, all others being ruled out by the analysis of Spitzer [10, 11, 7]. The problem for v_1 is also relatively straightforward: it satisfies

$$\frac{\partial v_1}{\partial \tau} = \frac{1}{2}\frac{\partial^2 v_1}{\partial x^2}, \qquad -\infty < x < \infty,$$

with $v_1(x,0) = -S^* x$ for $x < x_0^*$, $v_1 \sim -S^* x + V_1^*$ as $x \to \infty$, and periodicity as introduced above. A particular solution is $v_1(x,\tau) = -S^* x$, and as Spitzer's results preclude any addition to this function, this is exactly what v_1 is. Thus, we conclude that $V_1^* = 0$, thereby deriving the first effective boundary condition for the outer expansion.

Before proceeding to v_2, we show that, in the outer solution, $V_1(S,t') \equiv 0$. This is so because V_1 vanishes both at $t' = 0$ and on the exercise boundary $S = S_{\mathrm{cont}}(t')$, and hence by standard parabolic theory vanishes identically. This simplifies matters, as some terms no longer appear in the matching.

Substituting for v_0 and v_1, the problem for v_2 is

$$\frac{\partial v_2}{\partial \tau} + \dot{S}^* = \frac{1}{2}\frac{\partial^2 v_2}{\partial x^2} - \frac{S^{*2}\gamma^*}{2}, \quad -\infty < x < \infty,$$

[2] Were we to go to $O(\epsilon^3)$, we should be able to recover the time evolution of the free boundary, showing that $x_1^* = \dot{S}^*$ corresponding to the expansion $S^*_{\mathrm{cont}}(t' + \epsilon^2) = S^*_{\mathrm{cont}}(t') + \epsilon^2 \dot{S}^* + o(\epsilon^2)$; this corresponds to saying that the inner problem 'repeats itself' in the moving frame. However, the detailed calculations involved are very lengthy.

with $v_2(x,0) = 0$ for $x < x_0^*$ and the matching condition

$$v_2(x,\tau) \sim -\dot{S}^*\tau + \frac{1}{2}S^{*2}\gamma^* x^2 + V_2^* \quad \text{as } x \to \infty.$$

A particular solution, which satisfies the initial condition for $x < x_0^*$, is

$$v_2(x,\tau) = -\dot{S}^*\tau + S^{*2}\gamma^* \left(h^{(1)}(x - x_0^*, \tau) - \frac{1}{2}\tau \right);$$

note that it (but not its derivative) is continuous at $x = x_0^*$. Its asymptotic behaviour as $x \to \infty$ is

$$-\dot{S}^*\tau + S^{*2}\gamma^* \left(\frac{1}{2}(x - x_0^* + \beta)^2 - \frac{1}{8} \right), \tag{3}$$

and so this matches with the terms $-\dot{S}^*\tau$ and $\frac{1}{2}S^{*2}\gamma^* x^2$ in the inner expansion of the outer solution. Consequently, what remains after subtracting off the particular solution must, by matching, tend to a constant, with no linear growth, as there is no linear term in the inner expansion of the outer solution at this order. This is the crux of the problem. It might appear possible to remove the linear growth in (3) by adding an appropriate multiple of $h(x - x_0^*, \tau)$, as was the case for barrier options [7]. However, the optimality condition precludes this course of action, because it would introduce a discontinuity in $v(x,\tau)$ at $x = x_0^*$, and this would correspond to suboptimal behaviour by the option holder. Instead, *we must choose $x_0^* = \beta$ to eliminate the linear term in (3), and in this way the leading-order correction to the optimal exercise boundary is determined*: it is at $S = S^*_{\text{cont}}(t'_i)(1 + \epsilon\beta + o(\epsilon))$. Once again, a version of the BGK correction is seen to apply, at least to $O(\epsilon)$, although a result essentially equivalent to this one was proved earlier in [4] for optimal stopping problems with Brownian motion.

We can now determine the effective boundary condition for $V_2(S,t')$, from the constant term in the matching, as

$$V_2^* = -\frac{S^{*2}\gamma^*}{8},$$

which in the original financial variables implies a boundary value for V_2 of

$$-\frac{1}{4}\left(\frac{rK}{\sigma^2} - \frac{qS^*_{\text{cont}}(t')}{\sigma^2} \right).$$

We show below how to calculate V_2.

3.3 The composite expansion

For practical purposes it is useful to construct a composite expansion, uniformly valid, rather than the separate inner and outer expansions. Our outer expansion has the form[3]

$$\left(V_{\text{cont}}(S,t') + \epsilon^2 V_2(S,t') + O(\epsilon^3) \right) \mathcal{H}(S - S^*_{\text{cont}}(t')),$$

[3] Again, we are neglecting the outer expansion that we should construct far below $S = S^*_{\text{cont}}$.

where $\mathcal{H}(\cdot)$ is the Heaviside function. The inner expansion is

$$
\begin{aligned}
v_0(x,\tau) &+ \epsilon v_1(x,\tau) + \epsilon^2 v_2(x,\tau) \\
&= K - S^* - \epsilon S^* x + \epsilon^2 \left(-\dot{S}^* \tau + S^{*2}\gamma^* \left(h^{(1)}(x-\beta,\tau) - \frac{1}{2}\tau \right) \right) \\
&\sim K - S + \epsilon^2 S^{*2}\gamma^* \left(h^{(1)} \left(\frac{S - S^*(1+\epsilon\beta)}{\epsilon S^*}, \frac{t'-t_i}{\epsilon^2} \right) - \frac{1}{2}\frac{t'-t_i}{\epsilon^2} \right) + O(\epsilon^3)
\end{aligned}
$$

where $t_i = \lfloor t' \rfloor$ is the exercise date immediately before t' (this is in scaled time; in calendar time, it is the reset date immediately after t). The common expansion is the outer limit of the inner solution or the inner limit of the outer solution, namely

$$
\left(K - S + \frac{1}{2}\gamma^* \left((S-S^*)^2 - \frac{1}{4}S^{*2} \right) \right) \mathcal{H}(S - S^*_{\text{cont}}(t')),
$$

and then the composite expansion, 'outer + inner − common', is

$$
\begin{aligned}
V_{\text{cont}}(S,t') &+ \epsilon^2 V_2(S,t')\mathcal{H}(S - S^*_{\text{cont}}(t')) \\
&- \frac{1}{2}\gamma^* \left((S-S^*)^2 - \frac{1}{4}S^{*2} \right) \mathcal{H}(S - S^*_{\text{cont}}(t')) \\
&+ \epsilon^2 S^{*2}\gamma^* \left(h^{(1)} \left(\frac{S - S^*(1+\epsilon\beta)}{\epsilon S^*}, \frac{t'-t_i}{\epsilon^2} \right) - \frac{1}{2}\frac{t'-t_i}{\epsilon^2} \right) + O(\epsilon^3),
\end{aligned}
$$

where $V_{\text{cont}}(S,t')$ is understood to be equal to the payoff for $S < S^*_{\text{cont}}(t')$.

Note that there is no need for the interval before the first reset time to be equal to the reset interval ΔT: if it is not, we apply the outer expansion unchanged, and the inner expansion with the appropriate value of τ (if this is greater than 1, we continue to solve the diffusion equation for $h(x,\tau)$, without resetting the values on the negative x axis to zero).

3.4 Calculating V_2; financial interpretation of the correction

We now show how to calculate $V_2(S,t)$ (we revert to calendar time and unscaled variables) in the case $r > q$, which is usual in practice. We showed above that, as far as the outer expansion is concerned, the correction to the continuously-sampled option value is the value of a contract that pays

$$
-\frac{\epsilon^2 S^{*2}(t)\gamma^*(t)}{8} = -\frac{\Delta T}{4}\left(rK - qS^*_{\text{cont}}(t) \right)
$$

on $S = S^*_{\text{cont}}(t)$, and vanishes at expiry. (The boundary value is apparently independent of σ; however, S^*_{cont} itself does depend on σ, as does V_{cont}.) We can write this boundary value as

$$
-\frac{\Delta T}{4}\left(r(K - S^*_{\text{cont}}(t)) + (r-q)S^*_{\text{cont}}(t) \right) = -\frac{\Delta T}{4}\left(rV_{\text{cont}}(S,t) - (r-q)S\frac{\partial V_{\text{cont}}}{\partial S} \right)_{S=S^*_{\text{cont}}(t)},
$$

and both the terms on the right are the values of tradable contracts (solutions of the Black–Scholes equation) that vanish at expiry, as does V_2. Consequently, we have

$$V_2(S,t) = -\frac{\Delta T}{4}\left(rV_{\rm cont}(S,t) - (r-q)S\frac{\partial V_{\rm cont}}{\partial S}(S,t)\right)$$

for all $S > S^*_{\rm cont}(t)$. (Note that when $q = 0$ the correction is equivalent to a barrier contract that pays a constant rebate of $-rK\Delta T/4$ (independently of σ) if the asset falls to the continuously-sampled optimal exercise boundary, and pays zero at expiry, i.e. a digital put with the exercise boundary as barrier). It is easy to show that V_2 is negative, which corresponds to the loss of value of a discretely sampled option compared with its continuously-sampled counterpart. Indeed, the expression for V_2 approximately represents a quarter of the return over the reset interval on a portfolio consisting of the value of the option in cash and its delta-hedge in stock, although the financial meaning of this is not clear.

It is also possible to relate V_2 to the strike-derivative of the American put. We have the scaling invariance

$$V_{\rm cont}(S,t) = Kw(S/K,t), \qquad S^*_{\rm cont}(t) = Ks^*(t),$$

from which we find that

$$V_{\rm cont} = K\frac{\partial V_{\rm cont}}{\partial K} + S\frac{\partial V_{\rm cont}}{\partial S};$$

both the terms on the right satisfy the Black–Scholes equation and are hence tradable. Hence, we can eliminate either $V_{\rm cont}$ or its Delta from the expression for V_2. For example, when $q = 0$ we have the expression

$$V_2(S,t) = -\frac{r\Delta T}{4}K\frac{\partial V_{\rm cont}}{\partial K},$$

a result bearing a distant analogy to the BGK correction for barrier options, which is expressible in terms of the 'barrier sensitivity' of the original option.

3.5 Numerical illustration

By way of numerical illustration, we show how the error of the composite approximation $V_{\rm comp}(S,t)$, relative to a numerically computed price $V_{\rm num}(S,t)$, for the Bermudan contract (we used explicit finite differences with a space step of 0.002) varies with the number of resets. Although it is simple to allow the time period until the first reset to be different from the reset interval, in these illustrations it is equal to ΔT. In Figure 3 we plot the relative error $(V_{\rm comp} - V_{\rm num})/V_{\rm num}$; the caption shows the parameter values used. We notice first the excellence of the approximation, even for the extreme case $N = 2$. We also notice the gradient discontinuity of the relative error at $S = S^*_{\rm cont}(t)$ which is an inevitable feature of the approximation (see [7] for a discussion of the corresponding situation for barrier options). It should be noted that the presentation of relative, rather than absolute, errors magnifies the apparent jump: the magnitude of the discontinuity of the gradient of the relative error shown in Figure 3 is about 10^{-2}, which translates into an actual Delta discontinuity of about 0.25 times this (0.25 is the numerical value of the option at $S = S^*_{\rm cont}(t)$), i.e. well under 1% error relative to the Delta of the continuously-sampled option, which is -1. (The corresponding error in the Gamma will, however, be larger.) Lastly we note that the relative (but not the absolute) error increases as S increases away from $S^*_{\rm cont}$; we have no explanation for this, but note that the decrease in accuracy is not significant.

We also computed the Bermudan exercise price numerically, and compared it with the approximate value $S^*_{\rm cont}(t)(1+\epsilon\beta)$. No results are presented, because for all $N \geq 2$ the difference between the two was smaller than the grid size of the numerical solution used to compute the continuously-sampled option (0.002). Even with just one reset (that is, with one reset at $t = 0.25$, and the optimality condition calculated as if $t = 0$ were an exercise date), the difference was only 0.0027. An illustration is given in Figure 4.

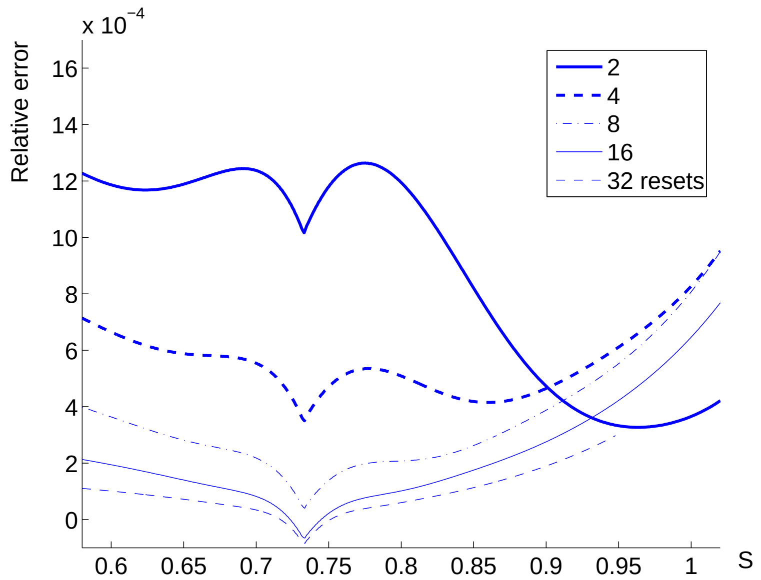

Figure 3: Variation of relative error with number N of resets. Lifetime is $T = 0.5$ and the initial period is the same as the reset interval; $r = 0.06$, $q = 0.02$, $\sigma = 0.3$, $K = 1$. For $N = 2$ we have $\epsilon = 0.1225$; for $N = 4$, $\epsilon = 0.0949$; for $N = 8$, $\epsilon = 0.0707$; for $N = 16$, $\epsilon = 0.0514$; for $N = 32$, $\epsilon = 0.0369$.

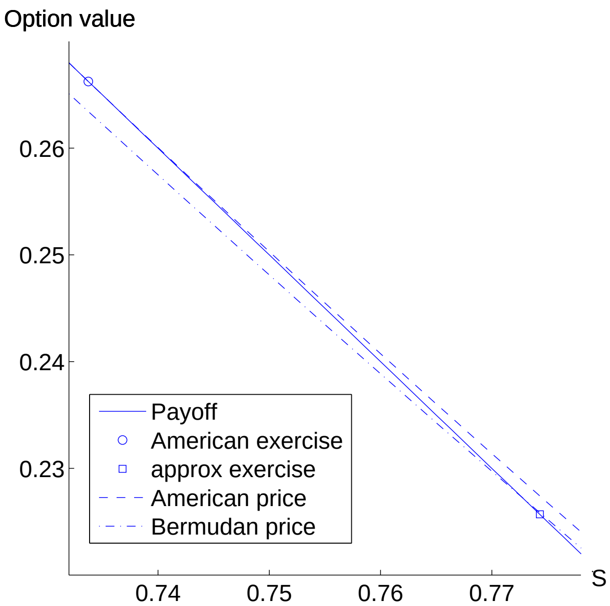

Figure 4: Detail near the exercise boundary, showing the American price and its exercise boundary S^*cont, the Bermudan price, the payoff, and the approximation $S^*_{\text{cont}}(1+\epsilon\beta)$ to the bermudan exercise price. Here $T = 0.5$ with 4 exercise dates; other parameters as above.

4 Discussion

We have shown how to calculate the value of a Bermudan option via an approximation based on the corresponding continuously-sampled American contract, using the method of matched asymptotic expansions; we have given an explicit expression for the correction. We stress that the approximation procedure is systematic rather than ad hoc, and should converge as $\epsilon \to 0$ in the usual asymptotic sense. The approximation is remarkably accurate, certainly good enough for any reasonable practical purposes.

One such purpose is the Monte-Carlo valuation of American options. As it stands, our method has the disadvantage that it requires a numerical computation of $S^*_{\text{cont}}(t)$ (for which there are now excellent approximations (see [3, 8] and references therein). Although we have used finite-difference methods, an increasingly popular alternative is Monte-Carlo valuation (see [6] for a review). Apart from duality based methods, the majority of Monte-Carlo schemes for American option valuation use some form of simulation of the exact asset price path between discrete exercise opportunities, the optimal exercise boundary being determined by backward recursion. This is essentially the valuation of a

Bermudan option. It is a consequence of our results that such an approximation procedure should be very accurate, even with a small number of relatively large timesteps. Furthermore, given a good Bermudan Monte-Carlo price V_{Ber} and exercise boundary $S^*_{\mathrm{Ber}}(t_i)$ (known only at discrete times), we can recover both the continuously sampled price and its exercise boundary as follows. To find the exercise boundary, we simply invert the formula $S^*_{\mathrm{Ber}}(t_i) = (1+\epsilon\beta)S^*_{\mathrm{cont}}(t_i)$ and interpolate between exercise dates.[4] To find the price, we calculate both the Bermudan option and either its Delta (probably needed for hedging purposes) or its strike derivative, the latter by valuing the option with strike K and strike $K + \delta K$ (this should be relatively cheap, as we use the same set of random numbers to calculate both prices). Assuming that the strike derivatives of the continuously and discretely sampled options differ only by $o(1)$ (in fact, by $O(\epsilon^2)$), we can invert the approximate formula for V_{Ber} in terms of V_{cont} and its strike derivative to find V_{cont} in terms of V_{Ber} and its strike derivative. It is also worth pointing out that, as the correction is determined entirely by the continuously-sampled contract, it can be used to compare Bermudan or Monte-Carlo prices with different reset intervals: the difference between such prices is proportional to the difference between their respective reset intervals, and this may be a useful consistency check on numerical approximations.

It is straightforward to extend the analysis to payoffs other than American puts. Suppose that the payoff is $P(S)$ and, for simplicity, that $P(S)$ is continuous, has an interval where it vanishes (for an American put, this is $[K,\infty)$), and is such that the continuously sampled exercise boundary emanates from one end of this interval. Then we only need modify various formulae as follows. The smooth pasting conditions are $V_{\mathrm{cont}}(S,t) = P(S)$ and $\partial V_{\mathrm{cont}}/\partial S = P'(S)$ at $S = S^*_{\mathrm{cont}}(t)$, and the boundary Gamma at time t_i is now

$$\gamma^* = 2\left(\alpha_2 P^* - \alpha_1 S^* P'^*\right)/S^{*2},$$

where $P^* = P(S^*)$, $P'^* = P'(S^*)$, and S^* is as before. The terms of the inner expansion are then

$$v_0(x,\tau) = P^*, \quad v_1(x,\tau) = S^* P'^* x, \quad v_2(x,\tau) = \dot{S}^* P^{*\prime}\tau + S^{*2}\gamma^*\left(h^{(1)}(x,\tau) - \frac{1}{2}\tau\right).$$

Lastly the effective boundary condition for the principal correction V_2 is

$$\epsilon^2 V_2^* = -\frac{\Delta T}{4}\left(rP^* - (r-q)S^* P'^*\right),$$

and as before the two parts of V_2 can be found in terms of the continuous contract and its Delta:

$$\epsilon^2 V_2(S,t) = -\frac{\Delta T}{4}\left(rV_{\mathrm{cont}}(S,t) - (r-q)S\frac{\partial V_{\mathrm{cont}}}{\partial S}\right).$$

[4]Although we have not proved it, we conjecture that we have the more accurate approximation $S^*_{\mathrm{Ber}}(t_{i+1}) = (1+\epsilon\beta+\epsilon^2\tau\dot{S}^*_{\mathrm{cont}}(t_i)/S^*_{\mathrm{cont}}(t_i))S^*_{\mathrm{cont}}(t_i)$, the additional term stemming from the coordinate change. Using this we have a discretisation of a first-order ordinary differential equation for $S^*_{\mathrm{cont}}(t)$.

It is also straightforward to allow for more than one free boundary (as, for example in the so-called Game option, which is simply an American-style option with both upper and lower optimal exercise boundaries). The modifications necessary to allow for local volatility or jump-diffusion models are as transparent as they are for barrier options, although again there are no explicit solutions in the continuous case. Finally, the extension to more than one asset dimension may be expected to have the same general structure, the inner region now being in a one-dimensional section normal to the optimal exercise surface, but the details remain to be investigated.

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