A Class of Exactly Solvable Free-Boundary Inhomogeneous Porous Medium Flows

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Abstract

We describe a class of inhomogeneous two-dimensional porous medium flows, driven by a finite number of multipole sources; the free boundary dynamics can be parametrized by polynomial conformal maps.

1 Inhomogeneous Porous-Medium Flows

A class of two-phase porous medium flows in two dimensions involves the dynamics of the boundary $\partial\Omega(t)$ in the (x, y) plane separating two disjoint, open regions, the liquid (saturated) region $\Omega = \Omega(t)$ and the unsaturated region $\mathbb{C}\backslash\bar{\Omega}$. The velocity of the liquid is proportional to the gradient of the pressure

$$v = -\kappa\nabla P, \tag{1}$$

where the permeability $\kappa = \kappa(z, \bar{z})$ is a real function, sufficiently regular in Ω, and $z = x + iy, \quad \bar{z} = x - iy$ are complex coordinates on the plane. The flow is incompressible and the velocity satisfies the continuity equation

$$\nabla \cdot v = 0. \tag{2}$$

The free boundary conditions are

$$P(\partial\Omega) = 0 \tag{3}$$

and the normal velocity of the boundary is

$$v_n = n \cdot v, \quad \text{for} \quad z \in \partial\Omega, \tag{4}$$

where n denotes the outward normal to the boundary.

The flow is driven by a point multipole source of order $k+1$ located at $z = z_1$, so that the pressure is singular at this point. From (1), (2), (7) it follows that, away from singularities, the pressure satisfies

$$\nabla \cdot (\kappa\nabla P) = 0, \quad z \neq z_1. \tag{5}$$

When z approaches the singular point z_1 the pressure tends to a linear combination of $\log|z-z_1|$ and its k first derivatives w.r.t. z and $\bar{z}$. For instance, in the case of the homogeneous problem

$$\kappa(z,\bar{z}) = 1, \tag{6}$$

$$P \sim \frac{-q(t)}{2}\log|z-z_1| + \sum_{j=1}^{k}\frac{(-1)^{j-1}}{(j-1)!}\left(\frac{q_j(t)}{(z-z_1)^j} + \frac{\bar{q}_j(t)}{(\bar{z}-\bar{z}_1)^j}\right) + O(1), \quad \text{as} \quad z \to z_1 \tag{7}$$

where $q(t)$ equals the time derivative of the area of Ω, while $q_j(t)$ is proportional to the time derivative of its jth harmonic moment $\int_\Omega (z-z_1)^j dxdy$.

In the case of simply-connected domains Ω that we deal with, the boundary conditions (3), (4), the elliptic equation (5) as well as the pressure asymptotics (e.g. (7)), together with the initial data $\Omega(0)$, constitute a free-boundary problem for this flow.

2 Main Result

The class of rational solutions to the homogeneous problem (6) has been considered in [3]. Here "rational" means that $\Omega(t)$ is the image of the unit circle under a rational conformal map. In [3] it is shown that the form of simply-connected fluid domains, resulting from the injection of fluid through monopole sources located at a finite number of points into an initially empty medium, is determined only by the fluxes injected at these points and is independent of the history of the sources.

In the homogeneous case, the multipole-driven flows (7) can be considered as limiting cases of the above rational solutions and the shape of the corresponding domains depends on the flux $Q = \int_0^t q(t')dt'$ as well as the "multipole fluxes"

$$Q_i = \int_0^t q_i(t')dt', \quad i = 1, ..., k.$$

The above property results in an infinite number of conservation laws. Similar conservation laws hold for an arbitrary inhomogeneous problem, with $\kappa = \kappa(z,\bar{z})$ that is regular in Ω [1].

The main result of the present paper establishes a correspondence between homogeneous problems and a class of inhomogeneous ones;

We first take a homogeneous medium flow driven by multipoles located at $z = z_1$ that admits polynomial solutions, by which we mean that the domain is the image of the unit disc $|w| < 1$ under the polynomial conformal map

$$z(w) = z_1 + rw + \sum_{j=1}^{k} u_j w^{1+j}, \tag{8}$$

and its boundary is the image of $|w| = 1$, where the coefficients

$$r = r(Q, Q_1, .., Q_k, \bar{Q}_1, .., \bar{Q}_k), \quad u_j = u_j(Q, Q_1, .., Q_k, \bar{Q}_1, .., \bar{Q}_k), \quad j = 1, .., k \tag{9}$$

are completely determined by the multipole fluxes.

Our main result is that the *same* fluid domain results from a special combination of multipole sources at $z = z_1$ into an initially empty inhomogeneous medium with the permeability

$$\kappa = \frac{1}{(z^s + \bar{z}^s)^{2n}(z^s - \bar{z}^s)^{2l}}, \quad n > l \geq 0, \tag{10}$$

where s, n, l are non-negative integers. This is a rational homogeneous function of x, y and an invariant of the group of a regular polygon ($4s$-gon if $l > 0$ or $2s$-gon otherwise). The multiplicity of the sources at $z = z_1$ is $(s(n+l)+1)(k+1)$.

Note that when stating that the domain dynamics are the same, we mean that the boundary evolutions coincide, while the flow dynamics differ in the homogeneous and inhomogeneous cases.

Obviously the same result holds for the fluid regions evolving from any initially non-empty regular "polynomial" domains.

The pressure satisfies the elliptic PDE

$$\nabla \cdot \left(\frac{1}{(z^s + \bar{z}^s)^{2n}(z^s - \bar{z}^s)^{2l}} \nabla P \right) = \pi \hat{\tilde{q}}[\delta(x - x_1)\delta(y - y_1)], \tag{11}$$

where $\hat{\tilde{q}}$ is the differential operator

$$\hat{\tilde{q}} = q + \sum_{i=1}^{\tilde{k}} \left(\tilde{q}_i \frac{\partial^i}{\partial z^i} + \bar{\tilde{q}}_i \frac{\partial^i}{\partial \bar{z}^i} \right), \quad \tilde{k} = (k+1)(s(n+l)+1) - 1.$$

The $\tilde{k}$ multipole fluxes of the non-homogeneous problem,

$$\widetilde{Q}_i = (-1)^i \int_0^t \tilde{q}_i(t')dt', \tag{12}$$

are fixed functions

$$\widetilde{Q}_i = \widetilde{Q}_i(Q, Q_1, ..., Q_k, \bar{Q}_1, ..., \bar{Q}_k, z_1, \bar{z}_1), \quad i = 1..\widetilde{k}$$

of the homogeneous problem multipole fluxes $Q_i, i = 1, .., k$ (which are proportional to the harmonic moments $\int_\Omega (z - z_1)^i dxdy$), as well as of the source position z_1.

The elliptic equation (11) is a counterpart of the Poisson equation

$$\Delta P = \pi \hat{q} \left[\delta(x - x_1)\delta(y - y_1) \right], \quad \hat{q} = q + \sum_{i=1}^{k} \left(q_i \frac{\partial^i}{\partial z^i} + \bar{q}_i \frac{\partial^i}{\partial \bar{z}^i} \right)$$

for the pressure in the corresponding homogeneous problem.

Before giving a sketch of the proof of the result, we give some examples.

3 Examples

Let us start with the simplest possible example, where the liquid is injected into an initially empty homogeneous porous medium (6) through the monopole source at $z = z_1 = x_1 + iy_1$.

By symmetry, the solution is a circular disc of radius $r(t)$, centered at $z = z_1$

$$|z - z_1| < r(t). \tag{13}$$

The pressure satisfies

$$\Delta P = -\pi q(t)\delta(x - x_1)\delta(y - y_1),$$

where the source power and total flux are

$$q(t) = \frac{d(r(t)^2)}{dt}, \quad Q = r^2$$

respectively.

According to Section 2, the variable-coefficient problem with permeability that varies as the inverse square of one Cartesian coordinate ($s = 1, n = 1, l = 0$ in (10))

$$\kappa = \frac{1}{x^2} \tag{14}$$

admits the same circular solution (13) if the flow is driven by a combination of the same monopole source (of strength $q(t)$) and a dipole source. To be precise, one has to add a dipole source located at the point $z = z_1$, of strength $-qQ/2x_1$ to the monopole source, to preserve the circular shape of the domain in the non-homogeneous case (14). The equation for the pressure distribution becomes

$$\nabla \cdot \left(\frac{1}{x^2}\nabla P\right) = -\pi\hat{\hat{q}}[\delta(x - x_1)\delta(y - y_1)], \quad \hat{\hat{q}} = \frac{dr^2}{dt}\left(1 - \frac{r^2}{2x_1}\frac{\partial}{\partial x}\right). \tag{15}$$

It is not difficult to check that the pressure

$$P = r\frac{dr}{dt}\left((2x_1x + \rho^2 + r^2)\log\rho - \frac{r^2x(x - x_1)}{\rho^2} - \rho^2 + x(x - x_1) - (2x_1x + \rho^2)\log(r)\right),$$

where $\rho = |z - z_1|$, is constant along the boundary. It satisfies (15) and the kinematic condition

$$\frac{dr}{dt} = -\frac{1}{x^2}\left(\frac{\partial P}{\partial n}\right)_{\rho=r}$$

holds at the disc boundary. Therefore, the above pressure distribution is indeed the solution of the inhomogeneous problem under consideration.

Next, consider an example of a homogeneous medium flow, again in an initially empty porous medium, driven by a combination of monopole and dipole sources, both located at $z = z_1$. The pressure satisfies the following equation

$$\Delta P = \pi \left(q(t) + q_1(t)\frac{\partial}{\partial z} + \bar{q}_1(t)\frac{\partial}{\partial \bar{z}} \right) [\delta(x - x_1)\delta(y - y_1)]. \qquad (16)$$

According to (8), (9) the boundary of the domain is the limacon

$$z = z_1 + rw + u_1 w^2, \quad w = e^{i\theta}, \quad 0 < \theta \le 2\pi, \qquad (17)$$

where r, u_1 are functions of $Q = \int_0^t q(t')dt', Q_1 = -\int_0^t q_1(t')dt', \bar{Q}_1 = -\int_0^t \bar{q}_1(t')dt'$, determined by the following equations:

$$Q = r^2 + 2|u_1|^2, \quad Q_1 = \bar{u}_1 r. \qquad (18)$$

The counterpart of (16) for the flow in a medium with permeability $k = 1/x^2$ is

$$\nabla \cdot \left(x^{-2}\nabla P\right) = \pi \left(q + \sum_{i=1}^{3} \left(\tilde{q}_i(t)\frac{\partial^i}{\partial z^i} + \bar{\tilde{q}}_i(t)\frac{\partial^i}{\partial z^i} \right) \right) [\delta(x - x_1)\delta(y - y_1)],$$

Now the parameters of the limacon map (17) are functions of

$$\begin{gathered} Q = r^2 + 2|u_1|^2, \quad \widetilde{Q}_1 = \frac{1}{4x_1}(r^4 + 4\bar{u}_1 x_1 r^2 + 6|u_1|^2 r^2 + 2|u_1|^4), \\ \widetilde{Q}_2 = \frac{\bar{u}_1 r^2}{48 x_1^2}(12 x_1 r^2 - \bar{u}_1 r^2 + 24|u_1|^2 x_1), \quad \widetilde{Q}_3 = \frac{1}{24 x_1}\bar{u}_1^2 r^4 \end{gathered} \qquad (19)$$

where the multipole fluxes are defined in (12). Eliminating r, u_1 from the union of equations (18) and (19), we can express the fluxes $\widetilde{Q}_i, i = 1, 2, 3$ as functions of $Q, Q_1, \bar{Q}_1$ (as well as x_1).

4 Main result, idea of the proof

In this section we give an idea of the proof of our main result. The proof relies on the fact that an arbitrary solutions $\phi(z, \bar{z})$, regular in Ω, of the elliptic PDE

$$\kappa^{-1} \cdot (\nabla \kappa \nabla \phi) = 0, \quad z \in \Omega \qquad (20)$$

satisfies the quadrature identity

$$\int_\Omega \phi dxdy = \pi \hat{\tilde{Q}}[\phi](z_1, \bar{z}_1) \qquad (21)$$

if Ω is a zero-initial condition solution to the inhomogeneous problem with

$$\nabla \cdot (\kappa \nabla P) = \pi \hat{\tilde{q}}[\delta(x - x_1)\delta(y - y_1)].$$

The above property is a generalization of quadrature identities for harmonic functions [4] to the case of variable-coefficient elliptic PDEs [1]. Therefore, for our purposes, the first chief idea is the correspondence between time-parametrized quadrature identities and the evolution in time of the fluid domain in the porous medium. In the proof, we obtain a complete set of solutions of (20) with κ given by (10) and then check that there exist values of multipole fluxes, such that corresponding $\phi(z,\bar{z})$ indeed satisfy (21) in domains (8) that are solutions for the homogeneous medium problem (these are special cases of algebraic domains, e.g. see [5]).

The second chief idea is to represent solutions to (10), (20) in the form [2]

$$\phi(z,\bar{z}) = T[f(z) + g(\bar{z})] \tag{22}$$

where T is a differential operator of order $s(n+l)$ with coefficients polynomial in $z, \bar{z}$, and f, g are analytic and antianalytic respectively in Ω. T is an intertwining operator that relates the elliptic equation (10), (20) with the Laplace equation via

$$T\Delta = \kappa^{-1}\nabla \cdot (\kappa\nabla)\, T$$

From (8), (22), the verification of the quadrature identity (21) for a polynomial map reduces to residue calculus, leading to an algebraic system of equations for the magnitudes of the fluxes (12) injected. This system has a unique solution [2] and yields our main result, given in Section 2.

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