

The motion of a viscous filament in a porous medium or Hele-Shaw cell: a physical realisation of the Cauchy–Riemann Equations

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Abstract

We consider the motion of a thin filament of viscous fluid in a Hele-Shaw cell. The appropriate thin film analysis and use of Lagrangian variables leads to the Cauchy-Riemann system in a surprisingly direct way. We illustrate the inherent ill-posedness of these equations in various contexts.

1 Introduction

The displacement of a fluid by another of lower viscosity in a two-dimensional porous medium or a Hele-Shaw cell is subject to the Saffman–Taylor instability, which in practice often leads to what is known as viscous fingering. (An early, and key paper in this subject was that of Saffman and Taylor [5], which concerned the Hele-Shaw version of the problem but the paper of Hill [3] even earlier had discussed the phenomenon in the context of miscible displacement in porous media. A review of the subject can be found in Homay [4].) In this paper, we relate this instability to the Cauchy–Riemann equations of complex analysis, for which the Cauchy problem is well known to be ill-posed. In particular, we show that the motion of a thin filament of viscous fluid between two effectively inviscid fluids gives a physical realisation of this Cauchy problem.

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In the Hele-Shaw problem with two incompressible fluids, of which one is inviscid and the other has viscosity μ , the standard model leads to a moving boundary problem with Laplace's equation to be solved in the viscous fluid and constant pressure in the inviscid fluid. The pressure p and velocity \mathbf{v} in the viscous fluid are, in suitable units, related by Darcy's law

$$\mathbf{v} = -\nabla p. \quad (1)$$

For incompressible flow we have

$$\nabla^2 p = 0$$

in the respective fluid regions. At interfaces separating the fluids, we assume the simple conditions

$$p = \text{constant} \quad (2)$$

and

$$-\frac{\partial p}{\partial n} = V_n, \quad (3)$$

where $\partial/\partial n$ is the derivative normal to the interface and V_n is its normal velocity. The effects of surface tension are ignored in these conditions. The model is completed by appropriate singularities representing the driving mechanism for the fluid motion, and by fixed boundary conditions as necessary.

The linear stability of a planar interface is a routine analysis. Suppose the viscous fluid is to the right of a slightly perturbed planar interface $x = Vt + \epsilon \tilde{x}(y, t)$ and the inviscid one to its left; when $\tilde{x}(y, t) = e^{\alpha t} \sin ky$ the result of a linear analysis about a travelling-wave solution is that

$$\alpha = V|k|. \quad (4)$$

An interface with $V > 0$ is therefore unstable if the less viscous fluid displaces the more viscous one, and the growth rate is proportional to the wavenumber. This dispersion relation is itself reminiscent of the Cauchy–Riemann equations. For example, suppose that we consider the Cauchy–Riemann system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

for $y > 0$, $-\infty < x < \infty$, thinking of y as a time variable and of data for u and v as being given on $y = 0$. Solutions in which $u(x, y) = U_0 e^{ikx} e^{\alpha y}$, $v(x, y) =$

$V_0 e^{ikx} e^{\alpha y}$ lead to the dispersion relation $\alpha^2 = k^2$ with catastrophic growth at large y , due to the root $\alpha = |k|$, similar to the large-time instability implied by (4) with $V = 1$, but also with one stable root $\alpha = -|k|$, corresponding to $V = -1$ in (4). In the next section we describe a physical realisation of this system. By using a thin filament of fluid with *two* free surfaces, we are able to have a flow which simultaneously has a stable free surface and an unstable one, and so is able to replicate the dispersion relation of the Cauchy–Riemann system; indeed, it replicates the Cauchy–Riemann system itself.

2 Motion of a filament in a porous medium or Hele-Shaw cell

Suppose that the Hele-Shaw cell or porous medium is divided into two parts by a thin filament of the viscous fluid, while the remainder is filled by the inviscid fluid, and that a pressure difference $\Delta P = 1$ (in dimensionless units) is applied across the filament. This situation is illustrated in Fig. 1 for flow in the region $-\infty < x < \infty$, $-1 \leq y \leq 1$ (in this case the filament is assumed to be orthogonal to $y = 0$ and $y = 1$). The location of the centreline of the filament is specified by the time-dependent curve

$$(x, y) = \mathbf{x}(\eta, t),$$

parametrised by a Lagrangian variable η , and its thickness, assumed small compared with the dimensions of the cell and the radius of curvature of the centreline, is denoted by $h(\eta, t)$.

The total mass (area) of the filament in any interval $[\eta_0, \eta_1]$ satisfies the integral balance

$$\int_{\eta_0}^{\eta_1} m(\eta) d\eta = \int_{\eta_0}^{\eta_1} h(\eta, t) \sqrt{x_\eta^2 + y_\eta^2} d\eta$$

where subscripts denote derivatives, and where the mass (area) per unit length of the filament at time $t = 0$ is $m(\eta)$. It follows that

$$h(\eta, t) \sqrt{x_\eta^2 + y_\eta^2} = m(\eta). \quad (5)$$

Because the fluid either side of the filament is inviscid, the pressure on one side of the filament, say to its right in the geometry of Figure 1, can be

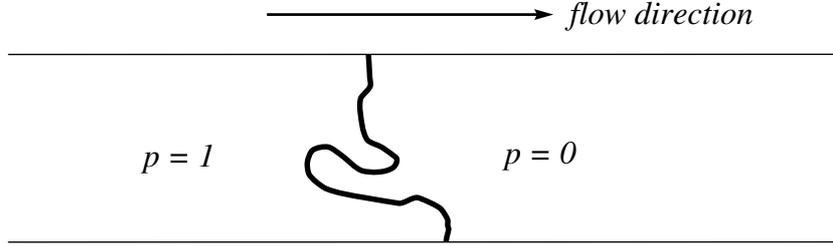


Figure 1: Geometry of a viscous filament

taken as everywhere equal to zero, while on the other side it is equal to 1. Because the filament is thin, pressure gradients across it are much bigger than those along it. Hence the pressure gradient is to leading order in the normal direction and the fluid in the filament moves approximately in a direction normal to the filament. By Darcy's law (1), a point $\mathbf{x}(\eta, t)$ moves according to

$$\mathbf{x}_t = -\frac{\mathbf{n}}{h}$$

where \mathbf{n} is the unit normal to the filament, given by

$$\mathbf{n} = \frac{(y_\eta, -x_\eta)}{\sqrt{x_\eta^2 + y_\eta^2}}.$$

Hence the equation of motion becomes

$$\mathbf{x}_t = \frac{(y_\eta, -x_\eta)}{h\sqrt{x_\eta^2 + y_\eta^2}}.$$

Using (5), it follows that

$$x_t = \frac{y_\eta}{m(\eta)}, \quad y_t = -\frac{x_\eta}{m(\eta)}.$$

The final transformation to a mass-weighted variable satisfying

$$d\xi = m(\eta) d\eta$$

leads directly to the Cauchy-Riemann equations

$$x_t = y_\xi, \quad y_t = -x_\xi,$$

stating that $x + iy$ is an analytic function of $\xi - it$. As stated earlier, these equations are well known to be ill-posed. Suppose, for example, that the filament initially lies along the y axis, that the high pressure region is to the left of the y axis, and that the filament has initial thickness $h_0(y)$. Defining $g(y)$ by $dg/dy = h_0(y)$, $g(0) = 0$, we have $x + iy = ig^{-1}(\xi - it)$, or $g(y - ix) = \xi - it$. Taking $h_0(y) = 1 - a^2 \operatorname{sech}^2 y$, with $0 < a < 1$ so that the filament is thinnest at $x = 0$, we have $g(y) = y - a^2 \tanh y$. Blow-up occurs at the first zero of the derivative of g as we move away from the real axis, namely when $\cosh^2(y - ix) = a^2$, and so the first blow-up, in which the filament develops a cusp whose tip moves with infinite speed, is at $x + iy = \cos^{-1} a$, corresponding to $t = \cos^{-1} a - (1 - a^2)^{\frac{1}{2}}$. (If $-1 < a^2 < 0$, so that the filament initially has a bulge at the origin, the blow-up occurs simultaneously at two points off the axis of symmetry.)

This analysis is much as one would expect given that one side of the filament is a retreating interface. Unlike the standard Hele-Shaw model in which time-reversal of an ill-posed problem leads to a well-posed one, here both the forward and backward problems are ill-posed: whichever way the filament moves, it always has a retreating interface. Extra physics, such as surface tension or miscibility with hydrodynamic dispersion, is needed to make a well-posed model.

3 Accelerating travelling-wave solutions, motion by mean curvature and the Saffman–Taylor finger

In this section, we consider filaments that propagate along a channel (the geometry of Figure 1) without changing their shape. We show that they have the shape of the ‘Grim Reaper’ of curvature flow, or the Saffman–Taylor ‘ $\lambda = \frac{1}{2}$ ’ finger.

It is more convenient here to use an Eulerian description of the motion of the filament. Suppose that its centreline is represented by $y = f(x, t)$, the thickness by $H(x, t)$, and that the angle between the normal to the curve and the positive x axis is θ , as shown in Figure 2.

With this description, the normal velocity of the filament is

$$V_n = \frac{f_t}{(1 + f_x^2)^{\frac{1}{2}}},$$

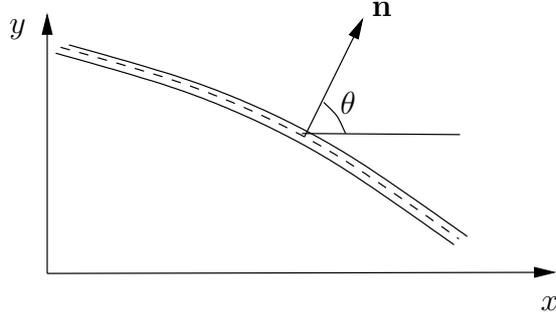


Figure 2: Eulerian description of a filament.

the angle θ satisfies

$$f_x = -\cot \theta, \quad \sin \theta = \frac{1}{(1 + f_x^2)^{\frac{1}{2}}}, \quad \cos \theta = -\frac{f_x}{(1 + f_x^2)^{\frac{1}{2}}},$$

and the local mass density is $H/\sin \theta$. Conservation of mass takes the form

$$\frac{\partial}{\partial t} \left(\frac{H}{\sin \theta} \right) + \frac{\partial}{\partial x} \left(\frac{H}{\sin \theta} V_n \cos \theta \right) = 0.$$

This equation can be rewritten in the equivalent forms

$$\frac{\partial}{\partial t} \left(\frac{1}{V_n \sin \theta} \right) + \frac{\partial}{\partial x} \cot \theta = 0, \quad (6)$$

$$\left(\frac{1 + f_x^2}{f_t} \right)_t - f_{xx} = 0,$$

and

$$f_t^2 f_{xx} - 2f_x f_t f_{xt} + (1 + f_x^2) f_{tt} = 0.$$

A linearisation of the last of these about the straight solution, writing $H(x, t) = 1 + \epsilon \tilde{H}(x, t)$, $f(x, t) = t + \epsilon \tilde{f}(x, t)$, leads to $\tilde{f}_{xx} + \tilde{f}_{tt} = 0$, in accordance with the results of the previous section. (It is remarkable that the transformations of the previous section, and the use of Lagrangian variables, reduce these nonlinear partial differential equations to the linear Cauchy–Riemann system.)

Suppose now that we look for a solution $f(x, t) = F(x - d(t))$, which propagates without change of shape. Thus θ is also a function of $x - d(t)$, say $\theta = \Theta(x - d(t))$, and $V_n = \dot{d} \cos \Theta$, where $\dot{} = d/dt$. Then (6) becomes

$$\frac{\partial}{\partial t} \left(\frac{1}{\dot{d} \cos \Theta \sin \Theta} \right) + \frac{\partial}{\partial x} \cot \Theta = 0,$$

and noting that $\sin \Theta \partial / \partial x = -\partial / \partial s$, where s is arclength moving with the curve, a short calculation shows that

$$\frac{d\Theta}{ds} = \frac{\ddot{d}}{\dot{d}^2} \cos \Theta. \quad (7)$$

Thus \ddot{d}/\dot{d}^2 is a constant which, by suitably scaling x , y and d may be set equal to one, and (7) becomes precisely the equation for the ‘ $\lambda = \frac{1}{2}$ ’ Saffman–Taylor finger. This curve also arises in analysis of motion by mean curvature, as (7) with $\ddot{d}/\dot{d}^2 = 1$ is the equation of motion for a curve whose normal velocity is equal to its curvature, and which propagates with unit velocity and constant shape. (This curve is known as the Grim Reaper, although we note that in our application the velocity is by *minus* the curvature, another manifestation of the ill-posed nature of our system. Note also that unsteady curvature flows lead to a parabolic equation rather than an elliptic one, so the coincidence is only in the travelling-wave case.)

Solving for $d(t)$, we find

$$d(t) = \text{constant} - \log(t - t^*)$$

where t^* is a blow-up time at which the accelerating filament reaches infinity. This blow-up time is determined by the thickness of the filament at the initial time.

The solution just derived is the limit of the solution presented by [2], who presents exact solutions for the evolution of a region of viscous fluid between two inviscid fluids in a channel, that is without the assumption of slenderness; these solutions have subsequently been generalised by [1]. Unfortunately the time-change employed by [2] obscures the interesting finite-time blow-up of his solutions.

4 Concluding Remarks

The Cauchy-Riemann equations are one of the canonical systems of partial differential equations. The phenomenon of viscous fingering is of funda-

mental interest in applied mathematics. Thus the heuristic direct physical interpretation of the Cauchy-Riemann equations as a model for the motion of a viscous filament may be of general interest. The model can be shown to be an exact limiting case when the thickness of a two-dimensional ribbon is taken to zero as the viscosity tends to infinity with the viscosity-thickness product held to unity. (It would be interesting to derive this result from a Birkhoff-Rott integral equation formulation for the evolution [7]; presumably the result is the Dirichlet-to-Neumann operator for $x + iy$ as an analytic function of $\xi - it$. It would be still more interesting to be able to shed light on the question of blow-up for the Muskat problem, in which the second fluid has non-zero viscosity; to date, all that is known is that curvature singularities can form in its interface [6].) The ill-posed nature of viscous fingering is thus clearly exposed, as the model is an initial value problem on the Cauchy-Riemann equations, a standard example of an ill-posed problem.

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