# Notes for a talk on the Hilbert-Mumford criterion

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## 1 Some results on reductive actions on affine schemes

**Lemma 1.** *Let $k$ be a field and let $G$ be a linea algebraic group over $k$, acting on an affine scheme $X = \operatorname{Spec}(A)$ of finite type over $k$. Let $S$ be a closed $G$-subscheme of $X$. Then*

1. *There is a $G$-equivariant closed embedding $f\colon X \to V$, where $V$ is a finite dimensional $G$-representation.*

2. *There is a $G$-equivariant morphism $f\colon X \to V$, where $V$ is a finite dimensional $G$-representation, such that $f^{-1}(0) = S$.*

*Proof.* Let $V^\vee \subset A$ be a finite dimensional $G$-subrep. This gives a map $\varphi\colon \operatorname{Sym} V^\vee \to A$ of $k$-algebras that is also a map of $G$-representations and hence a $G$-equivariant morphism $f\colon X \to V$. If $V$ generates $A$ as a $k$-algebra, then $\varphi$ is surjective and $f$ is a closed immersion. This proves the first part. In any case, $f^{-1}(0)$ is the closed subscheme of $X$ whose ideal is that geneated by $V$, so we can choose $V$ to generate the ideal of $S$, and hence the second part is proven. The crucial fact that we are using is that the $G$-subrepresentation (in this case of $A$) generated by a finite number of elements is finite dimensional. □

**Theorem 2.** *Let $k$ be an algebraically closed field, let $G$ be a connected reductive group over $k$ and let $X$ be an affine finite-type $G$-scheme over $k$. Let $x \in X(k)$ and let $S$ be a closed $G$-subscheme of $X$. Suppose that $S \cap \overline{Gx} \neq \varnothing$. Then there is a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ of $G$ such that the limit $\lim_{t\to 0} \lambda(t)x$ exists and is in $S$.*

*Proof.* Choose a point $y \in (S \cap \overline{Gx})(k)$. Take an integral curve $C_1 \subset \overline{Gx}$ containing both $x$ and $y$. Consider the orbit map $h\colon G \to X\colon g \mapsto gx$ and take a smooth projective curve $C$ and a rational map $a\colon C \dashrightarrow G$ such that the composition $f = h \circ a$ dominates $C_1$. There is a point $\sigma \in C(k)$ such that $f(\sigma) = y$. Let $R = k[\![t]\!]$ and $K = k((t))$. Since the completion $\widehat{\mathcal{O}}_{C,\sigma} \cong R$, we get a diagram

$$\begin{array}{ccccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R & & 0 \\ {\scriptstyle g}\downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{h} & X & & y \end{array} \tag{1}$$

To continue, we need Cartan-Iwahori decomposition, which states that

$$G(K) = G(R)\operatorname{Hom}(\mathbb{G}_m, G)G(R).$$

More precisely, for all $g \in G(K)$, there are $h_1, h_2 \in G(R)$ and a cocharacter $\lambda\colon \mathbb{G}_m \to G$ such that $g = h_{1|\operatorname{Spec} K} \cdot \lambda_{|\operatorname{Spec} K} \cdot h_{2|\operatorname{Spec} K}$. Note that we can regard $G(R)$ as a subgroup of $G(K)$.

Using this for our particular $g \in G(K)$ above, we get

$$y = \lim_{t \to 0} g(t)x = \lim_{t \to 0} h_1(t)\lambda(t)h_2(t)x$$

and

$$h_1(0)^{-1}y = \lim_{t \to 0} h_1(t)^{-1}g(t)x = \lim_{t \to 0} \lambda(t)h_2(t)x.$$

Replacing $x$ by $h_2(0)$ we may assume $h_2(0) = e \in G(k)$, the identity element.

*Claim* 3. $\lim \lambda(t)x$ exists.

*Proof of Claim.* By the first part of Lemma 1, we may assume that $X = V$ a vector space and that $G = \mathrm{GL}(V)$. Choosing coordinates on which $\lambda$ acts diagonally, we write. We also write $\lambda(t) = \operatorname{diag}(t^{n_1}, \ldots, t^{n_l})h_2(t) = (a_{ij})$ with $a_{ii} = 1$ and $a_{ij} \in (t) \subset k[\![t]\!]$ if $i \neq j$. Therefore $\lambda(t)h_2(t)x = (t^{n_i}x_i + t^{n_i}(t))$, and this limits exists when $t \to 0$. So if $x_i \neq 0$, then $n_i \geq 0$. Therefore $\lim_{t \to 0} \lambda(t)x$ also exists. □

*Claim* 4. $\lim \lambda(t)x$ equals $y$.

*Proof of Claim.* By the second part of Lemma 1 we may assume $V = X$ is a $G$-representation, $S = \{0\}$ and $G = \mathrm{GL}(V)$. Since $\lim_{t \to 0} \lambda(t)h_2(t) = 0$, in the expression above we must have that if $x_i \neq 0$ then $n_i > 0$. Therefore $\lim_{t \to 0} \lambda(t)x = 0$ also. □

□

*Remark* 5. We may reformulate the Theorem in stacky language as follows. Consider $\mathcal{X} = X/G$ and $\mathcal{Z} = S/G$ which is a closed substack of $\mathcal{X}$ and let $x \in \mathcal{X}(k)$. If $\mathcal{Z}$ intersects the closure of $\{x\}$, then there is a map $\lambda\colon \Theta_k \to \mathcal{X}$ such that $\lambda(1) = x$ and $\lambda(0)$ is in $\mathcal{Z}$. Here $\Theta_k = \mathbb{A}^1_k/\mathbb{G}_{m,k}$. One advantage of this is that we may reduce to the case where the group is $\mathrm{GL}_n$. Indeed, we can always find a closed embedding $G \to \mathrm{GL}_n$, and then $X/G = X \times^G \mathrm{GL}_n / \mathrm{GL}_n$ and $G \times^G \mathrm{GL}_n$ is affine. We also see that connectedness of $G$ is actually not important.

While Cartan-Iwahori decomposition can be difficult to prove in general, it is easy for $\mathrm{GL}_n$. Indeed, let $g \in \mathrm{GL}_n(K)$. We can find $g_1, g_2 \in \mathrm{GL}_n(R)$ such that $g_1 g g_2$ is diagonal (we can do this over PID's). An element of the diagonal is $a \in K^\times$, so there is a unique $n \in \mathbb{Z}$ such that $t^{-n}a \in R^\times$. The $t^n$ give a cocharacter $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ so that $g_1 g g_2 \lambda^{-1} \in \mathrm{GL}_n(R)$, as desired.

**Lemma 6.** *Let $G$ be linearly reductive over a field $k$ acting on an affine scheme $X = \operatorname{Spec} A$. Suppose $Z_1, Z_2$ are disjoint closed $G$-subschemes of $X$. Then there is $a \in A^G$ such that $a_{|Z_1} = 1$ and $a_{|Z_2} = 0$.*

*Proof.* Write $Z_i = \operatorname{Spec} A/I_i$. We have that $I_1 \oplus I_2 \to I_1 + I_2$ is surjective, so $I_1^G \oplus I_2^G \to (I_1+I_2)^G$ is surjective too, by exactness of invariants, that is, $(I_1+I_2)^G = I_1^G + I_2^G$. Now $1 \in I_1 + I_2$, so $1 \in (I_1+I_2)^G$ too and there are elements $g \in I_1^G$ and $f \in I_2^G$ such that $1 = g + f$. Then $f = a \in A^G$ is the sought-after element. □

## 2 Hilbert-Mumford criterion

We place ourselves in the following setup. We work over an algebraically closed field $k$, and consider a linearly reductive group $G$ over $k$. Let $X$ be a projective-over-affine finite-type scheme over $k$, endowed with an action of $G$ and an ample linearisation $\mathcal{L}$.

Let $x \in X(k)$. Let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup such that the limit $\lim_{t\to 0} \lambda(t)x = y$ exists. The multiplicative group $\mathbb{G}_m$ acts on $X$ via $\lambda$ and $y\colon \operatorname{Spec} k \to X$ is a fixed point, so $y^*\mathcal{L}$ is a $\mathbb{G}_m$-quivariant line bundle on $\operatorname{Spec} k$. Thus $\Gamma(\operatorname{Spec} k, y^*\mathcal{L}) \cong k \cdot s$ and $\lambda(t) \cdot s = t^n s$ for a unique $n \in \mathbb{Z}$ (the *weight*).

**Definition 7.** We denote

$$m(x,\lambda) = n$$

and call it the *Hilbert-Mumford* weight.

Recall the following definitions in Geometric Invariant Theory. The point $x$ is

1. *semistable* if there is $n > 0$ and $a \in \Gamma(X, \mathcal{L}^{\otimes n})^G$ such that $a(x) \neq 0$. There is a $G$-equivariant open subscheme $X^{\mathrm{ss}}$ consisting of the semistable points;
2. *polystable* if it is semistable and $Gx$ is closed in $X^{\mathrm{ss}}$;
3. *stable* if it is polystable and $\dim G_x = 0$, where $G_x$ is the stabiliser of $x$.

**Theorem 8** (Hilbert-Mumford criterion)**.** *The point $x$ is* $\begin{cases} \textit{semistable} \\ \textit{polystable} \\ \textit{stable} \end{cases}$ *if and only if for all $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)x$ exists, we have that*

$$\begin{cases} m(x,\lambda) \leq 0 \\ m(x,\lambda) \leq 0 \text{ with equality } \iff \exists g \in P(\lambda),\ \lambda^g \text{ is in } G_x \\ m(x,\lambda) \leq 0 \text{ with equality } \iff \lambda = 0. \end{cases}$$

Above $P(\lambda)$ is the *parabolic subgroup* of $\lambda$, whose $k$-points are those $g \in G(k)$ such that $\lim_{t\to 0} \lambda(t) g \lambda(t)^{-1}$ exists in $G$.

*Proof.* Let $A = \bigoplus_{n\in N} A_n = \bigoplus_{n\in\mathbb{N}} H^0(X, \mathcal{L}^{\otimes n})$, so $X = \operatorname{Proj} A$ because $X$ is projective-over-affine. By replacing $\mathcal{L}$ by a high enough tensor power, we may assume $A$ is generated in degree 1. Therefore, the projectivisation of the normal cone of $\operatorname{Spec} A$ at $\operatorname{Spec} A_0$ is isomorphic to $X$, and we have a cartesian diagram

$$\begin{array}{ccc} \mathbb{A}(\mathcal{L}^\vee) & \xrightarrow{\ f\ } & \operatorname{Spec} A \\ \sigma\Big\uparrow & \llcorner & \Big\uparrow \\ X & \longrightarrow & \operatorname{Spec} A_0, \end{array}$$

of $G$-equivariant morphisms, where $\sigma$ is the zero section and $f$ is identified with the blow-up of $\operatorname{Spec} A$ along $\operatorname{Spec} A_0$ and it is thus proper.

Let $x^* \in \mathbb{A}(\mathcal{L}^\vee)(k)$ be a lift of $x$ to the total space $\mathbb{A}(\mathcal{L}^\vee) = \operatorname{Spec} \bigoplus_{n\in\mathbb{N}} \mathcal{L}^{\otimes n}$ of $\mathcal{L}^\vee$ not in the zero section. Let $\lambda\colon \mathbb{G}_m \to G$ be such that $\lim_{t\to 0} \lambda(t)x$ exists in $X$.

*Claim* 9. We have $m(x, \lambda) > 0$ if and only if $\lim_{t\to 0} \lambda(t)x^*$ exists and lies in $\sigma(X)$.

Indeed, since the limit $\lim_{t\to 0} \lambda(t)x$ exists in $X$, by pulling back $\mathbb{A}(\mathcal{L}^\vee)$ along the induced $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_k \to X$ we may assume that $X = \mathbb{A}^1_k$ and $G = \mathbb{G}_m$ acts by scaling. Then $\lim_{t\to 0} \lambda(t)x^*$ exists and lies in the zero section if and only if $\mathcal{L}^\vee$ has an invariant section that vanishes at 0 but not at 1, and this is equivalent to $m(x, \lambda) > 0$. The claim follows. The figures below representing the trajectories of $\lambda(t)x^*$ in the three cases $m(x, \lambda) < 0, = 0$ and $> 0$ provide a good intuition for the claim.

Now, we have that the following statements are equivalent:

1. there is $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)x$ exists and $m(x, \lambda) > 0$;
2. there is $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)x^*$ exists and belongs to $\sigma(X)$;
3. there is $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)f(x^*)$ exists and belongs to $\operatorname{Spec} A_0$;
4. $\overline{Gf(x*)} \cap \operatorname{Spec} A_0 \neq \varnothing$;
5. there is no $a \in A^G$ with $a(f(x^*)) \neq 0$ and $a_0 = a_{|\operatorname{Spec} A_0} = 0$;
6. there is no $n > 0$, no $a \in A^G_n$ with $a(f(x^*)) \neq 0$;
7. $x$ is not semistable.

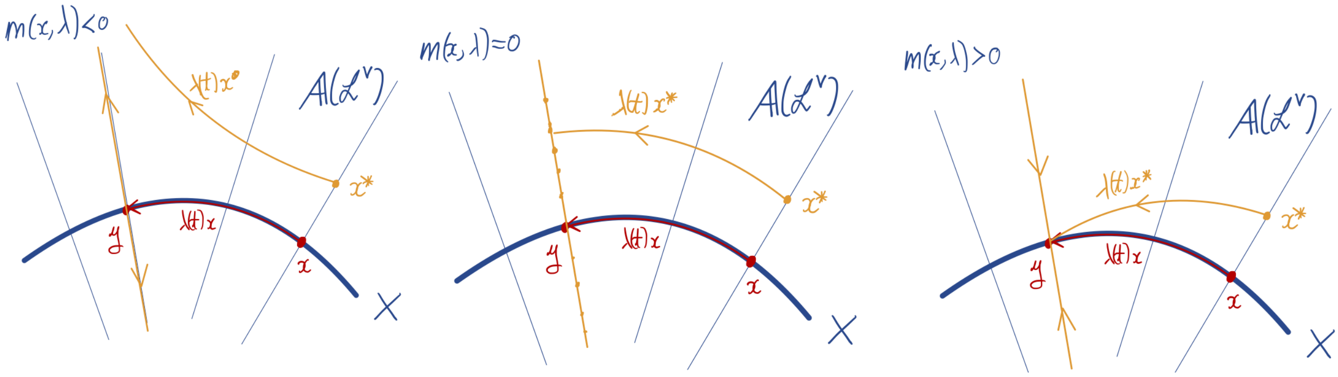
Indeed, 1 is equivalent to 2 by the Claim; 2 is equivalent to 3 by properness of $f$ and cartesianity of the square above; 3 is equivalent to 4 by Theorem 2; 4 is equivalent to 5 by Lemma 6; 5 is equivalent to 6 because if $a \in A^G$ satisfies the condition in 5, then the degree $n$ part $a_n$ of $a$ will also satisfy it for some $n > 0$; and, finally, 6 is equivalent to 7 because $A^G_n = H^0(X, \mathcal{L}^n)^G$ and $a(f(x^*)) \neq 0$ if and only if $a(x) \neq 0$. This finishes the proof of the Hilbert-Mumford criterion in the semistable case.

Assume now that $x$ is semistable. By covering $X^{\mathrm{ss}}$ by $G$-equivariant open subsets that are saturated with respect the good quotient $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ G$ and applying Theorem 2 on each of them, we see that the orbit $Gx$ is closed in $X^{\mathrm{ss}}$ if and only if for every $\lambda\colon \mathbb{G}_m \to G$ such that $y = \lim_{t\to 0} \lambda(t)x$ exists and lies in $X^{\mathrm{ss}}$, we have that actually $y$ is in the orbit $Gy$. As it can be seen, for example, by considering the stack $Gx/G \cong BG_x$, the statement that $y$ lies in $Gx$ is equivalent to the existence of an element $g \in P(\lambda)(k)$ such that $\lambda^g$ is in $G_x$. The Hilbert-Mumford criterion in the polystable case now follows from the following key fact that is of independent interest.

*Claim* 10. If $\lambda\colon \mathbb{G}_m \to G$ is such that $y = \lim_{t\to 0} \lambda(t)x$ exists in $X$, then $y$ lies in $X^{\mathrm{ss}}$ if and only if $m(x,\lambda) = 0$.

We sketch a proof of the Claim. If $y$ is in $X^{\mathrm{ss}}$, we can take $s \in H^0(X, \mathcal{L}^{\otimes n})^G$ such that $s(y) \neq 0$. Then $s(x) \neq 0$ and thus $m(x,\lambda) = 0$. Conversely, suppose that $y$ is not in $X^{\mathrm{ss}}$ and assume that $m(x,\lambda) = 0$. We can find $\mu\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \mu(t)y$ exists and $m(y,\mu) > 0$. By conjugating $\lambda$ and $\mu$ inside their parabolic subgroups, we may assume that $\lambda$ and $\mu$ commute. Then $\lim_{t\to 0} \lambda^n\mu(t)x$ exists for big enough $n$ and $m(x, \lambda^n\mu) > 0$, a contradiction with semistability of $x$.

If $x$ is polystable, then the stabiliser $G_x$ is reductive. Thus it has dimension 0 if and only if it has no nontrivial one-parameter subgroups. This proves the stable case. $\square$




## 3 Examples

We assume that $k$ is an algebraically closed field of characteristic 0.

### 3.1 Relation to convex geometry

Let $T = \mathbb{G}_m^l$ be a torus acting linearly on a finite dimensional vector space $V$, and let $x \in \mathbb{P}(V)(k)$ and $x^* \in V \setminus \{0\}$ a lift of $x$ to $V$. We consider the linearlisation $\mathcal{O}(1)$ on $\mathbb{P}(V)$. We choose coordinates on which $T$ acts diagonally by characters $\alpha_1, \ldots, \alpha_n\colon T \to \mathbb{G}_m$. If $x^* = (x_1, \ldots, x_n)$, then $tx^* = (\alpha_1(t)x_1, \ldots, \alpha_n(t)x_n)$.

Let $\Xi = \{\alpha_i \mid\ x_i \neq 0\} \subset \Gamma_{\mathbb{Z}}(T) \subset \Gamma_{\mathbb{R}}(T) := \Gamma_{\mathbb{Z}}(T) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\Gamma_{\mathbb{Z}}(T)$ is the group of characters of $T$. For $\lambda\colon \mathbb{G}_m \to T$, we have that $m(x,\lambda) = \min_{\alpha\in\Xi}\langle \lambda, \alpha\rangle$. Therefore, $x$ is semistable if and only if 0 is in the convex hull $\mathrm{conv}(\Xi)$ of $\Xi$. We can also check that $x$ is polystable if and only if 0 is in the relative interior of $\mathrm{conv}(\Xi)$ (relative to the vector subspace of $\Gamma_{\mathbb{R}}(T)$ generated by $\Xi$) and that $x$ is stable if and only if 0 is in the interior of $\mathrm{conv}(\Xi)$.

More generally, if $T$ is a maximal torus of a reductive group $G$ acting on $V$, then $x$ is semistable for the action of $G$ if and only if $gx$ is semistable for the action of $T$ for every $g \in G(k)$.

### 3.2 Tuples of points in the projective line

Consider the action of $\mathrm{SL}(2)$ on $(\mathbb{P}^1)^n$ with the linearisation $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\lambda\colon \mathbb{G}_m \to \mathrm{SL}(2)\colon t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$, which is a maximal torus of $\mathrm{SL}(2)$.

Assume first that $n = 1$ and take $z = [a, b] \in \mathbb{P}^1(k)$. We have that

$$\lim_{t \to 0} \lambda(t) z = \lim_{t \to 0} [ta, t^{-1}b] = \begin{cases} 0, & z \neq \infty \\ \infty, & z = \infty, \end{cases}$$

where $\infty = [1, 0]$ and $0 = [0, 1] \in \mathbb{P}^1(k)$. We thus have

$$m(z, \lambda) = \begin{cases} 1, & z = \infty, \\ -1, & z \neq \infty \end{cases}$$

and

$$m(z, \lambda^{-1}) = \begin{cases} 1, & z = 0 \\ -1, & z \neq 0. \end{cases}$$

For general $n$, let $(z_1, \ldots, z_n) \in (\mathbb{P}^1)^n(k)$ and denote $k = \#\{i \mid z_i = \infty\}$ and $l = \#\{i \mid z_i = 0\}$ we have

$$m((z_1, \ldots, z_n), \lambda) = \sum_i m(z_i, \lambda) = \#\{i \mid z_i = \infty\} - \#\{i \mid z_i \neq \infty\} = 2k - n$$

and

$$m((z_1, \ldots, z_n), \lambda^{-1}) = \sum_i m(z_i, \lambda^{-1}) = 2l - n.$$

Thus $(z_1, \ldots, z_n)$ is semistable for $\lambda(\mathbb{G}_m)$ if and only if $k, l \leq n/2$. We have that $(z_1, \ldots, z_n)$ is semistable for SL(2) if and only if $g(z_1, \ldots, z_n)$ for all $g \in \mathrm{SL}(2)(k)$. Since the action of SL(2) on $\mathbb{P}^1$ is 3-transitive, this happens if and only if no more than $n/2$ of the $z_i$ are equal. If $n$ is odd, any semistable point is stable. If $n$ is even, the strictly semistable points have half of the $z_i$ equal to one point of $\mathbb{P}^1$ and the other half equal to a different point of $\mathbb{P}^1$. The stabiliser of such points is $\mathbb{G}_m$ which is reductive. Hence all strictly semistable points are polystable.

### 3.3 Grassmannians as an affine GIT quotient

Let $V$ and $W$ be two finite-dimensional vector spaces, and conside the obvious action of $\mathrm{GL}(V)$ on $\mathrm{Hom}(V, W)$. We choose the linearisation of the action corresponding to the character $\det\colon \mathrm{GL}(V) \to \mathbb{G}_m$. Choose some coordinates on $V$ and $W$ so that we write elements of $\mathrm{Hom}(V, W)$ with matrix notation. Let $T = \{\mathrm{diag}(t_1, \ldots, t_n)\}$ be the maximal torus corresponding to the choice of coordinates.

Let $x = (a_{ij}) \in \mathrm{Hom}(V, W)$. For $\lambda\colon \mathbb{G}_m \to T$, write $\lambda(t) = \mathrm{diag}(t^{m_1}, \ldots, t^{m_n})$, where $n$ is the dimension of $V$. We have $\lambda(t)x = (t^{m_i} a_{ij})$, and the limit when $t$ tends to 0 exists if and only if we have $m_i \geq 0$ whenever the $i$th row of $x$ is not 0. In that case, the Hilbert-Mumford weight is $m(\lambda, x) = (\det^{-1}, \lambda) = -\sum_i m_i$. Thus $x$ is semistable for the action of $T$ if and only if every row of $x$ is nonzero. In that case, $x$ is stable. The point $x$ is semistable for the action of $\mathrm{GL}(V)$ if and only if each $gx$ is semistable for the action of $T$, for all $g \in \mathrm{GL}(V)(k)$. This happens precisely when $x$ is full rank, and in that case the stabiliser group $G_x$ is trivial.

Hence the GIT quotient $\mathrm{Hom}(V, W)^{\mathrm{ss}} /\!\!/ \mathrm{GL}(V)$ is isomorphic to the Grassmannian $\mathrm{Gr}_{n,W}$ of $n$-dimensional subspaces of $W$. Since all stabilisers are trivial, the GIT quotient $\mathrm{Hom}(V, W)^{\mathrm{ss}} /\!\!/ \mathrm{GL}(V)$ equals the stack quotient $\mathrm{Hom}(V, W)^{\mathrm{ss}} / \mathrm{GL}(V)$.