Andrew Graham. Week 4

Quasi modular forms

- Part I : definitions
- Part II : why number theorists care about them.

$\mathbb{H} = \{ \tau \in \mathbb{C} : \operatorname{Im}\tau > 0 \}$, $\tau = x+iy$

$\Gamma \subseteq SL_2(\mathbb{Z})$ finite index.

Definition An almost holomorphic modular form is $f: \mathbb{H} \to \mathbb{C}$ of weight k, level Γ, order $\leq r$

- f is real analytic + no poles at cusps
- $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
- $f(\tau) = \sum_{i=0}^{r} f_i(\tau)\, y^{-i}$ $\quad y = \operatorname{Im}\tau$

$f_i(\tau)$: holomorphic.

$r=0$: usual modular forms.

<u>Definition</u> A <u>quasi-modular form</u> is f_0 for some nearly holomorphic form f

[The holomorphic part of f.]

— This uniquely determines f.

— In Number Theory often work with nearly holomorphic quasi modular forms, but in literature often see

$N_k^r(\Gamma)$: vector space of weight k, level Γ order $\leq r$ f.

$$N_k(\Gamma) = \bigcup_{r \geq 0} N_k^r(\Gamma)$$

$$N(\Gamma) = \bigoplus_{k \in \mathbb{Z}} N_k(\Gamma)$$

Algebra: closed under multiplication.

Examples.

(1) Holomorphic modular forms $(r=0)$.

$$M_k(\Gamma) = N_k^0(\Gamma)$$ space of modular forms.

— Classical.

(2) Two important operators act on nearly holomorphic forms:

(i) Maass – Shimura operator δ_k

Different notations

$$\delta_k f = \delta f = Rf = R_k f$$
$$:= \frac{1}{2\pi i}\left(\frac{d}{d\tau} + \frac{k}{\tau-\bar\tau}\right) f \in N_{k+2}^{r+1}(\Gamma)$$

when f has weight k.

– Weight-raising operator.

— This takes modular forms → quasi modular forms

Weight lowering operator:

$$L_k f = Lf = -2iy^2 \frac{\partial}{\partial \bar\tau} f \in N_{k-2}^{r-1}(\Gamma)$$

(modular forms: the kernel of L_k).

③ Eisenstein series. Let $N \geq 1$,

ε Dirichlet character modulo N

i.e. $\varepsilon : \left(\mathbb{Z}/N\mathbb{Z}\right)^\times \to \mathbb{C}$ character.

extend by zero to $\mathbb{Z}/N\mathbb{Z}$

$\varepsilon : \mathbb{Z} \to \mathbb{C}$ lift to $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/N\mathbb{Z}$.

$\bar\varepsilon(n)$ inverse Dirichlet

$k \geq 1$. $\in \mathbb{C}$

$$E_{k,\varepsilon}(\tau; s) = \frac{\Gamma(s+\frac{k}{2})}{(-2\pi i)^k\, \pi^{s-k/2}} \sum_{\substack{(m,n) \in N\mathbb{Z}\times\mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{\bar\varepsilon(n)\, y^{s-k/2}}{(m\tau+n)^k\, |m\tau+n|^{2s-k}}$$

— Converges for $\operatorname{Re} s \gg 0$. Extends to $s \in \mathbb{C}$ by analytic continuation. Might have poles.

$0 \leq j \leq k-1$

Thm: $E_{k,\varepsilon}\left(\tau; \frac{k}{2} - j\right) \in N_k(\Gamma_1(N))$

$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) :$

$c \equiv 0 \mod N$

$d \equiv 1 \mod N.$

e.g. $N=1$, $\varepsilon = 1$, $k \geq 4$

$E_{k,1}(\tau; k/2) =$ usual weight k Eisenstein series.

holomorphic, modular

$\Gamma = SL(2,\mathbb{Z})$.

$N=1$, $\varepsilon=1$, $k=2$

$E_{2,\varepsilon}(\tau; 1) = E_2$ "weight 2 Eisenstein series"

Nearly holomorphic of order $= 1$.

$$\varepsilon : \frac{\Gamma_0(N)}{\Gamma_1(N)} \longrightarrow \mathbb{C}^*$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \varepsilon(d)$$

As well as being $\Gamma_1(N)$-modular, satisfies a more general transformation property wrt $\Gamma_0(N)$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 \bmod N \right\}.$$

<u>Properties:</u> $N(SL(2,\mathbb{Z}))$ or $N(\Gamma_1(N))$ have 'polynomial' q-expansions:

$$q = e^{2\pi i \tau} \qquad f(q, X) = \sum_{i, n=0}^{\infty} a_{n,i} q^n X^i$$

cuspidal if $a_{0,i} = 0 \ \forall i$

$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $x \in \mathbb{Z}$ lies in $\Gamma_1(N), \Gamma_0(N)$ <u>ie</u> zero if $q = 0$.

$$\in \mathbb{C}[[q]][X].$$

$$X = \frac{1}{y} = \frac{1}{\operatorname{Im}\tau}$$

polynomial in X, coefficients q-expansions.

Structure Theorem:

$N(\Gamma)$ is generated as an algebra by images E_2 and $M(\Gamma)$ under iterates of R.

— for any Γ finite index in $SL(2,\mathbb{Z})$.

All as a vector space, standard

E_2: never in image of R.

An algebra: $N(SL_2(\mathbb{Z}))$ gen by E_2, E_4, E_6.

Given $f \in S_k(\Gamma)$ (cuspidal hol. modular form.)

$k \geq 2$

vanishes at all cusps (finitely many)

Cusp: compactification point for $\mathbb{H}/\Gamma$.

$$\Rightarrow \quad \Phi_f(g) = (ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right)$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

This is an
Automorphic form

$$\phi_f \in \mathcal{A}(\Gamma) = \left\{ \phi : \frac{SL_2(\mathbb{R})}{\Gamma} \to \mathbb{C} : \right.$$

C^∞ (real analytic)

K_∞-finite, Z-finite

slowly increasing

$(\mathfrak{sl}_2, K_\infty)$-module.

center of universal algebra of $\mathfrak{sl}(2, \mathbb{R})$

$\}$

Subrepresentation generated by ϕ_f inside $\mathcal{A}(\Gamma)$ can be described using nearly holomorphic forms

$$R \longleftarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}) \qquad L \longleftarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

Automorphic forms very important for study of L-functions.

Algebraic interpretation: Algebraic curve.

$Y = Y_1(N)$ modular curve $/ \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$

smooth, quasiprojective 1-dim'l scheme.

$$Y(\mathbb{C}) = \Gamma_1(N) \backslash \mathbb{H}.$$

$\boxed{\text{Fix } \Gamma = \Gamma_1(N)}$

$Y(S)$

$S \to \operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ parametrizes pairs (E, P)

E elliptic curve with origin, scheme.

$P \in E(N)(S)$ order exactly N $\qquad E(N) = \ker(E \xrightarrow{\times N} E)$

size N^2.

$Y(S) \quad S \to$

$E/\mathbb{C}, \quad E = \mathbb{C}/\Lambda \qquad E(N)_{\mathbb{C}} = \dfrac{N^{-1}\Lambda}{\Lambda} \subseteq \mathbb{C}/\Lambda$

$\pi : \mathcal{E} \to Y$ universal elliptic curve.

$\omega_{\mathcal{E}} = \pi_* \Omega^1_{\mathcal{E}/Y}. \qquad \mathcal{H}_{\mathcal{E}} = H^1_{dR}(\mathcal{E}/Y)$

line bundle on Y $\qquad$ vector bundle of rank 2 on Y

$$0 \to \omega_E \to H_E \to \omega_E^{-1} \to 0$$ Hodge filtration.

real analytic splitting involves $\frac{1}{\tau - \bar\tau}$.

How to interpret nearly holomorphic forms:

$$N_k^r(\Gamma, M) \subseteq H^0\left(Y(\Gamma)_{\mathbb{C}}, \omega_E^{k-r} \otimes \mathrm{Sym}^r H_E\right)$$

$$\uparrow$$

algebraic interpretation as module over $\mathbb{Z}\left(\frac{1}{N}\right)$-module.

$r = 0$: ordinary holomorphic modular forms

To go from section of vector bundle on r.h.s. apply splitting of exact sequence above, involves $\frac{1}{\tau - \bar\tau}$, only exists in non-holomorphic world.

— Apply splitting to get a <u>non-holomorphic</u> section of line bundle ω_E^{k+2r}

$$\omega_\Sigma^{k+r} \otimes \mathrm{Sym}^r \mathcal{H}_\Sigma$$

$$\Big\downarrow \quad \text{splitting, non-holomorphic.}$$

$$\omega_\Sigma^{k+2r}$$

δ has a algebraic interpretation via Gauss-Manin connection:

$\mathcal{H}_\Sigma$ carries a connection

$$\nabla : \mathcal{H}_\Sigma \to \mathcal{H}_\Sigma \otimes \Omega^1_Y$$

<u>Note:</u> This holds over $\mathbb{Z}$, not just over $\mathbb{C}$, so also over any ring / field lying between $\mathbb{Z}$ and $\mathbb{C}$.

Special values of L-functions

Why do Number Theorists care?

{ Special values of L-functions } —— transcendental numbers / Automorphic periods ——→ { algebraicity properties }

- BSD Conjecture
- Bloch-Kato

Beilinson Conjectures

— uses nearly holomorphic forms

— including: algebraic interpretation.

p-adic congruences

{ Special arithmetic quantities

e.g. class group of a number field

e.g. rational points of an elliptic curve } ←——→ { p-adic L-function }

arithmetic both sides

— Aim is to prove by going around square

— Nearly holomorphic forms end up being an ingredient in this strategy.

Go back to Eisenstein series. $K \hookrightarrow \mathbb{C}$ fixed embedding.

$K/\mathbb{Q}$ imaginary quadratic. Class number 1.

$K = \mathbb{Q}(\sqrt{-D})$ $D >> 0$ N given

assume every prime dividing N splits in $K/\mathbb{Q}$

(Heegner hypothesis)

$\Rightarrow \exists \ \underset{\text{ideal}}{\mathfrak{n}} \subset \mathcal{O}_K$, $\mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$.

p prime, $p \nmid N$, splits in $K/\mathbb{Q}$.

Fix ε Dirichlet character modulo N.

— Think of it as a character of $\mathcal{O}_K$.

Attached to ε, consider the L series

$$L(\varepsilon, k_1, k_2) = \sum_{0 \neq (\alpha) \subset \mathcal{O}_K} \frac{\overline{\varepsilon}(\alpha)}{\alpha^{k_1} \overline{\alpha}^{k_2}}$$

converges $k_1 + k_2 > 2$

complex conjugate

Analytically continue to other values.

nonzero principal ideals

$k_1, k_2 \in \mathbb{C}$.

sum in $\mathbb{C}$: $L(\varepsilon, k_1, k_2) \in \mathbb{C}$.

$$L(\varepsilon, s) = \frac{1}{2} L(\varepsilon, s, s)$$

Interested in values when plug in $k_1, k_2 \in \mathbb{Z}$.

$j \geq 0$ integer

$$L(\varepsilon, 2+j, -j) \simeq \underbrace{\left(\partial^j E_{2,\varepsilon}(-;1)\right)}_{\text{algebraic number}}(\tau_n)$$

section of a line bundle.

r.h.s. can be made algebraic

particular point $\in \mathbb{H}$.

– The value at a point lies in a number field.

$\tau_n \in \mathbb{H} \cap K$ CM point

$\mathbb{Z} + \mathbb{Z}\tau_n = \overline{n}^{-1}$.

corresponds to elliptic curve E with complex multiplication

$$E \xrightarrow{N:1} E$$

$\mathbb{C}/\Lambda \qquad \mathbb{C}/\Lambda$

$\times \overline{n}^{-1}$

elt of K multiplies $\Lambda \to \Lambda$.

Period $\Omega \in \mathbb{C}^\times$ indep of j

$$\frac{L(\xi, 1+j, -j)}{\Omega}$$ algebraic integers.

$$\Uparrow$$

$L^{alg}(\xi, 1+j, j)$

CM elliptic curves

$$T_n \in \mathcal{Y}(N)(\mathbb{C})$$

$$\Uparrow$$

$$\mathcal{Y}_1(N)(K(N))$$

$K(N)$ ray class field of conductor N.

$|$ class field

K

— Once you have the algebraicity properties, can study congruences.

p-adic Kronecker limit formula (Katz)

$$\lim_{j \to -1} L^{alg}(\xi, 1+j, -j)$$

cannot plug $j=-1$ in. but can do it p-adically.

p-adically. $\simeq \log_p(u_\xi)$

p-adic logarithm

unit in $u_\xi \in \mathcal{O}_{K(N)}^\times$ elliptic unit.

= value of modular fn.

e) $p^n(p-1)-1$

$g_\xi \in \mathcal{O}(\mathcal{Y}, W)^\wedge$

at CM point τ_A.

— Need p-adic congruences between nearly holomorphic forms.

Relation, L-series $\longleftarrow$ elliptic units

Led to first proved cases of BSD conjecture.

Theorem (Coates–Wiles)

E/K CM elliptic curve.

crucial assumption.

$L(E/K, 1) \neq 0 \Rightarrow E(K)$ are finite.

$\uparrow$

K-rational points of E.

— Can generalize to Shimura varieties

L-fns for automorphic reps of unitary groups.