

Recap (from Dominic Joyce's lecture) ①

Let X : smooth, projective surface

fix $\bullet r \in \mathbb{Z}_{>0}$

$\bullet \alpha \in H^2(X, \mathbb{Z})$

$\bullet k \in \mathbb{Z}$

$\bullet L$: line bundle on X with

$$c_1(L) = \alpha$$

$\bullet \mathcal{O}_X(1)$: ample line bundle on X

$$M^{ss}(r, L, k) = \left\{ (E, \phi) : \begin{array}{l} E \text{ is a torsion-free sheaf on } X \text{ with } \underline{\det(E)} = L \\ c_2(E) = k \text{ and} \\ \bullet \phi \in \text{Hom}(E, E \otimes k_X) \text{ s.t. } \text{Tr } \phi = 0 \\ \bullet (E, \phi) \text{ is } \underline{\text{Gieseker semistable w.r.t. } \mathcal{O}_X(1)} \end{array} \right\}$$

$\bullet$ If E were a vector bundle
 $\det E = \wedge^r E$

$\bullet$ A coherent sheaf E on a smooth, projective surface X admits a resolution locally free

$$0 \leftarrow v_2 \leftarrow v_1 \leftarrow v_0 \rightarrow E \leftarrow 0 \quad (2)$$

$$\det E = \bigotimes_{i=0}^2 \left(\wedge^{\text{rk}(v_i)} v_i \right) (-1)^{i+1}$$

Diff. geo $\leadsto$ for any compact Lie group G , there should be VN m.s.

$\hookrightarrow$ for, algebra-geometrically,

- $G = U(r) \leadsto$ no need the data of L

$$M_{(r, \alpha, \beta)}^{ss} = (E, \Phi) \dots$$

$$G(E) = \sigma$$

- $G = SU(r)$: most standard

$$\text{fix } L = \partial_X$$

$\sim$ If $L \neq \partial_X$ (naïve guess) is to get VN $\#$ for $SU(r)$ / finite group?

$$L = \mathbb{C}X \Rightarrow SU(r) / \mathbb{Z}/r\mathbb{Z}$$

(E, ϕ) is Gieseker (semi)-stable ②
 if for every proper, ϕ -invariant
 subsheaf $F \subsetneq E$ on X , we have: $(\phi(F) \hookrightarrow F \otimes \mathcal{O}_X)$

$$\boxed{\frac{\chi(F(n))}{\text{rank}(F)} \leq \frac{\chi(E(n))}{\text{rank}(E)}} \quad \left(\begin{array}{l} \text{Eqn 2.8} \\ \text{r-1} \end{array} \right)$$

Here, $\bullet E(n) = E \otimes \underbrace{\mathcal{O}_X(n)}_{\mathcal{O}_X(1)^{\otimes n}}$, $F(n) = F \otimes \mathcal{O}_X(n)$

$$\bullet \chi(E) = \sum_i (-1)^i \cdot \dim H^i(X, E)$$

$$\bullet \chi(E(n)) = \text{rank}(E) \cdot \frac{n^2}{2} + \deg(E) \cdot n + \chi(E)$$

degree 2 polynomial in n
"Hilbert polynomial"

can be written
 as a func. of
 rank, degree, $\chi(E)$
 by Riemann-Roch

... Gieseker (semi) stability

(4)

$$\underbrace{\frac{n^2}{2} + \frac{\deg F}{\text{rank } F} \cdot n + \frac{\chi(F)}{\text{rank } F}}_I \leq \underbrace{\frac{n^2}{2} + \frac{\deg E}{\text{rank } E} \cdot n + \frac{\chi(E)}{\text{rank } E}}_{II}$$

This holds either if

① $\frac{\deg F}{\text{rank } F} < \frac{\deg E}{\text{rank } E}$ ($\Rightarrow II > I$ for $n \gg 0$)
 $\Rightarrow$ Gieseker stable)

② $\frac{\deg F}{\text{rank } F} = \frac{\deg E}{\text{rank } E} \geq \frac{\chi(F)}{\text{rank}(F)} \leq \frac{\chi(E)}{\text{rank}(E)}$

Recall from Dominic's talk:

(E, ϕ) is slope (semi) stable if

$$\text{slope}(F) = \frac{\deg F}{\text{rank } F} \leq \frac{\deg E}{\text{rank } E} = \text{slope } E$$

Hence;

slope stable $\Rightarrow$ Gieseker stable

$\Downarrow$

Gieseker ss $\Rightarrow$ Slope ss

DDJ's talk:

(5)

$$\mathbb{C}^* \begin{array}{c} \searrow \\ \downarrow_{ss} \\ M_{(r,L|k)} \end{array} \quad (E, \emptyset) \xrightarrow{\substack{\in \mathbb{C}^*}} (E, +\emptyset)$$

$M_{(r,L|k)} \leftarrow \text{non-compact}$

$$(M_{(r,L|k)}^{ss})^{\mathbb{C}^*} \leftarrow \text{compact}$$

$\hookrightarrow$ Carries a 2 term P.O.T. $E(-1) \rightarrow E(0)$

(for $\beta_g > 0$: $\exists$ a "reduced" POT
(remove bad piece in $E(-1)$)

$$VW_{(r,\alpha|k)} := \int \frac{1}{e^{LW_{vir}}} \in \mathbb{Q}$$

$$\left[(M_{(r,L|k)}^{ss})^{\mathbb{C}^*} \right]^{vir}$$

$$M_{(r,L|k)}^{ss} = M_{(r,L|k)}^{inst}$$

$$\downarrow$$

$$\emptyset = \emptyset$$

$$\coprod \left\{ M_{(r,L|k)}^{monop.} \right.$$

$$\downarrow$$

$$\emptyset \neq k$$

⑥

• We assume (T-T § 8) :

• X has $pg > 0$

• $r=2$

• $L = K_X$

① $\dim H^1(X, \mathcal{O}_X) = 0$

② $\exists$ smooth, connected curve

$0 \neq C \in |K_X|$

③ $L = \mathcal{O}_X$ is the only line bundle

with $0 \leq \deg L \leq \frac{1}{2} \deg K_X$

where $\deg L = c_1(L) \cdot c_1(\mathcal{O}_X(1))$

Examples:

1) Generic quintic surface in $\mathbb{P}^3$

2) Generic double cover of $\mathbb{P}^2$ branched over an octic curve.

3) General type surfaces with $\text{Pic}(X) = \mathbb{Z} \cdot K_X$

4) Blow-up of $K3$ at a point.

* If X : deg 5 hypersurface in $\mathbb{P}^3$ (7)

1) • Lefschetz hyperplane thm $\Rightarrow H^1(X, \mathcal{O}_X) = 0$

2) • Adjunction formula: $X \subset \mathbb{P}^3$

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^3}|_X \rightarrow N_{X/\mathbb{P}^3} \rightarrow 0$$

Dualize:

$$0 \rightarrow N_{X/\mathbb{P}^3}^\vee \rightarrow T_{\mathbb{P}^3}|_X^\vee \rightarrow T_X^\vee \rightarrow 0$$

$$\Rightarrow \det(T_{\mathbb{P}^3}|_X^\vee) = \det(N_{X/\mathbb{P}^3}^\vee) \otimes \det(T_X^\vee)$$

$$\parallel$$

$$K_{\mathbb{P}^3}|_X$$

$$\parallel \begin{matrix} \uparrow \text{line bundle} \\ \text{X: surface} \end{matrix}$$

$$N_{X/\mathbb{P}^3}^\vee$$

$$\parallel$$

$$K_X$$

Tensor both sides with $N_{X/\mathbb{P}^3}$

$\downarrow$

$$K_X = K_{\mathbb{P}^3}|_X \otimes N_{X/\mathbb{P}^3}$$

$$\parallel \qquad \qquad \parallel$$

$$\mathcal{O}(-4) \qquad \mathcal{O}(5)$$

$$\Rightarrow K_X = \mathcal{O}_{\mathbb{P}^2}(1)|_X \rightarrow \text{very ample} \textcircled{2}$$

Bertini's Thm: A generic section of K_X is a smooth, connected $\deg 5$ curve in $\mathbb{P}^2$

3) Lefschetz - Noether Thm: $X \subset \mathbb{P}^3$

Every line bundle on X is a $\deg=0$ $\deg=5$ restriction of a line bundle $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots)$ on $\mathbb{P}^3$.

So, $X \subset \mathbb{P}^3$ satisfies all assumptions

① W.B. Under all these assumptions
 we have : Helps classifying how
sheaves look like

Gieseker ss $\Rightarrow$ Slope stable
 (hence Gieseker stable)

Let $F \subseteq E$ a ϕ -invariant subsheaf
 Then, either:

① If $\deg(\det(F)) < 0$:

By assumption : $\det E = L \cdot K_X$
 $\Rightarrow \deg(E) = \deg K_X > 0$

By assumption $\exists$
 conn. curve in $|K_X|$

$$\Rightarrow \boxed{\deg F < \frac{\deg E}{2}} \sim \text{slope stable}$$

② If $\deg(\det F) \geq 0$:

(E, ϕ) : Gieseker ss $\Rightarrow$ Slope ss $\Rightarrow \deg F \leq \frac{\deg E}{2}$

By assumption

$$0 \leq \deg F \leq \frac{\deg K_X}{2} = F = \mathcal{O}_X \Rightarrow \deg F = 0$$

$$\Rightarrow \deg F < \frac{\deg E}{2}$$

Goal: Understand concretely the monodromy
 branch contribution to $V(N)$ # in terms of
 "Nested Hilbert schemes of points"

Let $(E, \phi) \in \mathcal{M}_{(r, LK)}^{\mathbb{C}^*}$

for $t \in \mathbb{C}^*$: $(E, \phi) \simeq (E, t\phi)$

$$\begin{array}{ccc}
 E & \xrightarrow{\psi_t} & E \\
 \downarrow \phi & & \downarrow t\phi \\
 E \otimes K_X & \xrightarrow{\psi_t \otimes \text{Id}_{K_X}} & E \otimes K_X
 \end{array}$$

In particular,

$$\mathbb{C}^* \curvearrowright E$$

$$\begin{array}{ccc}
 \mathbb{C}^* & \longrightarrow & \text{Aut}(E) \\
 t & \longmapsto & \psi_t
 \end{array}$$

- Any representation of $\mathbb{C}^*$ decomposes into weight spaces:

$$\boxed{E = \bigoplus_{i \in \mathbb{Z}} E_i} \quad \sim$$

where E_i : weight i space
 (+ acts by multiplication by t^i)

Claim (§7.3 2-1)

$$\phi|_{E_i}: E_i \rightarrow E_{i-1} \otimes K_x$$

Pf: Let $x \in E_i$ be a local section of E_i
 Want to show $\phi(x)$ is a local section of E_{i-1} .

$$\text{i.e. } \underbrace{\psi_t(\phi(x))}_{\text{want}} = t^{i-1} \cdot \phi(x)$$

$$\parallel$$

$$t \cdot \phi(\psi_t(x))$$

$$\parallel$$

$$t \cdot \phi(tx) = t^{i+1} \phi(x)$$

Fix by saying E_i : weight $-i$ space

Want: $\phi \neq 0$ (monopole brack) (2)
 • Assume $E = E_i \oplus E_j$, for $i \neq j$
 with $\text{rank } E_i = \text{rank } E_j = 1$

$\phi \neq 0$: w.l.o.g. assume $i > j$
 $j = i - 1$

$$\phi: E_i \rightarrow E_{i-1} \otimes K_X.$$

Q: What are E_{i-1} , E_i ?
 from now on: $\begin{array}{ccc} & \downarrow & \downarrow \\ & E_{-1} & E_0 \end{array}$

Lemma: $\det(E_{-1}) = \mathcal{O}_X$

① $E_{-1} \subset E \hookrightarrow \text{slope stable}$
 $\hookrightarrow \phi$ -invariant since
 $\phi: E_{-1} \rightarrow 0$

$$\text{slope}(E_{-1}) < \text{slope}(E)$$

$$\parallel$$

$$\deg(E_{-1}) < \frac{\deg E}{2} = \frac{\deg K_X}{2}$$

(12)

$$② \quad \deg(E_{-1}) \geq 0:$$

$$0 \neq \phi : E_0 \rightarrow E_{-1} \otimes K_X$$

$$0 \neq \det \phi : \det(E_0) \rightarrow \det(E_{-1}) \otimes K_X$$

$\Rightarrow \exists$ a non-zero map

$$\mathcal{O}_X \rightarrow \det(E_0)^{-1} \otimes \det(E_{-1}) \otimes K_X$$

line bd. with a nonzero section
needs to have non-negative
degree

$$\Rightarrow \deg(E_{-1}) + \deg(K_X) - \deg(E_0) \geq 0$$

$$\Rightarrow \deg(K_X) + \deg(E_{-1}) \geq \deg(E_0)$$

But $E = E_0 \oplus E_{-1} \Rightarrow \deg E = \deg E_0 + \deg E_{-1}$

$\parallel$
 $\deg(K_X)$

$$\Rightarrow \deg(E_0) = \deg K_X - \deg E_{-1}$$

$\parallel$
 $\deg(\det(E_0))$

(14)

$$\Rightarrow 0 \leq \deg(\det(E_{-1})) \stackrel{\text{slope stability}}{\leq} \frac{\deg K_X}{2}$$

$\Rightarrow$ By assumption @:

$$\underline{\det(E_{-1}) = \mathcal{O}_X}$$

Cor: $\det(E_0) = K_X$

Since $E = E_0 \oplus E_{-1}$

$$\det(E) \underset{K_X}{=} \det E_0 \otimes \underbrace{\det(E_{-1})}_{\mathcal{O}_X}$$

$$\therefore \boxed{\begin{array}{l} \det(E_{-1}) = \mathcal{O}_X \\ \det(E_0) = K_X \end{array}}$$

N.B. On a complex surface any sheaf ⁽¹⁵⁾
 which is $\left. \begin{array}{l} \bullet \text{ rank} = 1 \\ \bullet \text{ torsion-free} \\ \bullet \text{ det} = 0 \end{array} \right\}$ is the ideal sheaf
 of a 0 dim'l subscheme.

why? because?

① A torsion free sheaf $\mathcal{F}$ embeds
 into its double dual:

$$\mathcal{F} \hookrightarrow (\mathcal{F}^\vee)^\vee$$

$$\mathcal{F}^\vee = \text{Hom}(\mathcal{F}, \mathcal{O}_X)$$

$$\textcircled{2} \mathcal{F}: \text{rank } 1 \Rightarrow (\mathcal{F}^\vee)^\vee = \text{det } \mathcal{F}$$

since $(\mathcal{F}^\vee)^\vee$ is a reflexive sheaf
 on a surface it needs to be $\text{det } \mathcal{F}$

③ $\text{det } \mathcal{F} = \mathcal{O}_X$ by assumption

we have a SES

$$\therefore 0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X / \mathcal{F} \rightarrow 0$$

$$\left. \begin{array}{l} X \text{ surface} \\ \text{det } \mathcal{F} = 0 \end{array} \right\} \Rightarrow \overset{15}{0} \cong: 0 \text{ dim'l.}$$

(16)

$$0 \rightarrow I_Z^{\leftarrow \text{ideal sheaf of } Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

$$\therefore E_{-1} = I_{Z_1} \text{ for } Z_1 \text{ for } Z_1 \subset X$$

0 dim'l subscheme

$$\det(E_0) = \mathcal{O}_X$$

$$\bullet E_0 \otimes \mathcal{O}_X^{-1} \text{ satisfies } (*)$$

$$\Rightarrow E_0 = \mathcal{O}_X \otimes I_{Z_0} \text{ where}$$

$Z_0 \subset X$ is a 0 dim'l subscheme.

$$\begin{array}{ccc} \varphi: E_0 & \xrightarrow{\text{injective as } I_{Z_0} \text{ is torsion-free}} & E_{-1} \otimes \mathcal{O}_X \\ \parallel & & \parallel \\ I_{Z_0} \otimes \mathcal{O}_X & \hookrightarrow & I_{Z_1} \otimes \mathcal{O}_X \end{array}$$

$$\Rightarrow I_{Z_0} \hookrightarrow I_{Z_1}$$

$$\Rightarrow Z_1 \subset Z_0$$

Q: What's the number of points in z_{-1}, z_0 ?

Grothendieck Riemann-Roch
 $\chi(E) = \int_X c_1(E) \cdot \text{ Todd}(X)$
 $\searrow E = \mathcal{O}_{z_1}$

$$c_2(I_{z_{-1}}) = |z_{-1}|$$

$$c_2(I_{z_0}) = |z_0|$$

$$\parallel E_0 = I_{z_0} \otimes \mathcal{O}_X$$

$$c_2(E_0)$$

Since $E = E_0 \tau E_{-1}$ by the Whitney formula for Chern classes

$$c_2(E) = c_2(E_0) + c_2(E_{-1}) + \underbrace{c_1(E_0) \cdot c_1(E_{-1})}_{=0}$$

since $\det(E_{-1}) = \mathcal{O}_X$
 $c_1(\mathcal{O}_X) = 0$

$$\therefore \boxed{c_2(E) = |z_0| + |z_{-1}|}$$

$\parallel$
 K

⑧

THM ($\leftarrow$ Lemma 8.3 T-T)

$$M_{2, k, k}^{\neq 0 \text{ mod } p} = \prod_{i=0}^{\lfloor k/2 \rfloor} \chi^{(i, k-i)}$$

where $\chi^{(i, k-i)} \rightsquigarrow$ nested Hilbert scheme

is the moduli space of pairs (z_0, z_1)

with $z_0 \subset X$: 0 dim'l subscheme of X
of length $k-1$

$z_1 \subset X$: 0 dim'l subscheme of X
of length i

$$z_1 \subset z_0$$

Eqn 8.39 T-T: $\left(\begin{smallmatrix} \text{up to} \\ \deg=3 \end{smallmatrix} \right) \left\{ \begin{array}{l} v = p+1 \\ g = \text{arith. genus of } C \in |kx| \end{array} \right.$

$$\sum_{k \in \mathbb{Z}} V W_{2, k, k}^{\neq 0} (X) \cdot q^k = \rightarrow$$

$$(-2)^{v-g+1} \left[\begin{array}{l} 1 - 2(g-1)q + (2(g-1)(g-3) + 12v)q^2 \\ - \frac{4}{3}(g-1)(18v + g^2 - 8g + 9)q^3 \end{array} \right] + O(4)$$

Let $z_{r,\alpha}^{\text{inst}} = \sum_{k \in \mathbb{Z}} v w_{r,\alpha,k}^{0=0} q^k$ → instanton
basic
partition fun

$z_{r,\alpha}^{\text{mon}} = \sum_{k \in \mathbb{Z}} v w_{r,\alpha,k}^{0 \neq 0} q^k$ → monodromy
branch
partition
function

$z_{r,\alpha} := z_{r,\alpha}^{\text{inst}} + z_{r,\alpha}^{\text{mon}}$ → total
partition fun.

Conjecture 1 $\left\{ \begin{array}{l} \text{Vafa-Witten, Tenebris-Thom} \\ \text{Göttsche-Kool} \end{array} \right.$

$$z_{2,\alpha}^{\text{mon}} = \begin{cases} \left(\frac{1}{4} G(q^2) \right)^{v/2} \frac{1}{(1-q)^v} \left(\frac{q_0}{\eta^2} \right)^{1-g} & \text{if } \alpha \in H_2(X, \mathbb{Z}) / 2H_2(X, \mathbb{Z}) \\ \left(\frac{1}{4} G(q^2) \right)^{v/2} \left(\frac{q_0}{\eta^2} \right)^{1-g} & \text{if } \alpha \in G(kx) \\ 0 & \text{elsewhere} \end{cases}$$

$$G(q) = q^{-1} \prod_{k \geq 1} (1 - q^k)^{-24} \quad (20)$$

$$\frac{1}{\eta(q)} = q^{-1/24} \prod_{k \geq 1} (1 - q^k)^{-1}$$

$$\Theta_0(q) = \sum_{k \in \mathbb{Z}} q^{k^2}$$

$$\Theta_1(q) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{k^2}$$

$\exists$ a similar conjecture for $Z_{2, \alpha}$ inst.
THM (V-W) If Cons (22) holds, then

$$Z_{2, \alpha} \left(-\frac{1}{z} \right) = 2^{\frac{1-\chi(\alpha)}{2}} \cdot (-1)^{\chi(\alpha)} \left(\frac{z}{i} \right)^{-\frac{\chi(\alpha)}{2}} \sum_{\substack{\alpha, \beta \\ \beta \in H^2(X, \mathbb{Z}) / 2H^2(X, \mathbb{Z})}} (-1)^{\alpha \cdot \beta} Z_{2, \beta}(z)$$

$\downarrow 2\pi i T$
 $q = e$