

Geometry Advanced Class TT26

Cohomological Hall Algebras

Week 1. Introduction, by Dominic.

First idea

Let \mathcal{A} be an abelian category over \mathbb{C} , and \mathcal{M} the moduli stack of objects in \mathcal{A} , an Artin \mathbb{C} -stack.

Then we can form the cohomology $H^*(\mathcal{M})$ (or K theory, Borel-Moore homology, ...).

We aim to make $H^*(\mathcal{M})$ into an associative algebra, a cohomological Hall Algebra (CoHA).

Writing exact for the stack of exact sequences $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$,

the multiplication $*$ in $H^*(M)$

$$i) (\pi_2)_* \circ (\pi_1 \otimes \pi_3)^*, \quad \pi_i: \text{Exact} \rightarrow M$$

map sequence to E_i .

This only works for nice categories,
A satisfying condition (needed to
define $(\pi_2)_*$).

There may be other moduli stacks
 N (e.g. moduli stack of framed
objects in A) such that

$H^*(N)$ is a module over $H^*(M)$.

If we are lucky, then $H^*(M)$
is an infinite-dimensional algebra
already understood in Representation Theory,
and then we can use this to understand
 $H^*(M)$ and $H^*(N)$.

Overview of the field

e.g. Schiffmann
"Lectures on
Hall Algebras",
2006.

1990s,
2000s.
E.g. Green,
1995, Hall Algebras
and quantum groups

① Hall Algebra
over finite fields

Kontsevich -
Soibelman
2010
§2-§3.

Latyntsev
2021:
compatible
with vertex
coalgebra
structure.

② CoHAs of 1-dimensional
abelian categories, over \mathbb{C} :
 $\text{coh}(\Sigma_g)$, Σ_g curve,
mod $\mathbb{C}Q$, Q quiver.

Defined on
 $H^*(M)$.

③ CoHAs of
 $\text{coh}_{cs}(S)$, S
complex quasi-
projective surface.

Defined on Bord-
Moore homology $H_*^{Bord}(M)$

Kapranov - Vasserot 2019.

④ CoHAs of a
3-Calabi-Yan category
 $\text{coh}(X)$, X Calabi-Yan
3-fold, or mod $\mathbb{C}Q/W$,
quiver with super-potential
 W . Defined on
critical cohomology
 $H^*(P^m)$.

Kontsevich - Soibelman
2010, §4.

④ Representations of associative algebras $A \simeq H^*(N)$, N moduli stack associated to A , which look like they ought to be representations of a CoHA $H^*(M)$.

— Generally defined using correspondences $i: N \times N$. A is not geometrically defined, i.e. not presented as $A = H^*(M)$, but defined by generators and relations.

— Example: Grojnowski, Nakajima 1994: action of Heisenberg algebra $\simeq H^*(\text{Hilb}^*(S))$, S complex surface. CoHA interpretation by Mellit - Minets - Schiffmann - Vasserot 2023. Lots of work on $H^*(\text{quiver moduli spaces})$ by Nakajima, ...

0. Hall Algebras over finite fields

Reference: Schiffmann, "Lectures on Hall algebras", 2006.

An abelian category \mathcal{A} is finitary if for all $E, F \in \mathcal{A}$, $|\text{Hom}(E, F)|, |\text{Ext}^1(E, F)| < \infty$

An important class of examples is if p is a prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the finite field, and \mathcal{A} is \mathbb{F}_p -linear with $\dim_{\mathbb{F}_p} \text{Hom}(E, F), \dim_{\mathbb{F}_p} \text{Ext}^1(E, F) < \infty$.

Write $\mathcal{H} = \mathbb{Q} \left\{ \begin{array}{l} \text{Iso. classes } [E] \\ \text{of objects } E \in \mathcal{A} \end{array} \right\}$.

Define a \mathbb{Q} -bilinear product $*$ on \mathcal{H} by

$$[E_1] * [E_3] = \sum \frac{1}{|\text{Aut}(E_1)|} \cdot \frac{1}{|\text{Aut}(E_3)|} \cdot [E_2].$$

(extra twisting factor).

iso. class

$$[E_2] \in A$$

{exact sequences}

$$0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0$$

(Hall, in ex. example,
Riyd in general.)

in A ,
i.e. # pairs (α, β) }

This is associative. To prove that

$$([E_1] * [E_2]) * [E_3] = [E_1] * ([E_2] * [E_3]),$$

we consider filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset F_3 = F \text{ in } A$$

with $F_i / F_{i-1} \cong E_i$; essentially,

$$[E_1] * [E_2] * [E_3] = \sum_{(F)} \frac{1}{\prod_{i=1}^3 |\text{Aut}(E_i)|} [F].$$

{filtrations, like this}

Actually, Schiffmann works over \mathbb{C} and includes an extra factor $\prod_{i \geq 0} |\text{Ext}^i(E_j, E_i)|^{(-1)^i}$. Does it change associativity but needed for compatibility with coproducts.

Green 1995: If $\text{Ext}^i(E, F) = 0$

for $i > 1$ (i.e. $\dim A = 1$) then
can define a coproduct $\Delta: H \rightarrow H \hat{\otimes} H$
making A into a bialgebra, roughly by
in fact, a Hopf algebra

$$\Delta(E_2) = \sum \frac{1}{|\text{Aut}(E_2)|} (E_1) \otimes (E_3)$$

. extra twisting factor

iso. classes

$$(E_1), (E_3) \in A \quad \# \{ \text{exact sequences} \}$$
$$0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0$$

in A ,
i.e. $\# \text{ pairs } (\alpha, \beta) \}$

There are interesting examples involving quivers Q
over \mathbb{F}_p in which one can make the
set of isomorphism classes $[E] \in A = \text{mod-}\mathbb{F}_p Q$
independent of p . In fact, the indecomposable

objects (E) correspond to positive roots of a Kac-Moody Lie algebra associated to Q . Then the structure constants

$$C_{(E_1)(E_3)}^{(E_2)} = [E_1] * [E_3] = \sum_{(E_2)} C_{(E_1)(E_3)}^{(E_2)}$$

become functions of p , $C_{(E_1)(E_3)}^{(E_2)}(p)$.

They often turn out to be polynomials in p .
(need extra twisting factor in *-)

Then as a Hopf algebra with the Green coproduct, H turns out to be isomorphic to the positive part of the quantum group of the Kac-Moody algebra associated to the quiver Q , at quantum parameter $v = p^{1/2}$. (Oversimplified a bit.)

Moral: Hall (bi) algebras can be computed explicitly, and even simple examples can give very interesting and deep algebras from Representation Theory.

Aside: it is an interesting problem to construct the full quantum group from a "Hall algebra" of $D^b \text{mod-}\mathbb{F}_p Q$. See Bridgeland arXiv: 1111.0745 for best known result.

① Co. HA, of 1-dimensional abelian categories, over \mathbb{C} .

Let A be an abelian category over \mathbb{C} , such that for all $E, F \in A$, $\text{Hom}(E, F)$, $\text{Ext}^1(E, F)$ are finite-dimensional over \mathbb{C} , and $\text{Ext}^i(E, F) = 0$ for $i > 1$.

Then we call A 1-dimensional.

The main examples are $A = \text{mod } \mathbb{C}Q$ for a finite quiver Q , and $A = \text{coh}(C)$ for $C = \Sigma_g$ a smooth projective complex curve.

Let \mathcal{M} be the moduli stack of objects in \mathcal{A} , a (smooth) Artin stack over \mathbb{C} .

Then we can form the cohomology

$H^*(\mathcal{M}, \mathbb{Q})$. Kontsevich - Soibelman

2010 §2-§3 explain how to make

$H^*(\mathcal{M}, \mathbb{Q})$ into an associative algebra, a Cohomological Hall Algebra (CoHA).

Write Exact for the moduli stack of short exact sequences

$0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$, and

$\pi_i: \text{Exact} \rightarrow \mathcal{M}$ for $i=1,2,3$ for the morphism mapping this stack to E_i .

Then we have a diagram

$$M \times M \xleftarrow{\pi_1 \times \pi_3} \text{Exact} \xrightarrow{\pi_2} M.$$

Kotsevich - Seibelman define a bilinear product $*$: $H^*(M) \times H^*(M) \rightarrow H^*(M)$ by the composition

$$\begin{array}{ccc}
 H^*(M) \times H^*(M) & \xrightarrow{\boxtimes} & H^*(M \times M) \\
 \downarrow * & & \downarrow (\pi_1 \times \pi_2)^* \\
 H^*(M) & \xleftarrow{(\pi_2)_*} & H^*(\text{Exact})
 \end{array}$$

Here \boxtimes is the external tensor product and $(\pi_1 \times \pi_2)^*$ the pull back, both natural in cohomology.

The subtle part is the "pushforward" $(\pi_2)_*$. This is not natural in cohomology. To define it, we have to use the assumption that A has dimension 1.

Since A has dimension 1, M and Exact are smooth Artin stacks, their cotangent complexes $\mathbb{L}_M, \mathbb{L}_{\text{Exact}}$ are perfect in $(0, 1)$. Thus $\Pi_2: \text{Exact} \rightarrow M$ is quasi-smooth — its cotangent complex is perfect in $(-1, 1)$, it has a relative perfect obstruction theory.

Ignoring properness for the moment, this implies as in A. Khan arXiv:1902.01332 that there is a pushforward map

$$(\Pi_2)_* : H^*(\text{Exact}) \rightarrow H^{*-2 \dim_{\mathbb{C}} \Pi_2}(M),$$

essentially a relative virtual class.

Think of $(\Pi_2)_*$ as integration over virtual classes of the fibers of $\Pi_2: \text{Exact} \rightarrow M$.

(This is not how Kontsevich-Schmidman explain it.)

There is another problem: $(\mathbb{T}^2)_*$ should be defined if \mathbb{T}^2 is quasi-smooth and proper. For $\text{coh}(\Sigma_g)$, if we fix the Chern character, $\alpha_i = \text{ch } E_i$, then $\mathbb{T}^2: \text{Exact}_{\alpha_1, \alpha_2, \alpha_3} \rightarrow M_{\alpha_2}$ is proper, but \mathbb{T}^2 is not in general.

Splitting, $M = \coprod_{\alpha \in \text{Ch}(\text{coh}(\Sigma_g))} M_{\alpha}$,

we should define $*$ *not on*
 \downarrow *allows infinitely nonzero terms*
 $H^*(M) = \prod_{\alpha \in \text{Ch}(\text{coh}(\Sigma_g))} H^*(M_{\alpha})$,

but on \downarrow *allows only finitely many nonzero terms*
 $\tilde{H}^*(M) = \bigoplus_{\alpha \in \text{Ch}(\text{coh}(\Sigma_g))} H^*(M_{\alpha})$,

so properness of $\mathbb{T}^2_{\alpha_1, \alpha_2, \alpha_3}$ is enough.

— Then one can prove that $*$ is associative. The argument uses the fact that i is a Cartesian square of Artin stacks,

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow e & & \downarrow h \\ X & \xrightarrow{g} & Z, \end{array}$$

with f, g proper and quasi-smooth,

the following commutes:

$$\begin{array}{ccc} H^*(W) & \xrightarrow{\quad} & H^*(Y) \\ e^* \uparrow & f_* & \uparrow h^* \\ H^*(X) & \xrightarrow{g_*} & H^*(Z). \end{array}$$

Latyntsev arXiv: 2110.14356:
 vertex coalgebra structure on $H^*(M)$,
 compatible with the $\text{co}H A$ multiplication.

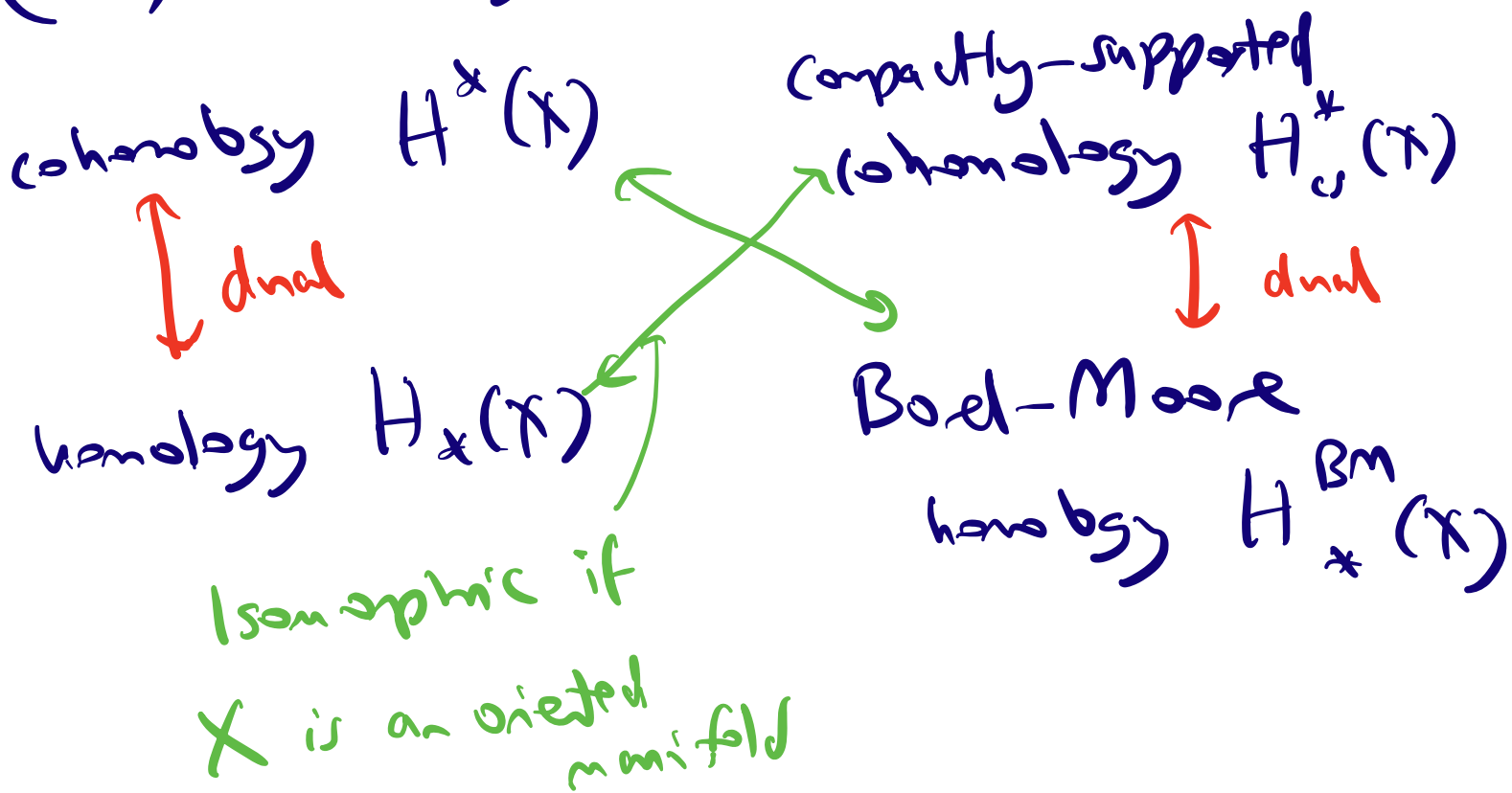
— Analogue of Green's coproduct in the finite field case.

② CoHA of complex surfaces

The construction in ① does not work when $A = \text{coh}(S)$ for S a complex projective surface, so that $\dim A = 2$, since now M , Σ_{ext} are not smooth (at least quasi-smooth), & $T_2: \Sigma_{\text{ext}} \rightarrow M$ is not quasi-smooth.

However, there is an alternative construction due to Kapranov-Vasserot 2019, which works on the Bred-Moore homology $H_*^{\text{BM}}(M)$.

(Co) homology theories in topology:



$H_c^*(X)$ and $H_*^{BM}(X)$ are not invariant under homotopy equivalence, so more difficult to work with.

$H_*^{BM}(X)$ has proper pushforward under $f: X \rightarrow Y$.

Borel-Moore homology can be defined for (nice) Artin stacks M :

$$H_k^{B\mathcal{M}}(M, \mathbb{Q}) = H^{-k}(M, \omega_M),$$

where ω_M is the dualizing

complex is ∞ stretchable

$$\text{sheaf, } D_c^b(M, \mathbb{Q}).$$

Note that it is defined for $k \in \mathbb{Z}$,
not $k \in \mathbb{N}$.

If M is a smooth Artin \mathbb{C} -stack
of \mathbb{C} -virtual dimension n then

$$\omega_M \cong \mathbb{Q}(n),$$

$$H_k^{B\mathcal{M}}(M, \mathbb{Q}) \cong H^{2n-k}(M, \mathbb{Q}),$$

and $H_k^{B\mathcal{M}}(M, \mathbb{Q}) = 0$ if $k \geq 2n$.

Let S be a projective complex surface and \mathcal{M} the moduli stack of objects in $\text{coh}(S)$. Then we can consider

$$\mathcal{M} \times \mathcal{M} \xleftarrow{\pi_1 \times \pi_3} \text{Exact} \xrightarrow{\pi_2} \mathcal{M}.$$

and define $*$: $H_x^{B_m}(\mathcal{M}) \times H_x^{B_m}(\mathcal{M}) \rightarrow H_x^{B_m}(\mathcal{M})$ by the composition

$$\begin{array}{ccc} H_x^{B_m}(\mathcal{M}) \times H_x^{B_m}(\mathcal{M}) & \xrightarrow{\otimes} & H_x^{B_m}(\mathcal{M} \times \mathcal{M}) \\ \downarrow * & & \downarrow (\pi_1 \times \pi_3)^* \\ H_x^{B_m}(\mathcal{M}) & \xleftarrow{(\pi_2)_*} & H_x^{B_m}(\text{Exact}). \end{array}$$

In this case the pushforward $(\pi_2)_*$ is functorial in homology (though we have a problem if π_2 is not proper, solved as before).

The pullback $(\pi_1 \times \pi_2)^*$ is
 now the subtle thing, as pullbacks
 are not natural on (Borel-Moore) homology.

But for sleeves & surface,

$$\pi_1 \times \pi_2: \text{Exact} \rightarrow M \times M$$

is quasi-smooth (has a perfect
 relative obstruction theory), so

there is a virtual pullback map

$$(\pi_1 \times \pi_2)^*: H_{lc}^{B\mathbb{M}}(M \times M) \rightarrow H_{k+2\text{vdim}_{\mathbb{C}}(\pi_1 \times \pi_2)}^{B\mathbb{M}}(\text{Exact}).$$

— defined by the same technology used
 to define virtual classes (see Khan paper).

The point that $*$ is associative
 are the same properties of
 pullback and pushforward, $\circlearrowleft H_*^{B\mathbb{M}}(-)$.

If M is the moduli stack
of objects in a 1-dimensional
category, as in (1), then

$$H^k(M) \cong H_{2 \dim M - k}^{BM}(M).$$

So we could write the coHAs
in (1) in terms of either homology

$H^*(M)$ or Bredon-Moore

homology $H_*^{BM}(M)$. The B-M

version is the one which extends
to sheaves on surfaces.