Critical COHA (26/05/2026)

Plan: 1/ Critical COHA of (Q,W)
2/ Factorization & BPS cohomology
3/ CY3 & dimensional reduction

1/ Critical COHA of (Q,W)

Original Ref: Kontsevich-Soibelman 1006.2706 §7.

Q : quiver finite oriented graph
Q_0: set of vertices
Q_1: set of arrows. $\Gamma = \mathbb{Z}^{Q_0}$ $\Gamma_+ = \mathbb{N}^{Q_0}$

$\gamma \in \Gamma_+$ $\mathfrak{M}_\gamma$: stack of quiver rep of dim vector γ

$$\mathfrak{M}_\gamma = \frac{\prod_{(\alpha: i \to j) \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma_i}, \mathbb{C}^{\gamma_j})}{\prod_{i \in Q_0} GL(\mathbb{C}^{\gamma_i})} = R_\gamma / G_\gamma$$

Affine space

$\mathfrak{M}_\gamma$ smooth Artin stack

$$\dim \mathfrak{M}_\gamma = \sum_{\alpha: i \to j} \gamma_i \gamma_j - \sum_i \gamma_i^2 = -\chi(\gamma,\gamma)$$

$\chi : \Gamma \times \Gamma \to \mathbb{Z}$ Euler form

$$(\gamma_i), (\gamma_i') \mapsto \sum_i \gamma_i \gamma_i' - \sum_{\alpha: i \to j} \gamma_i \gamma_j'$$

$\mathcal{A}_Q$ abelian category of quiver rep $=$ $\mathbb{C}Q$-mod

Homological dim 1 $\quad \mathrm{Ext}^0 \quad \mathrm{Ext}^1$

$\sim$ Behaves like $\mathrm{Coh}(\text{Smooth Curve})$

$$\chi = \dim \mathrm{Ext}^0 - \dim \mathrm{Ext}^1$$

$\to$ Could define COHA for $\mathcal{A}_Q$ as in Dominic's lecture.

More interesting:

Add a potential $W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$

linear combination of cycles.

$W = a_1 \cdots a_\ell$ cyclic

$$\frac{\partial W}{\partial a} = \sum_{a_i = a} a_{i+1} \cdots a_\ell a_1 \cdots a_{i-1}$$

$$\mathrm{Jac}(Q, W) = \mathbb{C}Q / \langle \partial W / \partial a \mid a \in Q_1 \rangle$$

Jacobi algebra

$\mathcal{A}_{(Q,W)}$: modules over (Q, W)

$\gamma \in \Gamma_+ \quad \mathfrak{M}_\gamma^{(Q,W)}$ = Critical locus of $\mathrm{Tr}(W): \mathfrak{M}_\gamma \longrightarrow \mathbb{C}$.

Naive guess for COHA:

$$H^{BM}_\bullet(\mathfrak{M}_\gamma^{(Q,W)}, \mathbb{Q})$$

<u>Pb</u>: No quasi-smoothness for pushforward.

Instead:

$$\mathcal{H}_\gamma := H^\bullet_c(\mathfrak{M}_\gamma, \underbrace{\phi_{\mathrm{Tr}(W)}\mathbb{Q}})^\vee$$

$\phi_{\mathrm{Tr}(W)}$ vanishing cycle functor

$$D^b_c(\mathfrak{M}_\gamma, \mathbb{Q}) \longrightarrow D^b_c(\mathfrak{M}_\gamma, \mathbb{Q})$$
$$F \longmapsto \phi_{\mathrm{Tr}(W)}(F)$$

preserves perverse t-structure

In particular, $\phi_{\mathrm{Tr}(W)}(\mathbb{Q})$ perverse sheaf.

$$\begin{array}{c} \mathfrak{M}_\gamma \\ {\scriptstyle \mathrm{Tr}W}\downarrow \\ \mathbb{C} \end{array}$$

Milnor fiber

$$\mathcal{H}^i(\phi_{\mathrm{Tr}(W)}(\mathbb{Q})|_p) = \tilde{H}^i(\text{Milnor fiber}, \mathbb{Q})$$

fibers near critical locus.

$$\begin{tikzcd} & \mathfrak{M}_{\gamma,\gamma'} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1\times\pi_3"] & \\ \mathfrak{M}_{\gamma+\gamma'} & & \mathfrak{M}_\gamma \times \mathfrak{M}_{\gamma'} \end{tikzcd}$$

Short exact seq

$$\phi_{\mathrm{Tr}W}\, \mathbb{D} = \mathbb{D}\, \phi_{\mathrm{Tr}(W)}$$

$\mathbb{D}$: Verdier dual

$\mathbb{D}\mathbb{Q} = \mathbb{Q}[2\dim]$ on smooth

$$H_c(\quad)^\vee [\chi(\gamma,\gamma)]$$

$$= H^\cdot(\quad)[-\chi(\gamma,\gamma)]$$

/ Smooth $\mathfrak{M}_\gamma$

CCL: Could also describe $\mathcal{H}_\gamma$ in terms of $H^\cdot$.

M smooth manifold $\dim_{\mathbb{R}} = n$

$$H^{n-i}(X) \simeq H^i_c(X)^\vee$$

$$H^{n-i} \times H^i_c \longrightarrow \mathbb{R}$$

$$(\alpha, \beta) \longmapsto \int_X \alpha \wedge \beta$$

Thom–Sebastiani

$f: X \to \mathbb{C}$

$g: Y \to \mathbb{C}$

$$H^\cdot(X, \phi_f \mathcal{F}) \otimes H^\cdot(Y, \phi_g \mathcal{G})$$

$$\simeq H^\cdot(X \times Y, \phi_{f \boxtimes g}(\mathcal{F} \boxtimes \mathcal{G}))$$

Analogue of Künneth formula in usual cohomology

$(\pi_1 \times \pi_3)^*$: Just pullback in cohomology

Pushforward ?

π_{2*} ?

π_2 proper

$$\begin{tikzcd} \mathfrak{M}_{\gamma,\gamma'} \arrow[d, "\pi_2"'] \\ \mathfrak{M}_{\gamma+\gamma'} \end{tikzcd}$$

In general

$\mathbb{D} f_! = f_* \mathbb{D}$

$\mathbb{Q} \to \pi_{2*}\mathbb{Q}$

Verdier dual: $\underline{\pi_2 \text{ proper}}$

$\pi_{2*}\mathbb{Q} \to \mathbb{Q}[\cdots]$

$\phi_{Tr(W)}$

Commute with proper pushforward

$$\pi_{2*}\phi_{Tr(W)}\mathbb{Q} \to \phi_{Tr(W)}\mathbb{Q}[\cdots]$$

$$H^\bullet(\quad) \to H^\bullet(\quad)$$

More on product? NS $\mathcal{M}_\gamma = R_\gamma / G_\gamma$

$$\mathcal{H}_\gamma = \left(H_{c,G_\gamma}(R_\gamma, W_\gamma)\right)^\vee$$

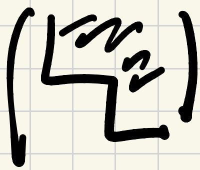
$$m^\vee_{\gamma_1,\gamma_2} : H_{c,G_\gamma}(\mathcal{M}_\gamma, W_\gamma) \qquad \gamma = \gamma_1 + \gamma_2$$

$$\longrightarrow H_{c,G_{\gamma_1}}(\mathcal{M}_{\gamma_1}, W_{\gamma_1}) \otimes H_{c,G_{\gamma_2}}(\mathcal{M}_{\gamma_2}, W_{\gamma_2})$$

M_{γ_1,γ_2} : rep of Q of dim $\gamma_1+\gamma_2$ $\subset M_\gamma$
s.t. standard coord subspace $\uparrow$ closed
of dim γ_2 = subrep

Affine $\circlearrowleft$ $\quad\quad$ $\circlearrowleft G_\gamma$

G_{γ_1,γ_2}



Without Φ:

Künneth

$$H^\bullet_{G_{\gamma_1}}(M_{\gamma_1}) \otimes H^\bullet_{G_{\gamma_2}}(M_{\gamma_2}) \longrightarrow H^\bullet_{G_{\gamma_1}\times G_{\gamma_2}}(M_{\gamma_1} \times M_{\gamma_2})$$

$$\xrightarrow{\sim} H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2})$$

$\mathrm{Tr} W|_{R_\gamma}$

vs

$\mathrm{Tr} W|_{R_{\gamma_1}} \boxtimes \mathrm{Tr} W|_{R_{\gamma_2}}$

$$\left.\begin{array}{l} M_{\gamma_1,\gamma_2} \xrightarrow{\sim} M_{\gamma_1} \times M_{\gamma_2} \\ G_{\gamma_1,\gamma_2} \xrightarrow{\sim} G_{\gamma_1} \times G_{\gamma_2} \end{array}\right\} \text{Homotopy equiv}$$

$$\longrightarrow H^{\bullet+2c_1}_{G_{\gamma_1,\gamma_2}}(M_\gamma) \xrightarrow{T_vM|_{R_\gamma}} H^{\bullet+2c_1+2c_2}_{G_\gamma}(M_\gamma)$$

Pushforward / Closed embedding G_{γ_1,γ_2}-equiv of $\mathbb{C}$-manifolds

Pushforward / Fund class of $G_\gamma / G_{\gamma_1,\gamma_2} = \prod_{i \in I} Gr(\gamma_1^i, \mathbb{C}^{\gamma_i})$

With ϕ

$$m^\vee : H_{c,G_\gamma}(M_\gamma, W_\gamma) \longrightarrow H_{c,G_{\gamma_1}}(M_{\gamma_1}, W_{\gamma_1}) \otimes H_{c,G_{\gamma_2}}(M_{\gamma_2}, W_{\gamma_2})$$

$$\downarrow$$

$$H_{c,G_{\gamma_1,\gamma_2}}(M_\gamma, W_\gamma)$$

Pullback associated to

$G_{\gamma_1,\gamma_2} \hookrightarrow G_\gamma$

with proper quotient.

$$G_\gamma / G_{\gamma_1,\gamma_2} = \prod_{i \in I} Gr(\gamma_2^i, \mathbb{C}^{\gamma_i})$$

$\to$ product $\mathcal{H}_\gamma \times \mathcal{H}_{\gamma'} \longrightarrow \mathcal{H}_{\gamma+\gamma'}$

$$\deg(\alpha \times \beta) = \deg(\alpha) + \deg(\beta) \overbrace{\underbrace{+\chi(\gamma,\gamma') - \chi(\gamma',\gamma)}_{\text{skew-symmetrized Euler form}}}^{\langle\gamma,\gamma'\rangle}$$

Associative.
COHA

$$\mathcal{H} = \bigoplus_{\gamma \in \Gamma_+} \mathcal{H}_\gamma$$

Degree preserving for symmetric quivers

associative algebra.

Critical COHA of (Q,W)

2/ Factorization & BPS cohomology

Davison - Meinhardt [1601.02679]

$\zeta \in \mathbb{Q}^{Q_0}$ stability slope $\mu(\gamma) = \dfrac{\sum_i \zeta_i \gamma_i}{\sum_i \gamma_i}$

$\mathfrak{M}_\gamma^{\zeta\text{-ss}}$

$$\mathcal{H}_\gamma^{\zeta\text{-ss}} = H^\bullet\left(\mathfrak{M}_\gamma^{\zeta\text{-ss}}, \phi_{Tr(W)}\big|_{\mathfrak{M}_\gamma^{\zeta\text{-ss}}}\right)$$

$\theta \in \mathbb{Q}$ $\quad \Gamma_\theta^{+\zeta} = \{\gamma \in \Gamma_+ \mid \mu(\gamma) = \theta\}$

$$\mathcal{H}_\theta^{\zeta\text{-ss}} = \bigoplus_{\gamma \in \Gamma_\theta^+} \mathcal{H}_\gamma^{\zeta\text{-ss}}$$

COHA algebra of semistable of slope θ

$\gamma \quad \gamma'$

$\mu(\gamma) = \mu(\gamma') = \theta$

$\mu(\gamma+\gamma') = ?$

$$\frac{\sum_i \zeta_i(\gamma_i + \gamma_i')}{\sum_i \gamma_i + \sum_i \gamma_i'} = \frac{\theta\left(\sum_i \gamma_i + \sum_i \gamma_i'\right)}{\text{—}}$$

$$= \theta.$$

Thm: $\displaystyle\bigotimes_{\theta} \mathcal{H}_\theta^{\zeta\text{-ss}} \xrightarrow{\sim} \mathcal{H}$ Multiplication

$\infty \to -\infty$ descending slopes.

Cohomological consequence

Harder-Narasimhan filtration

$\forall\, V \quad \exists!\ 0 = V_0 \subset V_1 \subset \dots \subset V_k = V$

V_i / V_{i-1} semistable

slopes decreasing

ζ θ-generic if $\gamma, \gamma' \in \Gamma^{\zeta}_{\theta+} \Rightarrow \langle \gamma, \gamma' \rangle = 0$

$$\mathcal{H}_\theta^{\zeta\text{-ss}} = \mathrm{Sym}\left(H^\bullet(\mathcal{M}_\theta^{\zeta\text{-ss}}, \mathrm{BPS}) \otimes H^\bullet(B\mathbb{C}^*)\right)$$

Isom of vector space

Not of algebras in <u>general</u>.

$\mathcal{M}_\gamma^{\zeta\text{-ss}}$ = GIT quotient of semistable

Scheme! Not stacky

Singular

BPS sheaf

$\mathrm{BPS} = \phi_{\mathrm{Tr}(W)} \mathrm{IC}$

Induced by multiplication from a canonical embedding

$$\underbrace{H^\bullet(\mathcal{M}_\theta^{\zeta\text{-ss}}, \mathrm{BPS}) \otimes H^*(B\mathbb{C}^*)}_{\substack{\text{Closed under} \\ \text{commutator } [\,,\,] \\ \to \text{BPS Lie algebra}}} \hookrightarrow \mathcal{H}_\theta^{\zeta\text{-ss}}$$

$\mathcal{H}_\theta^{\zeta\text{-ss}}$ not supercommutative in general

PBW $\mathfrak{g}$ Lie algebra

$U(\mathfrak{g}) \simeq \mathrm{Sym}\,\mathfrak{g}$

$$\mathfrak{M}_\gamma^{\zeta\text{-ss}} \to \mathcal{M}_\gamma^{\zeta\text{-ss}}$$

Decomposition Thm, Perverse filtration

Ex: $Q = \bullet$

∃ filtration s.t. $\mathrm{gr} = \mathrm{Sym} \ldots$

$\hookrightarrow$ Filtration $\mathrm{gr}_F U(\mathfrak{g}) \cong \mathrm{Sym}\,\mathfrak{g}$

Isom of alg

$$\bigoplus_n H^\bullet(\mathbb{A}^{\cdots}/GL_n) \cong \mathrm{Sym}(H^\bullet(B\mathbb{C}^*))$$

Weight polynomial

Poincaré polynomial for $H_c^\bullet$

BPS $= \mathbb{C}$.

$$\sum_{n\geq 0} \frac{(-q^{1/2})^{n^2}}{(q^n-1)\cdots(q^n-q^{n-1})}$$

$$= \exp\left(-\sum_{n\geq 1} \frac{1}{n} \frac{x^n}{q^{n/2}-q^{-n/2}}\right)$$

"Quantum dilogarithm".

$$H^\bullet(\mathcal{M}_\theta^{\zeta\text{-ss}}, \mathrm{BPS}) \longrightarrow \mathrm{BPS}\ \#$$

χ DT # ...

Wall-Crossing

3/ CY3 & dim reduction

Ginzburg dg-algebra $\Gamma_3(Q,W)$ $H^0 = \mathrm{Jac}(Q,W)$

≤ 0-graded dg-algebra

= stable dg-category CY3

t-structure Heart = $\mathrm{Rep}(Q,W)$

$\mathbb{C}Q$ deg 0 $da = 0$

$a : i \to j$: $a^* : j \to i$ deg -1 $da^* = \dfrac{\partial w}{\partial a}$

loop $t_i : i \to i$ deg -2 $\forall i$ $dt_i = e_i \left(\sum_{a \in Q_1} [a, a^*] \right) e_i$

Γ <u>smooth</u>

$$\left.\begin{array}{l} D^f\Gamma \text{ dg-modules } M \\ \dim H^*(M) < \infty \end{array}\right] \text{CY3}$$

$\mathcal{C}$ stable ∞-CY3-category of finite type

Joyce & collaborators.

$\mathcal{A}$ Heart of reasonable t-structure

$\mathfrak{M}$ (-1)-shifted symplectic derived stack

Darboux thm

locally critical locus of function on smooth scheme

+ Orientation data

ϕ Global version of sheaf of vanishing cycles.

$\mathcal{H} := H^{\circ}(\mathfrak{M}, \phi)$

COHA product: Kinjo-Park-Safronov 2406.12838

Descombes 2506.22600

Factorization & BPS cohomology

Bu-Davison-Ibanez Nunez-Kinjo-Padurariu

2502.04253

Dimensional reduction?

Ninja 2102.01568

$$\begin{tikzcd} & E \arrow[d, shift right] \\ \mathrm{Tot}(E^\vee) \arrow[r, "\pi"'] & U \arrow[u, shift right, "s"'] \end{tikzcd}$$

E vector bundle

s section

U scheme

$\bar{s}$: corresponding function on $\mathrm{Tot}(E^\vee)$

$\bar{s} = s(x)p$

$Z = s^{-1}(0)$
$= \mathrm{Crit}(\bar{s})$

$$\pi_! \phi_{\bar{s}} \mathbb{Q}_{\mathrm{Tot}(E^\vee)} = \mathbb{Q}_Z[-2\,\mathrm{rk}\,E]$$

Davison (2017)

Global version:

$\mathcal{X}$ quasi-smooth derived Artin stack.

$\pi \uparrow T^*[-1]\mathcal{X}$ $\begin{cases} \text{canonical } (-1)\text{-shifted symp} \\ + \text{ canonical orientation} \end{cases}$

$\hookrightarrow \varphi$

$$\pi_! \varphi = \mathbb{Q}_{\mathcal{X}}[\mathrm{vdim}\,\mathcal{X}]$$

S quasiproj surface $\quad X = K_S \quad \mathcal{M}_X = T^*[-1]\mathcal{M}_S$

Compactly supported Coh sheaves on X

--- on S

$$\Rightarrow \quad \underbrace{H^\bullet(\mathcal{M}_X, \varphi)}_{\text{CY3 Critical COHA}} = \underbrace{H^{BM}_{\mathrm{vdim}\,\mathcal{M}_S - \bullet}(\mathcal{M}_S)}_{\text{Kapranov–Vasserot COHA}}$$

CY3 Critical COHA $\longleftrightarrow$ Kapranov–Vasserot COHA

Comparism ?

Unpublished draft of Ninjo - Khan

$$\pi_! \varphi = \mathbb{Q}[\ ]$$

$$H_c(X, \varphi) = H_c(\mathbb{Q})$$

$$H_c(X,\varphi)^\vee = \underbrace{H_c^\vee(\mathbb{Q})}_{= H^{BM}}$$

$$\| $$

$$H^\bullet(X, \varphi)$$

<u>CCL:</u>

Surface COHA should be a special case of CY3 COHA

$\mathbb{D}\varphi = \varphi\mathbb{D}$

Minets Conjecture for COHA of 0-dim sheaves on CY3 generalizing Minets - Schiffmann - Vasserot for surfaces (as in Hulya's Talk).