Week 5 — Pierrick

Critical CoHA

1. Kontsevich–Soibelman CoHA, for quivers with potentials
2. Factorization and BPS cohomology.
3. CY3 case, dimensional reduction

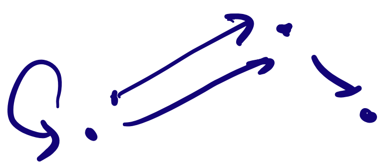
1. Quivers with potential

Q: finite quiver.

Q_0: set of vertices

Q_1: set of arrows

$h, t: Q_0 \to Q_1$



$\mathcal{A}_Q$: abelian category of finite dimensional reps of Q over $\mathbb{C}$.

$\iff$ module over path algebra $\mathbb{C}Q$.

Associative algebra.

$$V = \left((V_i)_{i \in Q_0}, (\varphi_e)_{e \in Q_1} \right) \qquad \dim V = (\dim V_i) \in \mathbb{N}^{Q_0}.$$

$\gamma \in \Gamma_T \implies \mathcal{M}_\gamma$ moduli stack of objects in $\mathcal{A}$ of dimension γ.

$$\mathcal{M}_\gamma = \frac{\overbrace{\bigoplus_{e \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\gamma(h(e))}, \mathbb{C}^{\gamma(t(e))}\right)}^{R_\gamma}}{\underbrace{\prod_{v \in Q_0} GL(\gamma(v), \mathbb{C})}_{G_\gamma}}$$

$\mathcal{M}_\gamma = [R_\gamma / G_\gamma]$

— very simple Artin stack. Smooth stack.

$$\text{dimension} = \sum_{e \in Q_1} \gamma(h(e))\gamma(t(e)) - \sum_{v \in Q_0} \gamma(v)^2$$

$$= \chi(\gamma, \gamma), \qquad \chi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$$

Euler form

$$\chi(\dim V, \dim W) = \dim \mathrm{Ext}^0(V, W) - \dim \mathrm{Ext}^1(V, W)$$

$\mathrm{Ext}^{\geq 2}(V, W) = 0.$

For A_Q, "Usual CoHA" on $\bigoplus_{\gamma\in\Gamma_+} H^*(M_\gamma, \mathbb{Q})$

$$H^*(M_\gamma, \mathbb{Q}) \cong H^*_{G_\gamma}(R_\gamma)$$

$$\begin{tikzcd}
& \mathrm{Exact}_{\gamma,\gamma'} \arrow[dl, "p"', "\pi_1\times\pi_2"] \arrow[dr, "\pi", "q"'] & \\
M_\gamma \times M_{\gamma'} & & M_{\gamma+\gamma'}
\end{tikzcd}$$

CoHA: $\underbrace{p_*}_{\text{wrong-way map}} \circ\, q^*$ $\qquad$ p is quasi-smooth

$\exists\ p_*$ virtual pushforward.

Kontsevich-Soibelman 2010 give explicit definition of p_*:

$$H^*_{G_{\gamma_1}}(R_{\gamma_1}, \mathbb{Q}) \otimes H^*_{G_{\gamma_2}}(R_{\gamma_2}, \mathbb{Q})$$

$$\cong \big\downarrow \text{ Künneth}$$

$$H^*_{G_{\gamma_1}\times G_{\gamma_2}}(R_{\gamma_1}\times R_{\gamma_2}, \mathbb{Q})$$

$$\big\downarrow$$

$$\cong \Big\downarrow q^*$$

$$\mathrm{Exact}_{\gamma_1,\gamma_2} = \left[\frac{R_{\gamma_1,\gamma_2}}{G_{\gamma_1,\gamma_2}}\right]$$

$$H^*_{G_{\gamma_1,\gamma_2}}(R_{\gamma_1,\gamma_2}, \mathbb{Q})$$

$\Big\downarrow$ Pushforward along closed embedding $R_{\gamma_1,\gamma_2} \hookrightarrow R_{\gamma_1+\gamma_2}$

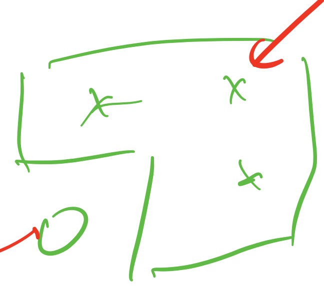
$$H^{*+?}_{G_{\gamma_1,\gamma_2}}(R_{\gamma_1+\gamma_2}, \mathbb{Q})$$

$\Big\downarrow$ Pushforward with fibre $G_{\gamma_1+\gamma_2}/G_{\gamma_1,\gamma_2}$: proper Grassmannian. Has a fundamental class.

$$H^{*+??}_{G_{\gamma_1+\gamma_2}}(R_{\gamma_1+\gamma_2}, \mathbb{Q})$$

$R_{\gamma_1,\gamma_2} \subset R_{\gamma_1+\gamma_2}$

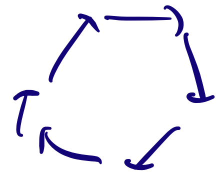
$\mathrm{Hom}(\gamma_2,\gamma_1)$

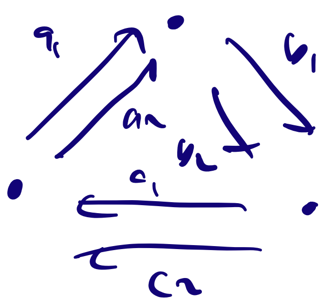


$\mathrm{Hom}(\gamma_1,\gamma_2)$

$G_{\gamma_1,\gamma_2} \subset G_{\gamma_1+\gamma_2}$

<u>Now:</u> Let's do critical CoHA. Choose a potential W. Linear combination of cycles in quiver.





$$W = ?\, a_1 b_1 c_1 + ?\, a_1 b_2 c_1 + \cdots$$

$$\mathbb{C}Q / \langle \frac{\partial W}{\partial e} : e \in Q_1 \rangle$$ Jacobi algebra $\mathrm{Jac}(Q,W)$

cycle $\alpha_1 \dots \alpha_n$

$$\frac{\partial W}{\partial \alpha} = \sum_{\substack{i: \\ \alpha_i = \alpha}} \alpha_{i+1} \dots \alpha_n \alpha_1 \dots \alpha_{i-1}$$

Abelian category of modules over $\mathrm{Jac}(Q,W)$

$\mathcal{A}_{(Q,W)}$. Moduli stacks of objects are very singular. $H^*(\mathcal{M}_{Q,W})$ is bad for purpose of defining CoHA.

Critical cohomology: moduli stack of modules on $\mathrm{Jac}(Q,W)$ of dim γ

= Critical locus of function on $\mathcal{M}_\gamma$

$$\mathrm{Crit}(\mathrm{Tr}(W) : \mathcal{M}_\gamma \to \mathbb{C})$$

Critical cohomology: $H^*(\mathcal{M}_\gamma, \phi_{\mathrm{Tr}W}(\mathbb{Q})(\dim \mathcal{M}_\gamma))$

perverse sheaf of vanishing cycles at $f=0$

$f : M \longrightarrow \mathbb{C}$

$\phi_f : D^b_c(M, \mathbb{Q}) \to D^b_c(M, \mathbb{Q})$ derived category of constructible sheaves

$F \mapsto \phi_f(F)\,[-1]$

$\phi_f(F)$ supported on critical locus

$\phi_f(F)_x =$ reduced cohomology of the milnor fiber of f at x with coefficients in F.

Alternate defn of critical cohomology:

$H^*(f^{-1}([0,\varepsilon]), f^{-1}(\varepsilon)) \quad 0 < \varepsilon \ll 1.$

Claim: if we use critical cohomology instead of naive cohomology, there will be a CoHA product This is a version of Borel Moore homology.

$\mathcal{H}_\gamma = H^*(M_\gamma, \phi_{Tr\,W}\mathbb{Q}).$

Best definition, more intrinsic, most generalizable.

$\cong \boxed{H_c^*(M_\gamma, \phi_{Tr\,W}\mathbb{Q})^\vee}$ up to shift $\chi(\gamma,\gamma)$

— Vanishing cycle functor commutes with Verdier duality.

— On M_γ, $(\mathcal{Q}_{M_\gamma})^\vee = \mathcal{Q}_{M_\gamma}[2\chi(\gamma,\gamma)]$

$\mathcal{H} = \bigoplus_{\gamma\in\Gamma_+} \mathcal{H}_\gamma$ has an associative product

— Same sequence of arrows, but apply vanishing cycles functor to everything.

Künneth $\Rightarrow$ Thom–Sebastiani.

Operations only on smooth ambient space.

Each $\mathcal{H}_\gamma$ is graded

$$\mathcal{H}_\gamma^i \otimes \mathcal{H}_{\gamma'}^j \longrightarrow \mathcal{H}_{\gamma+\gamma'}^{i+j+\chi(\gamma,\gamma')-\chi(\gamma',\gamma)}$$

skew-symmetrized Euler form

NB. Not commutative, even in degree 0.

<u>Remark</u>. If quiver is symmetric the multiplication preserves cohomological degree.

2. Factorization and BPS Cohomology

Next part: Davison–Meinhardt 1601.02479.

Goes beyond Kontsevich–Soibelman.

— Define stability condition on Q, W.

$\zeta \in \mathbb{Q}^{Q_0}$ stability

$$\mu(\gamma) = \frac{\sum_{v \in Q_0} \zeta_v \gamma_v}{\sum_{v \in Q_0} \gamma_v}$$ slope function.

— Then define stability + semistability in the usual way. Now can look at stack of semistable objects of given slope.

$\theta \in \mathbb{Q} \Rightarrow \Gamma_\theta^{+,\zeta} = \{\gamma \in \Gamma_T : \mu(\gamma) = \theta\}$

$\mathcal{M}_\gamma^{\zeta\text{-ss}}$ stack of ζ-semistable objects of dim γ

$\mathcal{M}_\gamma^{\zeta\text{-ss}} \underset{\text{open}}{\subset} \mathcal{M}_\gamma$

Can look at $\bigoplus_{\gamma \in \Gamma_\theta^{+,f}} H^*(\mathcal{M}_\gamma^{f-ss}, \phi_{\mathrm{Tr}(W)}(\mathbb{Q}))$

$$= \mathcal{H}_\theta^{f-ss}$$

- This still has a CoHA product.
- Since we fix slope θ, extensions of semistables are semistable.

Factorization Theorem (Davison–Meinhardt)

$$\bigotimes_{\theta \in \mathbb{Q}} \mathcal{H}_\theta^{f-ss} \xrightarrow{\sim} \mathcal{H}.$$

iso. of vector spaces.

ascending

Explanation: every object has a unique Harder–Narasimhan filtration.

This works by: include each factor into $\mathcal{H}$, then multiply in $\mathcal{H}$ CoHA

$$\mathcal{H}_\theta^{f-ss} \dashrightarrow \mathcal{H}$$

$$\xleftarrow{\text{split}}$$

Assume θ is $\underline{\text{$\delta$-generic}}$, $\underline{\text{ie}}$

$\forall\ \gamma, \gamma' \in \Gamma_\theta^{t, \delta} \Rightarrow \langle \gamma, \gamma' \rangle = 0$.

$\Leftrightarrow \chi(\gamma, \gamma') = \chi(\gamma', \gamma)$.

Then $\exists$ algebra filtration on $\mathcal{H}_\theta^{\delta\text{-}ss}$ s.t.

$$gr_F(\mathcal{H}_\theta^{\delta\text{-}ss}) = \mathrm{Sym}\Big(H^*(\mathcal{M}_\theta^{\delta\text{-}ss}, BPS) \otimes H^*(B\mathbb{C}^*)\Big)$$

$\mathcal{M}_\gamma^{\delta\text{-}ss}$

$\downarrow \pi_\gamma$

$\mathcal{M}_\gamma^{\delta\text{-}ss}$ coarse moduli space : perverse sheaf.

Generic stability $\Rightarrow$ some property of π_γ

Filtration used for this: $\pi_\gamma(\phi_\gamma \cdots)$ has filtration by t-structure ...

$BPS := \phi_{Tr(W)}(IC)$ of intersection cohomology sheaf of $\mathcal{M}_\gamma^{\delta\text{-}ss}$

Claim:

$$\underbrace{\underbrace{H^\bullet(\mathcal{M}_{\theta,1}^{\mathcal{J}\text{-}ss}, \mathbb{Q})}_{\text{has Lie algebra structure}} \otimes H^\bullet(BC^*)}$$

PBW theorem:

$gr_F(\mathcal{H}_\theta^{\mathcal{J}\text{-}ss})$ is the universal envelop

$gr\, U(g) = \mathrm{Sym}\, g$? How are $V(g)$ and $\mathcal{H}_\theta^{\mathcal{J}\text{-}ss}$ related.

3. (Q, W) is supposed to be Calabi-Yau 3-fold toy model for

Step. Given a dg-algebra $\Gamma(Q, W)$ has $\{g\}$

dg-modules on dg-algebra

t-structure with heart properly mod $\mathrm{Jac}(Q, W)$.

So: can we start with

$\mathcal{C}$ stable CY3 category

$\mathcal{A}$ heart of t-structure.

CoHA: see Kinjo–Park–Safronov 2024.

December, 2025