Kleinian orbifolds, COHAs, and Yangians. [2511.08576 Sala, Schiffmann, Shimpi)

[COHA GAC 16/06/2026, P. Bousseau]

Motivation: $\mathcal{A} \longmapsto H_{\mathcal{A}}$

COHA: abelian category $\to$ Associative algebra.

Often: COHA = universal enveloping algebra (or interesting quantum deformation of $\uparrow$) of a Lie algebra: $U(\mathfrak{n})$

$$H_{\mathcal{A}} \simeq U(\mathfrak{n})$$

Often: $\mathfrak{n}$ looks like the "positive part" of a bigger Lie algebra, looking like a simple Lie algebra:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$$

$\mathfrak{n}$ graded by $K_0^{num}(\mathcal{A})$

$\mathfrak{h} = \mathrm{Hom}(K_0^{num}(\mathcal{A}), \mathbb{R}) \Leftarrow K_0^{num}(\mathcal{A})$

Roots (positive)

<u>Ex:</u> $\mathfrak{g} = \mathfrak{sl}_{n+1}$ finite dim simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_- \qquad \mathfrak{h} = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \qquad \mathfrak{n} = \begin{pmatrix} 0 & * \cdots & * \\ & \ddots & \vdots \\ & & 0 \end{pmatrix}$$

$\mathfrak{g}$: $(n+1)\times(n+1)$ matrices, $\mathrm{Tr}=0$

$$\mathfrak{h} \simeq \mathbb{R}^n \qquad \mathfrak{n}_- = \begin{pmatrix} 0 & & \\ * & \ddots & \\ \vdots & \ddots & \\ * \cdots & * & 0 \end{pmatrix}$$

$e_i \in \mathfrak{h}^*$ Roots $\alpha_{ij} = e_i - e_j$ $\quad >0 : i<j$

Δ^+ $\quad <0 : i>j$

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{n}_\alpha \qquad \mathfrak{n}_\alpha = \begin{array}{c c} & j \\ i & \begin{pmatrix} 0 & \cdots & \ast \\ & \ddots & \\ & & 0 \end{pmatrix} \end{array}$$

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{n}_\alpha$$

Eigenspace decomposition for $\mathfrak{h} \curvearrowright \mathfrak{g}$ by $[h, -]$.

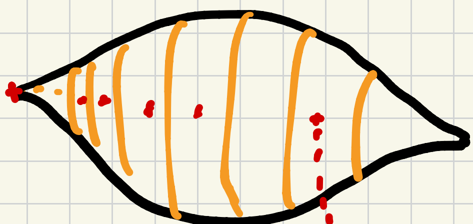
COHA: $\mathfrak{g}$ infinite-dimensional Lie algebra.

Hope: $U(\mathfrak{g}) \supset H_{\mathcal{A}}$

$\uparrow$ attached in some way to triangulated category $\mathcal{E} \supset \mathcal{A}$

$\mathcal{A}$ abelian category

Concrete example of the following picture:



$\mathcal{A}$

$H_{\mathcal{A}} = U(\mathfrak{n}_{\mathcal{A}})$

$\mathfrak{n}_{\mathcal{A}} \subset \mathfrak{g}$

Space of Bridgeland stability conditions on $\mathcal{E}$

$\mathcal{E} = D^b Coh(X)$ X = minimal resolution of ADE surface singularity $\mathbb{C}^2/\Gamma$

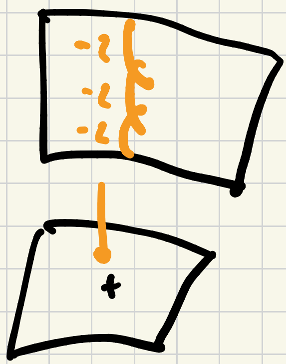
$\Gamma \subset SU(2)$ finite

e.g. $\Gamma = \mathbb{Z}/(n+1)\mathbb{Z}$ $\begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \in SU(2)$

A_n $\zeta = e^{2i\pi/(n+1)}$

$X_\emptyset = \mathbb{C}^2/\Gamma$ $\{xy = z^{n+1}\}$

$$\begin{array}{c} X \\ \big\downarrow \pi \\ X_\emptyset \end{array}$$



Chain of (-2)-curves

In general dual graph = ADE Dynkin diagram

$Coh(X) \subset D^b Coh(X)$

$\mathcal{X}_\emptyset$: $X_\emptyset$ viewed as smooth Deligne-Mumford stack

$Coh(\mathcal{X}_\emptyset) \subset D^b Coh(X_\emptyset)$

$I_f = \{1, \dots, n\}$ n (-2)-curves (or generally vertices of Dynkin diagram)

$\forall J \subset I_f$ Contract (-2)-curves indexed by $I \setminus J_f$

$X \to X_J \to X_\emptyset$ 2^n partial resolutions

Kapranov – Vasserot (~2000)

$$\begin{array}{ccccc} \mathcal{E} = D^b Coh(X) & \simeq & D^b Coh(X_J) & \simeq & D^b Coh(X_\emptyset) \\ \cup & & \cup & & \cup \\ Coh(X) & & Coh(X_J) & & Coh(X_\emptyset) \end{array}$$

ρ_0 = Trivial $\quad S_i \leftrightarrow \rho_i \otimes \mathcal{O}_0 \longrightarrow \mathbin{/\!/}$ ← derived McKay correspondence

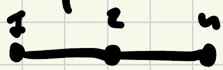
Irrep of $\Gamma \subset SU(2)$ $\qquad$ mod(Π)

Π = preprojective algebra of Q
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \hat{A}_n$ quiver

<u>affine</u> Dynkin diag with any orientation

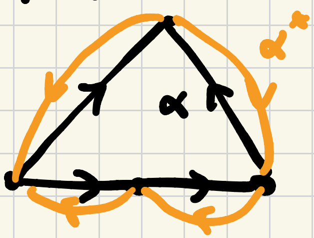
$$= \mathbb{C}\overline{Q} \Big/ \Big(\sum_{\substack{\alpha \text{ arrow} \\ \text{of } Q}} [\alpha, \alpha^*] = 0 \Big)$$

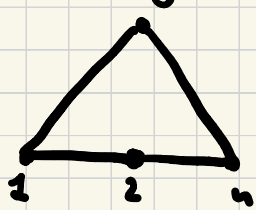
$Q_f = A_n$



$\overline{Q}$: double quiver of Q

$Q : \hat{A}_n$





$I := \{0, 1, \dots, n\} = \{0\} \cup I_f$

$$K_0(\mathcal{E}) = K_0(\mathrm{mod}(\Pi)) = \mathbb{Z}^I \simeq \mathbb{Z}^{n+1}$$
$\qquad \| \; K_0$

$$D^b Coh(X) \simeq D^b(\text{mod-}\Pi)$$

$C = \pi^{-1}(0)$ $\quad \exists \quad S_0, S_1, \dots, S_n$ simple rep

$C_1, \dots, C_n$ $\quad$ s.t. $\mathcal{O}_x \leftrightarrow \delta$-dim Rep

$$\mathcal{O}_C \leftrightarrow S_0$$

$$\mathcal{O}_{C_i}(-1)[1] \leftrightarrow S_i \qquad 1 \leqslant i \leqslant n$$

X = moduli space of semistable rep of $\dim(\delta_i)$

$P = \oplus P_i$ $\quad [1, \dots, 1]$ in A-cate

P tautological vector bundle $\quad RHom(P, -)$

—

$$\mathcal{E} = D^b Coh(X) \simeq D^b Coh(X_J) \simeq D^b Coh(X_\emptyset)$$

$$\begin{array}{ccc} \cup & \cup & \cup \\ Coh(X) & Coh(X_J) & Coh(X_\emptyset) \\ & & = \text{mod}(\Pi) \end{array}$$

"geometric" $\quad$ Mix of "geometric" & "algebraic" $\quad$ "algebraic"

2^n abelian hearts

More hearts?

Autoequivalences of $\mathcal{E}$?

$K_0 = \mathbb{Z}^I$

$\chi(-,-)$

$\delta = \text{Ker}$

$\updownarrow$

$\mathcal{O}_x$

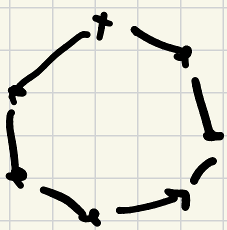
$\chi(\mathcal{O}_x, \mathcal{O}_x) = 0$

$RHom(\mathcal{O}_x, \mathcal{O}_x) = \mathbb{C} \oplus \mathbb{C}^2[-1] \oplus \mathbb{C}$

B : braid group on $(n+1)$-strands

Γ : group of outer automorphisms of Q

Dihedral group



$RHom(P, \mathcal{O}_x) = P_x$

S: spherical

x spherical $\mathrm{RHom}(x,x) = \mathbb{C} \oplus \mathbb{C}[-2]$

Spherical Twist $T_x(y) = \mathrm{Cone}(\mathrm{RHom}(x,y) \otimes x \xrightarrow{ev} y)$

$T_{S_0}, \dots, T_{S_n}$ generate braid group $\subset \mathrm{Aut}(\mathcal{C})$

$\Gamma \curvearrowright$ permutation of S_i

$B_{ex} := \Gamma \ltimes B$ extended braid group

$\sigma \in B_{ex}$ $\curvearrowright \mathrm{Aut}(\mathcal{C})$

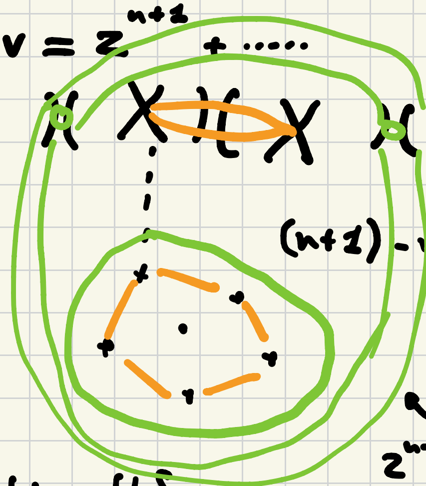
b.t: much more abelian categories.

<u>Rem:</u> $\mathrm{Coh}(X_J)$ not related by the action of B_{ex}.

Mirror of A_n $\quad$ Fuk(Y)

Y $\quad uv = z^{n+1} + \dots\dots$

$Y \xrightarrow{z} \mathbb{C}$



← Draw $z^{n+1} + Ct$

$S_0 \cup \dots \cup S_n$

Braid group $\leftrightarrow$ Dehn twists / Lag spheres.

Action of B_{ex} on K_0?

$K_0 \quad (-,-) = \chi(-,-)$

Symmetric bilinear form

$[S_i] = \alpha_i$

$(\alpha_i, \alpha_i) = 2$

$K_0 = K_0(D^b_C\, Coh(X))$

set-theoretically supported on $C = \pi^{-1}(0)$

$[O_x] = \delta = \sum_i \delta_i \alpha_i$

$(\delta, \delta) = 0$

$K_0 = \mathbb{Z}\delta \oplus (K_0)_f$

$\alpha_1, \ldots, \alpha_n$

$(K_0, (-,-))$

root lattice of affine lie algebra $\mathfrak{g}$

$\mathfrak{g}_f$: finite dim simple lie algebra $\leftrightarrow Q_f$ ADE

Eg $\mathfrak{g}_f = \mathfrak{sl}_{n+1}$ $\qquad \alpha_i = e_i - e_{i+1}$

$\alpha_1, \ldots, \alpha_n$ simple roots

Affine Kac-Moody lie algebra Central extension.

$\to \mathfrak{g} = \mathfrak{g}_f[s^{\pm}] \oplus \mathbb{C}c$

Central

$$[xs^m, ys^n] = [x,y]s^{m+n} + (x,y)m\delta_{m+n,0}\, c$$

$(-,-) \geq 0 \qquad$ Real roots $\Delta^{re} = [\alpha \mid (\alpha,\alpha) = 2]$

Imaginary roots : $\Delta^{im} = \{\alpha \mid (\alpha,\alpha) = 0\}$

$\Delta^{im} = \mathbb{Z}\delta$

$\alpha_0, \ldots, \alpha_n$ simple real roots

$\Delta = \Delta^{re} \cup \Delta^{im}$ $\quad \alpha + n\delta \quad n \in \mathbb{Z}$

∞ many roots $\quad \alpha \in \Delta_f$

$B_{ex} \curvearrowright K_0$

$B \curvearrowright$ Reflections / Roots $\quad W$: affine Weyl root

$W = W_f \ltimes \mathbb{Z}^n$

finite Weyl group

$$\begin{tikzcd} B_{ex} \arrow[r] \arrow[d, phantom, "\circlearrowright"] & W_{ex} = \Gamma \ltimes W \arrow[d, phantom, "\circlearrowright"] \\ \mathcal{E} \arrow[r] & K_0(\mathcal{E}) \end{tikzcd}$$

$s_i \quad \alpha_i \longmapsto -\alpha_i$

$\alpha_j \longmapsto \alpha_j \quad j \neq i$

$s_i^2 = 1$ in W

Spherical twist: $s_i^2 \neq \mathrm{Id}$

dual $\mathbb{R}$ space:

$\mathrm{Hom}(K_0(\mathcal{E}), \mathbb{R})$ $\quad \Delta$ roots $\to$ Hyperplane arrangements in $\mathrm{Hom}(K_0(\mathcal{E}), \mathbb{R})$

Focus on $D^b_c\,\mathrm{Coh}(X) = D^b(\mathrm{nil}\text{-}\Pi)$

$$\begin{matrix} \mathcal{E}_c = & D^b_c\,\mathrm{Coh}(X) = D^b(\mathrm{nil}\text{-}\Pi) \\ & \cap \qquad\qquad \cap \\ & D^b\,\mathrm{Coh}(X) = D^b(\mathrm{mod}\text{-}\Pi) \end{matrix} \qquad \begin{matrix} \mathrm{Coh}_c(\mathcal{X}_J) \\ \cap \\ \mathcal{E}_c \end{matrix}$$

All preserved by $B_{ex} \subset \mathrm{Aut}(\mathcal{E})$

$\subset \mathrm{Aut}(\mathcal{E}_c)$

stability conditions: easy to construct stability conditions with heart nilp-Π

$\mathrm{Stab}^\circ(X)$ connected component of $\mathrm{Stab}(D^b_c\,\mathrm{Coh}(X))$

containing those (Type A: known to be all)

<u>Thm</u> (Bridgeland) $\mathrm{Stab}^\circ(X) \twoheadrightarrow h_{reg}/W$

covering group of deck Transfo generated by $B \times \mathbb{Z}[2]$

Universal cover

$$h_{reg} = \mathrm{Hom}(K_0, \mathbb{C}) \setminus \bigcup_{\alpha\in\Delta} \{Z \mid Z(\alpha) = 0\}$$

$B_{ex} = \mathrm{Aut}(\mathcal{E}_c)$ preserving $\mathrm{Stab}^\circ(X)$

$Z(\alpha_i)$ Volume of spheres on Mirror

<u>Lem</u>: Every heart appearing in $\mathrm{Stab}^\circ(X)$

is $b.\mathrm{Coh}_c(X_J)$ $\quad b \in B_{ex}$

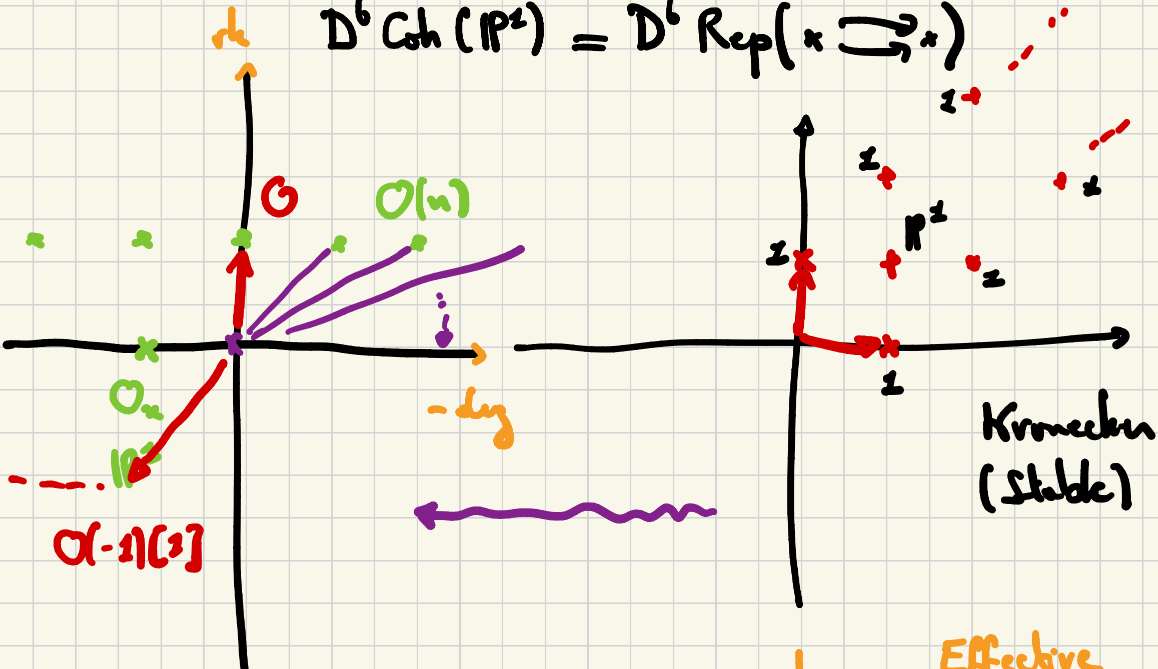
or $b.\overline{\mathrm{Coh}_c(X_J)}$ [tilt]

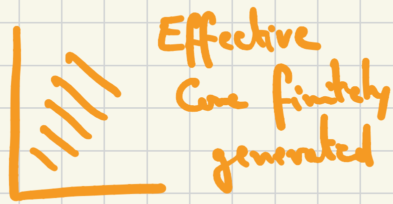
$\exists b_n \in B_{ex}$ $\quad b_n.\mathrm{Coh}_c(X_\emptyset) \longrightarrow \mathrm{Coh}_c(X_J)$

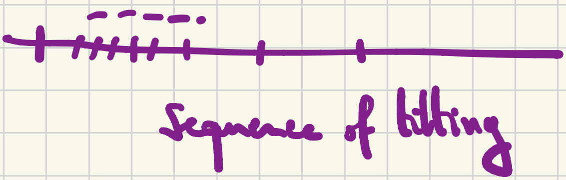
"limit"

Ex: Simpler situation: $\mathcal{O}, \mathcal{O}(1)$ Simple: $\mathcal{O}(-1)[1], \mathcal{O}$

$$D^b \mathrm{Coh}(\mathbb{P}^1) = D^b \mathrm{Rep}(\bullet \rightrightarrows \bullet)$$







$\sim A_1^{(1)}$

A_1 case

$T^*\mathbb{P}^1$

COHA: HA_J HA_ϕ : usual nil Π_Q

$J \neq \phi$

Formal thickening along curves: not finite type / limiting COHA

Λ_d moduli stack finite type pure dim $= -(d,d)$

definition: HA_J topological algebra

Claim $\longleftrightarrow$ $\neq$ sets of positive roots in a Lie algebra

Δ root system:

A set of positive root is a subset $\Delta_+ \subset \Delta$ s.t.

1) $\alpha, \beta \in \Delta_+$ s.t. $\alpha+\beta \in \Delta \Rightarrow \alpha+\beta \in \Delta_+$

2) $\forall \alpha \in \Delta$, $\alpha \in \Delta_+$ or $-\alpha \in \Delta_+$

3) $\forall \alpha \in \Delta_+$, $-\alpha \notin \Delta_+$

$\mathfrak{g}_f$: simple finite dim Lie algebra. All sets of positive roots are related by the (simply transitive) action of the Weyl group W_f.

$\mathfrak{g}$: affine Kac-Moody algebra: not true!

(Jacobsen-Kac) W fixes δ. Even mod $W \times \{\pm 1\}$

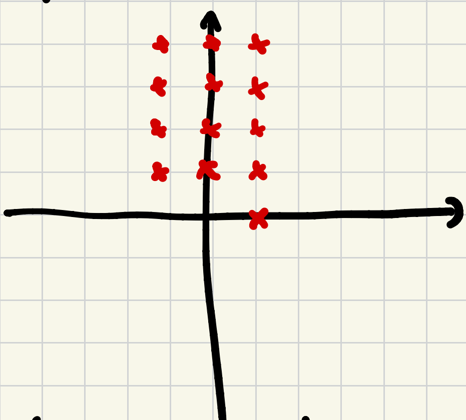
Standard: $\{\sum_{i=0}^{n} k_i \alpha_i \mid k_i \geq 0\} \setminus \{0\} =: \Delta_\phi$

Non-standard:

$$\forall J \subset I_f \quad \Delta_J^+ = \{\alpha + n\delta \mid \alpha \in \Delta_f^+ \cap J \quad n \in \mathbb{Z}\} \cup \Delta_f^+ \cup \{\alpha + n\delta \mid \alpha \in (\Delta_f \setminus J) \cup \{0\},\ n > 0\}$$

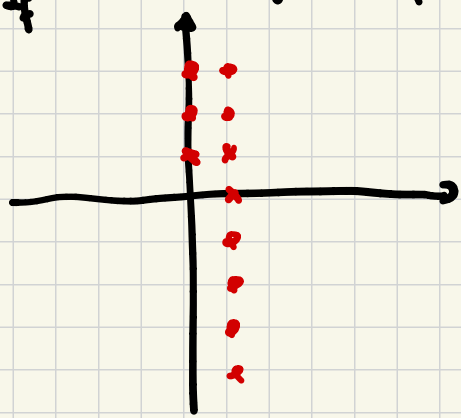
Ex:

$$\Delta_\phi^+ = \Delta_f^+ \cup \{\alpha + n\delta \mid \alpha \in \Delta_f^+ \cup \{0\},\ n > 0\}$$



~ Quiver

$$\Delta_{I_f}^+ = \{\alpha + n\delta \mid \alpha \in \Delta_f^+ \quad n \in \mathbb{Z}\} \cup \Delta_f^+ \cup \{n\delta \mid n > 0\}$$



Claim: All sets of positive roots are related by $W \times \{\pm\}$.

~ Geometric

Relevant Lie algebra for CoHA:

$$\mathfrak{g}_{ell} := \mathfrak{g}_f[s^{\pm}, t] \oplus K \qquad \begin{matrix} K \\ \text{central} \end{matrix}$$

$$K = \bigoplus_{\ell \in \mathbb{N}} \mathbb{Q} c_\ell \oplus \bigoplus_{\substack{\ell \in \mathbb{N}, \ell \geq 1 \\ k \in \mathbb{Z}\ k \neq 0}} \mathbb{Q} c_{k,\ell}$$

$$[x \otimes s^k t^\ell, y \otimes s^h t^n] = \begin{cases} [x,y]\, t^{\ell+n} + k(x,y) c_{\ell+n} & \text{if } k+h=0 \\ [x,y]\, s^{k+h} t^{\ell+n} + (kh - \ell n)(x,y) c_{k+h,\ell+n} & \text{if } k+h \neq 0 \end{cases}$$

$\mathbb{Z} \times \mathbb{Z}^I$-grading

$\deg(x s^k t^\ell) = (-2\ell, d + k\delta)$ $\quad x \in (\mathfrak{g}_f)_d$

$\deg(c_{k,\ell}) = (-2\ell, k\delta)$

$\deg c_\ell = (-2\ell, 0)$

$J \subset I_f \qquad \mathfrak{n}^+_{ell,J} := \bigoplus_{\beta \in \Delta_J} \mathfrak{n}_\beta[t] \oplus K_-$

$K_- = \bigoplus_{k<0} \mathbb{Q} c_{k,\ell}$

Thm: $HA_J \xrightarrow{\sim} \hat{U}(\mathfrak{n}^+_{ell,J})$

appropriate completion

$\ell = 0$

$\sim \mathfrak{g}\,\Big|^{\deg 0}_{\underline{\text{coh.}}}$

$J = \emptyset$

$x^+_{i,\ell} \longleftrightarrow (z_{i,1})^\ell \cap [\Lambda_{\alpha_i}]$

z c_1 of tautological bundle on Λ_{α_i}

Equivariant story $(\mathbb{C}^*)^2 \curvearrowright \mathbb{C}^2, X \dots$

$\varepsilon_1, \varepsilon_2$

$Y_{\varepsilon_1,\varepsilon_2}(\hat{\mathfrak{g}})$ affine Yangian

defo of $U(\mathfrak{g}_{ell})$