Plan of talk:

3. $\mathcal{C}^\infty$-Algebraic Geometry
   3.1 $\mathcal{C}^\infty$-rings
   3.2 Sheaves
   3.3 $\mathcal{C}^\infty$-schemes
3. $\mathcal{C}^\infty$-Algebraic Geometry

Our goal is to define the 2-category of d-manifolds $\text{dMan}$. First consider an algebro-geometric version of what we want to do. A good algebraic analogue of smooth manifolds are complex algebraic manifolds, that is, separated smooth $\mathbb{C}$-schemes $S$ of pure dimension. These form a full subcategory $\text{AlgMan}_C$ in the category $\text{Sch}_C$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects $S$ in $\text{Sch}_C$ whose cotangent complex $L_S$ is a vector bundle (i.e. perfect in the interval $[0,0]$).

To make a derived version of this, we first define an $\infty$-category $\text{DerSch}_C$ of derived $\mathbb{C}$-schemes, and then define the $\infty$-category $\text{DerAlgMan}_C$ of derived complex algebraic manifolds to be the full $\infty$-subcategory of objects $S$ in $\text{DerSch}_C$ which are quasi-smooth (have cotangent complex $L_S$ perfect in the interval $[-1,0]$), and satisfy some other niceness conditions (separated, etc.).

Thus, we have ‘classical’ categories $\text{AlgMan}_C \subset \text{Sch}_C$, and related ‘derived’ $\infty$-categories $\text{DerAlgMan}_C \subset \text{DerSch}_C$.

David Spivak, a student of Jacob Lurie, defined an $\infty$-category $\text{DerMan}_{\text{Spi}}$ of ‘derived smooth manifolds’ using a similar structure: he considered ‘classical’ categories $\text{Man} \subset \text{C}^\infty\text{Sch}$ and related ‘derived’ $\infty$-categories $\text{DerMan}_{\text{Spi}} \subset \text{DerC}^\infty\text{Sch}$. Here $\text{C}^\infty\text{Sch}$ is $\text{C}^\infty$-schemes, and $\text{DerC}^\infty\text{Sch}$ derived $\text{C}^\infty$-schemes. That is, before we can ‘derive’, we must first embed $\text{Man}$ into a larger category of $\text{C}^\infty$-schemes, singular generalizations of manifolds. Our set-up is a simplification of Spivak’s. I consider ‘classical’ categories $\text{Man} \subset \text{C}^\infty\text{Sch}$ and related ‘derived’ 2-categories $\text{dMan} \subset \text{dSpa}$, where $\text{dMan}$ is $d$-manifolds, and $\text{dSpa}$ $d$-spaces. Here $\text{dMan}, \text{dSpa}$ are roughly 2-category truncations of Spivak’s $\text{DerMan}, \text{DerC}^\infty\text{Sch}$ — see Borisov arXiv:1212.1153. This lecture will introduce classical $\text{C}^\infty$-schemes.
Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes. In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, $C^\infty$-schemes, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s. $C^\infty$-schemes are essentially algebraic objects, on which smooth real functions and calculus make sense.

The theory works by replacing commutative rings or $\mathbb{K}$-algebras in algebraic geometry by algebraic objects called $C^\infty$-rings.

Definition 3.1 (First definition of $C^\infty$-ring)

A $C^\infty$-ring is a set $\mathcal{C}$ together with $n$-fold operations $\Phi_f : \mathcal{C}^n \to \mathcal{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 0$, satisfying:

Let $m, n \geq 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathcal{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have

$$\Phi_\pi_j : (c_1, \ldots, c_n) \mapsto c_j.$$

A morphism of $C^\infty$-rings is a map of sets $\phi : \mathcal{C} \to \mathcal{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathcal{C}^n \to \mathcal{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $C^\infty$-Rings for the category of $C^\infty$-rings.
Definition 3.2 (Second definition of $C^\infty$-ring)

Write $\text{Euc}$ for the full subcategory of manifolds $\text{Man}$ with objects $\mathbb{R}^n$ for $n = 0, 1, \ldots$. That is, $\text{Euc}$ is the category with objects Euclidean spaces $\mathbb{R}^n$, and morphisms smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$. A $C^\infty$-ring is a product-preserving functor $F : \text{Euc} \to \text{Sets}$. That is, $F$ is a functor with functorial identifications $F(\mathbb{R}^{m+n}) = F(\mathbb{R}^m \times \mathbb{R}^n) \cong F(\mathbb{R}^m) \times F(\mathbb{R}^n)$ for all $m, n \geq 0$.

A morphism $\phi : F \to G$ of $C^\infty$-rings $F, G$ is a natural transformation of functors $\phi : F \Rightarrow G$.

Definitions 3.1 and 3.2 are equivalent as follows. Given $F : \text{Euc} \to \text{Sets}$ as above, define a set $\mathcal{C} = F(\mathbb{R})$. As $F$ is product-preserving, $F(\mathbb{R}^n) \cong F(\mathbb{R})^n = \mathcal{C}^n$ for all $n \geq 0$. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth then $F(f) : F(\mathbb{R}^n) \to F(\mathbb{R})$ is identified with a map $\Phi_f : \mathcal{C}^n \to \mathcal{C}$. Then $(\mathcal{C}, \Phi_f, f : \mathbb{R}^n \to \mathbb{R})$ is a $C^\infty$-ring as in Definition 3.1. Conversely, given $\mathcal{C}$ we define $F$ with $F(\mathbb{R}^n) = \mathcal{C}^n$.

Manifolds as $C^\infty$-rings

Let $X$ be a manifold, and write $\mathcal{C} = C^\infty(X)$ for the set of smooth functions $c : X \to \mathbb{R}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by $\Phi_f(c_1, \ldots, c_n)(x) = f(c_1(x), \ldots, c_n(x))$ for $x \in X$. These make $C^\infty(X)$ into a $C^\infty$-ring as in Definition 3.1. Define $F : \text{Euc} \to \text{Sets}$ by $F(\mathbb{R}^n) = \text{Hom}_{\text{Man}}(X, \mathbb{R}^n)$ and $F(f) = f \circ : \text{Hom}_{\text{Man}}(X, \mathbb{R}^m) \to \text{Hom}_{\text{Man}}(X, \mathbb{R}^n)$ for $f : \mathbb{R}^m \to \mathbb{R}^n$ smooth. Then $F$ is a $C^\infty$-ring as in Definition 3.2.

If $f : X \to Y$ is smooth map of manifolds then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of $C^\infty$-rings; conversely, if $\phi : C^\infty(Y) \to C^\infty(X)$ is a morphism of $C^\infty$-rings then $\phi = f^*$ for some unique smooth $f : X \to Y$. This gives a full and faithful functor $F : \text{Man} \to C^\infty\text{Rings}^{\text{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$. Thus, we can think of manifolds as examples of $C^\infty$-rings. But there are many more $C^\infty$-rings than manifolds. For example, $C^0(X)$ is a $C^\infty$-ring for any topological space $X$. 
Every $\mathcal{C}^\infty$-ring $\mathcal{C}$ is a commutative $\mathbb{R}$-algebra, where addition is $c + d = \Phi_f(c, d)$ for $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x + y$, and multiplication is $c \cdot d = \Phi_g(c, d)$ for $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = xy$, multiplication by $\alpha \in \mathbb{R}$ is $\alpha c = \Phi_h(c)$ for $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \alpha x$.

An ideal $I \subseteq \mathcal{C}$ in a $\mathcal{C}^\infty$-ring $\mathcal{C}$ is an ideal in $\mathcal{C}$ as an $\mathbb{R}$-algebra. Then the quotient vector space $\mathcal{C}/I$ is a commutative $\mathbb{R}$-algebra.

**Proposition 3.3**

If $\mathcal{C}$ is a $\mathcal{C}^\infty$-ring and $I \subseteq \mathcal{C}$ an ideal, then there is a unique $\mathcal{C}^\infty$-ring structure on $\mathcal{C}/I$ such that the projection $\pi : \mathcal{C} \to \mathcal{C}/I$ is a morphism of $\mathcal{C}^\infty$-rings.

**Definition**

A $\mathcal{C}^\infty$-ring $\mathcal{C}$ is called finitely generated if $\mathcal{C} \cong \mathcal{C}^\infty(\mathbb{R}^n)/I$ for some ideal $I \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$.

**Proof of Proposition 3.3**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and $c_1 + I, \ldots, c_n + I \in \mathcal{C}/I$. For $\pi : \mathcal{C} \to \mathcal{C}/I$ to be a morphism of $\mathcal{C}^\infty$-rings, we are forced to set

$$
\Phi_f(c_1 + I, \ldots, c_n + I) = \Phi_f(c_1, \ldots, c_n) + I,
$$

which determines the $\mathcal{C}^\infty$-ring structure on $\mathcal{C}/I$ completely. The only thing to prove is that this is well-defined. That is, if $c'_1, \ldots, c'_n \in \mathcal{C}$ with $c_i - c'_i \in I$, so that $c_1 + I = c'_1 + I, \ldots, c_n + I = c'_n + I$, we must show that

$$
\Phi_f(c_1, \ldots, c_n) - \Phi_f(c'_1, \ldots, c'_n) \in I.
$$
Proof of Proposition 3.3

Lemma 3.4 (Hadamard’s Lemma)

Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is smooth. Then there exist smooth \( g_i : \mathbb{R}^{2n} \to \mathbb{R} \) for \( i = 1, \ldots, n \) such that for all \( x_j, y_j \) we have

\[
f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) = \sum_{i=1}^{n} g_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \cdot (x_i - y_i).
\]

Note that \( g_i(x_1, \ldots, x_n, x_1, \ldots, x_n) = \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) \), so Hadamard’s Lemma gives an algebraic interpretation of partial derivatives. The definition of \( C^\infty \)-ring implies that

\[
\Phi f(c_1, \ldots, c_n) - \Phi f(c'_1, \ldots, c'_n) = \sum_{i=1}^{n} \Phi g_i(c_1, \ldots, c_n, c'_1, \ldots, c'_n) \cdot (c_i - c'_i),
\]

which lies in \( I \) as \( c_i - c'_i \in I \), as we have to prove.

Example 3.5 (Finitely presented \( C^\infty \)-rings. Compare Example 1.1.)

Suppose \( p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R} \) are smooth functions. Then \( C^\infty(\mathbb{R}^n) \) is a \( C^\infty \)-ring, and so an \( \mathbb{R} \)-algebra. Write \( I = (p_1, \ldots, p_k) \) for the ideal in \( C^\infty(\mathbb{R}^n) \) generated by \( p_1, \ldots, p_k \). Then \( C^\infty(\mathbb{R}^n)/(p_1, \ldots, p_k) \) is a \( C^\infty \)-ring, by Proposition 3.3. We think of it as the \( C^\infty \)-ring of functions on the smooth space \( X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : p_i(x_1, \ldots, x_n) = 0, i = 1, \ldots, k\} \). Note that \( X \) may be singular.

Example 3.6

Let \( I \subset C^\infty(\mathbb{R}) \) be the ideal of all smooth \( f : \mathbb{R} \to \mathbb{R} \) with \( f(x) = 0 \) for all \( x \geq 0 \). Then \( I \) is not finitely generated, so \( C^\infty(\mathbb{R}) \) is not noetherian as an \( \mathbb{R} \)-algebra. This is one way in which \( C^\infty \)-algebraic geometry behaves worse than ordinary algebraic geometry. We think of \( C^\infty(\mathbb{R})/I \) as the \( C^\infty \)-ring of smooth functions \( f : [0, \infty) \to \mathbb{R} \).
Definition

A $C^\infty$-ring $\mathcal{C}$ is a $C^\infty$-local ring if as an $\mathbb{R}$-algebra, $\mathcal{C}$ has a unique maximal ideal $m$, with $\mathcal{C}/m \cong \mathbb{R}$.

Example 3.7

Let $X$ be a manifold, and $x \in X$. Write $C^\infty_x(X)$ for the $C^\infty$-ring of germs of smooth functions $f : X \to \mathbb{R}$ at $x$. That is, elements of $C^\infty_x(X)$ are $\simeq$-equivalence classes $[U, f]$ of pairs $(U, f)$, where $x \in U \subseteq X$ is open and $f : U \to \mathbb{R}$ is smooth, and $(U, f) \sim (U', f')$ if there exists open $x \in U'' \subseteq U \cap U'$ with $f|_{U''} = f'|_{U''}$. Then $C^\infty_x(X)$ is a $C^\infty$-local ring.

Definition

An ideal $I \subseteq C^\infty(\mathbb{R}^n)$ is called fair if for $f \in C^\infty(\mathbb{R}^n)$, $\pi_x(f) \in \pi_x(I)$ for all $x \in \mathbb{R}^n$ implies that $f \in I$, where $\pi_x : C^\infty(\mathbb{R}^n) \to C^\infty_x(\mathbb{R}^n)$ is the projection. A $C^\infty$-ring $\mathcal{C}$ is called fair if $\mathcal{C} \cong C^\infty(\mathbb{R}^n)/I$ for $I \subseteq C^\infty(\mathbb{R}^n)$ a fair ideal.

Modules over $C^\infty$-rings

Definition

Let $\mathcal{C}$ be a $C^\infty$-ring. A module over $\mathcal{C}$ is a module over $\mathcal{C}$ as an $\mathbb{R}$-algebra.

You might expect that the definition of module should involve the operations $\Phi_f$ as well as the $\mathbb{R}$-algebra structure, but it does not.

Example 3.8

Let $X$ be a manifold, and $E \to X$ a vector bundle. Then $C^\infty(X)$ is a $C^\infty$-ring, and the vector space $C^\infty(E)$ of smooth sections of $E$ is a module over $C^\infty(X)$. 
Cotangent modules

Definition

Let $\mathcal{C}$ be a $C^\infty$-ring, and $M$ a $\mathcal{C}$-module. A $C^\infty$-derivation is an $\mathbb{R}$-linear map $d : \mathcal{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathcal{C}$, we have

$$d \Phi_f(c_1, \ldots, c_n) = \sum_{i=1}^n \Phi \frac{\partial f}{\partial x_i}(c_1, \ldots, c_n) \cdot dc_i.$$

Note that $d$ is not a morphism of $\mathcal{C}$-modules. We call such a pair $\Omega_{\mathcal{C}}, d_{\mathcal{C}}$ a cotangent module for $\mathcal{C}$ if it has the universal property that for any $\mathcal{C}$-module $M$ and $C^\infty$-derivation $d : \mathcal{C} \to M$, there is a unique morphism of $\mathcal{C}$-modules $\phi : \Omega_{\mathcal{C}} \to M$ with $d = \phi \circ d_{\mathcal{C}}$.

Every $C^\infty$-ring has a cotangent module, unique up to isomorphism.

Example 3.9

Let $X$ be a manifold, with cotangent bundle $T^*X$. Then $C^\infty(T^*X)$ is a cotangent module for the $C^\infty$-ring $C^\infty(X)$.

3.2. Sheaves

Sheaves are a central idea in algebraic geometry.

Definition

Let $X$ be a topological space. A presheaf of sets $\mathcal{E}$ on $X$ consists of a set $\mathcal{E}(U)$ for each open $U \subseteq X$, and a restriction map $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for all open $V \subseteq U \subseteq X$, such that:

(i) $\mathcal{E}(\emptyset) = \ast$ is one point;
(ii) $\rho_{UU} = \text{id}_{\mathcal{E}(U)}$ for all open $U \subseteq X$; and
(iii) $\rho_{UV} = \rho_{VV} \circ \rho_{UV}$ for all open $W \subseteq V \subseteq U \subseteq X$.

We call $\mathcal{E}$ a sheaf if also whenever $U \subseteq X$ is open and $\{V_i : i \in I\}$ is an open cover of $U$, then:

(iv) If $s, t \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = \rho_{UV_i}(t)$ for all $i \in I$, then $s = t$;
(v) If $s_i \in \mathcal{E}(V_i)$ for all $i \in I$ with $\rho_{V_i(V_i \cap V_j)}(s_i) = \rho_{V_j(V_i \cap V_j)}(s_j)$ in $\mathcal{E}(V_i \cap V_j)$ for all $i, j \in I$, then there exists $s \in \mathcal{E}(U)$ with $\rho_{UV_i}(s) = s_i$ for all $i \in I$. This $s$ is unique by (iv).
**Definition**

Let $\mathcal{E}, \mathcal{F}$ be (pre)sheaves on $X$. A morphism $\phi : \mathcal{E} \to \mathcal{F}$ consists of a map $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ for all open $U \subseteq X$, such that $\rho_{UV} \circ \phi(U) = \phi(V) \circ \rho_{UV} : \mathcal{E}(U) \to \mathcal{F}(V)$ for all open $V \subseteq U \subseteq X$. Then sheaves form a category.

If $\mathcal{C}$ is any category in which direct limits exist, such as the categories of sets, rings, vector spaces, $C^\infty$-rings, ..., then we can define (pre)sheaves $\mathcal{E}$ of objects in $\mathcal{C}$ on $X$ in the obvious way, and morphisms $\phi : \mathcal{E} \to \mathcal{F}$ by taking $\mathcal{E}(U)$ to be an object in $\mathcal{C}$, and $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$, $\phi(U) : \mathcal{E}(U) \to \mathcal{F}(U)$ to be morphisms in $\mathcal{C}$, and $\mathcal{E}(\emptyset)$ to be a terminal object in $\mathcal{C}$ (e.g. the zero ring). So in particular, we can define sheaves of $C^\infty$-rings on $X$.

Almost any class of functions on $X$, or sections of a bundle on $X$, will form a sheaf on $X$. To be a sheaf means to be ‘local on $X$', determined by its behaviour on any cover of small open sets.

**Stalks of sheaves**

**Definition**

Let $X$ be a topological space, and $\mathcal{E}$ a (pre)sheaf of sets (or $C^\infty$-rings, etc.) on $X$, and $x \in X$. The stalk $\mathcal{E}_x$ of $\mathcal{E}$ at $x$ is

$$\mathcal{E}_x = \lim_{\xrightarrow{x \in U \subseteq X}} \mathcal{E}(U),$$

where the direct limit (as a set, or $C^\infty$-ring, etc.) is over all open $U \subseteq X$ with $x \in U$ using $\rho_{UV} : \mathcal{E}(U) \to \mathcal{E}(V)$ for open $x \in V \subseteq U \subseteq X$. That is, for all open $x \in U \subseteq X$ we have a morphism $\pi_x : \mathcal{E}(U) \to \mathcal{E}_x$, such that for all $x \in V \subseteq U \subseteq X$ we have $\pi_x = \pi_x \circ \rho_{UV}$, and $\mathcal{E}_x$ is universal with this property.

**Example 3.10**

Let $X$ be a manifold. Define a sheaf of $C^\infty$-rings $O_X$ on $X$ by $O_X(U) = C^\infty(U)$ for all open $U \subseteq X$, as a $C^\infty$-ring, and $\rho_{UV} : C^\infty(U) \to C^\infty(V)$, $\rho_{UV} : f \mapsto f|_V$ for all open $V \subseteq U \subseteq X$. The stalk $O_{X,x}$ at $x \in X$ is $C^\infty_x(X)$ from Example 3.7.
Sheafification and pullbacks

**Definition**

Let $X$ be a topological space and $\mathcal{E}$ a presheaf (of sets, $C^\infty$-rings, etc.) on $X$. A sheafification of $\mathcal{E}$ is a sheaf $\mathcal{E}'$ and a morphism of presheaves $\pi : \mathcal{E} \to \mathcal{E}'$, with the universal property that any morphism $\phi : \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ a sheaf factorizes uniquely as $\phi = \phi' \circ \pi$ for $\phi' : \mathcal{E}' \to \mathcal{F}$.

Any presheaf $\mathcal{E}$ has a sheafification $\mathcal{E}'$, unique up to canonical isomorphism, and the stalks satisfy $\mathcal{E}_x \cong \mathcal{E}'_x$.

**Definition**

Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf on $Y$. Define a presheaf $\mathcal{P}f^{-1}(\mathcal{E})$ on $X$ by

$$\mathcal{P}f^{-1}(\mathcal{E}) = \lim_{\longrightarrow} V \supseteq f(U) \mathcal{E}(V),$$

where the direct limit is over open $V \subseteq Y$ with $f(U) \subseteq V$. Define the *pullback sheaf* $f^{-1}(\mathcal{E})$ to be the sheafification of $\mathcal{P}f^{-1}(\mathcal{E})$.

We can now define $C^\infty$-schemes almost exactly as for schemes in algebraic geometry, but replacing rings or $K$-algebras by $C^\infty$-rings.

**Definition**

A $C^\infty$-ringed space $X = (X, \mathcal{O}_X)$ is a topological space $X$ with a sheaf of $C^\infty$-rings $\mathcal{O}_X$. It is called a *local $C^\infty$-ringed space* if the stalks $\mathcal{O}_{X,x}$ are $C^\infty$-local rings for all $x \in X$.

A morphism $f : X \to Y$ of $C^\infty$-ringed spaces is $f = (f, f^\#)$, where $f : X \to Y$ is a continuous map of topological spaces, and $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of $C^\infty$-rings on $X$.

Write $C^\infty\text{RS}$ for the category of $C^\infty$-ringed spaces, and $LC^\infty\text{RS}$ for the full subcategory of local $C^\infty$-ringed spaces.
Definition

The global sections functor \( \Gamma : \mathcal{L}C^\infty RS \to C^\infty \text{Rings}^{\text{op}} \) maps \( \Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X) \). It has a right adjoint, the spectrum functor \( \text{Spec} : C^\infty \text{Rings}^{\text{op}} \to \mathcal{L}C^\infty RS \). That is, for each \( C^\infty \)-ring \( \mathcal{C} \) we construct a local \( C^\infty \)-ringed space \( X = \text{Spec} \mathcal{C} \). Points \( x \in X \) are \( \mathbb{R} \)-algebra morphisms \( x : \mathcal{C} \to \mathbb{R} \) (this implies \( x \) is a \( C^\infty \)-ring morphism). Then each \( c \in \mathcal{C} \) defines a map \( c : X \to \mathbb{R} \). We give \( X \) the weakest topology such that these \( c : X \to \mathbb{R} \) are continuous for all \( c \in \mathcal{C} \). We don’t use prime ideals.

In algebraic geometry, \( \text{Spec} : \text{Rings}^{\text{op}} \to \mathcal{L}RS \) is full and faithful. In \( C^\infty \)-algebraic geometry, it is full but not faithful, that is, \( \text{Spec} \) forgets some information, as we don’t use prime ideals. But on the subcategory \( C^\infty \text{Rings}^{\text{fa}} \) of fair \( C^\infty \)-rings, \( \text{Spec} \) is full and faithful.

Definition

A local \( C^\infty \)-ringed space \( X \) is called an affine \( C^\infty \)-scheme if \( X \cong \text{Spec} \mathcal{C} \) for some \( C^\infty \)-ring \( \mathcal{C} \). We call \( X \) a \( C^\infty \)-scheme if \( X \) can be covered by open subsets \( U \) with \( (U, \mathcal{O}_X|_U) \) an affine \( C^\infty \)-scheme. Write \( C^\infty \text{Sch} \) for the full subcategory of \( C^\infty \)-schemes in \( \mathcal{L}C^\infty RS \).

If \( X \) is a manifold, define a \( C^\infty \)-scheme \( \bar{X} = (X, \mathcal{O}_X) \) by \( \mathcal{O}_X(U) = C^\infty(U) \) for all open \( U \subseteq X \). Then \( \bar{X} \cong \text{Spec} C^\infty(X) \). This defines a full and faithful embedding \( \text{Man} \hookrightarrow C^\infty \text{Sch} \). So we can regard manifolds as examples of \( C^\infty \)-schemes.
Think of a $C^\infty$-ringed space $X$ as a topological space $X$ with a notion of ‘smooth function’ $f : U \to \mathbb{R}$ for open $U \subseteq X$, i.e.
$f \in \mathcal{O}_X(U)$. If $X$ is a local $C^\infty$-ringed space then the notion of ‘value of $f$ in $\mathbb{R}$ at a point $x \in U$’ makes sense, since we can compose the maps $f \in \mathcal{O}_X(U) \xrightarrow{\pi_x} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m \cong \mathbb{R}$. If $X$ is a $C^\infty$-scheme, then for small open $U \subseteq X$ we can locally reconstruct the sheaf $\mathcal{O}_X|_U$ from the $C^\infty$-ring $\mathcal{O}_X(U)$.

All fibre products exist in $C^\infty\text{Sch}$. In manifolds $\text{Man}$, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When $g$, $h$ are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $C^\infty\text{Sch}$, but may not be a manifold.

We also define vector bundles and quasicoherent sheaves on a $C^\infty$-scheme $X$, as sheaves of $\mathcal{O}_X$-modules, and write $\text{qcoh}(X)$ for the abelian category of quasicoherent sheaves. A $C^\infty$-scheme $X$ has a well-behaved cotangent sheaf $T^*_X$.

Differences with ordinary Algebraic Geometry

- In algebraic geometry, central examples of schemes such as $\mathbb{CP}^n$ are not affine. In $C^\infty$-algebraic geometry, most interesting $C^\infty$-schemes are affine (e.g. all manifolds), except for non-Hausdorff $C^\infty$-schemes. But scheme theory is still useful, to glue things from local data.
- The topology on $C^\infty$-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.
- Can find smooth functions supported on (almost) any open set.
- (Almost) any open cover has a subordinate partition of unity.
- Our $C^\infty$-rings $\mathcal{C}$ are generally not noetherian as $\mathbb{R}$-algebras. So ideals $I$ in $\mathcal{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$. This means there is not a well-behaved notion of coherent sheaf.
Plan of talk:

4 2-categories, d-spaces, and d-manifolds

4.1 2-categories

4.2 Differential graded $C^\infty$-rings

4.3 D-spaces

4.4 D-manifolds
4. 2-categories, d-spaces, and d-manifolds

Our goal is to define the 2-category of d-manifolds $\text{dMan}$. To do this we will define a 2-category $\text{dSpa}$ of ‘d-spaces’, a kind of derived $C^\infty$-scheme, and then define d-manifolds $\text{dMan} \subset \text{dSpa}$ to be a special kind of d-space, just as manifolds $\text{Man} \subset C^\infty\text{Sch}$ are a special kind of $C^\infty$-scheme.

First we introduce 2-categories. There are two kinds, strict 2-categories and weak 2-categories. We will meet both, as d-manifolds and d-orbifolds $\text{dMan}, \text{dOrb}$ are strict 2-categories, but Kuranishi spaces $\text{Kur}$ are a weak 2-category. Every weak 2-category $\mathcal{C}$ is equivalent as a weak 2-category to a strict 2-category $\mathcal{C}'$ (weak 2-categories can be ‘strictified’), so there is no fundamental difference, but weak 2-categories have more notation.

4.1. 2-categories

A 2-category $\mathcal{C}$ has objects $X, Y, \ldots$, 1-morphisms $f, g : X \to Y$ (morphisms), and 2-morphisms $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

Example 4.1

(a) The strict 2-category $\text{Cat}$ has objects categories $\mathcal{C}, \mathcal{D}, \ldots$, 1-morphisms functors $F, G : \mathcal{C} \to \mathcal{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.

(b) The strict 2-category $\text{Top}^{\text{ho}}$ of topological spaces up to homotopy has objects topological spaces $X, Y, \ldots$, 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies $H$ from $f$ to $g$. That is, $H : X \times [0, 1] \to Y$ is continuous with $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, and $H, H' : X \times [0, 1] \to Y$ are isotopic if there exists continuous $l : X \times [0, 1]^2 \to Y$ with $l(x, s, 0) = H(x, s)$, $l(s, x, 1) = H'(x, s)$, $l(x, 0, t) = f(x)$, $l(x, 1, t) = g(x)$. 
Definition

A (strict) 2-category $\mathcal{C}$ consists of a proper class of objects $\text{Obj}(\mathcal{C})$, for all $X, Y \in \text{Obj}(\mathcal{C})$ a category $\text{Hom}(X, Y)$, for all $X$ in $\text{Obj}(\mathcal{C})$ an object $\text{id}_X$ in $\text{Hom}(X, X)$ called the identity $1$-morphism, and for all $X, Y, Z$ in $\text{Obj}(\mathcal{C})$ a functor $\mu_{X,Y,Z} : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \to \text{Hom}(X, Z)$. These must satisfy the identity property, that

$$\mu_{X,X,Y}(\text{id}_X, -) = \mu_{X,Y,Y}(-, \text{id}_Y) = \text{id}_{\text{Hom}(X,Y)}$$ (4.1)

as functors $\text{Hom}(X, Y) \to \text{Hom}(X, Y)$, and the associativity property, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \text{id}) = \mu_{W,X,Z} \circ (\text{id} \times \mu_{X,Y,Z})$$ (4.2)

as functors $\text{Hom}(W, X) \times \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \to \text{Hom}(W, X)$.

Objects $f$ of $\text{Hom}(X, Y)$ are called 1-morphisms, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \text{Hom}_{\text{Hom}(X,Y)}(f, g)$ are called 2-morphisms, written $\eta : f \Rightarrow g$.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f, g)$ is the horizontal composition of 1-morphisms, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of $\eta, \zeta$ in $\text{Hom}(X, Y)$ gives the vertical composition of 2-morphisms of $\eta, \zeta$, written $\zeta \circ \eta : f \Rightarrow h$, as a diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta & & \downarrow \zeta \\
\downarrow g & \Rightarrow & \downarrow h \\
X & \xrightarrow{\zeta \circ \eta} & Y.
\end{array}
$$ (4.3)
And if \( f, \tilde{f} : X \to Y \) and \( g, \tilde{g} : Y \to Z \) are 1-morphisms and \( \eta : f \Rightarrow \tilde{f}, \zeta : g \Rightarrow \tilde{g} \) are 2-morphisms then \( \mu_{X,Y,Z}(\eta, \zeta) \) is the horizontal composition of 2-morphisms, written \( \zeta \ast \eta : g \circ f \Rightarrow \tilde{g} \circ \tilde{f} \), as a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta & & \downarrow \zeta \\
\tilde{f} & \Rightarrow & \tilde{g}
\end{array}
\quad \sim \quad
\begin{array}{ccc}
X & \xrightarrow{g \circ f} & Z \\
\downarrow \zeta \ast \eta & & \\
\tilde{g} \circ \tilde{f}
\end{array}
\]

(4.4)

There are also two kinds of identity: identity 1-morphisms \( \text{id}_X : X \to X \) and identity 2-morphisms \( \text{id}_f : f \Rightarrow f \).

A 2-morphism is a 2-isomorphism if it is invertible under vertical composition. A 2-category is called a (2,1)-category if all 2-morphisms are 2-isomorphisms. For example, stacks in algebraic geometry form a (2,1)-category.

In a 2-category \( \mathcal{C} \), there are three notions of when objects \( X, Y \) in \( \mathcal{C} \) are ‘the same’: equality \( X = Y \), and isomorphism, that is we have 1-morphisms \( f : X \to Y, g : Y \to X \) with \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \), and equivalence, that is we have 1-morphisms \( f : X \to Y, g : Y \to X \) and 2-isomorphisms \( \eta : g \circ f \Rightarrow \text{id}_X \) and \( \zeta : f \circ g \Rightarrow \text{id}_Y \). Usually equivalence is the correct notion.

Commutative diagrams in 2-categories should in general only commute up to (specified) 2-isomorphisms, rather than strictly. A simple example of a commutative diagram in a 2-category \( \mathcal{C} \) is

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \eta & & \downarrow g \\
\downarrow h & & \\
& & Z,
\end{array}
\]

which means that \( X, Y, Z \) are objects of \( \mathcal{C} \), \( f : X \to Y, g : Y \to Z \) and \( h : X \to Z \) are 1-morphisms in \( \mathcal{C} \), and \( \eta : g \circ f \Rightarrow h \) is a 2-isomorphism.
Definition (Fibre products in 2-categories. Compare §2.3.)

Let $\mathcal{C}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathcal{C}$. A fibre product $X \times_Z Y$ in $\mathcal{C}$ is an object $W$, 1-morphisms $\pi_X : W \to X$ and $\pi_Y : W \to Y$ and a 2-isomorphism $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in $\mathcal{C}$ with the following universal property: suppose $\pi'_X : W' \to X$ and $\pi'_Y : W' \to Y$ are 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ is a 2-isomorphism in $\mathcal{C}$. Then there exists a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X$, $\zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram commutes:

$$
\begin{array}{ccc}
W & \xrightarrow{g \circ \pi_X \circ b} & h \circ \pi_Y \circ b \\
\downarrow \text{id}_g \ast \zeta_X & & \downarrow \text{id}_h \ast \zeta_Y \\
W' & \xrightarrow{g \circ \pi'_X} & h \circ \pi'_Y.
\end{array}
$$

Furthermore, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of $b, \zeta_X, \zeta_Y$ then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with $\tilde{\zeta}_X = \zeta_X \circ (\text{id} \circ \pi_X \ast \theta)$ and $\tilde{\zeta}_Y = \zeta_Y \circ (\text{id} \circ \pi_Y \ast \theta)$.

If a fibre product $X \times_Z Y$ exists, it is unique up to equivalence.

Weak 2-categories

A weak 2-category, or bicategory, is like a strict 2-category, except that the equations of functors (4.1), (4.2) are required to hold not up to equality, but up to specified natural isomorphisms. That is, a weak 2-category $\mathcal{C}$ consists of data $\text{Obj}(\mathcal{C}), \text{Hom}(X, Y), \mu_{X,Y,Z}, \text{id}_X$ as above, but in place of (4.1), a natural isomorphism

$$
\alpha : \mu_{W,X,Y} \circ (\mu_{W,X,Y} \times \text{id}) \Rightarrow \mu_{W,X,Z} \circ (\text{id} \times \mu_{X,Y,Z}),
$$

and in place of (4.2), natural isomorphisms

$$
\beta : \mu_{X,Y} (\text{id}_X, -) \Rightarrow \text{id}, \quad \gamma : \mu_{X,Y} (-, \text{id}_Y) \Rightarrow \text{id},
$$

satisfying some identities. That is, composition of 1-morphisms is associative only up to specified 2-isomorphisms, so for 1-morphisms $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ we have a 2-isomorphism

$$
\alpha_{g \circ f, e} : (g \circ f) \circ e \Rightarrow g \circ (f \circ e).
$$

Similarly identities $\text{id}_X, \text{id}_Y$ work up to 2-isomorphism, so for each $f : X \to Y$ we have 2-isomorphisms

$$
\beta_f : f \circ \text{id}_X \Rightarrow f, \quad \gamma_f : \text{id}_Y \circ f \Rightarrow f.
$$
4.2. Differential graded \( \mathbb{C}^\infty \)-rings

As in \( \S 2 \), to define derived \( \mathbb{K} \)-schemes, we replaced commutative \( \mathbb{K} \)-algebras by commutative differential graded \( \mathbb{K} \)-algebras (or simplicial \( \mathbb{K} \)-algebras). So, to define derived \( \mathbb{C}^\infty \)-schemes, we should replace \( \mathbb{C}^\infty \)-rings by differential graded \( \mathbb{C}^\infty \)-rings (or perhaps simplicial \( \mathbb{C}^\infty \)-rings, as in Spivak and Borisov–Noël).

**Definition**

A differential graded \( \mathbb{C}^\infty \)-ring (or \( \text{dg} \, \mathbb{C}^\infty \)-ring) \( \mathcal{C}^\bullet = (\mathcal{C}^*, \partial) \) is a commutative differential graded \( \mathbb{R} \)-algebra \( (\mathcal{C}^*, \partial) \) in degrees \( \leq 0 \), as in \( \S 2.2 \), together with the structure \( (\Phi_f)_{f : \mathbb{R}^n \to \mathbb{R} \mathbb{C}^\infty} \) of a \( \mathbb{C}^\infty \)-ring on \( \mathcal{C}^0 \), such that the \( \mathbb{R} \)-algebra structures on \( \mathcal{C}^0 \) from the \( \mathbb{C}^\infty \)-ring and the cdga over \( \mathbb{R} \) agree.

A morphism \( \phi : \mathcal{C}^\bullet \to \mathcal{D}^\bullet \) of \( \text{dg} \, \mathbb{C}^\infty \)-rings is maps \( \phi^k : \mathcal{C}^k \to \mathcal{D}^k \) for all \( k \leq 0 \), such that \( (\phi^k)_{k \leq 0} \) is a morphism of cdgas over \( \mathbb{R} \), and \( \phi^0 : \mathcal{C}^0 \to \mathcal{D}^0 \) is a morphism of \( \mathbb{C}^\infty \)-rings.

Then \( \text{dg} \, \mathbb{C}^\infty \)-rings form an (\( \infty \)-)category \( \text{DGC}^\infty \text{Rings} \).

One could use \( \text{dg} \, \mathbb{C}^\infty \)-rings to define ‘derived \( \mathbb{C}^\infty \)-schemes’ and ‘derived \( \mathbb{C}^\infty \)-stacks’ as functors \( F : \text{DGC}^\infty \text{Rings} \to \text{SSets} \). An alternative is to use simplicial \( \mathbb{C}^\infty \)-rings \( \text{SC}^\infty \text{Rings} \), as in Spivak 2008, Borisov–Noël 2011, and Borisov 2012.

**Example 4.2** (Kuranishi neighbourhoods. Compare Example 2.1.)

Let \( V \) be a smooth manifold, and \( E \to V \) a smooth real vector bundle of rank \( n \), and \( s : V \to E \) a smooth section. Define a dg \( \mathbb{C}^\infty \)-ring \( \mathcal{C}^\bullet \) as follows: take \( \mathcal{C}^0 = \mathbb{C}^\infty(V) \), with its natural \( \mathbb{R} \)-algebra and \( \mathbb{C}^\infty \)-ring structures. Set \( \mathcal{C}^k = \mathbb{C}^\infty(\Lambda^{-k} E^*) \) for \( k = -1, -2, \ldots, -n \), and \( \mathcal{C}^k = 0 \) for \( k < -n \). The multiplication \( \mathcal{C}^k \times \mathcal{C}^l \to \mathcal{C}^{k+l} \) are multiplication by functions in \( \mathbb{C}^\infty(V) \) if \( k = 0 \) or \( l = 0 \), and wedge product \( \wedge : \Lambda^{-k} E^* \times \Lambda^{-l} E^* \to \Lambda^{-k-l} E^* \) if \( k, l < 0 \). The differential \( \partial : \mathcal{C}^k \to \mathcal{C}^{k+1} \) is contraction with \( s \), \( s : \Lambda^{-k} E^* \to \Lambda^{-k-1} E^* \).
We will use only a special class of dg $\mathcal{C}^\infty$-rings called square zero dg $\mathcal{C}^\infty$-rings, which form a 2-category $\mathbf{SZC}^\infty \mathbf{Rings}$.

**Definition**

A dg $\mathcal{C}^\infty$-ring $\mathcal{C}^\bullet$ is square zero if $\mathcal{C}^i = 0$ for $i < -1$ and $\mathcal{C}^{-1} \cdot d[\mathcal{C}^{-1}] = 0$. Then $\mathcal{C}$ is $\mathcal{C}^{-1} \xrightarrow{d} \mathcal{C}^0$, and $d[\mathcal{C}^{-1}]$ is a square zero ideal in the (ordinary) $\mathcal{C}^\infty$-ring $\mathcal{C}^0$, and $\mathcal{C}^{-1}$ is a module over the ‘classical’ $\mathcal{C}^\infty$-ring $H^0(\mathcal{C}^{-1}) = \mathcal{C}^0/d[\mathcal{C}^{-1}]$.

A 1-morphism $\alpha^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ in $\mathbf{SZC}^\infty \mathbf{Rings}$ is maps $\alpha^0 : \mathcal{C}^0 \to \mathcal{D}^0$, $\alpha^{-1} : \mathcal{C}^{-1} \to \mathcal{D}^{-1}$ preserving all the structure. Then $H^0(\alpha^\bullet) : H^0(\mathcal{C}) \to H^0(\mathcal{D})$ is a morphism of $\mathcal{C}^\infty$-rings.

For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ a 2-morphism $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathcal{C}^0 \to \mathcal{D}^{-1}$ with $\beta^0 = \alpha^0 + d \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ d$. There is an embedding of (2-)categories $\mathbf{C}^\infty \mathbf{Rings} \subset \mathbf{SZC}^\infty \mathbf{Rings}$ as the (2-)subcategory of $\mathcal{C}^\bullet$ with $\mathcal{C}^{-1} = 0$.

There is a truncation functor $T : \mathbf{DGC}^\infty \mathbf{Rings} \to \mathbf{SZC}^\infty \mathbf{Rings}$, where if $\mathcal{C}^\bullet$ is a dg $\mathcal{C}^\infty$-ring, then $\mathcal{D}^\bullet = T(\mathcal{C}^\bullet)$ is the square zero $\mathcal{C}^\infty$-ring with

\[
\mathcal{D}^0 = \mathcal{C}^0/[d\mathcal{C}^{-1}]^2, \quad \mathcal{D}^{-1} = \mathcal{C}^{-1}/[d\mathcal{C}^{-2} + (d\mathcal{C}^{-1}) \cdot \mathcal{C}^{-1}]].
\]

Applied to Example 4.2 this gives:

**Example 4.3 (Kuranishi neighbourhoods. Compare Example 4.2.)**

Let $V$ be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Associate a square zero dg $\mathcal{C}^\infty$-ring $\mathcal{C}^{-1} \xrightarrow{d} \mathcal{C}^0$ to the ‘Kuranishi neighbourhood’ $(V, E, s)$ by

\[
\mathcal{C}^0 = C^\infty(V)/I_s^2, \quad \mathcal{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*), \quad d(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,
\]

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by $s$.

These will be the local models for d-manifolds.
Cotangent complexes in the 2-category setting

Let $\mathcal{C}^\bullet$ be a square zero dg $C^\infty$-ring. Define the cotangent complex $L_{-1}^\mathcal{C} \xrightarrow{d_{\mathcal{C}}} L_0^\mathcal{C}$ to be the 2-term complex of $H^0(\mathcal{C}^\bullet)$-modules $\mathcal{C}^{-1} \xrightarrow{d_{\mathcal{C}} \circ d^{\mathcal{C}}} \Omega_{\mathcal{C}^0} \otimes_{\mathcal{C}^0} H^0(\mathcal{C}^\bullet)$, regarded as an element of the 2-category of 2-term complexes of $H^0(\mathcal{C}^\bullet)$-modules, with $\Omega_{\mathcal{C}^0}$ the cotangent module of the $C^\infty$-ring $\mathcal{C}^0$, as in §3.1. Let $\alpha^\bullet, \beta^\bullet : \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ be 1-morphisms and $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ a 2-morphism in $SZC^\infty$Rings. Then $H^0(\alpha^\bullet) = H^0(\beta^\bullet)$, so we may regard $\mathcal{D}^{-1}$ as an $H^0(\mathcal{C}^\bullet)$-module. And $\eta : \mathcal{C}^0 \to \mathcal{D}^{-1}$ is a derivation, so it factors through an $H^0(\mathcal{C}^\bullet)$-linear map $\hat{\eta} : \Omega_{\mathcal{C}^0} \otimes_{\mathcal{C}^0} H^0(\mathcal{C}^\bullet) \to \mathcal{D}^{-1}$. We have a diagram

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of cotangent complexes.

4.3. D-spaces

D-spaces are our notion of derived $C^\infty$-scheme:

Definition

A d-space $X$ is a topological space $X$ with a sheaf of square zero dg-$C^\infty$-rings $\mathcal{O}^\bullet_X = \mathcal{O}^{-1}_X \xrightarrow{d} \mathcal{O}^0_X$, such that $X = (X, H^0(\mathcal{O}^\bullet_X))$ and $(X, \mathcal{O}^0_X)$ are $C^\infty$-schemes, and $\mathcal{O}^{-1}_X$ is quasicoherent over $X$. We call $X$ the underlying classical $C^\infty$-scheme.

We require that the topological space $X$ should be Hausdorff and second countable, and the underlying classical $C^\infty$-scheme $X$ should be locally fair, i.e. covered by open $\text{Spec} \mathcal{C} \cong U \subseteq X$ for $\mathcal{C}$ a fair $C^\infty$-ring. Basically this means $X$ is locally finite-dimensional.

Note that $\mathcal{O}^\bullet_X$ is an ordinary (strict) sheaf of square zero dg $C^\infty$-rings, using only the objects and 1-morphisms in $SZC^\infty$Rings, and not (as usual in DAG) a homotopy sheaf using 2-isomorphisms $\rho_{VW} \circ \rho_{UV} \Rightarrow \rho_{UV}$ for open $W \subseteq V \subseteq U \subseteq X$. 
A 1-morphism \( f : X \to Y \) of d-spaces \( X, Y \) is \( f = (f, f^\#) \), where \( f : X \to Y \) is a continuous map of topological spaces, and \( f^\# : f^{-1}(\mathcal{O}_Y^\bullet) \to \mathcal{O}_X^\bullet \) is a morphism of sheaves of square zero dg \( \mathcal{C}^\infty \)-rings on \( X \). Then \( f = (f, H^0(f^\#)) : X \to Y \) is a morphism of the underlying classical \( \mathcal{C}^\infty \)-schemes.

Let \( f, g : X \to Y \) be 1-morphisms of d-spaces, and suppose the continuous maps \( f, g : X \to Y \) are equal. We have morphisms \( f^\#, g^\# : f^{-1}(\mathcal{O}_Y^\bullet) \to \mathcal{O}_X^\bullet \) of sheaves of square zero dg \( \mathcal{C}^\infty \)-rings. That is, \( f^\#, g^\# \) are sheaves on \( X \) of 1-morphisms in \( \text{SZC}^\infty \text{Rings} \).

A 2-morphism \( \eta : f \Rightarrow g \) is a sheaf on \( X \) of 2-morphisms \( \eta : f^\# \Rightarrow g^\# \) in \( \text{SZC}^\infty \text{Rings} \). That is, for each open \( U \subseteq X \), we have a 2-morphism \( \eta(U) : f^\#(U) \Rightarrow g^\#(U) \) in \( \text{SZC}^\infty \text{Rings} \), with \( \text{id}_{\rho_{UV}} \ast \eta(U) = \eta(V) \ast \text{id}_{\rho_{UV}} \) for all open \( V \subseteq U \subseteq X \).

With the obvious notions of composition of 1- and 2-morphisms, and identities, d-spaces form a strict 2-category \( \text{dSpa} \), in which all 2-morphisms are 2-isomorphisms.

\( \mathcal{C}^\infty \)-schemes include into d-spaces as those \( X \) with \( \mathcal{O}_X^{-1} = 0 \).

Thus we have inclusions of (2-)categories \( \text{Man} \subset \mathcal{C}^\infty \text{Sch} \subset \text{dSpa} \), so manifolds are examples of d-spaces.

The \textit{cotangent complex} \( \mathbb{L}^\bullet_X \) of \( X \) is the sheaf of cotangent complexes of \( \mathcal{O}_X^\bullet \), a 2-term complex \( \mathbb{L}_X^{-1} \xrightarrow{d_X} \mathbb{L}_X^0 \) of quasicoherent sheaves on \( X \). Such complexes form a 2-category \( \text{qcoh}[-1,0](X) \).

\textbf{Theorem 4.4}

All fibre products exist in the 2-category \( \text{dSpa} \).

The proof is by construction: given 1-morphisms \( g : X \to Z \) and \( h : Y \to Z \), we write down an explicit d-space \( W \), 1-morphisms \( e : W \to X, f : W \to Y \) and 2-isomorphism \( \eta : g \circ e \Rightarrow h \circ f \), and verify by hand that it satisfies the universal property in \( \S 4.1 \).
Gluing d-spaces by equivalences

Theorem 4.5

Let $X, Y$ be d-spaces, $\emptyset \neq U \subseteq X, \emptyset \neq V \subseteq Y$ open d-subspaces, and $f : U \to V$ an equivalence in the 2-category $d\text{Spa}$. Suppose the topological space $Z = X \cup_{U=V} Y$ made by gluing $X, Y$ using $f$ is Hausdorff. Then there exist a d-space $Z$, unique up to equivalence in $d\text{Spa}$, open $\mathring{X}, \mathring{Y} \subseteq Z$ with $Z = \mathring{X} \cup \mathring{Y}$, equivalences $g : X \to \mathring{X}$ and $h : Y \to \mathring{Y}$, and a 2-morphism $\eta : g|_{U} \Rightarrow h \circ f$.

The proof is again by explicit construction. First we glue the classical $C^\infty$-schemes $X, Y$ on $U \subseteq X, V \subseteq Y$ by the isomorphism $f : U \to V$ to get a $C^\infty$-scheme $Z$. The definition of $Z$ involves choosing a smooth partition of unity on $Z$ subordinate to the open cover $\{U, V\}$. This is possible in the world of $C^\infty$-schemes, but would not work in conventional (derived) algebraic geometry.

Theorem 4.6

Suppose $I$ is an indexing set, and $<$ is a total order on $I$, and $X_i$ for $i \in I$ are d-spaces, and for all $i < j$ in $I$ we are given open d-subspaces $U_{ij} \subseteq X_i, U_{ji} \subseteq X_j$ and an equivalence $e_{ij} : U_{ij} \to U_{ji}$, such that for all $i < j < k$ in $I$ we have a 2-commutative diagram

\[ e_{ij}|_{U_{ij} \cap U_{ik}} \Rightarrow U_{ij} \cap U_{jk} \xrightarrow{\eta_{ijk}} U_{ki} \cap U_{kj}. \]  

(4.5)

Define the quotient topological space $Z = (\bigsqcup_{i \in I} X_i)/\sim$, where $\sim$ is generated by $x_i \sim x_j$ if $i < j$, $x_i \in U_{ij} \subseteq X_i$ and $x_j \in U_{ji} \subseteq X_j$ with $e_{ij}(x_i) = x_j$. Suppose $Z$ is Hausdorff and second countable. Then there exist a d-space $Z$ and a 1-morphism $f_i : X_i \to Z$ which is an equivalence with an open d-subspace $\mathring{X}_i \subseteq Z$ for all $i \in I$, where $Z = \bigcup_{i \in I} \mathring{X}_i$, such that $f_i(U_{ij}) = \mathring{X}_i \cap \mathring{X}_j$ for $i < j$ in $I$, and there exists a 2-morphism $\zeta_{ij} : f_j \circ e_{ij} \Rightarrow f_i|_{U_{ij}}$. The d-space $Z$ is unique up to equivalence, and is independent of choice of $\eta_{ijk}$.
Theorem 4.6 generalizes Theorem 4.5 to gluing many d-spaces by equivalences. It is important that the 2-isomorphisms \( \eta_{ijk} \) in (4.5) are only required to exist, they need not satisfy any conditions on quadruple overlaps, etc., and \( Z \) is independent of the choice of \( \eta_{ijk} \). Because of this, Theorem 4.6 actually makes sense as a statement in the homotopy category \( \text{Ho}(d\text{Spa}) \). The analogue is false for gluing by equivalences for orbifolds \( \text{Orb} \), d-orbifolds \( d\text{Orb} \), and d-stacks \( d\text{Sta} \).

### 4.4. D-manifolds

**Definition**

A **d-manifold** \( X \) of **virtual dimension** \( n \in \mathbb{Z} \) is a d-space \( X \) such that \( X \) is covered by open d-subspaces \( Y \subset X \) with equivalences

\[
Y \simeq U \times_{g,W,h} V,
\]

where \( U, V, W \) are manifolds with \( \dim U + \dim V - \dim W = n \), regarded as d-spaces by \( \text{Man} \subset C^\infty \text{Sch} \subset d\text{Spa} \), and \( g : U \to W, h : V \to W \) are smooth maps, and \( U \times_{g,W,h} V \) is the fibre product in the 2-category \( d\text{Spa} \). Write \( d\text{Man} \) for the full 2-subcategory of d-manifolds in \( d\text{Spa} \).

Note that the fibre product \( U \times_W V \) exists by Theorem 4.4, and must be taken in \( d\text{Spa} \) as a 2-category, not as an ordinary category. Alternatively, we can write the local models as \( Y \simeq V \times_{0,E,s} V \), where \( V \) is a manifold, \( E \to V \) a vector bundle, \( s : V \to E \) a smooth section, and \( n = \dim V - \text{rank } E \). Then \( (V, E, s) \) is a **Kuranishi neighbourhood** on \( X \), as in Fukaya–Oh–Ohta–Ono.
Thus, a d-manifold \( X \) is a ‘derived’ geometric space covered by simple, differential-geometric local models: they are fibre products \( U \times_g W, h V \) for smooth maps of manifolds \( g : U \to W, h : V \to W \), or they are the zeroes \( s^{-1}(0) \) of a smooth section \( s : V \to E \) of a vector bundle \( E \to V \) over a manifold \( V \).

However, as usual in derived geometry, the way in which these local models are glued together (by equivalences in the 2-category \( \text{dSpa} \)) is more mysterious, is weaker than isomorphisms, and takes some work to understand. We discuss this later in the course.

If \( g : X \to Z, h : Y \to Z \) are 1-morphisms in \( \text{dMan} \), then Theorem 4.4 says that a fibre product \( W = X \times_g Z, h Y \) exists in \( \text{dSpa} \). If \( W \) is a d-manifold (which is a local question on \( W \)) then \( W \) is also a fibre product in \( \text{dMan} \). So we will give be able to give useful criteria for existence of fibre products in \( \text{dMan} \).

Theorems 4.5 and 4.6 immediately lift to results on gluing by equivalences in \( \text{dMan} \), taking \( U, V, X_i \) to be d-manifolds of a fixed virtual dimension \( n \in \mathbb{Z} \). Thus, we can define d-manifolds by gluing together local models by equivalences. This is very useful, as natural examples (e.g. moduli spaces) are often presented in terms of local models somehow glued on overlaps.

I chose to use square zero dg \( C^\infty \)-rings to define \( \text{dSpa}, \text{dMan} \) (rather than, say, general dg \( C^\infty \)-rings) as they are very ‘small’ — they are essentially the minimal extension of classical \( C^\infty \)-rings which remembers the ‘derived’ information I care about (in particular, sufficient to form virtual cycles for derived manifolds). This has the advantage of making the theory simpler than it could have been, e.g. by using 2-categories rather than \( \infty \)-categories, whilst still having good properties, e.g. ‘correct’ fibre products and gluing by equivalences. A possible disadvantage is that they forget ‘higher obstructions’, which occur in some moduli problems.