Lecture 1: Vertex algebra and Lie algebra structures on the homology of moduli spaces

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(See also arXiv:2005.05637 with Jacob Gross and Yuuji Tanaka.)

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These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`.

1. Introduction

Vertex algebras are a very complicated, very rich algebraic structure coming from Conformal Field Theory in Physics, which also occur in Moonshine and other areas of Representation Theory. All interesting vertex algebras are infinite-dimensional.

I will explain a new construction of (graded) vertex algebra structures on the homology $H_*(\mathcal{M})$ of certain moduli stacks $\mathcal{M}$. It is extraordinarily general, and produces a huge number of examples. There are versions in Algebraic Geometry, Differential Geometry (using topological stacks), and Representation Theory.

There is a functor from (graded) vertex algebras V to (graded) Lie algebras V_{Lie}. Roughly, the Lie algebra $H_*(\mathcal{M})_{\mathrm{Lie}}$ is the homology $H_*(\mathcal{M}^{\mathrm{pl}})$ of the associated 'projective linear' moduli stack $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$. Thus, we have a parallel new construction of infinite-dimensional (graded) Lie algebras $H_*(\mathcal{M}^{\mathrm{pl}})$. These have important applications in enumerative invariants (next lecture).

1.1. Vertex algebras (don't try to understand this slide.)

Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)}: V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k\geqslant 0}(-1)^{k+n+1}D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n\binom{l}{n}\left(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\right)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u,z) : V \longrightarrow V[[z,z^{-1}]], \quad Y(u,z) : v \longmapsto \textstyle\sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u,z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u,z)$. One interesting property is this: for all $u,v,w \in V$ there exist $N \gg 0$ depending on u,v such that

$$(y-z)^N Y(u,y)Y(v,z)w = (y-z)^N Y(v,z)Y(u,y)w. \qquad (1)$$

There may be a V-valued rational function $R(y,z)$ with poles when $y=0$, $z=0$ and $y=z$, such that the l.h.s. of (1) is a formal Laurent series convergent to $R(y,z)$ when $0<|y|<|z|$, and the r.h.s. converges to $R(y,z)$ when $0<|z|<|y|$.

Think of $u *_z v = Y(u,z)v$ as a multiplication on V depending on a complex variable z, with poles at $z=0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Any commutative algebra $(V, \mathbb{1}, \cdot)$ with derivation D is a vertex algebra, with $Y(u,z)v = (e^{zD}u) \cdot v$, so no poles, where $u_n(v) = \left(\frac{1}{(n+1)!}D^{n+1}u\right) \cdot v$ for $n \geqslant -1$, and $u_n(v) = 0$ for $n < -1$. We call such V a *commutative vertex algebra*. All non-commutative vertex algebras are infinite-dimensional, so even the simplest nontrivial examples are large, complicated objects, which are difficult to write down.

Let R be a field of characteristic zero. A *vertex operator algebra* (*VOA*) over R is a vertex algebra V over R, with a distinguished *conformal element* $\omega \in V$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, the L_n define an action of the *Virasoro algebra* on V_*, with central charge c_V, and $L_{-1} = D^{(1)}$. VOAs are important in Physics. We will give a geometric construction of vertex algebras, but often they will *not* be VOAs.

Lie algebras from vertex algebras

If V is a (graded/super) vertex algebra then $V/\langle D^{(k)}(V),\ k\geqslant 1\rangle$ is a (graded/super) Lie algebra, with Lie bracket

$$\bigl[u+\langle D^{(k)}(V),\ k\geqslant 1\rangle, v+\langle D^{(k)}(V),\ k\geqslant 1\rangle\bigr]=u_0(v)+\langle D^{(k)}(V),\ k\geqslant 1\rangle.$$

Vertex algebras were introduced in mathematics by Borcherds, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/\langle D^{(k)}(V),\ k\geqslant 1\rangle$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.
Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

2. Vertex algebras on homology of moduli stacks

Let $\mathcal{A}$ be a $\mathbb{C}$-linear abelian or triangulated category from Algebraic Geometry or Representation Theory, e.g. $\mathcal{A} = \mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$ for X a smooth projective $\mathbb{C}$-scheme, or $\mathcal{A} = \mathrm{mod}\text{-}\mathbb{C}Q$ or $D^b \mathrm{mod}\text{-}\mathbb{C}Q$. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, which is an Artin $\mathbb{C}$-stack in the abelian case, and a higher $\mathbb{C}$-stack in the triangulated case.

Given some extra data on $\mathcal{M}$, we will define a graded vertex algebra structure on the $\mathbb{Q}$-homology $H_*(\mathcal{M})$. We also define a graded Lie bracket $[\,,\,]$ on either a quotient $H_*(\mathcal{M})/D(H_*(\mathcal{M})$ of $H_*(\mathcal{M})$, or the $\mathbb{Q}$-homology $H_*(\mathcal{M}^{\mathrm{pl}})$ of a modification $\mathcal{M}^{\mathrm{pl}} = \mathcal{M}/[*/\mathbb{G}_m]$ of $\mathcal{M}$, making $H_*(\mathcal{M}^{\mathrm{pl}})$ into a *graded Lie (super)algebra* (with a nonstandard grading).

The extra data we need

We need some extra data, a perfect complex $\Theta^\bullet$ on $\mathcal{M}\times\mathcal{M}$ satisfying some assumptions; the formulae for for the vertex and Lie algebra structures involve $\mathrm{rank}\,\Theta^\bullet$ and Chern classes $c_i(\Theta^\bullet)$.
We also need signs $\epsilon_{\alpha,\beta}$ related to 'orientation data' for $\mathcal{A}$.
For graded antisymmetry of $[\,,\,]$ we need $\sigma^*(\Theta^\bullet)\cong(\Theta^\bullet)^\vee[2n]$ for some $n\in\mathbb{Z}$, where $\sigma:\mathcal{M}\times\mathcal{M}\to\mathcal{M}\times\mathcal{M}$ exchanges the factors, as then $c_i(\sigma^*(\Theta^\bullet))=(-1)^i c_i(\Theta^\bullet)$.
In our examples there is a natural perfect complex $\mathcal{E}xt^\bullet$ on $\mathcal{M}\times\mathcal{M}$ with $H^i(\mathcal{E}xt^\bullet|_{([E],[F])})\cong\mathrm{Ext}^i_{\mathcal{A}}(E,F)$ for $E,F\in\mathcal{A}$ and $i\in\mathbb{Z}$. If $\mathcal{A}$ is a $2n$-Calabi–Yau category then $\sigma^*((\mathcal{E}xt^\bullet)^\vee)\cong\mathcal{E}xt^\bullet[2n]$, and we put $\Theta^\bullet=(\mathcal{E}xt^\bullet)^\vee$. Otherwise we put $\Theta^\bullet=(\mathcal{E}xt^\bullet)^\vee\oplus\sigma^*(\mathcal{E}xt^\bullet)[2n]$.
Thus examples split into 'even Calabi–Yau' and 'general' vertex algebras.

More detail on the basic set-up

Let $K(\mathcal{A})$ be a quotient group of the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ such that $\mathcal{M}=\coprod_{\alpha\in K(\mathcal{A})}\mathcal{M}_\alpha$, with $\mathcal{M}_\alpha$ the moduli stack of objects $E\in\mathcal{A}$ in class α in $K(\mathcal{A})$, an open and closed substack in $\mathcal{M}$. We suppose we are given a biadditive map $\chi:K(\mathcal{A})\times K(\mathcal{A})\rightarrow\mathbb{Z}$ called the *Euler form*, with $\chi(\alpha,\beta)=\chi(\beta,\alpha)$. The restriction $\Theta^\bullet_{\alpha,\beta}=\Theta^\bullet|_{\mathcal{M}_\alpha\times\mathcal{M}_\beta}$ should have $\operatorname{rank}\Theta^\bullet_{\alpha,\beta}=\chi(\alpha,\beta)$.
There should be an Artin stack morphism $\Phi:\mathcal{M}\times\mathcal{M}\rightarrow\mathcal{M}$ mapping $\Phi(\mathbb{C}):([E],[F])\mapsto[E\oplus F]$ on $\mathbb{C}$-points, from direct sum in $\mathcal{A}$. It is associative and commutative. In perfect complexes on $\mathcal{M}_\alpha\times\mathcal{M}_\beta\times\mathcal{M}_\gamma$ for $\alpha,\beta,\gamma\in K(\mathcal{A})$ we should have

$$(\Phi_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma})\cong\Pi^*_{\mathcal{M}_\alpha\times\mathcal{M}_\gamma}(\Theta^\bullet_{\alpha,\gamma})\oplus\Pi^*_{\mathcal{M}_\beta\times\mathcal{M}_\gamma}(\Theta^\bullet_{\beta,\gamma}),$$

needed for the graded Jacobi identity for $[\,,\,]$, and corresponding to

$$\mathrm{Ext}^i_{\mathcal{A}}(E\oplus F,G)^*\cong\mathrm{Ext}^i_{\mathcal{A}}(E,G)^*\oplus\mathrm{Ext}^i_{\mathcal{A}}(F,G)^*.$$

The stack $[*/\mathbb{G}_m]$ and morphism Ψ

Write $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ as an algebraic $\mathbb{C}$-group under multiplication, and $[*/\mathbb{G}_m]$ for the quotient stack, where $* = \operatorname{Spec}\mathbb{C}$ is the point. If S is an Artin $\mathbb{C}$-stack and $s \in S(\mathbb{C})$ a $\mathbb{C}$-point there is an *isotropy group* $\mathrm{Iso}_S(s)$, an algebraic $\mathbb{C}$-group. We have $\mathrm{Iso}_{\mathcal{M}}([E]) \cong \mathrm{Aut}(E)$ for $E \in \mathcal{A}$. There is a natural morphism $\mathbb{G}_m \to \mathrm{Aut}(E)$ mapping $\lambda \mapsto \lambda \cdot \mathrm{id}_E \in \mathrm{Aut}(E) \subset \mathrm{Hom}_{\mathcal{A}}(E, E)$. There should be an Artin stack morphism $\Psi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ mapping $(*, [E]) \mapsto [E]$ on $\mathbb{C}$-points, and acting on isotropy groups by

$$\Psi_* : \mathrm{Iso}_{[*/\mathbb{G}_m]\times\mathcal{M}}(*, [E]) \cong \mathbb{G}_m \times \mathrm{Aut}(E) \longrightarrow \mathrm{Iso}_{\mathcal{M}}([E]) \cong \mathrm{Aut}(E),$$

$$\Psi_* : (\lambda, \mu) \longmapsto (\lambda \cdot \mathrm{id}_E) \circ \mu.$$

Here $[*/\mathbb{G}_m]$ is a *group stack*, and Ψ is an *action of* $[*/\mathbb{G}_m]$ *on* $\mathcal{M}$, which is free except over $[0] \in \mathcal{M}$. This Ψ encodes the natural morphisms $\mathbb{G}_m \to \mathrm{Iso}_{\mathcal{M}}([E])$ for all $[E] \in \mathcal{M}(\mathbb{C})$.

We require a compatibility between Ψ and $\Theta^\bullet$, roughly that

$$(\Psi\times\mathrm{id}_{\mathcal{M}})^*(\Theta^\bullet)\cong\Pi^*_{[*/\mathbb{G}_m]}(L)\otimes\Pi^*_{\mathcal{M}\times\mathcal{M}}(\Theta^\bullet)$$

where L is the line bundle on $[*/\mathbb{G}_m]$ associated to the obvious representation of $\mathbb{G}_m$ on $\mathbb{C}$. This corresponds to $\lambda\,\mathrm{id}_E\in\mathrm{Aut}(E)$ acting by multiplication by $\lambda\in\mathbb{G}_m$ on $\mathrm{Ext}^i(E,F)^*$.

We should be given $\epsilon_{\alpha,\beta}=\pm 1$ for $\alpha,\beta\in K(\mathcal{A})$ satisfying

$$\epsilon_{\alpha,\beta}\cdot\epsilon_{\beta,\alpha}=(-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)},$$

$$\epsilon_{\alpha,\beta}\cdot\epsilon_{\alpha+\beta,\gamma}=\epsilon_{\alpha,\beta+\gamma}\cdot\epsilon_{\beta,\gamma}.$$

They are needed to correct signs in defining $[\,,\,]$. Such $\epsilon_{\alpha,\beta}$ always exist. If we have chosen 'orientations' for $\mathcal{M}_\alpha,\mathcal{M}_\beta,\mathcal{M}_{\alpha+\beta}$ (in particular, in the Calabi–Yau 4-fold case), then $\epsilon_{\alpha,\beta}$ should be the natural sign comparing the orientations at $[E]\in\mathcal{M}_\alpha(\mathbb{C})$, $[F]\in\mathcal{M}_\beta(\mathbb{C})$ and $[E\oplus F]=\Phi([E],[F])\in\mathcal{M}_{\alpha+\beta}(\mathbb{C})$.
Define a *shifted grading* $\hat{H}_*(\mathcal{M})$ on $H_*(\mathcal{M})$ by $\hat{H}_n(\mathcal{M}_\alpha)=H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$ for $\alpha\in K(\mathrm{coh}(X))$.

Definition of the vertex algebra structure on $\hat{H}_*(\mathcal{M})$

Writing $H_*([*/\mathbb{G}_m])=\mathbb{Q}[t]$ with $\deg t=2$, the state-field correspondence $Y(z)$ is given by, for $u\in H_a(\mathcal{M}_\alpha)$, $v\in H_b(\mathcal{M}_\beta)$

$$Y(u,z)v=\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)}z^{\chi(\alpha,\beta)}\cdot H_*\bigl(\Phi\circ(\Psi\times\mathrm{id})\bigr)\\ \Bigl\{\Bigl(\sum_{i\geqslant 0}z^it^i\Bigr)\boxtimes\Bigl[(u\boxtimes v)\cap\exp\Bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)!z^{-j}\,\mathrm{ch}_j([\Theta^\bullet])\Bigr)\Bigr]\Bigr\}. \tag{2}$$

The identity $\mathbb{1}$ is $1\in H_0(\mathcal{M}_0)$. Define $e^{zD}:\hat{H}_*(\mathcal{M})\to\hat{H}_*(\mathcal{M})[[z]]$ by $Y(v,z)\mathbb{1}=e^{zD}v$. Then $(\hat{H}_*(\mathcal{M}),\mathbb{1},e^{zD},Y)$ is a graded vertex algebra.

Remark

This should be seen as an example of a *Borcherds bicharacter construction*. $H_*(\mathcal{M})$ is naturally a *bialgebra*, with product from $\Phi:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ and coproduct from $\Delta_\mathcal{M}:\mathcal{M}\to\mathcal{M}\times\mathcal{M}$. The morphism $\Psi:[*/\mathbb{G}_m]\times\mathcal{M}\to\mathcal{M}$ induces a derivation of this bialgebra, making $H_*(\mathcal{M})$ into a commutative vertex algebra. We twist this by a *bicharacter* from $\Theta^\bullet$ to make it noncommutative.

Explicit description of the vertex algebras

Theorem 1 (Jacob Gross arXiv:1907.03269)

Let X be a complex projective curve, surface, or smooth toric variety. Write $\mathcal{M}$ for the moduli stack of objects in $D^b\operatorname{coh}(X)$ and $K^0_{\rm sst}(X)$ for the **semi-topological K-theory** *of X (equal to $\operatorname{Image}(K^0(\operatorname{coh}(X))\rightarrow K^0_{\rm top}(X))$ for X a surface). Then $\mathcal{M}=\coprod_{\kappa\in K^0_{\rm sst}(X)}\mathcal{M}_\kappa$ with $\mathcal{M}_\kappa$ connected, and*

$$H_*(\mathcal{M}_\kappa,\mathbb{Q})\cong\operatorname{Sym}^*\bigl(H^{\rm even}(X,\mathbb{Q})\otimes_{\mathbb{Q}}t^2\mathbb{Q}[t^2]\bigr)\otimes_{\mathbb{Q}} \textstyle\bigwedge^*\bigl(H^{\rm odd}(X,\mathbb{Q})\otimes_{\mathbb{Q}}t\mathbb{Q}[t^2]\bigr). \qquad (3)$$

A similar equation holds for cohomology $H^*(\mathcal{M}_\kappa,\mathbb{Q})$.

This says we can describe $H_*(\mathcal{M})$ completely explicitly in the derived category case, which (surprisingly) is easier than the abelian category case. We can also describe $H_*(\mathcal{M})$ explicitly when $\mathcal{A}=\text{mod-}\mathbb{C}Q$ or $D^b\,\text{mod-}\mathbb{C}Q$.

Definition (Formal variables for (co)homology of $\mathcal{M}$.)

Let $X, \mathcal{M}, \mathcal{M}_\kappa$ be as in Theorem 1, and write $\mathcal{U}_\kappa^\bullet \to X \times \mathcal{M}_\kappa$ for the universal complex. Write $b^k = b^k(X)$ for $0 \leqslant k \leqslant 2m$, $m = \dim_{\mathbb{C}} X$, and choose bases $(e_{jk})_{j=1}^{b^k}$ for $H_k(X, \mathbb{Q})$. Write $(\epsilon_{jk})_{j=1}^{b^k}$ for the dual basis for $H^k(X, \mathbb{Q})$. For $l > k/2$ define $S_{jkl} \in H^{2l-k}(\mathcal{M}_\kappa)$ by $S_{jkl} = \mathrm{ch}_l(\mathcal{U}_\kappa^\bullet)\backslash e_{jk}$. Regard S_{jkl} as of degree $2l-k$, and as an even (odd) variable if k is even (odd). Then Theorem 1 shows $H^*(\mathcal{M}_\kappa)$ is the graded polynomial superalgebra

$$H^*(\mathcal{M}_\kappa) \cong \mathbb{Q}[S_{jkl} : 0 \leqslant k \leqslant 2m,\ 1 \leqslant j \leqslant b^k,\ l > k/2]. \tag{4}$$

We also give a dual description of homology $H_*(\mathcal{M}_\kappa)$ by

$$H_*(\mathcal{M}_\kappa) \cong e^\kappa \otimes \mathbb{Q}[s_{jkl} : 0 \leqslant k \leqslant 2m,\ 1 \leqslant j \leqslant b^k,\ l > k/2], \tag{5}$$

where e^κ is a symbol to remember κ, with $\deg e^\kappa = -\chi(\kappa,\kappa)$, and

$$\Bigl(\prod_{j,k,l} S_{jkl}^{m_{jkl}}\Bigr) \cdot \Bigl(e^\kappa \prod_{j,k,l} s_{jkl}^{m'_{jkl}}\Bigr) = \begin{cases} \pm \prod_{j,k,l} m_{jkl}!, & m_{jkl} = m'_{jkl} \text{ all } j,k,l, \\ 0, & \text{otherwise.} \end{cases}$$

The vertex algebra in formal coordinates s_{jkl}

Thus we identify

$$H_*(\mathcal{M}) \cong \bigoplus_{\kappa\in K^0_{\rm sst}(X)} e^\kappa\otimes\mathbb{Q}[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2]. \quad (6)$$

In these coordinates the vertex algebra structure is given by

$$\begin{aligned} &Y\bigl(e^\alpha u(s_{jkl}),z\bigr)\bigl(e^\beta v(s'_{j'k'l'})\bigr) = \\ &\epsilon_{\alpha,\beta}\, z^{\chi(\alpha,\beta)}\cdot\Bigl\{\exp\Bigl(z\Bigl(\sum_{j,k,l} s_{jk(l+1)}\frac{\partial}{\partial s_{jkl}}\Bigr)\Bigr)\circ \qquad (7)\\ &\exp\Bigl(-\sum_{\substack{j,k,j',k',\\ l\geqslant k/2,\ l'\geqslant k'/2}} (-1)^l(l+l'-(k+k')/2-1)!\, z^{(k+k')/2-l-l'}\cdot \\ &\mathrm{N}_{jk}^{j'k'}\frac{\partial^2}{\partial s_{jkl}\partial s'_{j'k'l'}}\Bigr)\bigl(e^\alpha u(s_{jkl})\cdot e^{\beta'} v(s'_{j'k'l'})\bigr)\Bigr\}\Big|_{s'_{jkl}=s_{jkl}}, \end{aligned}$$

where $\mathrm{N}_{jk}^{j'k'}=\mathrm{M}_{jk}^{j'k'}+(-1)^{kk'+(k+k')/2}\mathrm{M}_{j'k'}^{jk}$ with $\mathrm{M}_{jk}^{j'k'}=\int_X\epsilon_{jk}\cup\epsilon_{j'k'}\cup\mathrm{td}(X)$, and we introduce extra variables $s_{jk(k/2)}$ with $\alpha=\sum_{j,k}\alpha_{jk}s_{jk(k/2)}$, so $\frac{\partial}{\partial s_{jk(k/2)}}e^\alpha=\alpha_{jk}e^\alpha$. This $H_*(\mathcal{M})$ is a variant of a *super-lattice vertex algebra*. Equation (7) is really complicated to do explicit calculations with.

Conformal elements and vertex operator algebras

This slide is based on Bojko–Lim–Moreira 2022.

Suppose X is a projective surface with $b^1=0$ and $h^{2,0}(X)=0$.

Let $(\nu_{jk}^{j'k'})$ be the inverse matrix of $(\mathrm{N}_{jk}^{j'k'})$. Define $\omega\in H_4(\mathcal{M}_0)$ by

$$\omega=\tfrac{1}{2}e^0\sum_{k=0,2,4}\,\sum_{j=1}^{b^k(X)}\nu_{jk}^{j'k'}\,s_{jk(k/2+1)}s_{j'k'(k'/2+1)}. \tag{8}$$

Then ω is a *conformal element* for the graded vertex algebra $H_*(\mathcal{M})$, making it into a *vertex operator algebra*.

I can also make $H_*(\mathcal{M})$ into a VOA without assuming $b^1=0$ and $h^{2,0}(X)=0$, but you have to use the Hodge decomposition $H^k(X,\mathbb{C})=\bigoplus_{p+q=k}H^{p,q}(X)$, and modify $H_*(\mathcal{M})$ by adding finitely many extra variables s_{jpql}, as (8) should be replaced by a sum of $s_{jpq(p+1)}s_{j'p'q'(p'+1)}$.

If we take $\mathcal{M}$ to be the moduli of objects in $\mathrm{coh}(X)$, not $D^b\,\mathrm{coh}(X)$, then $H_*(\mathcal{M})$ is *not* a VOA, as $\mathcal{M}_0=*$ and $H_4(\mathcal{M}_0)=0$.

Graded Lie algebras on homology of moduli spaces

By a construction of Borcherds, as $\hat{H}_*(\mathcal{M})$ is a graded vertex algebra, $\hat{H}_{*+2}(\mathcal{M})/D(\hat{H}_*(\mathcal{M}))$ is a graded Lie algebra, where $D:\hat{H}_*(\mathcal{M})\rightarrow \hat{H}_{*+2}(\mathcal{M})$ is the translation operator, with (super) Lie bracket

$$\bigl[u+D(\hat{H}_*(\mathcal{M})),v+D(\hat{H}_*(\mathcal{M}))\bigr]=u_0(v)+D(\hat{H}_*(\mathcal{M})).$$

This has a geometrical interpretation. Recall that $[*/\mathbb{G}_m]$ is a group stack, and $\Psi:[*/\mathbb{G}_m]\times\mathcal{M}\rightarrow\mathcal{M}$ is an action of $[*/\mathbb{G}_m]$ on $\mathcal{M}$, which is free except over 0. We can form a quotient $\mathcal{M}^{\rm pl}=\mathcal{M}/[*/\mathbb{G}_m]$ called the 'projective linear moduli stack', with a morphism $\Pi^{\rm pl}:\mathcal{M}\rightarrow\mathcal{M}^{\rm pl}$ which is a principal $[*/\mathbb{G}_m]$-bundle except over 0. Then $\mathbb{C}$-points of $\mathcal{M}^{\rm pl}$ are isomorphism classes $[E]$ of $E\in\mathcal{A}$, and isotropy groups are

$$\mathrm{Iso}_{\mathcal{M}^{\rm pl}}([E])\cong\mathrm{Aut}(E)/(\mathbb{G}_m\cdot\mathrm{id}_E).$$

That is, we make $\mathcal{M}^{\rm pl}$ from $\mathcal{M}$ by quotienting out $\mathbb{G}_m$ from each isotropy group, a process called 'rigidification'.

Then $\Pi_*^{\rm pl}:H_*(\mathcal{M})\to H_*(\mathcal{M}^{\rm pl})$ has $D(H_*(\mathcal{M}))$ in its kernel, and descends to $\Pi_*^{\rm pl}:H_*(\mathcal{M})/D(H_*(\mathcal{M}))\to H_*(\mathcal{M}^{\rm pl})$, which is an isomorphism $H_*(\mathcal{M}_\alpha)/D(H_*(\mathcal{M}_\alpha))\to H_*(\mathcal{M}_\alpha^{\rm pl})$ for all $\alpha\in K(\mathcal{A})$ except $\alpha=0$. Write $\check H_*(\mathcal{M}^{\rm pl})$ for $H_*(\mathcal{M}^{\rm pl})$ with shifted grading $\check H_n(\mathcal{M}_\alpha^{\rm pl})=H_{n+2-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha^{\rm pl})$. There is a graded Lie bracket on $\check H_*(\mathcal{M}^{\rm pl})$, defined using the 'projective Euler class' (see Upmeier 2021), such that $\Pi_*^{\rm pl}$ is a Lie algebra morphism, and an isomorphism except on $\check H_*(\mathcal{M}_0^{\rm pl})$. This Lie bracket on $\check H_*(\mathcal{M}^{\rm pl})$ will be very important for enumerative invariants (next lecture).

We can give an explicit formulae for $\check H_*(\mathcal{M}^{\rm pl}),[\,,\,]$ if we restrict to sheaves of *positive rank*. Write $\mathcal{M}_{\rm rk>0}\subset\mathcal{M}$, $\mathcal{M}^{\rm pl}_{\rm rk>0}\subset\mathcal{M}^{\rm pl}$ for the open substacks of sheaves and complexes of positive rank. Then $\Pi^{\rm pl}_{\rm rk>0}:\mathcal{M}_{\rm rk>0}\to\mathcal{M}^{\rm pl}_{\rm rk>0}$ induces a surjective morphism $H_*(\mathcal{M}_{\rm rk>0})\to H_*(\mathcal{M}^{\rm pl}_{\rm rk>0})$. Supposing X connected, it turns out this induces an isomorphism from $\mathrm{Ker}(-\cap S_{101})$ to $H_*(\mathcal{M}^{\rm pl}_{\rm rk>0})$, where $\mathrm{Ker}(-\cap S_{101})$ is functions independent of s_{101}.

Explicit form for positive rank Lie algebra

Thus we identify

$$\check{H}_*(\mathcal{M}^{\rm pl}_{\rm rk>0}) \cong \bigoplus_{\kappa\in K^0_{\rm sst}(X):{\rm rk}\,\kappa>0} e^\kappa\otimes\mathbb{Q}[s_{jkl}:0\leqslant k\leqslant 2m,\ 1\leqslant j\leqslant b^k,\ l>k/2,\ (j,k,l)\neq(1,0,1)], \qquad (9)$$

where $\deg e^\kappa=2-\chi(\kappa,\kappa)$, $\deg s_{jkl}=2l-k$, and $\kappa=\sum_{j,k}\kappa_{jk}s_{jk(k/2)}$.

In this representation, we write the Lie bracket on $\check{H}_*(\mathcal{M}^{\rm pl}_{\rm rk>0})$ as

$$\begin{aligned}
&\bigl[e^\alpha u(s_{jkl}),e^\beta v(s'_{j'k'l'})\bigr]_{\rm rk>0}={\rm Res}_z\Bigl[\epsilon_{\alpha,\beta}\,z^{\chi(\alpha,\beta)}\cdot\\
&\Bigl\{\exp\Bigl(z\frac{{\rm rk}\,\beta}{{\rm rk}(\alpha+\beta)}\Bigl(\sum_{j,k,l}s_{jk(l+1)}\frac{\partial}{\partial s_{jkl}}\Bigr)\Bigr)\circ\\
&\exp\Bigl(-z\frac{{\rm rk}\,\alpha}{{\rm rk}(\alpha+\beta)}\Bigl(\sum_{j',k',l'}s'_{j'k'(l'+1)}\frac{\partial}{\partial s'_{j'k'l'}}\Bigr)\Bigr)\circ \qquad (10)\\
&\exp\Bigl(-\sum_{\substack{j,k,j',k',\\ l\geqslant k/2,\ l'\geqslant k'/2}}(-1)^l(l+l'-(k+k')/2-1)!\,z^{(k+k')/2-l-l'}\cdot\\
&\qquad {\rm N}^{j'k'}_{jk}\frac{\partial^2}{\partial s_{jkl}\partial s'_{j'k'l'}}\Bigr)\bigl(e^\alpha u(s_{jkl})\cdot e^{\beta'}v(s'_{j'k'l'})\bigr)\Bigr\}\Big|_{s'_{jkl}=s_{jkl}}\Bigr].
\end{aligned}$$

Remarks

- Theorem 1, giving an explicit description of $H_*(\mathcal{M})$ for $\mathcal{M}$ the moduli stack of objects in $D^b\,\mathrm{coh}(X)$, applies only for special X (curves, surfaces, toric varieties, Grassmannians, ...). However, for all smooth projective X, and for $\mathcal{M}$ the moduli of objects in either $\mathrm{coh}(X)$ or $D^b\,\mathrm{coh}(X)$, there is a natural vertex algebra morphism from $\hat{H}_*(\mathcal{M})$ to the explicit vertex algebra in e^κ, s_{jkl} above, and similarly for $\check{H}_*(\mathcal{M}^{\mathrm{pl}}_{\mathrm{rk}>0})$ and the explicit Lie algebras.
- We can also give similar explicit forms for the vertex algebras and Lie algebras from the moduli stacks of objects in both $\mathrm{mod}\text{-}\mathbb{C}Q$ and in $D^b\,\mathrm{mod}\text{-}\mathbb{C}Q$ for a quiver Q. Here $D^b\,\mathrm{mod}\text{-}\mathbb{C}Q$ yields a lattice vertex algebra.

The graded degree 0 part of the Lie algebra $\check{H}_0(\mathcal{M}^{\mathrm{pl}}_{D^b\,\mathrm{mod}\text{-}\mathbb{C}Q})$ is the Kac–Moody algebra associated to the underlying undirected graph of Q, considered as a Dynkin diagram. So for Q an ADE quiver, we get a finite-dimensional ADE Lie algebra.

3. Generalizations of the construction
Nonlocal vertex algebras

The vertex algebra construction admits many variations:

- To get an ordinary vertex algebra we assumed that the complex $\Theta^\bullet \to \mathcal{M} \times \mathcal{M}$ had the symmetry property $\sigma^*(\Theta^\bullet) \cong (\Theta^\bullet)^\vee[2n]$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ exchanges the factors. To get this when $\mathcal{M}$ is moduli of objects in $\mathrm{coh}(X)$ or $D^b \mathrm{coh}(X)$, we either assume that X is a Calabi–Yau $2n$-fold and take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, or else we put $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee \oplus \sigma^*(\mathcal{E}xt^\bullet)[2n]$.

If we take $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$ for X not $2n$-Calabi–Yau we get a *nonlocal vertex algebra*, or *quantum vertex algebra*, which are to vertex algebras as noncommutative algebras are to commutative algebras.

Extension to equivariant homology

- Let an algebra $\mathbb{C}$-group act on $\mathcal{A}$, so that G acts on $\mathcal{M},\mathcal{M}^{\mathrm{pl}}$.

Then for a *nonstandard* notion of equivariant homology we can extend the vertex algebra/Lie algebra theory to $H_*^G(\mathcal{M}),H_*^G(\mathcal{M}^{\mathrm{pl}})$. This is the right context to extend my enumerative invariant programme to invariants in equivariant homology.

Here $H_k^G(X)$ is not just $H_k([X/G])$. It is defined for $k\in\mathbb{Z}$, and if G acts trivially on X then

$$H_k^G(X)=\bigoplus_{i,j\geqslant 0:k=i-j}H_i(X)\otimes H^j([*/G]).$$

Vertex Lie algebras from odd-Calabi–Yau categories

- A (*graded*) *vertex Lie algebra* $(V_*, e^{zD}, Y_{<0})$ is a weakening of (graded) vertex algebras $(V_*, \mathbb{1}, e^{zD}, Y)$, which remembers only the poles in $z^{<0}$ in the state-field correspondence Y.

If in the situation of §1 we instead suppose that $\sigma^*(\Theta^\bullet) \cong (\Theta^\bullet)^\vee[2n+1]$, we can define a graded vertex Lie algebra structure on $H_*(\mathcal{M})$ by, for $u \in H_*(\mathcal{M}_\alpha)$, $v \in H_*(\mathcal{M}_\beta)$

$$Y_{<0}(u,z)v = Y_{<0}(z)(u \otimes v) = (-1)^{\chi(\alpha,\beta)} \sum\nolimits_{i,j\geqslant 0} (-1)^i i! z^{-1-i+j} \cdot \\ \bigl(\Phi \circ (\Psi \times \mathrm{id}_{\mathcal{M}})\bigr)_* \bigl(t^j \boxtimes ((u \boxtimes v) \cap \mathrm{ch}_i([\Theta^\bullet])\bigr) \bmod O(z^{\geqslant 0}). \quad (11)$$

This is natural when $\mathcal{A}$ is a $(2n+1)$-Calabi–Yau category, and $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$. We also get a graded Lie algebra structure on $H_*(\mathcal{M}^{\rm pl})$. I expect interesting applications to *equivariant Donaldson–Thomas invariants* counting compactly-supported coherent sheaves on a local Calabi–Yau 3-fold.

Complex oriented generalized (co)homology theories

- A *complex oriented generalized (co)homology theory* is a generalized (co)homology theory with a notion of Chern class. Examples include ordinary (co)homology, K-theory, elliptic (co)homology, and unitary (co)bordism.
  A complex oriented generalized (co)homology theory over a commutative ring R has an associated *formal group law* F over R, with (co)homology associated to $F(s,t)=s+t$ (the *additive formal group law*), and K-theory to $F(s,t)=s+t+st$.
  There is a notion of *vertex algebra over a formal group law* F, where ordinary vertex algebras are vertex algebras over $F(s,t)=s+t$.
  Suppose $E_*(-), E^*(-)$ are complex oriented generalized (co)homology theories of Artin or higher $\mathbb{C}$-stacks, with formal group law F. Then my construction generalizes to give a vertex algebra over the formal group law F on $E_*(\mathcal{M})$. See Gross–Upmeier 2021.

Self-dual categories

- Suppose $\mathcal{A}$ is a $\mathbb{C}$-linear additive category, such as $\mathrm{Vect}(X)$ or $D^b\,\mathrm{coh}(X)$, and $\delta:\mathcal{A}\to\mathcal{A}^{\mathrm{op}}$ is an equivalence of categories with $\delta^2\simeq\mathrm{id}_{\mathcal{A}}$, such as $E\mapsto E^*$ or $\mathcal{E}^\bullet\mapsto(\mathcal{E}^\bullet)^*$. Fixed points of δ are *self-dual* objects E of $\mathcal{A}$ with an isomorphism $\varphi:E\to\delta(E)$ with $\varphi^2\simeq\mathrm{id}_E$. Examples include vector bundles on X with an orthogonal or symplectic structure. Write $\mathcal{M}$ for the moduli stack of objects in $\mathcal{A}$, and $\mathcal{M}^{\mathrm{sd}}$ for the moduli stack of self-dual objects in $\mathcal{A}$. Work by my student Chenjing Bu, 2023, shows that we can give $H_*(\mathcal{M}^{\mathrm{sd}})$ the structure of a 'twisted module' over the graded vertex algebra $\hat H_*(\mathcal{M})$, and also of a representation of the graded Lie algebra $\check H_*(\mathcal{M}^{\mathrm{pl}})^{\delta=-1}$, which is the -1-eigenspace of δ in the usual graded Lie algebra $\check H_*(\mathcal{M}^{\mathrm{pl}})$.
  This is important in the study of enumerative invariants counting self-dual objects, e.g. for counting principal $\mathrm{O}(n,\mathbb{C})$-bundles or principal $\mathrm{Sp}(2n,\mathbb{C})$-bundles on projective surfaces.