Bordism categories and orientations of moduli spaces

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These slides available at
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1. Introduction

There are many examples of moduli spaces $\mathcal{M}$ in geometry in which either $\mathcal{M}$ is a smooth real manifold, or it behaves like one (it may be a *derived manifold*, or *Kuranishi space*), so that $\mathcal{M}$ has a notion of *orientation*. If $\mathcal{M}$ is compact and oriented of dimension d, it will have a *fundamental class* $[\mathcal{M}]_{\rm fund} \in H_d(\mathcal{M},\mathbb{Z})$ (called a *virtual class* for derived manifolds). Such fundamental classes are used to define *enumerative invariants*, as integrals $\int_{[\mathcal{M}]_{\rm fund}} \alpha$ of natural cohomology classes α. For example:

- Let (X,g) be a compact, oriented, Riemannian 4-manifold, G a Lie group, and $P \to X$ a principal G-bundle. The moduli space $\mathcal{M}_P$ of irreducible connections ∇ on P with anti-self-dual curvature (*instantons*) is a derived manifold, and a manifold if g is generic. Integrals over $\mathcal{M}_P$ are used to define *Donaldson invariants* of X.
- If (X,φ,g) is a compact coclosed G_2-manifold in 7 dimensions, we can consider moduli spaces of G_2-*instantons* on $P \to X$.
- If (X,Ω,g) is a compact $\mathrm{Spin}(7)$-manifold in 8 dimensions, we can consider moduli spaces of $\mathrm{Spin}(7)$-*instantons* on $P \to X$.

• Let X be a projective Calabi–Yau 4-fold over $\mathbb{C}$ and $\mathcal{M}_\alpha^{\rm st}(\tau)$ be a moduli scheme of Gieseker-stable coherent sheaves on X with Chern character α. Then Borisov–Joyce 2017 give $\mathcal{M}_\alpha^{\rm st}(\tau)$ the structure of a derived manifold. If there are no strictly semistable sheaves in class α then $\mathcal{M}_\alpha^{\rm st}(\tau)$ is compact. So if we can find an orientation on $\mathcal{M}_\alpha^{\rm st}(\tau)$ we have a virtual class $[\mathcal{M}_\alpha^{\rm st}(\tau)]_{\rm virt}$. These are used to define Donaldson–Thomas type *DT4 invariants* of X.

In all these examples, we have a class of moduli spaces $\mathcal{M}$ which behave enough like real manifolds to have a notion of *orientation*. So we can ask whether $\mathcal{M}$ is *orientable*, and if so, whether we can construct a *canonical orientation* on $\mathcal{M}$. Such orientations are important for enumerative invariant theories.

This talk will outline a general theory for studying orientability and canonical orientations of moduli spaces, using a new tool called *bordism categories*. Our main applications are in 7 and 8 dimensions, to orientations for moduli spaces of G_2- and $\mathrm{Spin}(7)$-instantons and of coherent sheaves on Calabi–Yau 4-folds.

Gauge theory moduli spaces and orientations

Let X be a compact manifold, G a Lie group, and $P \to X$ a principal G-bundle. Write $\mathcal{A}_P$ for the moduli space of all connections ∇ on P, an infinite-dimensional affine space, and $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$ for the moduli space of connections on P modulo gauge transformations, as a topological stack, where $\mathcal{G}_P = \mathrm{Aut}(P)$. Let $E^\bullet = \bigl(D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)\bigr)$ be an elliptic operator on X, for example, the Dirac operator on X if X is spin. Then for each $\nabla \in \mathcal{A}_P$ we have a twisted elliptic operator $D_\nabla : \Gamma^\infty(E_0 \otimes \mathrm{ad}(P)) \to \Gamma^\infty(E_1 \otimes \mathrm{ad}(P))$. There is a determinant line bundle $\hat{L}_P \to \mathcal{A}_P$ with fibre $\det D_\nabla = \det \mathrm{Ker}(D_\nabla) \otimes \det \mathrm{Coker}(D_\nabla)^*$ at $\nabla \in \mathcal{A}_P$, and a principal $\mathbb{Z}_2$-bundle $\hat{O}_P \to \mathcal{A}_P$ of orientations on the fibres of $\hat{L}$. These are $\mathcal{G}_P$-equivariant, and descend to $L_P \to \mathcal{B}_P$ and $O_P \to \mathcal{B}_P$.

An *orientation* on $\mathcal{B}_P$ is an isomorphism $O_P \cong \mathcal{B}_P \times \mathbb{Z}_2$.

Moduli spaces $\mathcal{M}_P$ of 'instantons' – connections on P satisfying a curvature condition – are subspaces $\mathcal{M}_P \subset \mathcal{B}_P$. In good cases, $\mathcal{M}_P$ is a smooth manifold, and $O_P|_{\mathcal{M}_P}$ is the principal $\mathbb{Z}_2$-bundle of orientations on $\mathcal{M}_P$ in the usual sense. So orientability / orientations for $\mathcal{B}_P$ give orientability / orientations for $\mathcal{M}_P$.

Here is how this relates to our examples:

- For moduli spaces of instantons $\mathcal{M}_P$ on a 4-manifold (X, g), orientations on $\mathcal{M}_P$ come from orientations on $\mathcal{B}_P$ with $E^\bullet = \bigl(\mathrm{d}^+ \oplus \mathrm{d}^* : \Gamma^\infty(T^*X) \to \Gamma^\infty(\Lambda^2_+ T^*X \oplus \Lambda^0 T^*X)\bigr)$.
- For moduli spaces of G_2-instantons $\mathcal{M}_P$ on a G_2-manifold (X, φ, g), orientations on $\mathcal{M}_P$ come from orientations on $\mathcal{B}_P$ with $E^\bullet$ the Dirac operator $\not{D}_X$.
- For moduli spaces of $\mathrm{Spin}(7)$-instantons $\mathcal{M}_P$ on a $\mathrm{Spin}(7)$-manifold (X, Ω, g), orientations on $\mathcal{M}_P$ come from orientations on $\mathcal{B}_P$ with $E^\bullet$ the positive Dirac operator $\not{D}_X^+$.
- For moduli spaces of coherent sheaves $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ on a Calabi–Yau 4-fold X, by Cao–Gross-Joyce 2020, orientations on $\mathcal{M}_\alpha^{\mathrm{st}}(\tau)$ can be pulled back from orientations on $\mathcal{B}_P$ with $E^\bullet$ the positive Dirac operator $\not{D}_X^+$ and $P \to X$ a principal $\mathrm{U}(m)$ bundle for $m \gg 0$.

The Cao–Gross–Joyce orientability theorem is wrong!

Theorem 1 (Cao–Gross–Joyce 2020)

Let (X, g) be a compact, oriented spin 8-manifold and $P \to X$ a principal G-bundle for $G = \mathrm{SU}(m)$ or $\mathrm{U}(m)$. Define orientations on $\mathcal{B}_P$ using the positive Dirac operator $\not{D}_X^+$. Then $\mathcal{B}_P$ is orientable.

Corollary 2 (Cao–Gross–Joyce 2020)

Let X be a projective Calabi–Yau 4-fold. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves or perfect complexes on X is orientable.

Unfortunately, there is a mistake in the proof of Theorem 1. The theorem is false, a counterexample is the compact spin manifold $X = \mathrm{SU}(3)$, and P the trivial $\mathrm{SU}(3)$-bundle over X. Corollary 2 may also be false, though we don't have a counterexample.

I apologize for this.

One of the goals of our new theory is to provide a corrected version of Theorem 1, under an extra condition on X.

2. First look at the methods in the theory

A principal G-bundle $P \to X$ is topologically equivalent to a map $\phi_P : X \to BG$, where BG is the classifying space of X. Thus $[X, \phi_P]$ is an element of the *spin bordism group* $\Omega_n^{\mathrm{Spin}}(BG)$. Orientability of $\mathcal{B}_P$ depends on the monodromy of $O_P \to \mathcal{B}_P$ around a loop $\gamma : \mathcal{S}^1 \to \mathcal{B}_P$. Then γ is equivalent to a principal G-bundle $Q \to X \times \mathcal{S}^1$, giving a map $\phi_Q : X \times \mathcal{S}^1 \to BG$, and a spin bordism class $[X \times \mathcal{S}^1, \phi_Q]$ in $\Omega_{n+1}^{\mathrm{Spin}}(BG)$. Now ϕ_Q is equivalent to a map $\psi_Q : X \to \mathcal{L}BG$, where $\mathcal{L}BG$ is the loop space of BG, so Q determines a bordism class $[X, \psi_Q]$ in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$, and $[X \times \mathcal{S}^1, \phi_Q]$ is the image of $[X, \psi_Q]$ under a natural map $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathrm{Spin}}(BG)$.

It turns out that orientation problems for $\mathcal{B}_P$ factor via $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ in a certain sense. For given X, we can show that $\mathcal{B}_P$ is orientable for all principal G-bundles $P \to X$ if and only if certain 'bad' classes α in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ cannot be written $\alpha = [X, \psi]$. If there are no bad classes we get orientability for all X, P (this often happens for $n = 7$). We need to compute $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ using algebraic topology.

If $\iota : G \to H$ is a morphism of Lie groups of 'complex type', and $P \to X$ is a principal G-bundle, then $Q = (P \times H)/G$ is a principal H-bundle, and an orientation for $\mathcal{B}_Q$ induces one for $\mathcal{B}_P$. Using complex type morphisms $\mathrm{SU}(8) \hookrightarrow E_8$ and $\mathrm{SU}(m) \hookrightarrow \mathrm{SU}(m')$ for $m \leqslant m'$, we can show that if X is a spin 8-manifold then orientability of $\mathcal{B}_Q$ for all principal E_8-bundles $Q \to X$ implies orientability of $\mathcal{B}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$. Thus, to solve the CY4 orientability problem, it is enough to understand orientability for E_8-bundles.

There is a 16-connected map $BE_8 \to K(\mathbb{Z}, 4)$, where $K(\mathbb{Z}, 4)$ is the Eilenberg–MacLane space classifying $H^4(-, \mathbb{Z})$, so $\Omega_n^{\mathrm{Spin}}(BE_8) \cong \Omega_n^{\mathrm{Spin}}(K(\mathbb{Z}, 4))$ for $n < 16$, and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BE_8) \cong \Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z}, 4))$ for $n < 15$. Using this, we can reduce orientability questions for E_8-bundles to conditions that can be computed using *cohomology* and *cohomology operations* on X, in particular Steenrod squares. The proofs involve lots of complicated calculations of bordism groups in Algebraic Topology, spectral sequences, etc.

3. Statement of main results: orientability

I'll explain only results in 8 dimensions relevant to $\mathrm{Spin}(7)$ instantons and DT4 invariants. They are part of a bigger theory, which also includes results on orientability of moduli spaces of submanifolds, such as Cayley 4-folds in $\mathrm{Spin}(7)$-manifolds.

Let X be a compact oriented spin 8-manifold. Impose the condition:

(*) Let $\alpha \in H^3(X, \mathbb{Z})$, and write $\bar{\alpha} \in H^3(X, \mathbb{Z}_2)$ for its mod 2 reduction, and $\mathrm{Sq}^2(\bar{\alpha}) \in H^5(X, \mathbb{Z}_2)$ for its Steenrod square. Then $\int_X \bar{\alpha} \cup \mathrm{Sq}^2(\bar{\alpha}) = 0$ in $\mathbb{Z}_2$ for all $\alpha \in H^3(X, \mathbb{Z})$.

Theorem 3

Suppose X satisfies condition $(*)$, and let G be a compact Lie group on the list, for all $m \geqslant 1$

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m). \qquad (1)$$

Then $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$. For $G = E_8$, this holds **if and only if** $(*)$ *holds.*

We do this by applying our general orientability theory for $G = E_8$ by studying $\Omega_n^{\mathrm{Spin}}(K(\mathbb{Z}, 4))$ and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z}, 4))$. The other cases are deduced from $G = E_8$ using complex type morphisms.

The case $G = U(m)$ and part of Cao–Gross–Joyce 2020 implies:

Corollary 4

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X is orientable in the sense of Borisov–Joyce 2017.

Example 5

Let $X \subset \mathbb{CP}^5$ be a smooth sextic. Then $H^3(X, \mathbb{Z}) = 0$ by the Lefschetz Hyperplane Theorem. So $(*)$ and Corollary 4 hold.

Corollary 6

Suppose a compact $\mathrm{Spin}(7)$-manifold (X, Ω) satisfies condition $(*)$, and G lies on the list (1), and $P \to X$ is a principal G-bundle. Then the moduli space $\mathcal{M}_P^{\mathrm{irr}}$ of irreducible $\mathrm{Spin}(7)$-instanton connections on P is orientable. (Here $\mathcal{M}_P^{\mathrm{irr}}$ is a smooth manifold if Ω is generic, and a derived manifold otherwise.)

4. Statement of main results: canonical orientations

Suppose now that $(*)$ holds, so we have orientability of moduli spaces $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$ on X. What extra choices do we need to make on X to define *canonical orientations* on $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$?

Definition

Let X be a spin 8-manifold, and $P \to X$ a principal G-bundle, and $O_P \to \mathcal{B}_P$ be the orientation bundle. Define the *normalized orientation bundle* $\check{O}_P \to \mathcal{B}_P$ by $\check{O}_P = O_P \otimes_{\mathbb{Z}_2} \mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$, where $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is the $\mathbb{Z}_2$-torsor of orientations of $\mathcal{B}_{X\times G}$ for the trivial G-bundle $X \times G \to X$ at the trivial connection ∇_0. A trivialization of $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is an orientation for $\mathrm{ind}(\not{D}_X^+) \otimes \mathfrak{g}$, where $\not{D}_X^+$ is the positive Dirac operator of X, $\mathrm{ind}(\not{D}_X^+)$ its orientation torsor as a Fredholm operator, $\mathfrak{g}$ the Lie algebra of G.

We show normalized orientations on $\mathcal{B}_P$ are determined by a choice of *flag structure* (next slide). Orientations on $\mathcal{B}_P$ also need an orientation on $\mathrm{ind}(\not{D}_X^+)\otimes\mathfrak{g}$. If X is a Calabi–Yau 4-fold, there is a natural orientation for $\mathrm{ind}(\not{D}_X^+)$, so we don't need this second choice.

Flag structures – first idea

Joyce 2018 and Joyce–Upmeier 2023 introduced *flag structures* on 7-manifolds, and used them to define orientations on moduli spaces of associative 3-folds and G_2-instantons on compact G_2-manifolds.
We define a related (but more complicated) notion of flag structure F for compact spin 8-manifolds X satisfying condition $(*)$, as a choice of natural trivialization of an orientation functor associated to X (more details later). We can write a flag structure F as $(F_\alpha : \alpha \in H^4(X,\mathbb{Z}))$, where each F_α lies in a $\mathbb{Z}_2$-torsor. Thus, the set of flag structures on X is a torsor for $\mathrm{Map}(H^4(X,\mathbb{Z}),\mathbb{Z}_2)$.
By imposing extra conditions we can cut this down to a finite choice of flag structures.
If X is a Calabi–Yau 4-fold, the orientation on $\boldsymbol{\mathcal{M}}$ at a perfect complex $[\mathcal{E}^\bullet] \in \boldsymbol{\mathcal{M}}$ depends on F_α for $\alpha = c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2$.
There is a canonical choice for F_0. Hence, if $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, there is a canonical choice of orientation on the connected component of $\boldsymbol{\mathcal{M}}$ containing $\mathcal{E}^\bullet$. Thus we deduce:

Theorem 7

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Choose a flag structure F on X. Then we can construct a canonical orientation on the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X. On the open and closed substack $\boldsymbol{\mathcal{M}}_{c_2-c_1^2=0} \subset \boldsymbol{\mathcal{M}}$ of perfect complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, we can define the canonical orientation without choosing a flag structure.

The second part resolves a paradox. There are several conjectures in the literature by Bojko, Cao, Kool, Maulik, Toda, ..., of the form

$$\text{Conventional invariants of } X \simeq \text{DT4 invariants of } X, \tag{2}$$

where the left hand side, involving Gromov–Witten invariants etc., needs no choice of orientation, but the right hand side needs a Borisov–Joyce orientation to determine the sign. All these conjectures are really about sheaves on points and curves — Hilbert schemes of points, MNOP, DT-PT, etc. — and so involve only complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$ in $H^4(X, \mathbb{Z})$.

Let $\boldsymbol{B}$ be a 'stable tangential structure' on manifolds, for example, $\mathbf{O}$ is 'unoriented', $\mathbf{SO}$ is 'oriented', $\mathbf{Spin}$ is 'spin' (which includes oriented). For each (nice) topological space T and $n \geqslant 0$, we define the *bordism group* $\Omega_n^{\boldsymbol{B}}(T)$ to be the set of $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact n-manifold with $\boldsymbol{B}$-structure with $\partial X = \emptyset$ and $f : X \to T$ is continuous. We write $(X_0, f_0) \sim (X_1, f_1)$ if there exists a compact $n+1$-manifold Y with $\boldsymbol{B}$-structure and a continuous map $g : Y \to T$, such that Y has boundary $\partial Y = -X_0 \amalg X_1$ with $\boldsymbol{B}$-structures, and $g|_{\partial Y} = f_0 \amalg f_1$. Here $-X_0$ is X_0 with the 'opposite $\boldsymbol{B}$-structure' (e.g. opposite orientation). Then $\Omega_n^{\boldsymbol{B}}(T)$ is an abelian group with addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$ and identity $0 = [\emptyset, \emptyset]$. Bordism $\Omega_*^{\boldsymbol{B}}(-)$ is a *generalized homology theory* – it satisfies all the Eilenberg–Steenrod axioms except the dimension axiom. There is an Atiyah–Hirzebruch spectral sequence $H_p(T, \Omega_q^{\boldsymbol{B}}(*)) \Rightarrow \Omega_{p+q}^{\boldsymbol{B}}(T)$. For T path-connected, *reduced bordism* is $\tilde{\Omega}_n^{\boldsymbol{B}}(T) = \Omega_n^{\boldsymbol{B}}(T, \{t_0\})$. Then $\Omega_n^{\boldsymbol{B}}(T) = \tilde{\Omega}_n^{\boldsymbol{B}}(T) \oplus \Omega_n^{\boldsymbol{B}}(*)$.

Spin bordism groups of classifying spaces

We will care about spin bordism groups $\Omega_n^{\mathbf{Spin}}(T)$ of classifying spaces such as BG for G a Lie group, or loops spaces $\mathcal{L}BG$. These can often be computed using Algebraic Topology (and a lot of work by Markus). The bordism groups of the point are

n	0	1	2	3	4	5	6	7	8	9
$\Omega_n^{\mathbf{Spin}}(*)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$

The homology $H_*(BG,\mathbb{Z})$ and $H_*(BG,\mathbb{Z}_2)$ is known for classical G. Using the A–H spectral sequence $\tilde H_p(BG,\Omega_q^{\boldsymbol{B}}(*))\Rightarrow\tilde\Omega_{p+q}^{\boldsymbol{B}}(BG)$, we can prove for example that $B\,\mathrm{SU}(m)$, $m\geqslant 5$ has reduced spin bordism

n	0,1,2,3,5,7	4	6	8	9
$\tilde\Omega_n^{\mathbf{Spin}}(B\,\mathrm{SU}(m))$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}_2$

We can give explicit basis elements for the groups, and describe the isomorphisms explicitly, e.g. $\tilde\Omega_8^{\mathbf{Spin}}(B\,\mathrm{SU}(m))\xrightarrow{\cong}\mathbb{Z}^3$ is

$$[X,P]\longmapsto\bigl(\textstyle\int_X[\frac{c_4(P)}{6}-\frac{c_2(P)^2}{12}-\frac{p_1(TX)c_2(P)}{24}],\int_X c_2(P)^2,\int_X c_4(P)\bigr).$$

Definition

A *Picard groupoid* $(\mathcal{G}, \otimes, \mathbb{1})$ is a groupoid $\mathcal{G}$ with a monoidal structure $\otimes : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ which is symmetric and associative up to coherent natural isomorphisms (not included in the notation), and an identity object $\mathbb{1}$ such that $\mathbb{1} \otimes X \cong X \otimes \mathbb{1} \cong X$ for all $X \in \mathcal{G}$, such that for every $X \in \mathcal{G}$ there exists $Y \in \mathcal{G}$ with $X \otimes Y \cong \mathbb{1}$.

Picard groupoids are classified up to equivalence by triples (π_0, π_1, q), where π_0, π_1 are abelian groups and $q : \pi_0 \to \pi_1$ is a map which is both linear and quadratic. To $(\mathcal{G}, \otimes, \mathbb{1})$ we associate the abelian groups π_0 of isomorphism classes $[X]$ of objects $X \in \mathcal{G}$ with multiplication $[X] \cdot [Y] = [X \otimes Y]$, and $\pi_1 = \mathrm{Aut}_{\mathcal{G}}(\mathbb{1})$.
Symmetric monoidal functors $F : (\mathcal{G}, \otimes, \mathbb{1}) \to (\mathcal{G}', \otimes, \mathbb{1}')$ are functors $F : \mathcal{G} \to \mathcal{G}'$ preserving all the structure. They are classified up to monoidal natural isomorphism by group morphisms $f_0 : \pi_0 \to \pi_0'$ and $f_1 : \pi_1 \to \pi_1'$ with $q' \circ f_0 = f_1 \circ q$.
We could call Picard groupoids *abelian 2-groups*, as they are a 2-categorical notion of abelian group.

Our theory uses special examples of Picard groupoids we call *bordism categories*. Here is an example.

Example

Let G be a Lie group, and $n \geqslant 0$. Define a Picard groupoid $\mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ to have objects pairs (X, P) of a compact spin n-manifold X and a principal G-bundle $P \to X$, and morphisms $[Y, Q] : (X_0, P_0) \to (X_1, P_1)$ to be equivalence classes $[Y, Q]$ of a compact spin $(n+1)$-manifold Y with boundary $\partial Y = -X_0 \amalg X_1$ and a principal G-bundle $Q \to Y$ with $Q|_{\partial Y} = P_0 \amalg P_1$, where the equivalence involves $(n+2)$-dimensional bordisms. The composition of $[Y, Q] : (X_0, P_0) \to (X_1, P_1)$ and $[Y', Q'] : (X_1, P_1) \to (X_2, P_2)$ is $[Y \amalg_{X_1} Y', Q \amalg_{P_1} Q']$. The monoidal structure is disjoint union, $(X, P) \otimes (X', P') = (X \amalg X', P \amalg P')$.

The classifying data is $\pi_0 = \Omega_n^{\mathbf{Spin}}(BG)$, $\pi_1 = \Omega_{n+1}^{\mathbf{Spin}}(BG)$, and $q : [X, P] \mapsto [X \times \mathcal{S}^1_{\mathrm{nb}}, P \times \mathcal{S}^1_{\mathrm{nb}}]$, where $\mathcal{S}^1_{\mathrm{nb}}$ is $\mathcal{S}^1$ with the non-bounding spin structure. Here $\Omega_*^{\mathbf{Spin}}(-)$ is *spin bordism*, a generalized homology theory, and BG is the classifying space.

Example

The groupoid $\mathbb{Z}_2\text{-tor}$ of $\mathbb{Z}_2$-*torsors* is a Picard groupoid with $\pi_0 = 0$ and $\pi_1 = \mathbb{Z}_2$.
The groupoid $\text{s-}\mathbb{Z}_2\text{-tor}$ of *super* $\mathbb{Z}_2$-*torsors* ($\mathbb{Z}_2$-graded $\mathbb{Z}_2$-torsors) is a Picard groupoid with $\pi_0 = \pi_1 = \mathbb{Z}_2$ and $q = \mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$.

Example

(a) Suppose $n \equiv 1, 7 \mod 8$. We can define a symmetric monoidal functor $F : \mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2\text{-tor}$ which maps (X, P) to the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ defined using the Dirac operator $\not{D}_X$.
(b) Suppose $n \equiv 0 \mod 8$. We can define a symmetric monoidal functor $F : \mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \to \text{s-}\mathbb{Z}_2\text{-tor}$ which maps (X, P) to the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ defined using the positive Dirac operator $\not{D}_X^+$, $\mathbb{Z}_2$-graded in degree $\mathrm{ind}(\not{D}_X^+ \otimes \mathrm{ad}(P)) \mod 2$.

Thus we can encode orientations of moduli spaces in *orientation functors* between Picard groupoids. This is not obvious. It depends on a bordism-invariance property of indices and determinants of Dirac operators proved in Upmeier arXiv:2312.06818.

Example

Let (X, g) be a compact spin n-manifold, and G be a Lie group. Define a subcategory $\mathfrak{Bord}_X(BG)$ of $\mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ to have objects (X, P) for X the fixed spin n-manifold and varying P, and to have morphisms $[X \times [0,1], Q]$ for $Y = X \times [0,1]$ the fixed spin $(n+1)$-manifold with boundary, and varying Q. Write $\mathrm{inc} : \mathfrak{Bord}_X(BG) \hookrightarrow \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ for the inclusion functor. Suppose $n \equiv 1, 7, 8 \mod 8$, and write $F_X = F \circ \mathrm{inc} : \mathfrak{Bord}_X(BG) \to \mathbb{Z}_2\text{-tor}$, where for $n \equiv 8$ we compose with $\text{s-}\mathbb{Z}_2\text{-tor} \to \mathbb{Z}_2\text{-tor}$ forgetting $\mathbb{Z}_2$-gradings.

Then a choice of orientation for $\mathcal{B}_P$ for each principal G-bundle $P \to X$, invariant under isomorphisms $P \cong P'$, is equivalent to a natural isomorphism $\eta : F_X \Rightarrow \mathbb{1}_X$, where $\mathbb{1}_X$ is the constant functor with value $\mathbb{Z}_2$. Hence, $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$ if and only if the functor $F_X : \mathfrak{Bord}_X(BG) \to \mathbb{Z}_2\text{-tor}$ is trivializable.

To see why this is true, note that $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$, where $\mathcal{A}_P$ is the infinite-dimensional affine space of connections on $P \to X$, and $\mathcal{G}_P = \mathrm{Aut}(P)$ is the gauge group. Here $\mathcal{A}_P$ is always orientable, with exactly two orientations, as it is contractible. So $\mathcal{B}_P$ is orientable if and only if the group $\mathcal{G}_P$ acts trivially on the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$.

Given an element $\gamma \in \mathrm{Aut}(P)$, we can define a morphism $[X \times [0,1], Q] : (X, P) \to (X, P)$ in $\mathfrak{Bord}_X(BG) \subset \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ by taking Q to be $P \times [0,1]$ with identifications $\mathrm{id}_P : P \times \{0\} \to P$ and $\gamma : P \times \{1\} \to P$. All morphisms $[X \times [0,1], Q] : (X, P) \to (X, P)$ are of this form. So $\mathcal{G}_P$ acts trivially on the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ if and only if F_X is trivializable over the object (X, P).

From this we easily deduce:

Theorem 8 (Joyce–Upmeier)

(a) *Let* $n \equiv 1, 7, 8 \mod 8$, *and* X *be a compact spin n-manifold and* G *a Lie group. Then* $\mathcal{B}_P$ *is orientable for all principal* G*-bundles* $P \to X$ *if and only if for all classes* $[X, \phi] \in \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG)$ *with domain* X *we have* $f_1 \circ \xi([X, \phi]) = 0$ *in* $\mathbb{Z}_2$, *where*

$$\Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \xrightarrow{\quad \xi \quad} \Omega_{n+1}^{\mathbf{Spin}}(BG) \xrightarrow{\quad f_1 \quad} \mathbb{Z}_2.$$

with $\xi : [X, \phi] \mapsto [X \times \mathcal{S}^1_{\mathrm{b}}, \phi']$ *and* f_1 *the classifying morphism for* F_G.

(b) $\mathcal{B}_P$ *is orientable for all compact spin n-manifolds* X *and all principal* G*-bundles* $P \to X$ *iff* $f_1 \circ \xi \equiv 0 : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \mathbb{Z}_2$.

We can use Algebraic Topology and spectral sequences to compute bordism groups such as $\Omega_n^{\mathbf{Spin}}(BG), \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG)$, and morphisms such as $\xi : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathbf{Spin}}(BG)$, and $f_0 : \Omega_n^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$ and $f_1 : \Omega_{n+1}^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$ which classify orientation functors. Then we can use these to prove theorems on orientability and canonical orientations.

Cohomology bordism categories

For R a commutative ring and $0 \leqslant k \leqslant n$, define the *cohomology bordism category* $\mathfrak{Bord}_n^{\boldsymbol{B}}(K(R,k))$ to have objects (X,γ) where X is a compact n-manifold with $\boldsymbol{B}$-structure with $\partial X = \emptyset$ and $\gamma \in C^k(X,R)$ with $\mathrm{d}\gamma = 0$ is a the k-cocycle in cohomology of X over R, and to have morphisms $[Y,\delta] : (X_0,\gamma_0) \to (X_1,\gamma_1)$ to be bordism/cohomology classes of pairs (Y,δ) of a compact $(n+1)$-manifold Y with $\boldsymbol{B}$-structure with boundary $\partial Y = -X_0 \amalg X_1$ and a k-cochain $\delta \in C^k(Y,R)$ with $\delta|_{\partial Y} = -\gamma_0 + \gamma_1$.
Then $\mathfrak{Bord}_n^{\boldsymbol{B}}(K(R,k))$ has invariants $\pi_i = \Omega_{n+i}^{\boldsymbol{B}}(K(R,k))$ for $i = 0,1$, where $K(R,k)$ is the Eilenberg–MacLane space classifying $H^k(-,R)$. We can often compute $\Omega_*^{\boldsymbol{B}}(K(R,k))$.
There is a 16-connected map $BE_8 \to K(\mathbb{Z},4)$, so $\Omega_n^{\boldsymbol{B}}(BE_8) \cong \Omega_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$ for $n < 16$. Thus we can define a symmetric monoidal functor $\mathfrak{Bord}_n^{\boldsymbol{B}}(BE_8) \to \mathfrak{Bord}_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$, which is an equivalence of categories for $n \leqslant 14$.
In this way we translate orientability questions for E_8 gauge theory into problems in cohomology and cohomology operations, such as Steenrod squares.

As we have an equivalence $\mathfrak{Bord}_n^{\boldsymbol{B}}(BE_8) \to \mathfrak{Bord}_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$ for $n \leqslant 14$, for $G = E_8$ we may replace BE_8 with $K(\mathbb{Z},4)$. Our calculations show that when $n = 7$, $f_1 \circ \xi \equiv 0 : \Omega_7^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4)) \to \mathbb{Z}_2$. Thus $\mathcal{B}_P$ is orientable for all principal E_8-bundles $P \to X$. However, when $n = 8$, $f_1 \circ \xi \not\equiv 0 : \Omega_8^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4)) \to \mathbb{Z}_2$. Elements of $\Omega_8^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4))$ may be written $[X,\alpha]$ for X a compact spin n-manifold and $\alpha \in H^4(X \times \mathcal{S}^1,\mathbb{Z})$. Then $\alpha = \beta \boxtimes \mathrm{Pd}[\mathcal{S}^1] + \gamma \boxtimes 1_{\mathcal{S}^1}$ for $\beta \in H^3(X,\mathbb{Z})$ and $\gamma \in H^4(X,\mathbb{Z})$, and $f_1 \circ \xi([X,\alpha]) = \int_X \bar\beta \cup \mathrm{Sq}^2(\bar\beta)$, where $\bar\beta \in H^2(X,\mathbb{Z}_2)$ is the mod 2 reduction of β and $\mathrm{Sq}^2(\bar\beta)$ is its Steenrod square. Thus by Theorem 8(a), if X is a compact spin 8-manifold, then $\mathcal{B}_P$ is orientable for all principal E_8-bundles $P \to X$ if and only if the following condition holds:

(*) Let $\alpha \in H^3(X,\mathbb{Z})$, and write $\bar\alpha \in H^3(X,\mathbb{Z}_2)$ for its mod 2 reduction, and $\mathrm{Sq}^2(\bar\alpha) \in H^5(X,\mathbb{Z}_2)$ for its Steenrod square. Then $\int_X \bar\alpha \cup \mathrm{Sq}^2(\bar\alpha) = 0$ in $\mathbb{Z}_2$ for all $\alpha \in H^3(X,\mathbb{Z})$.

Using 'complex type' morphisms, we deduce that if $(*)$ holds then $\mathcal{B}_P$ is orientable for all G-bundles $P \to X$ for any G on the list

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m). \qquad (3)$$

Definition

Let X be a compact spin n-manifold for $n = 7$ or 8. A *flag structure* on X is a natural isomorphism ζ in the diagram

$$\begin{tikzcd}
\mathfrak{Bord}_X^{\mathbf{Spin}}(K(\mathbb{Z},4)) \arrow[r, "\mathbb{1}_X"'] \arrow[d, "\mathrm{inc}"] \arrow[dr, phantom, "\Downarrow\zeta"] & \mathbb{Z}_2\text{-tor} \\
\mathfrak{Bord}_n^{\mathbf{Spin}}(K(\mathbb{Z},4)) \arrow[r, "F_{K(\mathbb{Z},4)}"] & \mathbb{Z}_2\text{-tor or } \mathbb{Z}_2\text{-tor}. \arrow[u, "\text{forget } \mathbb{Z}_2\text{-grading}"]
\end{tikzcd}$$

If $n = 7$, flag structures on X always exist.
If $n = 8$, flag structures on X exist iff X satisfies condition $(*)$.
If X has a flag structure, then we can construct canonical (normalized) orientations on $\mathcal{B}_P$ for all G-bundles $P \to X$ for any G on the list (3).