Introduction to Differential Geometry

Lecture 3 of 10: Tensors

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These slides available at
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Plan of talk:

3 Tensors
   3.1 Some linear algebra
   3.2 Operations on vector bundles; tensors
   3.3 Index notation for tensors
   3.4 The Lie bracket of vector fields
   3.5 Exponentiating vector fields
3. Tensors and exterior forms
3.1. Some linear algebra

We start with a reminder on some basic operations on vector spaces. For simplicity, all vector spaces will be finite-dimensional over \( \mathbb{R} \). If \( U \) is a vector space, the dual vector space is \( U^* = \text{Hom}(U, \mathbb{R}) \), with \( \dim U^* = \dim U \). If \( u^1, \ldots, u^m \) is a basis of \( U \), there is a dual basis \( u_1, \ldots, u_m \) of \( U^* \), with \( u_i(u^j) = \delta_{ij} \) for \( i, j = 1, \ldots, m \). We can identify \( U = (U^*)^* \).

If \( U, V \) are vector spaces, the direct sum is
\[
U \oplus V = U \times V = \{(u, v) : u \in U, \ v \in V\}.
\]
It is a vector space of dimension \( \dim U + \dim V \). If \( u^1, \ldots, u^m \) and \( v^1, \ldots, v^n \) are bases of \( U, V \), then \( u^1, \ldots, u^m, v^1, \ldots, v^n \) is a basis of \( U \oplus V \). So we can add vector spaces. Direct sum is associative and commutative, \( U \oplus V = V \oplus U \), \( U \oplus (V \oplus W) = (U \oplus V) \oplus W \).

We can also multiply vector spaces. For \( U, V \) vector spaces, the tensor product \( U \otimes V \) is a natural vector space with
\[
\dim(U \otimes V) = \dim U \cdot \dim V.
\]
There is a bilinear operation
\[
\otimes : U \times V \longrightarrow U \otimes V, \quad (u, v) \longmapsto u \otimes v.
\]
If \( u^1, \ldots, u^m \) and \( v^1, \ldots, v^n \) are bases of \( U, V \), then
\[
\{u^i \otimes v^j : i = 1, \ldots, m, j = 1, \ldots, n\}
\]
is a basis of \( U \otimes V \).

Formally, we may define
\[
U \otimes V = \{\text{bilinear maps } \alpha : U^* \times V^* \longrightarrow \mathbb{R}\},
\]
and for \( u \in U, \ v \in V \), define \( u \otimes v \in U \otimes V \) to be the bilinear map
\[
u \otimes v : U^* \times V^* \longrightarrow \mathbb{R}, \quad u \otimes v : (\alpha, \beta) \longmapsto \alpha(u) \cdot \beta(v).
\]
Tensor products are associative and commutative and distributive over direct sum, \( U \otimes V = V \otimes U \), \( U \otimes (V \oplus W) = (U \otimes V) \otimes W \), \( U \otimes (V \oplus W) = (U \otimes V) \oplus (U \otimes W) \), just as you would expect.
Symmetric and exterior (antisymmetric) products

Let $V$ be a vector space. Then we can form the \textit{n-fold tensor product} $\bigotimes^n V = V \otimes \cdots \otimes V$. The symmetric group $S_n$ acts on $\bigotimes^n V$ by permutations on the $n$ factors, so that $\sigma \in S_n$ acts by

$$\sigma : v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for $v_1, \ldots, v_n \in V$. The \textit{$n^{th}$ symmetric power} $S^n V$ is the subspace of $\bigotimes^n V$ invariant under $S_n$, with $\dim S^n V = \binom{\dim V + n - 1}{n}$.

The \textit{$n^{th}$ exterior power} $\Lambda^n V$ is the subspace of $\bigotimes^n V$ anti-invariant under $S_n$, with $\dim \Lambda^n V = \binom{\dim V}{n}$.

For $n = 2$ we have $\bigotimes^2 V = S^2 V \oplus \Lambda^2 V$.

We can identify $\bigotimes^2 \mathbb{R}^n$ with $n \times n$ matrices, $S^2 \mathbb{R}^n$ with symmetric matrices, and $\Lambda^2 \mathbb{R}^n$ with antisymmetric matrices.

There are projections $\Pi^S : \bigotimes^n V \to S^n V$ and $\Pi^\Lambda : \bigotimes^n V \to \Lambda^n V$ by symmetrization and antisymmetrization, given by

$$\Pi^S(v) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(v), \quad \Pi^\Lambda(v) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma(v).$$

The \textit{symmetric product} $\odot$ is tensor product $\otimes$ followed by symmetrization $\Pi^S$, so for example $v_1 \odot \cdots \odot v_n = \Pi^S(v_1 \otimes \cdots \otimes v_n)$ for $v_1, \ldots, v_n \in V$.

The \textit{exterior product} or \textit{wedge product} $\wedge$ is tensor product $\otimes$ followed by antisymmetrization $\Pi^\Lambda$, so for example we have $\wedge : \Lambda^m V \times \Lambda^n V \to \Lambda^{m+n} V$, $\alpha \wedge \beta = \Pi^\Lambda(\alpha \otimes \beta)$.

Both $\odot$, $\wedge$ are associative. We have $\beta \odot \alpha = \alpha \odot \beta$ and $\beta \wedge \alpha = (-1)^{\deg \alpha \deg \beta} \alpha \wedge \beta$. 
3.2. Operations on vector bundles; tensors

Now let $X$ be a smooth manifold, and as in §1.5 consider vector bundles $E \to X$, $F \to X$, so that for all $x \in X$ the fibres $E_x, F_x$ are vector spaces. The operations on vector spaces in §3.1 all make sense for vector bundles. So we can form the dual vector bundle $E^*$ with $\text{rank } E^* = \text{rank } E$ and fibres $(E^*)_x = (E_x)^*$, the direct sum vector bundle $E \oplus F \to X$, with $\text{rank } (E \oplus F) = \text{rank } E + \text{rank } F$ and fibres $(E \oplus F)_x = E_x \oplus F_x$, the tensor product bundle $E \otimes F \to X$, with $\text{rank } (E \otimes F) = \text{rank } E \cdot \text{rank } F$ and fibres $(E \otimes F)_x = E_x \otimes F_x$. Given $E \to X$, we can form the $n$-fold tensor product $\bigotimes^n E \to X$, the $n^{\text{th}}$ symmetric power $S^n E \to X$ and the $n^{\text{th}}$ exterior power $\Lambda^n E \to X$, with fibres $\bigotimes^n (E_x)$, $S^n(E_x)$, $\Lambda^n(E_x)$. We can take direct sums and tensor products of sections: if $e \in C^\infty(E)$, $f \in C^\infty(F)$ then $e \oplus f \in C^\infty(E \oplus F)$, and so on.

Tensor bundles and tensors

As in §2.1, any manifold $X$ has two natural vector bundles, the tangent bundle $TX \to X$ and cotangent bundle $T^*X \to X$. So we can make many more bundles by direct sums, tensor products, symmetric products, and exterior products, of $TX$, $T^*X$. The tensor bundles on $X$ are $\bigotimes^k TX \otimes \bigotimes^l T^*X$ for $k, l \geq 0$ (where if $k = 0$ or $l = 0$ we omit that term). They are vector bundles on $X$, of rank $(\dim X)^{k+l}$.

A tensor $T$ on $X$ is a smooth section of some tensor bundle, $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$. This is very general, and includes many interesting geometric structures.
Examples of interesting classes of tensors

**Example**

A vector field \( v \) on \( X \) is a section of \( TX \). This is a tensor with \( k = 1 \) and \( l = 0 \).

**Example**

An \( l \)-form for \( l \geq 0 \), or exterior form, on \( X \), is a section of \( \Lambda^l T^*X \). As \( \text{rank} \Lambda^l T^*X = \left( \dim X \right)^l \), this is only nonzero for \( l = 0, \ldots, \dim X \). Since \( \Lambda^l T^*X \) is a subbundle of \( \bigotimes^l T^*X = \bigotimes^0 TX \otimes \bigotimes^l T^*X \), \( l \)-forms are tensors with \( k = 0 \).

**Example**

A Riemannian metric \( g \) is a smooth section of \( S^2 T^*X \) such that \( g|_x \in S^2 T^*_x X \) is a positive definite quadratic form on \( T_x X \) for all \( x \in X \). As \( S^2 T^*X \subset \bigotimes^2 T^*X \), this is a tensor with \( k = 0, l = 2 \).

3.3. Index notation for tensors

Here is some useful notation for tensors, introduced by physicists. Let \( X \) be an \( n \)-manifold, and \( T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X) \) a tensor of type \((k, l)\) on \( X \). Let \((x^1, \ldots, x^n)\) be local coordinates on an open set \( U \subseteq X \). (For consistent notation, we use superscripts \( x^i \) rather than subscripts \( x_i \); \( x^i \) means the \( i \)th variable, not a power of \( x \).) Then \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \) are a basis of sections of \( TX \) on \( U \), and \( dx^1, \ldots, dx^n \) a basis of sections of \( T^*X \) on \( U \). Hence we may write

\[
T \big|_U = \sum_{a_1, \ldots, a_k=1, \ldots, n}^{a_1, \ldots, a_k=1, \ldots, n} T^{a_1 a_2 \ldots a_k}_{b_1 b_2 \ldots b_l} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{a_k}} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_l}. \tag{3.1}
\]

Here \( T^{a_1 a_2 \ldots a_k}_{b_1 b_2 \ldots b_l} : U \to \mathbb{R} \) is a smooth function for all values of \( a_1, \ldots, a_k, b_1, \ldots, b_l \in \{1, \ldots, n\} \).
Thus, on $U$ the tensor $T$ is uniquely determined by the real functions $T_{b_1 \ldots b_l}^{a_1 \ldots a_k}$ for all $a_i, b_j$, and vice versa. So we can identify $T$ with such $n^{k+l}$-tuples of functions $(T_{b_1 \ldots b_l}^{a_1 \ldots a_k})_{a_1, \ldots, a_k = 1, \ldots, n}$, which we can think of as a kind of generalized matrix.

If $(\tilde{x}^1, \ldots, \tilde{x}^n)$ is another coordinate system on $\tilde{U} \subseteq X$, and $\tilde{T}_{b_1 \ldots b_l}^{a_1 \ldots a_k}$ the corresponding functions from $T|_{\tilde{U}}$, then using

$$\frac{\partial}{\partial \tilde{x}^i} = \sum_{j=1}^n \frac{\partial x^j}{\partial \tilde{x}^i} \cdot dx^j, \quad d\tilde{x}^i = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j} \cdot dx^j,$$

on $U \cap \tilde{U}$ we have

$$\tilde{T}_{b_1 \ldots b_l}^{a_1 \ldots a_k} = \sum_{c_1, \ldots, c_k = 1, \ldots, n} \frac{\partial \tilde{x}^{a_1}}{\partial x^{c_1}} \cdots \frac{\partial \tilde{x}^{a_k}}{\partial x^{c_k}} \cdot \frac{\partial x^{d_1}}{\partial \tilde{x}^{b_1}} \cdots \frac{\partial x^{d_l}}{\partial \tilde{x}^{b_l}} \cdot T_{d_1 \ldots d_l}^{c_1 \cdots c_k}. \quad (3.2)$$

This tells you how the tuples $(T_{b_1 \ldots b_l}^{a_1 \ldots a_k})_{a_1, \ldots, a_k = 1, \ldots, n}$ transform under change of coordinates.

Upper indices $T^a$ are called **contravariant (vector) indices**. Lower indices $T_b$ are called **covariant (1-form) indices**.

In the index notation, we write the tensor $T$ (on all of $X$, not just on one coordinate chart $U \subseteq X$) as $T_{b_1 \ldots b_l}^{a_1 \ldots a_k}$. We could interpret this in several ways. We could view it just as a formal symbol, telling us that $T$ is a section of $\bigotimes^k TX \otimes \bigotimes^l T^*X$. Or, we could understand it to mean ‘every time we have coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, then we get an $n^{k+l}$-tuple $(T_{b_1 \ldots b_l}^{a_1 \ldots a_k})_{a_1, \ldots, a_k = 1, \ldots, n}$ of smooth functions $T_{b_1 \ldots b_l}^{a_1 \ldots a_k} : U \to \mathbb{R}$ as in (3.1), and under change of coordinates, these $n^{k+l}$-tuples transform as in (3.2)’.
Examples of tensor notation

**Example**

A vector field $v$ on $X$ is written $v^a$. In coordinates $(x^1, \ldots, x^n)$ this means functions $(v^1, \ldots, v^n)$ with $v = v^1 \frac{\partial}{\partial x^1} + \cdots + v^n \frac{\partial}{\partial x^n}$.

**Example**

An $l$-form on $X$ is a tensor $\alpha_{b_1 \cdots b_l}$ with

$$\alpha_{b_1 \cdots b_{i-1} b_j b_{i+1} \cdots b_{l-1} b_l} = -\alpha_{b_1 \cdots b_l}$$

for all $1 \leq i < j \leq l$. So a 2-form is $\alpha_{ab}$ with $\alpha_{ba} = -\alpha_{ab}$.

**Example**

A Riemannian metric is a tensor $g_{ab}$ with $g_{ab} = g_{ba}$, with $(g_{ab})_{a,b=1,\ldots,n}$ a positive definite $n \times n$ matrix of functions.

Index notation makes it easy to describe (anti)symmetries of tensors, by permuting indices.

The Einstein summation convention

As $TX$, $T^*X$ are dual, there is a dual pairing $TX \times T^*X \rightarrow \mathbb{R}$. This induces vector bundle morphisms

$$\bigotimes^{k+1} TX \otimes \bigotimes^{l+1} T^*X \rightarrow \bigotimes^k TX \otimes \bigotimes^l T^*X$$

by contracting together a $TX$ and a $T^*X$ factor (need to specify which factors). In index notation, this is done by the Einstein summation convention: if an index $c$ occurs twice in a tensor in a formula, once as an upper and once as a lower index, then (thinking in terms of tuples of functions) we are to sum the index $c$ from $1, \ldots, n = \dim X$, even though the sum $\sum_{c=1}^n$ is not written.
3.4. The Lie bracket of vector fields

In the next sections we will discuss various ways in which we can differentiate tensors, or more general sections of vector bundles. One of the simplest of these is the Lie bracket of vector fields.

Definition

Let $X$ be a manifold, and $v, w \in C^\infty(TX)$ be vector fields on $X$. We will define a vector field $[v, w] \in C^\infty(TX)$ called the Lie bracket of $v$ and $w$. In local coordinates $(x^1, \ldots, x^n)$ on $U \subseteq X$, this is given in index notation by the formula

$$[v, w]^a = v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \quad (3.3)$$

That is, if $v = v^1 \frac{\partial}{\partial x^1} + \cdots + v^n \frac{\partial}{\partial x^n}$ and $w = w^1 \frac{\partial}{\partial x^1} + \cdots + w^n \frac{\partial}{\partial x^n}$, then $[v, w] = u^1 \frac{\partial}{\partial x^1} + \cdots + u^n \frac{\partial}{\partial x^n}$, where

$$u^a = \sum_{b=1}^n v^b \frac{\partial w^a}{\partial x^b} - w^b \frac{\partial v^a}{\partial x^b}. \quad (3.4)$$
Exercises 3.1

Show that the Lie bracket \([v, w]\) in (3.3) is well-defined. That is, as a vector field it is independent of the choice of local coordinates \((x^1, \ldots, x^n)\) used to define it.

Proposition 3.2

The Lie bracket of vector fields satisfies

\[
[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0
\]

for all vector fields \(u, v, w \in C^\infty(TX)\).

Equation (3.5) is called the Jacobi identity. It means that vector fields \(C^\infty(TX)\) are an (infinite-dimensional) Lie algebra.

Lie derivatives of tensors

Definition

Let \(X\) be a manifold, \(v \in C^\infty(TX)\) be a vector field, and \(T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)\) a tensor. We will define a tensor \(L_v T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)\) called the Lie derivative of \(T\) along \(v\). In local coordinates \((x^1, \ldots, x^n)\) on \(U \subseteq X\), this is given in index notation by the formula

\[
(L_v T)^{a_1 \cdots a_k}_{b_1 \cdots b_l} = v^c \frac{\partial}{\partial x^c} T^{a_1 \cdots a_k}_{b_1 \cdots b_l} - \sum_{i=1}^k T^{a_i a_{i-1} a_{i+1} \cdots a_k}_{b_{i-1} b_i b_{i+1} \cdots b_k} \frac{\partial v^a_i}{\partial x^c} \frac{\partial v^c}{\partial b_i} + \sum_{j=1}^l T^{a_1 \cdots a_k}_{b_1 \cdots b_{j-1} c b_{j+1} \cdots b_l} \frac{\partial v^c}{\partial x^b_j}.
\]

This is well-defined, i.e. independent of the choice of coordinates \((x^1, \ldots, x^n)\). If \(T = w\) is a vector field then \(L_v w = [v, w]\).
We can think of $\mathcal{L}_v T$ as ‘the derivative of $T$ in the direction $v$’. But note that (3.6) involves derivatives of $v$ as well as $T$, so $\mathcal{L}_v T$ is not pointwise linear in $v$. That is, in general $\mathcal{L}_{f v} + g w T \neq f \mathcal{L}_v T + g \mathcal{L}_w T$ for vector fields $v, w$ and functions $f, g : X \to \mathbb{R}$.

**Example**

In coordinates $(x^1, \ldots, x^n)$, take $v = \frac{\partial}{\partial x^i}$, so that $v^1, \ldots, v^n$ are $v^a = 1$ for $a = i$ and $v^a = 0$ otherwise. Then (3.6) becomes

$$(\mathcal{L}_v T)^{a_1 \ldots a_k}_{b_1 \ldots b_l} = \frac{\partial}{\partial x^i} T^{a_1 \ldots a_k}_{b_1 \ldots b_l},$$

as you would expect.

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**3.5. Exponentiating vector fields**

Let $X$ be a compact manifold (for simplicity), and $v \in C^\infty(TX)$ a vector field. A *flow-line* of $v$ is a smooth map $\gamma : \mathbb{R} \to X$ satisfying the differential equation $\frac{d\gamma}{dt}(t) = v|_{\gamma(t)} \in T_{\gamma(t)}V$ for all $t \in \mathbb{R}$.

Results on o.d.e.s imply that for each $x \in X$, there is a unique flow-line $\gamma_x$ with $\gamma_x(0) = x$. Here we need $X$ compact so that flow-lines cannot ‘fall off the edge of $X$’, so that $\gamma$ could only be defined on an open interval, not all of $\mathbb{R}$. (Consider $X = (0, 1)$, noncompact and $v = \frac{\partial}{\partial x}$. Then $\gamma$ is only defined on $(-x, 1 - x)$.) Define $\exp(tv) : X \to X$ for $t \in \mathbb{R}$ by $\exp(tv) : x \mapsto \gamma_x(t)$, for $\gamma_x$ the flow-line of $v$ with $\gamma_x(0) = x$ as above. Then $\exp(tv)$ is a diffeomorphism of $X$ depending smoothly on $t$, with $\exp(0) = \text{id}_X$ and $\exp(sv) \circ \exp(tv) = \exp((s + t)v)$ for $s, t \in \mathbb{R}$. 
If $T \in C^\infty(\bigotimes^k TX \otimes \bigotimes^l T^*X)$ is a tensor on $X$, then $\exp(tv)^*(T)$ is a tensor depending smoothly on $t \in \mathbb{R}$. One can show that

$$\mathcal{L}_v T = \frac{d}{dt} \left[ \exp(tv)^*(T) \right] \bigg|_{t=0}. $$

That is, $\mathcal{L}_v T$ measures the infinitesimal change of $T$ under the flow of $v$. 
Plan of talk:

4. Exterior forms
   4.1. Exterior forms and the de Rham differential
   4.2. Homology and cohomology
   4.3. Examples

4. Exterior forms
   4.1. Exterior forms and the de Rham differential

Let $X$ be a manifold, of dimension $n$. Then we have vector bundles $\Lambda^k T^* X$ for $k = 0, 1, \ldots, n$ (note that $\Lambda^k T^* X = 0$ for $k > n$). Sections $\alpha$ of $\Lambda^k T^* X$ are called $k$-forms, and form a (generally infinite-dimensional) vector space $C^\infty(\Lambda^k T^* X)$. In index notation $\alpha = \alpha_{a_1 \cdots a_k}$, and is antisymmetric in the indices $a_1, \ldots, a_k$ (i.e. if you exchange any two $a_i, a_j$, you change the sign).

As in §3.1–§3.2 we have the exterior product (wedge product)

$$\wedge : C^\infty(\Lambda^k T^* X) \times C^\infty(\Lambda^l T^* X) \longrightarrow C^\infty(\Lambda^{k+l} T^* X),$$

acting in index notation by

$$(\alpha \wedge \beta)_{a_1 \cdots a_{k+l}} = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \text{sign}(\sigma) \alpha_{a_{\sigma(1)} \cdots a_{\sigma(k)}} \beta_{a_{\sigma(k+1)} \cdots a_{\sigma(k+l)}}. \quad (4.1)$$
Pullback of forms by smooth maps

Let \( f : X \to Y \) be a smooth map of manifolds. As in §2.2 we have \( Tf : TX \to TY \), which can be interpreted as a vector bundle morphism \( df : TX \to f^*(TY) \) on \( X \), with a dual morphism \( (df)^* : f^*(T^*Y) \to T^*X \). Taking exterior powers gives vector bundle morphisms on \( X \)

\[
\Lambda^k (df)^* : f^*(\Lambda^k T^*Y) \longrightarrow \Lambda^k T^*X.
\]

Let \( \alpha \in C^\infty(\Lambda^k T^*Y) \) be a \( k \)-form on \( Y \). Then we have a pullback \( f^{-1}(\alpha) \in C^\infty(f^*(\Lambda^k T^*Y)) \) on \( X \). Define the pullback \( k \)-form to be

\[
f^*(\alpha) = \Lambda^k (df)^* [f^{-1}(\alpha)] \in C^\infty(\Lambda^k T^*X).\]

Pullback is (contravariantly) functorial, \( (g \circ f)^*(\beta) = f^* \circ g^*(\beta) \) for smooth \( g : Y \to Z \) and \( \beta \in C^\infty(\Lambda^k T^*Z) \).

If \( X \subseteq Y \) is a submanifold, we write \( \alpha|_X \) for \( i^*(\alpha) \), with \( i : X \hookrightarrow Y \) the inclusion.

**Definition**

The de Rham differential \( \partial : C^\infty(\Lambda^k T^*X) \longrightarrow C^\infty(\Lambda^{k+1} T^*X) \) for \( k \geq 0 \) is defined in local coordinates \( (x^1, \ldots, x^n) \) on \( U \subseteq X \), using index notation, by the formula

\[
(d\alpha)_{a_1 \cdots a_{k+1}} = \sum_{i=1}^{k+1} (-1)^{i-1} \frac{\partial}{\partial x^{a_i}} \alpha_{a_1 \cdots a_{i-1} a_{i+1} \cdots a_{k+1}}.
\]

**Exercise 4.1**

Show that the de Rham differential is well-defined. That is, as a \( k + 1 \)-form, \( d\alpha \) is independent of the choice of local coordinates \( (x^1, \ldots, x^n) \) used to define it.
Properties of the de Rham differential

From equations (4.1) and (4.2) we can prove:

**Proposition 4.2**

For all forms $\alpha, \beta, \gamma$ on $X$, the de Rham differential satisfies

$$d \circ d \alpha = 0, \quad d(\beta \wedge \gamma) = (d\beta) \wedge \gamma + (-1)^{\deg \beta} \beta \wedge (d\gamma). \quad (4.3)$$

**Proposition 4.3**

Let $f : X \to Y$ be smooth map of manifolds and $\alpha \in C^\infty(\Lambda^k T^* Y)$. Then

$$d(f^*(\alpha)) = f^*(d\alpha). \quad (4.4)$$

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### 4.2. Homology and cohomology

A reminder of some algebraic topology: let $X$ be a topological space, and $\mathbb{F}$ a field (for simplicity). Then we can define the **homology groups** $H_k(X; \mathbb{F})$ and **cohomology groups** $H^k(X; \mathbb{F})$ for $k \in \mathbb{N}$, which are vector spaces over $\mathbb{F}$, with $H^k(X; \mathbb{F}) \cong H_k(X; \mathbb{F})^*$. If $f : X \to Y$ is continuous there are functorial **pushforward maps** $f_* : H_k(X; \mathbb{F}) \to H_k(Y; \mathbb{F})$ on homology, and **pullback maps** $f^* : H^k(Y; \mathbb{F}) \to H^k(X; \mathbb{F})$ on cohomology. There are **cup products** $\cup : H^k(X; \mathbb{F}) \times H^l(X; \mathbb{F}) \to H^{k+l}(X; \mathbb{F})$ making $H^*(X; \mathbb{F})$ into a supercommutative graded algebra.

If $X$ is a compact, oriented manifold of dimension $n$, then **Poincaré duality** says that $H^k(X; \mathbb{F}) \cong H_{n-k}(X; \mathbb{F})$.

The **Betti numbers** of $X$ are $b^k(X) = \dim H^k(X; \mathbb{R})$.

Homology and cohomology are important topological invariants of a space, one of the most basic things you can compute.
De Rham cohomology

**Definition**

Let $X$ be a smooth manifold. The *de Rham cohomology group* $H^k_{dR}(X; \mathbb{R})$ of $X$, for $k = 0, \ldots, \dim X$, is

$$H^k_{dR}(X; \mathbb{R}) = \frac{\ker(d : C^\infty(\Lambda^k T^*X) \to C^\infty(\Lambda^{k+1} T^*X))}{\operatorname{Im}(d : C^\infty(\Lambda^{k-1} T^*X) \to C^\infty(\Lambda^k T^*X))}.$$

This makes sense as $d \circ d = 0$, by Proposition 4.2. The second equation of (4.3) implies that we can define a *cup product*

$$\cup : H^k_{dR}(X; \mathbb{R}) \times H^l_{dR}(X; \mathbb{R}) \to H^{k+l}_{dR}(X; \mathbb{R}),$$

$$(\beta + \operatorname{Im} d) \cup (\gamma + \operatorname{Im} d) \mapsto \beta \wedge \gamma + \operatorname{Im} d,$$

which is associative and supercommutative as $\wedge$ is.

If $X$ is compact then $H^k_{dR}(X; \mathbb{R})$ is finite-dimensional.

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If $f : X \to Y$ is a smooth map of manifolds then Proposition 4.3 implies that we can define *pullback maps*

$$f^* : H^k_{dR}(Y; \mathbb{R}) \to H^k_{dR}(X; \mathbb{R}), \quad f^*(\alpha + \operatorname{Im} d) = f^*(\alpha) + \operatorname{Im} d.$$

These pullback maps are *independent of $f : X \to Y$ up to smooth (or continuous) deformation*. That is, if $g : X \times [0, 1] \to Y$ is smooth and $f_0, f_1 : X \to Y$ are $f_0(x) = g(x, 0)$, $f_1(x) = g(x, 1)$ then $f_0^* = f_1^* : H^k_{dR}(Y; \mathbb{R}) \to H^k_{dR}(X; \mathbb{R})$.

**Theorem (The de Rham Theorem)**

There are natural isomorphisms $H^k_{dR}(X; \mathbb{R}) \cong H^k(X; \mathbb{R})$, where $H^k(X; \mathbb{R})$ is the $k^{\text{th}}$ real cohomology group of the underlying topological space $X$. These isomorphisms are compatible with cup products and pullbacks on $H^*_dR(-; \mathbb{R})$ and $H^*(-; \mathbb{R})$. 
Let $X$, $Y$ be topological spaces, and $\mathbb{F}$ a field. We have a product topological space $X \times Y$ with projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$.

**Theorem (The Künneth Theorem)**

For each $k \geq 0$ there is an isomorphism

$$
\bigoplus_{i,j \geq 0 : i+j=k} : H^i(X; \mathbb{F}) \otimes_{\mathbb{F}} H^j(Y; \mathbb{F}) \longrightarrow H^k(X \times Y; \mathbb{F})
$$

acting by $\bigoplus_{i+j=k} \alpha^i \otimes \beta^j \longmapsto \sum_{i+j=k} \pi^*_X(\alpha^i) \cup \pi^*_Y(\beta^j)$, for $\alpha^i \in H^i(X; \mathbb{F})$ and $\beta^j \in H^j(Y; \mathbb{F})$.

In particular, this applies to de Rham cohomology of products of manifolds.

Let $X$ be a manifold (usually compact). The **Betti numbers** of $X$ are $b^k(X) = \dim H^k_{dR}(X; \mathbb{R})$. The **Euler characteristic** is

$$
\chi(X) = \sum_{k=0}^{\dim X} (-1)^k b^k(X).
$$

They are topological invariants of $X$. If $X$ is compact then $H^k_{dR}(X; \mathbb{R})$ is finite-dimensional, so these are well defined. If $X$ is compact and odd-dimensional then $\chi(X) = 0$.

The Künneth Theorem implies that $\chi(X \times Y) = \chi(X) \chi(Y)$.

The Euler characteristic is very important, and crops up in many different places. For example, if $X$ is a compact manifold then the number of zeroes of a generic vector field $v$ on $X$, counted with multiplicity, is $\chi(X)$.

The Gauss–Bonnet Theorem says that if $(X, g)$ is a compact Riemannian 2-manifold with Gaussian curvature $\kappa$ then

$$
\int_X \kappa \, dV_g = 2\pi \chi(X).
$$
4.3. Examples

Example

The de Rham cohomology of $\mathbb{R}^n$ for $n \geq 0$ is

$$H^k_{dR}(\mathbb{R}^n; \mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

When $n = 0$, so that $\mathbb{R}^0 = \ast$ is a point, this is immediate from the definitions. To prove it when $n > 0$, consider the smooth maps $i : * \rightarrow \mathbb{R}^n$, $i : * \mapsto (0, \ldots, 0)$, and $\pi : \mathbb{R}^n \rightarrow *$.

These induce maps $i^* : H^k_{dR}(\mathbb{R}^n; \mathbb{R}) \rightarrow H^k_{dR}(\ast; \mathbb{R})$ and $\pi^* : H^k_{dR}(\ast; \mathbb{R}) \rightarrow H^k_{dR}(\mathbb{R}^n; \mathbb{R})$.

Since $\pi \circ i = \text{id} : * \rightarrow *$ we see that $i^* \circ \pi^*$ is the identity on $H^k_{dR}(\ast; \mathbb{R})$. Conversely, although $i \circ \pi \neq \text{id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we can smoothly deform $i \circ \pi$ to $\text{id}$, so $\pi^* \circ i^*$ is the identity on $H^k_{dR}(\mathbb{R}^n; \mathbb{R})$. Hence $i^*, \pi^*$ are inverse, and $H^k_{dR}(\mathbb{R}^n; \mathbb{R}) \cong H^k_{dR}(\ast; \mathbb{R})$.

Example

The de Rham cohomology of $S^n$ for $n > 0$ is

$$H^k_{dR}(S^n; \mathbb{R}) \cong \begin{cases} \mathbb{R}, & k = 0 \text{ or } k = n, \\ 0, & \text{otherwise.} \end{cases} \quad (4.5)$$

Example

The de Rham cohomology of $T^n$ for $n \geq 0$ is

$$H^k_{dR}(T^n; \mathbb{R}) \cong \mathbb{R}^{n \choose k}.$$ 

This follows from (4.5) for $H^k_{dR}(S^1; \mathbb{R})$ and the Künneth Theorem.

Considering $H^1_{dR}(-; \mathbb{R})$ we see that:

Corollary

There is no diffeomorphism $S^n \cong T^n$ for $n \geq 2$.

De Rham cohomology is useful for distinguishing manifolds.