Complex manifolds and Kähler Geometry

Lecture 1 of 16: Complex manifolds

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Plan of talk:

1. Complex manifolds
   1.1 Complex manifolds
   1.2 Holomorphic functions and holomorphic maps
   1.3 Complex submanifolds
   1.4 Projective complex manifolds
1.1. Complex manifolds

We will give two definitions of complex manifolds. This lecture, we use complex charts and holomorphic transition functions. Next lecture, in a more Differential Geometric style, we use (almost) complex structures on a real manifold. The two points of view are equivalent, by the Newlander–Nirenberg Theorem.

Recall the definition of a (smooth, real) manifold: a topological space $X$ with an atlas of charts $(U_i, \phi_i)$ with transition functions $\phi_{ij}$ diffeomorphisms between open sets in $\mathbb{R}^n$. We can instead require other conditions on $\phi_{ij}$, e.g. $\phi_{ij}$ continuous gives you topological manifolds, or we could require $\phi_{ij}$ to be $C^k$, or real analytic. Requiring the $\phi_{ij}$ to be holomorphic gives you complex manifolds.

**Definition**

Let $X$ be a topological space, and fix $n \geq 0$. A (complex) chart on $X$ is $(U, \phi)$, where $U \subseteq \mathbb{C}^n$ is open and $\phi : U \to X$ is a homeomorphism from $U$ to an open subset $\phi(U)$ in $X$. Let $(U, \phi), (V, \psi)$ be charts. The transition function between them is

$$\psi^{-1} \circ \phi : \phi^{-1}(\phi(U) \cap \psi(V)) \to \psi^{-1}(\phi(U) \cap \psi(V)).$$

It is automatically a homeomorphism between open subsets of $\mathbb{C}^n$. We call $(U, \phi), (V, \psi)$ compatible if $\psi^{-1} \circ \phi$ is a biholomorphism between open subsets of $\mathbb{C}^n$, i.e. holomorphic with holomorphic inverse.

A (complex) atlas on $X$ is a system $\{(U_i, \phi_i) : i \in I\}$ of pairwise compatible charts on $X$ with $X = \bigcup_{i \in I} \phi_i(U_i)$. We may write $\phi_{ij}$ for the transition function $\phi_j^{-1} \circ \phi_i$. 

Definition (Continued)

An atlas is called \textit{maximal} if it is not a proper subset of any other atlas. Every atlas \( \{(U_i, \phi_i) : i \in I\} \) is contained in a unique maximal atlas, the set of all charts \((U, \phi)\) compatible with \((U_i, \phi_i)\) for all \(i \in I\).

An \textit{(n-dimensional) complex manifold} is a paracompact, Hausdorff topological space \(X\) together with a maximal atlas \(\{(U_i, \phi_i) : i \in I\}\) of \(n\)-dimensional complex charts \((U_i, \phi_i)\). Here \textit{paracompact} is to avoid pathological examples from topology; sometimes one asks for \textit{second countable} instead.

Usually we refer to \(X\) as the complex manifold, and suppress the atlas. Taking the atlas \textit{maximal} makes it independent of choices.

What a complex atlas on \(X\) gives you is a notion of \textit{local holomorphic coordinates}. Let \(x \in X\). Then we can choose a chart \((U_i, \phi_i)\) with \(x \in \phi_i(U)\), since \(X = \bigcup_{i \in I} \phi_i(U_i)\). Then we think of \(\phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}^n\) as holomorphic coordinates \((z_1, \ldots, z_n)\) defined on an open neighbourhood \(\phi_i(U_i)\) of \(x\). We can do a lot of definitions and proofs using local holomorphic coordinates.

Example

The simplest complex manifold is \(\mathbb{C}^n\). \((U, \phi) = (\mathbb{C}^n, \text{id}_{\mathbb{C}^n})\) is a chart on \(\mathbb{C}^n\), and \(\{(\mathbb{C}^n, \text{id}_{\mathbb{C}^n})\}\) is an atlas on \(\mathbb{C}^n\). This is contained in a unique maximal atlas, which makes \(\mathbb{C}^n\) into a complex manifold.
1.2. Holomorphic functions and holomorphic maps

Let $X$ be a complex manifold, and $f : X \to \mathbb{C}$ a function. We call $f$ \textit{holomorphic} if for all charts $(U, \phi)$ in the (maximal) atlas on $X$, $f \circ \phi$ is a holomorphic function $U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^n$ is open. It is enough to check this on the charts of any atlas on $X$.

Let $X, Y$ be complex manifolds of dimensions $m, n$, and $f : X \to Y$ a continuous function. We call $f$ \textit{holomorphic} if whenever $(U, \phi)$ and $(V, \psi)$ are charts from the atlases on $X, Y$, the map

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}(f(\phi(U)) \cap \psi(V)) \to V$$

is a holomorphic map from an open subset of $\mathbb{C}^m$ to an open subset of $\mathbb{C}^n$. Complex manifolds and holomorphic maps form a \textit{category}. A \textit{biholomorphism} $f : X \to Y$ is a holomorphic map with a holomorphic inverse.
1.3. Complex submanifolds

Let $X$ be a complex manifold of dimension $n$, and $Y \subseteq X$. We call $Y$ an \textit{(embedded) complex submanifold of} $X$ of dimension $k$, for $0 \leq k \leq n$, if for each $y \in Y$ there exist local holomorphic coordinates $(z_1, \ldots, z_n)$ on $X$ such that $Y$ is locally of the form $z_{k+1} = \cdots = z_n = 0$. That is, we have a chart $(U, \phi)$ on $X$ with $y \in \phi(U)$ such that $Y \cap \phi(U) = \phi(\mathbb{C}^k \cap U)$, where $\mathbb{C}^k = \{(z_1, \ldots, z_k, 0, \ldots, 0) \in \mathbb{C}^n\}$. Usually we want $Y$ closed in $X$. We can give a complex submanifold $Y$ of $X$ the structure of a complex $k$-manifold: for $(U, \phi)$ as above, $(\mathbb{C}^k \cap U, \phi|_{\mathbb{C}^k \cap U})$ is a $k$-dimensional chart on $Y$, and the set of such charts is an atlas on $Y$. The inclusion $i_Y : Y \hookrightarrow X$ is holomorphic.

Conversely, a holomorphic map $f : Y \to X$ is called an \textit{embedding} if it is injective, locally closed, and on tangent spaces $df|_y : T_y Y \to T_{f(y)} X$ is injective for all $y \in Y$. If $f$ is an embedding then $f(Y)$ is a complex submanifold of $X$ biholomorphic to $Y$.

1.4. Projective complex manifolds

Let $\mathbb{CP}^n$ have homogeneous coordinates $[z_0, \ldots, z_n]$. Let $p(z_0, \ldots, z_n)$ be a complex polynomial in $n+1$ variables, which is homogeneous of order $k$. Then $p(\lambda z_0, \ldots, \lambda z_n) = \lambda^k p(z_0, \ldots, z_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Hence $p(\lambda z_0, \ldots, \lambda z_n) = 0$ if and only if $p(z_0, \ldots, z_n) = 0$. Thus, for $[z_0, \ldots, z_n] \in \mathbb{CP}^n$, the condition $p(z_0, \ldots, z_n) = 0$ is independent of the choice of representative $(z_0, \ldots, z_n)$ for $[z_0, \ldots, z_n]$.

A \textit{projective variety} is a subset $X$ of $\mathbb{CP}^n$ which is defined by the vanishing of finitely many homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_d(z_0, \ldots, z_n)$, that is,

$$X = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : p_i(z_0, \ldots, z_n) = 0, \ i = 1, \ldots, d\}.$$

Then $X$ is closed in $\mathbb{CP}^n$, and so compact. We call $X$ a \textit{projective complex manifold} if $X$ is also a complex submanifold of $\mathbb{CP}^n$. 
Example

Let \( p(z_0, \ldots, z_n) \) be a nonzero homogeneous complex polynomial, and define
\[
X = \{ [z_0, \ldots, z_n] \in \mathbb{CP}^n : p(z_0, \ldots, z_n) = 0 \}.
\]
Then \( X \) is a complex submanifold of \( \mathbb{CP}^n \), of dimension \( n - 1 \), provided the following condition holds: let \((z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \) with \( p(z_0, \ldots, z_n) = 0 \). Then \( \frac{\partial p}{\partial z_i}(z_0, \ldots, z_n) \neq 0 \) for some \( i = 0, \ldots, n \). This holds for generic homogeneous polynomials \( p \).

Example

For \( d = 1, 2, \ldots \), \( X = \{ [z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^d + z_1^d + z_2^d = 0 \} \) is a projective complex 1-manifold, a Riemann surface of genus \( g = \frac{1}{2}(d - 1)(d - 2) \).

Example

\( X = \{ [z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0^2 + \cdots + z_3^2 = 0 \} \) is a projective complex 2-manifold biholomorphic to \( \mathbb{CP}^1 \times \mathbb{CP}^1 \).

Example

Let \( p_1, \ldots, p_k(z_0, \ldots, z_n) \) be homogeneous polynomials for \( k \leq n \). Suppose that whenever \((z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \) with \( p_i(z_0, \ldots, z_n) = 0 \) for all \( i \), then \( \frac{\partial p_1}{\partial z_i}(z_0, \ldots, z_n), \ldots, \frac{\partial p_k}{\partial z_i}(z_0, \ldots, z_n) \) are linearly independent in \( (\mathbb{C}^{n+1})^* \). Then
\[
X = \{ [z_0, \ldots, z_n] \in \mathbb{CP}^n : p_i(z_0, \ldots, z_n) = 0, \ i = 1, \ldots, k \}
\]
is a projective complex manifold of dimension \( n - k \), called a complete intersection.
Most projective complex manifolds are not complete intersections.
Projective complex manifolds give a huge number of interesting examples of complex manifolds. As they are defined using polynomials, one can study and classify them using algebraic techniques – Complex Algebraic Geometry.

Also, under some conditions one can guarantee that a compact complex manifold $X$ has an embedding $X \hookrightarrow \mathbb{CP}^n$ making it into a projective complex manifold. This is due to two important results, Chow’s Theorem and the Kodaira Embedding Theorem.

**Theorem 1.1 (Chow’s Theorem)**

Suppose $X$ is a compact complex submanifold in $\mathbb{CP}^n$. Then $X$ is a projective complex manifold, that is, $X$ may be defined as a subset of $\mathbb{CP}^n$ by the vanishing of homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_k(z_0, \ldots, z_n)$.

Thus, compact submanifolds of $\mathbb{CP}^n$ are algebraic objects. For a proof, see Griffiths and Harris, *Principles of Algebraic Geometry*. As $\mathbb{CP}^n$ is compact, $X$ compact is equivalent to $X$ closed.

We will cover the Kodaira Embedding Theorem later in the course. In brief, it says that if $X$ is a compact complex manifold and $L \to X$ is an ‘ample line bundle’ then we can use $L$ to construct an embedding $f : X \hookrightarrow \mathbb{CP}^n$ for some $n \gg 0$. Then $X$ is biholomorphic to $f(X)$, which is a compact complex submanifold of $\mathbb{CP}^n$, so by Chow’s Theorem, $f(X)$ is algebraic, and $X$ is biholomorphic to a projective complex manifold.
Projective complex manifolds are also closely connected to compact Kähler manifolds (next week). Every projective complex manifold is Kähler. But also, if $X$ is a compact Kähler manifold, then under mild topological conditions on $X$ one can show that $X$ possesses many ample line bundles $L \hookrightarrow X$, and then the Kodaira Embedding Theorem applies, and $X$ is biholomorphic to a projective complex manifold.
Plan of talk:

2 Complex manifolds as real manifolds; almost complex structures

2.1 Almost complex structures

2.2 The Nijenhuis tensor

2.3 Another definition of complex manifolds

2.4 More on almost complex geometry

2.1. Almost complex structures

We now explain a second way to define complex manifolds. To see the point simply, suppose $V$ is a complex vector space, of complex dimension $n$. Underlying $V$ is a real vector space $V_R$, of real dimension $2n$. Given $V_R$, what extra information do we need to reconstruct $V$? The only thing we are missing is multiplication by $i \in \mathbb{C}$. This induces a real linear map $J : V_R \to V_R$ with $J^2 = -\text{id}_{V_R}$.

Conversely, given a real vector space $V_R$ and $J \in \text{End}(V_R)$ with $J^2 = -\text{id}_{V_R}$, we make $V_R$ into a complex vector space by setting $(a + ib) \cdot v = a \cdot v + b \cdot J(v)$, for $a, b \in \mathbb{R}$ and $v \in V_R$; note that $\dim_{\mathbb{R}} V_R$ must be even. So, complex vector spaces are equivalent to real vector spaces with an endomorphism $J$ with $J^2 = -\text{id}$. 
If $X$ is a complex $n$-manifold in the sense of §1, then underlying $X$ is a real $2n$-manifold $X\mathbb{R}$. It has a tangent bundle $TX\mathbb{R}$, whose fibres $T_xX\mathbb{R}$ for $x \in X$ are real vector spaces of real dimension $2n$. Since $X$ is a complex $n$-manifold, they are also complex vector spaces of dimension $n$. So they have $J_x \in \text{End}(T_xX\mathbb{R})$ with $J_x^2 = -\text{id}_{T_xX\mathbb{R}}$. Over all $x \in X\mathbb{R}$, these $J_x$ form a tensor $J^b_a$ with $J^b_a J^c_b = -\delta^c_a$, using index notation.

**Definition**

Let $X$ be a real $2n$-manifold. An almost complex structure $J$ on $X$ is a tensor $J^b_a$ in $C^\infty(T^*X \otimes TX)$ with $J^b_a J^c_b = -\delta^c_a$. For a vector field $v \in C^\infty(TX)$, define $(Jv)^b = J^b_a v^a$. Then $J^2 = -1$, so $J$ makes the tangent spaces $T_xX$ into complex vector spaces.

Any complex manifold in the sense of §1 yields a real manifold $X$ with an almost complex structure $J$. But not all $(X, J)$ come from complex manifolds: we must impose extra conditions on $J$.

**Holomorphic functions**

**Definition**

Suppose $X$ is a $2n$-manifold, and $J$ an almost complex structure on $X$. Let $f : X \to \mathbb{C}$ be smooth, and write $f = u + iv$. Then $du, dv$ are 1-forms on $X$, so in index notation $du = du_a, dv = dv_b$. We call $f$ holomorphic if $du_a = J^b_a dv_b$. Since $J^2 = -\text{id}$, this is equivalent to $dv_a = -J^b_a du_b$. Hence in complex 1-forms we have

$$J^b_a (du_b + i dv_b) = i(du_a + i dv_a),$$

that is, $J^b_a df_b = i df_a$. 
### Example

Let $\mathbb{R}^2$ have coordinates $(x, y)$, and let $J = dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$ in $C^\infty(T^*\mathbb{R}^2 \otimes T\mathbb{R}^2)$. Then the equation $du_a = J_a^b dv_b$ becomes

$$\frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy = -\frac{\partial v}{\partial x} \cdot dy + \frac{\partial v}{\partial y} \cdot dx,$$

or equivalently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the Cauchy–Riemann equations for $u(x, y) + iv(x, y)$ to be a holomorphic function of $x + iy$.

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### 2.2. The Nijenhuis tensor

It turns out that when $n > 1$, for some almost complex structures on $X$ there may be few holomorphic functions locally on $X$ — in extreme cases, all holomorphic functions are constant. This is because the equations are *overdetermined*: there are $2n$ equations on 2 functions. We can express this in terms of an *obstruction* to the existence of holomorphic functions locally on $X$, called the Nijenhuis tensor.

#### Definition

Write $[v, w]$ for the *Lie bracket* of vector fields $v, w$ on $X$. The *Nijenhuis tensor* $N = N^a_{bc}$ of $J$ satisfies

$$N^a_{bc} v^b w^c = ([v, w] + J([Jv, w] + [v, Jw]) - [Jv, Jw])^a \quad (2.1)$$

for all $v, w \in C^\infty(TX)$. 

The point is that the r.h.s. of (2.1) is pointwise linear in $v, w$ (exercise): if we replace $v, w$ by $f \cdot v, g \cdot w$ for smooth $f, g : X \to \mathbb{R}$, then the r.h.s. is multiplied by $fg$, with no terms in derivatives of $f, g$.

Let $s + it : X \to \mathbb{C}$ be holomorphic. Then using (2.1) one can show that for all vector fields $v, w$ we have

$$N_{bc}^a v^b w^c ds_a \equiv N_{bc}^a v^b w^c dt_a \equiv 0 \quad \text{(exercise).}$$

Hence

$$N_{bc}^a ds_a \equiv N_{bc}^a dt_a \equiv 0 \quad \text{in} \ C^\infty(\Lambda^2 T^*X).$$

Thus, the Nijenhuis tensor constrains the possible first derivatives of holomorphic functions.

For $(X, J)$ to be a complex manifold, we want there to exist a system of holomorphic coordinates $(z_1, \ldots, z_n)$ near each point $x$ in $X$, that is, $(z_1, \ldots, z_n)$ are complex coordinates defined on open $x \in U \subseteq X$, and $z_j : U \to \mathbb{C}$ is holomorphic. If $z_j = s_j + it_j$ then $ds_1, \ldots, ds_n, dt_1, \ldots, dt_n$ span $T^*X$ on $U$. So

$$N_{bc}^a (ds_j)_a \equiv N_{bc}^a (dt_j)_a \equiv 0$$

imply that $N \equiv 0$. Thus, holomorphic coordinates $(z_1, \ldots, z_n)$ can exist locally on $X$ only if the Nijenhuis tensor $N \equiv 0$. 
The converse is the difficult Newlander–Nirenberg Theorem:

**Theorem 2.1 (Newlander–Nirenberg)**

Suppose $J$ is an almost complex structure on $X$ with Nijenhuis tensor $N \equiv 0$. Then near each $x \in X$ there exist holomorphic coordinates $(z_1, \ldots, z_n)$.

The point is to show that the first derivatives of holomorphic functions near $x$ span $T^*_x X$; then choosing any $(z_1, \ldots, z_n)$ whose derivatives span $T^*_x X$, they will be holomorphic coordinates in a small open neighbourhood of $x$.

Think of the Nijenhuis tensor $N$ as being like the ‘curvature’ of $J$, and the condition $N \equiv 0$ as a ‘flatness condition’. If $g = g_{ab}$ is a Riemannian metric, the Riemann curvature $R^i_{jkl}$ is a tensor defined using $g$ and its derivatives, in a similar way to $N^a_{bc}$, and $R^i_{jkl} \equiv 0$ if $g$ is flat. (Actually, $N$ is a *torsion* rather than a curvature, as it depends on one derivative of $J$, not two.)

### 2.3. Another definition of complex manifolds

Here is our second definition of complex manifold:

**Definition**

Let $X$ be a $2n$-manifold, and $J$ an almost complex structure on $X$ with Nijenhuis tensor $N$. We call $J$ an *integrable almost complex structure*, or just a *complex structure*, if $N \equiv 0$, and then we call $(X, J)$ a *complex manifold*.

This is equivalent to the definition of complex manifolds using complex atlases in §1. Here is why.
Suppose \((X, J)\) is a complex manifold in the sense above. Then by the Newlander–Nirenberg theorem, there exist holomorphic coordinates \((z_1, \ldots, z_n)\) near each \(x \in X\). Using these we define an atlas of charts \((U, \phi)\) on \(X\). The transition functions are automatically holomorphic. Extending to the unique maximal atlas defines a complex structure on \(X\) in the sense of §1.

Conversely, given a complex manifold \(X_C\) in the sense of §1, there is a natural underlying real manifold \(X_R\), and a unique almost complex structure \(J\) on \(X_R\) for which all local coordinate functions \(z_j\) are holomorphic, and \(N \equiv 0\), so \(J\) is a complex structure.

### Definition

Let \((X, I)\) and \((Y, J)\) be complex manifolds, and \(f : X \to Y\) a smooth map. We call \(f\) **holomorphic** if for all \(x \in X\) with \(y = f(x) \in Y\), so that \(df|_x : T_x X \to T_y Y\) is a linear map, we have \(df|_x \circ I|_x = J|_y \circ df|_x\). That is, \(df|_x : T_x X \to T_y Y\) is a complex linear map, regarding \(T_x X, T_y Y\) as complex vector spaces using \(I|_x, J|_y\).

This agrees with the definition of holomorphic maps in §1, under the correspondence between the two definitions of complex manifold. If \(g : Y \to \mathbb{C}\) is a holomorphic function then \(g \circ f : X \to \mathbb{C}\) is a holomorphic function. In fact, a smooth map \(f : X \to Y\) is holomorphic if and only if for all local holomorphic functions \(g : V \to \mathbb{C}\) for \(V \subseteq Y\) open, \(g \circ f : U = f^{-1}(V) \to \mathbb{C}\) is a local holomorphic function on \(X\).
Complex submanifolds

Definition

Let \((X, J)\) be a complex manifold, and \(Y\) a submanifold of \(X\). We call \(Y\) a complex submanifold if for each \(y \in Y\) we have 
\[J(T_yY) = T_yY,\]
as subspaces of \(T_yX\).

Then \(J_Y = J|_{TY}\) is an almost complex structure on \(Y\). The
Nijenhuis tensor \(N_Y\) of \(J_Y\) is the restriction to \(Y\) of the Nijenhuis

Real dimension two

Let \(J\) be an almost complex structure on \(X\), with Nijenhuis tensor 
\[N = N^a_{bc}.
\] Then \(N\) has natural symmetries 
\[N^a_{cb} = -N^a_{bc},\]
and 
\[J^d_b J^e_c N^a_{de} = -N^a_{bc}\]
(exercise). Using these one can show that \(N \equiv 0\) when \(\dim X = 2\). So almost complex 2-manifolds are complex, that
is, they are Riemann surfaces. This corresponds to the fact that for 
\(f : X \to \mathbb{C}\) to be holomorphic is \(2n\) equations on 2 functions,
which is overdetermined when \(n > 1\), but determined when \(n = 1\).
2.4. More on almost complex geometry

Consider the question: how much of complex geometry also works for non-integrable almost complex structures $J$ on $X$ with $\dim X > 2$?

We already know there are few holomorphic functions $f : X \to \mathbb{C}$ even locally. There are also few complex submanifolds $Y \subset X$ with $2 < \dim Y < \dim X$. However, 2-dimensional complex submanifolds $Y$ in $X$ ($J$-holomorphic curves) are well-behaved. This is important in Symplectic Geometry.

**Definition**

Let $X$ be a $2n$-manifold. A symplectic form $\omega$ on $X$ is a 2-form $\omega$ with $d\omega \equiv 0$, such that $\omega|_x^n$ is nonzero in $\Lambda^2 T^*_x X$ for all $x \in X$. Then $(X, \omega)$ is a symplectic manifold.

**Darboux’ Theorem** says that near each point $x$ in a symplectic manifold $(X, \omega)$ we can choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $X$ with $\omega = \sum_{j=1}^n dx_j \wedge dy_j$. So all symplectic manifolds are locally the same as the standard model $(\mathbb{R}^{2n}, \omega_0)$.

Similarly, the Newlander–Nirenberg Theorem shows that if $J$ is an almost complex structure on $X$ with Nijenhuis tensor $N \equiv 0$, then near each $x \in X$ we can choose coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $X$ with $J = \sum_{j=1}^n dx_j \otimes \frac{\partial}{\partial y_j} - dy_j \otimes \frac{\partial}{\partial x_j}$.

Thus, all complex manifolds are locally the same as the standard model $(\mathbb{R}^{2n}, J_0)$.
Let \((X, \omega)\) be symplectic. An almost complex structure \(J\) on \(X\) is \textit{compatible with} \(\omega\) if \(\omega(Jv, Jw) = \omega(v, w)\) for all vector fields \(v, w\) on \(X\), and \(\omega(v, Jv) > 0\) if \(v \neq 0\). Every symplectic manifold admits compatible almost complex structures. Many important areas of Symplectic Geometry — Gromov-Witten invariants, Lagrangian Floer cohomology, Fukaya categories, . . . — depend on choosing a compatible \(J\) on \((X, \omega)\) and then ‘counting’ \(J\)-holomorphic curves in \(X\). Often one can make the ‘number’ independent of the choice of \(J\).