Plan of talk:

5. **Hodge theory for Kähler manifolds**
   - 5.1 Hodge theory for compact Riemannian manifolds
   - 5.2 Hodge theory for compact Kähler manifolds
   - 5.3 The Kähler cone
   - 5.4 Lefschetz operators, the Hard Lefschetz Theorem
5.1. Hodge theory for compact Riemannian manifolds

We first recall Hodge theory for ordinary Riemannian manifolds. Let \((X, g)\) be a compact, oriented Riemannian \(n\)-manifold. Then the \textit{Hodge star} \(*\) acts on \(k\)-forms

\[
* : C^\infty(\Lambda^k T^*X) \rightarrow C^\infty(\Lambda^{n-k} T^*X).
\]

It satisfies \(*^2 = (-1)^k(n-k)\), so \(*^{-1} = \pm *\). We define

\[
d^* : C^\infty(\Lambda^k T^*X) \rightarrow C^\infty(\Lambda^{k-1} T^*X)
\]

by \(d^* = (-1)^k *^{-1} d\), and the \textit{Laplacian} on \(k\)-forms

\[
\Delta = dd^* + d^*d.
\]

Forms \(\alpha\) with \(\Delta_d \alpha = 0\) are called \textit{harmonic}. Later we will see this is equivalent to \(d \alpha = d^* \alpha = 0\) (for \(X\) compact). It is helpful to think about all this in terms of the \(L^2\)-\textit{inner product on forms}. If \(\alpha, \beta \in C^\infty(\Lambda^k T^*X)\) we define

\[
\langle \alpha, \beta \rangle_{L^2} = \int_X (\alpha, \beta) dV_g,
\]

where \((\alpha, \beta)\) is the pointwise inner product of \(k\)-forms using \(g\), and \(dV_g\) the volume form of \(g\). The Hodge star is defined so that if \(\alpha, \beta\) are \(k\)-forms then \(\alpha \wedge (* \beta) = (\alpha, \beta) dV_g\). Thus

\[
\langle \alpha, \beta \rangle_{L^2} = \int_X \alpha \wedge * \beta.
\]
Now let $\alpha$ be a $(k - 1)$-form and $\beta$ a $k$-form. Then we have
\[
\langle \alpha, d^* \beta \rangle_{L^2} = \langle \alpha, (-1)^k *^{-1} d * \beta \rangle_{L^2} \\
= (-1)^k \int_X (\alpha, *^{-1} d * \beta) dV_g \\
= (-1)^k \int_X \alpha \wedge *(\omega^{-1} d * \beta) \\
= (-1)^k \int_X \alpha \wedge d(\omega). 
\]

But by Stokes’ Theorem,
\[
0 = \int_X d[\alpha \wedge (\omega^*)] = \int_X (d\alpha) \wedge (\omega^*) + (-1)^{k-1} \int_X \alpha \wedge d(\omega^*). 
\]

Hence
\[
\langle \alpha, d^* \beta \rangle_{L^2} = \int_X (d\alpha) \wedge (\omega^*) = \int_X (d\alpha, \beta) dV_g = \langle d\alpha, \beta \rangle_{L^2}. 
\]

Thus $\langle \alpha, d^* \beta \rangle_{L^2} = \langle d\alpha, \beta \rangle_{L^2}$ for all $\alpha, \beta$, so $d^*$ behaves like the adjoint of $d$ under the $L^2$-inner product; we call $d^*$ the formal adjoint of $d$. One consequence is that $d^* \beta = 0$ if and only if $\langle d\alpha, \beta \rangle_{L^2} = 0$ for all $\alpha$. That is, $\text{Ker } d^* = (\text{Im } d)^\perp$, the kernel of $d^*$ in $C^\infty(\Lambda^k T^* X)$ is the subspace of $C^\infty(\Lambda^k T^* X)$ which is $L^2$-orthogonal to the image of $d : C^\infty(\Lambda^{k-1} T^* X) \to C^\infty(\Lambda^k T^* X)$.

We expect an orthogonal splitting
\[
C^\infty(\Lambda^k T^* X) = \text{Im } d \oplus \text{Ker } d^*. 
\]
(This is not a proof, though.)
Some more notation: write \( d_k, d^*_k \) for \( d, d^* \) acting on \( k \)-forms, and \( \mathcal{H}^k \) for \( \text{Ker} \Delta_d \) on \( k \)-forms. Then:

**Theorem 5.1 (Hodge Decomposition Theorem)**

Let \((X, g)\) be a compact, oriented Riemannian manifold. Then

\[
C^\infty(\mathcal{A}^k T^* M) = \mathcal{H}^k \oplus \text{Im}(d_{k-1}) \oplus \text{Im}(d^*_{k+1}).
\]

Moreover \( \text{Ker} d_k = \mathcal{H}^k \oplus \text{Im}(d_{k-1}) \) and \( \text{Ker} d^*_k = \mathcal{H}^k \oplus \text{Im}(d^*_{k+1}) \).

Hodge’s Theorem

So de Rham cohomology satisfies

\[
\mathcal{H}^k_{dR}(X; \mathbb{R}) = \text{Ker} d_k / \text{Im} d_{k-1} = (\mathcal{H}^k \oplus \text{Im}(d_{k-1})) / \text{Im} d_{k-1} \cong \mathcal{H}^k.
\]

This gives **Hodge’s Theorem:**

**Theorem 5.2 (Hodge’s Theorem)**

Every de Rham cohomology class on \( X \) contains a unique harmonic representative.

So \( \mathcal{H}^k \) is finite-dimensional (this also follows as it is the kernel of an elliptic operator on a compact manifold). The Hodge star gives an isomorphism \( * : \mathcal{H}^k \to \mathcal{H}^{n-k} \). Thus \( \mathcal{H}^k_{dR}(X; \mathbb{R}) \cong \mathcal{H}^{n-k}_{dR}(X; \mathbb{R}) \), a form of Poincaré duality.
We defined $\mathcal{H}^k$ as the kernel of $\Delta_d = dd^* + d^*d$. But if $\alpha \in \mathcal{H}^k$ then

$$0 = \langle \alpha, (dd^* + d^*d)\alpha \rangle_{L^2}$$

$$= \langle d^*\alpha, d^*\alpha \rangle_{L^2} + \langle d\alpha, d\alpha \rangle_{L^2}$$

$$= \|d^*\alpha\|_{L^2}^2 + \|d\alpha\|_{L^2}^2,$$

so $\|d^*\alpha\|_{L^2} = \|d\alpha\|_{L^2} = 0$, and $d^*\alpha = d\alpha = 0$. Thus

$$\mathcal{H}^k = \{ \alpha \in C^\infty(\Lambda^k T^*X) : d\alpha = d^*\alpha = 0 \}.$$

### 5.2. Hodge theory for compact Kähler manifolds

Now let $(X, J, g)$ be a compact Kähler manifold, with Kähler form $\omega$, of real dimension $2n$. Work now with complex forms, so that $d_k, d_k^*$ act on $C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C})$, and $\mathcal{H}^k$ is the kernel of $\Delta_d$ on complex forms. By the Kähler identities (§4.4) we have $\Delta_d = \Delta_\bar{d} = \frac{1}{2} \Delta_d$. But $\Delta_d, \Delta_\bar{d}$ both take $(p, q)$-forms to $(p, q)$-forms, so $\Delta_d$ also takes $(p, q)$-forms to $(p, q)$-forms.

Suppose $\alpha$ is a $k$-form with $\Delta_d\alpha = 0$, and write $\alpha = \sum_{p+q=k} \alpha_{p,q}$ with $\alpha_{p,q}$ of type $(p, q)$. Then the component of $\Delta_d\alpha = 0$ in type $(p, q)$ is $\Delta_d \alpha_{p,q} = 0$, as $\Delta_d$ takes $(p, q)$-forms to $(p, q)$-forms. So each $\alpha_{p,q}$ lies in $\mathcal{H}^k$. Define $\mathcal{H}^{p,q}$ to be the kernel of $\Delta_d$ on $(p, q)$-forms. We have shown that

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}.$$
Here is a version of the Hodge decomposition theorem for the $\bar{\partial}$ operator on $(p, q)$-forms. Write $\bar{\partial}_{p,q}, \bar{\partial}_{p,q}^*$ for $\bar{\partial}, \bar{\partial}^*$ on $(p, q)$-forms.

**Theorem 5.3**

Let $(X, J, g)$ be a compact Kähler manifold. Then

$$C^\infty(\Lambda^{p,q}M) = \mathcal{H}^{p,q} \oplus \text{Im}(\bar{\partial}_{p,q-1}) \oplus \text{Im}(\bar{\partial}_{p,q+1}).$$

Also $\text{Ker} \bar{\partial}_{p,q} = \mathcal{H}^{p,q} \oplus \text{Im}(\bar{\partial}_{p,q-1})$ and $\text{Ker} \bar{\partial}^*_{p,q} = \mathcal{H}^{p,q} \oplus \text{Im}(\bar{\partial}^*_{p,q+1}).$

So Dolbeault cohomology satisfies

$$H^p_{\bar{\partial}}(X) = \text{Ker} \bar{\partial}_{p,q} / \text{Im} \bar{\partial}_{p,q-1} = (\mathcal{H}^{p,q} \oplus \text{Im}(\bar{\partial}_{p,q-1})) / \text{Im} \bar{\partial}_{p,q-1} \cong \mathcal{H}^{p,q}.$$  

Write $H^{p,q}(X)$ for the subspace of $H^{p+q}(X; \mathbb{C})$ represented by forms in $\mathcal{H}^{p,q}$. Then we have

$$H^k(X; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad (5.1)$$

and $H^{p,q}(X) \cong H^p_{\bar{\partial}}(X)$. Hence

$$H^k(X; \mathbb{C}) \cong \bigoplus_{p+q=k} H^p_{\bar{\partial}}(X). \quad (5.2)$$

We can describe $H^{p,q}(X)$ as the subspace of $H^{p+q}(X; \mathbb{C})$ represented by closed $(p, q)$-forms. This is independent of the Kähler metric on $X$. But (5.1) and (5.2) fail for general compact complex manifolds.
Observe that complex conjugation takes $\mathcal{H}^{p,q}$ to $\mathcal{H}^{q,p}$ and $H^{p,q}(X)$ to $H^{q,p}(X)$. Since $\mathcal{H}^{p,q} \cong H^{p,q}_\partial(X)$, this implies that

$$H^{p,q}_\partial(X) \cong H^{q,p}_\partial(X).$$

This need not be true for general compact complex manifolds; $H^{p,q}_\partial(X)$ and $H^{q,p}_\partial(X)$ need not have the same dimension. Also $*$ gives

$$*: \mathcal{H}^{p,q} \xrightarrow{\cong} H^{n-p,n-q}.$$  

This gives Poincaré duality style isomorphisms

$$H^{p,q}(X) \cong H^{n-p,n-q}(X)^*, \quad H^{p,q}_\partial(X) \cong H^{n-p,n-q}_\partial(X)^*.$$

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This gives Poincaré duality style isomorphisms

$$H^{p,q}(X) \cong H^{n-p,n-q}(X)^*, \quad H^{p,q}_\partial(X) \cong H^{n-p,n-q}_\partial(X)^*.$$
The Betti numbers of $X$ are $b^k(X) = \dim_{\mathbb{C}} H^k_{dR}(X; \mathbb{C})$, and the Hodge numbers of $X$ are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X)$. From above we have

$$b^k(X) = \sum_{p+q=k} h^{p,q}(X),$$

$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X).$$

So in particular

$$b^{2k+1}(X) = 2 \sum_{j=0}^k h^{j,2k+1-j}(X).$$

**Corollary 5.4**

Let $(X, J, g)$ be a compact Kähler manifold. Then the odd Betti numbers $b^{2k+1}(X)$ for $k = 0, 1, \ldots$ are even.

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**A complex manifold with no Kähler metrics**

Let $n > 1$, and let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n \setminus \{0\}$ by $d: (z_1, \ldots, z_n) \mapsto (\lambda^d z_1, \ldots, \lambda^d z_n)$. Define $X = (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}$. Then $X$ is a compact complex manifold diffeomorphic to $S^1 \times S^{2n-1}$. By the Künneth theorem we find that the Betti numbers of $X$ are $b^k(X) = 1$ for $k = 0, 1, 2n-1, 2n$ and $b^k(X) = 0$ otherwise.

Thus $b^1(X)$ and $b^{2n-1}(X)$ are odd. If $X$ had a Kähler metric this would contradict Corollary 5.4. Hence $X$ has no Kähler metrics.

For Dolbeault cohomology, it turns out that $H^{1,0}_{\bar{\partial}}(X) = 0$, but $H^{0,1}_{\bar{\partial}}(X) \cong \mathbb{C}$, where $\bar{\partial} \log(|z_1|^2 + \cdots + |z_n|^2)$ represents a nontrivial class. So

$$H^{p,q}_{\bar{\partial}}(X) \neq H^{q,p}_{\bar{\partial}}(X),$$

in this example.
5.3. The Kähler cone

Let \((X, J)\) be a compact complex manifold, admitting Kähler metrics. Then we have
\[
H^2_{\text{dR}}(X; \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).
\]
If \(g\) is a Kähler metric on \((X, J)\) with Kähler form \(\omega\) then \(\omega\) is a real closed \((1,1)\)-form, so that
\[
[\omega] \in H^2_{\text{dR}}(X; \mathbb{R}) \cap H^{1,1}(X),
\]
with intersection in \(H^2_{\text{dR}}(X; \mathbb{C})\).

**Definition**

Define the Kähler cone \(\mathcal{K}\) of \((X, J)\) to be the set of all Kähler classes \([\omega]\) of Kähler metrics \(g\) on \((X, J)\).

Two important facts about \(\mathcal{K}\):

(a) \(\mathcal{K}\) is open in \(H^2_{\text{dR}}(X; \mathbb{R}) \cap H^{1,1}(X)\).

(b) \(\mathcal{K}\) is a convex cone.

For (a), note that if \(\omega\) is the Kähler form of \(g\) and \(\eta\) is a closed real \((1,1)\)-form with \(\|\eta\|_{C^0} < 1\), where \(\|\cdot\|_{C^0}\) is computed using \(g\), then \(\omega' = \omega + \eta\) is the Kähler form of \(g'\). Hence if \([\omega]\in\mathcal{K}\) and \([\eta]\in H^2_{\text{dR}}(X; \mathbb{R}) \cap H^{1,1}(X)\) is sufficiently small then \([\omega] + [\eta]\in\mathcal{K}\).

For (b), if \(g, g'\) are Kähler metrics on \((X, J)\) and \(s, s' > 0\) then \(sg + s'g'\) is also Kähler. Thus \([\omega], [\omega']\in\mathcal{K}\) implies that \(s[\omega] + s'[\omega']\in\mathcal{K}\).
Suppose $\Sigma \subset X$ is a compact complex curve (1-dimensional complex submanifold) in $X$. Then for any Kähler $g, \omega$ we have

$$[\omega] \cdot [\Sigma] = \int_{\Sigma} \omega = \text{vol}_g(\Sigma) > 0,$$

where $[\Sigma] \in H_2(X; \mathbb{Z})$ is the homology class. Hence

$$K \subseteq \{ \alpha \in H^2_{dR}(X; \mathbb{R}) \cap H^{1,1}(X) : \alpha \cdot [\Sigma] > 0, \Sigma \subset X \text{ curve} \}.$$

One can often describe $K$; in simple examples it is a polyhedral cone.

5.4. Lefschetz operators, the Hard Lefschetz Theorem

Let $(X, J, g)$ be compact Kähler, with Kähler form $\omega$. As in §4.4 we have operators on forms

$$L : C^\infty (\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty (\Lambda^{k+2} T^*X \otimes_{\mathbb{R}} \mathbb{C}),$$

$$\Lambda : C^\infty (\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty (\Lambda^{k-2} T^*X \otimes_{\mathbb{R}} \mathbb{C}),$$

given by $L(\alpha) = \alpha \wedge \omega$ and $\Lambda = (-1)^k * L *$. These also work on cohomology. Since $[\Delta_d, L] = [\Delta_d, \Lambda] = 0$ by the Kähler identities, $L, \Lambda$ take $\text{Ker} \Delta_d$ to $\text{Ker} \Delta_d$. So $L$ maps $\mathcal{H}^k \to \mathcal{H}^{k+2}$, $\Lambda$ maps $\mathcal{H}^k \to \mathcal{H}^{k-2}$. 
Define the **Lefschetz operator**

\[ L : H^k_{dR}(X; \mathbb{C}) \to H^{k+2}_{dR}(X; \mathbb{C}) \]

and the **dual Lefschetz operator**

\[ \Lambda : H^k_{dR}(X; \mathbb{C}) \to H^{k-2}_{dR}(X; \mathbb{C}) \]

to correspond to \( L : \mathcal{H}^k \to \mathcal{H}^{k+2} \) and \( \Lambda : \mathcal{H}^k \to \mathcal{H}^{k-2} \) under the isomorphisms \( \mathcal{H}^k \cong H^k_{dR}(X; \mathbb{C}) \). Then \( L(\alpha) = \alpha \wedge [\omega] \), so \( L \) depends only on the Kähler class \([\omega]\) of \( g \). We can reconstruct \( \Lambda \) from \( L \), so \( \Lambda \) also depends only on \([\omega]\). Then \( L, \Lambda \) map

\[
L : H^{p,q}(X) \to H^{p+1,q+1}(X) \quad \text{and} \\
\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X).
\]

As for the decomposition of forms on Kähler manifolds in §4.4, we have:

**Theorem 5.5 (The Hard Lefschetz Theorem)**

Let \((X, J, g)\) be a compact Kähler manifold with \(\dim_{\mathbb{C}} X = n\). Then \( L^k : H^{n-k}_{dR}(X; \mathbb{C}) \to H^{n+k}_{dR}(X; \mathbb{C}) \) is an isomorphism for \( k = 0, \ldots, n \).

Define the primitive cohomology \( H^k_0(X; \mathbb{C}) \) for \( k \leq n \) by

\[
H^k_0(X; \mathbb{C}) = \ker L^{n-k+1} : (H^k_{dR}(X; \mathbb{C}) \to H^{2n-k+2}_{dR}(X; \mathbb{C})) \\
= \ker (\Lambda : H^k_{dR}(X; \mathbb{C}) \to H^{k-2}_{dR}(X; \mathbb{C})).
\]

Then for \( k = 0, \ldots, 2n \) we have

\[
H^k_{dR}(X; \mathbb{C}) = \bigoplus_{j: 0 \leq 2j \leq k, k \leq n+j} L^j H^{k-2j}_0(X; \mathbb{C}).
\]
The proof is not hard. For the first part, we have
\( \Delta_d(\omega \wedge \alpha) = \omega \wedge (\Delta_d \alpha) \), so \( \Delta_d(\omega^k \wedge \alpha) = \omega^k \wedge (\Delta_d \alpha) \). Thus
\( \omega^k \wedge - \) maps \( \text{Ker} \, \Delta_d \) to \( \text{Ker} \, \Delta_d \), that is, \( \alpha \mapsto \omega^k \wedge \alpha \) maps \( \mathcal{H}^{n-k} \) to \( \mathcal{H}^{n+k} \). But \( \alpha \mapsto \omega^k \wedge \alpha \) is a (pointwise) isomorphism from \( (n-k) \)-forms to \( (n+k) \)-forms, so \( \alpha \mapsto \omega^k \wedge \alpha \) is an isomorphism \( \mathcal{H}^{n-k} \to \mathcal{H}^{n+k} \). Using isomorphisms \( \mathcal{H}^\ast \cong H^\ast_{dR}(X; \mathbb{C}) \) shows that
\( L^k : H^{n-k}_{dR}(X; \mathbb{C}) \to H^{n+k}_{dR}(X; \mathbb{C}) \) is an isomorphism.

Let \( (X, J, g) \) be a compact Kähler \( 2n \)-manifold, and \( Y \subset X \) a closed \( 2k \)-submanifold. It has a homology class \([Y] \in H_{2k}(X; \mathbb{Q})\). Poincaré duality gives an isomorphism
\( \text{Pd} : H^\ast_\ast(X; \mathbb{Q}) \to H^{2n-\ast}(X; \mathbb{Q}) \), so
\( \text{Pd}([Y]) \in H^{2n-2k}(X; \mathbb{Q}) \subset H^{2n-2k}(X; \mathbb{C}) \).

As \( Y \) is a complex submanifold, \( \text{Pd}([Y]) \in H^{n-k,n-k}(X) \). Thus
\( \text{Pd}([Y]) \in H^{2n-2k}(X; \mathbb{Q}) \cap H^{n-k,n-k}(X) \),
where the intersection is taken in \( H^{2n-2k}(X; \mathbb{C}) \).
We can also allow $Y$ to be a complex $k$-submanifold with singularities — a ‘$k$-cycle’.

**Conjecture (The Hodge Conjecture.)**

Let $(X, J, g)$ be a projective Kähler $2n$-manifold. Then for each $k = 0, \ldots, n$, $H^{2n-2k}(X; \mathbb{Q}) \cap H^{n-k,n-k}(X)$ is spanned over $\mathbb{Q}$ by $P^d([Y])$ for $k$-cycles $Y$ in $X$.

This is known for $k = 0, 1, n - 1, n$, and so for $n \leq 3$. There is a $1,000,000 prize for proving it.
Plan of talk:

6 Holomorphic vector bundles

6.1 Vector bundles

6.2 $\bar{\partial}$-operators and connections

6.3 Chern classes

6.4 Holomorphic line bundles

6.1. Vector bundles

Let $X$ be a real manifold. A (real) vector bundle $E \to X$ on $X$ of rank $k$ is a family of real $k$-dimensional vector spaces $E_x$ for $x \in X$, depending smoothly on $x$. Formally, a vector bundle is a manifold $E$ with a projection $\pi : E \to X$ which is a submersion, such that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is given the structure of a real $k$-dimensional vector space.

This must satisfy the condition (local triviality) that $X$ may be covered by open sets $U$ for which there is a diffeomorphism $\pi^{-1}(U) \cong \mathbb{R}^k \times U$ which identifies $\pi : \pi^{-1}(U) \to U$ with $\pi_U : \mathbb{R}^k \times U \to U$ and the vector space structure on $E_u$ with that on $\mathbb{R}^k \times \{u\}$ for $u \in U$. 
Some examples: trivial vector bundles $\mathbb{R}^k \times X \to X$, (co)tangent bundles $TX, T^*X$, exterior forms $\wedge^k T^*X$, and tensor bundles $\bigotimes^k TX \otimes \bigotimes^l T^*X$.

A \textit{complex vector bundle} on $X$ is the same, but with fibres $E_x$ complex vector spaces. Note that we will distinguish between complex vector bundles (on any manifold) and holomorphic vector bundles (on a complex manifold).

A \textit{(smooth) section} of $E \to X$ is a smooth map $e : X \to E$ with $\pi \circ e \equiv \text{id}_X$. The set $C^\infty(E)$ of smooth sections of $E$ has the structure of an (infinite-dimensional) vector space.

We can add other structures to vector bundles. For example, a \textit{metric $h$ on the fibres of $E$} is a family of Euclidean metrics $h_x$ on $E_x$ which vary smoothly with $x$. That is, $h$ is a smooth, positive definite section of $S^2E^*$. A \textit{connection} $\nabla$ on $E$ is a linear map

$$\nabla : C^\infty(E) \to C^\infty(E \otimes T^*X)$$

satisfying the Leibnitz rule

$$\nabla(fe) = f \cdot \nabla e + e \otimes df$$

for all $e \in C^\infty(E)$ and smooth $f : X \to \mathbb{R}$. A connection $\nabla$ has \textit{curvature} $F_\nabla \in C^\infty(\text{End}(E) \otimes \wedge^2 T^*X)$.

We can require $\nabla$ to preserve a metric $h$ on $E$ by

$$h(\nabla e_1, e_2) + h(e_1, \nabla e_2) = \text{d}h(e_1, e_2)$$

for all $e_1, e_2 \in C^\infty(E)$. 
Holomorphic vector bundles

We define holomorphic vector bundles by replacing real manifolds by complex manifolds and smooth maps by holomorphic maps in the definition of real vector bundles. So, if $(X, J)$ is a complex manifold, then a holomorphic vector bundle of rank $k$ is a family of complex $k$-dimensional vector spaces $E_x$ for $x \in X$ varying holomorphically with $x$.

Formally, a holomorphic vector bundle is a complex manifold $(E, K)$ with a projection $\pi : E \to X$ which is a holomorphic submersion, such that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is given the structure of a complex $k$-dimensional vector space, and $X$ may be covered by open sets $U$ for which there is a biholomorphism $\pi^{-1}(U) \cong \mathbb{C}^k \times U$ which identifies $\pi : \pi^{-1}(U) \to U$ with $\pi_U : \mathbb{C}^k \times U \to U$ and the vector space structure on $E_u$ with that on $\mathbb{C}^k \times \{u\}$ for each $u \in U$.

If $E \to X$ is a holomorphic vector bundle, then a map $e : X \to E$ with $\pi \circ e \equiv \text{id}_X$ is called a smooth section if $e$ is smooth, and a holomorphic section if $e$ is holomorphic. We write $C^\infty(E)$ for the complex vector space of smooth sections of $E$, and $H^0(E)$ for the complex vector space of holomorphic sections of $E$.

Algebraic operations on vector spaces have counterparts on holomorphic vector bundles: if $E, F$ are holomorphic vector bundles then the dual $E^*$, the exterior powers $\Lambda^k E$, the tensor product $E \otimes F$, etc., are all holomorphic vector bundles.
6.2. $\bar{\partial}$-operators and connections

In terms of real differential geometry, a holomorphic vector bundle $E$ over a complex manifold $(X, J)$ has the structure of a complex vector bundle over the underlying real manifold $X$. However, it also has more structure: we have a notion of \textit{holomorphic section} of holomorphic vector bundle, but there is no intrinsic notion of when a section of a complex vector bundle is holomorphic.

If $f : X \to \mathbb{C}$ is smooth, then $f$ is holomorphic iff $\bar{\partial} f = 0$ in $C^\infty(\Lambda^{0,1} X)$. In the same way, if $E$ is a holomorphic vector bundle, there is a natural $\bar{\partial}$-operator

$$\bar{\partial}_E : C^\infty(E) \to C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} X)$$

such that $e \in C^\infty(E)$ is holomorphic iff $\bar{\partial}_E e = 0$. It satisfies the Leibnitz rule

$$\bar{\partial}_E(fe) = f \cdot \bar{\partial}_E e + e \otimes_{\mathbb{C}} \bar{\partial} f$$

for all $e \in C^\infty(E)$ and smooth $f : X \to \mathbb{C}$.

Given $\bar{\partial}_E$ satisfying the Leibnitz rule, there are unique extensions

$$\bar{\partial}^{p,q}_E : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q} X) \to C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1} X)$$

with $\bar{\partial}_E = \bar{\partial}^{0,0}_E$, such that

$$\bar{\partial}^{p,q}_E(e \otimes \alpha) = \bar{\partial}_E e \wedge \alpha + e \otimes_{\mathbb{C}} \bar{\partial} \alpha$$

for $e \in C^\infty(E)$ and $\alpha \in C^\infty(\Lambda^{p,q} X)$.

On a complex manifold we have $\bar{\partial}^2 = 0$. Similarly, if $\bar{\partial}_E$ comes from a holomorphic vector bundle then $\bar{\partial}^{p,q+1}_E \circ \bar{\partial}^{p,q}_E = 0$ for all $p, q$. 
Thus we can give a differential-geometric definition of holomorphic vector bundle: a holomorphic vector bundle on \((X, J)\) is a complex vector bundle \(E \to X\) together with a \(\bar{\partial}\)-operator
\[
\bar{\partial}_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_C \Lambda^{0,1}X)
\]
satisfying the Leibnitz rule, such that the extensions \(\bar{\partial}_E^{p,q}\) satisfy
\[
\bar{\partial}_E^{p,q+1} \circ \bar{\partial}_E^{p,q} = 0.
\]
In fact it is enough that \(\bar{\partial}_E^{0,1} \circ \bar{\partial}_E = 0\). We define \(e \in C^\infty(E)\) to be a holomorphic section if \(\bar{\partial}_E e = 0\).

It turns out that this is equivalent to the first definition of holomorphic vector bundle. That is, using \(\bar{\partial}_E\) we can define a unique almost complex structure \(K\) on \(E\) such that \(\pi : E \to X\) is holomorphic, and \(K|_{E_x}\) comes from the complex vector space structure of \(E_x\), and the graphs of holomorphic sections are complex submanifolds of \((E, K)\). The condition that the Nijenhuis tensor of \(K\) vanishes, so that \((E, K)\) is a complex manifold, is equivalent to \(\bar{\partial}_E^{0,1} \circ \bar{\partial}_E = 0\).

\(\bar{\partial}\)-operators are closely related to connections. Let \((X, J)\) be a complex manifold, \(E \to X\) a complex vector bundle, and \(\nabla\) a connection on \(E\). Then \(\nabla\) is a map
\[
\nabla : C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{R}} T^*X)
\]
\[
\cong C^\infty(E \otimes_{\mathbb{C}} (T^*X \otimes_{\mathbb{R}} \mathbb{C}))
\]
\[
= C^\infty(E \otimes_{\mathbb{C}} (\Lambda^{1,0}X \oplus \Lambda^{0,1}X))
\]
\[
= C^\infty(E \otimes_{\mathbb{C}} \Lambda^{1,0}X) \oplus C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}X).
\]

So we may write \(\nabla = \partial_E \oplus \bar{\partial}_E\), where
\[
\partial_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_C \Lambda^{1,0}X),
\]
\[
\bar{\partial}_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_C \Lambda^{0,1}X).
\]
As $\nabla$ satisfies a Leibniz rule, both $\partial_E, \bar{\partial}_E$ satisfy Leibniz rules, and $\bar{\partial}_E$ is a $\bar{\partial}$-operator. Thus, a $\bar{\partial}$-operator is half of a connection. The condition $\bar{\partial}^{0,1}_E \circ \bar{\partial}_E = 0$ for a $\bar{\partial}$-operator to give a holomorphic vector bundle is a curvature condition. For any $\bar{\partial}_E$, the operator

$$\bar{\partial}^{0,1}_E \circ \bar{\partial}_E : C^\infty(E) \longrightarrow C^\infty(E \otimes \Lambda^{0,2}X)$$

is of the form $e \mapsto F^{0,2}_E \cdot e$ for unique $F^{0,2}_E \in C^\infty(\text{End}(E) \otimes \Lambda^{0,2}X)$ which we call the $(0, 2)$-curvature. If $\bar{\partial}_E$ is half of a connection $\nabla$, then $F^{0,2}_E$ is the $(0, 2)$-component of the curvature $F_\nabla$.

Let $E$ be a complex vector bundle over $(X, J)$, and $h$ a Hermitian metric on the fibres of $E$. Then there is a 1-1 correspondence between $\bar{\partial}$-operators $\bar{\partial}_E$ on $E$, and connections $\nabla = \partial_E \oplus \bar{\partial}_E$ on $E$ preserving $h$. That is, for each $\bar{\partial}$-operator $\bar{\partial}_E$, there is a unique $\partial_E$ so that $\nabla = \partial_E \oplus \bar{\partial}_E$ preserves $h$.

Let $E$ be a holomorphic vector bundle on $(X, J)$, with $\bar{\partial}$-operator $\bar{\partial}_E$. Choose a Hermitian metric $h$ on $E$. Then $\bar{\partial}_E$ extends uniquely to $\nabla = \partial_E \oplus \bar{\partial}_E$ on $E$ preserving $h$. Consider the curvature of $\nabla$,

$$F_\nabla \in C^\infty(\text{End}(E) \otimes \Lambda^2 T^*X).$$

The $(0, 2)$-component of $F_\nabla$ is $F^{0,2}_E = 0$ as $E$ is holomorphic. As $\nabla$ preserves $h$,

$$F_\nabla \in C^\infty(\text{Herm}^-(E) \otimes \Lambda^2 T^*X),$$

where $\text{Herm}^-(E) \subset \text{End}(E)$ are the anti-Hermitian transformations w.r.t. $h$. 

This implies that the \((2,0)\)-component of \(F_\nabla\) is is conjugate to the \((0,2)\)-component, so is also zero. Hence \(F_\nabla\) is of type \((1,1)\).

Thus, every holomorphic vector bundle \(E\) on \(X\) admits a Hermitian metric \(h\) and compatible connection \(\nabla\) with \(F_\nabla\) of type \((1,1)\).

Conversely, if \(E\) is a complex vector bundle on \(X\) with Hermitian metric \(h\) and compatible connection \(\nabla\) with \(F_\nabla\) of type \((1,1)\), then the \(\bar{\partial}\)-operator of \(\nabla\) makes \(E\) into a holomorphic vector bundle.

### 6.3. Chern classes

There is a lot of interesting algebraic topology associated to complex vector bundles – K-theory, Chern classes. (See e.g. Milnor and Stasheff, ‘Characteristic classes’.)

If \(X\) is a topological space and \(E \to X\) is a complex vector bundle of rank \(k\), then the Chern classes \(c_j(E) \in H^{2j}(X; \mathbb{Z})\) for \(j = 1, \ldots, k\) are topological invariants of \(E\).

Let \(X\) be a manifold. Choose a Hermitian metric \(h\) on \(E\) and a connection \(\nabla\) on \(E\) preserving \(h\). Then

\[
F_\nabla \in C^\infty(\text{Herm}^-(E) \otimes_{\mathbb{R}} \Lambda^2 T^*X) .
\]

There are ‘polynomials’ \(p_1, \ldots, p_k\) in \(F_\nabla\) such that \(p_j(F_\nabla)\) is a closed \(2j\)-form and

\[
[p_j(F_\nabla)] = c_j(E) \in H_{dR}^{2j}(X; \mathbb{R}) .
\]

To define \(p_j(F_\nabla)\), take

\[
F_\nabla \wedge \cdots \wedge F_\nabla \in C^\infty(\text{Herm}^-(E)^{\otimes j} \otimes \Lambda^{2j} T^*X) ,
\]

and then apply a natural linear map \(\text{Herm}^-(E)^{\otimes j} \to \mathbb{R}\), which can be thought of as a \(U(k)\)-invariant degree \(j\) homogeneous polynomial on the Lie algebra \(u(k)\).
Observe that the cohomology class \([p_j(F_\nabla)]\) is \(c_j(E)\), and so is independent of the choice of metric \(h\) and connection \(\nabla\).

Now suppose \(E\) is a holomorphic vector bundle on a complex manifold \((X, J)\). Then as in §6.2 we can choose \(h\) and \(\nabla\) on \(E\) with \(F_\nabla\) of type \((1, 1)\). Therefore \(p_j(F_\nabla)\) is a closed form of type \((j, j)\). If \((X, J, g)\) is compact Kähler, this gives \([p_j(F_\nabla)] \in H^{i+j}(X)\).

Hence

\[c_j(E) \in H^{2j}(X; \mathbb{Z}) \cap H^{i+j}(X),\]

with intersection in \(H^{2j}_{dR}(X; \mathbb{C})\).

Note the similarity to the Hodge Conjecture in §5.4. This gives obstructions to the existence of holomorphic vector bundles on \(X\): a rank \(k\) complex vector bundle \(E\) can admit a holomorphic structure only if \(c_j(E)\) lies in \(H^{i+j}(X)\) for \(j = 1, \ldots, k\).

### 6.4. Holomorphic line bundles

A **holomorphic line bundle** on \((X, J)\) is a rank 1 holomorphic vector bundle, with fibre \(\mathbb{C}\). An example: if \(\dim_{\mathbb{C}} X = n\) then as \(T^*X\) is a holomorphic vector bundle of rank \(n\), the top exterior power \(\Lambda^n_{\mathbb{C}} T^*X\) is a holomorphic vector bundle of rank \(\binom{n}{n} = 1\), that is, a line bundle. We call \(\Lambda^n_{\mathbb{C}} T^*X\) the **canonical bundle** of \(X\), written \(K_X\).

Here \(T^*X\) as a holomorphic vector bundle is really \(T^{*(1,0)}X\), so \(K_X\) is \(\Lambda^n_{\mathbb{C}} T^{*(1,0)}X = \Lambda^{n,0}X\). That is, \(K_X\) is the holomorphic line bundle of \((n, 0)\)-forms on \(X\).
Let $L \to X$ be a holomorphic line bundle. Choose a Hermitian metric $h$ on $L$. As in §6.2 we get a connection $\nabla$ on $L$ preserving $h$, with curvature $F_\nabla \in C^\infty(\text{Herm}^{-}(L) \otimes_\mathbb{R} \Lambda^2 T^*X)$ of type (1,1). But as $L$ is a line bundle, there are natural identifications $\text{End}(L) \cong \mathbb{C}$ and $\text{Herm}^{-}(L) \cong i\mathbb{R} \subset \mathbb{C}$. Thus we have $F_\nabla = i\eta$ for $\eta$ a real 2-form. In fact $\eta$ is a closed real (1,1)-form, and $p_1(F_\nabla) = \frac{1}{2\pi}\eta$, so that $[\eta] = 2\pi c_1(L)$ in $H^2_{d\bar{\partial}}(X; \mathbb{R})$.

If $\tilde{h}$ is an alternative choice of Hermitian metric on $L$ then $\tilde{h} = e^f \cdot h$ for some smooth $f : X \to \mathbb{R}$. If $\tilde{\nabla}$ and $\tilde{\eta}$ are $\nabla, \eta$ for this $\tilde{h}$ then we find that $\tilde{\eta} = \eta - \frac{1}{2}d\bar{\partial}c f$.

Let $h, \nabla, \eta$ be as above. If $(X, J, g)$ is compact Kähler, and $\hat{\eta}$ is a closed real (1,1)-form on $X$ with $[\hat{\eta}] = 2\pi c_1(L)$, then $\hat{\eta} - \eta$ is an exact real (1,1)-form on $X$, so $\hat{\eta} - \eta = -\frac{1}{2}d\bar{\partial}c f$ for some smooth $f : X \to \mathbb{R}$ by the Global $d\bar{\partial}c$-Lemma in §4.2, with $f$ unique up to addition of constants. Then $\hat{h} = e^f \cdot h$ is a Hermitian metric on $L$ yielding $\hat{\eta}$ as its curvature form. Thus, all closed real (1,1)-forms in the cohomology class $2\pi c_1(L)$ can be realized as curvature 2-forms of a metric $h$ on $L$, uniquely up to rescaling.