Miniprojects, for assessment of course during Easter vacation

General guidance: I am expecting you to produce an essay-type answer, of length perhaps 5 pages, ideally written in \LaTeX{} and submitted as a PDF file by e-mail to joyce@maths.ox.ac.uk, by end April 2020. You could spend perhaps three days on the project, of which half might be reading references. If you were to go into some of these projects in depth then you would probably spend much longer. But I don't want you to waste your time on this; from the point of view of assessment, all you have to demonstrate is that you've learnt something and understood it, and can string a sentence together. These are just suggestions; feel free to choose your own topic.

Project 1. Let \((X,J,g)\) be a Kähler manifold, and \(G\) a Lie group which acts on \(X\) by diffeomorphisms fixing \(J\) and \(g\), with Lie algebra \(g\). In the Kähler quotient construction, one defines a moment map \(\mu : X \to g^\ast\), and then for \(c \in \mathbb{Z}(g^\ast)\), under good conditions \(X//G := \mu^{-1}(c)/G\) is a new Kähler manifold, of real dimension \(\dim X - 2 \dim G\).

Explain all this. You could illustrate your discussion in terms of the quotient of \(\mathbb{C}^4\) by \(U(1)\) acting by

\[u : (z_1, z_2, z_3, z_4) \mapsto (uz_1, uz_2, u^{-1}z_3, u^{-1}z_4).\]

There are two different nonsingular cases and one singular case, according to the value of the moment map in \(Z(u(1)^\ast)\); make sure you cover them all. Can you describe the quotients explicitly?

Other topics you could look at (pick one):

- The hyperkähler quotient construction for hyperkähler manifolds (holonomy \(Sp(m)\)).
- Geometric Invariant Theory, and quotients in algebraic geometry. This is related to the question: in what sense can we regard the Kähler quotient \(X//G\), as a complex manifold, as a complex quotient \(X/G^c\) by the complexification of \(G\)?
- Marsden–Weinstein symplectic reduction for symplectic manifolds.

Useful references:
**Project 2.** Write about what is known about the moduli space of $K3$ surfaces (Calabi–Yau 2-folds), both as complex surfaces, and as hyperkähler 4-manifolds. Include Torelli Theorems, etc., and discuss why some $K3$ surfaces are projective and some are not.

**Useful references:**
W. Barth, C. Peters and A. van de Ven, *Compact complex surfaces*, Springer, 1984, §VIII.

**Project 3.** Outline the famous String Theory calculation of:


which computes (conjectural) numbers of holomorphic curves on the quintic in $\mathbb{CP}^4$ (Gromov–Witten invariants), in terms of the variation of Hodge structure of a ‘mirror’ Calabi–Yau 3-fold.

Say something about the mathematics that is needed to make some part of this calculation mathematically rigorous, for instance, how Gromov–Witten invariants are defined, or what variation of Hodge structure is.

**Useful references:**

**Project 4.** *Sheaves* are a fundamental idea in modern Algebraic Geometry, for instance, a *scheme* is a special kind of topological space with a sheaf of rings. Discuss sheaves of abelian groups on a topological space and their cohomology. As examples you could consider:

- The constant sheaf $\mathbb{R}$ with fibre $\mathbb{R}$ on a (sufficiently nice) topological space $X$, whose sheaf cohomology $H^k(X;\mathbb{R})$ is the Čech cohomology $H^k_{\text{Čech}}(X;\mathbb{R})$ of $X$;
- The sheaves of de Rham $k$-forms on a manifold $X$;
- The sheaf of holomorphic functions on a complex manifold $(X,J)$; and
- The sheaf of holomorphic sections of a holomorphic vector bundle $E$ on a complex manifold $(X,J)$.

Outline the proof that Čech cohomology $H^k_{\text{Čech}}(X;\mathbb{R})$ of a smooth manifold $X$ is isomorphic to de Rham cohomology $H^k_{\text{dR}}(X;\mathbb{R})$. The same method is used to
show that Dolbeault-style cohomology of a holomorphic vector bundle $H^q(E)$ on $(X,J)$, as defined in the lectures, is isomorphic to the Čech cohomology of the sheaf of holomorphic sections of $E$.

**Useful references:**
P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1994, §0.3.

**Project 5.** Find out and write about *moduli spaces* in Algebraic Geometry.
(You will need some background on schemes.)
Some topics you could consider:
- moduli of curves (Riemann surfaces);
- moduli of (algebraic) vector bundles (or more generally coherent sheaves), and Grothendieck’s Quot scheme;
- Geometric Invariant Theory, and quotients in algebraic geometry.

**Useful references:**