

Complex manifolds and Kähler Geometry

Lecture 11 of 16: Curvature of Kähler manifolds

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Plan of talk:

- 11 Curvature of Kähler manifolds
 - 11.1 Riemann curvature of Riemannian manifolds
 - 11.2 Decomposition of tensors on complex manifolds
 - 11.3 The Ricci form
 - 11.4 Ricci flat Kähler manifolds and Calabi–Yau manifolds

11.1. Riemann curvature of Riemannian manifolds

Let (X, g) be a Riemannian manifold, with metric $g = g_{ab}$. The *Riemann curvature tensor* $R^a{}_{bcd}$ of g is the curvature of the Levi-Civita connection ∇ of g . We also write $R_{abcd} = g_{ae}R^e{}_{bcd}$. In local coordinates (x^1, \dots, x^n) on X we have

$$R^a{}_{bcd} = \frac{\partial}{\partial x^c} \Gamma^a{}_{bd} - \frac{\partial}{\partial x^d} \Gamma^a{}_{bc} + \Gamma^a{}_{ce} \Gamma^e{}_{bd} - \Gamma^a{}_{de} \Gamma^e{}_{bc},$$

where

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(\frac{\partial}{\partial x^b} g_{cd} + \frac{\partial}{\partial x^c} g_{bd} - \frac{\partial}{\partial x^d} g_{bc} \right).$$

We call g *flat* if $R^a{}_{bcd} \equiv 0$.

The Riemann curvature has the symmetries

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, \quad (11.1)$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \quad (11.2)$$

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0, \quad (11.3)$$

where (11.2) and (11.3) are the *first* and *second Bianchi identities*.

The *Ricci curvature* is $R_{ab} = R^c_{acb}$, a trace of the Riemann curvature. It is symmetric, $R_{ab} = R_{ba}$. We call g *Einstein* if $R_{ab} \equiv \lambda g_{ab}$ for $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} \equiv 0$.

The *scalar curvature* of g is $s = g^{ab} R_{ab} = g^{ab} R^c_{acb}$.

Riemannian, Ricci and scalar curvature are important for many reasons. Einstein's equations in General Relativity prescribe the Ricci curvature of a (pseudo)metric g of type $(3,1)$. Einstein metrics are a class of 'best' metrics on manifolds X . They are used in the solution of the 3-d Poincaré conjecture. The Yamabe problem concerns metrics with prescribed scalar curvature in a conformal class of metrics. To have metrics with positive or zero Ricci curvature, or positive or zero scalar curvature, imposes interesting topological conditions on X .

Riemannian curvature of Kähler metrics

Now suppose (X, J, g) is a Kähler manifold. Then $\nabla J \equiv 0$, where ∇ is the Levi-Civita connection.

The defining property of Riemann curvature $R^a{}_{bcd}$ is that

$$\begin{aligned} R^a{}_{bcd} u^b v^c w^d &= v^c \nabla_c (w^d \nabla_d u^a) \\ &\quad - w^d \nabla_d (v^c \nabla_c u^a) - [v, w]^e \nabla_e u^a, \end{aligned} \tag{11.4}$$

for all vector fields u^a, v^b, w^d on X . Replace u^a by $(Ju)^a = J_e^a u^e$. Since $\nabla J \equiv 0$ we have $v^c \nabla_c (Ju)^a = J_b^a (v^c \nabla_c u^b)$. So J commutes with each ∇ in (11.4).

This implies that

$$R^a{}_{bcd}(J_e^b u^e)v^c w^d = J_e^a(R^e{}_{bcd}u^b v^c w^d)$$

for all vector fields u, v, w on X . That is, the Riemann curvature of a Kähler metric X satisfies

$$J_b^e R^a{}_{ecd} \equiv J_f^a R^f{}_{bcd}. \quad (11.5)$$

Together with (11.1), equation (11.5) implies other symmetries of $R^a{}_{bcd}$, as we will see in §11.2.

11.2. Decomposition of tensors on complex manifolds

On a complex manifold (X, J) , we work with complex k -forms rather than real k -forms, and split them into (p, q) -forms for $p + q = k$. In a similar way, we can split general tensors $T_{b_1 \dots b_l}^{a_1 \dots a_k}$ on X into components using J . We use the following notation. To the Roman indices a, b, c, d, e correspond Greek indices $\alpha, \beta, \gamma, \delta, \epsilon$ and their complex conjugates $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\epsilon}$.

We work with complex tensors on X . Let $T = T_{\dots}^a \dots$ be a complex tensor with one chosen contravariant index a , and other indices we don't care about. Define

$$T_{\dots}^{\alpha \dots} = \frac{1}{2}(T_{\dots}^a \dots - iJ_b^a T_{\dots}^b \dots), \quad T_{\dots}^{\bar{\alpha} \dots} = \frac{1}{2}(T_{\dots}^a \dots + iJ_b^a T_{\dots}^b \dots),$$

so that $T_{\dots}^a \dots = T_{\dots}^{\alpha \dots} + T_{\dots}^{\bar{\alpha} \dots}$. Similarly, if $T = T_{a \dots}$ define

$$T_{\alpha \dots} = \frac{1}{2}(T_{a \dots} - iJ_a^b T_{b \dots}), \quad T_{\bar{\alpha} \dots} = \frac{1}{2}(T_{a \dots} + iJ_a^b T_{b \dots}).$$

so that $T_{a \dots} = T_{\alpha \dots} + T_{\bar{\alpha} \dots}$.

These operations are projections on complex tensors on X . Think of indices $\alpha, \beta, \gamma, \dots$ as indices of type $(1,0)$, and indices $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \dots$ as indices of type $(0,1)$. For example, a general Riemannian metric $g = g_{ab}$ on X splits as

$$g_{ab} = g_{\alpha\beta} + g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}}.$$

However, if g is Hermitian then $g_{ab} = J_a^c J_b^d g_{cd}$. We have

$$J_a^c = i\delta_{\alpha}^{\gamma} - i\delta_{\bar{\alpha}}^{\bar{\gamma}}.$$

Substituting this into $g_{ab} = J_a^c J_b^d g_{cd}$ gives

$$g_{\alpha\beta} + g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}} + g_{\bar{\alpha}\bar{\beta}} = -g_{\alpha\beta} + g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}} - g_{\bar{\alpha}\bar{\beta}}$$

so that $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. That is, g is Hermitian if and only if

$$g_{ab} = g_{\bar{\alpha}\beta} + g_{\alpha\bar{\beta}}. \quad (11.6)$$

Essentially this says that g is of type $(1,1)$.

Decomposition of Riemann curvature of Kähler metrics

Let (X, J, g) be Kähler, with Riemann curvature $R^a{}_{bcd}$. Then (11.5) gives $J_b^e R^a{}_{ecd} \equiv J_f^a R^f{}_{bcd}$. This implies that $R^{\alpha}{}_{\bar{\beta}cd} = R^{\bar{\alpha}}{}_{\beta cd} = 0$. Hence

$$R^a{}_{bcd} = R^{\alpha}{}_{\beta cd} + R^{\bar{\alpha}}{}_{\bar{\beta}cd}. \quad (11.7)$$

Contracting (11.7) with $g_{ab} = g_{\alpha\beta} + g_{\alpha\bar{\beta}}$ by (11.6) yields

$$R_{abcd} = R_{\alpha\beta cd} + R_{\alpha\bar{\beta}cd}. \quad (11.8)$$

Combining (11.8) with $R_{abcd} = R_{cdab}$ from (11.1) gives

$$R_{abcd} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} + R_{\bar{\alpha}\beta\gamma\bar{\delta}} + R_{\alpha\bar{\beta}\bar{\gamma}\delta} + R_{\bar{\alpha}\beta\bar{\gamma}\delta}. \quad (11.9)$$

That is, of the 16 components of $R^a{}_{bcd}$ or R_{abcd} for the curvature of a Kähler metric, 12 are automatically zero. This is not true for general Hermitian metrics.

Taking the trace in (11.9) shows that the Ricci curvature satisfies

$$R_{ab} = R_{\alpha\bar{\beta}} + R_{\bar{\alpha}\beta}. \quad (11.10)$$

Comparing (11.6) and (11.10), we see R_{ab} has the same type decomposition as a Hermitian metric.

11.3. The Ricci form

From a Hermitian metric g on (X, J) , we may define the Hermitian form $\omega_{ab} = J_a^c g_{cb}$. In the same way, from the Ricci curvature R_{ab} of a Kähler metric, we define the *Ricci form* $\rho_{ab} = J_a^c R_{cb}$. Then (11.10) implies that

$$\rho_{ab} = iR_{\alpha\bar{\beta}} - iR_{\bar{\alpha}\beta}. \quad (11.11)$$

As $R_{ab} = R_{ba}$ we have $R_{\alpha\bar{\beta}} = R_{\bar{\beta}\alpha}$. Combining this with (11.11) shows that $\rho_{ab} = -\rho_{ba}$, that is, ρ is a 2-form. So ρ is a real (1,1)-form, as for Hermitian forms ω .

Using the second Bianchi identity (11.3), one can show that ρ is *closed*, so it is a closed real (1,1)-form, as for the Kähler form ω . The cohomology class $[\rho] \in H_{\text{dR}}^2(X; \mathbb{R})$ turns out to be $2\pi c_1(X)$, where the *first Chern class* $c_1(X)$ of X is

$$c_1(X) := c_1(TX) = -c_1(T^*X) = -c_1(K_X).$$

Thus $[\rho]$ depends only on (X, J) , not on g .

This has a natural interpretation: the Levi-Civita connection ∇ induces connections on the vector bundles of (p, q) -forms $\Lambda^{p,q}X$, and in particular on $\Lambda^{n,0}X = K_X$, where $n = \dim_{\mathbb{C}} X$. Then $-\rho$ is the curvature 2-form of this connection on the holomorphic line bundle K_X , as in §6.4.

The Ricci form has a nice expression in coordinates. Let (z_1, \dots, z_n) be local holomorphic coordinates on $U \subset X$. Then we may write

$$\omega^n = i^n n! e^f dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

for smooth $f : U \rightarrow \mathbb{R}$. Then

$$\rho|_U \equiv -i\partial\bar{\partial}f = -\frac{1}{2}dd^c f.$$

We may also write

$$f = \log \det(g_{\alpha\bar{\beta}}),$$

where we regard $(g_{\alpha\bar{\beta}})_{\alpha,\bar{\beta}=1}^n$ as an $n \times n$ Hermitian matrix of complex functions, and

$$f = -2 \log |dz_1 \wedge \cdots \wedge dz_n|_g.$$

Note that $dz_1 \wedge \cdots \wedge dz_n$ is a nonvanishing holomorphic $(n, 0)$ -form on $U \subseteq X$. More generally, if Ω is a nonvanishing holomorphic $(n, 0)$ -form on $U \subseteq X$ then

$$\rho|_U \equiv dd^c(\log |\Omega|_g).$$

11.4. Ricci flat Kähler manifolds and Calabi–Yau manifolds

Now suppose (X, J, g) is a Kähler manifold and the Ricci curvature R_{ab} of X is zero. Then the Ricci form ρ of g is zero. But ρ is the curvature 2-form of a connection on the canonical line bundle K_X . Thus, $c_1(X) = 0$ in $H_{\text{dR}}^2(X; \mathbb{R})$. (Note this does not imply that $c_1(K_X) = 0$ in $H^2(X; \mathbb{Z})$, but rather that $c_1(K_X)$ is a torsion element (has finite order) in $H^2(X; \mathbb{Z})$.)

As $\rho \equiv 0$, the connection on K_X is *flat*. Thus, locally there exist *constant sections* of K_X , that is, we may cover X by open subsets U with a nonzero $(n, 0)$ -form $\Omega \in C^\infty(K_X|_U)$ such that $\nabla\Omega \equiv 0$. Globally, flat connections on (Hermitian) line bundles are classified by group morphisms $\mu: \pi_1(X) \rightarrow \text{U}(1)$, where $\pi_1(X)$ is the fundamental group of X . Since $\text{U}(1)$ is abelian, this factors through the projection $\pi_1(X) \rightarrow H_1(X; \mathbb{Z})$.

Thus, (K_X, ∇) is classified up to isomorphism by a morphism $\mu : H_1(X; \mathbb{Z}) \rightarrow U(1)$. If $H_1(X; \mathbb{Z}) = 0$ this is trivial, and K_X is isomorphic to the trivial line bundle \mathcal{O}_X and connection, so that K_X admits a global constant section Ω with $\nabla\Omega = 0$. This implies that $\bar{\partial}\Omega = 0$, so Ω is a holomorphic section of K_X , i.e. $\Omega \in H^0(K_X)$. So Ω defines an isomorphism of holomorphic line bundles $K_X \cong \mathcal{O}_X$, that is, *the canonical bundle of X is trivial*. Also $\nabla\Omega = \nabla g = 0$ implies that $|\Omega|_g$ is constant. Conversely, suppose (X, J, g) is compact, Ricci-flat and Kähler, and there exists a nonvanishing holomorphic section Ω of K_X . Then

$$dd^c(\log |\Omega|_g) = \rho \equiv 0.$$

Taking the trace shows that $\Delta(\log |\Omega|_g) \equiv 0$, so $\log |\Omega|_g$ is constant by the maximum principle as X is compact, and thus $|\Omega|_g$ is constant.

Hence

$$\begin{aligned}0 &= \nabla_a |\Omega|_g^2 = \nabla_a (\Omega \otimes \bar{\Omega}) \\ &= (\nabla_a \Omega) \otimes \bar{\Omega} + \Omega \otimes (\nabla_a \bar{\Omega}) \\ &= (\nabla_\alpha \Omega) \otimes \bar{\Omega} + \Omega \otimes (\nabla_{\bar{\alpha}} \bar{\Omega}),\end{aligned}$$

since $\nabla_{\bar{\alpha}} \Omega = \bar{\partial} \Omega = 0$ as Ω is holomorphic, so $\nabla_\alpha \bar{\Omega} = 0$. Therefore

$$(\nabla_\alpha \Omega) \otimes \bar{\Omega} = \Omega \otimes (\nabla_{\bar{\alpha}} \bar{\Omega}) = 0,$$

so $\nabla_\alpha \Omega = 0$, and $\nabla \Omega = 0$. Thus, if (X, J, g) is compact, Ricci-flat and Kähler then any $\Omega \in H^0(K_X)$ is constant.

Calabi–Yau manifolds

Definition

A *Calabi–Yau manifold* is a compact Ricci-flat Kähler manifold (X, J, g) with trivial canonical bundle K_X .

— Actually, definitions differ. Some people also require $H^{p,0}(X) = 0$ for $0 < p < \dim_{\mathbb{C}} X$; some people require g to have Riemannian holonomy group $SU(m)$ (explained in §13–§14); some people do not require K_X trivial, only that g be Ricci-flat; some people take (X, J) to be Calabi–Yau, without a particular choice of Kähler metric g .

Suppose (X, J) is a compact complex manifold admitting Kähler metrics, with trivial canonical bundle. Then by Yau's solution of the Calabi Conjecture (next lecture), every Kähler class on X contains a unique Ricci-flat Kähler metric g , and then (X, J, g) is Calabi–Yau.

We can produce many examples of such (X, J) using algebraic geometry. For example, let (X, J) be a smooth hypersurface of degree $n + 2$ in $\mathbb{C}P^{n+1}$. Then using the adjunction formula as in §10.3 we find that $K_X \cong \mathcal{O}_X$, so X can be made into a Calabi–Yau n -fold (X, J, g) . So smooth quintics in $\mathbb{C}P^4$ are Calabi–Yau 3-folds, for instance.

Kähler–Einstein metrics

A (compact) Kähler manifold (X, J, g) is called *Kähler–Einstein* if g is Einstein, i.e. $R_{ab} = \lambda g_{ab}$ for some $\lambda \in \mathbb{R}$. The case $\lambda = 0$ is the Calabi–Yau case, so take $\lambda \neq 0$. Rescaling g by a constant we can take $\lambda = \pm 1$.

We have $\rho_{ab} = \lambda \omega_{ab}$, so $\omega = \lambda^{-1} \rho$, and $[\omega] = \lambda^{-1} [\rho] = 2\pi \lambda^{-1} c_1(X)$. Thus the Kähler class of g is determined by (X, J) and λ .

The cases $\lambda > 0$ and $\lambda < 0$ are rather different. In dimension 1, $\mathbb{C}P^1$ is Kähler–Einstein with $\lambda > 0$, and Riemann surfaces of genus $g > 1$ are Kähler–Einstein with $\lambda < 0$.

Kähler–Einstein manifolds with $\lambda > 0$ are *Fano manifolds*. Products of projective spaces are examples. The underlying complex manifolds are classified in dimension ≤ 3 . They can have positive-dimensional symmetry groups. A fixed (X, J) can have a family of Kähler–Einstein metrics g . The case $\lambda < 0$ is more rigid: the Aubin–Calabi–Yau theorem says that if (X, J) is a compact complex manifold and K_X is a positive line bundle, then there is a unique Kähler–Einstein metric g on X with $\lambda = -1$. They have no continuous symmetries.

Complex manifolds and Kähler Geometry

Lecture 12 of 16: The Calabi Conjecture

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Plan of talk:

12 The Calabi Conjecture

12.1 Rewriting the Conjecture

12.2 Sketch of the proof

12.3 Existence of Calabi–Yau metrics

12.4 Weitzenbock formulae, the Bochner argument

12. The Calabi Conjecture

This was posed by Calabi in 1954, and proved by Yau in 1976. Aubin also deserves credit.

Conjecture (The Calabi Conjecture)

Let X be a compact, complex manifold admitting Kähler metrics. Suppose ρ is a real, closed $(1, 1)$ -form on X with $[\rho] = 2\pi c_1(X)$ in $H_{\text{dR}}^2(X; \mathbb{R})$. Then in each Kähler class on X there exists a unique Kähler metric g with Ricci form ρ .

12.1. Rewriting the Conjecture

Suppose g, g' are Kähler metrics on (X, J) with Kähler forms ω, ω' and Ricci forms ρ, ρ' . Let (z_1, \dots, z_n) be local holomorphic coordinates on $U \subseteq X$. Then from §11.2, we may write

$$\begin{aligned}\omega^n &= i^n n! e^f dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n, \\ (\omega')^n &= i^n n! e^{f'} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,\end{aligned}$$

for smooth $f, f' : U \rightarrow \mathbb{R}$. Then

$$\rho|_U = -\frac{1}{2} dd^c f, \quad \rho'|_U = -\frac{1}{2} dd^c f'.$$

Thus on U we have $(\omega')^n \equiv e^{f'-f} \omega^n$ and $\rho' \equiv \rho - \frac{1}{2} \text{dd}^c(f' - f)$. This is true globally, so we deduce:

Lemma 12.1

Suppose g, g' are Kähler metrics on (X, J) , with Kähler forms ω, ω' and Ricci forms ρ, ρ' . Let $f : X \rightarrow \mathbb{R}$ be the unique smooth function with

$$(\omega')^n \equiv e^f \omega^n,$$

where $n = \dim_{\mathbb{C}} X$. Then

$$\rho' \equiv \rho - \frac{1}{2} \text{dd}^c f.$$

Now let (X, J, g) be compact Kähler, with Kähler and Ricci forms ω, ρ . Let ρ' be a closed real (1,1)-form with $[\rho'] = 2\pi c_1(X)$ in $H_{\text{dR}}^2(X; \mathbb{R})$. To solve the Calabi Conjecture, we seek a Kähler metric g' in the Kähler class of g with Ricci form ρ' .

As $[\rho'] = [\rho]$ in $H_{\text{dR}}^2(X; \mathbb{R})$, $\rho' - \rho$ is an exact real (1,1)-form on the compact Kähler manifold X . So by the Global dd^c -Lemma in §4.2 there exists smooth $f : X \rightarrow \mathbb{R}$, unique up to $f \mapsto f + c$, such that

$$\rho' \equiv \rho - \frac{1}{2} \text{dd}^c f.$$

By Lemma 12.1, g' has Ricci form ρ' if $(\omega')^n \equiv e^f \omega^n$. Now f is unique up to $f \mapsto f + c$. If g' is in the same Kähler class as g then we can fix this additive constant c . For then

$\int_X (\omega')^n = [X] \cdot [\omega']^n = [X] \cdot [\omega]^n = \int_X \omega^n$ as $[\omega'] = [\omega]$ in $H_{\text{dR}}^2(X; \mathbb{R})$. Hence we want $\int_X e^f \omega^n = \int_X \omega^n$, or equivalently $\int_X e^f dV_g = \text{vol}_g(X)$. This determines c uniquely.

Thus the Calabi Conjecture is equivalent to:

Conjecture (Calabi Conjecture, version 2)

Let (X, J) be a compact complex manifold, and g a Kähler metric on X , with Kähler form ω . Let $f : X \rightarrow \mathbb{R}$ be smooth with $\int_X e^f \omega^n = \int_X \omega^n$. Then there exists a unique Kähler metric g' in the Kähler class of g , with Kähler form ω' , such that $(\omega')^n \equiv e^f \omega^n$.

This is about finding Kähler metrics with *prescribed volume forms* on X . Note that prescribing ρ' is a 2nd order p.d.e. on g' , but prescribing $(\omega')^n$ is a 0th order p.d.e. on g' .

As $[\omega'] = [\omega]$, by the Global dd^c -Lemma in §4.2 we may write $\omega' = \omega + \frac{1}{2}dd^c\phi$ for $\phi : X \rightarrow \mathbb{R}$, unique up to $\phi \mapsto \phi + c$. We fix c by requiring

$$\int_X \phi \, dV_g = 0.$$

Then $(\omega')^n = e^f \omega^n$ becomes

$$\left(\omega + \frac{1}{2}dd^c\phi\right)^n = e^f \omega^n.$$

Conversely, if g, ω, ϕ are given, then $\omega' = \omega + \frac{1}{2}dd^c\phi$ is the Kähler form of a Kähler metric g' in the Kähler class of g iff ω' is a positive $(1,1)$ -form.

Therefore the Calabi Conjecture is equivalent to:

Conjecture (Calabi Conjecture, version 3)

Let (X, J) be a compact complex manifold, and g a Kähler metric on X , with Kähler form ω . Let $f : X \rightarrow \mathbb{R}$ be smooth with $\int_X e^f \omega^n = \int_X \omega^n$. Then there exists a unique smooth $\phi : X \rightarrow \mathbb{R}$ such that:

- (i) $\omega + \frac{1}{2}dd^c\phi$ is a positive $(1, 1)$ -form,
- (ii) $\int_X \phi dV_g = 0$, and
- (iii) $(\omega + \frac{1}{2}dd^c\phi)^n = e^f \omega^n$ on X .

We have

$$(\omega + \frac{1}{2}dd^c\phi)^n = \omega^n + \frac{n}{2}(dd^c\phi) \wedge \omega^{n-1} + \dots .$$

But $(dd^c\phi) \wedge \omega^{n-1} = -\frac{2}{n}(\Delta\phi)\omega^n$, where Δ is the Laplacian on functions on X . Hence we may rewrite $(\omega + \frac{1}{2}dd^c\phi)^n = e^f\omega^n$ as

$$(1 - \Delta\phi + \dots)\omega^n = (1 + f + \dots)\omega^n.$$

Thus, for small ϕ, f , the nonlinear equation $(\omega + \frac{1}{2}dd^c\phi)^n = e^f\omega^n$ on ϕ is approximated by the linear equation $\Delta\phi = -f$.

12.2. Sketch of the proof

We have reduced the Calabi Conjecture to a problem in analysis, of showing a nonlinear elliptic p.d.e. has a unique solution. To prove it, Yau used the *continuity method*.

Define a smooth family of functions $f_t : X \rightarrow \mathbb{R}$ for $t \in [0, 1]$ with $f_0 \equiv 0$ and $f_1 \equiv f$ and $\int_X e^{f_t} \omega^n = \int_X \omega^n$ for all $t \in [0, 1]$. Define S to be the set of $t \in [0, 1]$ for which there exists $\phi : X \rightarrow \mathbb{R}$ satisfying (i),(ii) and

$$(\omega + \frac{1}{2}dd^c\phi)^n = e^{f_t}\omega^n. \quad (12.1)$$

We must show that S is both *open* and *closed* in $[0,1]$. When $t = 0$, $\phi \equiv 0$ is a solution, so $0 \in S$. Thus S open and closed forces $S = [0, 1]$, so $1 \in S$, and this proves the Calabi Conjecture.

Showing S is open is fairly easy, and was done by Calabi. It depends on the fact that (12.1) is an *elliptic* p.d.e. – basically, $\phi \mapsto (\omega + \frac{1}{2}dd^c\phi)^n$ is like a nonlinear Laplacian – and uses only standard facts about elliptic operators.

Proving S is closed is much more difficult. One must show that that S contains its limit points. That is, if $(t_n)_{n=1}^\infty$ is a sequence in S with $t_n \rightarrow t' \in [0, 1]$, we want the corresponding solutions ϕ_n of (12.1) with $t = t_n$ to converge to a solution ϕ with $t = t'$, so that $t' \in S$. To show $(\phi_n)_{n=1}^\infty$ converges to a smooth limit you need difficult *a priori estimates* on $|\nabla^k \phi_n|$ for all k, n .

12.3. Existence of Calabi–Yau metrics

When $c_1(X) = 0$ in $H^2(X; \mathbb{R})$ we can take $\rho \equiv 0$, and then g is Ricci-flat. So the Calabi Conjecture gives:

Corollary 12.2

Let (X, J) be a compact complex manifold with $c_1(X) = 0$ in $H_{\text{dR}}^2(X, \mathbb{R})$. Then every Kähler class on X contains a unique Ricci-flat Kähler metric g .

If also K_X is trivial – automatically true if $\pi_1(X) = \{1\}$, for instance – then (X, J, g) is a *Calabi–Yau manifold*.

So, to construct Calabi–Yau manifolds we need only find examples of compact complex manifolds (X, J) satisfying some simple topological conditions, e.g. smooth hypersurfaces of degree $n + 2$ on $\mathbb{C}P^{n+1}$, and then by the Calabi Conjecture there exist metrics g on X such that (X, J, g) is Calabi–Yau.

All we know about g is that it exists. No explicit nontrivial examples of Calabi–Yau metrics on compact manifolds are known.

Ricci flatness and fundamental groups

Having positive or zero Ricci curvature has implications for the topology of the manifold. The next result follows from the *Cheeger–Gromoll Splitting Theorem*.

Theorem 12.3

Suppose (X, g) is a compact Riemannian manifold. If g is Ricci-flat then X admits a finite cover \tilde{X} isometric to $T^k \times N$, where T^k has a flat metric and N is a simply-connected Ricci-flat manifold.

If g has positive definite Ricci curvature then $\pi_1(X)$ is finite.

If g is Kähler then T^k and N are also Kähler. So combining Theorem 12.3 with the Calabi Conjecture gives:

Corollary 12.4

Let (X, J) be a compact complex manifold admitting Kähler metrics. If $c_1(X) = 0$ in $H_{\text{dR}}^2(X; \mathbb{R})$ then X admits a finite cover biholomorphic to $T^{2j} \times N$, where T^{2j} is a complex torus, and N a simply-connected complex manifold with $c_1(N) = 0$.

Also if (X, J) is a compact complex manifold with K_X a negative line bundle, then X admits Kähler metrics, and as

$$[\rho] = 2\pi c_1(X) = -2\pi c_1(K_X),$$

by the Calabi Conjecture X has Kähler metrics with positive Ricci curvature. Thus the last part of Theorem 12.3 gives:

Corollary 12.5

Let (X, J) be a compact complex manifold with negative canonical bundle K_X . Then $\pi_1(X)$ is finite.

12.4. Weitzenböck formulae, the Bochner argument

Let (X, g) be a Riemannian manifold. Let ∇ be the Levi-Civita connection, and R^i_{jkl} be the Riemann curvature of g . Then one can form two Laplacians on k -forms $C^\infty(\Lambda^k T^*M)$, $\nabla^*\nabla$ and $\Delta_d = dd^* + d^*d$. The *Weitzenböck formula* is

$$\Delta_d \alpha = \nabla^* \nabla \alpha + \tilde{R}(\alpha), \quad (12.2)$$

where $\tilde{R}(\alpha)$ is a natural contraction of R^i_{jkl} , g^{ab} and $\alpha_{a_1 \dots a_k}$:

$$\begin{aligned} \tilde{R}(\alpha)_{a_1 \dots a_k} &= \sum_{j=1}^k g^{bc} R_{ajb} \alpha_{a_1 \dots a_{j-1} c a_{j+1} \dots a_k} \\ &\quad - 2 \sum_{1 \leq i < j \leq k} g^{bc} R^d_{a_i c a_j} \alpha_{a_1 \dots a_{i-1} d a_{i+1} \dots a_{j-1} b a_{j+1} \dots a_k}. \end{aligned}$$

If α is a 1-form then \tilde{R} involves only the Ricci curvature R_{ab} of g , and (12.2) becomes

$$(\Delta_d \alpha)_a = (\nabla^* \nabla \alpha)_a + R_{ab} g^{bc} \alpha_c. \quad (12.3)$$

Suppose X is compact. Take the inner product of (12.3) with α in L^2 and integrate by parts. We get:

$$\|d^*\alpha\|_{L^2}^2 + \|d\alpha\|_{L^2}^2 = \|\nabla\alpha\|_{L^2}^2 + \int_X R_{ab}g^{bc}g^{ad}\alpha_c\alpha_d dV_g. \quad (12.4)$$

In particular, if $\alpha \in \mathcal{H}^1 \cong H^1(X; \mathbb{R})$, so that $d^*\alpha = d\alpha = 0$, then (12.4) becomes

$$\|\nabla\alpha\|_{L^2}^2 + \int_X R_{ab}g^{bc}g^{ad}\alpha_c\alpha_d dV_g = 0.$$

Hence if g is Ricci flat and X is compact then $d^*\alpha = d\alpha = 0$ for a 1-form α implies $\nabla\alpha = 0$. This is related to Corollary 12.4.

If R_{ab} is positive definite then $d^*\alpha = d\alpha = 0$ for a 1-form α implies $\alpha = 0$. Hence $\mathcal{H}^1 = 0$, so $H_{\text{dR}}^1(X; \mathbb{R}) = 0$; this follows from Corollary 12.5.

This kind of proof is called the *Bochner argument*.

If g is Kähler, we have $\Delta_d = 2\Delta_{\bar{\partial}}$ by the Kähler identities. If also α is a $(p, 0)$ -form for $p > 0$, the action of \tilde{R} on α again involves only the Ricci curvature, and (12.2) becomes

$$\begin{aligned} 2(\bar{\partial}^* \bar{\partial} \alpha)_{a_1 \dots a_p} &= (\nabla^* \nabla \alpha)_{a_1 \dots a_p} \\ &+ \sum_{j=1}^p R_{a_j b} g^{bc} \alpha_{a_1 \dots a_{j-1} c a_{j+1} \dots a_p}. \end{aligned} \tag{12.5}$$

Thus, if $\bar{\partial} \alpha = 0$, the Bochner argument gives

$$\|\nabla \alpha\|_{L^2}^2 + \langle \alpha, \tilde{R} \alpha \rangle_{L^2} = 0,$$

where $\langle \alpha, \tilde{R} \alpha \rangle_{L^2} = 0$ if $R_{ab} \equiv 0$, and is positive if R_{ab} is positive definite and $\alpha \neq 0$.

So as for Corollaries 12.4 and 12.5, using the Calabi Conjecture for Corollary 12.7, we have:

Corollary 12.6

Suppose (X, J, g) is a compact Ricci-flat Kähler manifold. Then every holomorphic $(p, 0)$ -form α on X is constant under the Levi-Civita connection.

Hence $\dim_{\mathbb{C}} H^{p,0}(X) \leq \binom{n}{p}$.

Corollary 12.7

Suppose (X, J) is a compact complex manifold, and K_X is a negative line bundle. Then $H^{p,0}(X) = 0$ for $p > 0$.

We can also apply this argument for vector fields v on a Kähler manifold (X, J, g) . In this case the analogue of (12.3) and (12.5) is

$$2(\bar{\partial}^* \bar{\partial} v)^a = (\nabla^* \nabla v)^a - R_{bc} g^{ab} v^c \quad (12.6)$$

— note the sign difference. Thus, if v is a holomorphic vector field ($\bar{\partial} v = 0$, or equivalently $\mathcal{L}_v J = 0$), taking L^2 inner products and integrating by parts gives

$$\|\nabla v\|_{L^2}^2 = \int_X R_{ab} v^a v^b dV_g.$$

So as for Corollaries 12.6 and 12.7, we get:

Corollary 12.8

Suppose (X, J, g) is a compact Ricci-flat Kähler manifold. Then every holomorphic vector field v on X is constant under the Levi-Civita connection.

Corollary 12.9

Suppose (X, J) is a compact complex manifold, and K_X is a positive line bundle. Then all holomorphic vector fields on X are zero. That is, X has no infinitesimal symmetries.

If (X, J) is a compact complex manifold, the vector space $H^0(TX)$ of holomorphic vector fields on X is a finite-dimensional complex Lie algebra. If K_X is positive then $H^0(TX) = 0$. If K_X is negative it can be nonzero, and is related to the existence of Kähler–Einstein metrics g on X . For such g to exist, it is necessary that $H^0(TX)$ be a *reductive* Lie algebra, as it must be the complexification of the Lie algebra of the compact Lie group of holomorphic isometries of (X, J, g) . When $\dim_{\mathbb{C}} X = 2$, this is also a sufficient condition.