Complex manifolds and Kähler Geometry

Lecture 5 of 16: Hodge theory for Kähler manifolds

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These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

5.1. Hodge theory for compact Riemannian manifolds

We first recall Hodge theory for ordinary Riemannian manifolds. Let (X, g) be a compact, oriented Riemannian n-manifold. Then the *Hodge star* $*$ acts on k-forms

$$*: C^\infty(\Lambda^k T^*X) \longrightarrow C^\infty(\Lambda^{n-k} T^*X).$$

It satisfies $*^2 = (-1)^{k(n-k)}$, so $*^{-1} = \pm *$. We define

$$\mathrm{d}^*: C^\infty(\Lambda^k T^*X) \longrightarrow C^\infty(\Lambda^{k-1} T^*X)$$

by $\mathrm{d}^* = (-1)^k *^{-1} \mathrm{d}*$, and the *Laplacian* on k-forms $\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

Forms α with $\Delta_{\mathrm{d}}\alpha = 0$ are called *harmonic*. Later we will see this is equivalent to $\mathrm{d}\alpha = \mathrm{d}^*\alpha = 0$ (for X compact). It is helpful to think about all this in terms of the *L^2-inner product on forms*. If $\alpha, \beta \in C^\infty(\Lambda^k T^*X)$ we define

$$\langle \alpha, \beta \rangle_{L^2} = \int_X (\alpha, \beta) \mathrm{d}V_g,$$

where (α, β) is the pointwise inner product of k-forms using g, and $\mathrm{d}V_g$ the volume form of g. The Hodge star is defined so that if α, β are k-forms then $\alpha \wedge (*\beta) = (\alpha, \beta)\mathrm{d}V_g$. Thus

$$\langle \alpha, \beta \rangle_{L^2} = \int_X \alpha \wedge *\beta.$$

Now let α be a $(k-1)$-form and β a k-form. Then we have

$$\begin{aligned}\langle\alpha,\mathrm{d}^*\beta\rangle_{L^2}&=\langle\alpha,(-1)^k*^{-1}\mathrm{d}*\beta\rangle_{L^2}\\&=(-1)^k\int_X(\alpha,*^{-1}\mathrm{d}*\beta)\mathrm{d}V_g\\&=(-1)^k\int_X\alpha\wedge*(*^{-1}\mathrm{d}*\beta)\\&=(-1)^k\int_X\alpha\wedge\mathrm{d}(*\beta).\end{aligned}$$

But by Stokes' Theorem,

$$0=\int_X\mathrm{d}\big[\alpha\wedge(*\beta)\big]=\int_X(\mathrm{d}\alpha)\wedge(*\beta)+(-1)^{k-1}\int_X\alpha\wedge\mathrm{d}(*\beta).$$

Hence

$$\langle\alpha,\mathrm{d}^*\beta\rangle_{L^2}=\int_X(\mathrm{d}\alpha)\wedge(*\beta)=\int_X(\mathrm{d}\alpha,\beta)\mathrm{d}V_g=\langle\mathrm{d}\alpha,\beta\rangle_{L^2}.$$

Thus $\langle\alpha,\mathrm{d}^*\beta\rangle_{L^2}=\langle\mathrm{d}\alpha,\beta\rangle_{L^2}$ for all α,β, so d^* behaves like the adjoint of d under the L^2-inner product; we call d^* the *formal adjoint* of d. One consequence is that $\mathrm{d}^*\beta=0$ if and only if $\langle\mathrm{d}\alpha,\beta\rangle_{L^2}=0$ for all α. That is, $\mathrm{Ker}\,\mathrm{d}^*=(\mathrm{Im}\,\mathrm{d})^\perp$, the kernel of d^* in $C^\infty(\Lambda^k T^*X)$ is the subspace of $C^\infty(\Lambda^k T^*X)$ which is L^2-orthogonal to the image of $\mathrm{d}:C^\infty(\Lambda^{k-1}T^*X)\to C^\infty(\Lambda^k T^*X)$.

We expect an orthogonal splitting

$$C^\infty(\Lambda^k T^*X)=\mathrm{Im}\,\mathrm{d}\oplus\mathrm{Ker}\,\mathrm{d}^*.$$

(This is not a proof, though.)

Some more notation: write $\mathrm{d}_k, \mathrm{d}_k^*$ for d, d^* acting on k-forms, and $\mathcal{H}^k$ for $\operatorname{Ker} \Delta_{\mathrm{d}}$ on k-forms. Then:

Theorem 5.1 (Hodge Decomposition Theorem)

Let (X, g) be a compact, oriented Riemannian manifold. Then

$$C^\infty(\Lambda^k T^* M) = \mathcal{H}^k \oplus \operatorname{Im}(\mathrm{d}_{k-1}) \oplus \operatorname{Im}(\mathrm{d}_{k+1}^*).$$

Moreover $\operatorname{Ker} \mathrm{d}_k = \mathcal{H}^k \oplus \operatorname{Im}(\mathrm{d}_{k-1})$ *and* $\operatorname{Ker} \mathrm{d}_k^* = \mathcal{H}^k \oplus \operatorname{Im}(\mathrm{d}_{k+1}^*)$.

Hodge's Theorem

So de Rham cohomology satisfies

$$
\begin{aligned}
H^k_{\mathrm{dR}}(X;\mathbb{R}) &= \operatorname{Ker} \mathrm{d}_k / \operatorname{Im} \mathrm{d}_{k-1} \\
&= \big(\mathcal{H}^k \oplus \operatorname{Im}(\mathrm{d}_{k-1})\big) / \operatorname{Im} \mathrm{d}_{k-1} \cong \mathcal{H}^k.
\end{aligned}
$$

This gives *Hodge's Theorem:*

Theorem 5.2 (Hodge's Theorem)

Every de Rham cohomology class on X contains a unique harmonic representative.

So $\mathcal{H}^k$ is finite-dimensional (this also follows as it is the kernel of an elliptic operator on a compact manifold). The Hodge star gives an isomorphism $* : \mathcal{H}^k \to \mathcal{H}^{n-k}$. Thus $H^k_{\mathrm{dR}}(X;\mathbb{R}) \cong H^{n-k}_{\mathrm{dR}}(X;\mathbb{R})$, a form of Poincaré duality.

We defined $\mathcal{H}^k$ as the kernel of $\Delta_{\mathrm{d}} = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. But if $\alpha \in \mathcal{H}^k$ then

$$\begin{aligned} 0 &= \langle \alpha, (\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\alpha \rangle_{L^2} \\ &= \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha \rangle_{L^2} + \langle \mathrm{d}\alpha, \mathrm{d}\alpha \rangle_{L^2} \\ &= \|\mathrm{d}^*\alpha\|_{L^2}^2 + \|\mathrm{d}\alpha\|_{L^2}^2, \end{aligned}$$

so $\|\mathrm{d}^*\alpha\|_{L^2} = \|\mathrm{d}\alpha\|_{L^2} = 0$, and $\mathrm{d}^*\alpha = \mathrm{d}\alpha = 0$. Thus

$$\mathcal{H}^k = \bigl\{ \alpha \in C^\infty(\Lambda^k T^*X) : \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0 \bigr\}.$$

5.2. Hodge theory for compact Kähler manifolds

Now let (X, J, g) be a compact Kähler manifold, with Kähler form ω, of real dimension $2n$. Work now with complex forms, so that $\mathrm{d}_k, \mathrm{d}_k^*$ act on $C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C})$, and $\mathcal{H}^k$ is the kernel of Δ_{d} on complex forms. By the *Kähler identities* (§4.4) we have $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta_{\mathrm{d}}$. But $\Delta_\partial, \Delta_{\bar\partial}$ both take (p,q)-forms to (p,q)-forms, so Δ_{d} also takes (p,q)-forms to (p,q)-forms.

Suppose α is a k-form with $\Delta_{\mathrm{d}}\alpha = 0$, and write $\alpha = \sum_{p+q=k} \alpha_{p,q}$ with $\alpha_{p,q}$ of type (p,q). Then the component of $\Delta_{\mathrm{d}}\alpha = 0$ in type (p,q) is $\Delta_{\mathrm{d}}\alpha_{p,q} = 0$, as Δ_{d} takes (p,q)-forms to (p,q)-forms. So each $\alpha_{p,q}$ lies in $\mathcal{H}^k$. Define $\mathcal{H}^{p,q}$ to be the kernel of Δ_{d} on (p,q)-forms. We have shown that

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}.$$

Here is a version of the Hodge decomposition theorem for the $\bar\partial$ operator on (p,q)-forms. Write $\bar\partial_{p,q}, \bar\partial^*_{p,q}$ for $\bar\partial, \bar\partial^*$ on (p,q)-forms.

Theorem 5.3

Let (X,J,g) be a compact Kähler manifold. Then

$$C^\infty(\Lambda^{p,q}M) = \mathcal{H}^{p,q} \oplus \mathrm{Im}(\bar\partial_{p,q-1}) \oplus \mathrm{Im}(\bar\partial^*_{p,q+1}).$$

Also $\mathrm{Ker}\,\bar\partial_{p,q} = \mathcal{H}^{p,q} \oplus \mathrm{Im}(\bar\partial_{p,q-1})$ *and* $\mathrm{Ker}\,\bar\partial^*_{p,q} = \mathcal{H}^{p,q} \oplus \mathrm{Im}(\bar\partial^*_{p,q+1})$.

So Dolbeault cohomology satisfies

$$\begin{aligned} H^{p,q}_{\bar\partial}(X) &= \operatorname{Ker}\bar\partial_{p,q} / \operatorname{Im}\bar\partial_{p,q-1} \\ &= \bigl(\mathcal{H}^{p,q} \oplus \operatorname{Im}(\bar\partial_{p,q-1})\bigr) / \operatorname{Im}\bar\partial_{p,q-1} \cong \mathcal{H}^{p,q}. \end{aligned}$$

Write $H^{p,q}(X)$ for the subspace of $H^{p+q}(X;\mathbb{C})$ represented by forms in $\mathcal{H}^{p,q}$. Then we have

$$H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{5.1}$$

and $H^{p,q}(X) \cong H^{p,q}_{\bar\partial}(X)$. Hence

$$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X). \tag{5.2}$$

We can describe $H^{p,q}(X)$ as the subspace of $H^{p+q}(X;\mathbb{C})$ represented by closed (p,q)-forms. This is independent of the Kähler metric on X. But (5.1) and (5.2) fail for general compact complex manifolds.

Observe that complex conjugation takes $\mathcal{H}^{p,q}$ to $\mathcal{H}^{q,p}$ and $H^{p,q}(X)$ to $H^{q,p}(X)$. Since $\mathcal{H}^{p,q} \cong H^{p,q}_{\bar\partial}(X)$, this implies that

$$H^{p,q}_{\bar\partial}(X) \cong \overline{H^{q,p}_{\bar\partial}(X)}.$$

This need not be true for general compact complex manifolds; $H^{p,q}_{\bar\partial}(X)$ and $H^{q,p}_{\bar\partial}(X)$ need not have the same dimension. Also $*$ gives

$$* : \mathcal{H}^{p,q} \xrightarrow{\cong} \overline{\mathcal{H}^{n-p,n-q}}.$$

This gives Poincaré duality style isomorphisms

$$H^{p,q}(X) \cong H^{n-p,n-q}(X)^*, \quad H^{p,q}_{\bar\partial}(X) \cong H^{n-p,n-q}_{\bar\partial}(X)^*.$$

The *Betti numbers* of X are $b^k(X) = \dim_{\mathbb{C}} H^k_{\mathrm{dR}}(X;\mathbb{C})$, and the *Hodge numbers* of X are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X)$. From above we have

$$b^k(X) = \sum_{p+q=k} h^{p,q}(X),$$
$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X).$$

So in particular

$$b^{2k+1}(X) = 2\sum_{j=0}^{k} h^{j,2k+1-j}(X).$$

Corollary 5.4

Let (X, J, g) be a compact Kähler manifold. Then the odd Betti numbers $b^{2k+1}(X)$ for $k = 0, 1, \ldots$ are even.

A complex manifold with no Kähler metrics

Let $n>1$, and let $\lambda\in\mathbb{C}$ with $|\lambda|>1$. Let $\mathbb{Z}$ act on $\mathbb{C}^n\setminus\{0\}$ by $d:(z_1,\ldots,z_n)\mapsto(\lambda^d z_1,\ldots,\lambda^d z_n)$. Define $X=(\mathbb{C}^n\setminus\{0\})/\mathbb{Z}$. Then X is a compact complex manifold diffeomorphic to $\mathcal{S}^1\times\mathcal{S}^{2n-1}$. By the Künneth theorem we find that the Betti numbers of X are $b^k(X)=1$ for $k=0,1,2n-1,2n$ and $b^k(X)=0$ otherwise.

Thus $b^1(X)$ and $b^{2n-1}(X)$ are odd. If X had a Kähler metric this would contradict Corollary 5.4. Hence X has no Kähler metrics.

For Dolbeault cohomology, it turns out that $H^{1,0}_{\bar\partial}(X)=0$, but $H^{0,1}_{\bar\partial}(X)\cong\mathbb{C}$, where $\bar\partial\log(|z_1|^2+\cdots+|z_n|^2)$ represents a nontrivial class. So

$$H^{p,q}_{\bar\partial}(X)\not\cong\overline{H^{q,p}_{\bar\partial}(X)}$$

in this example.

5.3. The Kähler cone

Let (X, J) be a compact complex manifold, admitting Kähler metrics. Then we have $H^2_{\mathrm{dR}}(X;\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$. If g is a Kähler metric on (X, J) with Kähler form ω then ω is a real closed (1,1)-form, so that

$$[\omega] \in H^2_{\mathrm{dR}}(X;\mathbb{R}) \cap H^{1,1}(X),$$

with intersection in $H^2_{\mathrm{dR}}(X;\mathbb{C})$.

Definition

Define the *Kähler cone* $\mathcal{K}$ of (X, J) to be the set of all Kähler classes $[\omega]$ of Kähler metrics g on (X, J).

Two important facts about $\mathcal{K}$:

(a) $\mathcal{K}$ is open in $H^2_{\rm dR}(X;\mathbb{R})\cap H^{1,1}(X)$.

(b) $\mathcal{K}$ is a convex cone.

For (a), note that if ω is the Kähler form of g and η is a closed real (1,1)-form with $\|\eta\|_{C^0}<1$, where $\|\,.\,\|_{C^0}$ is computed using g, then $\omega'=\omega+\eta$ is the Kähler form of g'. Hence if $[\omega]\in\mathcal{K}$ and $[\eta]\in H^2_{\rm dR}(X;\mathbb{R})\cap H^{1,1}(X)$ is sufficiently small then $[\omega]+[\eta]\in\mathcal{K}$.
For (b), if g,g' are Kähler metrics on (X,J) and $s,s'>0$ then $sg+s'g'$ is also Kähler. Thus $[\omega],[\omega']\in\mathcal{K}$ implies that $s[\omega]+s'[\omega']\in\mathcal{K}$.

Suppose $\Sigma \subset X$ is a compact complex curve (1-dimensional complex submanifold) in X. Then for any Kähler g, ω we have

$$[\omega] \cdot [\Sigma] = \int_\Sigma \omega = \mathrm{vol}_g(\Sigma) > 0,$$

where $[\Sigma] \in H_2(X;\mathbb{Z})$ is the homology class. Hence

$$\mathcal{K} \subseteq \bigl\{\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{R}) \cap H^{1,1}(X) : \\ \alpha \cdot [\Sigma] > 0, \ \ \Sigma \subset X \text{ curve}\bigr\}.$$

One can often describe $\mathcal{K}$; in simple examples it is a polyhedral cone.

5.4. Lefschetz operators, the Hard Lefschetz Theorem

Let (X, J, g) be compact Kähler, with Kähler form ω. As in §4.4 we have operators on forms

$$L : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k+2} T^*X \otimes_{\mathbb{R}} \mathbb{C}),$$
$$\Lambda : C^\infty(\Lambda^k T^*X \otimes_{\mathbb{R}} \mathbb{C}) \to C^\infty(\Lambda^{k-2} T^*X \otimes_{\mathbb{R}} \mathbb{C}),$$

given by $L(\alpha) = \alpha \wedge \omega$ and $\Lambda = (-1)^k * L *$. These also work on cohomology. Since $[\Delta_{\mathrm{d}}, L] = [\Delta_{\mathrm{d}}, \Lambda] = 0$ by the Kähler identities, L, Λ take $\mathrm{Ker}\,\Delta_{\mathrm{d}}$ to $\mathrm{Ker}\,\Delta_{\mathrm{d}}$. So L maps $\mathcal{H}^k \to \mathcal{H}^{k+2}$, Λ maps $\mathcal{H}^k \to \mathcal{H}^{k-2}$.

Define the *Lefschetz operator*

$$L : H^k_{\mathrm{dR}}(X;\mathbb{C}) \longrightarrow H^{k+2}_{\mathrm{dR}}(X;\mathbb{C})$$

and the *dual Lefschetz operator*

$$\Lambda : H^k_{\mathrm{dR}}(X;\mathbb{C}) \longrightarrow H^{k-2}_{\mathrm{dR}}(X;\mathbb{C})$$

to correspond to $L : \mathcal{H}^k \to \mathcal{H}^{k+2}$ and $\Lambda : \mathcal{H}^k \to \mathcal{H}^{k-2}$ under the isomorphisms $\mathcal{H}^k \cong H^k_{\mathrm{dR}}(X;\mathbb{C})$. Then $L(\alpha) = \alpha \wedge [\omega]$, so L depends only on the Kähler class $[\omega]$ of g. We can reconstruct Λ from L, so Λ also depends only on $[\omega]$. Then L, Λ map

$$L : H^{p,q}(X) \longrightarrow H^{p+1,q+1}(X) \quad \text{and}$$
$$\Lambda : H^{p,q}(X) \longrightarrow H^{p-1,q-1}(X).$$

As for the decomposition of forms on Kähler manifolds in §4.4, we have:

Theorem 5.5 (The Hard Lefschetz Theorem)

Let (X, J, g) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Then $L^k : H^{n-k}_{\mathrm{dR}}(X;\mathbb{C}) \to H^{n+k}_{\mathrm{dR}}(X;\mathbb{C})$ is an isomorphism for $k = 0, \ldots, n$.

Define the primitive cohomology $H^k_0(X;\mathbb{C})$ for $k \leqslant n$ by

$$\begin{aligned} H^k_0(X;\mathbb{C}) &= \mathrm{Ker}\, L^{n-k+1} : \big(H^k_{\mathrm{dR}}(X;\mathbb{C}) \to H^{2n-k+2}_{\mathrm{dR}}(X;\mathbb{C})\big) \\ &= \mathrm{Ker}\big(\Lambda : H^k_{\mathrm{dR}}(X;\mathbb{C}) \to H^{k-2}_{\mathrm{dR}}(X;\mathbb{C})\big). \end{aligned}$$

Then for $k = 0, \ldots, 2n$ we have

$$H^k_{\mathrm{dR}}(X;\mathbb{C}) = \bigoplus_{\substack{j:0\leqslant 2j\leqslant k,\\ k\leqslant n+j}} L^j H^{k-2j}_0(X;\mathbb{C}).$$

The proof is not hard. For the first part, we have $\Delta_{\rm d}(\omega\wedge\alpha)=\omega\wedge(\Delta_{\rm d}\alpha)$, so $\Delta_{\rm d}(\omega^k\wedge\alpha)=\omega^k\wedge(\Delta_{\rm d}\alpha)$. Thus $\omega^k\wedge -$ maps $\mathrm{Ker}\,\Delta_{\rm d}$ to $\mathrm{Ker}\,\Delta_{\rm d}$, that is, $\alpha\mapsto\omega^k\wedge\alpha$ maps $\mathcal{H}^{n-k}$ to $\mathcal{H}^{n+k}$. But $\alpha\mapsto\omega^k\wedge\alpha$ is a (pointwise) isomorphism from $(n-k)$-forms to $(n+k)$-forms, so $\alpha\mapsto\omega^k\wedge\alpha$ is an isomorphism $\mathcal{H}^{n-k}\to\mathcal{H}^{n+k}$. Using isomorphisms $\mathcal{H}^*\cong H^*_{\rm dR}(X;\mathbb{C})$ shows that $L^k:H^{n-k}_{\rm dR}(X;\mathbb{C})\to H^{n+k}_{\rm dR}(X;\mathbb{C})$ is an isomorphism.

The Hodge Conjecture

Let (X, J, g) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$, and $Y \subset X$ a closed complex submanifold with $\dim_{\mathbb{C}} Y = k$. It has a homology class $[Y] \in H_{2k}(X; \mathbb{Q})$. Poincaré duality gives an isomorphism $\mathrm{Pd} : H_*(X; \mathbb{Q}) \to H^{2n-*}(X; \mathbb{Q})$, so

$$\mathrm{Pd}([Y]) \in H^{2n-2k}(X; \mathbb{Q}) \subset H^{2n-2k}(X; \mathbb{C}).$$

As Y is a complex submanifold, $\mathrm{Pd}([Y]) \in H^{n-k,n-k}(X)$. Thus

$$\mathrm{Pd}([Y]) \in H^{2n-2k}(X; \mathbb{Q}) \cap H^{n-k,n-k}(X),$$

where the intersection is taken in $H^{2n-2k}(X; \mathbb{C})$.

The Hodge Conjecture

We can also allow Y to be a complex k-submanifold with singularities — a 'k-cycle'.

Conjecture (The Hodge Conjecture.)

Let (X, J, g) be a projective Kähler 2n-manifold. Then for each $k = 0, \ldots, n$, $H^{2n-2k}(X; \mathbb{Q}) \cap H^{n-k,n-k}(X)$ is spanned over $\mathbb{Q}$ by $\mathrm{Pd}([Y])$ for k-cycles Y in X.

This is known for $k = 0, 1, n-1, n$, and so for $n \leqslant 3$. There is a $1,000,000 prize for proving it.

Complex manifolds and Kähler Geometry

Lecture 6 of 16: Holomorphic vector bundles

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Spring 2022

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Plan of talk:

6.1. Vector bundles

Let X be a real manifold. A (*real*) *vector bundle* $E \to X$ on X of *rank* k is a family of real k-dimensional vector spaces E_x for $x \in X$, depending smoothly on x. Formally, a vector bundle is a manifold E with a projection $\pi : E \to X$ which is a submersion, such that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is given the structure of a real k-dimensional vector space.

This must satisfy the condition (local triviality) that X may be covered by open sets U for which there is a diffeomorphism $\pi^{-1}(U) \cong \mathbb{R}^k \times U$ which identifies $\pi : \pi^{-1}(U) \to U$ with $\pi_U : \mathbb{R}^k \times U \to U$ and the vector space structure on E_u with that on $\mathbb{R}^k \times \{u\}$ for $u \in U$.

Some examples: trivial vector bundles $\mathbb{R}^k \times X \to X$, (co)tangent bundles TX, T^*X, exterior forms $\Lambda^k T^*X$, and tensor bundles $\bigotimes^k TX \otimes \bigotimes^l T^*X$.

A *complex vector bundle* on X is the same, but with fibres E_x complex vector spaces. Note that we will distinguish between complex vector bundles (on any manifold) and holomorphic vector bundles (on a complex manifold).

A (*smooth*) *section* of $E \to X$ is a smooth map $e : X \to E$ with $\pi \circ e \equiv \mathrm{id}_X$. The set $C^\infty(E)$ of smooth sections of E has the structure of an (infinite-dimensional) vector space.

We can add other structures to vector bundles. For example, a *metric h on the fibres of E* is a family of Euclidean metrics h_x on E_x which vary smoothly with x. That is, h is a smooth, positive definite section of S^2E^*. A *connection* ∇ on E is a linear map

$$\nabla : C^\infty(E) \longrightarrow C^\infty(E \otimes T^*X)$$

satisfying the Leibnitz rule

$$\nabla(fe) = f \cdot \nabla e + e \otimes \mathrm{d}f$$

for all $e \in C^\infty(E)$ and smooth $f : X \to \mathbb{R}$. A connection ∇ has *curvature* $F_\nabla \in C^\infty(\mathrm{End}(E) \otimes \Lambda^2 T^*X)$.
We can require ∇ to preserve a metric h on E by
$h(\nabla e_1, e_2) + h(e_1, \nabla e_2) = \mathrm{d}h(e_1, e_2)$ for all $e_1, e_2 \in C^\infty(E)$.

Holomorphic vector bundles

We define holomorphic vector bundles by replacing real manifolds by complex manifolds and smooth maps by holomorphic maps in the definition of real vector bundles. So, if (X, J) is a complex manifold, then a holomorphic vector bundle of rank k is a family of complex k-dimensional vector spaces E_x for $x \in X$ varying holomorphically with x.

Formally, a holomorphic vector bundle is a complex manifold (E, K) with a projection $\pi : E \to X$ which is a holomorphic submersion, such that for each $x \in X$ the fibre $E_x = \pi^{-1}(x)$ is given the structure of a complex k-dimensional vector space, and X may be covered by open sets U for which there is a biholomorphism $\pi^{-1}(U) \cong \mathbb{C}^k \times U$ which identifies $\pi : \pi^{-1}(U) \to U$ with $\pi_U : \mathbb{C}^k \times U \to U$ and the vector space structure on E_u with that on $\mathbb{C}^k \times \{u\}$ for each $u \in U$.

If $E \to X$ is a holomorphic vector bundle, then a map $e : X \to E$ with $\pi \circ e \equiv \mathrm{id}_X$ is called a *smooth section* if e is smooth, and a *holomorphic section* if e is holomorphic. We write $C^\infty(E)$ for the complex vector space of smooth sections of E, and $H^0(E)$ for the complex vector space of holomorphic sections of E.

Algebraic operations on vector spaces have counterparts on holomorphic vector bundles: if E, F are holomorphic vector bundles then the dual E^*, the exterior powers $\Lambda^k E$, the tensor product $E \otimes F$, etc., are all holomorphic vector bundles.

6.2. $\bar\partial$-operators and connections

In terms of real differential geometry, a holomorphic vector bundle E over a complex manifold (X, J) has the structure of a complex vector bundle over the underlying real manifold X. However, it also has more structure: we have a notion of *holomorphic section* of holomorphic vector bundle, but there is no intrinsic notion of when a section of a complex vector bundle is holomorphic.

If $f : X \to \mathbb{C}$ is smooth, then f is holomorphic iff $\bar\partial f = 0$ in $C^\infty(\Lambda^{0,1}X)$. In the same way, if E is a holomorphic vector bundle, there is a natural $\bar\partial$-*operator*

$$\bar\partial_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}X)$$

such that $e \in C^\infty(E)$ is holomorphic iff $\bar\partial_E e = 0$. It satisfies the Leibniz rule

$$\bar\partial_E(fe) = f \cdot \bar\partial_E e + e \otimes_{\mathbb{C}} \bar\partial f$$

for all $e \in C^\infty(E)$ and smooth $f : X \to \mathbb{C}$.

Given $\bar\partial_E$ satisfying the Leibniz rule, there are unique extensions

$$\bar\partial_E^{p,q} : C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q}X) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{p,q+1}X)$$

with $\bar\partial_E = \bar\partial_E^{0,0}$, such that

$$\bar\partial_E^{p,q}(e \otimes \alpha) = \bar\partial_E e \wedge \alpha + e \otimes_{\mathbb{C}} \bar\partial\alpha$$

for $e \in C^\infty(E)$ and $\alpha \in C^\infty(\Lambda^{p,q}X)$.
On a complex manifold we have $\bar\partial^2 = 0$. Similarly, if $\bar\partial_E$ comes from a holomorphic vector bundle then $\bar\partial_E^{p,q+1} \circ \bar\partial_E^{p,q} = 0$ for all p, q.

Thus we can give a *differential-geometric definition* of holomorphic vector bundle: a holomorphic vector bundle on (X, J) is a complex vector bundle $E \to X$ together with a $\bar\partial$-operator

$$\bar\partial_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1} X)$$

satisfying the Leibnitz rule, such that the extensions $\bar\partial_E^{p,q}$ satisfy $\bar\partial_E^{p,q+1} \circ \bar\partial_E^{p,q} = 0$. In fact it is enough that $\bar\partial_E^{0,1} \circ \bar\partial_E = 0$. We define $e \in C^\infty(E)$ to be a *holomorphic section* if $\bar\partial_E e = 0$.

It turns out that this is equivalent to the first definition of holomorphic vector bundle. That is, using $\bar\partial_E$ we can define a unique almost complex structure K on E such that $\pi : E \to X$ is holomorphic, and $K|_{E_x}$ comes from the complex vector space structure of E_x, and the graphs of holomorphic sections are complex submanifolds of (E, K). The condition that the Nijenhuis tensor of K vanishes, so that (E, K) is a complex manifold, is equivalent to $\bar\partial_E^{0,1} \circ \bar\partial_E = 0$.

$\bar\partial$-operators and connections

$\bar\partial$-operators are closely related to connections. Let (X, J) be a complex manifold, $E \to X$ a complex vector bundle, and ∇ a connection on E. Then ∇ is a map

$$\begin{aligned}\nabla : C^\infty(E) \longrightarrow{}& C^\infty\bigl(E \otimes_{\mathbb{R}} T^*X\bigr)\\ \cong{}& C^\infty\bigl(E \otimes_{\mathbb{C}} (T^*X \otimes_{\mathbb{R}} \mathbb{C})\bigr)\\ ={}& C^\infty\bigl(E \otimes_{\mathbb{C}} (\Lambda^{1,0}X \oplus \Lambda^{0,1}X)\bigr)\\ ={}& C^\infty\bigl(E \otimes_{\mathbb{C}} \Lambda^{1,0}X\bigr) \oplus C^\infty\bigl(E \otimes_{\mathbb{C}} \Lambda^{0,1}X\bigr).\end{aligned}$$

So we may write $\nabla = \partial_E \oplus \bar\partial_E$, where

$$\begin{aligned}\partial_E &: C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{1,0}X),\\ \bar\partial_E &: C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,1}X).\end{aligned}$$

As ∇ satisfies a Leibnitz rule, both $\partial_E, \bar\partial_E$ satisfy Leibnitz rules, and $\bar\partial_E$ is a $\bar\partial$-operator. Thus, *a $\bar\partial$-operator is half of a connection*. The condition $\bar\partial_E^{0,1} \circ \bar\partial_E = 0$ for a $\bar\partial$-operator to give a holomorphic vector bundle is a curvature condition. For any $\bar\partial_E$, the operator

$$\bar\partial_E^{0,1} \circ \bar\partial_E : C^\infty(E) \longrightarrow C^\infty(E \otimes_{\mathbb{C}} \Lambda^{0,2} X)$$

is of the form $e \mapsto F_E^{0,2} \cdot e$ for unique $F_E^{0,2} \in C^\infty(\mathrm{End}(E) \otimes_{\mathbb{C}} \Lambda^{0,2} X)$ which we call the $(0,2)$-*curvature*. If $\bar\partial_E$ is half of a connection ∇, then $F_E^{0,2}$ is the $(0,2)$-component of the curvature F_∇.

Let E be a complex vector bundle over (X, J), and h a Hermitian metric on the fibres of E. Then there is a 1-1 correspondence between $\bar\partial$-operators $\bar\partial_E$ on E, and connections $\nabla = \partial_E \oplus \bar\partial_E$ on E preserving h. That is, for each $\bar\partial$-operator $\bar\partial_E$, there is a unique ∂_E so that $\nabla = \partial_E \oplus \bar\partial_E$ preserves h.

Let E be a holomorphic vector bundle on (X, J), with $\bar\partial$-operator $\bar\partial_E$. Choose a Hermitian metric h on E. Then $\bar\partial_E$ extends uniquely to $\nabla = \partial_E \oplus \bar\partial_E$ on E preserving h. Consider the curvature of ∇,

$$F_\nabla \in C^\infty(\mathrm{End}(E) \otimes_{\mathbb{R}} \Lambda^2 T^*X).$$

The $(0, 2)$-component of F_∇ is $F_E^{0,2} = 0$ as E is holomorphic. As ∇ preserves h,

$$F_\nabla \in C^\infty(\mathrm{Herm}^-(E) \otimes_{\mathbb{R}} \Lambda^2 T^*X),$$

where $\mathrm{Herm}^-(E) \subset \mathrm{End}(E)$ are the anti-Hermitian transformations w.r.t. h.

This implies that the $(2,0)$-component of F_∇ is is conjugate to the $(0,2)$-component, so is also zero. Hence F_∇ is of type (1,1).

Thus, every holomorphic vector bundle E on X admits a Hermitian metric h and compatible connection ∇ with F_∇ of type $(1,1)$.

Conversely, if E is a complex vector bundle on X with Hermitian metric h and compatible connection ∇ with F_∇ of type $(1,1)$, then the $\bar{\partial}$-operator of ∇ makes E into a holomorphic vector bundle.

6.3. Chern classes

There is a lot of interesting algebraic topology associated to complex vector bundles – K-theory, Chern classes. (See e.g. Milnor and Stasheff, 'Characteristic classes'.)

If X is a topological space and $E \to X$ is a complex vector bundle of rank k, then the *Chern classes* $c_j(E) \in H^{2j}(X;\mathbb{Z})$ for $j = 1,\ldots,k$ are topological invariants of E.

Let X be a manifold. Choose a Hermitian metric h on E and a connection ∇ on E preserving h. Then $F_\nabla \in C^\infty(\mathrm{Herm}^-(E) \otimes_{\mathbb{R}} \Lambda^2 T^*X)$. There are 'polynomials' $p_1,\ldots,p_k$ in F_∇ such that $p_j(F_\nabla)$ is a closed $2j$-form and $[p_j(F_\nabla)] = c_j(E) \in H^{2j}_{\mathrm{dR}}(X;\mathbb{R})$. To define $p_j(F_\nabla)$, take

$$F_\nabla \wedge \cdots \wedge F_\nabla \in C^\infty\bigl(\mathrm{Herm}^-(E)^{\otimes^j} \otimes \Lambda^{2j} T^*X\bigr),$$

and then apply a natural linear map $\mathrm{Herm}^-(E)^{\otimes^j} \to \mathbb{R}$, which can be thought of as a $\mathrm{U}(k)$-invariant degree j homogeneous polynomial on the Lie algebra $\mathfrak{u}(k)$.

Observe that the cohomology class $[p_j(F_\nabla)]$ is $c_j(E)$, and so is independent of the choice of metric h and connection ∇.

Now suppose E is a holomorphic vector bundle on a complex manifold (X, J). Then as in §6.2 we can choose h and ∇ on E with F_∇ of type $(1,1)$. Therefore $p_j(F_\nabla)$ is a closed form of type (j,j). If (X, J, g) is compact Kähler, this gives $[p_j(F_\nabla)] \in H^{j,j}(X)$. Hence

$$c_j(E) \in H^{2j}(X; \mathbb{Z}) \cap H^{j,j}(X),$$

with intersection in $H^{2j}_{\mathrm{dR}}(X; \mathbb{C})$.

Note the similarity to the Hodge Conjecture in §5.4. This gives obstructions to the existence of holomorphic vector bundles on X: a rank k complex vector bundle E can admit a holomorphic structure only if $c_j(E)$ lies in $H^{j,j}(X)$ for $j = 1, \ldots, k$.

6.4. Holomorphic line bundles

A *holomorphic line bundle* on (X, J) is a rank 1 holomorphic vector bundle, with fibre $\mathbb{C}$. An example: if $\dim_{\mathbb{C}} X = n$ then as T^*X is a holomorphic vector bundle of rank n, the top exterior power $\Lambda^n_{\mathbb{C}} T^*X$ is a holomorphic vector bundle of rank $\binom{n}{n} = 1$, that is, a line bundle. We call $\Lambda^n_{\mathbb{C}} T^*X$ the *canonical bundle* of X, written K_X.

Here T^*X as a holomorphic vector bundle is really $T^{*(1,0)}X$, so K_X is $\Lambda^n_{\mathbb{C}} T^{*(1,0)}X = \Lambda^{n,0}X$. That is, K_X is the holomorphic line bundle of $(n, 0)$-forms on X.

Let $L \to X$ be a holomorphic line bundle. Choose a Hermitian metric h on L. As in §6.2 we get a connection ∇ on L preserving h, with curvature $F_\nabla \in C^\infty(\mathrm{Herm}^-(L) \otimes_{\mathbb{R}} \Lambda^2 T^*X)$ of type (1,1). But as L is a line bundle, there are natural identifications $\mathrm{End}(L) \cong \mathbb{C}$ and $\mathrm{Herm}^-(L) \cong i\mathbb{R} \subset \mathbb{C}$. Thus we have $F_\nabla = i\eta$ for η a real 2-form. In fact η is a closed real (1,1)-form, and $p_1(F_\nabla) = \frac{1}{2\pi}\eta$, so that $[\eta] = 2\pi\, c_1(L)$ in $H^2_{\mathrm{dR}}(X;\mathbb{R})$.

If $\tilde{h}$ is an alternative choice of Hermitian metric on L then $\tilde{h} = e^f \cdot h$ for some smooth $f : X \to \mathbb{R}$. If $\tilde{\nabla}$ and $\tilde{\eta}$ are ∇, η for this $\tilde{h}$ then we find that $\tilde{\eta} = \eta - \frac{1}{2}\mathrm{dd}^c f$.

Let h, ∇, η be as above. If (X, J, g) is compact Kähler, and $\hat\eta$ is a closed real (1,1)-form on X with $[\hat\eta] = 2\pi\, c_1(L)$, then $\hat\eta - \eta$ is an exact real (1,1)-form on X, so $\hat\eta - \eta = -\frac{1}{2}\mathrm{dd}^c f$ for some smooth $f : X \to \mathbb{R}$ by the Global dd^c-Lemma in §4.2, with f unique up to addition of constants. Then $\hat h = e^f \cdot h$ is a Hermitian metric on L yielding $\hat\eta$ as its curvature form. Thus, all closed real (1,1)-forms in the cohomology class $2\pi\, c_1(L)$ can be realized as curvature 2-forms of a metric h on L, uniquely up to rescaling.