Complex manifolds and Kähler Geometry

Lecture 9 of 16: Vanishing theorems and the Kodaira Embedding Theorem

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http://people.maths.ox.ac.uk/~joyce/
Plan of talk:

9 Vanishing theorems and the Kodaira Embedding Theorem

9.1 Vanishing theorems

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9.4 The Kodaira Embedding Theorem
9.1. Vanishing theorems

Let \((X, J)\) be a compact complex manifold, and \(E \rightarrow X\) a holomorphic vector bundle. A \textit{vanishing theorem} says that under some assumptions \(H^q(E) = 0\) for \(q > 0\). Then

\[
\chi(X, E) = \sum_{q=0}^{n} (-1)^q \dim_{\mathbb{C}} H^q(E) = \dim_{\mathbb{C}} H^0(E),
\]

so the Hirzebruch–Riemann–Roch Theorem in §8.2 gives

\[
\dim_{\mathbb{C}} H^0(E) = \int_X \text{ch}(E) \text{td}(X).
\]

So we can compute the number of holomorphic sections of \(E\). This can be a powerful tool.
Definition

Let \((X, J)\) be a complex manifold, and \(L\) a holomorphic line bundle. We call \(L\) positive if \(c_1(L)\) in \(H^2_{dR}(X; \mathbb{R})\) can be represented by a positive closed real \((1,1)\)-form \(\eta\), where \(\eta\) positive means that \(\eta(v, Jv) > 0\) for nonzero vectors \(v\).

Then \(g(v, w) = \eta(v, Jw)\) is a Kähler metric on \(X\), with Kähler form \(\eta\). So if \(X\) has a positive line bundle then \(X\) admits Kähler metrics.

The converse is not true, e.g. there exist Kähler K3 surfaces with no positive line bundles. We call \(L\) negative if \(L^{-1}\) is positive.
Recall from §5.3 that the \textit{Kähler cone} $\mathcal{K}$ of $X$ is the set of Kähler classes of Kähler metrics on $X$, an open convex cone in $H^2_{\text{dR}}(X; \mathbb{R}) \cap H^{1,1}(X)$. A holomorphic line bundle $L \to X$ is positive iff $c_1(L) \in \mathcal{K}$.

From this and §7, we see that if $(X, J)$ is compact and admits Kähler metrics, then $X$ has positive line bundles $L$ iff

$$H^2(X; \mathbb{Z}) \cap \mathcal{K} \neq \emptyset,$$

with intersection in $H^2_{\text{dR}}(X; \mathbb{C})$.

As any element of $H^2(X; \mathbb{Q})$ has a positive multiple in $H^2(X; \mathbb{Z})$, this is equivalent to

$$H^2(X; \mathbb{Q}) \cap \mathcal{K} \neq \emptyset.$$
The tautological line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$ is positive. If $(X, J)$ is a projective complex manifold then $X$ is (isomorphic to) a complex submanifold of some $\mathbb{CP}^n$, and then $\mathcal{O}(1)|_X$ is a positive line bundle on $X$. Thus, all projective complex manifolds admit positive line bundles. The Kodaira Embedding Theorem (later) shows that if a compact complex manifold $(X, J)$ admits positive line bundles, then it is projective.
9.2. The Kodaira and Serre Vanishing Theorems

**Theorem 9.1 (Kodaira Vanishing Theorem)**

*Let $N$ be a positive line bundle on a compact complex manifold $(X, J)$ of complex dimension $n$. Then*

$$H^q(N \otimes \Lambda^p T^*X) = 0 \text{ for } p + q > n.$$  

We sketch a proof. As $N$ is positive, we may choose a positive closed real $(1,1)$-form $\omega$ with $[\omega] = 2\pi c_1(N)$. Let $g$ be the Kähler metric on $X$ with Kähler form $\omega$.

From §6.2 we may choose a Hermitian metric $h$ on $N$, such that if $\nabla$ is the connection on $N$ preserving $h$ and inducing the $\bar{\partial}$-operator $\bar{\partial}_N$ of $N$, then $F_\nabla = -i \omega$. Write $\nabla = \partial_N + \bar{\partial}_N$ and $\nabla^* = \partial_N^* + \bar{\partial}_N^*$. From §8.1 we have

$$H^q(N \otimes \Lambda^p T^*X) \cong \mathcal{H}^{p,q}(N),$$

where $\mathcal{H}^{p,q}(N) = \text{Ker } \Delta_N^{p,q}$. 
As in §4.4 we have operators $L, \Lambda, \partial, \bar{\partial}, \partial^*, \bar{\partial}^*$ on $(p, q)$-forms on $X$ satisfying the Kähler identities. Another identity that we did not mention is that $[\Lambda, L] = (n - (p + q)) \cdot \text{id}$ on $(p, q)$-forms. This extends to $N$-valued $(p, q)$-forms using $\partial_N, \bar{\partial}_N$. Also $[\Lambda, \bar{\partial}] = -i\partial^*$ extends to $N$-valued $(p, q)$-forms. If $\alpha$ is an $N$-valued $(p, q)$-form we have

$$L(\alpha) = \omega \wedge \alpha = iF_N \wedge \alpha$$

$$= i(\nabla \wedge \nabla)\alpha = i(\bar{\partial}_N + \partial_N)^2 \alpha$$

$$= i(\bar{\partial}_N \partial_N + \partial_N \bar{\partial}_N)\alpha.$$
Suppose $\alpha \in \mathcal{H}^{p,q}(N)$, so that $\bar{\partial}_N \alpha = \bar{\partial}_N^* \alpha = 0$. Then
\[
\langle L \circ \Lambda \alpha, \alpha \rangle_{L^2} = \langle i(\bar{\partial}_N \partial_N + \partial_N \bar{\partial}_N) \Lambda \alpha, \alpha \rangle_{L^2} \\
= \langle \partial_N \Lambda \alpha, -i \bar{\partial}_N^* \alpha \rangle_{L^2} + \langle \bar{\partial}_N^* \Lambda \alpha, -i \partial_N^* \alpha \rangle_{L^2} \\
= 0 + \langle \bar{\partial}_N \Lambda \alpha, [\Lambda, \bar{\partial}_N] \alpha \rangle_{L^2} \\
= \langle \bar{\partial}_N \Lambda \alpha, \Lambda \bar{\partial}_N \alpha \rangle_{L^2} - \langle \bar{\partial}_N \Lambda \alpha, \bar{\partial}_N \Lambda \alpha \rangle_{L^2} \\
= 0 - \| \bar{\partial}_N \Lambda \alpha \|^2_{L^2},
\]
using $\bar{\partial}_N^* \alpha = \bar{\partial}_N \alpha = 0$ and $[\Lambda, \bar{\partial}_N] = -i \partial_N^*$. Similarly
\[
\langle \Lambda \circ L \alpha, \alpha \rangle_{L^2} = \| \partial_N \alpha \|^2_{L^2}.
\]
But $[\Lambda, L] = (n - (p + q)) \cdot \text{id}$ on $N$-valued $(p, q)$-forms. Hence
\[
(n - (p + q)) \| \alpha \|^2_{L^2} = \langle [\Lambda, L] \alpha, \alpha \rangle_{L^2} \\
= \| \partial_N \alpha \|^2_{L^2} + \| \bar{\partial}_N \Lambda \alpha \|^2_{L^2}.
\]
If $p + q > n$ then the l.h.s. is $\leq 0$ and the r.h.s. $\geq 0$, so $\alpha = 0$, and $\mathcal{H}^{p,q}(N) = 0$, giving $H^q(N \otimes \Lambda^p T^* X) = 0$ as we want.
In the case \( p = n \) we have \( \Lambda^p T^*X = K_X \) and \( p + q > n \) becomes \( q > 0 \), giving:

**Corollary 9.2**

Suppose \( L \) is a positive line bundle on a compact complex manifold \((X, J)\). Then

\[
H^q(L \otimes K_X) = 0 \quad \text{for all} \quad q > 0.
\]

Equivalently, if \( L \) is a line bundle with \( L \otimes K_X^{-1} \) positive then \( H^q(L) = 0 \) for \( q > 0 \), so that

\[
\dim_{\mathbb{C}} H^0(L) = \int_X \text{ch}(L) \text{td}(X)
\]

by the Hirzebruch–Riemann–Roch Theorem.
A similar proof to the Kodaira Vanishing Theorem yields:

**Theorem 9.3 (Serre Vanishing Theorem)**

Let $L$ be a positive line bundle on a compact complex manifold $(X, J)$, and $E$ any holomorphic vector bundle on $X$. Then there exists $m_0 \in \mathbb{Z}$ such that $H^q(E \otimes L^m) = 0$ for all $q > 0$ and $m \geq m_0$.

This also holds for coherent sheaves $E$, using sheaf cohomology.
Let $E$ be a holomorphic vector bundle of rank $k > 0$, and consider $\chi(X, E \otimes L^m)$ as a function of $m$ in $\mathbb{Z}$. The H–R–R Theorem gives

$$
\chi(X, E \otimes L^m) = \int_X \text{ch}(E \otimes L^m) \, \text{td}(X)
$$

$$
= \int_X \text{ch}(E) \exp(m \, c_1(L)) \, \text{td}(X).
$$

Here $\exp(m \, c_1(L)) = 1 + mc_1(L) + \frac{m^2}{2!} c_1(L)^2 + \cdots + \frac{m^n}{n!} c_1(L)^n$, where $n = \dim_{\mathbb{C}} X$. Thus $\chi(X, E \otimes L^m)$ is a polynomial in $m$ of degree $n$, with leading term

$$
\chi(X, E \otimes L^m) = \frac{k}{n!} \int_X c_1(L)^n \, m^n + \cdots.
$$
As $L$ is positive, $c_1(L)$ is represented by the Kähler form $\omega$ of a Kähler metric $g$ on $X$, and then
\[
\int_X c_1(L)^n = \int_X \omega^n = n! \text{vol}_g(X) > 0.
\]
Thus the leading term of $\chi(X, E \otimes L^m)$ is positive, proving:

**Lemma 9.4**

Let $(X, J)$ be a compact complex manifold, $L$ a positive line bundle on $X$, and $E$ a holomorphic vector bundle on $X$ of positive rank. Then $\chi(X, E \otimes L^m) \gg 0$ for $m \gg 0$. Hence $\dim H^0(E \otimes L^m) \gg 0$ for $m \gg 0$ by the Serre Vanishing Theorem.
9.3. Application to line bundles and divisors

Recall from §7 that if \((X, J)\) is a compact complex manifold then the Picard group \(\text{Pic}(X)\) is the group of holomorphic line bundles up to isomorphism, and \(\text{Div}(X)/\sim\) is the group of divisors on \(X\) up to equivalence, and there is an injective morphism \(\mu : (\text{Div}(X)/\sim) \to \text{Pic}(X)\) whose image is the subgroup of \([L] \in \text{Pic}(X)\) for which \(L\) admits meromorphic sections.

Suppose \(X\) has a positive line bundle \(\tilde{L}\). We will show that any line bundle \(L\) on \(X\) has a meromorphic section. Applying Lemma 9.4 to \(L\) and \(O_X\) shows that \(\dim H^0(L \otimes \tilde{L}^m) \gg 0\) and \(\dim H^0(O_X \otimes \tilde{L}^m) \gg 0\) when \(m \gg 0\). So we can choose \(m \gg 0\) and \(0 \neq s \in H^0(L \otimes \tilde{L}^m), 0 \neq t \in H^0(O_X \otimes \tilde{L}^m)\). Then \(s \otimes t^{-1}\) is a meromorphic section of \((L \otimes \tilde{L}^m) \otimes (O_X \otimes \tilde{L}^m)^* \simeq L\).
This proves:

**Theorem 9.5**

*Suppose $(X, J)$ is a compact complex manifold which admits positive line bundles (equivalently, $(X, J)$ is projective). Then $\mu : (\text{Div}(X) / \sim) \to \text{Pic}(X)$ in §7.4 is an isomorphism.*

As in §7.2, we can describe $\text{Pic}(X)$ very precisely in terms of $H_1(X; \mathbb{Z})$, $H^2(X; \mathbb{Z})$, and $H^{1,1}(X)$. So we get a description of $\text{Div}(X) / \sim$. In particular, this proves the existence of many (possibly singular) complex hypersurfaces in projective complex manifolds. This proves the case $k = n - 1$ of the Hodge Conjecture in §5.4.
The base locus, morphisms to projective spaces

Definition

Let \((X, J)\) be a compact complex manifold, and \(L\) a holomorphic line bundle on \(X\). Then \(H^0(L)\) is a finite-dimensional vector space. The base locus of \(L\) is

\[
B = \{ x \in X : s(x) = 0 \ \forall s \in H^0(L) \}.
\]

It is a closed subset of \(X\), algebraic when \(X\) is algebraic.

Theorem 9.6 (Bertini’s Theorem)

Let \((X, J)\) be a compact complex manifold, and \(L\) a holomorphic line bundle on \(X\). Then for generic \(s \in H^0(L)\), the zeroes \(s^{-1}(0)\) are a smooth hypersurface in \(X\) away from \(B\).
In particular, if $B = \emptyset$, which is often true, then $Y = s^{-1}(0)$ is a compact complex submanifold of $X$ of dimension $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X - 1$, whose homology class $[Y]$ is Poincaré dual to $c_1(L)$. So we can prove the existence of many compact hypersurfaces in $X$, and by induction, of many compact submanifolds of any codimension.

If $L$ has base locus $B$, we can define a natural holomorphic map $\Phi_L : X \setminus B \to \mathbb{P}(H^0(L)^*)$ as follows: for $x \in X \setminus B$, choose an isomorphism $\phi_x : L_x \to \mathbb{C}$, and define $\psi_x : H^0(L) \to \mathbb{C}$ by $\psi_x(s) = \phi_x(s(x))$. Then $\psi_x \in H^0(L)^*$, with $\psi_x \neq 0$ as $x \notin B$, so $[\psi_x] \in \mathbb{P}(H^0(L)^*)$. We define $\Phi_L(x) = [\psi_x]$. This is independent of the choice of $\phi_x$. 
9.4. The Kodaira Embedding Theorem

**Definition**

Let $L$ be a holomorphic line bundle on a compact complex manifold $(X, J)$. We call $L$ *very ample* if the base locus $B$ of $L$ is $\emptyset$, and the map $\Phi_L : X \to \mathbb{P}(H^0(L)^*)$ is an embedding of complex manifolds. We call $L$ *ample* if $L^k$ is very ample for some positive integer $k$.

If $L$ is very ample then choosing a basis for $H^0(L)$ gives an embedding $\Phi_L : X \to \mathbb{C}P^N$, where $N + 1 = \dim H^0(L)$, which identifies $X$ with a complex submanifold of $\mathbb{C}P^N$. One can show that $L \cong \Phi_L^*(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the usual line bundle on $\mathbb{C}P^N$. But $\mathcal{O}(1)$ is a positive line bundle on $\mathbb{C}P^N$, so $\Phi_L^*(\mathcal{O}(1))$ is positive. So any very ample line bundle on $X$ is positive.
Also if $L^k$ is positive for $k > 0$, so that $c_1(L^k)$ is represented by a positive $(1,1)$-form $\omega$, then $c_1(L)$ is represented by $\frac{1}{k}\omega$, so $L$ is positive. Thus, if $L$ is ample, then $L$ is positive. The important Kodaira Embedding Theorem is a converse to this:

**Theorem 9.7 (Kodaira Embedding Theorem)**

*Let $(X, J)$ be a compact complex manifold, and $L$ a positive line bundle on $X$. Then $L$ is ample.*

The proof is complicated. A partial explanation is that as $\dim H^0(L^k) \gg 0$ for $k \gg 0$ by Lemma 9.4, when $k$ is large there are many sections of $L^k$, and these are enough both to force $B = \emptyset$, and to embed $X$ in $\mathbb{P}(H^0(L^k)^*) \cong \mathbb{CP}^N$. 
Given a positive line bundle $L$, a multiple $L^k$ induces an embedding of $X$ in a projective space, giving:

**Corollary 9.8**

Suppose $(X, J)$ is a compact complex manifold admitting positive line bundles. Then $X$ is projective, that is, $X$ is isomorphic to a complex submanifold of $\mathbb{CP}^N$ for some $N \gg 0$. Conversely, if $X \subset \mathbb{CP}^N$ is projective then it admits positive line bundles, e.g. $\mathcal{O}(1)|_X$ is positive.
From §9.1, if $(X, J)$ is a compact complex manifold admitting Kähler metrics, then $X$ admits positive line bundles if and only if

$$H^2(X; \mathbb{Q}) \cap \mathcal{K} \neq \emptyset,$$

with intersection in $H^2_{dR}(X; \mathbb{C})$. So we deduce:

**Corollary 9.9**

*Let $(X, J)$ be a compact complex manifold admitting Kähler metrics. Then $X$ is projective if and only if*

$$H^2(X; \mathbb{Q}) \cap \mathcal{K} \neq \emptyset.$$
In particular, if $H^{2,0}(X) = 0$ then $H^{1,1}(X) = H^2_{dR}(X; \mathbb{C})$, so

$$H^2(X; \mathbb{Q}) \cap H^{1,1}(X) = H^2(X; \mathbb{Q}),$$

which is dense in $H^2(X; \mathbb{R})$. Also $\mathcal{K}$ is a nonempty open set in $H^{1,1}(X) \cap H^2(X; \mathbb{R}) = H^2(X; \mathbb{R})$, so $H^2(X; \mathbb{Q}) \cap \mathcal{K} \neq \emptyset$. Thus we have:

**Corollary 9.10**

*Let $(X, J)$ be a compact complex manifold admitting Kähler metrics with $H^{2,0}(X) = 0$. Then $X$ is projective.*

So under mild conditions, compact Kähler manifolds are projective, and can be studied using complex algebraic geometry.
Complex manifolds and Kähler Geometry

Lecture 10 of 16: Topics on line bundles and divisors

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Plan of talk:

10. Topics on line bundles and divisors

10.1 Finite covers

10.2 The Lefschetz Hyperplane Theorem

10.3 The adjunction formula

10.4 Blow-ups
10.1. Finite covers

From the Kodaira Embedding Theorem, a compact complex manifold is projective iff it admits positive line bundles. We use this to prove:

Proposition 10.1

Let \((X, J)\) be a compact complex manifold, and \((\tilde{X}, \tilde{J})\) a finite cover of \(X\), with covering map \(\pi : \tilde{X} \to X\). Then \(\tilde{X}\) is projective iff \(X\) is projective.
Proof of Proposition 10.1

Suppose $X$ is projective. Then there exists a positive line bundle $L$ on $X$, so $c_1(L)$ is represented by a positive, closed, real $(1,1)$-form $\eta$. The pullback $\pi^*(L)$ has $c_1(\pi^*(L))$ represented by $\pi^*(\eta)$, which is positive as $\pi$ is a local diffeomorphism, so $\pi^*(L)$ is positive, and $\tilde{X}$ is projective.

Conversely, suppose $\tilde{X}$ is projective, so there exists $\tilde{L}$ on $\tilde{X}$ positive, with $c_1(\tilde{L})$ represented by $\tilde{\eta}$ positive.

Define a line bundle $L$ on $X$ to have fibre $L|_x = \bigotimes_{\tilde{x} \in \tilde{X} : \pi(\tilde{x}) = x} \tilde{L}|_{\tilde{x}}$. Then $L$ is holomorphic (it is the determinant line bundle of the push-forward sheaf $\pi_*(\tilde{L})$) and $c_1(L)$ is represented by $\eta$, where

$$\eta|_x = \sum_{\tilde{x} \in X : \pi(\tilde{x}) = x} d\pi_*(\tilde{\eta}|_{\tilde{x}}).$$

This is locally a sum of positive forms, so is positive, and $L$ is positive, and $X$ is projective.
Example: complex tori

Let $n \geq 2$, and consider the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, where $\mathbb{R}^{2n}$ has coordinates $(x_1, \ldots, x_{2n})$. Let $J = J^b_a$ be a complex structure and $g = g_{ab}$ a compatible Kähler metric on $\mathbb{R}^{2n}$ (not necessarily the standard ones), where $J^b_a$ and $g_{ab}$ are constant in coordinates $(x_1, \ldots, x_{2n})$. That is, $J$ is an element of $\text{GL}(2n, \mathbb{R})$ with $J^2 = -1$. The set of such $J$ is $\mathcal{M}_n \cong \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$, a complex manifold with $\dim_{\mathbb{C}} \mathcal{M}_n = n^2$. Then $J, g$ both descend to $T^{2n}$, to make $(T^{2n}, J, g)$ a compact Kähler manifold.

Under what conditions is $(T^{2n}, J)$ projective? Well, if $\alpha \in H^2(T^{2n}; \mathbb{Z}) \cong \mathbb{Z}^{n(2n-1)}$ then $\alpha$ is $c_1(L)$ for a holomorphic line bundle $L$ iff $\pi_{2,0}(\alpha) = 0$, where $\pi_{2,0} : H^2(T^{2n}; \mathbb{Z}) \to H^{2,0}(T^{2n})$ is projection to the $(2,0)$-component in $H^2(T^{2n}; \mathbb{C})$.

We have $H^{2,0}(T^{2n}) \cong \mathbb{C}^{n(n-1)/2}$. So the subset of $J$ for which $(T^{2n}, J)$ has a holomorphic line bundle $L$ with $c_1(L) = \alpha$ is a subvariety $\mathcal{N}_\alpha$ in $\mathcal{M}_n$ of codimension $\frac{1}{2}n(n-1)$. 
Example: complex tori

In particular, \( \mathcal{M}_n \setminus \bigcup_{0 \neq \alpha \in \mathbb{Z}^{n(2n-1)}} \mathcal{N}_\alpha \) is nonempty, and if \( J \) lies in this subset of \( \mathcal{M}_n \) then \((T^{2n}, J)\) has no holomorphic line bundles \( L \) with \( c_1(L) \neq 0 \), so no positive line bundles, and \((T^{2n}, J)\) is not projective. Thus, generic complex tori \((T^{2n}, J)\) for \( n \geq 2 \) are not projective; the family of projective complex tori are of complex codimension \( \frac{1}{2} n(n-1) \) in the family of all complex tori.
Let \((X, J)\) be a compact complex manifold, and \(Y\) a hypersurface in \(X\), that is, \(Y\) is a closed, embedded complex submanifold of \(X\) with \(\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X - 1\). Then \(Y\) is a divisor in \(X\). (We assume \(Y\) is nonsingular, though divisors can be singular). By the correspondence between line bundles and divisors in §7, there exists a line bundle \(L_Y\), and \(s \in H^0(L_Y)\) with \(Y = s^{-1}(0)\), and \(s = 0\) with multiplicity 1 on \(Y\).

How are the cohomologies of \(X\) and \(Y\) related? Well, restriction of \(k\)-forms on \(X\) to \(Y\) induces a map \(\rho : H^k_{dR}(X; \mathbb{C}) \to H^k_{dR}(Y; \mathbb{C})\). If \(X\) admits Kähler metrics then so does \(Y\), and \(H^k_{dR}(X; \mathbb{C})\) splits into \(H^{p,q}(X)\). As the restriction of a \((p, q)\)-form on \(X\) to \(Y\) is a \((p, q)\)-form on \(Y\), we see that \(\rho\) maps \(H^{p,q}(X) \to H^{p,q}(Y)\).

The Lefschetz Hyperplane Theorem gives conditions for these \(\rho\) to be isomorphisms.
Theorem 10.2 (Lefschetz Hyperplane Theorem)

Let \((X, J)\) be a compact complex manifold with \(\dim \mathbb{C} X = n\), and \(Y\) a smooth hypersurface in \(X\). Suppose the induced line bundle \(L_Y\) on \(X\) is positive. Then the restriction maps

\[ \rho : H^k_{dR}(X; \mathbb{C}) \to H^k_{dR}(Y; \mathbb{C}) \]

are isomorphisms for \(k \leq n - 2\) and injective for \(k = n - 1\). Hence \(\rho : H^{p,q}(X) \to H^{p,q}(Y)\) is an isomorphism for \(p + q \leq n - 2\) and injective for \(p + q = n - 1\).

Also, if \(n \geq 3\) then \(\pi_1(X) \cong \pi_1(Y)\).
Sketch proof.

In the case $p = 0$, using sheaf cohomology ideas, one can show that there is a long exact sequence

$$
\cdots \rightarrow H^q(L^*_X) \rightarrow H^{0,q}(X) \xrightarrow{\rho} H^{0,q}(Y) \rightarrow H^{q+1}(L^*_Y) \rightarrow \cdots.
$$

By Serre duality in §8.3 we have $H^q(L^*_Y) \cong H^{n-q}(L_Y \otimes \Lambda^n T^*X)^*$. So by the Kodaira Vanishing Theorem in §9.2 and $L_Y$ positive we have $H^q(L^*_Y) = 0$ for $q < n$. Hence $\rho : H^{0,q}(X) \rightarrow H^{0,q}(Y)$ is an isomorphism for $q < n - 1$, and injective for $q = n - 1$. The case $p > 0$ is more complicated, with two long exact sequences.
The Lefschetz Hyperplane Theorem is a useful computational tool. Usually we use it when we understand the topology of $X$ well, e.g. $X = \mathbb{CP}^n$, and we want to compute $H^*(Y)$. The Lefschetz Hyperplane Theorem gives $H^k(Y) \cong H^k(X)$ for $k < n - 1$. Then Poincaré duality gives $H^k(Y)$ for $k > n - 1$. It remains only to compute $H^{n-1}(Y)$, the middle dimension. For instance, if we can compute $\chi(Y)$ then as we know $b^k(Y)$ for $k \neq n - 1$, we can deduce $b^{n-1}(Y)$. 
Example 10.3

Consider the line bundle $\mathcal{O}(k)$ on $\mathbb{CP}^n$ for $k > 0$. For every $0 \neq z \in \mathbb{C}^{n+1}$, there is a homogeneous order $k$ polynomial $p$ with $p(z) \neq 0$. This corresponds to $s \in H^0(\mathcal{O}(k))$ with $s([z]) \neq 0$. Hence the base locus of $\mathcal{O}(k)$ is empty. Let $s \in H^0(\mathcal{O}(k))$ be generic. Then $s^{-1}(0)$ is smooth by Bertini’s theorem in §9.3.

Let $X = \mathbb{CP}^n$ and $Y = s^{-1}(0)$. The line bundle $L_Y$ is $\mathcal{O}(k)$, which is positive, so the Lefschetz Hyperplane Theorem applies. Hence $H^j(Y; \mathbb{C}) = \mathbb{C}$ if $0 \leq j < n - 1$ is even, and $H^j(Y; \mathbb{C}) = 0$ if $0 \leq j < n - 1$ is odd, and $\pi_1(Y) = \{1\}$ if $n \geq 3$. 
Example 10.4

Let \( X = \mathbb{CP}^1 \times \mathbb{CP}^1 \) and \( Y = \{[1, 0], [0, 1]\} \times \mathbb{CP}^1 \). Then the line bundle \( L_Y \) on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) is \( \mathcal{O}(2, 0) \), where
\[
\mathcal{O}(k, l) = \pi_1^*(\mathcal{O}(k)) \otimes \pi_2^*(\mathcal{O}(l)).
\]
Here \( \mathcal{O}(k, l) \) is positive iff \( k, l > 0 \), so \( \mathcal{O}(2, 0) \) is not positive, and the Lefschetz Hyperplane Theorem does not apply. In fact \( H^0(X; \mathbb{C}) \cong \mathbb{C} \) and \( H^0(Y; \mathbb{C}) \cong \mathbb{C}^2 \), so \( \rho : H^0(X; \mathbb{C}) \to H^0(Y; \mathbb{C}) \) is not an isomorphism, and the conclusions of the Lefschetz Hyperplane Theorem do not hold.
10.3. The adjunction formula

Let \((X, J)\) be a compact complex manifold, and \(Y\) a hypersurface in \(X\), that is, \(Y\) is a closed complex submanifold of \(X\) with \(\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X - 1\). Then \(Y\) induces a holomorphic line bundle \(L_Y\) on \(X\), with a holomorphic section \(s\) vanishing on \(Y\). The normal bundle \(\nu_Y\) of \(Y\) in \(X\) is \(TX|_Y / TY\), a holomorphic line bundle on \(Y\). As \(s|_Y \equiv 0\) but \(\nabla s \neq 0\) on \(Y\), the derivative of \(s\) in the normal directions to \(Y\) gives an isomorphism of line bundles \(ds|_Y : \nu_Y \to L_Y|_Y\).
The adjunction formula

We have an exact sequence of holomorphic vector bundles on $Y$

$$0 \rightarrow TY \rightarrow TX|_Y \rightarrow \nu_Y \rightarrow 0.$$ 

Using $\nu_Y \cong L_Y|_Y$ and dualizing gives

$$0 \rightarrow L^*_Y|_Y \rightarrow T^*X|_Y \rightarrow T^*Y \rightarrow 0.$$ 

Thus taking top exterior powers gives an isomorphism

$$\Lambda^n T^*X|_Y \cong \Lambda^{n-1} T^*Y \otimes L^*_Y|_Y,$$

where $n = \dim \mathbb{C} X$. Therefore

$$K_Y \cong (K_X \otimes L_Y)|_Y. \quad (10.1)$$

This is the adjunction formula.

We often use the adjunction formula when we understand $X$ and $K_X$ – e.g. $X = \mathbb{CP}^n$ – and we want to compute $K_Y$. 

**Example 10.5**

Suppose $Y$ is a smooth degree $k$ hypersurface in $X = \mathbb{CP}^n$. That is, $Y = s^{-1}(0)$ for $s \in H^0(\mathcal{O}(k))$. Then $L_Y \cong \mathcal{O}(k)$. Also $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$, as in §7.2. So the adjunction formula gives

$$K_Y \cong (\mathcal{O}(-n-1) \otimes \mathcal{O}(k))|_Y = \mathcal{O}(k-n-1)|_Y.$$

In particular, if $k = n+1$ then $K_Y \cong \mathcal{O}(0)|_Y \cong \mathcal{O}_Y$, that is, the canonical bundle of $Y$ is trivial. Then $Y$ is called *Calabi–Yau*. So, for example, a smooth quartic in $\mathbb{CP}^3$ is a Calabi–Yau 2-fold (*$K3$ surface*), and a smooth quintic in $\mathbb{CP}^4$ is a Calabi–Yau 3-fold. If $k < n+1$ then $K_Y$ is a negative line bundle (*$Y$ is a *Fano manifold*). If $k > n+1$ then $K_Y$ is a positive line bundle (*$Y$ is of general type*).
10.4. Blow-ups

Let $(X, J)$ be a complex $n$-manifold, and $Y$ a closed, embedded complex $k$-submanifold in $X$. The **blow-up of $X$ along $Y$** is a complex manifold $	ilde{X}$ with a proper holomorphic map $\pi : \tilde{X} \to X$, such that $\pi^{-1}(Y)$ is a smooth, closed hypersurface $D$ in $\tilde{X}$ called the **exceptional divisor**, and $\pi : \tilde{X} \setminus D \to X \setminus Y$ is a biholomorphism. Thus, $\tilde{X}$ is made by cutting the $k$-submanifold $Y$ out of $X$ and replacing it by the $(n - 1)$-submanifold $D$. If $X$ is compact then $\tilde{X}$ is compact.

Blow-ups also work in the worlds of varieties and schemes – basically, singular complex manifolds. One can define the blow-up of a scheme at a closed subscheme, which is another scheme. Blow-ups are often used to resolve singularities. That is, if $X$ is a singular complex manifold (scheme), then by (repeatedly) blowing up $X$ at its singularities, we can define a nonsingular complex manifold $\tilde{X}$. 
The next example defines the blow-up of $\mathbb{C}^n$ at 0.

**Example 10.6**

Let $\tilde{X}$ be the subset of points $((x_1, \ldots, x_n), [y_1, \ldots, y_n])$ in $\mathbb{C}^n \times \mathbb{CP}^{n-1}$ such that $x_j = \lambda y_j$ for $j = 1, \ldots, n$, for some $\lambda \in \mathbb{C}$. That is, either $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$ and $[y_1, \ldots, y_n] = [x_1, \ldots, x_n]$, or $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and $[y_1, \ldots, y_n]$ is arbitrary. Then $\tilde{X}$ is a complex submanifold of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$, with complex dimension $n$. Define $\pi : \tilde{X} \to \mathbb{C}^n$ by

$$
\pi : ((x_1, \ldots, x_n), [y_1, \ldots, y_n]) \mapsto (x_1, \ldots, x_n).
$$

Then $\pi$ is holomorphic. If $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$ then $\pi^{-1}(x_1, \ldots, x_n)$ is the point $((x_1, \ldots, x_n), [x_1, \ldots, x_n])$. Also $\pi^{-1}(0) = \{0\} \times \mathbb{CP}^{n-1}$ is a smooth hypersurface $D$ in $\tilde{X}$, and $\pi : \tilde{X} \setminus D \to \mathbb{C}^n \setminus \{0\}$ is biholomorphic. The other projection $\pi_2 : \tilde{X} \to \mathbb{CP}^{n-1}$ identifies $\tilde{X}$ with the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^{n-1}$.
In the same way, the blow-up $\tilde{X}$ of a complex manifold $X$ at a point $x$ replaces $x$ by the projective space $D = \mathbb{P}(T_Xx)$. The blow-up $\tilde{X}$ of $X$ along a complex submanifold $Y$ replaces $Y$ by $D = \mathbb{P}(\nu)$, where $\nu = T_X|_Y/TY$ is the normal bundle of $Y$ in $X$. That is, we have $\pi : D \to Y$ with $\pi^{-1}(y) = \mathbb{P}(T_yX/T_yY)$ for $y \in Y$. 
We can consider holomorphic line bundles on blow-ups. If $\tilde{X}$ is the blow-up of $X$ along $Y$, with exceptional divisor $D$, and $L \to X$ is a holomorphic line bundle on $X$, then $\pi^*(L)$ is a holomorphic line bundle on $\tilde{X}$.

We also have the holomorphic line bundle $L_D$ on $\tilde{X}$ associated to $D$. A calculation similar to the adjunction formula shows that

$$K_{\tilde{X}} \cong L_D^{n-k-1} \otimes \pi^*(K_X),$$

where $n = \dim_{\mathbb{C}} X$, $k = \dim_{\mathbb{C}} Y$.

**Proposition 10.7**

*Suppose $(X, J)$ is a compact complex manifold, $Y$ a closed complex submanifold in $X$, $\pi : \tilde{X} \to X$ the blow-up of $X$ along $Y$ with exceptional divisor $D$, and $L$ a positive line bundle on $X$. Then $L_D^{-1} \otimes \pi^*(L)^k$ is a positive line bundle on $\tilde{X}$ for $k \gg 0$.*
Sketch proof.

The projection $\pi : D \to Y$ has fibre $\mathbb{C}P^{n-k-1}$ over $y \in Y$, where $n = \dim_{\mathbb{C}} X$, $k = \dim_{\mathbb{C}} Y$. One can show that $L_D|_{\pi^{-1}(y)}$ is the line bundle $\mathcal{O}(-1) \to \mathbb{C}P^{n-k-1}$. Thus $L^{-1}_D|_{\pi^{-1}(y)}$ is $\mathcal{O}(1)$, which is positive. We can choose a closed real $(1,1)$-form $\eta$ on $\tilde{X}$ representing $c_1(L^{-1}_D)$, such that $\eta|_{\pi^{-1}(y)}$ is positive on $\pi^{-1}(y) \cong \mathbb{C}P^{n-k-1}$ for each $y \in Y$.

As $L$ is positive on $X$ we can choose a closed, real, positive $(1,1)$-form $\zeta$ on $X$ representing $c_1(L)$. Then $\pi^*(\zeta)$ represents $c_1(\pi^*(L))$, and $\eta + k\pi^*(\zeta)$ represents $c_1(L^{-1}_D \otimes \pi^*(L)^k)$. We claim $\eta + k\pi^*(\zeta)$ is a positive $(1,1)$-form for $k \gg 0$, so that $L^{-1}_D \otimes \pi^*(L)^k$ is positive. To see this, note that $\pi^*(\zeta)$ is nonnegative on $\tilde{X}$, and zero only on the tangent bundles of $\pi^{-1}(y)$ for $y \in Y$; also, $\eta$ is positive on the tangent bundles of $\pi^{-1}(y)$, though it may be negative in other directions.
By a corollary of the Kodaira Embedding Theorem, a compact complex manifold is projective iff it admits positive line bundles. So we deduce.

**Corollary 10.8**

Let \((X, J)\) be a projective complex manifold, \(Y\) a closed complex submanifold of \(X\), and \(\tilde{X}\) the blow-up of \(X\) along \(Y\). Then \(\tilde{X}\) is projective.