Orientations on moduli spaces of coherent sheaves on Calabi–Yau 4-folds. I

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These slides available at
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References

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Let X be a projective Calabi–Yau 4-fold over $\mathbb{C}$, and $\boldsymbol{\mathcal{M}}$ be a derived moduli scheme or stack of coherent sheaves (or perfect complexes) on X, in the sense of Toën–Vezzosi. Then $\boldsymbol{\mathcal{M}}$ has a -2-*shifted symplectic structure* ω in the sense of Pantev–Toën–Vaquié–Vezzosi arXiv:1111.3209. Writing $\mathbb{L}_{\boldsymbol{\mathcal{M}}}, \mathbb{T}_{\boldsymbol{\mathcal{M}}}$ for the (co)tangent complex of $\boldsymbol{\mathcal{M}}$, the inner product with ω gives a quasi-isomorphism $\omega\cdot : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[-2]$. This is a geometric incarnation of Serre duality: for a point $E \in \boldsymbol{\mathcal{M}}$ corresponding to $E \in \mathrm{coh}(X)$, we have $H^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}}|_E) \cong \mathrm{Ext}^{i+1}(E,E)$ and $H^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}|_E) \cong \mathrm{Ext}^{1-i}(E,E)^*$, and the isomorphism $\mathbb{T}_{\boldsymbol{\mathcal{M}}} \cong \mathbb{L}_{\boldsymbol{\mathcal{M}}}[-2]$ corresponds at E to the Serre duality isomorphism $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{4-i}(E,E)^*$, since $K_X \cong \mathcal{O}_X$.

Let $(\mathcal{S},\omega)$ be a -2-shifted symplectic derived $\mathbb{C}$-scheme. Brav–Bussi–Joyce arXiv:1305.6302 proved a 'Darboux Theorem' showing that $(\mathcal{S},\omega)$ is Zariski locally described by charts of the form (V,E,Q,s), where V is a smooth $\mathbb{C}$-scheme, $E\rightarrow V$ a vector bundle, $Q\in H^0(S^2E^*)$ is a non-degenerate quadratic form on E, and $s\in H^0(E)$ is a section of E which is isotropic, that is, $Q(s,s)=0$. Then $\mathcal{S}$ is locally modelled on $X=s^{-1}(0)\subset V$, with

$$\mathbb{L}_{\mathcal{S}}|_X\simeq\ \bigl[\underset{-2}{TV|_X}\xrightarrow{\mathrm{d}s}\underset{-1}{E|_X}\xrightarrow{(\mathrm{d}s)^*\circ Q}\underset{0}{T^*V|_X}\bigr].$$

Borisov–Joyce arXiv:1504.00690 defined a notion of *orientation* for a -2-shifted symplectic derived $\mathbb{C}$-scheme or stack $(\mathcal{S},\omega)$. The equivalence $\omega\cdot\ :\mathbb{T}_{\mathcal{S}}\rightarrow\mathbb{L}_{\mathcal{S}}[-2]$ induces an isomorphism of determinant line bundles $\det\omega:\det\mathbb{T}_{\mathcal{S}}\rightarrow\det\mathbb{L}_{\mathcal{S}}$, where $\det\mathbb{T}_{\mathcal{S}}=(\det\mathbb{L}_{\mathcal{S}})^*$. An orientation is an isomorphism $\phi:\mathcal{O}_{\mathcal{S}}\rightarrow\det\mathbb{L}_{\mathcal{S}}$ with $\det\omega=\phi\circ\phi^*$. On a chart (V,E,Q,s), this corresponds to an orientation of the quadratic form (E,Q) on $s^{-1}(0)$.

Borisov–Joyce showed that if $(\boldsymbol{\mathcal{S}},\omega)$ is a separated -2-shifted symplectic derived $\mathbb{C}$-scheme, we can give the complex analaytic space $\mathcal{S}_{\rm an}$ the structure of a C^∞ *derived manifold* $\mathcal{S}_{\rm dm}$, of dimension $\mathrm{vdim}_{\mathbb{R}}\,\mathcal{S}_{\rm dm}=\mathrm{vdim}_{\mathbb{C}}\,\boldsymbol{\mathcal{S}}=\frac{1}{2}\,\mathrm{vdim}_{\mathbb{R}}\,\boldsymbol{\mathcal{S}}$. Orientations for $(\boldsymbol{\mathcal{S}},\omega)$ correspond to orientations of $\mathcal{S}_{\rm dm}$. If $\boldsymbol{\mathcal{S}}$ is also proper then $\mathcal{S}_{\rm dm}$ is compact, and has a *virtual class* $[\mathcal{S}_{\rm dm}]_{\rm virt}\in H_{\mathrm{vdim}_{\mathbb{C}}\boldsymbol{\mathcal{S}}}(\mathcal{S}_{\rm an},\mathbb{Z})$. They proposed to use this to define Donaldson–Thomas type 'DT4 invariants' of Calabi–Yau 4-folds, 'counting' semistable moduli schemes $\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ of coherent sheaves on a Calabi–Yau 4-fold X. For an oriented -2-shifted symplectic derived $\mathbb{C}$-scheme $(\boldsymbol{\mathcal{S}},\omega)$, Oh–Thomas arXiv:2009.05542 gave an alternative definition of the virtual class $[\boldsymbol{\mathcal{S}}]_{\rm virt}$ in Chow homology $A_{\frac{1}{2}\mathrm{vdim}_{\mathbb{C}}\boldsymbol{\mathcal{S}}}(S)$, in the style of Behrend–Fantechi. In charts (V,E,Q,s), this involves taking the Euler class of (E,Q), and showing it can be localized to $s^{-1}(0)$. DT4 invariants are now a very active field, see work by Bojko, Cao, Kiem, Kool, Leung, Maulik, Oberdieck, Park, Toda, ...

These two talks will discuss the following:

Question

Let X be a projective Calabi–Yau 4-fold over $\mathbb{C}$, and $\boldsymbol{\mathcal{M}}$ the moduli stack of coherent sheaves (or perfect complexes) on X, with its -2-shifted symplectic structure ω. Is $(\boldsymbol{\mathcal{M}},\omega)$ orientable, in the sense of Borisov–Joyce? If so, maybe after choosing some data on X, is there some way to construct a canonical orientation on $\boldsymbol{\mathcal{M}}$?

This is important as without such orientations, we cannot define DT4 invariants of Calabi–Yau 4-folds.

It makes sense to study orientations on the full moduli stack $\boldsymbol{\mathcal{M}}$, and then restrict them to the substacks $\boldsymbol{\mathcal{M}}_\alpha^{\rm st}(\tau)\subset\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)\subset\boldsymbol{\mathcal{M}}$ of Gieseker (semi)stable sheaves in Chern character α. If $\boldsymbol{\mathcal{M}}_\alpha^{\rm st}(\tau)=\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ then $\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ is a proper -2-shifted symplectic derived $\mathbb{C}$-scheme, so given an orientation, we can form a virtual class $[\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)]_{\rm virt}$, and use this to define DT4 invariants as $\int_{[\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)]_{\rm virt}}\Phi$ for Φ some natural cohomology class on $\boldsymbol{\mathcal{M}}$.

The Cao–Gross–Joyce orientability theorem is wrong!

Theorem (Cao–Gross–Joyce 2020)

Let X be a compact Calabi–Yau 4-fold. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X is orientable.

Unfortunately, there is a mistake in the proof. The theorem itself may be false, though we don't have a counterexample. I apologize for this.

Outline of proof in Cao–Gross–Joyce:

Step 1: Let $P \to X$ be a principal $\mathrm{U}(m)$-bundle, $m \geqslant 4$. Define moduli spaces $\mathcal{B}_P$ of all connections on P. Define a principal $\mathbb{Z}_2$-bundle $O_P \to \mathcal{B}_P$ of orientations on $\mathcal{B}_P$, using gauge theory. Prove O_P is trivializable, that is, $\mathcal{B}_P$ is orientable. (This proof wrong.)

If X is a $\mathrm{Spin}(7)$-manifold, orientations of $\mathcal{B}_P$ restrict to orientations of moduli spaces $\mathcal{M}_P$ of $\mathrm{Spin}(7)$-*instantons* on P.

Step 2: Define map of topological classifying spaces $\Psi : \boldsymbol{\mathcal{M}}^{\mathrm{cla}}_{\mathrm{ch}=\mathrm{ch}\,P} \to \mathcal{B}^{\mathrm{cla}}_P$. Show orientations of $\mathcal{B}_P$ pull back along Ψ to orientations of $\boldsymbol{\mathcal{M}}_{\mathrm{ch}=\mathrm{ch}\,P}$. Hence $\mathcal{B}_P$ orientable implies $\boldsymbol{\mathcal{M}}$ orientable. (This proof is correct, as far as we know.)

Gauge theory moduli spaces and orientations

Let X be a compact manifold, G a Lie group, and $P \to X$ a principal G-bundle. Write $\mathcal{A}_P$ for the moduli space of all connections ∇ on P, an infinite-dimensional affine space, and $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$ for the moduli space of connections on P modulo gauge transformations, as a topological stack, where $\mathcal{G}_P = \mathrm{Aut}(P)$. Let $E^\bullet = \bigl(D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)\bigr)$ be an elliptic operator on X, for example, the Dirac operator on X if X is spin. Then for each $\nabla \in \mathcal{A}_P$ we have a twisted elliptic operator $D_\nabla : \Gamma^\infty(E_0 \otimes \mathrm{ad}(P)) \to \Gamma^\infty(E_1 \otimes \mathrm{ad}(P))$. There is a determinant line bundle $\hat{L}_P \to \mathcal{A}_P$ with fibre $\det D_\nabla = \det \mathrm{Ker}(D_\nabla) \otimes \det \mathrm{Coker}(D_\nabla)^*$ at $\nabla \in \mathcal{A}_P$, and a principal $\mathbb{Z}_2$-bundle $\hat{O}_P \to \mathcal{A}_P$ of orientations on the fibres of $\hat{L}$. These are $\mathcal{G}_P$-equivariant, and descend to $L_P \to \mathcal{B}_P$ and $O_P \to \mathcal{B}_P$. An *orientation* on $\mathcal{B}_P$ is an isomorphism $O_P \cong \mathcal{B}_P \times \mathbb{Z}_2$. Moduli spaces $\mathcal{M}_P$ of 'instantons' – connections on P satisfying a curvature condition – are subspaces $\mathcal{M}_P \subset \mathcal{B}_P$. In good cases, $\mathcal{M}_P$ is a smooth manifold, and $O_P|_{\mathcal{M}_P}$ is the principal $\mathbb{Z}_2$-bundle of orientations on $\mathcal{M}_P$ in the usual sense. So orientability / orientations for $\mathcal{B}_P$ give orientability / orientations for $\mathcal{M}_P$.

How to fix the mistake in Cao–Gross–Joyce

Markus Upmeier and myself have developed a new theory for studying orientability and canonical orientations for moduli spaces $\mathcal{B}_P$, where X is a compact spin n-manifold with $n \equiv 1, 7, 8 \mod 8$, and G is a Lie group, and $P \to X$ is a principal G-bundle, and $\mathcal{B}_P$ is the moduli space (topological stack) of all connections ∇ on P, and orientations on $\mathcal{B}_P$ mean orientations of the (positive) Dirac operator on X twisted by $(\mathrm{ad}(P), \nabla)$. If X is a $\mathrm{Spin}(7)$-manifold, orientations on $\mathcal{B}_P$ restrict to orientations on moduli spaces of $\mathrm{Spin}(7)$-instantons on X. If X is a Calabi–Yau 4-fold and $G = \mathrm{U}(m)$, orientations on $\mathcal{B}_P$ restrict to Borisov–Joyce orientations on moduli spaces of rank m algebraic vector bundles on X.

When $n = 8$ (also $n = 7$) we give sufficient conditions on X for orientability of $\mathcal{B}_P$ for many G, including $G = \mathrm{U}(m)$ (necessary and sufficient if $G = E_8$). If these sufficient conditions hold, the problem with Step 1 of Cao–Gross–Joyce is fixed, and we deduce the Cao–Gross–Joyce orientability theorem under this extra condition. We also specify data (a *flag structure*) which determines canonical orientations.

2. First look at the methods in the proof

A principal G-bundle $P \to X$ is topologically equivalent to a map $\phi_P : X \to BG$, where BG is the classifying space of X. Thus $[X, \phi_P]$ is an element of the *spin bordism group* $\Omega_n^{\mathrm{Spin}}(BG)$. Orientability of $\mathcal{B}_P$ depends on the monodromy of $O_P \to \mathcal{B}_P$ around a loop $\gamma : \mathcal{S}^1 \to \mathcal{B}_P$. Then γ is equivalent to a principal G-bundle $Q \to X \times \mathcal{S}^1$, giving a map $\phi_Q : X \times \mathcal{S}^1 \to BG$, and a spin bordism class $[X \times \mathcal{S}^1, \phi_Q]$ in $\Omega_{n+1}^{\mathrm{Spin}}(BG)$. Now ϕ_Q is equivalent to a map $\psi_Q : X \to \mathcal{L}BG$, where $\mathcal{L}BG$ is the loop space of BG, so Q determines a bordism class $[X, \psi_Q]$ in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$, and $[X \times \mathcal{S}^1, \phi_Q]$ is the image of $[X, \psi_Q]$ under a natural map $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathrm{Spin}}(BG)$.

It turns out that orientation problems for $\mathcal{B}_P$ factor via $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ in a certain sense. For given X, we can show that $\mathcal{B}_P$ is orientable for all principal G-bundles $P \to X$ if and only if certain 'bad' classes α in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ cannot be written $\alpha = [X, \psi]$. If there are no bad classes we get orientability for all X, P (this often happens for $n = 7$). We need to compute $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ using algebraic topology.

If $\iota : G \to H$ is a morphism of Lie groups of 'complex type', and $P \to X$ is a principal G-bundle, then $Q = (P \times H)/G$ is a principal H-bundle, and an orientation for $\mathcal{B}_Q$ induces one for $\mathcal{B}_P$. Using complex type morphisms $\mathrm{SU}(8) \hookrightarrow E_8$ and $\mathrm{SU}(m) \hookrightarrow \mathrm{SU}(m')$ for $m \leqslant m'$, we can show that if X is a spin 8-manifold then orientability of $\mathcal{B}_Q$ for all principal E_8-bundles $Q \to X$ implies orientability of $\mathcal{B}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$. Thus, to solve the CY4 orientability problem, it is enough to understand orientability for E_8-bundles.

There is a 16-connected map $BE_8 \to K(\mathbb{Z}, 4)$, where $K(\mathbb{Z}, 4)$ is the Eilenberg–MacLane space classifying $H^4(-, \mathbb{Z})$, so $\Omega_n^{\mathrm{Spin}}(BE_8) \cong \Omega_n^{\mathrm{Spin}}(K(\mathbb{Z}, 4))$ for $n < 16$, and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BE_8) \cong \Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z}, 4))$ for $n < 15$. Using this, we can reduce orientability questions for E_8-bundles to conditions that can be computed using *cohomology* and *cohomology operations* on X, in particular Steenrod squares. The proofs involve lots of complicated calculations of bordism groups in Algebraic Topology, spectral sequences, etc.

3. Statement of main results: orientability

I'll explain only results in 8 dimensions relevant to DT4 invariants, and a bit extra on $\mathrm{Spin}(7)$ instantons. They are part of a bigger theory, which also includes results on orientability of moduli spaces of submanifolds, such as Cayley 4-folds in $\mathrm{Spin}(7)$-manifolds.

Let X be a compact oriented spin 8-manifold. Impose the condition:

(*) Let $\alpha \in H^3(X,\mathbb{Z})$, and write $\bar\alpha \in H^3(X,\mathbb{Z}_2)$ for its mod 2 reduction, and $\mathrm{Sq}^2(\bar\alpha) \in H^5(X,\mathbb{Z}_2)$ for its Steenrod square. Then $\int_X \bar\alpha \cup \mathrm{Sq}^2(\bar\alpha) = 0$ in $\mathbb{Z}_2$ for all $\alpha \in H^3(X,\mathbb{Z})$.

Theorem 1

Suppose X satisfies condition $(*)$, and let G be a compact Lie group on the list, for all $m \geqslant 1$

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m). \tag{1}$$

Then $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$. For $G = E_8$, this holds **if and only if** $(*)$ *holds.*

We do this by applying our general orientability theory for $G = E_8$ by studying $\Omega_n^{\mathrm{Spin}}(K(\mathbb{Z},4))$ and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z},4))$. The other cases are deduced from $G = E_8$ using complex type morphisms.

The case $G = \mathrm{U}(m)$ and Step 2 of Cao–Gross–Joyce implies:

Corollary 2

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X is orientable in the sense of Borisov–Joyce 2017.

Example

Let $X \subset \mathbb{CP}^5$ be a smooth sextic. Then $H^3(X, \mathbb{Z}) = 0$ by the Lefschetz Hyperplane Theorem. So $(*)$ and Corollary 2 hold.

Corollary 3

Suppose a compact $\mathrm{Spin}(7)$-manifold (X, Ω) satisfies condition $(*)$, and G lies on the list (1), and $P \to X$ is a principal G-bundle. Then the moduli space $\mathcal{M}_P^{\mathrm{irr}}$ of irreducible $\mathrm{Spin}(7)$-instanton connections on P is orientable. (Here $\mathcal{M}_P^{\mathrm{irr}}$ is a smooth manifold if Ω is generic, and a derived manifold otherwise.)

4. Statement of main results: canonical orientations

Suppose now that $(*)$ holds, so we have orientability of moduli spaces $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$ on X. What extra choices do we need to make on X to define *canonical orientations* on $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$?

Definition

Let X be a spin 8-manifold, and $P \to X$ a principal G-bundle, and $O_P \to \mathcal{B}_P$ be the orientation bundle. Define the *normalized orientation bundle* $\check{O}_P \to \mathcal{B}_P$ by $\check{O}_P = O_P \otimes_{\mathbb{Z}_2} \mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$, where $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is the $\mathbb{Z}_2$-torsor of orientations of $\mathcal{B}_{X\times G}$ for the trivial G-bundle $X \times G \to X$ at the trivial connection ∇_0. A trivialization of $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is an orientation for $\mathrm{ind}(\not{D}_X^+) \otimes \mathfrak{g}$, where $\not{D}_X^+$ is the positive Dirac operator of X, $\mathrm{ind}(\not{D}_X^+)$ its orientation torsor as a Fredholm operator, $\mathfrak{g}$ the Lie algebra of G.

We show normalized orientations on $\mathcal{B}_P$ are determined by a choice of *flag structure* (next slide). Orientations on $\mathcal{B}_P$ also need an orientation on $\mathrm{ind}(\not{D}_X^+)\otimes\mathfrak{g}$. If X is a Calabi–Yau 4-fold, there is a natural orientation for $\mathrm{ind}(\not{D}_X^+)$, so we don't need this second choice.

Flag structures – first idea

Joyce 2018 and Joyce–Upmeier 2023 introduced *flag structures* on 7-manifolds, and used them to define orientations on moduli spaces of associative 3-folds and G_2-instantons on compact G_2-manifolds.

We define a related (but more complicated) notion of flag structure F for compact spin 8-manifolds X satisfying condition $(*)$, as a choice of natural trivialization of an orientation functor associated to X (more details later). We can write a flag structure F as $(F_\alpha : \alpha \in H^4(X,\mathbb{Z}))$, where each F_α lies in a $\mathbb{Z}_2$-torsor. Thus, the set of flag structures on X is a torsor for $\mathrm{Map}(H^4(X,\mathbb{Z}),\mathbb{Z}_2)$. By imposing extra conditions we can cut this down to a finite choice of flag structures.

If X is a Calabi–Yau 4-fold, the orientation on $\boldsymbol{\mathcal{M}}$ at a perfect complex $[\mathcal{E}^\bullet] \in \boldsymbol{\mathcal{M}}$ depends on F_α for $\alpha = c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2$. There is a canonical choice for F_0. Hence, if $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, there is a canonical choice of orientation on the connected component of $\boldsymbol{\mathcal{M}}$ containing $\mathcal{E}^\bullet$. Thus we deduce:

Theorem 4

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Choose a flag structure F on X. Then we can construct a canonical orientation on the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X. On the open and closed substack $\boldsymbol{\mathcal{M}}_{c_2-c_1^2=0} \subset \boldsymbol{\mathcal{M}}$ of perfect complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, we can define the canonical orientation without choosing a flag structure.

The second part resolves a paradox. There are several conjectures in the literature by Bojko, Cao, Kool, Maulik, Toda, . . . , of the form

$$\text{Conventional invariants of } X \simeq \text{DT4 invariants of } X, \qquad (2)$$

where the left hand side, involving Gromov–Witten invariants etc., needs no choice of orientation, but the right hand side needs a Borisov–Joyce orientation to determine the sign. All these conjectures are really about sheaves on points and curves — Hilbert schemes of points, MNOP, DT-PT, etc. — and so involve only complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$ in $H^4(X, \mathbb{Z})$.

5. Picard groupoids and bordism categories

Definition

A *Picard groupoid* $(\mathcal{G}, \otimes, \mathbb{1})$ is a groupoid $\mathcal{G}$ with a monoidal structure $\otimes : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ which is symmetric and associative up to coherent natural isomorphisms (not included in the notation), and an identity object $\mathbb{1}$ such that $\mathbb{1} \otimes X \cong X \otimes \mathbb{1} \cong X$ for all $X \in \mathcal{G}$, such that for every $X \in \mathcal{G}$ there exists $Y \in \mathcal{G}$ with $X \otimes Y \cong \mathbb{1}$.

Picard groupoids are classified up to equivalence by triples (π_0, π_1, q), where π_0, π_1 are abelian groups and $q : \pi_0 \to \pi_1$ is a map which is both linear and quadratic. To $(\mathcal{G}, \otimes, \mathbb{1})$ we associate the abelian groups π_0 of isomorphism classes $[X]$ of objects $X \in \mathcal{G}$ with multiplication $[X] \cdot [Y] = [X \otimes Y]$, and $\pi_1 = \mathrm{Aut}_{\mathcal{G}}(\mathbb{1})$.
Symmetric monoidal functors $F : (\mathcal{G}, \otimes, \mathbb{1}) \to (\mathcal{G}', \otimes, \mathbb{1}')$ are functors $F : \mathcal{G} \to \mathcal{G}'$ preserving all the structure. They are classified up to monoidal natural isomorphism by group morphisms $f_0 : \pi_0 \to \pi_0'$ and $f_1 : \pi_1 \to \pi_1'$ with $q' \circ f_0 = f_1 \circ q$.
We could call Picard groupoids *abelian 2-groups*, as they are a 2-categorical notion of abelian group.

Our theory uses special examples of Picard groupoids we call *bordism categories*. Here is an example.

Example

Let G be a Lie group, and $n \geqslant 0$. Define a Picard groupoid $\mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ to have objects pairs (X, P) of a compact spin n-manifold X and a principal G-bundle $P \to X$, and morphisms $[Y, Q] : (X_0, P_0) \to (X_1, P_1)$ to be equivalence classes $[Y, Q]$ of a compact spin $(n+1)$-manifold Y with boundary $\partial Y = -X_0 \amalg X_1$ and a principal G-bundle $Q \to Y$ with $Q|_{\partial Y} = P_0 \amalg P_1$, where the equivalence involves $(n+2)$-dimensional bordisms. The composition of $[Y, Q] : (X_0, P_0) \to (X_1, P_1)$ and $[Y', Q'] : (X_1, P_1) \to (X_2, P_2)$ is $[Y \amalg_{X_1} Y', Q \amalg_{P_1} Q']$. The monoidal structure is disjoint union, $(X, P) \otimes (X', P') = (X \amalg X', P \amalg P')$.

The classifying data is $\pi_0 = \Omega_n^{\mathbf{Spin}}(BG)$, $\pi_1 = \Omega_{n+1}^{\mathbf{Spin}}(BG)$, and $q : [X, P] \mapsto [X \times \mathcal{S}^1_{\rm nb}, P \times \mathcal{S}^1_{\rm nb}]$, where $\mathcal{S}^1_{\rm nb}$ is $\mathcal{S}^1$ with the non-bounding spin structure. Here $\Omega_*^{\mathbf{Spin}}(-)$ is *spin bordism*, a generalized homology theory, and BG is the classifying space.

Example

The groupoid $\mathbb{Z}_2\text{-tor}$ of $\mathbb{Z}_2$-*torsors* is a Picard groupoid with $\pi_0 = 0$ and $\pi_1 = \mathbb{Z}_2$.
The groupoid $\text{s-}\mathbb{Z}_2\text{-tor}$ of *super* $\mathbb{Z}_2$-*torsors* ($\mathbb{Z}_2$-graded $\mathbb{Z}_2$-torsors) is a Picard groupoid with $\pi_0 = \pi_1 = \mathbb{Z}_2$ and $q = \mathrm{id} : \mathbb{Z}_2 \to \mathbb{Z}_2$.

Example

(a) Suppose $n \equiv 1, 7 \mod 8$. We can define a symmetric monoidal functor $F : \mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2\text{-tor}$ which maps (X, P) to the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ defined using the Dirac operator $\not{D}_X$.
(b) Suppose $n \equiv 0 \mod 8$. We can define a symmetric monoidal functor $F : \mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \to \text{s-}\mathbb{Z}_2\text{-tor}$ which maps (X, P) to the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ defined using the positive Dirac operator $\not{D}_X^+$, $\mathbb{Z}_2$-graded in degree $\mathrm{ind}(\not{D}_X^+ \otimes \mathrm{ad}(P)) \mod 2$.

Thus we can encode orientations of moduli spaces in *orientation functors* between Picard groupoids. This is not obvious. It depends on a bordism-invariance property of indices and determinants of Dirac operators proved in Upmeier arXiv:2312.06818.

Example

Let (X, g) be a compact spin n-manifold, and G be a Lie group. Define a subcategory $\mathfrak{Bord}_X^{\mathbf{Spin}}(BG)$ of $\mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ to have objects (X, P) for X the fixed spin n-manifold and varying P, and to have morphisms $[X \times [0,1], Q]$ for $Y = X \times [0,1]$ the fixed spin $(n+1)$-manifold with boundary, and varying Q. Write $\mathrm{inc} : \mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \hookrightarrow \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ for the inclusion functor. Suppose $n \equiv 1, 7, 8 \mod 8$, and write $F_X = F \circ \mathrm{inc} : \mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2\text{-tor}$, where for $n \equiv 8$ we compose with $\text{s-}\mathbb{Z}_2\text{-tor} \to \mathbb{Z}_2\text{-tor}$ forgetting $\mathbb{Z}_2$-gradings.

Then a choice of orientation for $\mathcal{B}_P$ for each principal G-bundle $P \to X$, invariant under isomorphisms $P \cong P'$, is equivalent to a natural isomorphism $\eta : F_X \Rightarrow \mathbb{1}_X$, where $\mathbb{1}_X$ is the constant functor with value $\mathbb{Z}_2$. Hence, $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$ if and only if the functor $F_X : \mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2\text{-tor}$ is trivializable.

To see why this is true, note that $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$, where $\mathcal{A}_P$ is the infinite-dimensional affine space of connections on $P \to X$, and $\mathcal{G}_P = \mathrm{Aut}(P)$ is the gauge group. Here $\mathcal{A}_P$ is always orientable, with exactly two orientations, as it is contractible. So $\mathcal{B}_P$ is orientable if and only if the group $\mathcal{G}_P$ acts trivially on the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$.

Given an element $\gamma \in \mathrm{Aut}(P)$, we can define a morphism $[X \times [0,1], Q] : (X,P) \to (X,P)$ in $\mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \subset \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ by taking Q to be $P \times [0,1]$ with identifications $\mathrm{id}_P : P \times \{0\} \to P$ and $\gamma : P \times \{1\} \to P$. All morphisms $[X \times [0,1], Q] : (X,P) \to (X,P)$ are of this form. So $\mathcal{G}_P$ acts trivially on the $\mathbb{Z}_2$-torsor of orientations on $\mathcal{A}_P$ if and only if F_X is trivializable over the object (X,P).

The orientation functor $F : \mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2\text{-tor}$ or $\text{s-}\mathbb{Z}_2\text{-tor}$ is classified by morphisms $F_0 : \Omega_n^{\mathbf{Spin}}(BG) \to 0$ or $\mathbb{Z}_2$ and $F_1 : \Omega_{n+1}^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$. For a fixed compact spin n-manifold X, we can show that F_X is trivializable (hence, $\mathcal{B}_P$ is orientable for all principal G-bundles $P \to X$) if and only if there does not exist an element of the form $[X \times \mathcal{S}^1_{\mathrm{b}}, Q] \in \Omega_{n+1}^{\mathbf{Spin}}(BG)$ with $F_1([X \times \mathcal{S}^1_{\mathrm{b}}, Q]) \neq 0$ in $\mathbb{Z}_2$. We can also write this in terms of $[X, Q] \in \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG)$ with $F_1 \circ \xi([X, Q]) \neq 0$, where $\mathcal{L}BG = \mathrm{Map}_{C^0}(\mathcal{S}^1, BG)$ is the free loop space and $\xi : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathbf{Spin}}(BG)$ maps $\xi : [X, Q] \mapsto [X \times \mathcal{S}^1_{\mathrm{b}}, Q]$. If $F_1 \circ \xi : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \mathbb{Z}_2$ is identically zero then $\mathcal{B}_P$ is orientable for all compact spin n-manifolds X and principal G-bundles $P \to X$. We can use Algebraic Topology and spectral sequences to compute bordism groups such as $\Omega_n^{\mathbf{Spin}}(BG), \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG)$, and morphisms such as $\xi : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathbf{Spin}}(BG)$, and $F_0 : \Omega_n^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$ and $F_1 : \Omega_{n+1}^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$ which classify orientation functors. Then we can use these to prove theorems on orientability and canonical orientations. I'll tell you more about all this next week.