Orientations on moduli spaces of coherent sheaves on Calabi–Yau 4-folds. II

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

References

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1. Recap from last lecture

Let X be a projective Calabi–Yau 4-fold over $\mathbb{C}$, and $\boldsymbol{\mathcal{M}}$ be the derived moduli stack of coherent sheaves or perfect complexes on X. Then $\boldsymbol{\mathcal{M}}$ has a -2-*shifted symplectic structure* ω in the sense of Pantev–Toën–Vaquié–Vezzosi 2011. This gives an equivalence $\omega\cdot\;:\mathbb{T}_{\boldsymbol{\mathcal{M}}}=(\mathbb{L}_{\boldsymbol{\mathcal{M}}})^\vee\rightarrow\mathbb{L}_{\boldsymbol{\mathcal{M}}}[-2]$, and an isomorphism of determinant line bundles $\det\omega:(\det\mathbb{L}_{\boldsymbol{\mathcal{M}}})^*\rightarrow\det\mathbb{L}_{\boldsymbol{\mathcal{M}}}$. An *orientation* on $\boldsymbol{\mathcal{M}}$ is an isomorphism $\phi:\mathcal{O}_{\boldsymbol{\mathcal{M}}}\rightarrow\det\mathbb{L}_{\boldsymbol{\mathcal{M}}}$ with $\det\omega=\phi\circ\phi^*$. We need orientations to define Donaldson–Thomas type *DT4 invariants* of X (Borisov–Joyce 2015, Oh–Thomas 2020).

Let $P\rightarrow X$ be a principal $\mathrm{U}(m)$-bundle, and $\mathcal{B}_P$ the topological moduli stack of all connections ∇ on P, modulo gauge equivalence. We define a principal $\mathbb{Z}_2$-bundle $O_P\rightarrow\mathcal{B}_P$ of *orientations* on $\mathcal{B}_P$, which orient the Fredholm twisted Dirac operators $\not{D}_X^+\otimes\mathrm{ad}(P)$. By Cao–Gross–Joyce 2018, orientations on $\mathcal{B}_P$ pull back to orientations on $\boldsymbol{\mathcal{M}}_{\mathrm{ch}=\mathrm{ch}\,P}\subset\boldsymbol{\mathcal{M}}$. So if we can prove orientability / define canonical orientations on spaces $\mathcal{B}_P$, we deduce orientability / define canonical orientations on $\boldsymbol{\mathcal{M}}$.

I defined a *bordism category* $\mathfrak{Bord}_8^{\mathbf{Spin}}(B\mathrm{U}(m))$, which is a *Picard groupoid* (*abelian 2-group*), and an *orientation functor* $F_{\mathrm{U}(m)} : \mathfrak{Bord}_8^{\mathbf{Spin}}(B\mathrm{U}(m)) \to \mathrm{s}\text{-}\mathbb{Z}_2\text{-}\mathrm{tor}$. I explained that choices of orientation for $\mathcal{B}_P$ for all $P \to X$, invariant under isomorphisms $P \cong P'$, are equivalent to a trivialization of $F_{\mathrm{U}(m)}$ on a subcategory $\mathfrak{Bord}_X^{\mathbf{Spin}}(B\mathrm{U}(m)) \subset \mathfrak{Bord}_8^{\mathbf{Spin}}(B\mathrm{U}(m))$.
We can understand $\mathfrak{Bord}_8^{\mathbf{Spin}}(B\mathrm{U}(m))$ and $F_{\mathrm{U}(m)}$ very explicitly by computation of spin bordism groups like $\Omega_n^{\mathbf{Spin}}(B\mathrm{U}(m))$ for $n = 8, 9$ in Algebraic Topology. Using this we can answer questions on orientability and canonical orientations for moduli spaces $\mathcal{B}_P$.
Today I will give more details of this picture.

2. Bordism

Let $\boldsymbol{B}$ be a 'stable tangential structure' on manifolds, for example, $\mathbf{O}$ is 'unoriented', $\mathbf{SO}$ is 'oriented', $\mathbf{Spin}$ is 'spin' (which includes oriented). For each (nice) topological space T and $n \geqslant 0$, we define the *bordism group* $\Omega_n^{\boldsymbol{B}}(T)$ to be the set of $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact n-manifold with $\boldsymbol{B}$-structure with $\partial X = \emptyset$ and $f : X \to T$ is continuous. We write $(X_0, f_0) \sim (X_1, f_1)$ if there exists a compact $n+1$-manifold Y with $\boldsymbol{B}$-structure and a continuous map $g : Y \to T$, such that Y has boundary $\partial Y = -X_0 \amalg X_1$ with $\boldsymbol{B}$-structures, and $g|_{\partial Y} = f_0 \amalg f_1$. Here $-X_0$ is X_0 with the 'opposite $\boldsymbol{B}$-structure' (e.g. opposite orientation). Then $\Omega_n^{\boldsymbol{B}}(T)$ is an abelian group with addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$ and identity $0 = [\emptyset, \emptyset]$. Bordism $\Omega_*^{\boldsymbol{B}}(-)$ is a *generalized homology theory* – it satisfies all the Eilenberg–Steenrod axioms except the dimension axiom. There is an Atiyah–Hirzebruch spectral sequence $H_p(T, \Omega_q^{\boldsymbol{B}}(*)) \Rightarrow \Omega_{p+q}^{\boldsymbol{B}}(T)$. For T path-connected, *reduced bordism* is $\tilde{\Omega}_n^{\boldsymbol{B}}(T) = \Omega_n^{\boldsymbol{B}}(T, \{t_0\})$. Then $\Omega_n^{\boldsymbol{B}}(T) = \tilde{\Omega}_n^{\boldsymbol{B}}(T) \oplus \Omega_n^{\boldsymbol{B}}(*)$.

Spin bordism groups of classifying spaces

We will care about spin bordism groups $\Omega_n^{\mathbf{Spin}}(T)$ of classifying spaces such as BG for G a Lie group, or loops spaces $\mathcal{L}BG$. These can often be computed using Algebraic Topology (and a lot of work by Markus). The bordism groups of the point are

n	0	1	2	3	4	5	6	7	8	9
$\Omega_n^{\mathbf{Spin}}(*)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$

The homology $H_*(BG,\mathbb{Z})$ and $H_*(BG,\mathbb{Z}_2)$ is known for classical G. Using the A–H spectral sequence $\tilde{H}_p(BG,\Omega_q^{\boldsymbol{B}}(*))\Rightarrow\tilde{\Omega}_{p+q}^{\boldsymbol{B}}(BG)$, we can prove for example that $B\,\mathrm{SU}(m)$, $m\geqslant 5$ has reduced spin bordism

n	0,1,2,3,5,7	4	6	8	9
$\tilde{\Omega}_n^{\mathbf{Spin}}(B\,\mathrm{SU}(m))$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}_2$

We can give explicit basis elements for the groups, and describe the isomorphisms explicitly, e.g. $\tilde{\Omega}_8^{\mathbf{Spin}}(B\,\mathrm{SU}(m))\xrightarrow{\cong}\mathbb{Z}^3$ is

$$[X,P]\longmapsto\Bigl(\int_X\bigl[\tfrac{c_4(P)}{6}-\tfrac{c_2(P)^2}{12}-\tfrac{p_1(TX)c_2(P)}{24}\bigr],\int_X c_2(P)^2,\int_X c_4(P)\Bigr).$$

3. Bordism categories and orientation functors

Recall that a *Picard groupoid* (or *abelian 2-group*) $(\mathcal{G}, \otimes, \mathbb{1})$ is a symmetric monoidal groupoid $(\mathcal{G}, \otimes)$ with an identity object $\mathbb{1} \in \mathcal{G}$ such that all objects in $\mathcal{G}$ are invertible. Picard groupoids are classified by abelian groups $\pi_0 = \{\text{iso. classes of objects in } \mathcal{G}\}$, $\pi_1 = \mathrm{Aut}_{\mathcal{G}}(\mathbb{1})$ and a linear quadratic map $q : \pi_0 \to \pi_1$. Symmetric monoidal functors between Picard groupoids, and monoidal natural isomorphisms, are all classified by group data.

We define *bordism categories*, which are geometrically-defined Picard groupoids. For example, if $\boldsymbol{B}$ is a stable tangential structure, $n \geqslant 0$, and G is a Lie group, then $\mathfrak{Bord}_n^{\boldsymbol{B}}(BG)$ has objects pairs (X, P) of a compact n-manifold X with $\partial X = \emptyset$ with $\boldsymbol{B}$-structure, and a principal G-bundle $P \to X$, and morphisms $[Y, Q] : (X_0, P_0) \to (X_1, P_1)$ to be relative bordism classes $[Y, Q]$ of a compact $(n+1)$-manifold Y with $\boldsymbol{B}$-structure with boundary $\partial Y = -X_0 \amalg X_1$ and a principal G-bundle $Q \to Y$ with $Q|_{\partial Y} = P_0 \amalg P_1$.

The classifying data is $\pi_i = \Omega_{n+i}^{\boldsymbol{B}}(BG)$, $i = 0, 1$, and $q : [X, P] \mapsto [X \times \mathcal{S}^1_{\mathrm{nb}}, P \times \mathcal{S}^1_{\mathrm{nb}}]$, where $\mathcal{S}^1_{\mathrm{nb}}$ is $\mathcal{S}^1$ with the non-bounding $\boldsymbol{B}$-structure. This can be computed explicitly.

We define *orientation functors*, which are symmetric monoidal functors from a bordism category to $\mathbb{Z}_2\text{-tor}$ or $\text{s-}\mathbb{Z}_2\text{-tor}$, the Picard groupoids of (super) $\mathbb{Z}_2$-torsors. These are defined using analysis of elliptic operators (usually twisted Dirac operators), and encode orientation problems on moduli spaces.

For example, the functors which control orientations of moduli spaces of $\mathrm{Spin}(7)$-instantons in 8 dimensions, and (for $G = \mathrm{U}(m)$, $m \gg 0$) orientations of moduli spaces of coherent sheaves on Calabi–Yau 4-folds, are functors $F_G : \mathfrak{Bord}_8^{\mathbf{Spin}}(BG) \to \text{s-}\mathbb{Z}_2\text{-tor}$ defined by twisting the positive Dirac operator $\not{D}_X^+$ on a compact spin 8-manifold X by connections ∇ on a principal G-bundle $P \to X$. Such F_G are classified by group morphisms $f_0 : \Omega_8^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$, $f_1 : \Omega_9^{\mathbf{Spin}}(BG) \to \mathbb{Z}_2$ with $f_1 \circ q = q' \circ f_0$. These are computable (with some work). For example, if $G = \mathrm{SU}(m)$, $m \geqslant 5$, then f_0, f_1 are zero on $\Omega_{8+i}^{\mathbf{Spin}}(*)$, and $f_0 : \tilde{\Omega}_8^{\mathbf{Spin}}(B\,\mathrm{SU}(m)) \cong \mathbb{Z}^3 \to \mathbb{Z}_2$ maps $(x, y, z) \mapsto y \mod 2$, and $f_1 : \tilde{\Omega}_9^{\mathbf{Spin}}(B\,\mathrm{SU}(m)) \cong \mathbb{Z}_2 \to \mathbb{Z}_2$ is $\mathrm{id}_{\mathbb{Z}_2}$. (We compute f_0 by the Atiyah–Singer Index Theorem. There is no Index Theorem to compute f_1, but it is determined by $f_1 \circ q = q' \circ f_0$.)

Factorizing orientation functors

Let $\rho : G \to H$ be a morphism of Lie groups. There is a monoidal functor $T_\rho : \mathfrak{Bord}_n^{\boldsymbol{B}}(BG) \to \mathfrak{Bord}_n^{\boldsymbol{B}}(BH)$ mapping $(X, P) \mapsto (X, P \times_G H)$. Call ρ *of complex type* if on Lie algebras $\rho_* : \mathfrak{g} \to \mathfrak{h}$ is injective, and the quotient G-representation $\mathfrak{m} = \mathfrak{h}/\rho_*(\mathfrak{g})$ is the underlying real representation of a complex G-representation. Then we have a 2-commuting diagram of symmetric monoidal functors and monoidal natural isomorphisms

$$\begin{tikzcd}
\mathfrak{Bord}_n^{\boldsymbol{B}}(BG) \arrow[d, "T_\rho"] \arrow[drr, bend left=10, "F_G"] \arrow[d, phantom, shift left=12ex, "\Downarrow"] & & \\
\mathfrak{Bord}_n^{\boldsymbol{B}}(BH) \arrow[rr, "F_H"] & & \text{s-}\mathbb{Z}_2\text{-tor}.
\end{tikzcd}$$

This will imply that if an n-dimensional moduli space problem is orientable for principal H-bundles, then it is orientable for principal G-bundles. Also if $G = H \times T$ for T abelian, the G-problem is orientable iff the H-problem is. If T_ρ is an equivalence of categories (e.g. for $\mathrm{SU}(m) \hookrightarrow \mathrm{SU}(m+1)$, $2m \geqslant n+1$), the G-problem is orientable iff the H-problem is.

Using complex type morphisms, abelian factors $G = H \times T$, and equivalences $T_\rho : \mathfrak{Bord}_n^{\boldsymbol{B}}(BG) \xrightarrow{\simeq} \mathfrak{Bord}_n^{\boldsymbol{B}}(BH)$, we can deduce orientability for many different G from orientability for a single G. For example, for n-dimensional moduli space problems for $n \leqslant 8$, if we have orientability for $G = E_8$, we have orientability for any G in the list, for all $m \geqslant 1$

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m).$$

Cohomology bordism categories

For R a commutative ring and $0 \leqslant k \leqslant n$, define the *cohomology bordism category* $\mathfrak{Bord}_n^{\boldsymbol{B}}(K(R,k))$ to have objects (X,γ) where X is a compact n-manifold with $\boldsymbol{B}$-structure with $\partial X = \emptyset$ and $\gamma \in C^k(X,R)$ with $\mathrm{d}\gamma = 0$ is a the k-cocycle in cohomology of X over R, and to have morphisms $[Y,\delta] : (X_0,\gamma_0) \to (X_1,\gamma_1)$ to be bordism/cohomology classes of pairs (Y,δ) of a compact $(n+1)$-manifold Y with $\boldsymbol{B}$-structure with boundary $\partial Y = -X_0 \amalg X_1$ and a k-cochain $\delta \in C^k(Y,R)$ with $\delta|_{\partial Y} = -\gamma_0 + \gamma_1$. Then $\mathfrak{Bord}_n^{\boldsymbol{B}}(K(R,k))$ has invariants $\pi_i = \Omega_{n+i}^{\boldsymbol{B}}(K(R,k))$ for $i = 0,1$, where $K(R,k)$ is the Eilenberg–MacLane space classifying $H^k(-,R)$. We can often compute $\Omega_*^{\boldsymbol{B}}(K(R,k))$.

There is a 16-connected map $BE_8 \to K(\mathbb{Z},4)$, so $\Omega_n^{\boldsymbol{B}}(BE_8) \cong \Omega_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$ for $n < 16$. Thus we can define a symmetric monoidal functor $\mathfrak{Bord}_n^{\boldsymbol{B}}(BE_8) \to \mathfrak{Bord}_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$, which is an equivalence of categories for $n \leqslant 14$.

In this way we translate orientability questions for E_8 gauge theory into problems in cohomology and cohomology operations, such as Steenrod squares.

Let X be a compact spin n-manifold for $n \equiv 1,7,8 \mod 8$, and G a Lie group. Define a subcategory $\mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \subset \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ to have objects (X,P) for X the fixed spin n-manifold and varying P, and to have morphisms $[X \times [0,1], Q]$ for $Y = X \times [0,1]$ the fixed spin $(n+1)$-manifold with boundary, and varying Q. Write $\mathrm{inc} : \mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \hookrightarrow \mathfrak{Bord}_n^{\mathbf{Spin}}(BG)$ for the inclusion functor.

For each principal G-bundle $P \to X$, we define a moduli stack $\mathcal{B}_P = \mathcal{A}_P / \mathcal{G}_P$ of connections ∇ on P modulo gauge, and a $\mathbb{Z}_2$-bundle $O_P \to \mathcal{B}_P$ of orientations for the twisted Dirac operators $\not{D}_X^{(+)} \otimes (\mathrm{ad}(P), \nabla)$. I explained last time that a choice of orientation for $\mathcal{B}_P$ for all $P \to X$, invariant under isomorphisms $P \cong P'$, is equivalent to a natural isomorphism η in the diagram

$$\begin{tikzcd}
\mathfrak{Bord}_X^{\mathbf{Spin}}(BG) \arrow[rr, "\mathbb{1}_X"'] \arrow[d, "\mathrm{inc}"] \arrow[drr, phantom, "\Downarrow\eta"] & & \mathbb{Z}_2\text{-tor} \\
\mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \arrow[rr, "F_G"] & & \text{s-}\mathbb{Z}_2\text{-tor}, \arrow[u, "\text{forget } \mathbb{Z}_2\text{-grading}"']
\end{tikzcd}$$

with $\mathbb{1}_X$ the constant functor with value $\mathbb{Z}_2$.

From this we easily deduce:

Theorem 1 (Joyce–Upmeier)

(a) *Let* $n \equiv 1, 7, 8 \mod 8$, *and* X *be a compact spin n-manifold and* G *a Lie group. Then* $\mathcal{B}_P$ *is orientable for all principal* G*-bundles* $P \to X$ *if and only if for all classes* $[X, \phi] \in \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG)$ *with domain* X *we have* $f_1 \circ \xi([X, \phi]) = 0$ *in* $\mathbb{Z}_2$, *where*

$$\Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \xrightarrow{\quad \xi \quad} \Omega_{n+1}^{\mathbf{Spin}}(BG) \xrightarrow{\quad f_1 \quad} \mathbb{Z}_2.$$

with $\xi : [X, \phi] \mapsto [X \times \mathcal{S}^1_{\mathrm{b}}, \phi']$ *and* f_1 *the classifying morphism for* F_G.

(b) $\mathcal{B}_P$ *is orientable for all compact spin n-manifolds* X *and all principal* G*-bundles* $P \to X$ *iff* $f_1 \circ \xi \equiv 0 : \Omega_n^{\mathbf{Spin}}(\mathcal{L}BG) \to \mathbb{Z}_2$.

As above, using complex type morphisms $\rho : G \to H$, if we know the conditions of Theorem 1(a) or (b) hold for some well-chosen Lie group G, such as $G = E_8$, we can deduce them for many other Lie groups G. By computing $\Omega_n^{\mathbf{Spin}}(\mathcal{L}BG), \Omega_{n+1}^{\mathbf{Spin}}(BG), \xi, f_1$ explicitly, we can answer many orientability problems.

If $f_1 \circ \xi \not\equiv 0$, it may not be easy to decide if some given X satisfies **(a)**.

The case of $G = E_8$ and $K(\mathbb{Z},4)$

As we have an equivalence $\mathfrak{Bord}_n^{\boldsymbol{B}}(BE_8) \to \mathfrak{Bord}_n^{\boldsymbol{B}}(K(\mathbb{Z},4))$ for $n \leqslant 14$, for $G = E_8$ we may replace BE_8 with $K(\mathbb{Z},4)$. Our calculations show that when $n = 7$, $f_1 \circ \xi \equiv 0 : \Omega_7^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4)) \to \mathbb{Z}_2$. Thus $\mathcal{B}_P$ is orientable for all principal E_8-bundles $P \to X$. However, when $n = 8$, $f_1 \circ \xi \not\equiv 0 : \Omega_8^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4)) \to \mathbb{Z}_2$. Elements of $\Omega_8^{\mathbf{Spin}}(\mathcal{L}K(\mathbb{Z},4))$ may be written $[X,\alpha]$ for X a compact spin n-manifold and $\alpha \in H^4(X \times \mathcal{S}^1,\mathbb{Z})$. Then $\alpha = \beta \boxtimes \mathrm{Pd}[\mathcal{S}^1] + \gamma \boxtimes 1_{\mathcal{S}^1}$ for $\beta \in H^3(X,\mathbb{Z})$ and $\gamma \in H^4(X,\mathbb{Z})$, and $f_1 \circ \xi([X,\alpha]) = \int_X \bar\beta \cup \mathrm{Sq}^2(\bar\beta)$, where $\bar\beta \in H^2(X,\mathbb{Z}_2)$ is the mod 2 reduction of β and $\mathrm{Sq}^2(\bar\beta)$ is its Steenrod square.

Thus by Theorem 1(a), if X is a compact spin 8-manifold, then $\mathcal{B}_P$ is orientable for all principal E_8-bundles $P \to X$ if and only if the following condition holds:

(*) Let $\alpha \in H^3(X,\mathbb{Z})$, and write $\bar\alpha \in H^3(X,\mathbb{Z}_2)$ for its mod 2 reduction, and $\mathrm{Sq}^2(\bar\alpha) \in H^5(X,\mathbb{Z}_2)$ for its Steenrod square. Then $\int_X \bar\alpha \cup \mathrm{Sq}^2(\bar\alpha) = 0$ in $\mathbb{Z}_2$ for all $\alpha \in H^3(X,\mathbb{Z})$.

Combining this with the previous material on complex type morphisms, etc., we prove:

Theorem 2

Suppose a compact spin 8-manifold X satisfies condition $(*)$, and let G be a compact Lie group on the list, for all $m \geqslant 1$

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m). \qquad (1)$$

Then $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$. For $G = E_8$, this holds **if and only if** $(*)$ *holds.*

The case $G = \mathrm{U}(m)$ is needed for DT4 theory for Calabi–Yau 4-folds. The compact 8-manifold $X = \mathrm{SU}(3)$ does not satisfy condition $(*)$. We show $\mathcal{B}_P$ is *not* orientable when $P \to X$ is the trivial G-bundle for G any of $\mathrm{SU}(m), \mathrm{U}(m)$, or $\mathrm{Spin}(2m)$ for $m \geqslant 3$, G_2, E_6, E_7, or E_8. For Lie groups G not on the list (1), different conditions apply. We have not done detailed calculations, but we have some examples of non-orientability for X in dimensions $7, 8$ for G on the list F_4, $\mathrm{Sp}(m+1)$, $\mathrm{Spin}(2m+3)$, $\mathrm{SO}(2m+3)$, $m \geqslant 1$, which would be orientable for G on the list (1).

We now consider the question: let X be fixed, and suppose we have proved orientability of moduli spaces $\mathcal{B}_P$ for all principal G-bundles $P \to X$. How can we construct *canonical orientations* on $\mathcal{B}_P$ for all $P \to X$, possibly after choosing some data on X? We now restrict to dimension $n = 7$ or 8. Then for any G in the list E_8, E_7, E_6, G_2, $\mathrm{Spin}(3)$, $\mathrm{SU}(m)$, $\mathrm{U}(m)$, $\mathrm{Spin}(2m)$ for $m \geqslant 1$, we have a 2-commutative diagram of Picard groupoids

$$\begin{tikzcd}
\mathfrak{Bord}_n^{\mathbf{Spin}}(BG) \arrow[d, "T_G"] \arrow[rr, bend left, "F_G"] \arrow[dr, phantom, "\Downarrow"] & & \\
\mathfrak{Bord}_n^{\mathbf{Spin}}(K(\mathbb{Z},4)) \arrow[rr, "F_{K(\mathbb{Z},4)}"] & & \mathbb{Z}_2\text{-tor or } \mathbb{Z}_2\text{-tor}.
\end{tikzcd}$$

Here F_G is the orientation functor controlling 'normalized orientations' on moduli spaces $\mathcal{B}_P$ for principal G-bundles $P \to X$, and $F_{K(\mathbb{Z},4)}$ is an explicitly defined symmetric monoidal functor. Also F_G determines orientations on moduli spaces of G_2-instantons when $n = 7$, and on moduli spaces of $\mathrm{Spin}(7)$-instantons and of coherent sheaves on Calabi–Yau 4-folds when $n = 8$.

Definition 3

Let X be a compact spin n-manifold for $n = 7$ or 8. A *flag structure* on X is a natural isomorphism ζ in the diagram

$$\begin{tikzcd} \mathfrak{Bord}_X^{\mathbf{Spin}}(K(\mathbb{Z},4)) \arrow[rr, "\mathbb{1}_X"'] \arrow[d, "\mathrm{inc}"] \arrow[drr, phantom, "\Downarrow\zeta"] & & \mathbb{Z}_2\text{-tor} \\ \mathfrak{Bord}_n^{\mathbf{Spin}}(K(\mathbb{Z},4)) \arrow[rr, "F_{K(\mathbb{Z},4)}"] & & \mathbb{Z}_2\text{-tor or } \mathbb{Z}_2\text{-tor}. \arrow[u, "\text{forget } \mathbb{Z}_2\text{-grading}"'] \end{tikzcd} \tag{2}$$

A different but equivalent geometric definition was given for flag structures when $n = 7$ in Joyce arXiv:1610.09836, which showed that a flag structure on a G_2-manifold determines orientations on moduli spaces of associative 3-folds. Joyce–Upmeier arXiv:1811.02405 show that a flag structure on a G_2-manifold determines orientations on moduli spaces of G_2-instantons for $G = \mathrm{SU}(m)$ or $\mathrm{U}(m)$. Definition 3 gives a generalization to $n = 8$.

If $n = 7$, flag structures on X always exist.

If $n = 8$, flag structures on X exist iff X satisfies condition $(*)$.

When flag structures exist, the set of flag structures on X is a torsor for $\mathrm{Map}(H^4(X,\mathbb{Z}),\mathbb{Z}_2)$ (meaning arbitrary maps, not group homomorphisms), which is a very large choice. That is, a flag structure is a $\mathbb{Z}_2$ choice for each element of $\pi_0(\mathfrak{Bord}_X^{\mathbf{Spin}}(K(\mathbb{Z},4))) \cong H^4(X,\mathbb{Z})$. We say that a flag structure ζ *factors via* $\mathbb{Z}_2$ if (2) can be factorized as

$$\begin{tikzcd}
\mathfrak{Bord}_X(K(\mathbb{Z},4)) \arrow[r] \arrow[d, "\mathrm{inc}"] \arrow[rr, bend left, "\mathbb{1}"] \arrow[rr, phantom, bend left=10, "\Downarrow\mathrm{id}"] \arrow[dr, phantom, "\mathrm{id}\Uparrow"] & \mathfrak{Bord}_X(K(\mathbb{Z}_2,4)) \arrow[r, "\mathbb{1}"'] \arrow[d, "\mathrm{inc}"] \arrow[dr, phantom, "\zeta'\Uparrow"] & \mathbb{Z}_2\text{-tor} \\
\mathfrak{Bord}_n^{\mathbf{Spin}}(K(\mathbb{Z},4))) \arrow[r] \arrow[rr, bend right, "F_{K(\mathbb{Z},4)}"'] \arrow[rr, phantom, bend right=10, "\Uparrow"] & \mathfrak{Bord}_n^{\mathbf{Spin}}(K(\mathbb{Z}_2,4))) \arrow[r, "F_{K(\mathbb{Z}_2,4)}"] & \text{s-}\mathbb{Z}_2\text{-tor.} \arrow[u, "\text{forget}"']
\end{tikzcd}$$

This is always possible when $n=7$. When $n=8$, it is possible iff

(†) There does not exist $\bar\alpha \in H^3(X,\mathbb{Z}_2)$ such that $\int_X \bar\alpha \cup \mathrm{Sq}^2(\bar\alpha) \neq 0$ in $\mathbb{Z}_2$.

The set of flag structures factoring via $\mathbb{Z}_2$ is a torsor for $\mathrm{Map}\big(\mathrm{Im}(H^4(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}_2)),\mathbb{Z}_2\big)$, which is a finite choice.

If G lies on the list (1), then a choice of flag structure ζ determines orientations on $\mathcal{B}_P$ for all principal G-bundles $P \to X$. We can decompose ζ as $(\zeta_\alpha)_{\alpha \in H^4(X,\mathbb{Z})}$, where each ζ_α lies in a $\mathbb{Z}_2$-torsor. In the case $G = \mathrm{U}(m)$, for a principal $\mathrm{U}(m)$-bundle $P \to X$, the orientation on $\mathcal{B}_P$ is determined by ζ_α for $\alpha = c_2(P) - c_1(P)^2$.

This shows how using flag structures cuts down choices of orientations: the orientation on $\mathcal{B}_P$ depends only on $c_2(P) - c_1(P)^2 \in H^4(X,\mathbb{Z})$, not on other Chern classes $c_i(P)$.

If ζ factorizes via $\mathbb{Z}_2$ then the orientation on $\mathcal{B}_P$ depends only on $c_2(P) - c_1(P)^2 \in H^4(X,\mathbb{Z}_2)$. There is a natural choice for ζ_0, so there is a canonical orientation for $\mathcal{B}_P$, independent of flag structure, if $c_2(P) - c_1(P)^2 = 0$ in $H^4(X,\mathbb{Z})$ or $H^4(X,\mathbb{Z}_2)$.

For studying DT4 invariants of Calabi–Yau 4-folds, it would be interesting to know how orientations of $\mathcal{B}_P$ behave under direct sums $(P,P') \mapsto P \oplus P'$, where $P \oplus P'$ is the associated $\mathrm{U}(m+m')$-bundle. Unfortunately, the flag structure approach is not well adapted for this, as in general

$$c_2(P \oplus P') - c_1(P \oplus P')^2 \neq \bigl(c_2(P) - c_1(P)^2\bigr) + \bigl(c_2(P') - c_1(P')^2\bigr).$$

We can also use our theory to study structures on moduli spaces which generalize orientations. One example is *gradings of Floer theories*. If X is a compact oriented 3-manifold and G a Lie group (usually $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$), one can study moduli spaces $\mathcal{M}_P^{\mathrm{flat}} \subset \mathcal{B}_P$ of flat connections on P, which are (roughly) critical loci of a functional on $\mathcal{B}_P$. By a kind of infinite-dimensional Morse theory, under good conditions one can define *instanton Floer homology groups* $HF_k(P)$. We ask: how is $HF_*(P)$ graded, i.e. where does the index k live? It turns out that if $\mathcal{B}_P$ is orientable, then $HF_*(P)$ is graded over $\mathbb{Z}_2$. But sometimes we can do better. The usual orientation principal $\mathbb{Z}_2$-bundle $O_P \to \mathcal{B}_P$ is the $\mathbb{Z}_2$ reduction of a principal $\mathbb{Z}$-bundle $O_P^{\mathbb{Z}} \to \mathcal{B}_P$. (This happens because the elliptic operator used to define $\mathcal{O}_P$ is self-adjoint.) This has a reduction $O_P^{\mathbb{Z}_k} \to \mathcal{B}_P$ to $\mathbb{Z}_k$ for any $k \geqslant 1$. If $O_P^{\mathbb{Z}_k}$ is trivializable as a $\mathbb{Z}_k$-bundle, then we can grade Floer homology $HF_*(P)$ over $\mathbb{Z}_k$. For $G = \mathrm{SU}(2)$, instanton Floer homology in 3 dimensions can be graded over $\mathbb{Z}_8$.

One can imagine trying to define an instanton Floer homology theory for G_2-manifolds (X,φ) in 7-dimensions, using G_2-instantons instead of flat connections, and using $\mathrm{Spin}(7)$-instantons on $X\times\mathbb{R}$ as the flow lines. This would raise many difficult analytic problems, which we do not discuss. But we can use our bordism category theory to answer questions on the grading of such a Floer theory, if it exists.

Let X be a compact spin 7-manifold, G a Lie group, and $P\to X$ a principal G-bundle. Since the Dirac operator $\not{D}_X$ in 7-dimensions is self-dual, the orientation $\mathbb{Z}_2$-bundle $O_P\to\mathcal{B}_P$ is the $\mathbb{Z}_2$ reduction of a principal $\mathbb{Z}$-bundle $O_P^{\mathbb{Z}}\to\mathcal{B}_P$, with $\mathbb{Z}_k$ reduction $O_P^{\mathbb{Z}_k}\to\mathcal{B}_P$. Trivializations of $O_P^{\mathbb{Z}}, O_P^{\mathbb{Z}_k}$ are controlled by orientation functors $F_G:\mathfrak{Bord}_7^{\mathbf{Spin}}(BG)\to\mathbb{Z}$-tor or $\mathbb{Z}_k$-tor, and we can use our theory to study orientability and canonical orientations.

We prove a negative result: if G is $\mathrm{SU}(m)$ or $\mathrm{Sp}(m)$ for $m\geqslant 2$ or E_8 and $k>2$, then there exists a compact spin 7-manifold X and a principal G-bundle $P\to X$ such that $O_P^{\mathbb{Z}}\to\mathcal{B}_P$ or $O_P^{\mathbb{Z}_k}\to\mathcal{B}_P$ is not trivializable. (Here $O_P^{\mathbb{Z}_2}\to\mathcal{B}_P$ is trivializable by our orientability theorem.)

7. Beyond orientations: orientation data

Let X be a compact, spin 6-manifold. Then the Dirac operator $\not{D}_X$ is $\mathbb{C}$-linear. Let G be a Lie group and $P \to X$ a principal G-bundle. Then there is a natural complex line bundle $L_P \to \mathcal{B}_P$, whose fibre at a connection ∇ on P is $\det_{\mathbb{C}} \not{D}_\nabla = \det_{\mathbb{C}} \mathrm{Ker}(\not{D}_\nabla) \otimes \det_{\mathbb{C}} \mathrm{Coker}(\not{D}_\nabla)^*$, where $D_\nabla = \not{D}_X \otimes (\mathrm{ad}(P), \nabla)$ is the ∇-twisted Dirac operator.

Orientation data on $\mathcal{B}_P$ is a choice of square root line bundle $L_P^{1/2}$ for L_P. Really this should be called a *spin structure* (a spin structure on a complex manifold Y is equivalent to a choice of square root $K_Y^{1/2}$ of the canonical bundle K_Y). The choice of name is due to Kontsevich–Soibelman 2008.

If X is a Calabi–Yau 3-fold and $G = \mathrm{U}(m)$ for $m \gg 0$, such orientation data can be used to construct square root line bundles on derived moduli stacks of coherent sheaves on Calabi–Yau 3-folds, which are important in Donaldson–Thomas theory.

In Joyce–Upmeier arXiv:1908.03524, arXiv:2001.00113 we prove existence of canonical orientation data for any Calabi–Yau 3-fold. (There are still important open questions, though.)

The basic idea is that orientation data for the 6-manifold X and $\mathcal{B}_P$ is closely connected to orientations in the usual sense on $\mathcal{B}_Q$ for principal G-bundles $Q \to X \times \mathcal{S}^1$ on the 7-manifold $X \times \mathcal{S}^1$ such that $Q|_{X\times\{1\}} \cong P$. A loop $\gamma : \mathcal{S}^1 \to \mathcal{B}_P$ induces such a principal bundle Q, and the two possible square roots for $L_P|_\gamma \to \mathcal{S}^1$ correspond to the two possible orientations for $\mathcal{B}_Q$. So, being able to choose canonical orientations for all $\mathcal{B}_Q$ determines square roots $(L_P|_\gamma)^{1/2}$ for all loops γ in $\mathcal{B}_P$, and we show these can be assembled into a global square root $L_P^{1/2}$.

We hope in future to include orientation data in the bordism categories framework, but we haven't worked out the details yet.