Fundamental groups

A reference for the following is, for instance:

C.R.F. Maunder, *Algebraic Topology* (1970), Chapter 3.

Let X be a Hausdorff topological space. We are interested in continous maps $\gamma : [0,1] \to X$. Suppose $\gamma_0, \gamma_1 : [0,1] \to X$ are continuous maps with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. Define an equivalence relation $\sim$ on such maps by $\gamma_0 \sim \gamma_1$ if and only if there exists a continuous map $\Gamma : [0,1] \times [0,1] \to X$ such that

$$\Gamma(0,t) = \gamma_0(t),\ \Gamma(1,t) = \gamma_1(t),\ \Gamma(s,0) = \gamma_0(0) = \gamma_1(0),\ \text{and}\ \Gamma(s,1) = \gamma_0(1) = \gamma_1(1),$$

for all $s, t \in [0,1]$.

First, we say X is *connected* (or *path-connected*) if whenever $x, y \in X$, there exists a continous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Any topological space is a disjoint union of a number of connected topological spaces, whaich are called the *connected components* of X.

Now let X be a connected topological space. Choose a point $x_0 \in X$. Define a *based loop* in X to be a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. We define the *fundamental group* $\pi_1(X)$ of X to be the set of equivalence classes $[\gamma]$ of based loops γ in X under the equivalence relation $\sim$.

There is a natural group structure on $\pi_1(X)$, defined as follows. If γ_0, γ_1 are based loops in X, define

$$\gamma_0\gamma_1(X) = \begin{cases} \gamma_0(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma_1(2t-1) & \text{if } \frac{1}{2} < t \leq 1, \end{cases} \qquad \text{and} \quad \gamma_0^{-1}(t) = \gamma_0(1-t).$$

Then $\gamma_0\gamma_1$ and γ_0^{-1} are also based loops in X. Define group multiplication in $\pi_1(X)$ by $[\gamma_0] \cdot [\gamma_1] = [\gamma_0\gamma_1]$ and inverses in $\pi_1(X)$ by $[\gamma_0]^{-1} = [\gamma_0^{-1}]$. Define $\alpha : [0,1] \to X$ by $\alpha(t) = x_0$. Then α is a based loop in X, and $[\alpha]$ is the identity in $\pi_1(X)$. With these definitions, $\pi_1(X)$ is a group.

We say that a connected topological space X is *simply-connected* if $\pi_1(X) = \{1\}$. Not all connected topological spaces are simply-connected. Let X be a connected topological space, and let $\tilde{X}$ be the set of equivalence classes under $\sim$ of continuous maps $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$. Define a map $\pi : \tilde{X} \to X$ by $\pi\bigl([\gamma]\bigr) = \gamma(1)$. Then $\tilde{X}$ is called the *universal cover* of X. It is naturally a connected and simply-connected topological space. The map $\pi : \tilde{X} \to X$ is continuous and surjective, and is called the *covering map*. Also, there is a natural, free action of $\pi_1(X)$ on $\tilde{X}$, and $X \cong \tilde{X}/\pi_1(X)$.