Dominic Joyce **Differential Geometry** **Nairobi 2019**

Problem Sheet 2

1(a) Let X be the sphere $\mathcal{S}^n = \big\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1} : x_0^2+\cdots+x_n^2=1\big\}$. Explain why we may identify

$$T_{(x_0,\ldots,x_n)}\mathcal{S}^n \cong \big\{(y_0,\ldots,y_n)\in\mathbb{R}^{n+1} : x_0y_0+\cdots+x_ny_n=0\big\}.$$

(b) By identifying $\mathbb{R}^{2k+2}\cong\mathbb{C}^{k+1}$, show that any odd-dimensional sphere $\mathcal{S}^{2k+1}$ has a nonvanishing vector field $v\in C^\infty(T\mathcal{S}^{2k+1})$ (i.e. $v\neq 0$ at every point).

For discussion: can the same thing hold for even-dimensional spheres $\mathcal{S}^{2k}$?

2*. Let X^n be a *compact* n-dimensional manifold covered by coordinate neighbourhoods U_α, with coordinate maps $\psi_\alpha : U_\alpha\to\mathbb{R}^n$. Show by using compactness that there exists a finite set of bump functions $\varphi_1,\ldots,\varphi_k$ such that for each $x\in X$ at least one φ_i is identically 1 in a neighbourhood of x, and the support of each φ_i is contained in $U_{\alpha(i)}$ for some $\alpha(i)$.
Now define $F : X\to\mathbb{R}^{k(n+1)}$ by

$$F(x)=\big(\varphi_1,\ldots,\varphi_k,\varphi_1\psi_{\alpha(1)},\varphi_2\psi_{\alpha(2)},\ldots\varphi_k\psi_{\alpha(k)}\big)(x).$$

Show that:

(i) F is injective.

(ii) the derivative of F is injective at each point $x\in X$.

(iii) the manifold topology is the induced topology.

Deduce that $F : X\to\mathbb{R}^{k(n+1)}$ is an embedding.
(This is a special case of the Whitney Embedding Theorem.)

3. Let X be a manifold and $v, w \in C^\infty(TX)$. Suppose $(x_1, \ldots, x_n)$ are local coordinates on an open set $U \subseteq X$, so that we may write $v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$ and $w = w_1 \frac{\partial}{\partial x_1} + \cdots + w_n \frac{\partial}{\partial x_n}$ on U, for $v_i, w_j : U \to \mathbb{R}$ smooth. Define the Lie bracket $[v, w] \in C^\infty(TX)$ by

$$[v, w] = \sum_{i,j=1}^{n} \Big(v_i \frac{\partial w_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial}{\partial x_i} \Big) \qquad \text{on } U. \tag{1}$$

Prove that this is independent of choice of local coordinates. That is, if $(y_1, \ldots, y_n)$ is another local coordinate system on $V \subseteq X$, then (1) and its analogue for $(y_1, \ldots, y_n), V$ define the same vector field on $U \cap V$.

4. Find the vector fields u, v, w on $\mathbb{R}^2$ given by the following three one-parameter groups of diffeomorphisms:

- $\varphi_t(x_1, x_2) = (x_1 + t, x_2)$.
- $\varphi_t(x_1, x_2) = (x_1, x_2 + t)$.
- $\varphi_t(x_1, x_2) = ((\cos t)x_1 + (\sin t)x_2, (-\sin t)x_1 + (\cos t)x_2)$.

Show that the Lie bracket of any pair of u, v, w is a linear combination of u, v, w (with constant coefficients).

5. Let A be an $n \times n$ matrix and consider the vector field v in $\mathbb{R}^n$ defined by

$$v = \sum_{i,j} A_{ij} x_j \frac{\partial}{\partial x_i}$$

Use the exponential of a matrix:

$$\exp C = I + C + \frac{C^2}{2} + \cdots + \frac{C^n}{n!} + \cdots$$

to integrate this vector field to a one-parameter group of diffeomorphisms.