Problem Sheet 3

1. Take $\alpha \in \Lambda^k V$ where $\dim V = n$ and consider the linear map $A_\alpha : \Lambda^{n-k} V \to \Lambda^n V$ defined by $A_\alpha(\beta) = \alpha \wedge \beta$.

(i) Show that if $\alpha \neq 0$, then $A_\alpha \neq 0$.

(ii) Prove that the map $\alpha \mapsto A_\alpha$ is an isomorphism from $\Lambda^k V$ to the vector space $\mathrm{Hom}(\Lambda^{n-k} V, \Lambda^n V)$ of linear maps from $\Lambda^{n-k} V$ to $\Lambda^n V$. Thus if we choose an isomorphism $\Lambda^n V \cong \mathbb{R}$ we get isomorphisms $\Lambda^k V \cong (\Lambda^{n-k} V)^*$.

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (xy, yz, zx)$. Calculate $F^*(x \,\mathrm{d}y \wedge \mathrm{d}z)$ and $F^*(x \,\mathrm{d}y + y \,\mathrm{d}z)$.

3. Show that the product $X \times Y$ of two orientable manifolds is orientable.

4. Is $\mathcal{S}^2 \times \mathbb{RP}^2$ orientable? What about $\mathbb{RP}^2 \times \mathbb{RP}^2$?

5. A *Riemann surface* is defined as a 2-dimensional manifold X with an atlas $\{(U_i, \phi_i) : i \in I\}$ whose transition maps $\phi_j^{-1} \circ \phi_i$ for $i, j \in I$ are maps from an open set $\phi_i^{-1}(\phi_j(U_j))$ of $\mathbb{C} = \mathbb{R}^2$ to another open set $\phi_j^{-1}(\phi_i(U_j))$ which are *holomorphic* and invertible. By considering the Jacobian of $\phi_j^{-1} \circ \phi_i$, show that a Riemann surface is orientable.

P.T.O.

6*. The objective of this question is to prove that for all $n > 0$

$$H^k(\mathcal{S}^0) \cong \begin{cases} \mathbb{R}^2, & k = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad H^k(\mathcal{S}^n) \cong \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

You may assume the following facts from lectures:

- $H^0(X) \cong \mathbb{R}^N$, where N is the number of connected components of X.
- $H^0(\mathbb{R}^n) = \mathbb{R}$ and $H^k(\mathbb{R}^n) = 0$, $k > 0$.
- $H^k(X \times \mathbb{R}^n) \cong H^k(X)$ for all manifolds X and $k, n \geq 0$.

Define $U = \mathcal{S}^n \setminus \{(1, 0, \ldots, 0)\}$, $V = \mathcal{S}^n \setminus \{(-1, 0, \ldots, 0)\}$ and $W = U \cap V$. Then U, V, W are open in $\mathcal{S}^n$ with $\mathcal{S}^n = U \cup V$, and we have diffeomorphisms

$$U \cong \mathbb{R}^n, \qquad V \cong \mathbb{R}^n, \qquad W \cong \mathcal{S}^{n-1} \times \mathbb{R}.$$

If $B \subseteq A \subseteq \mathcal{S}^n$ are open, write $\rho_{AB} : \Omega^k(A) \to \Omega^k(B)$ for the restriction map. Then we have an *exact sequence*

$$0 \longrightarrow \Omega^k(\mathcal{S}^n) \xrightarrow{\rho_{\mathcal{S}^n U} \oplus \rho_{\mathcal{S}^n V}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\rho_{UW} \oplus -\rho_{VW}} \Omega^k(W) \longrightarrow 0.$$

(a) Suppose $\alpha \in \Omega^k(\mathcal{S}^n)$ for $k > 1$ with $\mathrm{d}\alpha = 0$. Show that there exist $\beta \in \Omega^{k-1}(U)$ and $\gamma \in \Omega^{k-1}(V)$ with $\alpha|_U = \mathrm{d}\beta$ and $\alpha|_V = \mathrm{d}\gamma$. Set $\delta = \beta|_W - \gamma|_W \in \Omega^{k-1}(W)$. Show that $\mathrm{d}\delta = 0$.

We have cohomology classes $[\alpha] \in H^k(\mathcal{S}^n)$ and $[\delta] \in H^{k-1}(W)$. Show that $[\delta]$ depends only on $[\alpha]$, not on the choices of α, β, γ. Thus we may define a linear map $\Phi : H^k(\mathcal{S}^n) \to H^{k-1}(W)$, $\Phi : [\alpha] \mapsto [\delta]$.

(b) Suppose $[\delta] = \Phi([\alpha]) = 0$. Then $\delta = \mathrm{d}\epsilon$ for $\epsilon \in \Omega^{k-2}(W)$. Prove that $[\alpha] = 0$ in $H^k(\mathcal{S}^n)$, so that Φ is injective.

Hint: Let $\{\eta_U, \eta_V\}$ be a partition of unity on $\mathcal{S}^n$ subordinate to $\{U, V\}$, and consider $\beta|_W - \mathrm{d}(\eta_V \epsilon)$ and $\gamma|_W + \mathrm{d}(\eta_U \epsilon)$ in $\Omega^{k-1}(W)$.

(c) Suppose $\delta \in \Omega^{k-1}(W)$ with $\mathrm{d}\delta = 0$. Show that we can choose α, β, δ in (a) giving this δ. Then $\Phi([\alpha]) = [\delta]$, so that Φ is surjective.

Hint: Choose α, β, γ such that $\alpha|_W = \mathrm{d}\eta_V \wedge \delta = -\mathrm{d}\eta_U \wedge \delta$.

(d) Use (a)–(c) and the facts above to show $H^k(\mathcal{S}^n) \cong H^{k-1}(\mathcal{S}^{n-1})$ if $k > 1$.

(e) What goes wrong in part (a) if $k = 1$? Adapt your arguments to show that $H^1(\mathcal{S}^1) \cong \mathbb{R}$, and $H^1(\mathcal{S}^n) = 0$ for $n > 1$.

(f) Deduce (1) by induction on n.